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TACHYON CONDENSATION IN STRING FIELD THEORY:

the tachyon potential in the conformal field theory approach

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"It does seem that these physicists go to some effort and expense to state what seems obvious."

William S. Burroughs

Dankwoord

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Chapter 1

Introduction

1.1 String theory

Unification

One of the most important concepts in the development of theoretical physics is the search for *unification*. The unified description of diverse physical phenomena in a single theory has in many cases proven to be more than just aesthetically appealing: it has often led to deep new insights in the fundamentals of physics. For example, Maxwell's unified description of electric and magnetic phenomena in the theory of electromagnetism lay at the heart of Einstein's formulation of the special theory of relativity in the beginning of the previous century.

This search for unification culminated in the formulation of the *Standard Model* of elementary particles and interactions in the seventies. This model successfully describes the known elementary particles as well as their interactions due to electromagnetic as well as weak and strong nuclear forces. Moreover, the model offers a microscopic description in accordance with the laws of quantum mechanics and has been verified in a huge number of experiments.

The main obstacle to the continuation of the ambitious project of unifying all known particles and forces into a single theory was the incorporation of the gravitational force. This force is described very elegantly by Einstein's theory of *general relativity* which has also been confirmed experimentally. This

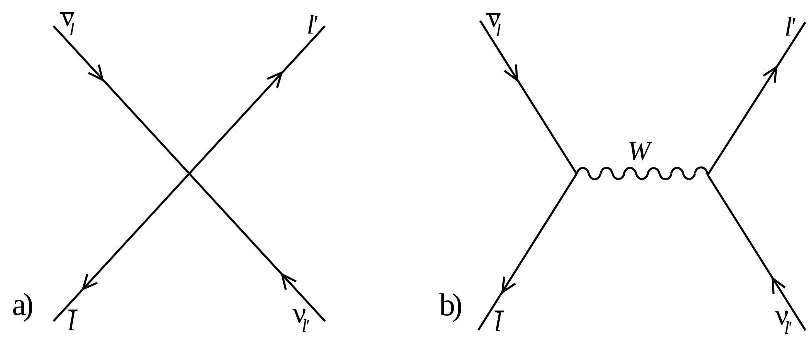


Figure 1.1: (a) A weak interaction process in the original four-fermi description. (b) The same process as described by the Standard Model: the interaction is 'smeared out' by the exchange of a W-boson.

theory, however, is in essence a classical theory, and is reliable only for the description of phenomena on a macroscopic scale. Fundamental problems arise when one attempts to apply the laws of quantum mechanics to the theory of general relativity: in technical terms, general relativity is a non-renormalisable theory, which means that applying quantum-mechanical perturbation theory leads to uncontrolled divergences in scattering amplitudes.

Non-renormalisability and effective theories

Let us illustrate the problem of non-renormalisability in a different example, the *four-fermi theory of weak interactions.* The weak interaction was originally described as an interaction of four fermionic fields at a spacetime point as depicted in figure 1.1(a). The strength of this interaction is determined by a coupling constant G_F which has the dimension[1] of $[\text{energy}]^{-2}$. This means that, in a process with characteristic energy E, the effective dimensionless coupling is of order $G_F E^2$. This coupling becomes arbitrarily large at high energies and leads to divergences in loop amplitudes which, in contrast to the infinities arising in renormalisable theories, cannot be absorbed in a redefinition of the physical parameters of the theory[2].

[1] We are working in units with $c = \hbar = 1$.

[2] More correctly, the removal of infinities in a nonrenormalisable theory would require the introduction of an *infinite* number of physical parameters whose values would have to be deter-

We have already mentioned that the Standard Model does give a good quantummechanical description of the weak interaction, and it is worthwhile to reflect on how the problem of nonrenormalisability in the four-fermi model is solved there. In figure 1.1(b) we depicted the same process as described by the Standard Model. Here we notice that, when we look at the process at a sufficiently small scale, the four-fermion interaction actually involves the exchange of a new particle, the W-boson. In this way, the interaction of the original theory is smeared out and the high energy (= small distance) divergences of the four-fermi model are absent. The original four-fermi model does however give a good approximation at sufficiently small energies (= sufficiently large distances). This is summarised in the statement that the four-fermi theory gives an *effective description* of the physics at low energies.

This example illustrates how non-renormalisable theories have come to be regarded in recent times: non-renormalisability is interpreted as a signal that the theory in question provides an effective description of a more fundamental theory. The latter theory should contain new degrees of freedom (such as the W-boson in our example) which come into play at sufficiently high energies and whose presence 'smears out' the interactions that gave rise to high-energy divergences.

String theory: a fundamental description of gravity

We return now to the case of general relativity. Here as well, the coupling constant of the theory, Newton's constant G_N, has the dimension of $[\text{energy}]^{-2}$. The same problems of non-renormalisability arise here and one expects that, here as well, one has to do with an effective description of a more fundamental theory. The search for a fundamental description of gravity in terms of a 'traditional' theory of interacting particles has failed, however. It therefore seemed likely that such a description would require a radical new approach. Such a radical new idea, and still the only known one that solves the aforementioned problems, came, now more than thirty years ago, to the attention of the scientific community in the form of *string theory*[3].

The fundamental objects in string theory are not particles, but submicroscopic vibrating strings. Seen from a sufficiently large distance, these vibrat-

mined experimentally. This of course severely restricts the predictive power of such theories.

[3]Although string theory was originally proposed as a model for the strong interaction, it was soon realised that its true promise lay elsewhere, as a candidate for the fundamental description of gravity.

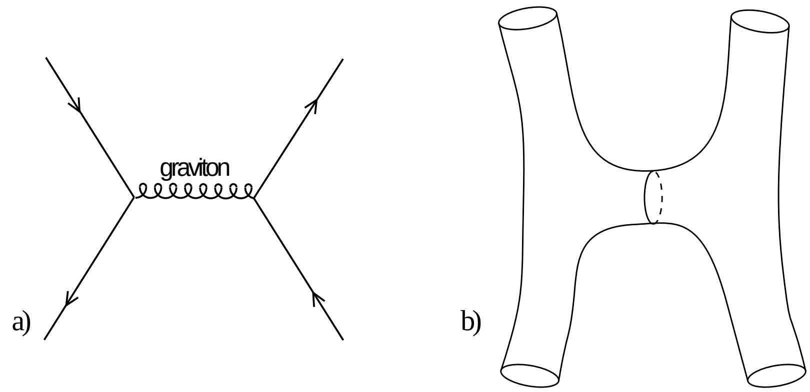


Figure 1.2: (a) A scattering process involving the exchange of a graviton. (b) The same process as described in string theory: the interaction is 'smeared out' as a consequence of replacing particles by strings.

ing strings look like particles, and each vibrational mode of the string corresponds to a particle with definite physical properties. Many particle theories can in this way be regarded as effective descriptions of a more fundamental theory of strings. One of the most important results in string theory is that one of the vibrational modes of the string has exactly the right properties to be identified as the particle which carries the gravitational force, the *graviton*. Hence string theory became a candidate for the fundamental theory for which general relativity provides an effective description (see figure 1.2) and, as it turned out, string theory does satisfy the main requirement for such a theory: it provides a perturbation expansion which is finite order by order.

String theory and unification

Besides gravity, string theory also contains a number of other ingredients which were proposed during the course of the twentieth century in the search for a unified description of nature. Here we just list the most prominent ones:

- String theory contains many other excitations besides the graviton, and some of those have the right properties to be identified as the matter particles and gauge bosons (which can be seen as the carriers of the various forces) of the Standard Model. In string models (see e.g. [1]), the

Standard Model symmetry group, $SU(3)\times SU(2)\times U(1)$ [2, 3], is typically embedded in a larger symmetry group such as $SU(5)$ or $SO(10)$. In this way, string theory unifies gravity with the Grand Unified Theory (GUT) extensions of the Standard Model proposed in the seventies.

- String theory also incorporates the concept of *extra spatial dimensions*. Indeed, the known consistent string theories require the presence of 9 spatial dimensions. This idea is not in contradiction with experimental data as long as 6 of these dimensions are 'curled up' to a sufficiently small size, and, in fact, this mechanism was proposed by Kaluza and Klein [4] as a mechanism for unification of gravity and electromagnetism as early as the 1920's. Traditionally, it was assumed that these extra dimensions would be of the size of the Planck length, $l_P = G_N^{-1/2} = 1.6\times 10^{-33}$ cm. Recently, it has been realized that dimensions can be much larger, up to the order of 1 mm, without conflicting with observations [5].

- A third important ingredient in string theory is *supersymmetry*, a symmetry which exchanges bosonic and fermionic degrees of freedom and which is present in all known consistent string theories. Supersymmetry was also proposed before as a possible extension of the Standard Model, amongst other reasons because it gives a natural explanation for difference in magnitude between the typical scales of Standard Model physics and the Planck scale l_P (see e.g. [6] for a recent discussion).

1.2 Why string field theory?

In the previous section we saw that string theory, in its conventional formulation, provides a formalism in which scattering amplitudes involving gravitons and and other string excitations can be calculated perturbatively. We therefore denote this conventional formulation of the theory, which has been extensively studied in the literature, by *string perturbation theory*. A number of applications however, one of which is the main subject of this thesis, require a more extensive formalism, that of *string field theory*. In order to understand the reasons for introducing such a formulation, we should first reflect on the limitations inherent in the formulation of perturbative string theory. Here too, we can gain some insight by comparing with the situation in elementary particle physics.

Field theory versus S-matrix theory

The natural framework for the description of interactions between elementary particles is *quantum field theory*, where particles are described as fluctuations of a *quantum field*. Quantum field theory gives a description of particle interactions, including those in which particles are created or annihilated, in a manner which is consistent with the requirements of quantum mechanics and special relativity. In particular, the formalism leads to a perturbative method for calculating transition amplitudes between initial and final states, each containing a definite number of particles. Such transition amplitudes are called *S-matrix*[4] elements and form the basis for the calculation of measurable quantities such as decay rates and cross-sections.

In this perturbative framework, S-matrix elements are approximated by a sum of contributions, each of which can be represented by a *Feynman diagram* (examples of such diagrams were already encountered in figures 1.1 and 1.2). These diagrams are built up according to a set of *Feynman rules* which can be read off in a straightforward manner from the quantum field theory action. We give a simple example: the theory of a scalar field ϕ with the action:

$$S[\phi] = \int d^4x [\frac{1}{2}\phi(\partial_\mu\partial^\mu - m^2)\phi + \frac{g}{3!}\phi^3]. \tag{1.2.1}$$

The first term in the action gives rise to the *propagator*, which represents free propagation of the particle and is represented in Feynman diagrams by a straight line (figure 1.3(a)), while the second term describes interactions in which two particles annihilate and a third particle is created. This interaction is represented in Feynman diagrams by a three-point *interaction vertex* (figure 1.3(b)). Hence, in elementary particle theory, the perturbative S-matrix expansion follows naturally from the field theory description. In perturbative string theory, the situation is more or less the reverse: here one has at one's disposal a perturbative expansion for the S-matrix but it is not a priori clear whether this expansion has a field-theoretic origin. The goal of string field theory is to provide this field-theoretic framework by introducing *string fields*, whose fluctuations correspond to strings, and by proposing a *string field theory action* for these fields which, through the resulting set of Feynman rules, generates the string theory S-matrix expansion.

[4] In fact, the physical restrictions on S-matrix elements, such as Lorentz invariance , unitarity and causality, are so stringent that they naturally lead to the introduction of quantum fields [18].

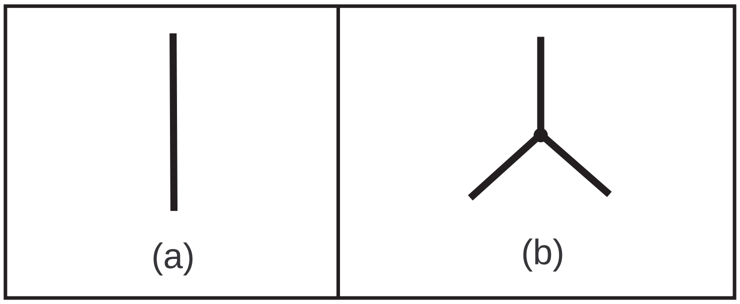


Figure 1.3: Feynman diagrams in scalar ϕ^3 theory are built up out of the elementary building blocks of (a) the propagator and (b) a three-point interaction vertex.

The advantages of a field theory description

A natural question to ask in this context is whether we can learn anything new from such a description. If all physically relevant questions could be answered within the framework of perturbative string theory, the introduction of string field theory would not be a necessity but merely amount to an alternative description of the same phenomena.

In particle theory, many applications are known for which the perturbative S-matrix description is insufficient but which can adequately be described by quantum field theory. We give a few examples:

- First of all, a perturbation expansion is an expansion in a coupling constant like the parameter g in the example 1.2.1, which is supposed to be small. To make matters worse, such perturbation expansions are generically (and, presumably, in the case of string perturbation theory as well), not convergent, but rather asymptotic series for small g. Such asymptotic series can give a very good approximation for small values of g, but can, for larger values of g, miss important aspects of the physics of the model. Examples include the description of *solitonic objects*, such as magnetic monopoles in gauge theories, which typically have a mass inversely proportional to g. Another example are the *instanton corrections* to certain processes, who are typically of order $e^{-1/g}$.

- For the description of processes in the presence of background fields as well as for the calculation of quantum corrections to the classical action, one needs to be able to calculate more general amplitudes than the ones incorporated in the S-matrix, the so-called *off-shell amplitudes*.
- The S-matrix approach also fails in the description of *collective phenomena* which involve large numbers of particles. An example of such a phenomenon, which plays a crucial role in the formulation of the Standard Model, is the Brout-Englert-Higgs-Kibble effect, in which a condensate of scalar particles is formed that is responsible for the generation of particle masses in the Standard Model.

The ambition of string field theory is to provide a framework to describe similar effects in string theory. Remarkably, in string theory, a number of of nonperturbative issues have found a description in string theory without going beyond perturbation theory. What happens here is that string theory at large coupling g is equivalent or *dual* to a different string theory at small coupling g' proportional to $1/g$[5]. By this mechanism, it becomes possible to study the strong coupling physics of a given string theory by doing perturbation theory in the dual theory.

Although string field theory has played little or no role in the discovery of these aspects of nonperturbative string theory, it has proven a necessary tool in a number of applications. One of these applications, tachyon condensation, will be the main subject of this thesis.

Witten's string field theory

The most successful string field theory to date was proposed by Witten in 1986 [8] and describes the interactions between open strings. The action has the following form:

$$S[\Psi] = \int [\frac{1}{2}\Psi \star Q\Psi + \frac{g}{3}\Psi \star \Psi \star \Psi] \tag{1.2.2}$$

Without going into the precise meaning of all the symbols (this will be the subject of chapter 3), we would like to discuss briefly the physical meaning of the two terms in the action. Comparing to the action (1.2.1) for the scalar

[5] This type of duality goes under the name of *S-duality* A different type of duality relation in string theory, called *T-duality*, was studied in our paper [76].

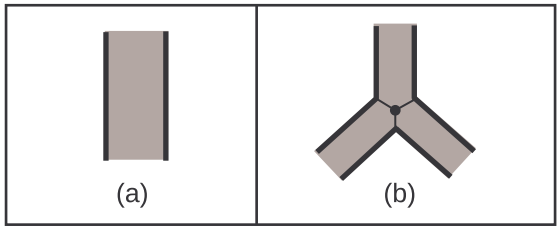


Figure 1.4: Feynman diagrams in open string theory are built up out of (a) a propagator and (b) a three-string interaction vertex.

field, we notice a number of similarities: the scalar field ϕ is replaced by the string field Ψ, the kinetic energy operator by the as yet undefined operator Q, and the multiplication and integration operations are by similar operations $\star$ and $\oint$ on string fields. The meaning of both terms in the action 1.2.2 is also comparable to the meaning of the terms in (1.2.1): the first term describes free string propagation (figure 1.4(a)) while the second term describes interactions in which two strings join to form a third string (figure 1.4(b)).

1.3 Unstable objects in string theory

D-branes

Of great importance in the recent developments in string theory has been the realization that string theory is more than a theory of strings alone: it also contains other extended objects, the so-called *D-branes*. These D-branes can come in various dimensionalities, and one distinguishes between the D-particle, the D-string, the D-membrane, D-3-brane and so on.

Despite the fact that D-branes are essentially a part of closed string theory, open strings play an important role in their description, for a D-brane can be seen as a hypersurface on which open strings can end. These open strings correspond to fluctuations of the D-brane.

The D-branes in closed superstring theory have an important physical property: they carry a certain type of charge, called Ramond-Ramond charge.

This property is closely related to the presence of supersymmetry: the fact that D-branes carry such charge identifies them as objects in whose presence the theory remains invariant under a number of supersymmetries. Such objects are also called *BPS-objects*. An important consequence of the BPS property is that D-branes are *stable* objects that cannot decay into lighter states.

Tachyons and instabilities in field theory

Before we turn to the discussion of unstable objects in string theory, we briefly mention some facts concerning instabilities in quantum field theory.

In quantum field theory, instabilities of the system are signalled by the presence of *tachyons* in the perturbative spectrum of the theory. By the term tachyon we mean a fluctuation of the field which, should one insist on giving it a particle interpretation, would represent a particle with imaginary rest mass (and, hence, moving at a speed greater than the speed of light). A small perturbation of such a system initiates a decay process to a stable configuration. This process is called *tachyon condensation* because, in the course of this process, a condensate of scalar particles is formed. The presence of such a condensate has important consequences for the physics of the model; we already mentioned in the previous section that such a condensate is responsible for the generation of particle masses in the Standard Model.

Unstable branes and Sen's conjecture

Besides the aforementioned stable branes, string theory contains *unstable branes* as well. Their existence was first established by Sen. The instability of these objects is, just as in field theory, signalled by the fact that the spectrum of fluctuations contains a tachyon. Here too, the system will decay to a stable configuration under tachyon condensation. An important question concerns the nature of the final state in this decay process.

Based on a number of arguments involving string dualities, Sen has proposed a hypothesis which provides the motivation for the work presented in this thesis. According to *Sen's conjecture*, the end product of tachyon condensation is the vacuum of closed string theory, or, in other words, the unstable brane 'decays to empty space'.

If this hypothesis is true, then all the energy contained in the mass of the unstable brane is used up in the process of tachyon condensation. This can be the case only if the potential for the tachyon satisfies the following requirement: the difference between the value of the potential at its maximum (which corresponds to the unstable brane) and the value at its minimum (which describes the stable configuration) should be precisely equal to the tension of the unstable brane.

The determination of the tachyon potential in string theory hence provides a concrete test of Sen's conjecture. Furthermore, such a calculation is out of reach for string perturbation theory and it has to be performed in the formalism of string field theory. This provides us with an excellent opportunity to test string field theory in a concrete application. Indeed, before string field theory was used to verify Sen's conjecture, it was widely believed that it had failed to achieve its goals because of its inability to produce results that were not already obtained in string perturbation theory. The study of the tachyon potential in string field theory, which is the subject of the present work, has, in our opinion, to a certain extent invalidated this point of view.

1.4 Outline of the thesis

Having provided the general context in which our work is situated, we now give an outline of the topics studied in the thesis.

Chapter 2 is devoted to perturbative string theory. A first purpose of this chapter is to familiarise the non-expert reader with the basic setup of string perturbation theory. For this reason, we have included a discussion of the Polyakov action in section 2.1 and of the construction of the string theory S-matrix in section 2.2 . Section 2.4 treats the physical spectrum of the bosonic string in the formalism of BRST-quantisation. Section 2.5 discusses the extension to superstrings, leading to the analysis of the physical spectrum in 2.5.5. We end by giving an overview of the known consistent string theories in 2.5.6.

We have also included in chapter 2 a number of technical developments that are necessary for a complete understanding of the technical aspects of chapters 3–7. These include the discussion of some tools in two-dimensional conformal field theory in section 2.3 and the treatment of some technicalities involving the ghosts in superstring theory in 2.5.4 and 2.5.5.

Chapter 3 deals with the formulation of string field theory. After introducing the concept of a string field in 3.1.1, we turn to a discussion of string field theory in Witten's original formulation, which makes contact with the intuitive notion that string interactions arise from the splitting and joining of strings, in sections 3.1.2 and 3.1.3. In section 3.1.4 we derive an equivalent formulation of string field theory in terms of conformal field theory correlators, which is more suited for performing concrete calculations and which will be used extensively in chapters 5 and 7. The derivation in 3.1.4 is based on the path-integral representation of the string field and has not appeared before in the literature. Although, in this thesis, we will be concerned with the classical aspects of string field theory, we comment on the quantisation of the theory in 3.1.7. In section 3.2, we discuss superstring field theory. Three different proposals for such a theory have been made in the literature, all of which are reviewed in this section. We conclude the chapter by listing some imperfections and open problems in the current formulation of string field theory (section 3.3).

In chapter 4, we provide the motivation for our work on the subject of tachyon condensation and place it in a broader context.

In section 4.1, we give some important examples of tachyon condensation in quantum field theory. Sections 4.2-4.4.2 give an overview of stable and unstable D-branes in string theory and their role in the context of string dualities. In sections 4.4.3 and 4.4.4, we formulate Sen's conjecture concerning the tachyon potential on unstable D-branes, and argue that the verification of this conjecture should take place in string field theory.

In chapter 5, we discuss tachyon condensation in bosonic string field theory. Although this chapter does not contain new results, we do make an effort to tie up some loose ends in the existing literature. At the same time, the bosonic model provides an ideal setting to illustrate some basic principles in a relatively simple context, which can later be transferred to the technically more involved case of superstrings.

In sections 5.1 and 5.2, we argue that the tachyon potential is a universal function independent of certain details of the model. Section 5.3. introduces the calculational method that will be used in all calculations in this thesis and goes under the name of level truncation. In sections 5.5 and 5.6, we review the calculation of the tachyon potential in the level $(4, 8)$ approximation, and find that the behaviour of the potential is in very good agreement with the predictions of Sen's conjecture. We end the chapter with a discussion of the physics described by the model after tachyon condensation in section 5.7, including the hypothesis that the model could give a description of closed strings. We consider this speculative idea to be one of the most promising directions for future research in this field.

In chapter 6, we make an effort to put the level truncation method, which is used in most string field theory calculations, on a sounder footing. This method gives a series of successive approximations that, in practice, has turned out to converge rapidly to the exact result, but a priori arguments for this fact are largely nonexistent. Therefore, we consider level truncation in a simplified model that is inspired by the full string field theory, where we are able to obtain exact results. These results have not appeared before in the literature.

In sections 6.1 and 6.2, we introduce the model and derive the resulting equations for the condensate. In section 6.3, we obtain an exact solution to the problem for a specific choice of the parameters of the model. In section 6.4, this exact solution is compared to the level truncation approximation, and it is shown that this approximation provides an algorithm that converges to the exact result in a manner which is exponential as a function of the level. This behaviour is in agreement with the one found 'experimentally' in the full string field theory.

In chapter 7, we present our results concerning the study of the tachyon potential in superstring field theory. We have already seen that there exist three different proposals for superstring field theory in the literature. The study of the tachyon potential in these theories should be seen in a different light from the analogous study in the bosonic model: in the superstring case, Sen's conjecture has been put on a firmer basis while it is less clear a priori which string field theory description is the correct one. The study of the tachyon potential should here be seen as a test of the string field theory description, rather than a test of Sen's conjecture.

In section 7.1, we discuss the behaviour of the tachyon potential in Witten's string field theory. These results have appeared in our paper [115]. We find that the tachyon potential in this theory has no minimum and conclude that there is no agreement with the behaviour predicted by Sen's conjecture. Section 7.2 treats the tachyon potential in modified cubic string field theory. Here too, our results force us to the conclusion that there is no agreement with the expected behaviour. In section 7.3, which collects the results of our paper [118], we study the tachyon potential in the theory proposed by Berkovits. The results in this theory turn out to be in good agreement with the predictions from Sen's conjecture.

Chapter 2

String perturbation theory

In this chapter, we will attempt to give an overview of string perturbation theory. The treatment will be rather sketchy at times, the main objective is to establish notation and emphasise those results that are needed for the discussion of our own work in the next chapters. As in most modern texts on the subject, our approach will rely heavily on the pioneering work [13] stressing the role of two-dimensional conformal field theories as the building blocks of string theory. The reader interested in studying a more self-contained introduction to the subject is referred to the textbooks by Polchinski [9], from which most of our conventions and notations are taken. Other good introductions to the subject are [10, 11, 12]. Good reviews of conformal field theory in two dimensions can be found in [14, 15].

In the following chapters, extensive use will be made of some technical aspects of conformal field theory presented in this chapter. The most important ones are: the state-operator mapping and the behaviour of vertex operators under finite conformal transformations (section 2.3.2), the state-operator mapping (section 2.3.3), the BPZ inner product (section 2.3.5), and the discussion of bosonised superghosts and pictures in (section 2.5.4).

2.1 Polyakov action

We want to describe the classical dynamics of a string moving in a space-time which we take to be D-dimensional. The motion of the string in space and time traces out a $1 + 1$ dimensional volume called the *world-sheet*. A natural action principle, that goes under the name of the *Nambu-Goto* action, is obtained by requiring that the motion is such that the world-sheet has minimal surface; this generalises the action principle for a massive point particle. Denoting the coordinates on the worldvolume M by σ^a, $a = 0, 1$, the coordinates of space-time by X^μ, $\mu = 0 \ldots D-1$ and the space-time metric (of signature $(- + + \ldots +)$) by $G_{\mu\nu}$ the action takes the form:

$$S_p[X^\mu] = -T \int_M d^2\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X^\nu G_{\mu\nu})}. \qquad (2.1.1)$$

The parameter T, which generalises the point-particle mass, is the *string tension*. The Nambu-Goto action has a global invariance under the isometries of space-time and is also invariant under local reparametrisations of the world-sheet.

Although it has a very simple physical interpretation, the fact that the action (2.1.1) is nonpolynomial in the fundamental variables makes it very hard to use as a starting point for constructing a quantum theory of strings.

It is possible to write down an action which is classically equivalent to (2.1.1) and which is quadratic in the embedding coordinates X^μ at the cost of introducing an auxiliary field. This field has the interpretation of a metric tensor on the world-volume and will be denoted by g_{ab}. The resulting action is the so-called *Polyakov action*:

$$S[X^\mu, g_{ab}] = -\frac{1}{2} T \int_M d^2\sigma \sqrt{-g} g^{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu} \qquad (2.1.2)$$

Upon the elimination of g_{ab} through its algebraic field equation one recovers (2.1.1). By introducing an auxiliary world-sheet metric, one has also introduced a new local symmetry: apart from invariance under global space-time isometries and local world-sheet reparametrisations, the action (2.1.2) is now also invariant under local rescalings of the metric; this symmetry is called *Weyl invariance*. The fact that there are as many independent local symmetries as there are independent components in g_{ab} means that, at least locally, gauge invariance can be used to bring g_{ab} into any preferred form. This will be important when we discuss gauge-fixing in section 2.2.2. It will not be possible to preserve both diffeomorphism and Weyl invariance in a quantum theory based on (2.1.2) in a general background. In particular, the preservation of these symmetries will put a restriction on the number of space-time dimensions. We will come back to this point in section 2.4.

2.2 String perturbation theory

2.2.1 Sum over histories

In the quantum mechanics of particles, it is possible to represent the propagator, i.e., the amplitude for the transition from one position to another, by a sum over particle histories that interpolate between the initial and final positions [16]. Each history is weighed by a factor $e^{iS_{cl}}$ where S_{cl} is the classical action for the history. Analogously, in string theory, one can represent transition amplitudes as a sum over string world-sheets. The sum-over-histories approach to string theory has the advantage that it naturally allows for the inclusion of interactions. This procedure leads to a perturbation expansion for scattering amplitudes between physical states; this expansion forms the basis of *string perturbation theory*. String perturbation theory does have its limitations however. In the theory of relativistic particles, the natural framework for the description of particle interactions is quantum field theory, which contains much more information than the perturbation expansion for S-matrix elements. A similar attempt to overcome the restrictions of string perturbation theory is provided by *string field theory*. In chapter 3, we will discuss classical string field theory, while in chapters 4, 5 and 7, we will discuss how string field theory can be used to obtain certain results that cannot be calculated in the framework of string perturbation theory.

We will now take a look at the quantisation of strings in a D-dimensional Minkowski space-time by summing over string world-sheets. Therefore, we take $G_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(- + \ldots +)$. The global symmetries of the action form the Poincaré group in D dimensions. It is also customary to analytically continue the world-sheet metric to Euclidean signature $(+,+)$. Contrary to the analytical continuations performed in quantum field theory, this step is hard to justify rigorously in the context of string theory[1], and the Euclidean theory is taken here simply as a starting point. Additional confidence is gained from the fact that this starting point leads to consistent quantum theory with transition amplitudes satisfying necessary physical properties such as unitarity.

In the path-integral approach to string theory, amplitudes for string propagation can be represented by a sum over all world-sheets interpolating between the initial and final string configurations. In the Euclidean theory, each

[1]The main problem in providing such a justification is that a Minkowski metric on a world-sheet with holes is necessarily singular and it is not known how to handle the integration over such singular metrics.

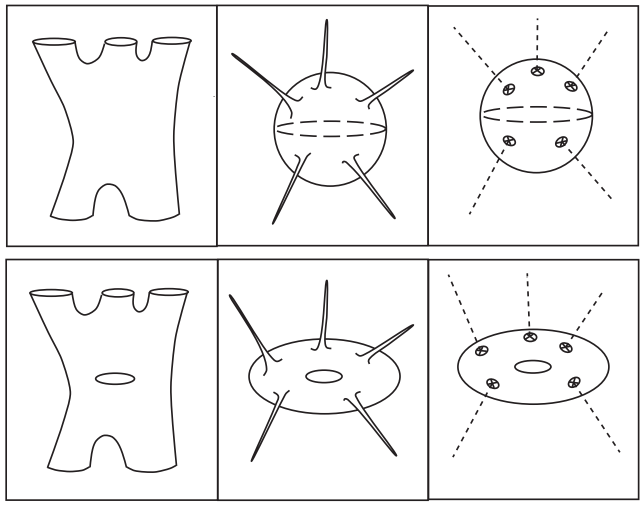


Figure 2.1: Tree level (upper figure) and 1-loop (lower figure) contributions to a closed string scattering amplitude.

world-sheet is weighed by a factor e^{-S_P} with the Polyakov action S_P given by:

$$S_P[X^\mu, g_{ab}] = \frac{1}{4\pi\alpha'} \int_M d^2\sigma \sqrt{g} \left[g^{ab} \partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \right], \qquad (2.2.1)$$

where, in keeping with tradition, we have written the string tension as $T \equiv \frac{1}{2\pi\alpha'}$. Scattering amplitudes between general string states are represented by a path integral with wave-function insertions for the asymptotic incoming and outgoing states.

As an example, consider the scattering amplitude for a process in which two closed strings join and split again into three strings in the final state. Figure 2.1 shows two such contributions to this process: Figure (2.1 (a)) represents a tree diagram and figure (2.1 (b)), where the world-volume has a hole, represents a 1-loop contribution to this process. Using the diffeomorphism and Weyl invariance of the theory, it is possible to deform these world-sheets into the equivalent representation of a sphere and a torus respectively where the external string states are reduced to marked points or punctures represented by a $\otimes$ in figure 2.1. After performing this transformation, the

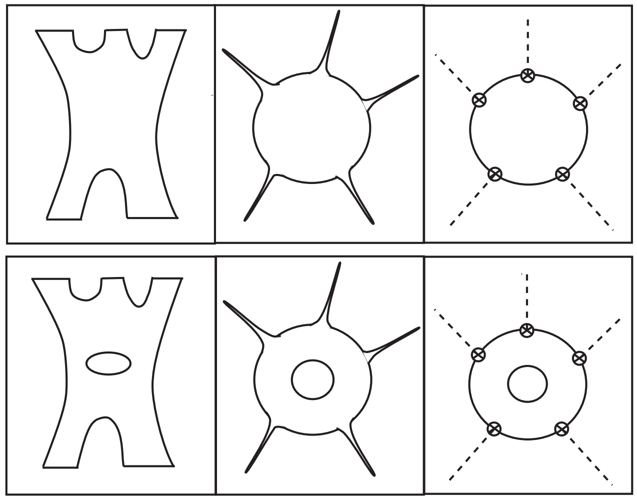


Figure 2.2: Tree level (upper figure) and 1-loop (lower figure) contributions to an open string scattering amplitude.

wave-function insertions representing the asymptotic states have become insertions of *local* operators at the marked points. These local operators representing asymptotic string states are called *vertex operators*. The marked points can occur at any point of the world-sheet, so the vertex operators should be integrated over the world-sheet.

A similar reasoning applies to open string amplitudes as well. This is illustrated in figure 2.2. Suitably chosen Weyl transformations reduce the world-sheet to a disc (at tree level) or an annulus (at 1-loop level) with vertex operators inserted on the boundary. These vertex operators should be integrated over the boundary.

Since a closed world-sheet containing h holes (h is called the *genus*) represents a closed string diagram with h loops, we see that the sum over world-sheets naturally represents an expansion in string loops, the sum over all world-sheets of genus h representing the contribution of diagrams with h loops. To interpret this series as a perturbation series, higher loop contributions should be accompanied by increasing powers of a (preferably small) *coupling constant*. In string theory, this is accomplished by adding an extra renormalisable term to the action:

$$S = S_P + \Phi_0 \chi \tag{2.2.2}$$

Φ_0 is a constant parameter and the quantity χ is the Euler number:

$$\chi = \frac{1}{4\pi} \int_M d^2\sigma \sqrt{g} R + \frac{1}{2\pi} \int_{\partial \mathrm{M}} ds\, k. \tag{2.2.3}$$

where k denotes the geodesic curvature of the boundary. The Euler number is a topological invariant so the extra term in the action respects diffeomorphism and Weyl invariance. For a compact surface obtained from the sphere by adding h handles and b boundaries the Euler number equals $2 - 2h - b$. By adding a loop to a closed string diagram, the path-integral weight factor $e^{-\Phi_0 \chi}$ changes by $e^{2\Phi_0}$, so the closed string coupling constant g_c should be proportional to e^{Φ_0}. For open strings, adding a loop means adding a boundary so the open string coupling g_o is proportional to $e^{\Phi_0/2}$.

Summarising, we can now tentatively write an expression for the scattering amplitude for a process involving n external string states:

$$S(1;\ldots;n) = \sum_{\text{compact topologies}} \int \frac{[dg][dX]}{\mathrm{Vol}_{\mathrm{diffxWeyl}}} e^{-S[X,g]} \prod_{i=1}^{n} V_i, \tag{2.2.4}$$

where we have denoted the vertex operator, (integrated over the world-sheet) corresponding to the ith external state by V_i. We have included a formal division by $\mathrm{Vol}_{\mathrm{diffxWeyl}}$ to indicate that we still have to make up for the overcounting of configurations (X, g) and (X', g') related by local diffeomorphism and Weyl transformations. This will be made more concrete in the following subsection.

2.2.2 Gauge-fixing

The standard way to deal with the overcounting of gauge equivalent configurations is by the introduction of *Fadeev-Popov ghosts*. We will use the gauge freedom to bring the world-sheet metric into a preferred form referred to as the *conformal gauge*; other gauge-fixings are of course possible and are sometimes used in the literature.[2] We will discuss the procedure for the tree-level

[2]For example the *light-cone gauge* where one fixes the form of one of the embedding coordinates.

contribution to the string S-matrix and then briefly touch upon the generalisation to higher loop diagrams. For closed strings the tree level contribution comes from surfaces with the topology of a sphere ($\chi = 2$). We have already remarked that the number of independent gauge transformations equals the number of independent components of the metric. This allows us to bring the metric locally into any preferred form; for example, one could locally choose a flat metric.

This is not possible globally, however: indeed, substituting a globally flat metric in (2.2.3) would give $\chi = 0$ which is incorrect. The closest we can come to a flat metric by using globally defined diffeomorphism and Weyl transformations is the standard Poincaré metric on the sphere (see e.g. [17], chapter IV) which is of the conformally flat form

$$ds^2 = \frac{dzd\bar{z}}{(1+|z|^2)^2}, \tag{2.2.5}$$

where the complex coordinates $z = \sigma^1 + i\sigma^2$, $\bar{z} = \sigma^1 - i\sigma^2$ run over the complex plane $\bar{\mathbf{C}}$ with the point at infinity included. In this gauge, the Polyakov action simplifies to

$$S_P = \frac{1}{2\pi\alpha'}\int d^2z\, \partial X^\mu \bar{\partial} X^\nu \eta_{\mu\nu}. \tag{2.2.6}$$

A straightforward application of the Fadeev-Popov procedure (see e.g [18]) leads to the introduction of anticommuting ghost fields: a traceless symmetric 2-tensor b_{ab} and a vector c^a [3] . The action for the ghosts takes the form

$$S_{gh} = \frac{1}{2\pi}\int d^2z \left[b\bar{\partial}c + \tilde{b}\partial\tilde{c}\right], \tag{2.2.7}$$

where we have used the abbreviations $b \equiv b_{zz}$; $c \equiv c^z$ and $\bar{b} \equiv b_{\bar{z}\bar{z}}$; $\bar{c} \equiv c^{\bar{z}}$.

The choice of gauge (2.2.5) does not fix the gauge freedom completely; indeed, the coordinate transformations acting as

$$z \to f(z) = \frac{az+b}{cz+d}, \quad \bar{z} \to \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}; \tag{2.2.8}$$

[3] Actually, one should also introduce a ghost field for the Weyl transformations; it couples to the trace part of b_{ab}. These fields turn out to be auxiliary (no derivatives acting on them) and can be eliminated (see e.g. [19]).

have the effect of rescaling the metric (2.2.5) and can be compensated by a Weyl transformation. The parameters a, b, c, d are complex numbers satisfying $ad - bc = 1$. These transformations form a group, the *conformal Killing group*, isomorphic to $SL(2, \mathbf{C})/\mathbf{Z}_2$. The overcounting due to these remaining symmetries can be remedied by using them to fix the position of three of the vertex operators in (2.2.4).

The result for the closed string tree level amplitudes is:

$$\begin{aligned} S_{\text{tree}}(1;\ldots;n) &= g_c^{-2}\prod_{i=4}^{n}\int d^2z_i\int[dXdbdc]e^{-S_{\text{P}}-S_{\text{gh}}}\prod_{j=1}^{n}\mathcal{V}_j(z_j,\bar{z}_j) \\ &\equiv g_c^{-2}\prod_{i=4}^{n}\int d^2z_i\Big\langle\prod_{j=1}^{n}\mathcal{V}_j(z_j,\bar{z}_j)\Big\rangle. \end{aligned} \tag{2.2.9}$$

For open string tree amplitudes, the same discussion applies with the restriction that the complex coordinates now range over the upper half plane (which is conformally equivalent to the disc, the boundary of the disc being mapped to the real line) only. The conformal Killing group acts as in (2.2.8), but the a, b, c, d are now restricted to be real. Fixing these symmetries can be accomplished by fixing the position of three vertex operators on the real line.

For loop diagrams, there is an additional complication: the gauge redundancy is not sufficient to eliminate the path integral over the metrics completely. What remains is a finite dimensional integral over parameters parametrising what is called the *moduli space*. The number of these parameters depends on the world-sheet topology. For example, a closed string 1-loop amplitude involves an integration over the moduli space of the torus, which is parametrised by a single complex parameter τ. The geometrical meaning of τ is the following: the torus with modulus τ can be represented as the complex plane with the identifications

$$z \cong z + 2\pi(m + n\tau),$$

with $m,\ n$ in $\mathbf{Z}$. The moduli space of the annulus, which represents an open string 1-loop diagram, consists of a real parameter, the difference of the radii of the outer and inner circles. Also, for string loop diagrams, the conformal Killing group will be different from the tree-level case.

We will not discuss in this thesis the resulting modifications for the S-matrix expression for loop amplitudes; more details can be found e.g. in [9], chapter 5. Suffice it to say at this point that the final result for the S-matrix satisfies the necessary physical requirements such as finiteness and unitarity. At first sight, this might not seem a spectacular result. What makes it so is the fact that, as we will see in section 2.4.2, one of the string excitations that can appear as an external state in the S-matrix can be identified as a graviton. So far, string theory is still the only known theory that provides a consistent framework for the quantum theory of gravitational interactions.

Summarising, we have seen that the calculation of string scattering amplitudes reduces to the evaluation of correlators of local operators in the 2-dimensional field theories of the X^μ and the ghosts $b,\ c$. These 2-dimensional theories are examples of *conformal field theories* [22] which we will now study in more detail.

2.3 Conformal field theory

2.3.1 Operator Product Expansions

As a first example of a conformal field theory (CFT) in 2 dimensions we will consider the the theory based on the gauge-fixed action for the X^μ obtained in (2.2.6). We will use this theory to illustrate some key concepts in conformal field theory that will be relevant in the discussion of our own work.

The action (2.2.6) describes D free massless fields propagating in a two dimensional space parametrised by $(z, \bar{z})$. We will, for the moment, consider the theory defined on $\bar{\mathbf{C}}$ as required for the discussion of closed strings, and come back to the theory on the upper half plane (open strings) later on. The classical equations of motion following from (2.2.6) are

$$\partial\bar{\partial}X^\mu(z,\bar{z}) = 0. \tag{2.3.1}$$

These equations state that the functions ∂X^μ and $\bar{\partial}X^\mu$ are holomorphic and anti-holomorphic respectively.

Expectation values are defined by the functional integral

$$\left\langle \mathcal{F}[X] \right\rangle = \int [dX]\, e^{-S_P} \mathcal{F}[X],$$

where $\mathcal{F}[X]$ is any functional of the X^μ. The equations of motion (2.3.1) no longer hold for the X^μ insertions in the path integral, i.e. they do not hold as operator equations. For example, the 2-point function or *propagator* following from (2.2.6) satisfies:

$$\frac{1}{\pi\alpha'}\partial\bar{\partial}\left\langle X^{\mu}(z,\bar{z})X^{\nu}(z',\bar{z}')\right\rangle = -\eta^{\mu\nu}\delta(z-z',\bar{z}-\bar{z}'), \tag{2.3.2}$$

The solution to this equation is:

$$\left\langle X^{\mu}(z,\bar{z})X^{\nu}(z',\bar{z}')\right\rangle = -\frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2. \tag{2.3.3}$$

Other correlation functions can be calculated using Wick's theorem by summing over all possible contractions with the propagator (2.3.3).

When dealing with composite operators, some form of ordering prescription is required. A convenient prescription is the so-called *conformal normal ordering*. Normal ordered operators will be denoted by $:\mathcal{F}[X]:$. Simply put, the ordering prescription states that, when encountering a normal ordered operator $:\mathcal{F}:$ in a correlation function, no contractions are to be made among the operators appearing in $\mathcal{F}[X]$ themselves. This means that the normal ordered operator $:\mathcal{F}[X]:$ is obtained from $\mathcal{F}[X]$ by subtracting all self-contractions. This can be summarised in a formal expression:

$$\begin{aligned} :\mathcal{F}[X]: &= \mathcal{F} - \sum(\text{self}-\text{contractions}) \\ &= \exp\left\{\frac{\alpha'}{4}\int d^2z\,d^2z'\,\ln|z-z'|^2\frac{\delta}{\delta X(z,\bar{z})}\frac{\delta}{\delta X(z',\bar{z}')}\right\}\mathcal{F}[X]. \end{aligned}$$

Similarly, for a pair of normal ordered operators, the prescription implies:

$$\begin{aligned} &:\mathcal{F}[X]\,\mathcal{G}[X]:=:\mathcal{F}[X]::\mathcal{G}[X]: - \sum(\text{cross}-\text{contractions}) \\ &= \exp\left\{\frac{\alpha'}{2}\int d^2z'\,d^2z''\,\ln|z'-z''|^2\frac{\delta_F}{\delta X(z',\bar{z}')}\frac{\delta_G}{\delta X(z'',\bar{z}'')}\right\}:\mathcal{F}[X]::\mathcal{G}[X]: \end{aligned} \tag{2.3.4}$$

A simple example is provided by:

$$:X^{\mu}(z,\bar{z})X^{\nu}(z',\bar{z}'):= X^{\mu}(z,\bar{z})X^{\nu}(z',\bar{z}') + \frac{\alpha'}{2}\eta^{\mu\nu}\ln|z-z'|^2. \tag{2.3.5}$$

From this formula, we see that X^{μ} insertions inside the normal ordering symbol do obey the classical equations of motion:

$$\partial\bar{\partial}:X^{\mu}(z,\bar{z})X^{\nu}(z',\bar{z}'):= 0 \tag{2.3.6}$$

The previous considerations provide a framework for calculating an important set of operator relations known as the *operator product expansion* (OPE). The idea is to expand a product of two local operators close to each other as a linear combination of local operators:

$$\mathcal{A}_i(z,\bar{z})\mathcal{A}_j(0,0) = \sum_k c^k{}_{ij}(z,\bar{z})\mathcal{A}_k(0,0). \tag{2.3.7}$$

The indices $i,\ j,\ k$ label basis elements in the space of local operators. The expansion coefficients $c^k{}_{ij}(z,\bar{z})$ are functions of $(z,\bar{z})$, usually they are simple powers of z and $\bar{z}$. For example, using (2.3.5) and (2.3.6) one derives

$$\begin{aligned} X^\mu(z,\bar{z})X^\nu(0,0) &= -\frac{\alpha'}{2}\ln|z|^2 \\ &+ \sum_{k=1}^{\infty}\frac{1}{k!}\left[z^k :X^\nu\partial^k X^\mu(0,0): + \bar{z}^k :X^\nu\bar{\partial}^k X^\mu(0,0):\right] \end{aligned}$$

2.3.2 Ward identities and conformal transformations

We have already mentioned that the global symmetry group of the action (2.2.6) consists of the conformal isometries that map the complex plane into itself:

$$z \to f(z) = \frac{az+b}{cz+d}, \quad \bar{z} \to \bar{f}(\bar{z}) = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}; \tag{2.3.8}$$

A central role is played by the local operators $\mathcal{A}$ that have simple transformation properties under the symmetry transformations (2.3.8):

$$\mathcal{A}(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{A}\left(f(z),\bar{f}(\bar{z})\right).$$

An operator transforming in this way is called a *quasi-primary operator* and the numbers $(h,\bar{h})$ are called *conformal weights.*

When we drop the requirement that transformations should be well-defined on the extended complex plane, the action becomes invariant under a larger set of transformations. Indeed, any holomorphic reparametrisation

$$z \to f(z), \quad \bar{z} \to \bar{f}(\bar{z}) \tag{2.3.9}$$

leaves the action invariant. These transformations are referred to as *conformal transformations* and theory with this invariance is termed a *conformal field theory (CFT)*. Only the subset of transformations (2.3.8) is well-defined on the whole extended complex plane. Nevertheless the transformations (2.3.9) play a very important role because they reflect the existence of an infinite number of conserved currents. The components of the energy-momentum tensor are given by

$$T(z) \equiv T_{zz} = -\frac{1}{\alpha'} :\partial X^\mu \partial X_\mu :, \qquad \tilde{T}(\bar{z}) = T_{\bar{z}\bar{z}} = -\frac{1}{\alpha'} :\bar{\partial} X \bar{\partial} X :, \quad T_{z\bar{z}} = T_{\bar{z}z} = 0. \tag{2.3.10}$$

The components $T(z)$ and $\tilde{T}(\bar{z})$ are separately conserved: $\bar{\partial} T(z) = 0$, $\partial \tilde{T}(\bar{z}) = 0$. This implies the existence of an infinite number of conserved currents

$$j(z) = i\varepsilon(z)T(z), \qquad \bar{j}(\bar{z}) = i\bar{\varepsilon}(\bar{z})\tilde{T}(\bar{z}). \tag{2.3.11}$$

for arbitrary holomorphic functions $\varepsilon(z)$. These currents are the Noether currents corresponding to infinitesimal conformal transformations $z \to z + \varepsilon(z)$, $\bar{z} \to \bar{z} + \bar{\varepsilon}(\bar{z})$. The infinitesimal version of the global transformations (2.3.8) corresponds to taking

$$\varepsilon(z) = \varepsilon_{-1} + \varepsilon_0 z + \varepsilon_1 z^2 \qquad \bar{\varepsilon}(\bar{z}) = \bar{\varepsilon}_{-1} + \bar{\varepsilon}_0 \bar{z} + \bar{\varepsilon}_1 \bar{z}^2.$$

Local operators $\mathcal{O}$ that transform as

$$\mathcal{O}(z,\bar{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \mathcal{O}\left(f(z), \bar{f}(\bar{z})\right) \tag{2.3.12}$$

under the full set of conformal transformations are called *primary fields* of weight $(h, \bar{h})$.

We will now discuss a method to calculate the conformal transformation properties of local operators. In field theory, the transformation properties of operators under symmetry transformations are encoded in *Ward identities*. In two dimensions, the Ward identity for the transformation of a local operator $\mathcal{A}$ under a general symmetry of the theory takes the form[4] (see e.g [9], chapter 2):

$$\delta \mathcal{A}(z_0, \bar{z}_0) = \frac{1}{2\pi i} \oint_C (dz\, j - d\bar{z}\, \tilde{j})\, \mathcal{A}(z_0, \bar{z}_0), \tag{2.3.13}$$

[4] We are assuming here that the symmetry transformation in question is a symmetry of the full quantum theory, i.e. that it is nonanomalous.

where j and $\tilde{j}$ denote the components of the Noether current associated with that symmetry. The contour C runs around the point z_0 in counterclockwise direction. The contour integral picks out the residues as $z \to z_0$ and $\bar{z} \to \bar{z}_0$ in the OPE's of j and $\tilde{j}$ with $\mathcal{A}$ respectively. So in order to calculate the transformation properties of a certain operator it suffices to calculate singular terms in the relevant OPE's.

Applying the Ward identity (2.3.13) one finds that a quasi-primary operator $\mathcal{A}$ is characterised by an OPE of the form

$$T(z)\mathcal{A}(0,0) \;=\; \ldots + \frac{h}{z^2}\mathcal{A}(0,0) + \frac{1}{z}\partial\mathcal{A}(0,0) + \ldots \tag{2.3.14}$$

while a primary field $\mathcal{O}$ satisfies

$$T(z)\mathcal{O}(0,0) = \frac{h}{z^2}\mathcal{O}(0,0) + \frac{1}{z}\partial\mathcal{O}(0,0) + \ldots\,, \tag{2.3.15}$$

(and similarly for $\tilde{T}(\bar{z})$).

For example, the operators ∂X^μ and $:e^{ik\cdot X}:$ are primary fields of weight $(1,0)$ and $\left(\frac{\alpha' k^2}{4}, \frac{\alpha' k^2}{4}\right)$ respectively. The OPE of the energy-momentum tensor with itself turns out to be:

$$T(z)T(0) \sim \frac{D}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0), \tag{2.3.16}$$

where the symbol $\sim$ denotes that we have only displayed the singular part of the OPE as $z \to 0$.

This shows that the energy-momentum tensor T is a quasi-primary field of weight $(2,0)$ but not a primary field. Instead, the OPE (2.3.19) implies the infinitesimal transformation law

$$\delta T(z) = \frac{D}{12}\partial^3\varepsilon(z) + 2\partial\varepsilon(z)T(z) + \varepsilon(z)\partial T(z). \tag{2.3.17}$$

Later on, we will also need the transformation law of T under finite transformations $z \to f(z)$ for which we will adopt the notation $f \circ T(z)$. One way to derive this is by using the definition of the normal ordered product:

$$T(z) = \lim_{\varepsilon\to 0}\left[-\frac{1}{\alpha'}\partial X^\mu(z+\varepsilon)\partial X_\mu(z) - \frac{D}{2\varepsilon^2}\right]$$

and the fact that ∂X^μ is a $(1,0)$ primary field:

$$\begin{aligned} f\circ T(z) &= \lim_{\varepsilon\to 0}\left[-\frac{1}{\alpha'}f'(z+\varepsilon)f'(z)\partial X^\mu(z+\varepsilon)\partial X_\mu(z) - \frac{D}{2\varepsilon^2}\right] \\ &= (f')^2 T(f(z)) + \frac{D}{12}\left(\frac{f'''f' - \frac{3}{2}(f'')^2}{(f')^2}\right). \end{aligned} \tag{2.3.18}$$

The second term in this expression arises from normal ordering and goes under the name of the *Schwarzian derivative* of f with respect to z.

For a general CFT, the TT OPE will take the form

$$T(z)T(0) \sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0), \tag{2.3.19}$$

with c a constant parameter known as the *central charge.* The transformation law will also remain of the form (2.3.18) with D replaced with c. A similar set of relations holds for the antiholomorphic component $\tilde{T}$ with, for general CFT's, a possibly different central charge parameter $\tilde{c}$.

2.3.3 Mode expansions and vertex operators

So far, we have not specified which directions in the complex plane we consider to be the time and space directions respectively. Since for closed strings the space direction should be compact it is a natural choice to take the time to run radially so that the curves of constant time are circles around the origin. The origin corresponds to the infinite past while the point at infinity corresponds to the infinite future. Usually, once a time direction is chosen, quantisation proceeds by imposing canonical equal-time commutation relations on the Fourier modes of the fields. This leads to the construction of an abstract space called the *Fock space* $\mathcal{F}$. In conformal field theory, $\mathcal{F}$ has a concrete representation: to every state in $\mathcal{F}$ corresponds a local operator, the vertex operator corresponding to that particular state. This correspondence is called the *state-operator correspondence* and will be emphasised in the approach we will follow. The vertex operators are precisely the wave-function insertions representing asymptotic string states that enter in the S-matrix expression (2.2.9).

A holomorphic or anti-holomorphic quasi-primary field of weight h or $\bar{h}$ can be expanded in a Laurent series around the origin:

$$\mathcal{A}(z) = \sum_{m\in Z} \frac{\mathcal{A}_m}{z^{m+h}} \qquad \tilde{\mathcal{A}}(\bar{z}) = \sum_{m\in Z} \frac{\tilde{\mathcal{A}}_m}{\bar{z}^{m+\bar{h}}}.$$

Conversely, the coefficients or *modes* are obtained from

$$\mathcal{A}_m = \frac{1}{2\pi i}\oint_C dz\, z^{m+h-1}\mathcal{A}(z), \qquad \tilde{\mathcal{A}}_m = -\frac{1}{2\pi i}\oint_C d\bar{z}\, \bar{z}^{m+h-1}\tilde{\mathcal{A}}(z).$$

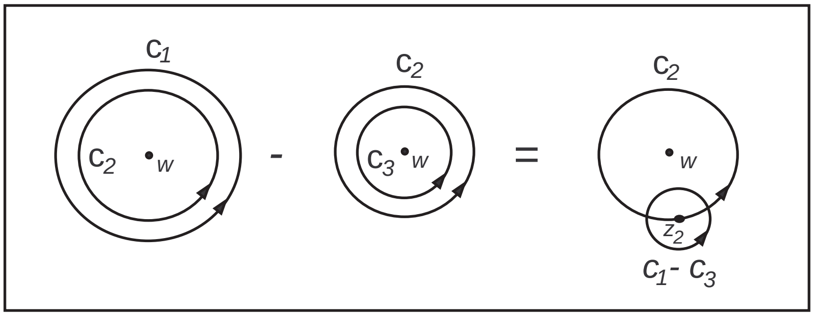


Figure 2.3: Contour deformation leading to formula (2.3.21).

By taking the contour C to be a circle centered around the origin, each mode $\mathcal{A}_m$ can be seen as a conserved charge Q corresponding to the conserved current $j = z^{m+h-1}\mathcal{A}(z)$ (compare (2.3.11)).

Any conserved charge Q acts in a natural way on the space of local operators as the infinitesimal generator of the corresponding symmetry transformation. Using the Ward identity (2.3.13) we can express the action $Q \cdot \Phi$ of Q on a local operator $\Phi(z)$ as

$$\begin{aligned} Q \cdot \Phi(0) &\equiv \delta\Phi(z) \\ &= \frac{1}{2\pi i}\oint_C dz\, j(z)\Phi(0). \end{aligned} \tag{2.3.20}$$

The contour integral picks out the coefficient of the pole term in the OPE of j with Φ. The resulting operation $\cdot$ is associative as a result of the associativity of the OPE.

Using a contour argument (see figure (2.3), the commutator of two charges can be expressed as:

$$[Q_1, Q_2] = \frac{1}{2\pi i}\oint_{C_2} dz_2\, \mathrm{Res}_{z\to z_2} j_1(z) j_2(z_2). \tag{2.3.21}$$

This can be calculated from the OPE between the currents.

As an example, consider the modes of the holomorphic component of the energy-momentum tensor:

$$T(z) = \sum_{m=-\infty}^{\infty} \frac{L_m}{z^{m+2}}, \qquad \tilde{T}(\bar{z}) = \sum_{m=-\infty}^{\infty} \frac{\tilde{L}_m}{\bar{z}^{m+2}}. \tag{2.3.22}$$

Using the formula (2.3.21) and the TT OPE to calculate the commutation relations yields the *Virasoro algebra*,

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{2.3.23}$$

The mode L_0 is the generator of time translations and can be interpreted as the Hamiltonian of the system. The modes of the anti-holomorphic component $\tilde{T}(\bar{z})$ form a second copy of the Virasoro algebra.

As a second example, consider the modes of the primary fields ∂X^μ and $\bar{\partial} X^\mu$:

$$\partial X^\mu(z) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\alpha_m^\mu}{z^{m+1}}, \qquad \bar{\partial} X^\mu(\bar{z}) = -i\sqrt{\frac{\alpha'}{2}} \sum_{m=-\infty}^{\infty} \frac{\tilde{\alpha}_m^\mu}{\bar{z}^{m+1}}. \tag{2.3.24}$$

These can be integrated to give a mode expansion for $X^\mu(z, \bar{z})$, which we split into a sum of holomorphic and anti-holomorphic parts:

$$X^\mu(z, \bar{z}) = X^\mu(z) + \tilde{X}^\mu(\bar{z})$$

with

$$X^\mu(z) = \frac{x^\mu}{2} - i\frac{\alpha'}{2} p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m} \alpha_m^\mu z^{-m}$$

$$\tilde{X}^\mu(\bar{z}) = \frac{x^\mu}{2} - i\frac{\alpha'}{2} p^\mu \ln \bar{z} + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{1}{m} \tilde{\alpha}_m^\mu \bar{z}^{-m}$$

where $p^\mu \equiv \sqrt{\frac{2}{a'}}\alpha_0 \equiv \sqrt{\frac{2}{\alpha'}}\tilde{\alpha}_0$. From (2.3.21) and the XX OPE the commutation relations can be calculated:

$$\begin{aligned} \left[\alpha_m^\mu, \alpha_n^\nu\right] &= \left[\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu\right] = m\delta_{m,-n}\eta^{\mu\nu} \\ \left[x^\mu, p^\nu\right] &= i\eta^{\mu\nu}. \end{aligned} \tag{2.3.25}$$

The operators $x^\mu and p^\nu$ represent the center-of-mass position and momentum of the string while the α_m^μ form an infinite set of harmonic oscillators representing all possible vibrational modes. A representation for the algebra

(2.3.25) can be constructed by starting from states $|0,k>$ that have momentum k^μ and are annihilated by all the lowering operators α_m^μ with $m > 0$. The states obtained by acting with raising operators form a vector space, the Fock space $\mathcal{F}$.

We will now discuss the correspondence between Fock space states and local operators. Consider first the Fock vacuum $|0,0>$. By definition, this is the state that is annihilated by p^μ and all α_m^μ with $m > 0$. It can be represented by the local operator 1. Indeed, applying the definition (2.3.20) one finds:

$$p^\mu \cdot 1 = 0; \quad \alpha_m^\mu \cdot 1 = 0 \ \text{ for } m > 0$$

so we find the state-operator correspondence

$$|0,0> \qquad \leftrightarrow \qquad 1.$$

Similarly, one can check that

$$|0,k> \qquad \leftrightarrow \qquad :e^{ik\cdot X}(0):.$$

The other states can be found by acting with the raising operators; for example

$$\alpha_{-m}^\mu |0,k> \qquad \leftrightarrow \qquad \sqrt{\frac{2}{\alpha'}}\frac{i}{(m-1)!}:\partial^m X^\mu e^{ik\cdot X}(0):.$$

We can also express the Virasoro generators in terms of the modes of X^μ by inserting the Laurent expansion (2.3.24) into the energy-momentum tensor and collecting the terms with the appropriate power of z. For example, the Virasoro generator L_0 can be expressed as

$$L_0 = \frac{\alpha' p^2}{4} + \sum_{n=1}^{\infty}(\alpha_{-n}^\mu \alpha_{\mu n}) + a^X. \tag{2.3.26}$$

Since, in this expression, we have moved the lowering operators to the left of the raising operators (also called *creation-annihilation normal ordering*), we have introduced a normal ordering constant a^X. It can be determined from the action of L_0 on the states, for instance from the action on the ground state $L_0 \cdot 1 = 0$ one sees that $a^X = 0$. Although creation-annihilation normal ordering coincides with conformal normal ordering in the free scalar theory we are presently considering, this will no longer be the case in general CFT's.

2.3.4 CFT on the upper half plane

As promised, we will give some comments on CFT on the upper half plane relevant for the description of open strings. First we need to specify boundary conditions; the appropriate choice consistent with translational invariance in the D-dimensional target space is for the fields to obey Neumann boundary conditions[5]:

$$X^\mu(z) = \tilde{X}^\mu(\bar{z}) \qquad \text{at } z = \bar{z}.$$

A convenient way to implement this boundary condition is to use the *doubling trick*: one can combine the holomorphic and anti-holomorphic fields $X^\mu(z)$ and $\tilde{X}^\mu(\bar{z})$ on the upper half plane into a single holomorphic field defined on the whole complex plane by defining, for $\mathrm{Im}\, z \leq 0$:

$$X^\mu(z) \equiv \tilde{X}^\mu(\bar{z}).$$

In this way, the Neumann boundary condition is automatically satisfied. So the CFT on the upper half plane can be reduced to the holomorphic sector of the CFT on the complex plane. The mode expansion for $X^\mu(z)$ takes the form

$$X^\mu(z) = \frac{x^\mu}{2} - i\alpha' p^\mu \ln z + i\sqrt{\frac{\alpha'}{2}} \sum_{\substack{m=-\infty \\ m\neq 0}}^{\infty} \frac{1}{m} \alpha^\mu_m z^{-m} \qquad (2.3.27)$$

and the Virasoro generator L_0 can be expressed as

$$L_0 = \alpha' p^2 + \sum_{n=1}^{\infty} (\alpha^\mu_{-n} \alpha_{\mu n}). \qquad (2.3.28)$$

2.3.5 Inner products

So far, we have looked at the Fock space $\mathcal{F}$ as a vector space over the complex numbers whose elements are denoted by ‘kets’ $|A>$. For the CFT we are considering (and for other CFT’s as well), there are two inner products worth mentioning (for more details, see e.g. [20, 21]). They will both play a role in the discussion of the reality condition in string field theory in chapter 3.

[5] Another choice, one that breaks space-time translation invariance, would be to take the some of the fields X^μ to satisfy Dirichlet boundary conditions. This choice is relevant for the description of *D-branes* and will be discussed in section 4.2.

The vector space $\mathcal{F}$ is naturally equipped with a nondegenerate bilinear inner product that was first introduced in [22] and goes under the name of *BPZ inner product*. Given two Fock-space states $|A>$, $|B>\in\mathcal{F}$ with vertex operators $\mathcal{V}_A$ and $\mathcal{V}_B$, their BPZ inner product is defined as the following correlation function on the sphere:

$$(|A>, |B>)_{\text{bpz}} \equiv \langle I \circ \mathcal{V}_A(0)\ \mathcal{V}_B(0)\rangle$$

where I is the conformal transformation $I(z) \equiv -1/z$. This inner product is linear in both arguments because the conformal mapping I doesn't act on the c-number coefficients of the operators. The BPZ inner product of two primary fields can also be extracted from their OPE. If $\mathcal{O}_1$ and $\mathcal{O}_2$ are primary operators of weight h_1 and h_2 respectively, their OPE takes the form

$$\mathcal{O}_1(z)\mathcal{O}_2(0) = \frac{(|\mathcal{O}_1>, |\mathcal{O}_2>)_{\text{bpz}}}{z^{h_1+h_2}}.$$

For example, one has[6]

$$(|0, k>, |0, k'>)_{\text{bpz}} = (2\pi)^d\delta(k+k').$$

The BPZ inner product associates to any operator $\mathcal{O}$ a conjugate operator, denoted by $\text{bpz}(\mathcal{O})$, through the relation

$$(|A>, \mathcal{O}|B>)_{\text{bpz}} = (\text{bpz}(\mathcal{O})|A>, |B>)_{\text{bpz}}.$$

for any $|A>$, $|B>\in\mathcal{F}$. For example, one has[7]

$$\text{bpz}(\alpha_n^\mu) = (-1)^{-n+1}\alpha_{-n}^\mu \tag{2.3.29}$$

and, for modes $\mathcal{O}_n$ of a primary field $\mathcal{O}$ of weight h, one finds

$$\text{bpz}(\mathcal{O}_n) = (-1)^{-n+h}\mathcal{O}_{-n}. \tag{2.3.30}$$

For any state $|A>\in\mathcal{F}$, there is an associated linear mapping $(|A>, \cdot\,)_{\text{bpz}}$, an element of the dual vector space. This element is called the *BPZ conjugate of* $|A>$ and we will often write it as a 'ket' $\ll A|$:

$$\ll A| : \quad \mathcal{F} \to \mathbf{C} : \quad |B>\in\mathcal{F} \mapsto (|A>, |B>)_{\text{bpz}}.$$

[6] Of course, the normalisation factor in this expression is a matter of convention

[7] Another feature, which is important when considering CFT's with Grassmann-valued objects, is that the conformal mapping I, and hence the BPZ conjugation as well, does not change the order of the oscillator modes [33].

Henceforth, we will also use a 'bracket' notation for the BPZ inner product:

$$\ll A|B> \equiv (|A>, |B>)_{\mathrm{bpz}}.$$

Although the BPZ inner product arises naturally in conformal field theory, some of its properties are undesirable for certain purposes. First of all, we would like to have an inner product which is antilinear in the first argument and linear in the second. Also, in analogy with action Hermitean conjugation for the quantum-mechanical harmonic oscillator, we would like the raising operators α^{μ}_{-n} and lowering operators α^{μ}_{n} to be conjugate to each other without the sign factor in (2.3.29). One therefore introduces another inner product called the *Hermitean inner product*, which does have the desired properties. On momentum states, it is defined by

$$(|0,k>, |0,k'>)_{\mathrm{h}} = (2\pi)^d \delta(k-k').$$

The action is extended to the whole Fock space by requiring antilinearity in the first argument and linearity in the second, and by requiring that it defines a conjugation operation, called *Hermitean conjugation* and denoted by hc, which acts on on the oscillator modes as

$$\mathrm{hc}(\alpha^{\mu}_{n}) = \alpha^{\mu}_{-n}.$$

The Hermitean inner product associates to every state $|A>$ a conjugate linear functional, also called the *Hermitean conjugate of* $|A>$, for which we will use the 'bra' notation $<A|$:

$$<A|: \quad \mathcal{F} \to \mathbf{C}: \quad |B> \in \mathcal{F} \mapsto (|A>, |B>)_{\mathrm{h}}.$$

We will also use the following bracket notation for the Hermitean inner product:

$$<A|B> \equiv (|A>, |B>)_{\mathrm{h}}.$$

2.3.6 Ghost CFT

The gauge-fixing procedure for the bosonic string led to the introduction of anticommuting ghost fields (b, c) and $(\tilde{b}, \tilde{c})$. This system is a conformal field theory in itself and can be treated with the methods outlined in the previous subsection.

The equations of motion following from the ghost action

$$S_{gh} = \frac{1}{2\pi} \int d^2 z [b\bar{\partial} c + \tilde{b} \partial \tilde{c}]. \tag{2.3.31}$$

state that $(b,\ c)$ are holomorphic fields while $(\tilde{b},\ \tilde{c})$ are anti-holomorphic. Once again we restrict attention to the holomorphic sector, the anti-holomorphic case proceeding analogously. OPE's can be calculated using the propagator:

$$\left\langle c(z) b(w) \right\rangle = \frac{1}{z - w}$$

The energy-momentum tensor is given by:

$$T(z) =: (\partial b) c : -2\partial(: bc :). \tag{2.3.32}$$

From the Tb and Tc OPE's it follows that b and c are primary fields of weight $(2, 0)$ and $(-1, 0)$ respectively. The central charge turns out to be

$$c = -26$$

The fields can be expanded in a Laurent series:

$$b(z) = \sum_n \frac{b_n}{z^{n+2}} \qquad ; \qquad c(z) = \sum_n \frac{c_n}{z^{n-1}} \ .$$

The modes satisfy the anticommutation relations

$$\{b_m, c_n\} = \delta_{m,-n}. \tag{2.3.33}$$

These represent an infinite number of fermionic harmonic oscillators, where the $n > 0$ modes can be seen as lowering operators and the $n < 0$ mode as raising operators. A Fock space can be built up by acting acting with raising modes on a ground state that is annihilated by all the lowering operators. In this case, there are two such ground states $|\uparrow>$ and $|\downarrow>$ satisfying

$$b_0 |\downarrow>= 0, \qquad c_0 |\uparrow>= 0. \tag{2.3.34}$$

They are related by the action of the zero modes b_0 and c_0: $|\uparrow>= c_0 |\downarrow>$, $|\downarrow>= b_0 |\uparrow>$.

We can now work out the state-operator mapping. Consider the vertex operator 1. Using the definition (2.3.20), we find that it has the properties

$$b_m \cdot 1 = 0, \quad m \geq -1, \qquad c_m \cdot 1 = 0, \quad m \geq 2.$$

So we see that the vertex operator 1 doesn't map to one of the ground states, but instead represents the state

$$b_{-1}|\downarrow\rangle \qquad \leftrightarrow \qquad 1.$$

So the ground states are represented by

$$|\downarrow\rangle \leftrightarrow c, \qquad |\uparrow\rangle \leftrightarrow \partial c\, c. \tag{2.3.35}$$

We will follow the generally adopted convention to consider b_0 to be a lowering operator and c_0 to be a raising operator, which singles out $|\downarrow\rangle$ as the ghost vacuum. For the other raising operators, we have the dictionary

$$b_{-m} \;\leftrightarrow\; \frac{1}{(m-2)!}\partial^{m-2}b, \qquad c_{-m} \;\leftrightarrow\; \frac{1}{(m+1)!}\partial^{m+1}c. \tag{2.3.36}$$

The mode expansion can be used to express the Virasoro generators in terms of the oscillator modes, for instance one finds:

$$L_0 = \sum_{n=1}^{\infty} n(b_{-n}c_n + c_{-n}b_n) - 1. \tag{2.3.37}$$

The last term in this expression is the normal ordering constant.

The action (2.3.31) has a global $U(1)$ symmetry under which the fields transform by a phase factor $b \to e^{i\theta}b,\ c \to e^{-i\theta}c$. The corresponding Noether current is

$$j_{gh}(z) = -:bc:$$

and the associated charge, the *ghost number* Q_{gh}, counts the number of c fields minus the number of b fields. From the Tj OPE it follows that j is not a quasi-primary operator:

$$T(z)j(0) \sim -\frac{3}{z^3} + \frac{j(0)}{z^2} + \frac{\partial j(0)}{z}.$$

It transforms covariantly under rescalings and translations, but the presence of the $-\frac{3}{z^3}$ term means that the transformation under $z \to f(z) = 1/z$ is anomalous. A calculation similar to the one performed in (2.3.18) shows that, under finite conformal transformations, the ghost number current transforms as

$$f \circ j(z) = f'(z) j(f(z)) - \frac{3}{2}\frac{f''}{f'}. \tag{2.3.38}$$

This has an important consequence for the correlation functions: *the only non-vanishing expectation values are the ones where the total ghost number equals* 3. One way to see this is by inserting into any amplitude a contour integral $\frac{1}{2\pi i}\oint_C j_{gh}$ where C encircles all the vertex operators. One can calculate this quantity in two coordinate frames z and u related by the conformal transformation $u = 1/z$. In the first frame, the result is the original amplitude times the total ghost number. Due to the transformation law (2.3.38), the result in the u frame is three times the original amplitude. These results should be equal, from which one finds that the total ghost number should equal three for the amplitude to be nonvanishing.

The fact that we need the (b, c) ghost number (and, for the antiholomorphic sector, the $(\tilde{b}, \tilde{c})$ ghost number as well) to be equal to three in order for amplitudes to be nonvanishing can be combined with the requirement that three of the vertex operators in the S-matrix expression (2.2.9) need to be fixed [8]. We define fixed and integrated vertex operators to be of the form:

$$\begin{aligned} V_{\text{fixed}}(z, \bar{z}) &= c(z)\tilde{c}(\bar{z})\mathcal{V}(z, \bar{z}) \\ V_{\text{integrated}} &= \int d^2 z \mathcal{V}(z, \bar{z}) \end{aligned}$$

where $\mathcal{V}(z, \bar{z})$ represents a local operator of ghost number zero. The determination of the precise form of the operators $\mathcal{V}(z, \bar{z})$ will be the subject of the next section.

2.4 Physical states

So far, we have described how the procedure of gauge-fixing has nicely reduced our initial system to a theory consisting of conformal building blocks for the matter fields and the ghosts. However, we expect that the physics does not depend on the gauge we have chosen. In particular, the physical spectrum of the quantised string theory should not depend on the details of

[8] In fact, the two requirements are related [23].

the gauge-fixing procedure. By going deeper into this question, we will also be able to fill in an important ingredient in our previous discussion: we have not yet specified which local operators $\mathcal{V}(z, \bar{z})$ in the CFT are suited to represent the asymptotic string states in the the S-matrix expression (2.2.9).

2.4.1 BRST formalism

These ideas can be made more precise in the formalism of *BRST quantisation*. For general systems possessing gauge symmetries, the gauge-fixed Lagrangian (including the Fadeev-Popov ghosts) will be invariant under a set of anticommuting *BRST transformations* representing small changes in the gauge-fixing conditions. The associated conserved charge is called the *BRST charge* Q_B. It has the property of being nilpotent:

$$Q_B \cdot Q_B = 0.$$

Physical states of the theory are characterised by the requirement that they are “BRST closed”, i.e. annihilated by the BRST charge

$$Q_B \cdot V = 0. \tag{2.4.1}$$

This is trivially true for the “BRST exact” states V of the form

$$V = Q_B \cdot \chi.$$

However, when such a state is present in an amplitude involving only physical states, this amplitude is constrained to vanish. Hence two physical states which differ by a BRST exact state will yield the same amplitudes and are physically equivalent.

Summarised, the physical states of the theory can be represented by the equivalence classes of BRST closed states modulo BRST exact states. The vector space formed by these equivalence classes is also called the *BRST cohomology*. We will now discuss the BRST cohomology for bosonic string theory [23].

BRST invariance leads to a conserved BRST current j_B, a primary field of weight $(1, 0)$:

$$j_B = cT_m + \frac{1}{2} : cT_{gh} : + \frac{3}{2}\partial^2 c.$$

and correspondingly for $\tilde{j}_B$. T_m and T_{gh} are the energy-momentum tensors of the matter and ghost theory given in (2.3.10) and (2.3.32) respectively. The BRST operator is

$$Q_B = \frac{1}{2\pi i}\oint (dz\, j_B - d\bar{z}\, \tilde{j}_B).$$

As we mentioned in section (2.1), it is not possible to preserve both diffeomorphism and Weyl symmetries in the quantum theory outside a *critical dimension* D. This anomaly in the gauge symmetry shows up in the BRST formalism in the failure of Q_B to be nilpotent. Indeed, from the $j_B j_B$ OPE

$$j_B(z) j_B(0) \sim -\frac{D-18}{2z^3} c\partial c(0) - \frac{D-18}{4z^2} c\partial^2 c(0) - \frac{D-26}{12z} c\partial^3 c(0).$$

and equation (2.3.21), one sees that Q_B is nilpotent only in the critical dimension $D = 26$. Another useful relation is

$$\{Q_B, b_0\} = L_0, \tag{2.4.2}$$

where L_0 denotes the zero-mode of the total (matter + ghost) energy-momentum tensor.

2.4.2 Physical vertex operators

We are now ready to take a look at the physical states of the theory. Let's start with open string theory.

Apart from the the BRST condition (2.4.1) there is another condition that can be imposed on physical states (see e.g. [9], chapter 4):

$$b_0 \cdot V = 0. \tag{2.4.3}$$

Due to the relation (2.4.2), the equations (2.4.1) and (2.4.3) imply $L_0 \cdot V = 0$. Using the mode expansions (2.3.28) and (2.3.37), we find the *mass-shell condition* restricting the masses of physical states:

$$m^2 = \frac{1}{\alpha'}(N^{\text{tot}} - 1), \tag{2.4.4}$$

where N^{tot} denotes the total (matter + ghost) excitation number or *level*.

At the lowest level, $N^{\text{tot}} = 0$, the physical state is a tachyon with vertex operator

$$\mathcal{V} =: e^{ik\cdot X} :, \qquad k^2 = \frac{1}{\alpha'}. \tag{2.4.5}$$

At $N^{\text{tot}} = 1$, the physical states are massless spacetime vectors:

$$\mathcal{V} = e_\mu : \partial X^\mu e^{ik\cdot X} :, \qquad k^2 = 0,\ k\cdot e = 0. \tag{2.4.6}$$

The vector e_μ plays the role of the polarisation and adding a BRST exact part amounts to the equivalence

$$e_\mu \sim e_\mu + k_\mu\, \Lambda(k).$$

This is the momentum-space form of the gauge transformation of a massless U(1) field in 26 dimensions.

The discussion of the physical spectrum of the closed string can be reduced to combining two copies of the open string spectrum. As in (2.4.3), one can restrict attention to the states V satisfying

$$b_0 \cdot V = \tilde{b}_0 \cdot V = 0$$

so that

$$L_0 \cdot V = \tilde{L}_0 \cdot V = 0.$$

These relations determine the mass-shell condition

$$m^2 = \frac{2}{\alpha'}(N^{\text{tot}} + \tilde{N}^{\text{tot}}) - \frac{4}{\alpha'}$$

as well as the *level matching condition*

$$N^{\text{tot}} = \tilde{N}^{\text{tot}}$$

where N^{tot} and $\tilde{N}^{\text{tot}}$ represent the total excitation numbers of the holomorphic and anti-holomorphic sector respectively.

At the lowest level, the spectrum again consists of a tachyon

$$\mathcal{V} =: e^{ik\cdot X} :, \qquad k^2 = \frac{4}{\alpha'}$$

while at the next level one finds massless states:

$$\mathcal{V} = e_{\mu\nu} : \partial X^\mu \bar{\partial} X^\nu e^{ik\cdot X} :, \qquad k^2 = 0,\ k^\mu e_{\mu\nu} = k^\nu e_{\mu\nu} = 0.$$

and BRST-equivalence reduces to

$$e_{\mu\nu} \sim e_{\mu\nu} + a_\mu k_\nu + k_\mu b_\nu, \quad a \cdot k = b \cdot k = 0. \tag{2.4.7}$$

In terms of transformation properties under the D-dimensional Lorentz-group, these states fall into three irreducible representations: the symmetric, traceless part of $e_{\mu\nu}$ corresponds to a massless spin-2 particle or *graviton*, the antisymmetric part corresponds to a massless 2-form referred to as the *B-field*, and the trace of $e_{\mu\nu}$ represents a massless Lorentz scalar called the *dilaton*. The equivalences (2.4.7) correspond to the appropriate gauge transformations of these massless fields. The massless spin-2 field can be identified as the *graviton*, the quantum that corresponds to fluctuations of the gravitational field. which represents fluctuations of the gravitational field. The appearance of this state in the physical spectrum of closed string theory and the fact that string perturbation theory provides a consistent framework for the calculation of S-matrix elements involving gravitons still constitutes one of the main triumphs of string theory. So far, no other framework is known in which quantum-mechanical gravitational interactions can be consistently described.

2.5 Superstrings

The bosonic string theories of the previous section are, despite having some favourable characteristics such as the consistent quantum description of gravitational interactions, rather flawed as theories of the real world.

First of all, both the closed and open string theories contain a tachyonic state in the perturbative spectrum. This is not a catastrophe in itself; in fact a lot of theories (including the standard model) appear to contain a tachyonic scalar field when perturbation theory is set up around a local maximum of the potential energy of this scalar field. We shall come back to this point in section 4.1. The tachyonic mode is absent when one considers fluctuations around a local *minimum* of the potential energy. The question whether such a local minimum exists in string theory as well is a central theme in the present work. For open strings, there is good evidence that this is indeed the case, as we will show in chapters 5 and 7.

For closed bosonic strings however, the question whether the tachyon potential has a minimum is still an open problem.

Another shortcoming of bosonic strings is the fact that the spectrum doesn't contain space-time fermions while, in the real world, most matter is made up out of fermions.

Both objections can be remedied in the theory of *superstrings*. The superstring action is obtained from the Polyakov action (2.2.1) by incorporating *local supersymmetry*. As a consequence, the embedding coordinates X^μ acquire fermionic partners ψ^μ and the world-sheet metric gets a supersymmetric partner as well, the gravitino. The resulting action, which we will not display here, is invariant under supersymmetric generalisations of diffeomorphism and Weyl transformations.

2.5.1 Gauge-fixed action

As in the bosonic case, the integral over metric and gravitino can be taken care of (again modulo global issues) by making a special gauge choice called the *superconformal gauge*. In this gauge, tree level scattering amplitudes can again be obtained by calculating correlation functions in the conformal field theory of matter and ghost fields.

In the matter sector, the gauge-fixed action reduces to

$$S[X^\mu,\psi^\mu] = \frac{1}{4\pi}\int d^2z\ \left[\frac{2}{\alpha'}\partial X^\mu\bar{\partial}X_\mu + \psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu\right] \tag{2.5.1}$$

where the ψ_μ are two dimensional Majorana-Weyl fermions. Just as the X^μ theory, the ψ^μ theory is a CFT in itself. We will presently discuss some of its properties.

2.5.2 Fermionic CFT

The equations of motion following from the fermionic action

$$S = \frac{1}{4\pi}\int d^2z[\psi^\mu\bar{\partial}\psi_\mu + \tilde{\psi}^\mu\partial\tilde{\psi}_\mu] \tag{2.5.2}$$

state that the ψ^μ are holomorphic fields while the $\tilde{\psi}^\mu$ are anti-holomorphic. The OPE's between the spinor fields are:

$$\psi^\mu(z)\psi^\nu(w) \sim \frac{\eta^{\mu\nu}}{z-w}, \tag{2.5.3}$$

and the energy-momentum tensor reads:

$$T(z) = -\frac{1}{2} :\psi^\mu \partial \psi_\mu: .$$

From the $T\psi$ OPE one infers that the ψ^μ are primary fields of weight $(\frac{1}{2}, 0)$ and from the TT OPE one finds that this system has central charge

$$c = \frac{D}{2}.$$

For fermions on the complex plane, there are two possible choices (also called *spin structures*) for the transformation under rotations over an angle 2π: they can be either periodic or anti-periodic. These possibilities go under the name of *Neveu-Schwarz (NS)* and *Ramond (R)* boundary conditions respectively:

$$\begin{aligned} \text{NS} &: \quad \psi^\mu(e^{2\pi i} z) = \psi^\mu(z), \\ \text{R} &: \quad \psi^\mu(e^{2\pi i} z) = -\psi^\mu(z). \end{aligned}$$

Correspondingly, there are two sets of mode expansions:

$$\begin{aligned} \text{NS} &: \quad \psi^\mu(z) = \sum_{r \in \mathbf{Z} + \frac{1}{2}} \frac{\psi^\mu_r}{z^{r+\frac{1}{2}}}, \\ \text{R} &: \quad \psi^\mu(z) = \sum_{r \in \mathbf{Z}} \frac{\psi^\mu_r}{z^{r+\frac{1}{2}}}. \end{aligned}$$

From the $\psi\psi$ OPE one learns that the modes form an infinite set of fermionic harmonic oscillator algebras:

$$\{\psi^\mu_r, \psi^\nu_s\} = \eta^{\mu\nu} \delta_{r,-s}. \tag{2.5.4}$$

The construction of the Fock space again proceeds by acting with raising operators on a suitably chosen vacuum state that is annihilated by all the lowering operators ψ^μ_r with $r > 0$.

In the NS sector, the Fock vacuum can again be represented by the operator 1

$$|0\rangle_{NS} \quad \leftrightarrow \quad 1$$

and the remaining state-operator correspondence can be worked out just as in the bosonic case.

In the R sector however, the ground state is degenerate due to the zero modes ψ_0^μ which map ground states into ground states. The zero modes themselves form a D-dimensional Clifford algebra and the degenerate ground states will form a representation space for this algebra. We will denote these degenerate ground states by $|0,\alpha>_R$. Under spacetime Lorentz transformations, these states transform as a Dirac spinor. The Fock space is now built up by acting with raising operators on the ground states $|0,\alpha>_R$ (with α an index in the representation space of the Clifford algebra). The operators S_α corresponding to the R ground states $|0,\alpha>_R$

$$|0,\alpha>_R \qquad \leftrightarrow \qquad S_\alpha$$

are called *spin fields*. They can't be expressed as local operators in terms of the ψ^μ but there exists an alternative representation of the fermionic system (called *bosonisation*) where the spin fields can be represented by local operators. We will not go into the details of this construction in this thesis[9] , we just mention that the conformal weight of the R ground states S_α turns out to be $\frac{D}{16}$. This gives the normal ordering constant in the mode expression of L_0:

$$L_0 = \sum_{r\in\mathbf{N}+\nu} \psi^\mu_{-r}\psi_{r\ \mu} + a^\psi,$$

where we have defined ν to take the value $\frac{1}{2}$ in the NS sector and 0 in the R sector.

$$a^\psi = 0 \qquad \text{NS sector}, \qquad a^\psi = \frac{D}{16} \qquad \text{R sector}.$$

2.5.3 Superconformal symmetry

As we have seen, the X and ψ theories are CFT's by themselves. The combined theory, however, has an even larger set of symmetries called *superconformal symmetries*. As in the bosonic case, these can be seen as the combined superdiffeomorphism and super-Weyl transformations that remain after fixing the superconformal gauge. The extended symmetry is reflected in the existence of extra conserved currents $G(z)$ and $\tilde{G}(\bar{z})$ called the *world-sheet supercurrents*:

$$G(z) = i(2/\alpha')^{\frac{1}{2}}\psi^\mu\partial X_\mu; \qquad \tilde{G}(\bar{z}) = i(2/\alpha')^{\frac{1}{2}}\tilde{\psi}^\mu\bar{\partial} X_\mu. \tag{2.5.5}$$

[9]We will, however, discuss the bosonisation procedure for the ghost fields of the superstring in section 2.5.4.

The OPE of $G(z)$ with the holomorphic component $T(z)$ of the energy-momentum tensor

$$T(z) =: \frac{1}{\alpha'}\partial X^\mu \partial X_\mu - \frac{1}{2}\psi^\mu \psi_\mu :$$

is of the form

$$\begin{aligned} T(z)T(0) &\sim \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}\partial T(0) \\ T(z)G(0) &\sim \frac{3}{2z^2}T_F(0) + \frac{1}{z}\partial T_F(0) \\ G(z)G(0) &\sim \frac{2c}{3z^3} + \frac{2}{z}T_B(0) \end{aligned} \tag{2.5.6}$$

where c is the central charge of the combined system

$$c = \frac{3D}{2}.$$

It follows that the operator $G(z)$ is a $(\frac{3}{2}, 0)$ primary field. Its mode expansion is given by

$$G(z) = \sum_{r \in Z+\nu} \frac{G_r}{z^{r+3/2}}$$

where the parameter ν is 0 in the R sector and $\frac{1}{2}$ in the NS sector. The OPE (2.5.6) implies that the modes of T and G form the *superconformal algebra*

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \\ [L_m, G_r] &= \frac{m-2r}{2}G_{m+r} \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2 - 1)\delta_{r+s,0}. \end{aligned} \tag{2.5.7}$$

The modes of the anti-holomorphic components $\tilde{T}$ and $\tilde{G}$ generate a second copy of the algebra (2.5.7). This is summarised in the statement that the theory is a $(N, \tilde{N}) = (1, 1)$ *superconformal field theory*.

2.5.4 Superghosts

Linear dilaton CFT

Before turning to the description of the ghost sector of the superstring, we will discuss a slight modification of the scalar field CFT that will be of use later on. It arises from starting with the action for a single massless scalar field φ

$$S = -\frac{\epsilon}{4\pi}\int d^2z\partial\varphi\bar{\partial}\varphi \tag{2.5.8}$$

(where $\epsilon = \pm 1$) with an extra term added to the energy-momentum tensor

$$T(z) =:\frac{1}{2}[\epsilon(\partial\varphi)^2 - Q\partial^2\varphi]:; \qquad \tilde{T}(\bar{z}) =:\frac{1}{2}[\epsilon(\bar{\partial}\varphi)^2 - Q\bar{\partial}^2\varphi]: \tag{2.5.9}$$

The central charge is equal to

$$c = \tilde{c} = 1 - 3\epsilon Q^2.$$

The extra term in the energy-momentum tensor modifies the conformal transformation properties of the fields. For example, the field $j(z) \equiv \epsilon\partial\varphi$ gets an extra term in the transformation law

$$f \circ j(z) = f'(z)j(f(z)) - \frac{\epsilon Q}{2}\frac{f''}{f'} \tag{2.5.10}$$

and is therefore no longer a primary field. On the other hand, the operators $:e^{q\varphi}:$ remain primary fields, albeit with modified weight $\frac{1}{2}\epsilon q(q+Q)$.

Ghost sector for the superstring

As was the case for the bosonic string, the gauge-fixing procedure for the superstring leads to the introduction of ghost fields. These consist of the anticommuting $(b,\ c)$, $(\tilde{b},\ \tilde{c})$ ghost theory already discussed in section 2.3.6 and a system of commuting *superghosts* $(\beta,\ \gamma)$ and $(\tilde{\beta},\ \tilde{\gamma})$ arising from gauge-fixing the fermionic reparametrisations:

$$S = \frac{1}{2\pi}\int d^2z[\beta\bar{\partial}\gamma + \tilde{\beta}\partial\tilde{\gamma}]. \tag{2.5.11}$$

The equations of motion state that $(\beta,\ \gamma)$ are holomorphic fields while $(\tilde{\beta},\ \tilde{\gamma})$ are anti-holomorphic. Once again we restrict attention to the holomorphic sector, the anti-holomorphic case proceeding analogously. Under conformal transformations, the fields β and γ transform as $(\frac{3}{2},0)$ and $(-\frac{1}{2},0)$ primary fields respectively and the action is conformally invariant. The fundamental fields obey the OPE

$$\beta(z)\gamma(w) = \frac{1}{z-w}.$$

The energy-momentum tensor is taken to be

$$T(z) =:(\partial\beta)\gamma: -\frac{3}{2}\partial(:\beta\gamma:). \tag{2.5.12}$$

This choice is consistent with the fact that β and γ are primary fields of weight $(\frac{3}{2},0)$ and $(\frac{1}{2},0)$. The central charge turns out to be

$$c = \tilde{c} = 11.$$

As was the case for the fermionic matter system, the superghosts can obey either Neveu-Schwarz (periodic) or Ramond (anti-periodic) boundary conditions. This leads to the Laurent expansions:

$$\beta(z) = \sum_{n\in\mathbf{Z}+\nu} \frac{\beta_n}{z^{n+\frac{3}{2}}} \qquad ; \qquad \gamma(z) = \sum_{n\in\mathbf{Z}+\nu} \frac{\gamma_n}{z^{n-\frac{1}{2}}} \ .$$

where $\nu = \frac{1}{2}$ in the NS sector and $\nu = 0$ in the R sector. These modes satisfy the commutation relations

$$[\gamma_n, \beta_m] = \delta_{n,-m}. \tag{2.5.13}$$

The action (2.5.11) has a global $U(1)$ symmetry under which the fields transform by phase factor $\beta \to e^{i\theta}\beta,\ \gamma \to e^{-i\theta}\gamma$. The corresponding Noether current is

$$j_{\beta\gamma}(z) = -:\beta\gamma:$$

and the associated charge, the *superghost number* $Q_{\beta\gamma}$, counts the number of γ fields minus the number of β fields. As was the case for the $(b,\ c)$ ghost system, the number current has an anomalous transformation law:

$$f \circ j(z) = f'(z)j(f(z)) - \frac{f''}{f'}. \tag{2.5.14}$$

An argument similar to the one following (2.3.38) shows that the only non-vanishing amplitudes in the $(\beta,\ \gamma)$ are the ones where the total superghost number is equal to -2.

One can now build up the Fock space by acting with raising operators on a vacuum state annihilated by the lowering operators. In fact, depending on how we split up the modes into creation and annihilation operators, we can define a whole family of vacuum states $|q>$ satisfying:

$$\begin{aligned} \beta_n|q> &= 0, \qquad n > -q - \frac{3}{2} \\ \gamma_n|q> &= 0, \qquad n \geq q + \frac{3}{2}. \end{aligned}$$

In the NS sector, q can take on integer values while in the R sector q is a half-integer number.

The vacuum states $|q>$ have a simple operator representation in terms of an equivalent formulation of the (β, γ) system known as 'bosonisation'. Indeed, the system can be equivalently represented by a single linear dilaton field φ with $\epsilon = -1$ and $\mathcal{Q} = -2$ and anticommuting ghost fields $(\eta,\ \xi)$ with conformal weights $(1,0)$ and $(0,0)$ respectively:

$$\beta =: \partial\xi e^{-\varphi}:, \qquad \gamma =: \eta e^{\varphi}: .$$

From these equivalences we see that, in order to describe the (β, γ) system, we do not really need the ξ zero-mode: we can construct the whole Fock space with the modes of $\rho \equiv \partial\xi$. This is called working in the *small Hilbert space*. However, for some applications, it is useful to keep the ξ zero mode and work in the *large Hilbert space*.

The superghost number current can be represented by

$$j_{\beta\gamma} = -\partial\varphi.$$

This has the correct transformation law as one can see by comparing (2.5.14) with (2.5.10). The vacuum states $|q>$ can now be explicitly represented by

$$|q> \qquad \leftrightarrow \qquad :e^{q\varphi}: .$$

The different vacua are distinguished by the fact that they have different superghost numbers and conformal weights:

$$\begin{aligned} Q_{\beta\gamma}|q> &= q|q> \\ L_0|q> &= -\frac{1}{2}q(q+2)|q> . \end{aligned}$$

For example, the operator 1 corresponds to the $|q = 0>$ vacuum in the NS sector and has superghost number and conformal weight zero. It is left invariant by the global symmetry group of $SL(2, \mathbf{C})$ transformations. On the other hand, the 'canonical' choice of vacuum, annihilated by all the positive modes, is

$$|q = -1> \quad \leftrightarrow \quad :e^{-\varphi}: \qquad \text{NS sector} \tag{2.5.15}$$

$$|q = -\frac{1}{2}> \quad \leftrightarrow \quad :e^{-\frac{1}{2}\varphi}: \qquad \text{R sector.} \tag{2.5.16}$$

With respect to these canonical vacuum states, the expression for the Virasoro generator L_0 reads:

$$L_0 = \sum_{r \in \mathbf{N}+\nu} [\beta_{-n}\gamma_n + \gamma_{-n}\beta_n] + a_{\beta\gamma}$$

where the normal ordering constant $a_{\beta\gamma}$ is determined by the conformal weight of the vacuum state:

$$\begin{aligned} a_{\beta\gamma} &= \frac{1}{2} \qquad \text{NS sector} \\ a_{\beta\gamma} &= \frac{3}{8} \qquad \text{R sector.} \end{aligned}$$

The arbitrariness in the choice of ground state for the (β, γ) ghost leads to the concept of *pictures*. The picture P of a vertex operator in bosonised form is defined as the φ charge plus the number of ξ's minus the number of η's, as measured by the operator

$$P = \frac{1}{2\pi i} \oint dz(-\partial\varphi - :\eta\xi:).$$

We will see in the next section that a physical state can be represented by vertex operators in different pictures. By acting with the (β, γ) modes on the canonical vacua (2.5.15, 2.5.16), one obtains vertex operators in the canonical picture which is -1 in the NS sector and $-\frac{1}{2}$ in the R sector.

The combined $(b,\ c)$ and $(\beta,\ \gamma)$ theories and their anti-holomorphic counterparts are again invariant under a larger set of symmetries than just the conformal transformations: they form a $(1,1)$ superconformal field theory with $c = \tilde{c} = -15$. The energy-momentum tensor and supercurrent are given by:

$$\begin{aligned} T(z) &= T_{bc} + T_{\eta\xi} + T_{\varphi} \\ &= :(\partial b)c - 2\partial(bc) + \partial\xi\eta - \frac{1}{2}\partial\varphi\partial\varphi - \partial^2\varphi: \\ G(z) &= :\partial^2\xi e^{-\varphi} - \partial\xi\partial\varphi e^{-\varphi} + \frac{3}{2}\partial c\partial\xi e^{-\varphi} - 2b\eta e^{\varphi}: \end{aligned}$$

where we have given the expressions in their bosonised form as we will consequently do from now on.

2.5.5 Physical states

In this section we will discuss the BRST cohomology for the superstring for both the NS and R sectors. Later, in the next section, we will discuss which of these sectors should be kept in order to obtain a consistent string theory. The BRST charge is given by

$$Q_B = \frac{1}{2\pi i}\oint (dz\, j_B - d\bar{z}\,\tilde{j}_B). \tag{2.5.17}$$

where

$$\begin{aligned} j_B &= \; :cT_m + \gamma G_m + \frac{1}{2}(cT_g + \gamma G_g) - \partial(c\gamma\beta): \\ &= \; :c(T_m + T_\varphi + T_{\eta\xi} + \partial c b) + \gamma G_m - \eta\partial\eta e^{2\varphi}: \end{aligned} \tag{2.5.18}$$

and similarly for $\tilde{j}_B$. As was the case for the bosonic string, the BRST charge is nilpotent only if the total (matter and ghost) central charge vanishes: $c = \tilde{c} = 0$. This happens in the critical dimension $D = 10$.

There is another important operator, denoted by $e^{i\pi F}$, which anticommutes with all world-sheet spinors (including the superghosts $(\beta,\ \gamma)$. This operator also commutes with the BRST charge and will play an important role when considering consistent truncations of the string spectrum. It is defined as follows. The *world-sheet fermion number F* counts the number of world-sheet spinor operators. The action of $e^{i\pi F}$ on the various ground states is defined as follows: on the spin fields S^α, it acts as the chirality $\Gamma^{11} \equiv \frac{1}{2^5}\prod_{\mu=0}^{9}\psi_0^\mu$, while on the superghost ground states $:e^{q\varphi}:$ it acts by a phase $e^{i\pi q}$.

Open strings

As for the bosonic string, one chooses, in each cohomology class, representatives satisfying extra conditions:

$$\begin{aligned} b_0\Phi = 0 \quad &\Rightarrow \quad L_0\Phi = 0 \\ \beta_0\Phi = 0 \quad &\Rightarrow \quad G_0\Phi = 0 \qquad \text{(R sector)}. \end{aligned} \tag{2.5.19}$$

The first condition determines the mass spectrum:

$$m^2 = \frac{1}{\alpha'} N^{tot} - \frac{\nu}{\alpha'}$$

We will now discuss the lowest lying physical states. In the NS sector, there is a tachyon with vertex operator

$$\mathcal{V} =: e^{ik\cdot X} e^{-\varphi} :, \qquad k^2 = \frac{1}{2\alpha'}$$

transforming as a scalar under the spacetime Lorentz group $SO(9,1)$. At the next level, one finds the massless states

$$\mathcal{V} = e_\mu : \psi^\mu e^{ik\cdot X} e^{-\varphi} :, \qquad k^2 = 0;\ e \cdot k = 0$$

with BRST equivalence $e^\mu \sim e^\mu + k^\mu$. These states form a massless vector with $U(1)$ gauge invariance .

In the R sector, the level zero states are massless:

$$\mathcal{V} = u_\alpha : S^a e^{ik\cdot X} e^{-\varphi/2} :, \qquad k^2 = 0,\ k_\mu \Gamma^\mu u = 0.$$

They form a massless Dirac spinor and can be decomposed into two Weyl spinors with chirality $e^{i\pi F}$ equal to plus or minus one.

These states are summarised in table (2.1). The different sectors are denoted by NS or R and by + or - for $e^{i\pi F}$ equal to plus or minus one. Their properties under Lorentz transformations are summed up by giving the irreducible representation of the little group $SO(8)$.

sector	$SO(8)$ representation	m^2
NS-	$\mathbf{1}$	$-\frac{1}{2\alpha'}$
NS+	$\mathbf{8}_v$	0
R+	$\mathbf{8}$	0
R-	$\mathbf{8}'$	0

Table 2.1: The lowest level physical states of the open superstring.

The labels $\mathbf{8}_v$, $\mathbf{8}$, $\mathbf{8}'$ denote the 8-dimensional vector, chiral spinor and antichiral spinor representations of $SO(8)$ respectively.

The vertex operators we constructed here were in the canonical picture. Vertex operators in different pictures can be obtained by applying the *picture-changing operator* Z:

$$Z(z) = Q_B \cdot \xi(z)$$

Since Z commutes with the BRST charge Q_B, applying Z on a physical state yields another physical state with picture number raised by 1. There also exists a so-called *inverse picture-changing operator* Y [24] with picture number -1 that commutes with Q_B and satisfies

$$\lim_{\varepsilon\to 0} Y(z+\varepsilon)Z(z) = 1.$$

The explicit form of these operators is

$$Z = :-\partial\xi c + e^{\phi}G_m - \partial\eta\, be^{2\phi} - \partial\left(\eta be^{2\phi}\right):, \tag{2.5.20}$$

$$Y = :-\partial\xi c e^{-2\phi}: . \tag{2.5.21}$$

Closed strings

For closed strings, the conditions (2.5.19) are enforced on both the holomorphic and antiholomorphic side of the spectrum, leading to the mass shell conditions

$$m^2 = \frac{4}{\alpha'}(N^{tot} - \nu) = \frac{4}{\alpha'}(\tilde{N}^{tot} - \tilde{\nu})$$

The closed string spectrum can be obtained by tensoring two copies of the open string spectrum. On the anti-holomorphic side, one defines an operator $e^{i\pi\tilde{F}}$ with eigenvalues ± 1 analogous to the one on the holomorphic side.

In the $(NS-,\ NS-)$ sector, the level 0 state is a tachyon with mass $-\frac{2}{\alpha'}$. The massless spectrum is summarised in table 2.2.

sector	$SO(8)$ representation		dimensions
(NS+,NS+)	$\mathbf{8}_v \times \mathbf{8}_v$	=	$\mathbf{1} + \mathbf{28} + \mathbf{35}$
(R+,R+)	$\mathbf{8} \times \mathbf{8}$	=	$\mathbf{1} + \mathbf{28} + \mathbf{35}_+$
(R+,R-)	$\mathbf{8} \times \mathbf{8}'$	=	$\mathbf{8}_v + \mathbf{56}_t$
(R-,R-)	$\mathbf{8}' \times \mathbf{8}'$	=	$\mathbf{1} + \mathbf{28} + \mathbf{35}_-$
(NS+,R+)	$\mathbf{8}_v \times \mathbf{8}$	=	$\mathbf{8}' + \mathbf{56}$
(NS+,R-)	$\mathbf{8}_v \times \mathbf{8}'$	=	$\mathbf{8} + \mathbf{56}'$

Table 2.2: The massless physical states of the closed superstring.

Inequivalent representations with the same dimension are distinguished by a subscript: $\mathbf{35}$ is a traceless symmetric 2-tensor, $\mathbf{35}_+$ and $\mathbf{35}_-$ denote self-dual and anti self-dual 4-forms, $\mathbf{56}_t$ is a 3-form while $\mathbf{56}$ and $\mathbf{56}'$ are vector-spinors of opposite chirality. The states coming from the (NS, R) sector are spacetime fermions while the other sectors contain space-time bosons.

Type II string theories

Not all of the states listed in table (2.2) can be present together in a consistent string theory (for a more precise definition of the term 'consistent' we refer to [9], chapter 10). Consistent theories are obtained by projecting the full spectrum down to eigenspaces of the operators $e^{i\pi F}$ and $e^{i\pi \tilde{F}}$. Such a projection is known as a *Gliozzi- Scherk - Olive (GSO) projection*.

One of the possible consistent truncations of the spectrum is obtained by keeping only the + sectors on the holomorphic and anti-holomorphic sides. The resulting theory is called the *type IIB superstring* and the remaining massless spectrum is

IIB :	(NS+,NS+),	(R+,NS+),	(NS+,R+),	(R+,R+)
	1 +**28**+**35**	**8'** +**56**	**8'** +**56**	**1** +**28**+$\mathbf{35}_+$

Another possibility is to keep the + sectors on the holomorphic side and the $NS+$ and $R-$ sectors on the anti-holomorphic side. This theory is called the *type IIA superstring*. The resulting massless spectrum is

IIA :	(NS+,NS+),	(R+,NS+),	(NS+,R-),	(R+,R-)
	1 +**28**+**35**	**8'** +**56**	**8** +**56'**	$\mathbf{8}_v + \mathbf{56}_t$

Both theories have the appealing property that the tachyon is projected out. Also, the number of space-time bosons in the spectrum matches the number of space-time fermions. This is an indication that the theory possesses *space-time supersymmetry*. In fact, the presence of two spin-$\frac{3}{2}$ fields or *gravitini* in the spectrum reflects the fact that the type II string theories possess two local spacetime supersymmetries.

2.5.6 Other consistent string theories

Apart from the type II closed string theories in flat 9 + 1 dimensional spacetime, many other consistent string theories can be constructed. First of all, one can replace the flat Minkowski space by a different background, as long as the matter sector of the theory remains a $(1, 1)$ superconformal field theory with central charge 15. This includes the possibility to take 6 of spatial directions to parametrise a compact manifold and end up with a theory in 3 + 1 noncompact dimensions.

It is also possible to 'mod out' a discrete symmetry, a procedure which goes under the name of *orbifolding*, and end up with a new consistent string theory. For example, one can start with the type IIB theory and mod out by a world-sheet parity transformation to obtain the *type I theory* which contains both open and closed unoriented strings. The open string sector contains a gauge field for the gauge group $SO(32)$. Other consistent string theories have been obtained by combining the holomorphic side of the bosonic string with the anti-holomorphic side of the superstring. These are the so-called *heterotic* string theories. There are only two such theories that are space-time supersymmetric: one of them contains a gauge field for the gauge group $SO(32)$, the other one has an $E_8 \times E_8$ gauge group.

2.6 Summary

In this chapter we have only been able to give a very brief outline of the vast subject of string perturbation theory. We pause for a moment to highlight those developments that will play a crucial role in the rest of the thesis.

- The determination of the conformal transformation properties of local operators using the OPE (section 2.3.2), will play a central role in the formulation of the string field theory action (section 3.1.4).
- The relation between Fock space states and local operators (section 2.3.3) is important to understand the relation between different formulations of string field theory (section 3.1.1).
- The inner products on the CFT Fock space (section 2.3.5) will play a role in defining a reality condition on string fields in section 3.1.5.
- The BRST operator in bosonic string theory (section 2.4) and well as for superstrings (section 2.5.5), is the object that governs the dynamics and gauge symmetries in covariant bosonic and supersymmetric string field theories (sections 3.1 and 3.2).
- The representation of the superghosts in bosonised form (section 2.5.4), as well as the picture-changing operations (2.5.20, 2.5.21) are necessary ingredients in the formulation of superstring field theory (section 3.2).

All these developments are indispensable for the concrete calculations in chapters 5 and 7.

Chapter 3

Open string field theory

In the previous chapter, we have given an outline of perturbative string theory, where amplitudes are represented by path-integrals over string worldsheets. This framework leads to a perturbative expansion for the scattering amplitudes between on-shell string states. Although this accomplishment should not be underestimated, especially since it provides a consistent framework for calculating graviton scattering amplitudes, it is still too restrictive for some purposes. As we have already mentioned in the Introduction, in the description of particle interactions, the formalism of quantum field theory provides more information than just the Feynman rules for calculating on-shell S-matrix elements. For many applications, such as the calculation of low-energy effective actions, one needs to be able to calculate off-shell matrix elements as well. Also, quantum field theory can provide information beyond perturbation theory.

Ideally, one would like to have a *string field theory* description that reproduces the perturbative S-matrix expansion. In such a description, the basic object is a ‘string wavefunctional’ or *string field*, whose fluctuations correspond to string states.

There have been many approaches to string field theory, essentially differing in the definition of the string wave-functional Ψ. As a first attempt, one could take Ψ to be a functional on the configuration space of the original Polyakov action, $\Psi[X^\mu, g_{ab}]$. This approach has not been very fruitful, mainly due to difficulties in implementing the gauge invariances of the Polyakov action. More successful have been the approaches that take as a starting point some gauge-fixed form of the Polyakov action. *Light-cone string field theory* is based on string wave-functionals on the configuration space of the Polyakov action in the light-cone gauge (see e.g. [25] and references therein). This approach has the advantage that it dispenses with the Fadeev-Popov ghosts (there are no propagating ghosts in this particular gauge), but the main drawback is that space-time covariance is lost. In this chapter, we will restrict our attention to the *space-time covariant* approach pioneered in [26, 27], where the starting point is the gauge-fixed action in the conformal gauge discussed in the previous chapter. The string field will now be a functional of the matter fields and the Fadeev-Popov ghosts, $\Psi[X^{\mu}, b, c]$. In this approach, the gauge invariances of the original Polyakov action show up in the form of BRST invariance, and it is this invariance which has been the guiding principle for Witten's proposal for an interacting open string field theory [8].

We will start this chapter by giving the definition of a string field and reviewing various different representations of this object. We then proceed to construct the string field theory action in the representation which was used in the original formulation by Witten and which makes clear the fact that string interactions arise from the splitting and joining of strings. We then proceed to reformulate the theory in a language which is more practical for concrete calculations such as the ones that will be performed in chapters 5 and 7. In this representation, the string field theory action can be evaluated by computing CFT correlation functions. Our derivation of the string field theory action in the CFT representation, given in section 3.1.4, has, to our knowledge, not appeared in the existing literature.

In this and the following chapters, we will take

$$\boxed{\alpha' = 1}$$

3.1 Bosonic open strings

3.1.1 The classical string field in various representations

The fundamental variables appearing in the classical string field theory action are *classical string fields*. To begin with, we will focus on their definition and discuss some of their representations. The defining characteristic of a classical string field Ψ is that it is a state in the Fock space of first-quantised string theory. We stress at this point that the string field can be *any* state in the first-quantised Fock space, i.e. it is not restricted to belong to the subspace of physical states satisfying (2.4.1). In the specific string field theories that will be discussed below, further restrictions are placed on the string field in the form of a restriction on the ghost number, the picture number (for superstrings) and a reality condition to ensure that the string field theory action is real. We presently discuss the various representations of the string field common in the literature. For simplicity, we restrict our attention to the matter sector of the bosonic string. We will comment on the inclusion of the ghost sector on p. 66.

In the X^μ sector of the bosonic string, a general Fock space state can be represented in a variety of ways:

1. as a Schrödinger wave-functional $\Psi[X^\mu]$, a mapping from the space of string configurations to the complex numbers;
2. as a vector $|\Psi>$ in Fock space;
3. as a local vertex operator $\mathcal{V}_\Psi$ in the CFT of matter and ghost fields.

We have already discussed the last two representations in chapter 2 and will now comment on their relation with the first, perhaps less familiar, representation.

As discussed in section 2.3.4, it is convenient to represent the open string world-sheet as the upper half plane. The original world-sheet, an infinite strip, is parametrised by (τ, σ) with $-\infty < \tau < \infty$, $0 \leq \sigma \leq \pi$. We use the conformal mapping $z = -e^{\tau - i\sigma}$ to map the strip onto the complex plane (the phase factor is introduced here for later convenience). Time runs radially in the new parametrisation (see figure 3.1).

On the real axis, the fields should obey Neumann boundary conditions. This is taken care of by using the doubling trick to extend all the fields to the whole complex plane as explained in section 2.3.4. We will restrict our attention the X^μ sector at first and comment on the extension to the ghost sector later.

We can now consider the position operator $\hat{X}^\mu$ (the hat symbol is introduced to distinguish the operator from its classical counterpart) on the constant time slice $|z| = 1$ ($\tau = 0$). To avoid later confusion, we introduce the parameter $\zeta \equiv -e^{-i\sigma}$ on the unit circle. Formula (2.3.27) gives the mode expansion of $\hat{X}^\mu(\zeta)$ as:

$$\hat{X}^\mu(\zeta) \quad = \quad \hat{x}^\mu + \sum_{n=1}^{\infty} \hat{X}^\mu_n(\zeta^n + \bar{\zeta}^n)$$

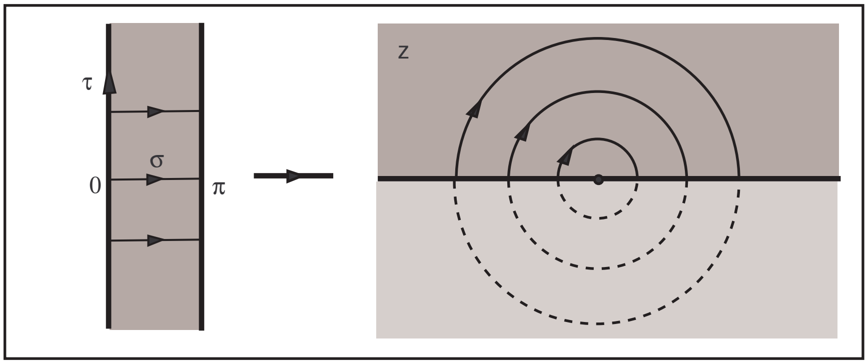


Figure 3.1: Relation between the coordinates $(\tau,\ \sigma)$ on the strip and the complex coordinate z.

$$= \quad \hat{x}^\mu + 2\sum_{n=1}^{\infty}(-1)^n \hat{X}_n^\mu \cos n\sigma,$$

where the operators $\hat{X}_n^\mu$, which are self-conjugate with respect to the Hermitean inner product defined in section 2.3.5, are given in terms of the previously defined oscillator modes as

$$\hat{X}_n^\mu = \frac{i}{\sqrt{2n}}(\alpha_n^\mu - \alpha_{-n}^\mu).$$

We can now consider a complete basis of 'position eigenstates' $|X^\mu(\zeta)>$ satisfying

$$\hat{X}^\mu(\zeta)|X^\mu(\zeta)>= X^\mu(\zeta)|X^\mu(\zeta)> .$$

These eigenstates can be expressed explicitly in the oscillator basis ([28]). The result is (up to an overall normalisation factor)

$$|X^\mu(\zeta)>= \exp\sum_{n=1}^{\infty}[-\frac{n}{2}X_n^\nu X_{\nu n} + i\sqrt{2}X_n^\nu \alpha_{\nu -n} + \frac{1}{2n}\alpha_{-n}^\nu \alpha_{\nu -n}]|x^\mu>, \quad (3.1.1)$$

where $|x^\mu>$ stands for an eigenstate of the center-of mass position $\hat{x}^\mu$.

Any state $|\Psi>$ can now be represented as a Schrödinger wave-functional

$\Psi[X^\mu(\zeta)]$ by expanding it in the basis of position eigenstates:

$$|\Psi> = \int_C [DX]\ \Psi[X^\mu(\zeta)]\ |X^\mu(\zeta)> .$$

with $\Psi[X^\mu(\zeta)] = <X^\mu(\zeta)|\Psi>$. The subscript C means that the functional integral runs over fields defined on the unit circle. This establishes the relation between the representations 2 and 1.

In chapter 2, we have argued that a local operator $\mathcal{V}_\Psi(0)$ can be seen as a Schrödinger wave-functional obtained by expanding $|\Psi>$ in eigenstates of the position operator in the limit $\tau \to -\infty$, in which the circle of constant τ shrinks to a single point (i.e. the origin). This provides the link between the representations 2 and 3.

We can also work out a direct relation between the representations 3 and 1 which will be of use to us in section 3.1.4. From the foregoing, it is clear that the wave-functional $\Psi[X^\mu(\zeta)]$ arises from the vertex operator $\mathcal{V}_\Psi(0)$ under time evolution from $\tau = -\infty$ to $\tau = 0$. In path-integral language, this means that $\Psi[X^\mu(\zeta)]$ can be written as a functional integral on the unit disc D with $\mathcal{V}_\Psi$ inserted at the origin:

$$\Psi[X^\mu(\zeta)] = \int_D [D\tilde{X}]_{|X^\mu} e^{-S[\tilde{X}^\mu]} \mathcal{V}_\Psi(0). \tag{3.1.2}$$

where the integral is subject to the boundary condition $\tilde{X}^\mu(\zeta) = X^\mu(\zeta)$ on the unit circle.

As an example, we will now work out the various representations of the Fock vacuum state $|0>$. By expanding into position eigenstates at $\tau = 0$ using (3.1.1), one gets the Schrödinger representation which is of the Gaussian form:

$$\Psi_0[X^\mu(\zeta)] = e^{-\frac{1}{2}\sum_{n=1}^{\infty} n X_n^\mu X_{n\mu}}. \tag{3.1.3}$$

We have argued in section 2.3.3 that $|0>$ is represented by the vertex operator 1. We can now explicitly verify this by showing that under time evolution to $\tau = 0$ it reproduces the wave-functional (3.1.3). To do this, we have to calculate the path-integral (3.1.2) with an insertion of the operator 1. The result is, up to a normalisation factor, given by the contribution from the action of the classical solution $\tilde{X}^\mu(z,\bar{z})$ on the disc that obeys the boundary condition $\tilde{X}^\mu(\zeta) = X^\mu(\zeta)$ on the boundary:

$$\Psi_0[X^\mu(\zeta)] = e^{-S[\tilde{X}^\mu(z,\bar{z})]}$$

where

$$S[\tilde{X}^\mu(z,\bar{z})] = \frac{1}{4\pi}\sum_{m,n=1}^{\infty} mn\, X_m^\mu X_{n\mu}\int_D d^2 z z^{m-1}\bar{z}^{n-1}$$
$$= \frac{1}{2}\sum_{n=1}^{\infty} nX_n^\mu X_{n\mu}$$

in agreement with (3.1.3).

The Fock-space raising and lowering operators act in the Schrödinger basis as:

$$\alpha_n^\mu = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial X_{n\mu}} + nX_n^\mu\right) \qquad n > 0$$
$$\alpha_{-n}^\mu = \frac{i}{\sqrt{2}}\left(-\frac{\partial}{\partial X_{n\mu}} + nX_n^\mu\right) \qquad n > 0$$

This follows from the commutation relations (2.3.25). Using this representation, the Schrödinger representation of an arbitrary string field can be constructed from the vacuum wave-functional as

$$\Psi[X^\mu(\zeta)] = \left[\phi(x) + A_\mu(x)\alpha_{-1}^\mu + B_\mu(x)\alpha_{-2}^\mu + \ldots + A_{\mu\nu}(x)\alpha_{-1}^\mu\alpha_{-1}^\mu + \ldots\right]\Psi_0[X^\mu(\zeta)]$$

The component fields ϕ, A_μ, $B_\mu, \ldots$ in this expansion depend only on the center-of-mass coordinate x and can be interpreted as ordinary space-time fields. From this fact we see that any string field theory is, from the space-time point of view, a field theory with an infinite number of fields, one for every vibrational mode of the string.

The three representations of a string field give rise to three equivalent formulations of string field theory. Witten's covariant string field theory, which will be described in the next sections, was originally written in terms of the Schrödinger wave-functional $\Psi[X^\mu(\zeta)]$. Various authors worked out the two other representations in the following order:

Schrödinger wavefunctionals $\Psi[X^\mu(\zeta)]$ [8]
↓
Fock space states $|\Psi>$ [28, 29, 30, 31]
↓
vertex operators $\mathcal{V}_\Psi$ [33, 34]

In section 3.1.3, we will review string field theory in the first representation, where the intuitive notion that string interactions arise from the splitting and joining of strings is most apparent. This representation is however not the one that is best suited for doing concrete calculations such as the ones we will perform in chapters 5 and 7. Therefore we will, in section 3.1.4, using the relations we have just derived, pass to the third representation of the string field, in which the string field theory action is conveniently represented in terms of CFT correlators. For a discussion of string field theory in the second representation, which uses to the so-called *Neumann coefficients*, we refer to the Ph.D. thesis of Pieter-Jan De Smet [35].

Before turning to the explicit form of the string field theory action, we first collect some facts about the algebraic structure that underlies it.

3.1.2 Formal algebraic structure

Witten's proposal for an interacting open string field theory [8] takes the form of a generalisation of Chern-Simons gauge field theory. We will first discuss the algebraic framework that underlies this type of theory and later realize and interpret the various quantities and operations in the specific context of string field theory.

Consider an algebra over the complex numbers whose elements we will call 'fields' A and where the multiplication operation will be denoted by $\star$. We suppose that the algebra comes equipped with a $\mathbf{Z}_2$ grading so that to every field A is associated a degree $(-1)^A$ which is ± 1. We further assume the existence of an 'integration' $\oint$ and an operation Q on the algebra satisfying the following axioms:

- The $\star$ *product* is associative and the degree $(-1)^A$ is multiplicative under the $\star$ operation:

$$(A \star B) \star C = A \star (B \star C) \tag{3.1.4}$$
$$(-1)^{A \star B} = (-1)^A \cdot (-1)^B \tag{3.1.5}$$

- The *operation* Q is nilpotent, has degree -1 and is a derivation of the $\star$ algebra:

$$Q^2 A = 0 \tag{3.1.6}$$
$$(-1)^{QA} = -(-1)^A \tag{3.1.7}$$
$$Q(A \star B) = (QA) \star B + (-1)^A A \star QB \tag{3.1.8}$$

- The *integration* $\oint$ is a linear functional on the algebra of fields and has the properties

$$\oint QA = 0 \tag{3.1.9}$$

$$\oint A \star B = (-1)^{AB} \oint B \star A \tag{3.1.10}$$

These axioms guarantee that one can write down a Chern-Simons-type *action functional* of the fields A

$$S[A] = \oint (\frac{1}{2} A \star QA + \frac{g}{3} A \star A \star A) \tag{3.1.11}$$

which is invariant under *'gauge transformations'* under which A transforms as

$$\delta A = Q\varepsilon + g\, A \star \varepsilon - g\, \varepsilon \star A \tag{3.1.12}$$

where ε is another field and where, in order for all terms in (3.1.12) to have the same degree, we will require A and ε to have degree -1 and $+1$ respectively. The parameter g plays the role of a coupling constant. The *field equations* following from (3.1.11) state that the 'field strength' F vanishes:

$$F \equiv QA + gA \star A = 0.$$

An important feature of this construction is that the $\star$ product is *not required to be commutative*.

An example in which all the axioms are satisfied is in the theory of gauge fields. In this case, the fields A are taken to be matrix-valued differential forms on a manifold, the degree of a k-form to be $(-1)^k$, the operator Q to be de Rham operator, $\star$ to be the wedge product combined with matrix multiplication and $\oint$ to be a combination of ordinary integration and matrix trace. On a three-dimensional manifold, the only fields contributing to the action are the 1-forms, while the gauge parameters are 0-forms and form a closed subalgebra under the $\star$-product. The action (3.1.11) is then the Chern-Simons action.

Although the properties (3.1.4-3.1.10) represent the actual algebraic structure underlying Witten's string field theory, we will not give a complete proof of them in this thesis. One can remark however that, in order to prove gauge invariance of the action (3.1.11), one only needs a weaker set of properties. In particular, it is sufficient for this purpose that the properties (3.1.4-3.1.8) hold within the integration $\oint$. We will use this fact in section 3.1.4 when discussing gauge invariance in string field theory.

3.1.3 The $\star$ product and midpoint interactions

For string theory, in view of (3.1.6-3.1.8), a natural candidate for the nilpotent operator Q is the *BRST charge* Q_B, while a natural grading is provided by $(-1)^{Q_{gh}}$ where Q_{gh} is the *ghost number*. As in the example above, it will turn out that only the fields of ghost number 1 contribute to the action, while the gauge parameters will have ghost number 0 and form a closed algebra under the $\star$-product.

We will now go into the definition of the $\star$ and $\oint$ operations as formulated in [8]. The intuitive picture underlying these definitions is the idea that the $\star$ operation should describe a joining of two strings into one, so that the cubic term in the action (3.1.11) can be interpreted as describing a three-string interaction vertex, similar to the situation in quantum field theory, where a cubic interaction term describes processes where two particles annihilate and a third particle is created (see the Introduction). Let's suppress the ghosts for the moment and try to find suitable $\star$ and $\oint$ operations in the X^μ sector.

$\star$ product

We can start by looking for an associative $\star$ operation on the position eigenstates $|X^\mu(\zeta)>$. This operation should combine two position eigenstates $|X_1^\mu(\zeta)>$ and $|X_2^\mu(\zeta)>$ into a new state $|X_1^\mu(\zeta)>\star|X_2^\mu(\zeta)>$.

A first attempt at defining a $\star$ product would be to take the operation of joining strings at their endpoints: the $\star$ product is zero unless the endpoint of the X_1 configuration coincides with the starting point of X_2, in which case the $\star$ product yields a new position eigenstate obtained by joining the two strings together (figure 3.2(a)). The resulting operation is almost, but not quite, associative (see figure 3.2(b)): if one compares the configurations corresponding to $(|X_1> \star|X_2>) \star |X_3>$ and $|X_1> \star(|X_2> \star|X_3>)$ one sees that they are parametrised differently. Since we are working in a gauge-fixed formalism in which the reparametrisation freedom is no longer present, these configurations are not equivalent.

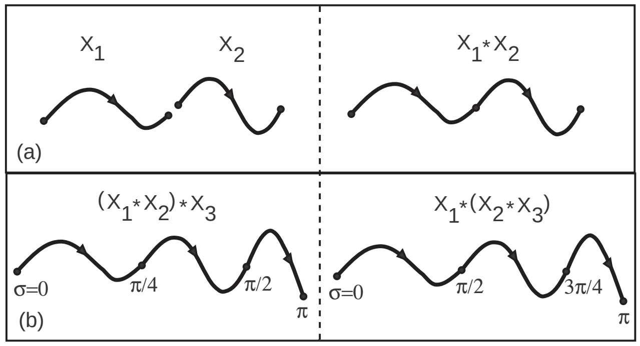


Figure 3.2: (a) A $\star$ product operation defined by joining strings at their end-points. (b) The resulting operation is not quite associative since $(X_1 \star X_2) \star X_3$ and $X_1 \star (X_2 \star X_3)$ are parametrised differently.

A proposal which leads to a truly associative $\star$ operation is the joining of strings at their mid-points. The mid-point of a string configuration is defined to be the space-time point corresponding to $\sigma = \pi/2$ or $\zeta = i$. Witten's definition of the $\star$ product is as follows (see figure 3.3(a)): the strings $|X_1>$ and $|X_2>$ can join only if the right part of $|X_1>$ (the image of the points ζ on the unit circle with Re $(\zeta) \geq 0$) coincides with the left part of $|X_2>$ (the image of the points on the unit circle with Re $(\zeta) \leq 0$). The result is a new position eigenstate $|X_1> \star |X_2>$ obtained by joining the left part of $|X_1>$ to the right part of $|X_2>$. Denoting the left and right parts of a string configuration by X_L and X_R respectively, the definition of the $\star$ product can be written as:

$$|(X_{1L}, X_{1R})> \star |(X_{2L}, X_{2R})>= |(X_{1L}, X_{2R})> \delta(X_{1R} - X_{2L}), \tag{3.1.13}$$

where $\delta(X_{1R} - X_{2L})$ stands for $\prod_\mu \prod_{\zeta, \text{Re } (\zeta) \leq 0} \delta\left(X_1^\mu(-1/\zeta) - X_2^\mu(\zeta)\right)$. The $\star$ operation defined in this way is associative as illustrated in figure 3.3(b).

Integration

We now turn to the definition of the $\oint$ operation. From formula (3.1.10) and associativity, $\oint$ should satisfy $\oint(|X_1> \star |X_2> \star |X_3>) = \oint(|X_3> \star |X_1>$

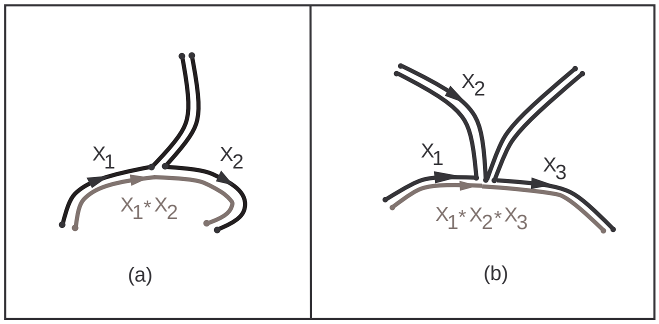


Figure 3.3: (a) In Witten's $\star$ operation, strings X_1 and X_2 can join only if the right part of X_1 coincides with the left part of X_2. (b) The resulting operation is associative.

$\star|X_2>)$. The way to accomplish this is by defining the integral of a position eigenstate $|X>$ to be 0 unless the left part X_L coincides with the right part X_R:

$$\oint |X^\mu(z)>= \delta(X_L - X_R) \tag{3.1.14}$$

as illustrated in figure (3.4).

From the definitions of the $\star$ and $\oint$ operations on position states (3.1.13, 3.1.14) we immediately deduce their action on general string fields in the

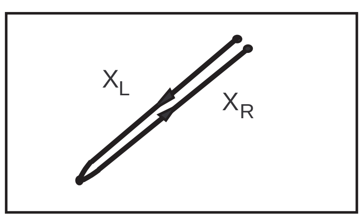


Figure 3.4: The integration operation identifies the left- and right halves of the string.

Schrödinger representation $\Psi[X^\mu(\zeta)]$:

$$\begin{aligned} \Psi_1 \star \Psi_2[(X_L, X_R)] &= \int_C [D\tilde{X}]\Psi_1[(X_L, \tilde{X}_R)]\Psi_2[(\tilde{X}_L, X_R)]\delta(\tilde{X}_R - \tilde{X}_L) \\ \oint \Psi[(X_L, X_R)] &= \int_C [DX]\Psi[(X_L, X_R)]\delta(X_L - X_R). \end{aligned} \quad (3.1.15)$$

Extension to the ghost sector

The definition of the $\star$ and $\oint$ operations can be generalised in a straightforward manner to include the ghost fields (b, c). This is most easily done[8, 28] by using the bosonised form of the ghost system in which the (b, c) fields are represented by a single linear dilaton field (see section 2.5.4) φ with $\varepsilon = 1$ and $Q = -3$:

$$b = :e^{-\varphi}: \qquad c = :e^{\varphi}: .$$

The extension of $\star$ and $\int$ to the ghost system now follows by treating the φ field as an extra scalar field in addition to the X^μ. For example, the string field in the Schrödinger representation becomes a functional $\Psi[X^\mu(\zeta), \varphi(\zeta)]$ of the X^μ fields as well as the the bosonised ghost φ. In the following, we will suppress the dependence on the φ field for notational simplicity.

Reality condition

The above definitions allow us to write down an action of the form (3.1.11) for the string field $\Psi[X]$:

$$S[\Psi] = \oint (\frac{1}{2}\Psi \star Q\Psi + \frac{g}{3}\Psi \star \Psi \star \Psi). \quad (3.1.16)$$

This action will not be real for a generic complex-valued string field $\Psi[X]$. To make it so, an extra *reality condition* on the fields is required. This condition is taken to be [36]

$$\Psi[X^\mu(\zeta)] = \Psi^*[X^\mu(-1/\zeta)]. \quad (3.1.17)$$

If this condition is satisfied, the integral of a star product of fields $\oint \Psi_1 \star \Psi_2$ reduces to an inner product for wave-functionals reminiscent of ordinary quantum mechanics:

$$\oint \Psi_1 \star \Psi_2 = \int_C [DX]\ \Psi_1^*[X]\Psi_2[X].$$

We will come back to the issue of reality of the action in the next section.

3.1.4 String field theory in the vertex operator representation

In order to make concrete calculations in string field theory, we need to make the operations defined in the previous section a bit more explicit. This will also be required to give a complete proof of invariance of the action (3.1.11) under the gauge transformation (3.1.12). In particular, we have not yet established the derivative property (3.1.8). For this purpose, we will use the vertex operator representation of the string field. This will allow us to express integrals of $\star$ products of an arbitrary number of fields as CFT correlators on the sphere. Gauge invariance of the action will then follow from the properties of those CFT correlators. We will also be able to translate the reality condition (3.1.17) into conformal field theory language.

Relation with CFT correlators

Consider first an integral of a product of two string fields $\oint \Psi_1 \star \Psi_2$. For notational simplicity, we will display only the dependence on the X^μ, the extension to the ghost sector being straightforward. As discussed in section 3.1.1, Ψ_1 and Ψ_2 can be represented as path-integrals on unit discs D_1 and D_2 with vertex operator insertions $\mathcal{V}_{\Psi_1}$ and $\mathcal{V}_{\Psi_2}$ respectively. We denote the coordinates on the discs D_1 and D_2 by w_1 and w_2 respectively. Combining the expressions (3.1.15) with the representation (3.1.2) gives

$$\oint \Psi_1 \star \Psi_2 = \int_{D_1} [DX_1] \int_{D_2} [DX_2]\, e^{-S[X_1]-S[X_2]} \prod_{\zeta \in C} \delta(X_1(-1/\zeta) - X_2(\zeta))$$

$$\times \mathcal{V}_{\Psi_1}(X_1(0))\mathcal{V}_{\Psi_2}(X_2(0)) \tag{3.1.18}$$

The delta-function implies that the discs D_1 and D_2 should be glued together along their boundaries with the identification

$$w_1 w_2 = -1, \qquad |w_1| = |w_2| = 1. \tag{3.1.19}$$

The resulting manifold is a sphere where the coordinate patches w_1 and w_2 each cover a hemisphere, the identification (3.1.19) providing the transition function on the equator where the patches overlap. Things become much simpler when written in terms of a coordinate $z \in \bar{\mathbf{C}}$ which is well-defined in the neighbourhood of the equator. The coordinate z is related to w_1 and w_2 by holomorphic transition functions

$$\begin{aligned} z &= t_1(w_1) \qquad \text{in patch 1,} \\ z &= t_2(w_2) \qquad \text{in patch 2.} \end{aligned}$$

In order for z to be well-defined, these should satisfy

$$t_1(w) = t_2(-1/w) \qquad \text{for} |w| = 1.$$

In terms of the new coordinate z, the path integral (3.1.18) becomes a CFT correlator on the 2-sphere:

$$\begin{aligned} \oint \Psi_1 \star \Psi_2 &= \int_{S_2} [DX]\, e^{-S[X]}\, t_1 \circ \mathcal{V}_{\Psi_1}(0)\; t_2 \circ \mathcal{V}_{\Psi_2}(0) \\ &= \left\langle t_1 \circ \mathcal{V}_{\Psi_1}(0) t_2 \circ \mathcal{V}_{\Psi_2}(0) \right\rangle. \end{aligned} \tag{3.1.20}$$

Various choices for the functions t_1 and t_2 are possible, related by global conformal transformations. The result is independent of the specific choice due to conformal invariance of the correlator (3.1.20). The simplest choice for t_1 and t_2 is to take

$$\begin{aligned} t_1(z) &= -1/z \equiv I(z), \\ t_2(z) &= z. \end{aligned}$$

With this choice, we see that the integral (3.1.20) reduces to the BPZ inner product we encountered before in 2.3.5:

$$\oint \Psi_1 \star \Psi_2 = \ll \Psi_1 | \Psi_2 > . \tag{3.1.21}$$

Another choice, easily generalised to integrals of star products of any number of fields, is to take

$$\begin{aligned} t_1(z) &= f_1^{(2)}(z) \equiv \left(\frac{1+iz}{1-iz}\right), \\ t_2(z) &= f_2^{(2)}(z) = -\left(\frac{1+iz}{1-iz}\right). \end{aligned}$$

These functions map D_1 and D_2 to the half-planes Re $z \geq 0$ and Re $z \leq 0$ respectively, with the transformed vertex operators inserted at $z = +1$ and $z = -1$ (see figure 3.5).

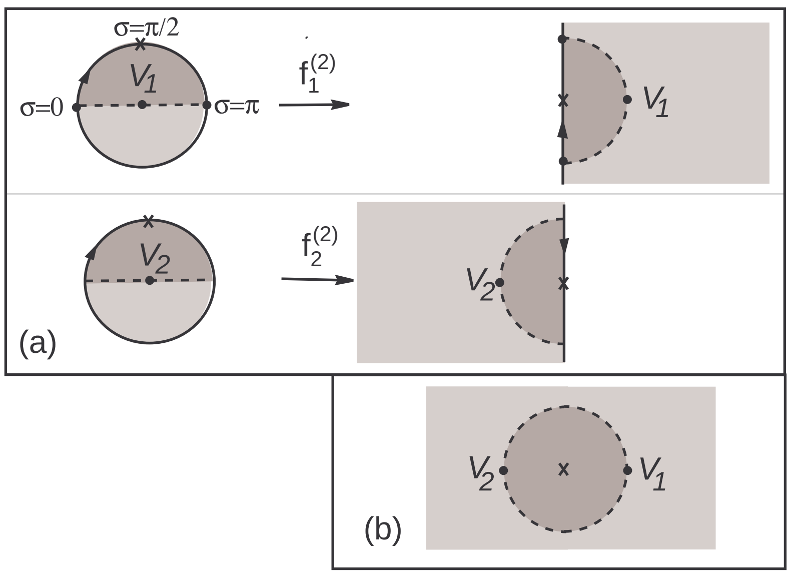


Figure 3.5: (a) The mappings $f_1^{(2)}$ and $f_1^{(2)}$ define a well-defined coordinate z on the sphere. (b) The integral $\int \Psi_1 \star \Psi_2$ reduces to a correlation function on the sphere.

A similar procedure can be followed to work out the three-point vertex $\oint(\Psi_1 \star \Psi_2 \star \Psi_3)$. The three string fields can be represented as path-integrals on unit discs D_1, D_2 and D_3 on which we choose coordinates w_1, w_2 and w_3. These should be glued together along their boundaries according to the identifications:

$$\begin{aligned} w_1 w_2 &= -1 \qquad \text{for } |w_1| = |w_2| = 1;\ \mathrm{Re}\ w_1 \geq 0;\ \mathrm{Re}\ w_2 \leq 0, \\ w_2 w_3 &= -1 \qquad \text{for } |w_2| = |w_3| = 1;\ \mathrm{Re}\ w_2 \geq 0;\ \mathrm{Re}\ w_3 \leq 0, \\ w_3 w_1 &= -1 \qquad \text{for } |w_3| = |w_1| = 1;\ \mathrm{Re}\ w_3 \geq 0;\ \mathrm{Re}\ w_1 \leq 0. \end{aligned}$$

The integral $\oint(\Psi_1 \star \Psi_2 \star \Psi_3)$ can once again be written as a correlator on the the two-sphere by transforming to a suitable coordinate z that is well-defined in the overlap region. The transition functions t_1, t_2, t_3 should obey:

$$\begin{aligned} t_1(-1/w) &= t_2(w) \qquad \text{for } |w| = 1;\ \mathrm{Re}\ w \geq 0, \\ t_2(-1/w) &= t_3(w) \qquad \text{for } |w| = 1;\ \mathrm{Re}\ w \geq 0, \\ t_3(-1/w) &= t_1(w) \qquad \text{for } |w| = 1;\ \mathrm{Re}\ w \geq 0. \end{aligned}$$

The following choice does the trick:

$$t_k(w_k) = f_k^{(3)}(w_k) \equiv e^{\frac{2\pi i(k-1)}{3}} \left(\frac{1+iw_k}{1-iw_k}\right)^{2/3} \qquad k = 1, \ldots, 3.$$

The branch for the fractional power $z = w^{2/3}$ is chosen to be $-\pi/3 \leq Arg(z) \leq \pi/3$. These functions map the unit discs D_k into wedge-shaped portions of the complex plane (see figure 3.6). The three-point vertex reduces to the correlator

$$\oint(\Psi_1 \star \Psi_2 \star \Psi_3) = \left\langle f_1^{(3)} \circ \mathcal{V}_{\Psi_1}(0)\ f_2^{(3)} \circ \mathcal{V}_{\Psi_2}(0)\ f_3^{(3)} \circ \mathcal{V}_{\Psi_3}(0) \right\rangle.$$

The choice of mappings $f_k^{(3)}$ is essentially unique in the sense that all other possible choices t_k are related to this one by a common conformal transformation [33]:

$$t_k = g \circ f_k^{(3)}.$$

For example, by choosing a suitable g, one obtains a representation which more closely resembles the intuitive picture of two strings joining to form a third.

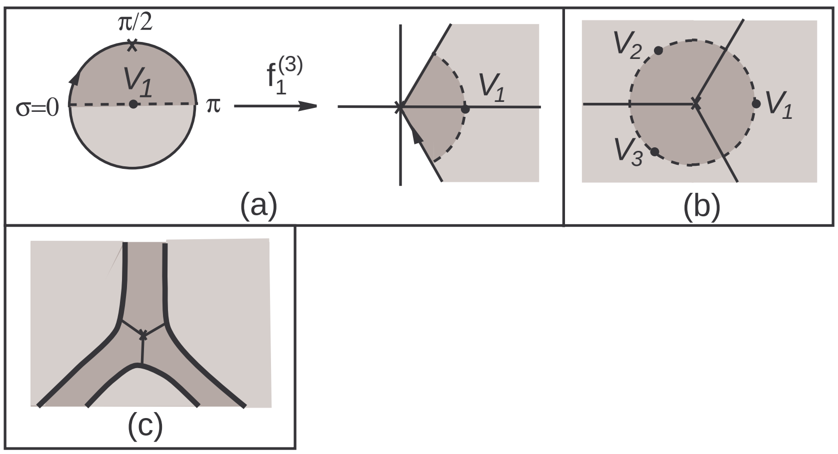


Figure 3.6: (a) The mappings $f_k^{(3)}$ map the unit disc to wedge-shaped portions covering the complex plane. (b) The three-point vertex reduces to a correlator on the sphere. (c) After performing a conformal transformation, we obtain a representation which more closely resembles the intuitive picture of two strings joining to form a third.

Similarly, a general integral $\oint(\Psi_1 \star \Psi_2 \star \dots \Psi_n)$ reduces to a correlator for which we will use the notation $\langle\langle\ \mathcal{V}_{\Psi_1}\ \mathcal{V}_{\Psi_2} \dots \mathcal{V}_{\Psi_n}\ \rangle\rangle$:

$$\begin{aligned}\oint \Psi_1 \star \Psi_2 \star \dots \Psi_n &= \langle\langle\ \Psi_1\ \Psi_2 \dots \Psi_n\ \rangle\rangle \\ &= \Big\langle f_1^{(n)} \circ \mathcal{V}_{\Psi_1}(0)\ f_2^{(n)} \circ \mathcal{V}_{\Psi_2}(0)\ \dots\ f_n^{(n)} \circ \mathcal{V}_{\Psi_n}(0) \Big\rangle.\end{aligned}$$

The conformal transformations $f_k^{(n)}$ are mappings from the unit disc to wedge-shaped portions of the complex plane:

$$f_k^{(n)}(z) = e^{\frac{2\pi i(k-1)}{n}} \left(\frac{1+iz}{1-iz}\right)^{2/n} \qquad k = 1, \dots, n. \tag{3.1.22}$$

The branch of $z = w^{2/n}$ is chosen to be $-\pi/n \leq Arg(z) \leq \pi/n$.

CFT representation of the $\star$ product

We have just presented a representation of integrals of $\star$ products in terms of correlators between vertex operators in CFT. One might wonder whether the $\star$ product itself can also be worked out in terms of manipulations in CFT such as taking OPE's and performing conformal transformations. It turns out that an expression for the star product of two string fields, $\Psi_1 \star \Psi_2$, can be derived from the formula for the three-point vertex derived above. Indeed, from equation (3.1.21) we know that the three-point vertex $\oint \Phi \star \Psi_1 \star \Psi_2$ can be written as the BPZ inner product $\ll \Phi | \Psi_1 \star \Psi_2 >$. Hence we should be able to derive an expression for $\Psi_1 \star \Psi_2$ by manipulating the three-point vertex $\oint \Phi \star \Psi_1 \star \Psi_2$, for arbitrary Φ, until it has the form of a BPZ inner product $\ll \Phi | \dots >$.

We start from the previously derived expression

$$\oint \Phi \star \Psi_1 \star \Psi_2 = \Big\langle f_1^{(3)} \circ \mathcal{V}_{\Phi}(0)\ f_2^{(3)} \circ \mathcal{V}_{\Psi_1}(0)\ f_3^{(3)} \circ \mathcal{V}_{\Psi_2}(0) \Big\rangle.$$

The first insertion $f_1^{(3)} \circ \mathcal{V}_\Phi(0)$ can be seen as the vertex operator representing a state $U_{f_1^{(3)}}|\Phi>$ where the operator $U_{f_1^{(3)}}$ implements the conformal transformation $f_1^{(3)}(z)$ in Fock space. $U_{f_1^{(3)}}$ can be written in terms of infinitesimal generators as

$$U_{f_1^{(3)}} = \exp \sum_{n \in \mathbf{Z}} v_n L_n, \tag{3.1.23}$$

where the L_n are the modes of the total (i.e. matter and ghost) energy-momentum tensor and the coefficients v_n can be computed from the Laurent coefficients of the function $f_1^{(3)}(z)$ through a recursive procedure [34, 37].

The next step is to perform an $SL(2,\mathbf{C})$ transformation by $I(z) = -1/z$:

$$\oint \Phi \star \Psi_1 \star \Psi_2 = \left\langle I \circ \mathcal{V}_{U_{f_1^{(3)}}\Phi}(0)\; I \circ f_2^{(3)} \circ \mathcal{V}_{\Psi_1}(0)\; I \circ f_3^{(3)} \circ \mathcal{V}_{\Psi_2}(0) \right\rangle.$$

We can now use the OPE to replace the last two vertex operators by some linear combination of vertex operators inserted at the origin. Introducing the notation $\mathcal{V}_\Psi$ for this particular linear combination of vertex operators, we have

$$I \circ f_2^{(3)} \circ \mathcal{V}_{\Psi_1}(0)\; I \circ f_3^{(3)} \circ \mathcal{V}_{\Psi_2}(0) = \mathcal{V}_\Psi(0) \tag{3.1.24}$$

We are now able to rewrite the three-point function as a BPZ inner product:

$$\oint \Phi \star \Psi_1 \star \Psi_2 = \ll \Phi\; U_{f_1^{(3)}} |\Psi> .$$

We can now use inverse BPZ conjugation to let the operator act on the 'ket' $|\Psi>$. Doing this, we obtain an expression for $|\Psi_1 \star \Psi_2>$:

$$|\Psi_1 \star \Psi_2> = \mathrm{bpz}(U_{f_1^{(3)}})|\Psi> . \tag{3.1.25}$$

Using (2.3.30) and (3.1.23), the operator $\mathrm{bpz}(U_{f_1^{(3)}})$ can be also written as

$$\mathrm{bpz}(U_{f_1^{(3)}}) = \exp \sum_{n \in \mathbf{Z}} (-1)^n v_n L_n.$$

The expression (3.1.25) makes it possible to calculate star products, at least in principle. In practice, as should be clear from the procedure we have just outlined, calculating star products is a difficult matter and few explicit examples are known. We will consider here only one rather trivial example.

Consider the $\star$ product of the $SL(2,\mathbf{C})$ invariant vacuum $|0>$ (corresponding to the vertex operator 1) with itself. The OPE in (3.1.24) is trivial in this case: the vertex operator $\mathcal{V}_\Psi$ on the right-hand side is again the operator 1. The formula (3.1.25) then yields

$$|0> \star |0> = \mathrm{bpz}(U_{f_1^{(3)}})|0> .$$

The resulting state is completely determined by the conformal transformation $f_1^{(3)}$; states with this property are also called *surface states*. Explicitly, the first few components are given by [38]:

$$|0> \star |0> = \exp\left(-\frac{5}{27}L_{-2} + \frac{13}{486}L_{-4} - \frac{317}{39366}L_{-6} + \ldots\right)|0> .$$

More examples of star products are worked out in [38].

Gauge invariance

Since nonvanishing CFT correlators require a total ghost number of three, we see that the string fields contributing to the action

$$S[\Psi] = \frac{1}{2}\langle\langle\, \mathcal{V}_\Psi Q_B \mathcal{V}_\Psi \,\rangle\rangle + \frac{g}{3}\langle\langle\, \mathcal{V}_\Psi\; \mathcal{V}_\Psi\; \mathcal{V}_\Psi \,\rangle\rangle \tag{3.1.26}$$

should have ghost number 1 (recall that this is the natural ghost number for vertex operators obtained by acting with raising operators on the ghost vacuum $|\downarrow>$ as in section 2.3.6). This means that gauge parameters have ghost number 0 and form a closed subalgebra under the $\star$ product.

In section 3.1.2, we summarised the algebraic structure of string field theory in the properties (3.1.4-3.1.10). Although these properties can be shown to hold (for example, we have established associativity of the $\star$ product in 3.1.3), but, for the purpose of proving gauge invariance of the action, it is sufficient for these properties to hold within the $\oint$ operation. A similar procedure will be followed when we will discuss gauge invariance in superstring field theory in section 3.2.

Translated into CFT language, gauge invariance of the action (3.1.26) can be established by proving the following properties of the CFT correlators (compare p. 62)

$$\begin{aligned}
\langle\langle \cdots Q_B^2(\Phi_1\cdots\Phi_n)\cdots \rangle\rangle &= 0\,, && (3.1.27)\\
\langle\langle\, Q_B(\Phi_1\cdots\Phi_n)\,\rangle\rangle &= 0\,, && (3.1.28)\\
\langle\langle \cdots Q_B(\Psi\Phi)\cdots \rangle\rangle &= \langle\langle \cdots (Q_B\Psi\,\Phi - \Psi\, Q_B\Phi)\cdots \rangle\rangle\,, &&\\
\langle\langle \cdots Q_B(\varepsilon\Phi)\cdots \rangle\rangle &= \langle\langle \cdots (Q_B\varepsilon\,\Phi + \varepsilon\, Q_B\Phi)\cdots \rangle\rangle\,, && (3.1.29)\\
\langle\langle\, \Phi_1\cdots\Phi_{n-1}\Phi_n \,\rangle\rangle &= \langle\langle\, \Phi_n\Phi_1\cdots\Phi_{n-1} \,\rangle\rangle && (3.1.30)
\end{aligned}$$

where Ψ and ε represent the string field (ghost number 1) and the gauge parameter (ghost number 0) respectively and the Φ_i represent CFT operators of arbitrary ghost number. The proof of the properties (3.1.27-3.1.30) goes as follows:

- Property (3.1.27) is nothing but the statement that Q_B is nilpotent, which, as we have seen in section 2.4, is the case in the critical dimension.
- Property (3.1.28) follows from writing Q_B as a contour integral of j_B where the contour encircles all the vertex operators Φ_i. By doing an $SL(2,\mathbf{C})$ transformation to a coordinate $u = 1/z$ and using that j_B is a weight 1 primary field, one sees that, in the u-plane, the operator j_B encircles the origin with no vertex operator insertions inside the contour. Hence, the result vanishes by analyticity.
- The first equality in (3.1.29) follows from writing Q_B as a contour integral of j_B encircling the operators Ψ and Φ. The contour integral can be written as a sum of two terms: an integral where the contour encircles only Ψ and one where the contour encircles only Φ. The second term comes with a minus sign since one needs to interchange j_B and Ψ and both are Grassmann odd fields. One then uses the fact that Q_B is a weight 0 primary field so that applying Q_B commutes with applying a conformal transformation.

  The second equality in (3.1.29) is proven in the same way, using the fact that ε is a Grassmann even field.
- To prove the property (3.1.30) we start by performing the $SL(2,\mathbf{C})$ transformation of rotating over an angle $2\pi/n$. The correlator reduces to

$$\langle f_2^{(n)} \circ \Phi_1(0) \cdots f_n^{(n)} \circ \Phi_{n-1}(0)\ R \circ f_1^{(n)} \circ \Phi_n(0)\rangle \tag{3.1.31}$$

  where R denotes a rotation over 2π. If Φ_n has conformal weight h, the rotation R yields a phase factor:

$$R \circ f_1^{(n)} \circ \Phi_n(0) = e^{2\pi i h} f_1^{(n)} \circ \Phi_n(0).$$

  Consider first the case that Φ_n has space-time momentum zero. Such a state is constructed from the Fock vacuum by applying integer-moded raising operators and its conformal weight h is an integer, hence $e^{2\pi i h} = 1$. We now look what happens if we move $f_1^{(n)} \circ \Phi_n(0)$ to the front in the correlator (3.1.31). Since a nonvanishing amplitude should have ghost number three and hence be Grassmann odd, we see that no phase is picked by moving $f_1^{(n)} \circ \Phi_n(0)$ past the other vertex operators and the property (3.1.30) holds.

If Φ_n is an operator of space-time momentum k^μ, the rotation R will yield a phase factor $e^{\pi i\alpha' k^2}$. On the other hand, another phase is picked up when moving $f_1^{(n)} \circ \Phi_n(0)$ to the front. Indeed, from the definition of normal ordering (2.3.4) we have

$$\begin{aligned} :e^{ik_1\cdot X}(z_1)::e^{ik_2\cdot X}(z_2): &= (z_1 - z_2)^{\alpha' k_1\cdot k_2} :e^{ik_1\cdot X}(z_1)e^{ik_2\cdot X}(z_2): \\ &= e^{\pi i\alpha' k_1\cdot k_2} :e^{ik_2\cdot X}(z_2)::e^{ik_1\cdot X}(z_1): . \end{aligned}$$

Hence, by moving $f_1^{(n)} \circ \Phi_n(0)$ to the front we pick up a phase $e^{i\pi k\cdot(k_{tot}-k)}$ where k^μ_{tot} is the total momentum of the amplitude. Due to momentum conservation nonvanishing amplitudes have $k^\mu_{tot} = 0$ and hence the phase picked by moving $f_1^{(n)} \circ \Phi_n(0)$ to the front cancels with the one from the rotation R. This concludes the proof of property (3.1.30). ∎

3.1.5 Reality condition

We can also work out the implications of the reality condition at the level of Fock space states $|\Psi>$ and their vertex operators $\mathcal{V}_\Psi$. Again, we restrict our attention to the X^μ sector for notational simplicity.

The reality condition (3.1.17)

$$\Psi[X(\zeta)] = \Psi^*[X(-1/\zeta)]$$

can be written as

$$<X(\zeta)|\Psi> = <X(-1/\zeta)|\Psi>^* .$$

From the discussion in section 2.3.5 one deduces that BPZ conjugation acts on position eigenstates as

$$\mathrm{bpz}(|X(\zeta)>) = <X(-1/\zeta)|.$$

In terms of Fock space states $|\Psi >$, the reality condition implies invariance under BPZ conjugation followed by the inverse of Hermitean conjugation:

$$\mathrm{hc}^{-1} \circ \mathrm{bpz}|\Psi> = |\Psi> . \tag{3.1.32}$$

Indeed, expanding $|\Psi>$ in position eigenstates one finds

$$\begin{aligned}
&\mathrm{hc}^{-1} \circ \mathrm{bpz}(|\Psi>) = \\
&= \int_C [DX(\zeta)] <X(\zeta)|\Psi>^* \ |X(-1/\zeta)> \\
&= \int_C [DX(\zeta)] <X(-1/\zeta)|\Psi>^* \ |X(\zeta)> \\
&= \int_C [DX(\zeta)] <X(\zeta)|\Psi> \ |X(\zeta)> \\
&= |\Psi>,
\end{aligned}$$

where, in the second step, we have performed a change of variables in the functional integral.

The reality condition on the string field can be interpreted as a reality condition on the space-time component fields. Consider a Fock space state of the form

$$|\Psi>= \phi(k)[\text{oscillator modes}]|0,k>,$$

where the coefficient $\phi(k)$ is the Fourier transform of a space-time component field $\phi(x)$. Under combined BPZ and inverse Hermitean conjugation, the state $|0,k>$ goes to $|0,-k>$ while the oscillator modes pick up a sign, say $(-1)^s$. The reality condition (3.1.32) then implies $\phi(k) = (-1)^s \phi^*(-k)$ so that $\phi(x)$ has to be real for $s = 1$ and imaginary for $s = -1$.

It is possible to argue that the reality condition (3.1.5) on the fields leads to a real action [21]. Here, we will restrict our attention to the kinetic term. It is easy to see that, for real string fields, the integral $\Psi_1 \star \Psi_2$ becomes equal to the Hermitean inner product $<\Psi_1|\Psi_2>$. The first term in the action is real because the BRST charge Q_B is a Hermitean operator. To prove the reality of the cubic term one has to show that the $\star$ product preserves the reality condition.

3.1.6 An example: the action for the lowest modes

As an example, consider a string field containing only excitations of the two lowest levels:

$$|\Psi>= \int \frac{d^{26}k}{(2\pi)^{26}} \left[t(k)c_1 + e_\mu(k)\alpha^\mu_{-1}c_1 + ih(k)c_0 \right] |0,k> .$$

The field $t(k)$ describes the off-shell tachyon, while e_μ corresponds to the massless vector field (compare (2.4.5, 2.4.6)). The field h will turn out to be an auxiliary field.

The reality condition (3.1.32) on these fields reads

$$\begin{aligned} t^*(k) &= t(-k), \\ e^*_\mu(k) &= e_\mu(-k), \\ h^*(k) &= h(-k). \end{aligned}$$

Substituting $|\Psi>$ in the kinetic term of the action (3.1.26) and Fourier transforming to position space gives

$$\frac{1}{2}\langle\langle\, \mathcal{V}_\Psi Q_B \mathcal{V}_\Psi\,\rangle\rangle = \int d^{26}x \left[\frac{1}{2}\partial_\mu t \partial^\mu t - \frac{1}{2}t^2 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}(h + \partial_\mu A^\mu)^2\right]. \tag{3.1.33}$$

where

$$t(x) = \int \frac{d^{26}k}{(2\pi)^{26}} t(k) e^{ik\cdot x} \quad ; \quad A_\mu(x) = \int \frac{d^{26}k}{(2\pi)^{26}} e_\mu(k) e^{ik\cdot x};$$
$$h(x) \quad = \quad \int \frac{d^{26}k}{(2\pi)^{26}} h(k) e^{ik\cdot x}.$$

We can also illustrate the transformation of the string field under linearised gauge transformations. The gauge parameter to this level is

$$|\varepsilon> = \int \frac{d^{26}k}{(2\pi)^{26}} i\Lambda(k)|0,k> .$$

and the linearised gauge transformation

$$\delta|\Psi> = Q_B|\varepsilon> = \int \frac{d^{26}k}{(2\pi)^{26}} i\Lambda(k)\left[k^2 c_0 + k_\mu c_1 \alpha^\mu_{-1}\right]|0,k> .$$

translates into a a gauge transformation on the space-time fields

$$\begin{aligned} \delta A_\mu(x) &= \partial_\mu \Lambda(x) \\ \delta h(x) &= -\partial_\mu \partial^\mu \Lambda(x). \end{aligned}$$

Clearly, these transformations leave the action (3.1.33) invariant.

3.1.7 Quantisation and modular invariance

Although in the following we will only be concerned with the evaluation of the *classical* string field theory action, we feel it is nevertheless important to say a few words about quantisation (see e.g [39] for more details). Indeed, the primary justification for any string field theory action is its ability to reproduce the perturbation expansion for on-shell string amplitudes given in section 2.2.2.

Gauge-fixing

The string field theory action (3.1.16) is an infinite-component gauge field theory of the cubic type. The first step towards quantisation is to deal with the gauge freedom (3.1.12). This is traditionally done by imposing the *Feynman-Siegel gauge*:

$$b_0|\Psi>= 0. \tag{3.1.34}$$

We will come back to the justification of this gauge choice in section 5.4. For the example considered in the previous section, this gauge choice puts the field h to zero. The resulting action for the massless field is known in quantum field theory as the gauge-fixed action in the Feynman gauge.

Feynman rules and integration over moduli space

Any string field obeying the gauge condition (3.1.34) satisfies $|\Psi>= b_0c_0|\Psi>$. Plugging this into the kinetic term, we get

$$\begin{aligned}
\int\Psi\star Q_B\Psi &= <\Psi|c_0b_0Q_bb_0c_0\Psi> \\
&= <\Psi|c_0L_0b_0c_0\Psi> \\
&= <\Psi|c_0L_0\Psi> \\
&\equiv <\Psi, L_0\Psi>,
\end{aligned}$$

where we have used $\{Q_B, b_0\} = L_0$ and, in the last line, defined yet another inner product $<\Phi,\Psi>\equiv<\Phi|c_0\Psi>$. This inner product has the property that it is positive definite when restricted to the subspace of physical states satisfying $Q_B|\Psi>= 0$ (see e.g. [9], chapter 4).

The kinetic energy operator, L_0, can be formally inverted[1] to give the *propagator*:

$$1/L_0 = \int_0^\infty d\tau e^{-\tau L_0}.$$

Since L_0 generates translations of the world-sheet time, the operator $e^{-\tau L_0}$ implements free propagation over a time τ (see also section 2.3.3): it builds up a world-sheet which is a rectangular strip of width π and length τ in the (σ, τ) plane (in the complex coordinate z, it generates a half disc of radius $e^{-\tau}$). The τ integral tells us to sum over all such worldsheets. The resulting expression for the propagator is comparable to the Schwinger proper time parametrisation for the propagator in ordinary field theory.

The cubic term in the action represents an interaction which can be viewed as a small perturbation if the coupling g is small. As in ordinary field theory, a perturbation expansion can be set up in terms of *Feynman diagrams*. A Feynman diagram in this theory is a world-sheet made up out of propagators, now represented by strips instead of lines, and vertices where three strips are joined. Loop diagrams correspond to world-sheets with holes.

At tree level, ample evidence has been gathered to show that Witten's action indeed reproduces the on-shell scattering amplitudes of open strings [8, 28, 30, 32].

In chapter 2, we have seen that loop diagrams in string theory require an integration over moduli space. The main result is that the sum over Feynman diagrams in string field theory exactly reproduces the correct integration measure, where each diagram covers a small part of the moduli space so that the sum over diagrams exactly covers the whole space [40]. The decomposition of the moduli space of higher genus Riemann surfaces in terms of a sum over Feynman graphs and integrals over strip lengths is a nontrivial result that was discovered in the mathematics literature only shortly before the advent of string field theory.

3.2 Open superstrings

We have seen that, in the bosonic theory, Witten's string field theory seems to pass all the requirements for being a consistent off-shell description of open

[1] Obviously, we are ignoring here the possibility of string fields of weight 0. We will come back to this subtlety in section 5.4.

string theory. In the case of superstrings, however, the situation is not so clear-cut. Various proposals for a field theory of open superstrings have been put forth in the literature, and all of them are claimed to pass at least some of the requirements for a consistent string field theory. At the moment, it is not completely clear whether these proposals provide different, but consistent, off-shell descriptions with the same on-shell behaviour, or whether closer scrutiny will point out inconsistencies in some descriptions[2]. In this section, we give an overview of the open superstring field theory actions that have been proposed in the literature. The three actions we will discuss here differ not only in form but also in the ghost and picture number restrictions imposed on the string field.

We begin by recalling some definitions from section (2.5.4) regarding the ghost system of the superstring. The superghosts are 'bosonised' as:

$$\beta = :\partial\xi e^{-\phi}: \qquad \gamma = :\eta e^{\phi}:.$$

The *ghost number* and *picture number* assignments of the various fields are summarised in table 3.1. Furthermore, we adopt the convention that, for q

field	ghost number	picture number
b	-1	0
c	1	0
η	1	-1
ξ	-1	1
$:e^{q\varphi}:$	0	q

Table 3.1: Ghost- and picture number assignments for the various fields.

odd, $:e^{q\varphi}:$ is a Grassmann odd field (anticommuting with *all* other Grassmann odd fields). This definition is necessary if we want the bosonised system to reproduce the amplitudes of the original (β, γ) system [41]. As far as the (η, ξ) system is concerned, one distinguishes between the *small Hilbert space* which excludes the zero mode ξ_0 of the ξ field, and the *large Hilbert space* which includes ξ_0. We adopt the following normalisation convention for the elementary correlators:

$$\langle c\partial c\partial^2 c(z_1) :e^{-2\varphi}(z_2):\rangle \equiv 2 \qquad \text{(small Hilbert space)} \quad (3.2.1)$$

[2] As we will argue in chapter 7, the calculations presented there do provide a testing ground for these proposals and, as we will see there, not all proposals pass such a test with flying colours.

$$\langle \xi(z_1) c\partial c\partial^2 c(z_2) :e^{-2\varphi}(z_3):\rangle \quad \equiv \quad 2 \qquad \text{(large Hilbert space)} \quad (3.2.2)$$

It follows that nonvanishing correlators in the small Hilbert space have an odd total Grassmann parity, while nonvanishing correlators in the large Hilbert space have an even total Grassmann parity.

Although we will keep using the symbol Ψ to represent the string field to avoid unnecessary propagation of symbols, it is important to keep in mind that the ghost and picture number assignments differ in each theory as summarised in table 3.2. For the moment, we will restrict the string field to be in the GSO+ sector containing fields with $e^{i\pi F} = +1$. Later, in chapter 7, we will extend the action to include fields in the GSO− sector (with $e^{i\pi F} = -1$), such as the tachyon state, as well.

action	ghost number	picture	Hilbert space
Witten	1	−1	small
modified cubic	1	0	small
Berkovits	0	0	large

Table 3.2: Constraints on the NS sector string field Ψ in the various actions proposed in the literature.

The reality condition on the string field is taken to be

$$\mathrm{hc}^{-1} \circ \mathrm{bpz}(|\Psi>) = |\Psi> \qquad (3.2.3)$$

in all three cases.

3.2.1 Witten's action

In [36], Witten proposed a field theory action for open superstrings of the Chern-Simons type (3.1.11) as well but with slightly modified definitions of the $\star$ and $\oint$ operations. We will restrict our attention to the NS sector of the theory since this is the relevant sector for the calculation of the tachyon potential in the next chapter. In this theory one considers string fields of ghost number 1 and picture number −1. Furthermore, the string field is restricted to live in the small Hilbert space of the bosonised system.

If we are to write down an action of the form (3.1.11) for the superstring, the operations $\oint$ and $\star$ should be suitably modified. In order to ensure that both terms in the action (3.1.11) are non-vanishing, they should both map to CFT correlators with total picture number -2. If we modify the $\star$ operation so that it raises the picture by n units and the integration $\oint$ to raise the picture with m units, we find the constraints

$$\begin{aligned} m + n &= 0 \\ 2m + n &= -1 \end{aligned}$$

from the requirement that both terms in the action (3.1.11) are nonvanishing. So we need $m = 1$ and $n = -1$. In ref. [36], this is accomplished in a manner that preserves gauge invariance as well as conformal invariance by modifying the $\star$ and $\oint$ operations to include insertions of picture-changing operators at the mid-point: the $\star$ operation is accompanied by an insertion of the picture-raising operator Z (2.5.20) and $\oint$ comes with an insertion of the picture-lowering operator Y (2.5.21). Translated into CFT language, this means that general integrals $\oint(\Psi_1 \star \Psi_2 \star \ldots \star \Psi_n)$ reduce to correlators

$$\begin{aligned} &\oint(\Psi_1 \star \Psi_2 \star \ldots \star \Psi_n) \equiv \langle\langle\, \mathcal{V}_{\Psi_1}\, \mathcal{V}_{\Psi_2} \ldots \mathcal{V}_{\Psi_n}\, \rangle\rangle \\ &\quad = \Big\langle Y(0) f_1^{(n)} \circ \mathcal{V}_{\Psi_1}(0)\ Z(0)\ f_2^{(n)} \circ \mathcal{V}_{\Psi_2}(0)\ Z(0) \ldots\ Z(0)\ f_n^{(n)} \circ \mathcal{V}_{\Psi_n}(0) \Big\rangle. \end{aligned} \tag{3.2.4}$$

The Chern-Simons action can again be written in terms of CFT correlators:

$$S = \oint(\frac{1}{2}\Psi \star Q\Psi + \frac{g}{3}\Psi \star \Psi \star \Psi) \tag{3.2.5}$$

$$= \frac{1}{2}\langle\langle\, \mathcal{V}_\Psi\ Q_B \mathcal{V}_\Psi\, \rangle\rangle + \frac{g}{3}\langle\langle\, \mathcal{V}_\Psi\ \mathcal{V}_\Psi\ \mathcal{V}_\Psi\, \rangle\rangle\,. \tag{3.2.6}$$

The action is invariant under the gauge transformations:

$$\delta\Psi = Q\varepsilon + g\ \Psi \star \varepsilon - g\ \varepsilon \star \Psi,$$

where the gauge parameter ε is an arbitrary string field of ghost number 0 and picture number -1. A formal proof of gauge invariance relies on the following properties of the correlator (3.2.4) (compare p. 61, 75):

$$\begin{aligned} \langle\langle\, Q_B(\Phi_1 \cdots \Phi_n)\, \rangle\rangle &= 0\,, \\ \langle\langle\, \cdots Q_B^2(\Phi_1 \cdots \Phi_n) \cdots\, \rangle\rangle &= 0\,, \end{aligned}$$

$$\begin{aligned}\langle\langle \cdots Q_B(\Psi\Phi)\cdots\rangle\rangle &= \langle\langle \cdots (Q_B\Psi\,\Phi - \Psi\, Q_B\Phi)\cdots\rangle\rangle\,,\\ \langle\langle \cdots Q_B(\varepsilon\Phi)\cdots\rangle\rangle &= \langle\langle \cdots (Q_B\varepsilon\,\Phi + \varepsilon\, Q_B\Phi)\cdots\rangle\rangle\,,\\ \langle\langle\,\Phi_1\cdots\Phi_{n-1}\Phi_n\,\rangle\rangle &= \langle\langle\,\Phi_n\Phi_1\cdots\Phi_{n-1}\,\rangle\rangle\,. \end{aligned} \tag{3.2.7}$$

where Ψ and ε represent the string field and the gauge parameter respectively and the Φ_i represent arbitrary operators in the NS, GSO+ sector. These properties can be proven formally in manner very similar to the proof of gauge invariance for the bosonic string (see p. 75). One should keep in mind that the ghost and picture number assignments imply that the string field Ψ is a Grassmann odd object while the gauge parameter is Grassmann even. To prove the last property, one needs to use the fact that the total Grassmann parity of any nonvanishing correlator is odd. Indeed, in the small Hilbert space, nonvanishing correlators are restricted to have (b, c) ghost number 3, (η, ξ) number 0 and φ charge -2 and an even number of ψ^μ insertions.

The properties (3.2.7) would normally suffice to prove gauge invariance of the action. However, there is a subtlety we have overlooked so far. From the definition (3.2.4), one sees that correlators involving more than three string fields contain two or more insertions of the picture-changing operator at the origin. These correlators are ill-defined because the OPE of the picture-changing operator Z with itself has a pole term, leading to divergences in such correlators. Hence the properties (3.2.7) also become meaningless from the moment they involve more than three string fields. This is the case in the proof of gauge invariance of the action which involves correlators with four string fields. Similar problems occur when one tries to verify associativity of the $\star$ product. These problems of the action (3.2.6) were found and addressed in [42]. A resolution of these difficulties requires dealing with the aforementioned divergences and hence a sort of renormalisation procedure is required, a rather dubious procedure since we are still working in a classical theory. The procedure proposed in [42] does however seem to lead to the correct on-shell scattering amplitudes.

3.2.2 Modified cubic actions

In answer to the difficulties with Witten's field theory discussed above, two groups [43, 44] have proposed a modification that does not suffer from problems with divergences already at the classical level. From the discussion in the previous section, it is clear that the origin of these divergences was the Z insertion in the definition of the star product. One would like a $\star$ operation that doesn't involve a picture-changing operation. This is possible when the string field is taken to be in the 0 picture instead of the canonical -1 picture as before. The fields are still restricted to be of ghost number 1 and to live in the small Hilbert space of the bosonised ghost system. In order to have a non-vanishing action, the $\oint$ operation should then be accompanied by an insertion of an operator that lowers the picture by two units. This *double-step picture lowering operator*, denoted by Y_{-2}, is again required to be a BRST-invariant primary field of weight 0. Two possible operators[3] satisfying these criteria were found. The first possibility is to take

$$Y_{-2} = \frac{1}{3}e^{-2\phi} + \frac{1}{15}\partial\xi c G^m e^{-3\phi}. \tag{3.2.8}$$

Another possible choice is provided[4] by

$$Y_{-2} = Y(0)Y(\infty). \tag{3.2.9}$$

Integrals of star products reduce to correlators:

$$\begin{aligned}\oint(\Psi_1 \star \Psi_2 \star \ldots \star \Psi_n) &\equiv \langle\langle\, \mathcal{V}_{\Psi_1}\, \mathcal{V}_{\Psi_2} \ldots \mathcal{V}_{\Psi_n}\,\rangle\rangle = \\ &= \Big\langle Y_{-2}(0) f_1^{(n)} \circ \mathcal{V}_{\Psi_1}(0)\;\; f_2^{(n)} \circ \mathcal{V}_{\Psi_2}(0)\; \ldots\; f_n^{(n)} \circ \mathcal{V}_{\Psi_n}(0)\Big\rangle.\end{aligned}$$

and the action reads

$$\begin{aligned} S &= \oint(\frac{1}{2}\Psi \star Q\Psi + \frac{g}{3}\Psi \star \Psi \star \Psi) \\ &= \frac{1}{2}\langle\langle\, \mathcal{V}_\Psi\, Q_B \mathcal{V}_\Psi\,\rangle\rangle + \frac{g}{3}\langle\langle\, \mathcal{V}_\Psi\, \mathcal{V}_\Psi\, \mathcal{V}_\Psi\,\rangle\rangle\,. \end{aligned} \tag{3.2.10}$$

The gauge invariance

$$\delta\Psi = Q\varepsilon + g\, \Psi \star \varepsilon - g\, \varepsilon \star \Psi \tag{3.2.11}$$

involves a gauge parameter of ghost and picture number 0. As was the case for the bosonic string, the gauge parameters form a closed subalgebra of the

[3] That is to say, up to the addition of a BRST exact part.

[4] Here we give the expression relevant to the specific conformal frame we are working in: the midpoint is at $z = 0$, the string boundary runs along the unit circle and $z = \infty$ is the point conjugate to $z = 0$ with respect to the boundary. The expression in an arbitrary conformal frame reads $Y_{-2} = Y(z_0)Y(z_0^*)$ where z_0 denotes the midpoint and z_0^* its conjugate point with respect to the boundary.

full $\star$ algebra. Gauge invariance follows from the properties (3.2.7) of the correlator. The proof of these properties is analogous to the proof in the bosonic case following (3.1.30).

In refs. [43, 44], it was verified that the modified action (3.2.10) reproduces the correct tree-level on-shell scattering amplitudes without suffering from the divergences that arose in Witten's proposal. In [45] however, it was suggested that the insertion of Y_{-2} leads to problems in defining an off-shell propagator for the theory.

Also, one might object to the appearance of picture changing operators in off-shell string field theory actions on general grounds. It is well-known that these operators provide a one-to-one a mapping between the BRST cohomology classes in different pictures, but that their action on general states is ambiguous [46]. More specifically, adding a BRST exact part to a picture changing operator has no effect on its action on the BRST cohomology, while it does affect its action on general states and leads to a different off-shell string field theory action.

3.2.3 Berkovits' action

Using the embedding of the $N = 1$ superstring into a critical $N = 2$ theory found in [47], Berkovits proposed a superstring field theory based on a non-commutative generalisation of the *Wess-Zumino-Witten action* [48, 49]. This action describes the NS-sector of the open superstring in a space-time covariant manner. The covariant extension to the R-sector fields is as yet unknown.

In Berkovits' formalism, an NS-sector string field is represented by an open string vertex operator Ψ of ghost number 0 and picture number 0. It is taken to live in the large Hilbert space of the bosonised superghost system containing the zero mode of the ξ field. In contrast to Witten's field, Ψ is a Grassmann even vertex operator.

The proposed action is of the Wess-Zumino-Witten type and can be formulated in terms of Witten's $\oint$ and $\star$ operations:

$$S[\Psi] = \frac{1}{2g^2}\oint\Big((e^{-\Psi}Q_B e^{\Psi})(e^{-\Psi}\eta_0 e^{\Psi}) - \int_0^1 dt\,(e^{-t\Psi}\partial_t e^{t\Psi})\{(e^{-t\Psi}Q_B e^{t\Psi}),(e^{-t\Psi}\eta_0 e^{t\Psi})\}\Big), \tag{3.2.12}$$

where the exponentials are defined by their series expansion and all products are to be interpreted as $\star$ products. Also, $\{A,B\} \equiv A \star B + B \star A$

and $e^{-t\Psi}\partial_t e^{t\Psi} = \Psi$ but has been written this way to emphasise the similarity with the action of the Wess-Zumino-Witten model [50]. We have denoted by $\eta_0 = \frac{1}{2\pi i}\oint dz\eta(z)$ the zero mode of the field η acting on the Hilbert space of matter and ghost CFT.

Translated into CFT language, the action reads:

$$\begin{aligned} S = \frac{1}{2g^2} \langle\langle\, & (e^{-\mathcal{V}_\Psi} Q_B e^{\mathcal{V}_\Psi})(e^{-\mathcal{V}_\Psi}\eta_0 e^{\mathcal{V}_\Psi}) \\ - & \int_0^1 dt (e^{-t\mathcal{V}_\Psi}\partial_t e^{t\mathcal{V}_\Psi})\{(e^{-t\mathcal{V}_\Psi} Q_B e^{t\mathcal{V}_\Psi}),(e^{-t\mathcal{V}_\Psi}\eta_0 e^{t\mathcal{V}_\Psi})\}\,\rangle\rangle\,. \end{aligned} \tag{3.2.13}$$

To evaluate the correlators, one has to expand all exponentials in formal Taylor series carefully preserving the order of the operators. The correlator $\langle\langle \cdots \rangle\rangle$ of an ordered sequence of arbitrary vertex operators $\mathcal{V}_1, \dots \mathcal{V}_n$ is defined as:

$$\langle\langle\, \mathcal{V}_1 \dots \mathcal{V}_n \,\rangle\rangle = \left\langle f_1^{(n)} \circ \mathcal{V}_1(0) \cdots f_n^{(n)} \circ \mathcal{V}_n(0) \right\rangle . \tag{3.2.14}$$

where, once more, we made use of the conformal transformations (3.1.22).

The action (3.2.12) is invariant under the gauge transformations:

$$\delta e^{\Psi} = (Q_B\Omega)e^{\Psi} + e^{\Psi}(\eta_0\Omega')\,, \tag{3.2.15}$$

where the gauge parameters Ω and Ω' represent arbitrary independent string fields of ghost and picture number equal to $(-1,0)$ and $(-1,1)$ respectively. These assignments imply that Ω and Ω' are Grassmann odd objects. The proof of gauge invariance relies on the following identities:

$$\begin{aligned} \{Q_B,\eta_0\} = 0, &\qquad Q_B^2 = \eta_0^2 = 0 \\ \langle\langle\, Q_B(...)\,\rangle\rangle &= \langle\langle\, \eta_0(...)\,\rangle\rangle = 0 \\ \langle\langle\, \dots Q_B(\Psi_1\Psi_2)\dots\,\rangle\rangle &= \langle\langle\, \dots (Q_B\Psi_1)\Psi_2 + \Psi_1(Q_B\Psi_2)\dots\,\rangle\rangle \\ \langle\langle\, \dots \eta_0(\Psi_1\Psi_2)\dots\,\rangle\rangle &= \langle\langle\, \dots (\eta_0\Psi_1)\Psi_2 + \Psi_1(\eta_0\Psi_2)\dots\,\rangle\rangle \\ \langle\langle\, \Phi_1\dots\Phi_{n-1}\Psi\,\rangle\rangle &= \langle\langle\, \Psi\Phi_1\dots\Phi_{n-1}\,\rangle\rangle \\ \langle\langle\, \Phi_1\dots\Phi_{n-1}Q_B\Psi\,\rangle\rangle &= -\langle\langle\, Q_B\Psi\Phi_1\dots\Phi_{n-1}\,\rangle\rangle \\ \langle\langle\, \Phi_1\dots\Phi_{n-1}\eta_0\Psi\,\rangle\rangle &= -\langle\langle\, \eta_0\Psi\Phi_1\dots\Phi_{n-1}\,\rangle\rangle\,. \end{aligned} \tag{3.2.16}$$

The proof of these properties [51] is again analogous to the proof in the bosonic case following (3.1.30). One should keep in mind that the field Ψ is

Grassmann even and that nonvanishing correlators in the large Hilbert space require the total Grassmann parity to be even as well.

We can now show that the action (3.2.12) is invariant under the gauge transformations (3.2.15). Denoting $G = e^{\Psi}$, one finds that under an arbitrary variation δG,

$$\delta S = \frac{1}{g^2} \langle\langle\, G^{-1}\delta G \eta_0 (G^{-1} Q_B G)\,\rangle\rangle \tag{3.2.17}$$

which can also be written as

$$\delta S = -\frac{1}{g^2} \langle\langle\, G \delta G^{-1} Q_B (G \eta_0 G^{-1})\,\rangle\rangle\,. \tag{3.2.18}$$

To prove invariance under $\delta G = (Q_B \Omega) G$, we use (3.2.17) and the fact that $\eta_0(G^{-1}\delta G) = 0$. To prove the second invariance, $\delta G = (Q_B\Omega)G$, we use (3.2.18) and $Q_B(G\delta G^{-1}) = -Q_B(\delta G G^{-1}) = 0$.

The equation of motion derived from (3.2.13) is:

$$\eta_0(e^{-\Psi} Q_B e^{\Psi}) = 0.$$

In [52], it was checked that the action (3.2.12) reproduces the correct four-point scattering amplitudes.

3.3 Open problems in present-day string field theory

A first shortcoming of string field theory is its limited success in the description of closed strings. A number of difficulties show up when trying to extend string field theory to *closed strings*.

A first difficulty arises when trying to write down a *kinetic term* of the form $\ll \Psi | Q_B \Psi >$ as in the open string case. Such a term vanishes due to ghost number conservation (the string field has ghost number $(1,1)$ while Q_B consists of a term of ghost number $(1,0)$ and a term with ghost number $(0,1)$). This can be remedied by imposing the conditions $L_0 - \tilde{L}_0 = 0$ and $b_0 - \tilde{b}_0 = 0$ on the string field Ψ as well as the gauge parameter ε. The kinetic term is then taken to be

$$S_0 = \frac{1}{2} \ll \Psi | (c_0 - \tilde{c}_0) Q_B \Psi >$$

which is invariant under $\delta\Psi = Q_B\varepsilon$.

Further complications arise when trying to describe interactions. One would be tempted to propose an action of the cubic type as for open strings. This would require an *associative $\star$ operation*, which is *not known* for closed strings. Furthermore, in order to construct a theory that reproduces on-shell amplitudes with the correct moduli space integration, a cubic interaction is insufficient [20, 53]: one needs to add an infinite number of higher vertices. The proposed action is nonpolynomial in the string field Ψ, making concrete calculations, such as the ones we are about to perform in the next chapter, rather hard (see however [54]).

Another idea regarding closed string field theory, which has been around for a long time, is that it might be possible to extract *closed string field theory from open string field theory*. The idea stems from the fact that open string theory automatically incorporates processes in which virtual closed strings are created. Recent developments, which we will relate in the following chapters, have brought this idea a step closer to a concrete realization.

Another rather unsatisfactory feature of the both open and closed string field theory is the *background dependence*. The string field theory action describes the dynamics of small fluctuations around the specific background of $9+1$-dimensional Minkowski space. On the other hand, we know that string perturbation theory continues to make sense if we replace the flat background by some other manifold as long as the matter theory remains a CFT of central charge 26. In a background-independent formulation of string field theory, one would expect these backgrounds on which strings can propagate to arise as classical solutions or vacua around which one can set up a perturbation theory for small fluctuations. The string field theories we have described should then be recovered by considering small fluctuations around the specific background of flat $9+1$-dimensional Minkowski space.

One could compare the present state of affairs to the following hypothetical situation: suppose one had somehow guessed the gravitational action for small perturbations $h_{\mu\nu}$ around the Minkowski metric $\eta_{\mu\nu}$, but doesn't know that the terms of the action can be summed to give the Einstein-Hilbert action for the metric $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$. One would be perfectly capable of describing gravitational phenomena around flat space, but it would be very hard to describe the physics of the gravitational field near a star or a black hole. What would be lacking in this description is the notion of the fundamental geometric object, i.e. the metric, and the symmetry principle, i.e. general covariance, which governs the theory. The object $h_{\mu\nu}$ on which the description is based, is an element of the tangent space to the space of metrics at the classical solution $\eta_{\mu\nu}$.

Similarly, string fields Ψ should probably be seen as elements of the tangent space to a space of more fundamental objects. In [55], it was suggested that this last space should be taken to be the 'space of 2-dimensional field theories' (although this remains a rather abstract object) and that a background-independent formulation should have the elements of this space as fundamental variables. The minima of the background-independent action should occur at conformal (matter+ghost) theories and the full gauge invariance should reduce to the familiar BRST invariance in the description of small fluctuations. A concrete proposal for a background-independent approach to open string field theory was also put forth in [55]. Although some specific calculations can be performed in this formalism [56]–[58], the range of applications remains rather limited (see however section 5.8).

Chapter 4

Tachyon condensation in string theory

One of the most important advances in string theory in recent years has been the discovery of the extended objects called D-branes. The D-branes of type II string theory are charged under the RR fields and are BPS states, meaning that in the presence of such a D-brane a part of the space-time supersymmetry is preserved. The type II string theories also contain various non-BPS branes. Generically, these are unstable and can decay into lighter states such as BPS branes. This instability is signalled by the presence of a tachyon in the spectrum of the open strings that describe the excitations of the brane. Such tachyonic instabilities are common in field theory and signal the fact that the ground state of the system is unstable and, under small fluctuations, it will decay in a process known as tachyon condensation. The fate of an unstable D-brane under tachyon condensation is the subject of a conjecture made by Sen. In this chapter, we will review some background material on tachyon condensation in string theory and Sen's conjecture, while in the next chapter we will present evidence for the validity of this conjecture. We begin by reviewing some important examples of tachyonic instabilities in field theory.

4.1 Tachyons and field theory instabilities

It is common to characterise particles according to their transformation properties under the symmetry group of Minkowkski space-time, the Poincaré group. One of the Casimir operators of the Poincaré group is the square of the momentum operator P^2, which in our conventions equals minus the rest-mass squared. Irreducible representations will be made up out of eigenstates of this operator with a fixed eigenvalue. Although all known particles fall into representations with P^2 smaller than or equal to zero, one can consider the possibility of particles with $P^2 > 0$. Such hypothetical particles of imaginary rest mass are called *tachyons* because their velocity exceeds the speed of light. In fact, since P^2 is Lorentz invariant, it would be impossible to 'slow down' a tachyon to a velocity smaller than the speed of light. However, as we will presently argue, the existence of tachyons in the particle spectrum of a theory points to a more fundamental problem: the system is unstable against small fluctuations and the basic premises for setting up quantum mechanical perturbation theory are violated. Many systems, however, do admit a sensible perturbation theory around a different, stable state, which can be seen as the endpoint of a decay process which is termed *tachyon condensation*. An overview of tachyonic instabilities in field theory and their applications can be found, for example, in [59], chapter 1, on which this section is based.

Consider first the example of the so-called ϕ^4 theory in 3 + 1 dimensions, consisting of a real scalar field ϕ with Lagrangian density

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 \tag{4.1.1}$$

where $m^2 \geq 0$. When $\lambda \ll 1$ and supposing ϕ is small, the last term in (4.1.1) can be neglected and the field equation reduces to the familiar Klein-Gordon equation $(\partial_\mu\partial^\mu - m^2)\phi = 0$. Any solution can be decomposed as a linear combination of plane wave solutions:

$$\phi(x) = (2\pi)^{-3/2}\int \frac{d^3k}{\sqrt{2k_0}}[a(\mathbf{k})e^{ik\cdot x} + a(\mathbf{k})^*e^{-ik\cdot x}], \tag{4.1.2}$$

where $k_0 \equiv \sqrt{\mathbf{k}^2 + m^2}$. The general solution represents fluctuations around $\phi = 0$, the latter configuration being the minimum of the potential energy density (see figure (4.1(a)))

$$V(\phi) = \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4.$$

The plane waves in the decomposition (4.1.2) are interpreted as particles of energy-momentum k_μ and, upon quantisation, the coefficients $a(\mathbf{k})^*$ and

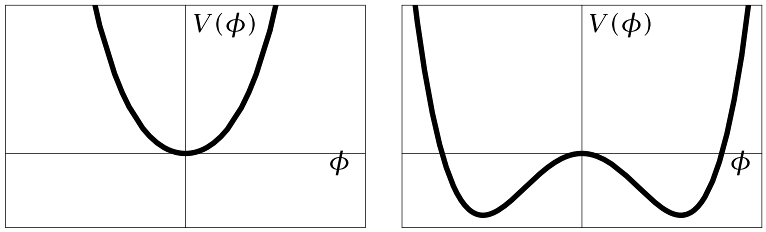


Figure 4.1: The potential energy density $V(\phi)$ in ϕ^4 theory: (a) for positive mass-squared; (b) for negative mass-squared.

$a(\mathbf{k})$ become operators which create or annihilate these particles. The rest mass of these particles can be read off from the quadratic term in the action (4.1.1). Scattering processes can be calculated by treating the interaction term in (4.1.1) as a perturbation, leading to the familiar Feynman diagram expansion.

Now consider a theory with a Lagrangian of the form (4.1.1), but with m^2 now taken to be a negative number, $m^2 = -\mu^2$:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4. \tag{4.1.3}$$

When attempting the same analysis as in the previous paragraph, one soon runs into trouble. Supposing again that ϕ is small so that interactions can be neglected in a first approximation, one finds two types of elementary solution to the free equation of motion. For $\mathbf{k}^2 > \mu^2$, we find plane-wave solutions $e^{\pm ik\cdot x}$ with $k_0 \equiv \sqrt{\mathbf{k}^2 - \mu^2}$. These represent particle-like excitations which are tachyonic.

For $\mathbf{k}^2 < \mu^2$ on the other hand, we find solutions which grow exponentially with time:

$$\phi(x) = e^{\pm\sqrt{\mu^2-\mathbf{k}^2}t\pm i\mathbf{k}\cdot\mathbf{x}}.$$

These solutions can no longer be seen as particles propagating on the fixed background $\phi = 0$, rather they represent perturbations leading to a rapid evolution away from the configuration $\phi = 0$. The time scale for the growth of these fluctuations is of the order of $1/\mu$. Hence the configuration $\phi = 0$

is unstable and small perturbations initiate a decay process. It is also important to note that, when describing this decay process, ϕ can no longer be considered to be small and the $\frac{\lambda}{4}\phi^4$ interaction term in (4.1.3) can no longer be neglected. Looking at the potential energy density, we see that $\phi = 0$ corresponds to a local maximum of the potential and that local minima occur at $\phi_0 = \pm\frac{\mu}{\sqrt{\lambda}}$ (see figure (4.1(b)). The decay process represents a transition from $\phi = 0$ to a stable ground state $\phi = \pm\phi_0$. The latter state is (classically) stable as we can see by considering small fluctuations around ϕ_0. Shifting variables to $\tilde{\phi} = \phi - \phi_0$, the Lagrangian density becomes:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\tilde{\phi}\partial^\mu\tilde{\phi} - \mu^2\tilde{\phi}^2 + \mathcal{O}(\tilde{\phi}^3).$$

The fluctuations of the new field $\tilde{\phi}$ have positive mass-squared, equal to $\sqrt{2}\mu^2$, and the general solution to the free equation of motion can be expanded into plane waves as before. In terms of the original field ϕ:

$$\phi(x) = \phi_0 + (2\pi)^{-3/2}\int\frac{d^3k}{\sqrt{2k_0}}[a(\mathbf{k})e^{ik\cdot x} + a(\mathbf{k})^* e^{-ik\cdot x}]. \tag{4.1.4}$$

The plane waves represent particles propagating on the $\phi = \phi_0$ background and it is now possible to set up a sensible perturbation theory. Upon quantisation, the expectation value of the field is $\langle\phi(x)\rangle = \phi_0$ and one says that ϕ has acquired a *vacuum expectation value (VEV)*. We will refer to the process in which one or more scalar fields undergo a transition from an unstable to a stable vacuum where the fields have nonzero vacuum expectation values as *tachyon condensation*. In many physical applications, this process also entails symmetry breaking[1] and is then also known as *spontaneous symmetry breaking*.

We now briefly discuss two examples of tachyon condensation which play an important role in the Standard Model of particles and interactions. First, consider the scalar field theory (4.1.3) coupled to a massless fermion ψ with a so-called Yukawa coupling of strength h:

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4}\phi^4 - \bar{\psi}(i\gamma^\mu\partial_\mu + h\phi)\psi.$$

After tachyon condensation, the fermionic field acquires a mass $m = h|\phi_0| = h\mu/\sqrt{\lambda}$. This mechanism is responsible for the generation of fermion masses in the Standard Model.

[1] In our example, the new vacuum is no longer invariant under the $\mathbf{Z}_2$ symmetry $\phi \to -\phi$

A second example, the Brout-Englert-Higgs-Kibble model [60], consists of a complex scalar field χ coupled to a $U(1)$ gauge field A_μ:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - (D_\mu\chi)^* D^\mu\chi + \mu^2|\chi|^2 - \lambda|\chi|^4,$$

where $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ and $D_\mu\chi \equiv (\partial_\mu - ieA_\mu)\chi$. Parametrizing the scalar field χ as

$$\chi(x) = \frac{1}{\sqrt{2}}\Big(\phi(x) + \phi_0 \exp\left(i\zeta(x)/\phi_0\right)\Big),$$

we see that the minimum of the scalar potential occurs at $\phi = 0$, $\phi_0 = \mu/\sqrt{\lambda}$. Defining $\tilde{A}_\mu = A_\mu - \frac{1}{e\phi_0}\partial_\mu\zeta$ and expanding the action to quadratic order in the fields, we obtain:

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu\tilde{A}_\nu - \partial_\nu\tilde{A}_\mu)^2 - \frac{e^2\phi_0^2}{2}\tilde{A}_\mu\tilde{A}^\mu - \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \mu^2\phi^2 + \text{cubic terms}.$$

We see that, after tachyon condensation, we end up with a real scalar ϕ with mass $\sqrt{2}\mu$ and a massive vector field $\tilde{A}_\mu$ with mass $e\mu/\sqrt{\lambda}$. This last field is made up out of the original massless vector field A_μ and the scalar component ζ. Also in this case, tachyon condensation goes hand in hand with symmetry breaking, this time of a local $U(1)$ symmetry. This example lies at the basis of mass generation for the $W_\pm$ and Z vector bosons in the theory of electroweak interactions.

In the presence of gravity, the process of tachyon condensation leads to important gravitational effects as well. In cosmological models, tachyon condensation induces a period in which space undergoes a rapid expansion known as *inflation*. This phenomenon may well have occured in the early stages of of the evolution of the universe [59].

4.2 D-branes in string theory

In the discussion of open string boundary conditions in (2.3.4), we have limited our attention to Neumann boundary conditions at the endpoints of the string:

$$\partial X^\mu(z) = \bar{\partial}\tilde{X}^\mu(\bar{z}).$$

Because this choice preserves space-time translation invariance, it was, for a long time, the only one that was thoroughly studied in the literature, although it was known that other boundary conditions also lead to consistent string theories [61]. The importance of *Dirichlet boundary conditions* in the description of extended objects in string theory became clear through the ground-breaking work of Polchinski [62]. Here we will just be able to convey some very basic notions and refer the reader to the reviews [63, 64, 65] for a more complete discussion.

Consider the bosonic open string with a Dirichlet boundary condition in the X^{25} direction (while keeping Neumann conditions along the other directions):

$$X^{25}(z,\bar{z}) = c \qquad \text{for } z = \bar{z}.$$

This condition means that the endpoints of the open string are stuck on the hyperplane $X^{25} = c$ and breaks translation invariance along this direction. The mode expansion for X^{25} becomes:

$$X^{25}(z,\bar{z}) = c + i\sqrt{\frac{\alpha'}{2}} \sum_{m\neq 0} \frac{\alpha^{25}{}_m}{m}(z^{-m} - \bar{z}^{-m}).$$

The massless spectrum now consists of the states

$$\begin{aligned} \mathcal{V}^\mu &= :\partial X^\mu e^{ik\cdot X}: \qquad \mu = 0\ldots 24 \\ \mathcal{V}^{25} &= :\partial X^{25} e^{ik\cdot X}:, \end{aligned}$$

where k_μ, $\mu = 0\ldots 24$ is a vector satisfying $k^\mu k_\mu = 0$. The first state represents a massless vector field localized on the hyperplane, while the second state represents a massless scalar field on the hyperplane. The existence of this last field points toward the interpretation that the hyperplane is not a completely rigid object, the scalar field representing small fluctuations in the geometry of the hyperplane itself. Such massless excitations, also called moduli or collective coordinates, arise also in the description of extended objects in other theories, for example when considering monopoles in Yang-Mills-Higgs theory [66]. Hyperplanes on which open strings end should hence be seen as dynamical objects and are called Dirichlet-branes or *D-branes*. The example we considered represents an object with 24 spatial dimensions and is called a D24-brane. The excitations of a Dp brane are described by an open string theory with Dirichlet boundary conditions along $25 - p$ directions. Apart from the open strings confined to the D-brane, we also consider closed strings propagating in the bulk. The D-brane degrees of freedom couple to the bulk degrees of freedom, including gravity, through open-closed string interactions. The tree-level amplitudes between the massless degrees of freedom can also be reproduced by a space-time effective action of the Dirac-Born-Infeld [67] type:

$$S_p = -T_p \int d^{p+1}\xi e^{-\phi}[-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})]^{\frac{1}{2}}, \qquad (4.2.1)$$

where the ξ^a are coordinates on the D-brane worldvolume and the fields G_{ab} and B_{ab} are the induced metric and antisymmetrc tensor on the brane. The dilaton coupling $e^{-\phi} \sim 1/g_c$ reflects the fact that these interactions arise from disc amplitudes (see section 2.2.1). This also shows that D-branes are solitonic objects in the following sense: their energy per unit volume appoaches infinity as $g_c \to 0$ and they are invisible in the perturbative spectrum of the theory.

We can also consider the presence of multiple D-branes. For example, we can take m parallel $D24$ branes, labelled by an index $i = 1 \dots n$ and located at $X^{25} = c_i$. We now also have to consider open strings running from brane i to brane j labelled by indices[2] ij. The mode expansion for strings of type ij is given by

$$X^{25}_{ij} = (z, \bar{z}) = c_i + \frac{i}{2\pi}(c_j - c_i)\ln(z/\bar{z}) + i\sqrt{\frac{\alpha'}{2}} \sum_{m \neq 0} \frac{\alpha^{25}{}_m}{m}(z^{-m} - \bar{z}^{-m}),$$

and the mass-shell condition (2.4.4) becomes:

$$m^2 = -p_\mu p^\mu = \left(\frac{c_j - c_i}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(N^{tot} - 1).$$

In the first term, one recognizes the square of the length of the stretched string times the string tension. When several D-branes approach each other, the length of the strings stretched between them goes to zero and we get extra massless fields. For n coinciding D-branes, there are n^2 massless vectors that can be combined into a single $U(n)$ gauge field. Similarly, there are n^2 massless scalars that can be combined into a single scalar field in the adjoint representation of $U(n)$. The system can be described by tensoring the open string degrees of freedom with matrices in the Lie algebra of $U(n)$. These 'internal' matrices are called *Chan-Paton factors*.

This story generalizes to the superstring case. Consider a type II closed string theory supplemented with open strings whose endpoints are stuck on

[2] We are considering oriented open strings, so that ij strings are considered different from ji strings for $i \neq j$.

a p+1 dimensional hyperplane. Here, one should supplement the Dirichlet boundary conditions on the bosonic fields with suitable boundary conditions for the fermions [65].

It is clear that, apart from translational invariance in the directions transverse to the brane, space-time supersymmetry will also be broken due to the open string boundary conditions. The remarkable feature here is that, for the even-dimensional branes in type IIA theory as well as for the odd-dimensional branes in type IIB, a certain amount of supersymmetry remains even when a D-brane is present. Indeed, in the presence of a D-brane, precisely half of the space-time supersymmetries are preserved. This identifies the D-branes in superstring theory as *BPS states*. Such states generally carry conserved charges, in this case a Dp brane carries charge under the $p+1$-form potential in the RR sector [62]. This is consistent with the restriction on the dimensions of the BPS-branes: the type IIA theory contains odd RR-forms which couple to the even-dimensional branes, while the even RR forms of type IIB couple to odd-dimensional branes. The effective action for the massless fields now includes a coupling to these RR fields called the *Wess-Zumino term*:

$$\begin{aligned} S_p &= -T_p \int d^{p+1}\xi e^{-\phi}[-\det(G_{ab} + B_{ab} + 2\pi\alpha' F_{ab})]^{\frac{1}{2}} \\ &\quad + i\mu_p \int_{p+1} \exp(B + 2\pi\alpha' F) \wedge \sum_q C_q, \end{aligned} \tag{4.2.2}$$

where, in the exponential, one should interpret all products as wedge products, the C_q denote the RR potentials and the integral picks out the $p+1$ form in the expansion of the integrand. The action (4.2.2) is known to receive further gravitational corrections as well, which we haven't displayed here. A more complete discussion can be found in [68].

4.3 Dualities and strings at strong coupling

D-branes have played an important role in the discovery of *dualities* relating the various superstring theories. A duality is by definition an equivalence relation between two seemingly different theories. The two theories can be mapped into each other by using some dictionary which relates quantities on both sides. Familiar examples of dualities in physics include electric-magnetic duality in free Maxwell theory, in which the roles of the electric and magnetic fields are interchanged (see e.g. [69]), and, in statistical mechanics, Kramers-Wannier duality [70] which relates the Ising model on a 2-dimensional lattice to an Ising model on the dual lattice at the inverse temperature. Another example of a duality, which relates two seemingly different 2-dimensional field theories, is the duality between the massive Thirring model and the sine-Gordon model ([71]).

In section 2.5.6 we saw that there exist five apparently different string theories in $9 + 1$ dimensions: the type IIA and IIB closed string theories, the type I open-closed string theory and the heterotic string with gauge group $SO(32)$ or $E_8 \times E_8$. The answers to physical questions in each of these theories depend on a number of parameters or *moduli* such as the value of the string coupling constant g and, if one allows more general backgrounds than the flat Minkowski space which we have considered so far, the geometry of the target space $\mathcal{M}$. A generic duality will relate theory A with moduli $(g, \mathcal{M})$ to theory B with different moduli $(g', \mathcal{M}')$. We now give some examples.

One of the best understood duality symmetries in string theory is target-space duality or *T-duality* for short (for a review, see [72, 73]). As the name suggests, it relates string theories on different target manifolds. T-duality was first discovered in the context of toroidal compactifications [74, 75]. Consider the type IIA theory with coupling g where one of the spatial directions, say X^9, is taken to be a circle of radius R. One then encounters an important new ingredient: the existence of states corresponding to strings wrapped around the compact circle. One hence distinguishes these *winding modes* and the usual unwrapped states referred to as *momentum modes*. T-duality relates this theory to the type IIB theory with a compact direction of radius α'/R and coupling $\frac{\sqrt{\alpha'}}{R}g$. Under the duality symmetry, winding and momentum get interchanged. The duality also relates D-branes in the two theories: if we start with a $2p$-brane in type IIA which is wrapped along the compact direction, we end up with a $2p - 1$ brane in type IIB. On the other hand, if we start with a $2p$-brane which is not wrapped on the compact circle, we end up with a wrapped $2p + 1$ brane in the dual theory. The reason why T-duality is so well-established is that it can be studied in a regime where both sides are weakly coupled and where perturbation theory is reliable. It can be extended to strings moving in more general backgrounds as well (see our paper [76] and references therein).

Another type of duality, which is by now widely believed to be present in string theory [77], is strong coupling duality or *S-duality* for short. S-duality relates a strongly coupled theory to a weakly coupled theory. This makes it a very useful kind of duality, since it allows strong coupling calculations in both theories, but at the same time makes it very hard to prove. A field theory example of S-duality is *electric-magnetic duality* proposed by Montonen and Olive [78]. Here the duality relates a weakly coupled supersymmetric non-abelian gauge theory to the same theory at strong coupling, but with the role of electric and magnetic charges interchanged. A similar situation arises in the type IIB theory: according to S-duality, the type IIB theory at coupling g is equivalent to type IIB at coupling $1/g$, but with the roles of the fundamental string and the D-string interchanged. The duality also interchanges the NS 5-brane (the object which couples to the 6-form dual to $B_{\mu\nu}$) and the D5-brane. The D3-brane is self-dual, and S-duality reduces to Montonen-Olive duality for the fields living on the D3-brane. Another example of S-duality in string theory is the duality between the type I theory and the heterotic string, both with gauge group $SO(32)$. Here, the D-string of type I gets interchanged with the fundamental string on the heterotic side.

Progress has also been made in the identification of the strong coupling limit of the two remaining string theories, the type IIA theory and the heterotic string with gauge group $E_8 \times E_8$. Both limits are related to an incompletely known theory in 11 dimensions that goes under the name of *M-theory*. This theory is conjectured to reduce to the type IIA string when compactified on a circle of radius $R_{10} = g\sqrt{\alpha'}$, so that in the limit $g \to \infty$ an extra dimension appears. The $E_8 \times E_8$ heterotic string is conjectured to arise from compactifying M-theory on a $\mathbf{Z}_2$ orbifold of the circle. As far as the description of M-theory itself is concerned, the only thing that is really known is the effective low-energy dynamics of the massless states in the theory, which is given by 11-dimensional supergravity. A proposal for a complete quantum-mechanical description of M-theory has been made in the form of the *matrix model* [79]; this model has passed some nontrivial tests such as its ability to describe gravitational scattering processes. On the other hand, an essential object in M-theory, the M5-brane, seems to be incompletely described by the matrix model.

These developments point towards the interpretation that there is a single theory from which all known string theories arise as limits in the space of moduli. The term M-theory, which originally stood for the specific eleven-dimensional limit of the previous paragraph, has now come to denote the full theory. So far, we only have an understanding of this theory in the special corners of the moduli space where a description is possible in terms of a weakly coupled string theory. Away from these special points, the description remains unknown.

4.4 Non-BPS states in string theory and Sen's conjecture

String theory also contains brane-like objects that don't possess the BPS property: a trivial example is provided by the D-branes of the bosonic string. The type II string theories also contain various non-BPS branes. Generically, these are unstable and can decay into lighter states such as BPS branes.

Sen's conjecture, which concerns the properties of the tachyon potential on unstable non-BPS branes, is closely related to these checks of the S-duality conjecture. In studying the non-BPS spectra in two S-dual theories, Sen found agreement provided that the tachyon potential on a non-BPS D-brane has a specific form. The calculations and results presented in the next chapters seem to confirm that the tachyon potential indeed has the required form and can be seen as providing further evidence for S-duality.

To place the calculations of the next chapter in their context, we now give a brief review of the properties of brane-antibrane systems and non-BPS D-branes. More details can be found in [80, 81] and references therein.

4.4.1 Brane-antibrane systems

We discussed in section 4.2 the existence of extended objects, called D-branes. Their defining property is that open strings can end on them. The type IIA and IIB string theories contain $2p$ and $2p+1$ branes respectively and, in their presence, half of the space-time supersymmetry transformations are preserved. A D-p brane carries charge under the $p+1$-form gauge field in the RR sector of the theory.

As can be seen from the effective action (4.2.2), D-branes are oriented objects. Depending on their orientation, D-branes fall into two classes with opposite RR charges, and one speaks of D-branes and anti-D-branes ($\bar{\text{D}}$-branes). Although the D-brane and $\bar{\text{D}}$-brane separately preserve half of the supersymmetries, the combined system breaks all of the space-time supersymmetry. Consider now the situation where a D-brane and a $\bar{\text{D}}$-brane coincide. In this case there are four different types of open strings as illustrated in figure 4.2. To label these four sectors, we introduce internal 2×2 matrices or Chan-Paton

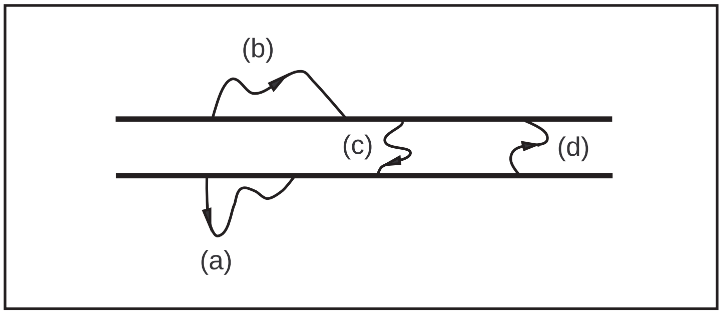


Figure 4.2: The four open string sectors arising in the description of the D$\bar{\text{D}}$ system.

(CP) factors. We assign the following CP factors to the four different sectors:

$$(a): \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \qquad (b): \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$
$$(c): \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad (d): \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

In sectors (a) and (b), the GSO projection keeps the states with $e^{i\pi F} = 1$ while in the sectors (c) and (d) one should project on states with $e^{i\pi F} = -1$ [80]. This means (see table 2.1) that the sectors (c) and (d) contain a tachyon. These can be combined into a complex scalar field.

4.4.2 Non-BPS D-branes

The non-BPS D-branes of type II string theory can be obtained from the D$\bar{\text{D}}$-system by an orbifold projection. In the bulk, the theory contains closed strings. In this sector, one can consider operators F_L and F_R which count the *space-time fermion number* coming from the holomorphic and anti-holomorphic sectors respectively. Their action on states is very simple: they give 0 in the NS sector and 1 in the R sector. These operators should not be confused with the operators counting the world-sheet fermion number which we denoted by F and $\tilde{F}$. The operator $e^{i\pi F_L}$ acts by changing the sign of the states in the holomorphic R sector and is a symmetry of both type II theories when no D-branes are present [89]. In particular, $e^{i\pi F_L}$ changes the sign of the fields in the RR sector. Since a D-brane carries charge under the RR field it follows that $e^{i\pi F_L}$ takes a D-brane to a D̄-brane and vice-versa. So $e^{i\pi F_L}$ is a symmetry of the type II theory in the presence of the coinciding D-brane and D̄-brane considered earlier, and we can consider the theory obtained by modding out this symmetry in an 'orbifold' construction.

For definiteness, let's start with a D2p-D̄2p brane pair in type IIA theory and take the orbifold of this configuration by $e^{i\pi F_L}$. On the bulk fields, the result is that type IIA becomes type IIB [89]. In the open string sector, $e^{i\pi F_L}$ acts by interchanging the D-brane with the D̄ brane, so it acts on a CP-matrix Λ as

$$\Lambda \to \sigma_1 \Lambda (\sigma_1)^{-1}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

After performing the orbifold projection, we are left with the states with CP factors I and σ_1, while those with CP factors σ_3 and $i\sigma_2$ are projected out. The resulting object is a *non-BPS D2p brane* of the type IIB theory. It should be seen as a single, rather than a composite, object, since the degree of freedom for separating the branes, residing in the sector with CP factor σ_3, is projected out after orbifolding.

In an analogous manner, we can construct non-BPS $(2p+1)$-branes in type IIA string theory by starting from a brane-antibrane pair in type IIB and orbifolding by $e^{i\pi F_L}$. Summarizing, we list some of the properties of non-BPS D-branes that will be relevant in what follows:

- A non-BPS Dp-brane in type II theory is described by an open string sector with Dirichlet boundary conditions in $9-p$ directions and Neumann boundary conditions along the $p+1$ world-volume directions.
- The open strings carry Chan Paton factors I or σ_1.
- The GSO projection singles out the states with $e^{i\pi F} = 1$ in the sector with CP factor I and the states with $e^{i\pi F} = -1$ in the sector with CP factor σ_1.
- The non-BPS branes are unstable objects. A first indication for this comes from the fact that the open string spectrum contains a tachyon in the σ_1 sector with mass $m^2 = -\frac{1}{2\alpha'}$.

In a series of papers [82]-[88], Sen has shown the existence of *stable* non-BPS branes as well. These are typically obtained by taking some orbifold such that the open string tachyon is projected out. Such states are stable when they are the lightest states carrying some conserved charge. Stable non-BPS states are interesting for a number of reasons: first of all, their world-volume theories are non-supersymmetric and provide string-theoretic models of non-supersymmetric field theory. Second, non-BPS states have played an important role in providing checks of S-duality: if the S-duality conjecture is correct, then also the spectrum of non-BPS states on both sides should match.

4.4.3 Sen's conjecture

Sen's conjecture concerns the properties of the potential for the tachyon on unstable non-BPS branes. It originated from considering a pair of orbifolds of type IIB theory which are conjectured to be S-dual, where it arose as a necessary condition for the non-BPS spectrum to match on both sides [83]. It was subsequently argued to hold more generally [84]. We now proceed to state the conjecture in its general form.

So far, we have encountered three types of branes in which the open string spectrum contained a tachyonic mode T: the D-branes of the bosonic string theory and the $D\bar{D}$ and non-BPS branes in the type II theories. According to Sen's conjecture, after tachyon condensation, whereby T acquires a vacuum expectation value $< T >= T_0$, the system is indistinguishable from the closed string vacuum without any D-branes. In particular, this implies that the energy that was contained in the mass of the unstable brane is precisely used up in the process of tachyon condensation. This can only be the case if the energy density $V(T)$ for the tachyon field satisfies the following property: the difference $V(0) - V(T_0)$ between the value at the local maximum $T = 0$ (corresponding to the unstable brane) and the value at the minimum T_0 (corresponding to the stable configuration) should be precisely equal to the tension T_p of the unstable brane.

4.4.4 Tachyon condensation and string field theory

When attempting to verify Sen's conjecture, we are immediately confronted with the limitations of string perturbation theory. To describe tachyon condensation in field theory, as in the example of ϕ^4 theory we studied in 4.1, we looked for a scalar field configuration which is constant and which minimizes the potential. Although this is a simple classical computation, it is also non-perturbative: by making a perturbation expansion in the coupling constant

λ, one would only find the unstable vacuum $\phi = 0$ and miss the solutions $\phi_0 = \pm\frac{\mu}{\sqrt{\lambda}}$ corresponding to the stable vacua.

This is why string perturbation theory is insufficient to describe tachyon condensation: it can only describe interactions between on-shell states, while we are interested in the physics around a constant tachyon which is not an on-shell state. In order to properly describe tachyon condensation, we need an off-shell description of string theory, including interactions. This is the realm of string field theory. In this way, Sen's conjecture provides an excellent testing ground for string field theory. As Polchinski observed in 1998 [9], string field theory had proven unsuccesful because 'it has not allowed us to calculate anything we did not know how to calculate already using string perturbation theory'. The study of the tachyon potential is an ideal setting to invalidate this point of view.

Chapter 5

Tachyon condensation in bosonic string field theory

In this chapter we will discuss the tachyon potential for D-branes in bosonic string theory. This is the context in which Sen's conjecture has been most intensely studied. Although our own work involves the study of the tachyon potential in supersymmetric theories, we include the bosonic case since it provides a simple setting to discuss various aspects of the calculation which can then be generalized to the technically more involved case of the superstring. Most material in this section is based on [90, 94].

5.1 The tachyon potential on a D25 brane

The open bosonic string field theory discussed in 3.1 describes open strings moving freely (i.e. with Neumann boundary conditions) in 26 flat dimensions. Or, in more modern language, it describes the dynamics of a $D25$ brane in bosonic string theory.

In this subsection, we will first discuss tachyon condensation on the $D25$ brane on general grounds. Not only the tachyonic component of the string field $c_1|0>$ will get a vacuum expectation value in this process, but, in order to satisfy the string field theory equations of motion, various other fields that couple to the tachyon will be switched on as well. Not all string field components will play a role in tachyon condensation however. For example, since, according to Sen's conjecture, the stable ground state is Lorentz invariant, we expect only fields that transform as scalars under spacetime Lorentz transformations to participate in the process of tachyon condensation. We will presently argue that the fields that acquire a vacuum expectation value under tachyon condensation belong to a subspace $\mathcal{H}_1$ (which, of course, includes the tachyonic mode). The properties of this subspace will then be used in the next section to show that the tachyon potential does not depend on certain details of the model.

Indeed, if we can decompose the space of string fields of ghost number 1 (denoted by $\mathcal{H}$) into two subspaces, a subspace $\mathcal{H}_1$ which contains the tachyonic state, and a subspace $\mathcal{H}_2$ such that the component fields of $\mathcal{H}_2$ couple at least quadratically to the component fields of $\mathcal{H}_1$ (i.e. there are no couplings involving just one $\mathcal{H}_2$ field and any number of $\mathcal{H}_1$ fields), we see that it is consistent with the equations of motion to put all the $\mathcal{H}_2$ components of the string field equal to zero. In other words, we obtain a consistent truncation of the theory by restricting the string field to lie in $\mathcal{H}_1$.

We will now describe such a decomposition of $\mathcal{H}$ into $\mathcal{H}_1$ and $\mathcal{H}_2$, $\mathcal{H}_1$ containing the zero momentum tachyon $c_1|0>$. In looking for the true vacuum of the theory, we can then restrict attention to the states in $\mathcal{H}_1$.

We include in $\mathcal{H}_1$ all the states obtained from the $SL(2,\mathbf{C})$ invariant vacuum $|0>$ (with vertex operator 1) by acting with the ghost modes b_n, c_n and the Virasoro generators L_n^m of the matter theory. In vertex operator language, these can be obtained as products of derivatives of $b(z)$, $c(z)$ and the matter energy-momentum tensor $T^m(z)$. Note that all these states are Lorentz scalars.

The subspace $\mathcal{H}_2$ then contains all the states in $\mathcal{H}$ not included in $\mathcal{H}_1$. More specifically, it consists of all states with non-zero momentum k, and the states of momentum 0 obtained by acting with the modes b_n, c_n and L_n^m on primary fields of weight > 0[1].

We still have to show that our decomposition satisfies the requirement formulated earlier. Consider first the kinetic term in the action. We need to prove that there is no coupling between a $\mathcal{H}_1$ field and a $\mathcal{H}_2$ field. First of all, the BRST charge Q_B is constructed out of the ghost oscillators and the matter Virasoro generators, hence it does not mix the states of $\mathcal{H}_1$ and $\mathcal{H}_2$. Since the BPZ inner product between descendants of different primaries vanishes ([9],

[1]Although we have worked with the modes L_n^m instead of the α_n^μ, it can be shown ([9], chapter 15) that all states can indeed be obtained this way.

chapter 15), it follows that there is no coupling between $\mathcal{H}_1$ and $\mathcal{H}_2$ from the kinetic term.

Consider now the cubic term in the action. We have to show that the coupling between two $\mathcal{H}_1$ fields and one $\mathcal{H}_2$ field vanishes. First of all, we note that conformal transformations take a state in $\mathcal{H}_1$ ($\mathcal{H}_2$) to a state in $\mathcal{H}_1$ ($\mathcal{H}_2$). It then suffices to show that the $\mathcal{H}_1$ fields form a subalgebra of the $\star$ algebra. From the construction on p. 72 we see that this is indeed the case, since the OPE between two $\mathcal{H}_1$ fields contains only states in $\mathcal{H}_1$.

The calculation of the tachyon potential now proceeds by substituting a general element $\mathcal{T}$ (which we will call 'tachyon field') of $\mathcal{H}_1$ into the string field theory action[2]:

$$S[\mathcal{T}] = -\frac{1}{g^2}\left(\frac{1}{2}\langle\langle\, \mathcal{V}_{\Psi_\mathcal{T}} Q_B \mathcal{V}_{\Psi_\mathcal{T}}\,\rangle\rangle + \frac{1}{3}\langle\langle\, \mathcal{V}_{\Psi_\mathcal{T}}\; \mathcal{V}_{\Psi_\mathcal{T}}\; \mathcal{V}_{\Psi_\mathcal{T}}\,\rangle\rangle\right).$$

The result will be of the form

$$S[\mathcal{T}] = -\frac{1}{g^2}\int d^{26}x\; h(\mathcal{T})$$

where $h(\mathcal{T})$ is space-time independent due to the fact that $\mathcal{T}$ is momentum independent. The combination $\frac{1}{g^2}h(\mathcal{T})$ can hence, up to an additive constant, be identified with the total potential energy of the configuration $\mathcal{T}$. Since $h(\mathcal{T} = 0) = 0$, this additive constant should be equal to the brane tension[3] T_{25}. The coupling constant g is related to the tension of the 25 brane as [90]

$$T_{25} = \frac{1}{2\pi^2 g^2}.$$

Defining the *tachyon potential* as $f(\mathcal{T}) = 2\pi^2 h(\mathcal{T})$, we can write the total energy density as

$$V_{25}(\mathcal{T}) = T_{25}(1 + f(\mathcal{T})). \tag{5.1.1}$$

The objective is now to find the ground state $\mathcal{T}_0$ for which $f(\mathcal{T})$ is minimal. If Sen's conjecture is correct, at the minimum one should have

$$f(\mathcal{T}_0) = -1. \tag{5.1.2}$$

[2]In comparing with (3.1.26), the reader will note that we have rescaled the string field $\Psi \to \Psi/g$ and introduced an overall minus sign. The latter step is performed in order for the kinetic term in the action to have the correct sign for our choice of signature (compare (3.1.33) to (4.1.3)).

[3]Of course, such an constant energy density has no physical meaning as long as one does not include gravity. We have used extra input from the coupling of the brane to gravity (4.2.1): $V(T = 0) = T_{25}$

We will discuss an approximation scheme to check the conjecture (5.1.2) in section 5.3.

5.2 Universality

The remarks of the previous paragraph allow us to argue that the result of the calculation for the D25 brane is *universal* in the sense that the form of the tachyon potential does not depend on certain details of the theory such as the dimension of the brane or whether the calculation is performed in a flat background [90].

From the definition of the subspace $\mathcal{H}_1$ and the form of the string theory action, it is clear that, to obtain the function $f(\mathcal{T})$, one only needs to calculate correlation functions involving the ghost fields and the matter energy-momentum tensor. This implies that $f(\mathcal{T})$ is a universal function in the sense that it is insensitive to the details of the matter theory but depends only on its conformal properties; these are in turn determined by the central charge which is 26. So, if we were to replace the X^μ theory with some other CFT of central charge 26, the form of the potential $f(\mathcal{T})$ will remain unchanged.

For instance, if we want to describe a lower-dimensional D-brane, say of spatial dimension p, we should use a matter theory in which the X^μ obey Dirichlet boundary conditions in the $25 - p$ transversal directions and Neumann boundary conditions along the $p+1$ world-volume directions. The total energy density will now take the form

$$V_p(\mathcal{T}) = T_p(1 + f(\mathcal{T}))$$

where T_p is the D-brane tension and $f(\mathcal{T})$ is the same function as in the D25 brane case. The function $f(\mathcal{T})$ is also insensitive to the details of the geometry of the background in which the string propagates as long as the string theory remains critical (i.e. the matter central charge is 26). So the form of the tachyon potential remains the same if we replace the 26 dimensional Minkowski background by some other manifold. The matter theory describing the dynamics of a D-brane is called a *boundary conformal field theory* (BCFT); the previous properties are summarised in the statement that the tachyon potential has a universal form, independent of the BCFT.

These properties imply that if Sen's conjecture (5.1.2) is found to hold for the D25 brane in Minkowski space, it will automatically hold for lower dimensional D-branes and in general backgrounds.

5.3 The level truncation method

We will now describe method for the calculation of $f(\mathcal{T})$ by successive approximations that was first proposed in [91]. It consists of truncating the tachyon field $\mathcal{T}$ to include only components up to a certain level n. Here it is customary to redefine the level of a state to be its eigenvalue under $L_0 + 1$ so that the purely tachyonic state (which has $L_0 = -1$) is of level zero. One can also truncate the function $f(\mathcal{T})$ to include only terms up to certain level m, where the level of a term in $f(\mathcal{T})$ is defined as the sum of the levels of the fields appearing in this term. The resulting approximation of the function $f(\mathcal{T})$ is called the *level* (n, m) *approximation*. Since the kinetic term in the action is already of level $2n$, it does not make much sense to take m smaller than $2n$. In the following, we will mostly consider level $(n, 2n)$ approximations to $f(\mathcal{T})$.

One can then find the configuration $\mathcal{T}_0$ which extremises the truncated potential $f(\mathcal{T})$ and compute the approximate value $f(\mathcal{T}_0)$ at the minimum. In practice, this value turns out to approach a limiting value quite rapidly as one includes more levels, although there doesn't seem to exist a convincing a priori argument that this should be the case. This is one of the issues we will address in chapter 6, where we will study a 'toy model', a truncated version of string field theory, where exact results can be derived and convergence of the level truncation method can be established.

5.4 Gauge-fixing and twist invariance

Before we give an explicit sample calculation of the tachyon potential in the level truncation method, we mention some facts which will allow us to further simplify the calculation.

First of all, in section 3.1.2 we saw that the string field theory action has a gauge invariance (3.1.12). This implies that the tachyon potential will have many flat directions and we can limit the number of fields we have to take into account by fixing the gauge. In our calculations, we will impose the Feynman-Siegel gauge already encountered in 3.1.7:

$$b_0|\mathcal{T}>= 0. \tag{5.4.1}$$

This gauge choice can be justified in perturbation theory (i.e. as long as the coupling g is small enough) [94].

First of all, the gauge (5.4.1) can (almost) always be reached by using a linearised gauge transformation. Suppose we start from a string field $|T_n>$ at level n not satisfying (5.4.1). Consider the gauge-equivalent state

$$|\tilde{T}_n>=|T_n>-\frac{1}{n-1}Q_B(b_0|T_n>).$$

This state does satisfy the condition (5.4.1) due to the commutation relation $\{Q_B, b_0\} = L_0$. Clearly, the argument fails for $n = 1$, so we can not impose the gauge condition (5.4.1) on these states.

Secondly, the Feynman-Siegel gauge fixes the gauge completely, i.e. there is no residual gauge freedom left after imposing (5.4.1). Suppose that both $b_0|T_n>=0$ and $b_0(|T_n>+Q_B|\Lambda_n>)=0$ with $|T_n>$ and $|\Lambda_n>$ states at level n, $n \neq 1$. Then $Q_B|\Lambda_n>$ vanishes, since:

$$(n-1)Q_B|\Lambda_n>=\{Q_B, b_0\}Q_B|\Lambda_n>=0.$$

The string field theory action, when restricted to string fields $\mathcal{T}$ in the subspace $\mathcal{H}_1$, has a global $\mathbf{Z}_2$ symmetry, called *twist symmetry*, which will allow us to restrict our attention to states at even levels. Consider the conformal transformation $M(z) = -z$. If $\mathcal{V}_{\mathcal{T}}$ is the vertex operator corresponding to the string field $\mathcal{T} \in \mathcal{H}_1$, we will show that the string field theory action is invariant under the transformation $\mathcal{V}_{\mathcal{T}} \to -(M \circ \mathcal{V}_{\mathcal{T}})$. This follows from the following properties of the conformal transformations (3.1.22):

$$\begin{aligned} f_1^{(n)} \circ M \circ \mathcal{O} &= \tilde{I} \circ R \circ f_1^{(n)} \circ \mathcal{O} \\ f_i^{(n)} \circ M \circ \mathcal{O} &= \tilde{I} \circ f_{n-i+2}^{(n)} \circ \mathcal{O} \qquad \text{for } 2 \leq i \leq n, \end{aligned} \tag{5.4.2}$$

where R denotes a rotation over 2π and $\tilde{I}$ is an $SL(2, \mathbf{C})$ transformation

$$\tilde{I}(z) = 1/z.$$

These relations can be verified by taking $\mathcal{O}$ to be a primary field. The transformation R has no effect on the states in $\mathcal{H}_1$ since these have integer conformal weight. Using the fact that CFT correlators are invariant under $\tilde{I}$, we see that under $\mathcal{V}_{\mathcal{T}} \to -(M \circ \mathcal{V}_{\mathcal{T}})$ the action transforms as

$$g^2 S[-(M \circ \mathcal{T})] \quad = \quad \frac{1}{2}\langle f_1^{(2)} \circ \mathcal{V}_{\mathcal{T}}(0) f_2^{(2)} \circ Q_B \mathcal{V}_{\mathcal{T}}(0)\rangle$$

$$
\begin{aligned}
& -\frac{1}{3}\langle f_1^{(3)} \circ \mathcal{V}_{\mathcal{T}}(0) f_3^{(3)} \circ \mathcal{V}_{\mathcal{T}}(0) f_2^{(3)} \circ \mathcal{V}_{\mathcal{T}}(0) \\
= \; & \frac{1}{2}\langle f_1^{(2)} \circ \mathcal{V}_{\mathcal{T}}(0) f_1^{(2)} \circ Q_B \mathcal{V}_{\mathcal{T}}(0)\rangle \\
& +\frac{1}{3}\langle f_2^{(3)} \circ \mathcal{V}_{\mathcal{T}}(0) f_1^{(3)} \circ \mathcal{V}_{\mathcal{T}}(0) f_3^{(3)} \circ \mathcal{V}_{\mathcal{T}}(0) \\
= \; & g^2 S[\mathcal{T}],
\end{aligned}
$$

where, in the last line, we have used that $\mathcal{V}_{\mathcal{T}}$ is Grassmann odd.

The twist symmetry transformation $\mathcal{V}_{\mathcal{T}} \to -(M \circ \mathcal{V}_{\mathcal{T}})$ works on level n states as $(-1)^n$. Under the action of the twist symmetry, the space $\mathcal{H}_1$ splits up in the *twist even* states (containing the pure tachyon) and the *twist odd* states. Twist invariance of the action implies that there are no linear couplings between twist even and twist odd states, hence it is a consistent truncation to take $\mathcal{T}$ to be a twist even field. So we can consistently put to zero the states at odd levels.

5.5 Level 4 string field

Combining the simplifications discussed in the previous section, one finds the states up to level four contributing to the tachyon potential listed in table 5.1. The level four string field $\mathcal{T}$ is given by

$$\mathcal{T} = tT + uU + vV + aA + bB + cC + dD + eE + fF. \tag{5.5.1}$$

5.6 Level (4, 8) tachyon potential

In order to evaluate the tachyon potential, one first has to compute the conformal transformations of the vertex operators in table 5.1. As discussed in section 2.3.2, a primary field $\mathcal{O}$ transforms as:

$$f \circ \mathcal{O}(z) = (f'(z))^h \mathcal{O}(f(z)), \tag{5.6.1}$$

and derivatives of primary fields transform according to:

$$f \circ \partial^n \mathcal{O}(z) = \frac{\partial^n}{\partial z^n} \left((f'(z))^h \mathcal{O}(f(z)) \right). \tag{5.6.2}$$

level	state	vertex operator
0	$c_1\|0>$	$T = c$
2	$c_{-1}\|0>$	$U = \frac{1}{2}\partial^2 c$
	$L^m_{-2}c_1\|0>$	$V = T^m c$
4	$L^m_{-4}c_1\|0>$	$A = \frac{1}{2}\partial^2 T^m c$
	$L^m_{-2}L^m_{-2}c_1\|0>$	$B = T^m T^m c$
	$c_{-3}\|0>$	$C = \frac{1}{24}\partial^4 c$
	$b_{-3}c_{-1}c_1\|0>$	$D = \frac{1}{2}\partial b \partial^2 cc$
	$b_{-2}c_{-2}c_1\|0>$	$E = \frac{1}{6}b\partial^3 cc$
	$L^m_{-2}c_{-1}\|0>$	$F = \frac{1}{2}T^m\partial^2 c$

Table 5.1: States up to level four contributing to the tachyon potential. The states at odd levels are left out due to twist invariance. States containing the c_0 mode are absent due to the Feynman-Siegel gauge condition (5.4.1). Recall that T^m denotes the energy-momentum tensor of the matter fields.

The transformation of a normal ordered product of primary fields can give rise to anomalous terms as was the case for the energy-momentum tensor (2.3.18) and the ghost number current (2.3.38).

One then has to calculate the two- and three-point correlators in the expression for the string field theory action (3.1.26). The level $(2,6)$ and $(4,12)$ approximations to the tachyon potential were calculated before the advent of D-branes [91, 92]. These results were subsequently reinterpreted in the context of Sen's conjecture [94]. For example, in the level $(0,0)$ and $(2,4)$ approximations one finds:

$$
\begin{aligned}
f_{(0,0)}(\mathcal{T}) &= 2\pi^2\Big(-\frac{1}{2}t^2 + \frac{3^3\sqrt{3}}{2^6}t^3\Big) \\
f_{(2,4)}(\mathcal{T}) &= f_{(0,0)}(\mathcal{T}) + 2\pi^2\Big(-\frac{1}{2}u^2 + \frac{1}{2}v^2 + \frac{11\cdot 3\sqrt{3}}{2^6}t^2 u \\
&\quad -\frac{5\cdot 3\sqrt{39}}{2^6}t^2 v + \frac{19}{2^6\sqrt{3}}t u^2 + \frac{7\cdot 83}{2^6\cdot 3\sqrt{3}}t v^2 - \frac{11\cdot 5\sqrt{13}}{2^5\cdot 3\sqrt{3}}t u v\Big).
\end{aligned}
$$

The coefficients of the level $(4,8)$ contribution to the tachyon potential are also listed in [94].

In determining the extrema of the the potential in these approximations, one finds, apart from the trivial solution with all coefficients equal to zero (corresponding to the unstable configuration), another solution in which the coefficients are nonzero. This is our candidate for the stable vacuum. One finds that the values of the coefficients (denoted by a subscript $_0$), converge quite rapidly as more levels are included , and that the value of the potential at the extremum also converges rapidly to the expected answer (see table 5.2). In the $(0,0)$ approximation, the value of the potential at the extremum con-

level	t_0	u_0	v_0	$f(\mathcal{T}_0)$
$(0,0)$	0.456	–	–	−0.684
$(2,4)$	0.542	0.173	0.187	−0.949
$(4,8)$	0.548	0.204	0.205	−0.986

Table 5.2: Values of some coefficients at the extremum of the tachyon potential in the level truncation method.

tributes already 68% of the value $f(\mathcal{T}_0) = -1$ expected from Sen's conjecture. At level $(2,4)$ and $(4,8)$ one obtains respectively 95% and 99% of the expected answer.

We should mention at this point that, in contrast to the field theory examples discussed in section 4.1, the extremum found here is not a local minimum of the potential including all the fields $t, u, v, \ldots$, but a saddle point. This does not necessarily imply physical instability of the new vacuum; to decide on this point one should analyse the spectrum of physical fluctuations around the new vacuum. Evidence for stability was provided in [93] using the level truncation method. Also, if Sen's conjecture holds in its most general form, there are no physical open string fluctuations around the new vacuum and it follows that this state is perturbatively stable. We will come back to the implications of Sen's conjecture for the physics around the new vacuum in section 5.7.

The field T however is a physical tachyon and we do require that its fluctuations around the new vacuum have positive mass-squared. For this reason, it is important to consider the effective potential $f(t)$ for the purely tachyonic mode by eliminating the other modes in terms of t through their field equations. The result in the various approximations is shown in figure 5.1. We see that the new vacuum corresponds to a local minimum of $f(t)$ so that the t-fluctuations around the new vacuum have positive mass-squared. At level $(2,4)$, the potential diverges at $t \approx -0.325$. In the level $(4,8)$ approximation, the fields we integrate out have quadratic field equations, so the tachyon potential has different branches. The branch which connects the un-

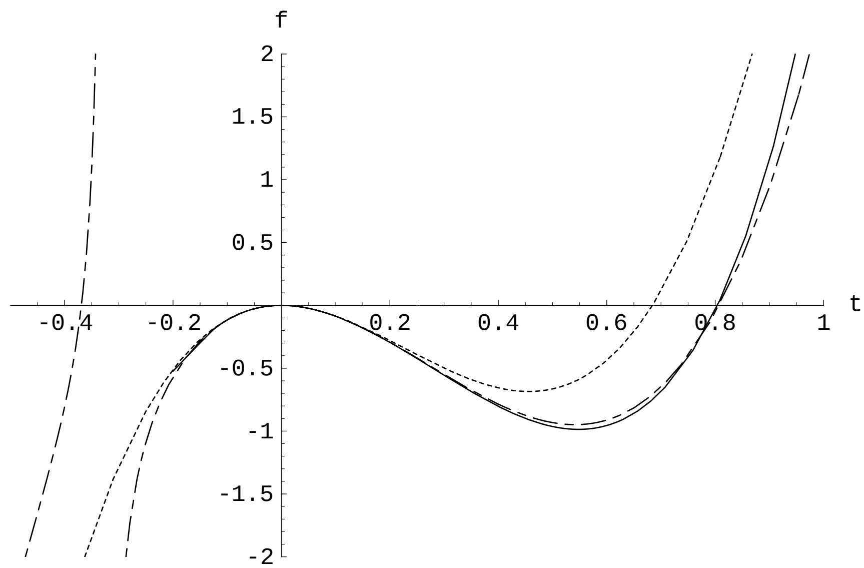


Figure 5.1: The effective tachyon potential $f(t)$ in the $(0,0)$ (dotted line), $(2,4)$ (dashed line) and $(4,8)$ (solid line) approximations.

stable and and stable vacua (shown in figure 5.1) is no longer divergent but ends at $t \approx -0.2$, at which point another branch takes over.

How should we interpret these different branches? One should note that we have been rather cavalier in our treatment of the gauge-fixing. A more careful treatment leads to an extra constraint which remains to be imposed. This can be seen as follows: an arbitrary string field Ψ can be written as a sum of two pieces, a part Ψ_1 that involves the zero mode b_0 ($\Psi_1 = b_0\Phi$ for some Φ) and a part Ψ_2 that doesn't involve b_0. Fixing the Siegel gauge (5.4.1) is equivalent to putting the Ψ_2 part to zero. What we have found so far is an approximate solution to the field equation for Ψ_1. However, from varying the action with respect to Ψ_2, we obtain a second equation $\frac{\delta S}{\delta \Psi_2}_{|\Psi_2=0} = 0$ which should be imposed as a constraint[4] if we want our solution to satisfy all the original field equations. In [99], it was checked that this constraint is satisfied for the unstable configuration and the stable vacuum, but not for the extrema of the other branches of the effective tachyon potential. Hence these extrema are unphysical.

The calculations described in the previous section have been pushed to even greater accuracy by including higher level fields. The authors of [100] have calculated the (10, 20) approximation to the tachyon potential, obtaining 99.912% of the conjectured exact value. The numerical evidence obtained in that paper also shows that the level-truncated value of the potential at the minimum converges to the exact value in an exponential manner: the difference between the conjectured exact minimum value and the value at level k was found to be proportional to $(1/3)^k$. Also, the point where the branch that connects the stable and unstable configurations ends was found to converge towards $t \approx -0.125$. In chapter 6, we will consider a truncated version of string field theory where an exact analysis is possible and where similar properties can be proven to hold. The calculations in [100] were performed using the method of Neumann coefficients which is more suited for automated computation. In particular, with this method, the process of finding all states at a given level and the calculation of the conformal transformations becomes a fully automated process. Unfortunately, as it stands, its extension to the case of superstrings [101], loses its effectiveness due to technical difficulties in the treatment of the superghosts. This is why we will stick with CFT methods in this thesis.

[4] This constraint can be restated as the requirement that the solution is BRST invariant, where the term 'BRST invariant' refers here to the BRST transformation coming from fixing the string field theory gauge invariance (3.1.12).

In conclusion, string field theory calculations provide strong evidence for the validity of Sen's conjecture. The level truncation method turns out to give an approximation scheme that, in practice, converges rapidly to the expected answer.

5.7 Physics around the stable vacuum

So far, we have focused on one particular aspect of Sen's conjecture, namely the energy difference between the unstable and the stable vacuum. In its more general form, the conjecture states that after tachyon condensation, the system is indistinguishable from the closed string vacuum. This would mean that physics around the stable vacuum is radically different from the physics around the unstable vacuum and some interesting new phenomena, are expected to arise. In particular, the conjecture implies that, after tachyon condensation, 26-dimensional Poincaré invariance is restored and that open string excitations have disappeared from the spectrum.

Let us first describe how these phenomena are expected to show up in the string field theory description of the system [102]. Suppose one would know the exact solution Ψ_0 to the string field theory equations of motion that corresponds to the stable vacuum. String field theory around the stable vacuum is then described by performing a shift $\Psi \to \Psi_0 + \Psi$ in the cubic action (3.1.11):

$$S[\Psi_0 + \Psi] = S[\Psi_0] - \frac{1}{g^2}\int (\frac{1}{2}\Psi \star \mathcal{Q}\Psi + \frac{1}{3}\Psi \star \Psi \star \Psi) \qquad (5.7.1)$$

where we have defined a new operation $\mathcal{Q}$

$$\mathcal{Q}\Psi = Q_B\Psi + \Psi_0 \star \Psi + \Psi \star \Psi_0.$$

Apart from the inconsequential constant term $S[\Psi_0]$ (which is equal to the mass of the unstable brane), this is again a cubic action of the Chern-Simons form with Q_B replaced by $\mathcal{Q}$. Extending the action of $\mathcal{Q}$ to string fields Φ of arbitrary ghost number by defining

$$\mathcal{Q}\Phi = Q_B\Phi + \Psi_0 \star \Phi - (-)^{\Phi}\Phi \star \Psi_0$$

and using the fact that Ψ_0 satisfies

$$Q_B\Psi_0 + \Psi_0 \star \Psi_0 = 0,$$

one can easily show that Q satisfies the properties (3.1.27- 3.1.30). This implies that the action (5.7.1) is invariant under the gauge transformations

$$\delta\Psi = Q\varepsilon + \Psi \star \varepsilon - \varepsilon \star \Psi.$$

For small fluctuations around the stable vacuum, we can neglect the interaction term and the field equation becomes $Q\Psi = 0$, while gauge transformations linearise to $\delta\Psi = Q\varepsilon$. Hence the perturbative spectrum around the stable vacuum is given by the cohomology of the operator Q. If Sen's conjecture is correct and there are no open string excitations around the stable vacuum, we expect the operator Q to have vanishing cohomology [102, 103]. It would be a very important check of the conjecture if this turned out to be the case. Unfortunately, despite many efforts, the exact solution Φ_0 remains elusive. Although in [104] a recursive procedure is outlined from which this solution could be obtained, an analytic expression for Ψ_0 remains unknown. In the level truncation approximation, a study of the perturbative spectrum around the stable vacuum has been performed in [105], supporting the expectation that Q has vanishing cohomology.

An important question is whether the theory described by (5.7.1) remains trivial when the interactions are included. It has been suggested [106] that the full interacting theory might be able to describe closed strings, giving a concrete realization of the old idea of 'closed strings from open strings' that we encountered in section 3.3. The intuitive picture is as follows. Suppose we start with a configuration consisting of a non-BPS D-brane and a fundamental string with both ends on the brane. The endpoints of the string act as oppositely charged sources for the electric field on the brane, so there will be electric flux lines running from one endpoint to the other (see figure 5.2(a). After tachyon condensation, the unstable brane has disappeared, but charge conservation dictates that the fundamental string and the flux lines cannot just vanish. The resulting configuration looks like a closed string where part of the closed loop is made up out of a piece of flux tube (see figure 5.2(b), and indeed, it has been argued that the energy density of the flux tube is the same as the tension of the string [107]. The argument described here remains highly speculative, but, if true, could provide a new approach to the elusive off-shell description of closed strings.

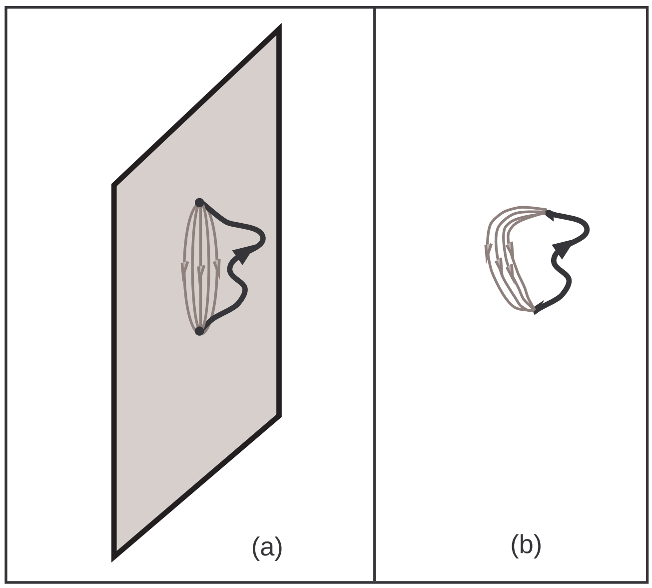


Figure 5.2: (a) An open string with endpoints on a non-BPS D-brane before tachyon condensation. (b) The configuration after tachyon condensation: open string and flux tube together form a closed string.

5.8 Omissions and further developments

In our discussion of tachyon condensation, we have limited our attention to those aspects that are necessary as background for the calculations described in the next chapter. In doing this, we have left out important related developments, on some of which we will now briefly comment. For a more complete review of the literature on this subject, see [108].

It is believed that a bosonic D-brane with a space-dependent tachyon profile in the form of a localised 'lump' of energy describes a D-brane of lower dimension [109]. The existence of lump solutions describing lower-dimensional D-branes has been shown in the level truncation method [110]. Their tensions have been found to agree with the known tensions of the lower-dimensional branes. Such descent relations between D-branes are expected to be present in superstring theory as well. There, a non-BPS Dp-brane in the presence of a tachyon profile in the form of a 'kink' interpolating between the two minima of the potential describes a stable BPS D$(p-1)$ brane. Existence of kink solutions with the expected behaviour in superstring field theory has recently been confirmed in [111]. These applications have also found a geometric interpretation in terms of K-theory [112].

The tachyon potential has also been studied in the presence of a B-field background. In the presence of a large, constant B-field, the tachyon potential has been found to simplify considerably, making it possible to obtain exact results [113].

In section 3.3, we have mentioned another approach to string field theory, called boundary string field theory (BSFT), which is independent of the background. In this approach, it is only the purely tachyonic mode that acquires a vacuum expectation value. This is fortunate, because in this approach it would be very difficult to describe a condensate of non-tachyonic modes. The BSFT approach is assumed to be related to the conventional string field theory through a field redefinition. BSFT has been used to confirm that the energy difference between the unstable and stable vacua is in exact agreement with Sen's conjecture [114]. However, it is not known at present how to give a full description of the physics around the stable vacuum in this approach.

Chapter 6

Toy model for tachyon condensation

In this chapter, we will discuss a toy model for tachyon condensation in bosonic string field theory. The full string field theory problem, which we discussed in chapter 5, consisted of extremising a complicated functional on the Fock space built up from an infinite number of matter and ghost oscillators. As a first simplification, one can consider the variational problem in the restricted Hilbert space of states generated by a single matter oscillator. This problem is still rather nontrivial because the restricted Hilbert space still contains an infinite number of states. The model we will consider here is precisely of this form and its behaviour closely resembles the one found in the full theory with level approximation methods. The main simplification lies in the limited number of degrees of freedom and the fact that we don't have to deal with the technicalities of the ghost system. Other toy models for tachyon condensation were considered in [95].

The motivation for considering such simplified models is twofold. First of all, the level approximation method to the full string theory problem remains largely 'experimental': there doesn't seem to be a convincing a priori reason why the approximation converges to the exact answer, nor do we have any information about the rate of convergence except the 'experimental' information we have from considering the first few levels. Our toy model will allow for the derivation of exact results on the convergence of the level truncation method albeit in a not fully realistic context.

The second reason for considering toy models is perhaps more fundamental: for various reasons discussed in section 5.7, it would be of considerable interest to obtain the exact solution for the stable vacuum in the full theory. However, despite many efforts, this solution is lacking at the present time. The model we will consider is in some sense the 'minimal' problem one should be able to solve if one hopes to find an analytic solution to the full problem.

6.1 The model

In the matter sector of bosonic string theory, we have encountered an infinite number of oscillator modes α_n^μ. Suppose we pick out one of these oscillators, which we call a, and its Hermitean conjugate $a^\dagger$, and normalise them so that the commutator has the standard form

$$[a, a^\dagger] = 1.$$

The zero momentum tachyon, which we will denote here by $|0>$, is annihilated by a:

$$a|0>= 0.$$

We will consider the restricted Hilbert space of states $|\psi>$ obtained by acting with $a^\dagger$ on the vacuum $|0>$. A state in this space can be expanded as

$$|\psi>= \sum_{n=0}^{\infty} \psi_n (a^\dagger)^n |0> . \tag{6.1.1}$$

When restricted to this subspace, the tachyon potential takes the form [28]-[34]:

$$V(\psi) = \frac{1}{2} <\psi|(L_0 - 1)|\psi> + \frac{1}{3} <\mathcal{V}_3|\psi>_{123} .$$

where $L_0 = a^\dagger a$. The first term comes from the kinetic term in the string field theory action, the -1 representing the normal ordering constant (or zero-point energy) responsible for the negative mass-squared of the tachyon state $|0>$. In the second term, coming from the interaction term in string field theory, we have introduced the following notation. We consider three copies of the harmonic oscillator Hilbert space, obtained by acting with oscillator modes $a_i^\dagger$, $i = 1\dots 3$ on vacuum states $|0>_i$. The state $|\psi>_{123}$ stands for the following element in the tensor product space of these three copies:

$$|\psi>_{123}=\left(\sum_{m=0}^{\infty}\psi_m(a_1^\dagger)^m|0>_1\right)\otimes\left(\sum_{n=0}^{\infty}\psi_n(a_2^\dagger)^n|0>_2\right)\otimes\left(\sum_{p=0}^{\infty}\psi_p(a_3^\dagger)^p|0>_3\right). \tag{6.1.2}$$

The three-point vertex $<\mathcal{V}_3|$ is then defined by

$$<\mathcal{V}_3| = {}_{123}<0|\exp\left(\sum_{i,j=1}^{3}N_{ij}a_ia_j\right).$$

Where N is a 3×3 matrix. The cyclicity property (3.1.30) translates into

$$<\mathcal{V}_3|\psi>|\eta>|\xi>=<\mathcal{V}_3|\xi>|\psi>|\eta>=<\mathcal{V}_3|\eta>|\xi>|\psi>,$$

which restricts the matrix N to be of the form

$$N=\begin{pmatrix}2\lambda & \mu & \mu\\ \mu & 2\lambda & \mu\\ \mu & \mu & 2\lambda\end{pmatrix}.$$

If one takes a to represent one of the lowest oscillator modes of the bosonic string, the appropriate matrix elements λ and μ are tabulated in [96].

From the three-point vertex, we can derive an expression for the $\star$ product in our restricted Hilbert space. By writing the three-point vertex as

$$<\mathcal{V}_3|\psi>|\eta>|\xi>=<\psi|(|\eta>\star|\xi>)$$

for general states $|\psi>$, $|\eta>$, $|\xi>$ we obtain a formula for the $\star$ product:

$$\begin{aligned}|\eta>\star|\xi> \quad = \quad & {}_{23}<0|\exp\left(\frac{1}{2}\sum_{i,j=2}^{3}N_{ij}a_ia_j+\sum_{i=2}^{3}N_{1i}a_1^\dagger a_i+\frac{1}{2}N_{11}(a_1^\dagger)^2\right)\\ & |0>_1|\eta>_2|\xi>_3 .\end{aligned} \tag{6.1.3}$$

6.2 Equation for the condensate

We will look for the extrema of the potential (6.1). These occur at the solutions of

$$(a^\dagger a-1)|\psi>+|\psi>\star|\psi>=0. \tag{6.2.1}$$

We will now write these equations in terms of the components ψ_n in the expansion (6.1.1). Let us first take a look at the potential (6.1) in components:

$$V(\psi) = \frac{1}{2}\sum_n n!(n-1)\psi_n^2 + \frac{1}{3}\sum_{m,n,p} m!n!p!\, G_{mnp}\psi_m\psi_n\psi_p \qquad (6.2.2)$$

where the coefficients G_{mnp} are generated by the function:

$$\begin{aligned} G(z_1, z_2, z_3) &= \exp(\frac{1}{2}\sum_{i,j=1}^{3} z_i N_{ij} z_j) \\ &\equiv \sum_{mnp} G_{mnp}(z_1)^m (z_2)^n (z_3)^p. \end{aligned}$$

Due to the form of the matrix N, the G_{mnp} are completely symmetric and are zero when the sum $(m+n+p)$ is odd. This last property guarantees that the potential possesses a $\mathbf{Z}_2$ *twist symmetry* just as in the full string field theory. This symmetry acts on the components as $\psi_n \to (-1)^n\psi_n$. As in the full string field theory, the components that are odd under the twist symmetry can be consistently put to zero:

$$\psi_{2n+1} = 0.$$

The equation 6.2.1 for the even components becomes

$$(2m-1)\psi_{2m} + \sum_{n,p=0}^{\infty} (2n)!(2p)! G_{2m,2n,2p}\psi_{2n}\psi_{2p} = 0. \qquad (6.2.3)$$

The trivial solution, $\psi_{2m} = 0$, has $V(\psi) = 0$ and is the one that will correspond to the unstable state. The solution we are looking for will have lower energy and will correspond to a local minimum of the potential.

6.3 Exact solution for $\mu = 0$

We were not able to solve the equation (6.2.3) for general values of the parameters λ, μ, although we did manage to obtain an exact solution for the special case $\mu = 0$. It is possible to show that, in this special case, the $\star$ product defined by (6.1.3) is associative, a property which is also satisfied by the star product in the full string field theory. In this way, we hope that what we learn from considering this special case will still be relevant to the full problem[1].

We now construct the exact solution in the case $\mu = 0$. The coefficients $G_{2m,2n,2p}$ are particularly simple:

$$G_{2m,2n,2p} = \frac{\lambda^{m+n+p}}{m!n!p!}.$$

The equation (6.2.3) reduces to

$$\psi_{2m} = \frac{\lambda^m}{(1-2m)m!} g(\lambda)^2 \tag{6.3.1}$$

where we have defined a function $g(\lambda)$ by

$$g(\lambda) = \sum_{n=0}^{\infty} \frac{(2n)!\lambda^n}{n!} \psi_{2n}(\lambda).$$

Multiplying equation (6.3.1) by $(2m)!\lambda^m/m!$ and summing over m we obtain $g(\lambda)$:

$$g(\lambda) = \left(\sum_n \frac{\lambda^{2n}(2n)!}{(n!)^2(1-2n)} \right)^{-1} = \frac{1}{\sqrt{1-4\lambda^2}}$$

Hence our candidate for the *stable vacuum* $|\mathrm{vac}>$ is

$$|\mathrm{vac}> = \frac{1}{1-4\lambda^2} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!(1-2n)} (a^\dagger)^{2n} |0>$$

The coefficients ψ_{2n} can be derived from a generating function[2] $F(x)$:

$$\begin{aligned} F(x) &= \frac{1}{1-4\lambda^2} \left(\exp \lambda x^2 + i\sqrt{\pi\lambda} x \; \mathrm{Erf}(i\sqrt{\lambda} x) \right) \\ &\equiv \sum_n \psi_{2n} x^{2n} \end{aligned}$$

[1] Another case which might prove relevant to the full string theory problem is $\lambda = -1/6$, $\mu = 2/3$. In this case as well, the star product defined by (6.1.3) is associative and, moreover, the matrix N satisfies $N^2 = 1$, a property which is also obeyed by its counterpart in the full string field theory. In this case, we have found that the equation for the condensate (6.2.1) reduces to an ordinary differential equation when we represent the state $|\psi>$ by its momentum-space wavefunction $\psi(p)$:

$$-\psi'' + (p^2 - 3)\psi + \pi^{\frac{1}{4}} \sqrt{6} \psi^2 = 0$$

We have, however, not been able to find the solution to this nonlinear equation using standard methods [98].

[2] In fact, the problem can also be recast into a differential equation for the generating function F.

The value of the potential (6.2.2) at the stable vacuum can be expressed entirely in terms of the function $g(\lambda)$:

$$V(\text{vac}) = -\frac{1}{6}g(\lambda)^3 = -\frac{1}{6}(1-4\lambda^2)^{-3/2},$$

which is indeed smaller than the value at the unstable vacuum $V(0) = 0$. It is clear that the stable vacuum only exists for $|\lambda| < \frac{1}{2}$ since the value of the potential becomes imaginary outside this range. Also, for $|\lambda| > \frac{1}{2}$, the state $|\text{vac}>$ is no longer normalisable.

We can also determine the exact *effective tachyon potential* $V(t)$ by solving for the $\psi_{2n},\ n > 0$ in terms of $t \equiv \psi_0$. The equation for these components becomes:

$$\psi_{2m}(\lambda, t) = \frac{\lambda^m}{(1-2m)m!}(t + h(\lambda, t))^2 \qquad \text{for } m > 0 \tag{6.3.2}$$

where we have defined

$$h(\lambda, t) = \sum_{n=1}^{\infty} \frac{(2n)!\lambda^n}{n!}\psi_{2n}(\lambda, t).$$

Multiplying equation (6.3.2) by $(2m)!\lambda^m/m!$ and summing over m we get a quadratic equation for $h(\lambda, t)$:

$$h(\lambda, t) = (\sqrt{1-4\lambda^2} - 1)(t + h(\lambda, t))^2.$$

The two solutions $h_\pm$

$$h_\pm = \frac{1}{2(1-\sqrt{1-4\lambda^2})}\left(-2t(1-\sqrt{1-4\lambda^2}) - 1 \pm \sqrt{4t(1-\sqrt{1-4\lambda^2}) + 1}\right)$$

will give rise to two branches of the effective potential. When we also impose the equation for t, we see that the unstable vacuum $t = 0$ and the stable vacuum $t = \frac{1}{1-4\lambda^2}$ lie on the same branch (i.e. the one determined by h_+) just as in the full theory (see chapter 5 p. 117). Substituting $h_\pm$ in (6.3.2) to obtain the coefficients $\psi_{2n\pm}(\lambda, t)$ and substituting those in (6.2.2) we find the exact form of the two branches of the effective potential $V_\pm(t)$:

$$V_\pm = -\frac{1}{2}t^2 + \frac{h_\pm^2}{2(1-\sqrt{1-4\lambda^2})} + \frac{1}{3}(t + h_\pm)^3.$$

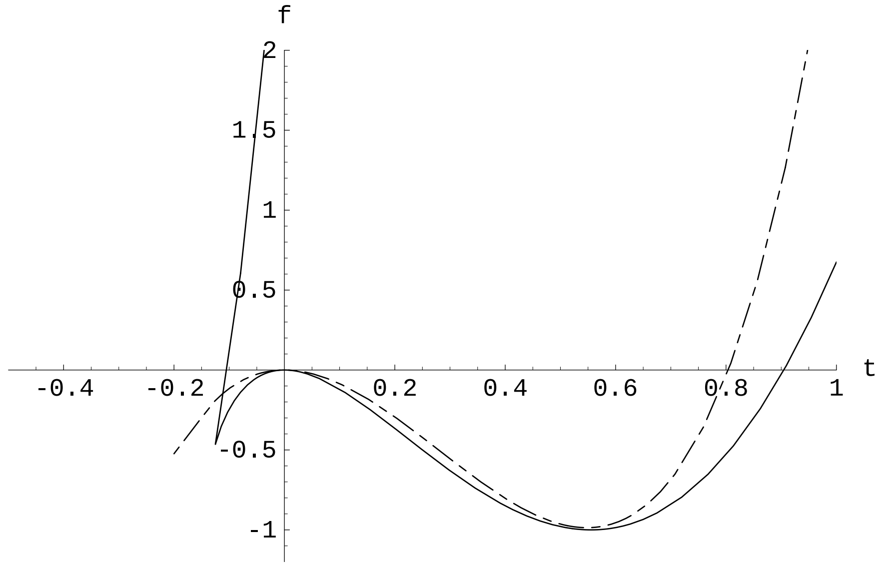


Figure 6.1: The two branches of the toy model tachyon potential in rescaled variables at $\lambda = 0.4$ (full line), as compared to the string field theory result in the level $(4, 8)$ approximation (dashed line).

As was the case in bosonic string field theory, the branch $V_+(t)$, which links the unstable and the stable vacuum, terminates at a finite negative value of t, given in this case by $t = -\frac{1}{4(1-\sqrt{1-4\lambda^2})}$. At this point, the two branches meet. It is also the only point where they intersect, since $V_- > V_+$ for all other values of t. It is interesting to see just how close this simple toy model comes to the real thing. Introducing a rescaled potential $f(t) = 6(1-4\lambda^2)^{3/2}V(t)$ so that the value at the minimum is $f = -1$ and a rescaled variable t in order for the branches of the potential to meet at $t = -0.125$ as conjectured to be the case in the full string field theory problem [100], we find that we should take $\lambda \approx 0.399$ in order for the minimum to occur at the value $t \approx 0.546$ as in the full theory. The behaviour of the effective potential is in good agreement with the one found in the full string field theory with the level truncation method. In figure 6.1, we have plotted both branches of the toy model potential as compared to the string field theory potential at level $(4, 8)$.

6.4 The level truncation method and convergence issues

We can also discuss the convergence of the level truncation method in this model. We will focus on the level $(2k, 6k)$ approximation to the tachyon potential. This means that we include the fields up to level $2k$ and keep all the terms in the potential. In this approximation, the equation for the extremum is just (6.2.3) with all sums now running from 0 to k. The solution proceeds just as in the previous section. First one solves for the function $g^{(k)}(\lambda)$:

$$g^{(k)}(\lambda) \equiv \left(\sum_{n=0}^{k} \frac{\lambda^{2n}(2n)!}{(n!)^2(1-2n)} \right)^{-1} = \left(\sqrt{1-4\lambda^2} + E(\lambda, k) \right)^{-1}.$$

The function $E(\lambda, k)$, which represents the error we make by truncating at level $2k$, can be expressed in terms of special functions

$$E(\lambda, k) = \frac{2^{1+2k}\lambda^{2(1+k)}\Gamma(\frac{1}{2}+k)\ {}_2F_1(1, \frac{1}{2}+k, 2+k; 4\lambda^2)}{\sqrt{\pi}(k+1)!}.$$

We are mainly interested in its asymptotic behaviour for large level k [97]:

$$E(\lambda, k) \sim \frac{2\lambda^2}{\sqrt{\pi}(1-4\lambda^2)} k^{-3/2}(4\lambda^2)^k[1+\mathcal{O}(k^{-1})] \qquad \text{for } k \to \infty.$$

The level-truncated expressions for the components of the approximate vacuum state $|\text{vac}^{(k)}>$ and the value of $f_{(2k,6k)}$ at the minimum are given by:

$$\begin{aligned} \psi_{2m}^{(k)} &= \frac{\lambda^m}{(1-2m)m!} g^{(k)}(\lambda)^2 \\ f_{(2k,6k)}(\text{vac}^{(k)}) &= -(1-4\lambda^2)^{3/2} g^{(k)}(\lambda)^3 \end{aligned}$$

so that the error we make in the level approximation goes like

$$\begin{aligned} \psi_{2m} - \psi_{2m}^{(k)} &\sim \frac{2^{2k+2}\lambda^{2k+m+2}k^{-3/2}}{\sqrt{\pi}m!(1-2m)(1-4\lambda^2)^{5/2}} \\ f(\text{vac}) - f_{(2k,6k)}(\text{vac}^{(k)}) &\sim -\frac{6\lambda^2 k^{-3/2}(4\lambda^2)^k}{\sqrt{\pi}(1-4\lambda^2)^{3/2}} \end{aligned}$$

for large level k. We see that, both for the components of the vacuum state and the value of the potential at the minimum, the level truncation method

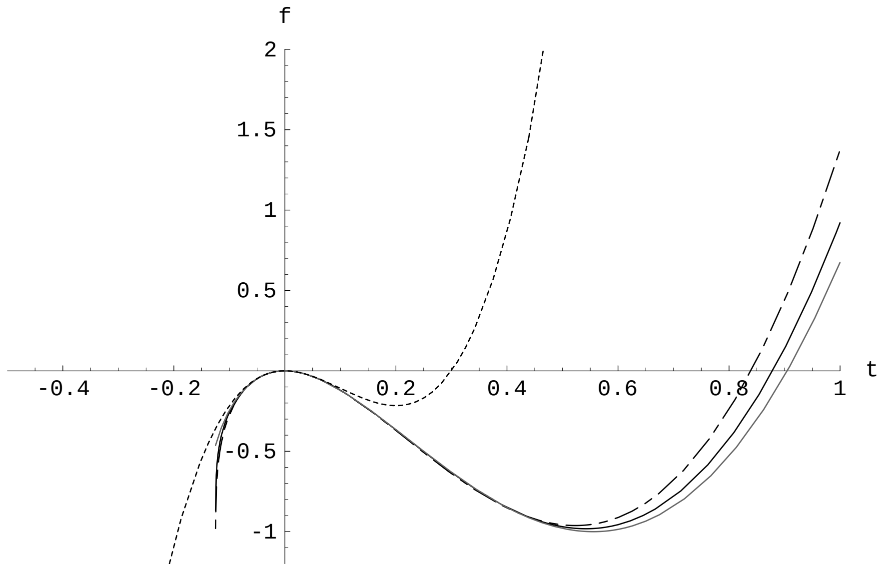


Figure 6.2: The level-truncated effective potential in rescaled variables for $\lambda = 0.4$ at level $(0,0)$ (dotted line), level $(2,6)$ (dashed line) and level $(4,12)$ (full line), as compared to the exact result (gray line).

converges to the exact answer in a manner which is exponential as a function of the level: it goes like $k^{-3/2}e^{-k|\ln 4\lambda^2|}$.

The determination of the level-truncated effective tachyon potential also proceeds as before. The result is

$$f_{(2k,6k)}(t) = 6(1-4\lambda^2)^{3/2}\left(-\frac{1}{2}t^2 + \frac{{h_\pm^{(k)}}^2}{2(1-\sqrt{1-4\lambda^2}-E(\lambda,k))} + \frac{1}{3}(t+h_\pm^{(k)})^3\right)$$

with

$$\begin{aligned} h_\pm^{(k)} &= \frac{1}{2(1-\sqrt{1-4\lambda^2}-E(\lambda,k))}\Big(-2t(1-\sqrt{1-4\lambda^2}-E(\lambda,k))-1 \\ &\quad \pm\sqrt{4t(1-\sqrt{1-4\lambda^2}-E(\lambda,k))+1}\Big). \end{aligned}$$

A plot of the + branch of the potential for $k = 0,\ 1,\ 2$ at $\lambda = 0.4$, as compared to the exact result, is shown in figure 6.2.

Summarising, we have introduced a class of toy models for tachyon condensation obtained by restricting the full string field theory problem to the subspace of states generated by a single oscillator mode. For a special value of one of the parameters of the model, we were able to obtain the exact solution for the stable vacuum state and the value of the potential at the minimum. By comparing with the results from the level truncation method, we showed that the error made in this method, for the coefficients of the stable vacuum as well as for the value of the potential at the minimum, goes like $k^{-3/2}(4\lambda^2)^k$ as a function of the level k. This exponential behaviour is comparable to the one found 'experimentally' in the full string field theory problem in [100]: there, the error was found to behave like $(\frac{1}{3})^k$.

Chapter 7

Tachyon condensation in superstring theory

In order to check Sen's conjecture for the non-BPS D-branes and brane-antibrane systems in superstring theory, one needs an off-shell description of the degrees of freedom living on these objects. In section 3.2 we gave an overview of the three proposed open string field theory actions. In modern language, these represent proposals for the off-shell description of the degrees of freedom on a D9-brane. We will now describe how these actions can be modified to describe the degrees of freedom on non-BPS D-branes and brane-antibrane systems, and discuss the tachyon potential in each of the three cases. These calculations should be seen in a somewhat different light than the calculations in the bosonic theory. There, the string field theory description was well-established and the calculations should be seen as a check of Sen's conjecture. For superstrings, the situation is more or less the reverse: here the conjecture can be put on a more sound footing using T-duality arguments [84], while the correct string field theory description remains disputed. In this case, the study of the tachyon potential should be seen as a check on the string field theory description rather than a check on the validity of Sen's conjecture.

7.1 The tachyon potential in Witten's superstring field theory

We now present our results concerning the tachyon potential in Witten's superstring field theory. This section is an expanded version of our paper [115].

7.1.1 Non-BPS D-branes in Witten's theory

In order to describe NS excitations on a non-BPS D-brane (see section 4.4.2), one should extend Wittens theory to include states with odd world-sheet fermion number as well. This can be accomplished by tensoring the fields with suitable internal Chan-Paton (CP) factors [116, 51]. The results of section 4.4.2 imply that the GSO+ sector string field Ψ_+ should be tensored with the 2×2 unit matrix I while the GSO− field Ψ_- is tensored with σ_1. The fields with CP factors attached are denoted with hats:

$$\hat{\Psi} = \Psi_+ \otimes I + \Psi_- \otimes \sigma_1.$$

In order to preserve (formal) gauge invariance of the action, we associate Chan-Paton factors with the gauge parameters ε, the BRST charge Q_B and the $\star$ operation. We also include a trace over the CP indices in the integration $\int$ We take the following assignments for these internal CP factors:

$$\begin{aligned} \hat{\varepsilon} &= \varepsilon_+ \otimes \sigma_3 + \varepsilon_- \otimes i\sigma_2 , \\ \hat{Q}_B &= Q_B \otimes \sigma_3 , \\ \hat{\star} &= \star \otimes \sigma_3 , \\ \hat{\int} &= \frac{1}{2}\mathrm{Tr}\int , \end{aligned} \tag{7.1.1}$$

We will come back to the motivation for this choice in the discussion following (7.1.4).

The action takes the form[1]

$$\begin{aligned} g^2 S[\hat{\Psi}] &= -\frac{1}{2}\hat{\int} \hat{\Psi} \hat{\star} \hat{Q}_B \hat{\Psi} - \frac{1}{3}\hat{\int} \hat{\Psi} \hat{\star} \hat{\Psi} \hat{\star} \hat{\Psi} \\ &= -\frac{1}{2} \langle\langle\, \hat{\Psi}\, \hat{Q}_B \hat{\Psi} \,\rangle\rangle - \frac{1}{3} \langle\langle\, \hat{\Psi}\hat{\Psi}\hat{\Psi} \,\rangle\rangle \end{aligned} \tag{7.1.2}$$

[1] Again, in comparing with (3.2.6), the reader will note that we have rescaled $\Psi \to \Psi/g$ and adjusted the overall sign in order to give the correct kinetic term for our choice of signature.

where the double brackets should now be interpreted as

$$\begin{aligned}\langle\langle\, \hat{\Phi}_1\hat{\Phi}_2\cdots\hat{\Phi}_{n-1}\hat{\Phi}_n\,\rangle\rangle \quad &= \quad \frac{1}{2}\mathrm{Tr}\Big\langle \hat{Y}(0) f_1^n\circ\hat{\Phi}_1(0)\hat{Z}(0) f_2^n\circ\hat{\Phi}_2(0)\hat{Z}(0)\\ &\quad \cdots \hat{Z}(0) f_{n-1}^n\circ\hat{\Phi}_{n-1}(0)\hat{Z}(0) f_n^n\circ\hat{\Phi}_n(0)\Big\rangle\,.\end{aligned}$$

Here, the trace runs over the internal CP indices and we have defined

$$\begin{aligned}\hat{Z} &= Z\otimes\sigma_3\\ \hat{Y} &= Y\otimes I.\end{aligned}$$

Also, the definition of the conformal transformations requires a bit more care when we include the GSO− sector. In the GSO+ sector, all states at zero momentum had integer conformal weight and so that the transformations (5.6.1) were well-defined. The fields in the GSO− sector have half-integer conformal weight, so we have to select a branch for the square root. Since

$$f_k^{(n)\,\prime}(0) = \frac{4i}{n}e^{2\pi i(\frac{k-1}{n})} = \frac{4}{n}e^{2\pi i(\frac{k-1}{n}+\frac{1}{4})}.$$

we make the following choice for the conformal transformation $f_k^{(n)}\circ\mathcal{O}(0)$ of a primary field $\mathcal{O}$ of weight h:

$$f_k^{(n)}\circ\mathcal{O}(0) = \left(\frac{4}{N}\right)^h e^{2\pi i h(\frac{k-1}{n}+\frac{1}{4})}\mathcal{O}\left(f_k^{(n)}(0)\right) \tag{7.1.3}$$

Since arbitrary vertex operators can be written as products of derivatives of primary fields, this uniquely determines the transformation $f_k^{(n)}\circ\mathcal{V}(0)$ for all operators $\mathcal{V}$.

The gauge transformations are

$$\delta\hat{\Psi} = \hat{Q}_B\hat{\varepsilon} + \hat{\Psi}\star\hat{\varepsilon} - \hat{\varepsilon}\star\hat{\Psi}\,. \tag{7.1.4}$$

The CP factor assignments (7.1.1) were chosen such that gauge transformations preserve the CP structure of the string field:

$$\delta\hat{\Psi} = \delta\Psi_+\otimes I + \delta\Psi_-\otimes\sigma_1$$

and such that the properties of the correlator (3.2.7) are preserved:

$$\langle\langle\, \hat{Q}_B(\hat{\Phi}_1\cdots\hat{\Phi}_n)\,\rangle\rangle \quad = \quad 0\,, \tag{7.1.5}$$

$$\begin{aligned}
\langle\langle \cdots \widehat{Q}_B^2(\widehat{\Phi}_1 \cdots \widehat{\Phi}_n) \cdots \rangle\rangle &= 0\,, && (7.1.6)\\
\langle\langle \cdots \widehat{Q}_B(\widehat{A}\widehat{\Phi}) \cdots \rangle\rangle &= \langle\langle \cdots (\widehat{Q}_B\widehat{A}\,\widehat{\Phi} - \widehat{A}\,\widehat{Q}_B\widehat{\Phi}) \cdots \rangle\rangle\,, && (7.1.7)\\
\langle\langle \cdots \widehat{Q}_B(\widehat{\varepsilon}\widehat{\Phi}) \cdots \rangle\rangle &= \langle\langle \cdots (\widehat{Q}_B\widehat{\varepsilon}\,\widehat{\Phi} + \widehat{\varepsilon}\,\widehat{Q}_B\widehat{\Phi}) \cdots \rangle\rangle\,, && (7.1.8)\\
\langle\langle\, \widehat{\Phi}_1 \cdots \widehat{\Phi}_{n-1}\widehat{\Phi}_n \,\rangle\rangle &= \langle\langle\, \widehat{\Phi}_n\widehat{\Phi}_1 \cdots \widehat{\Phi}_{n-1} \,\rangle\rangle && (7.1.9)
\end{aligned}$$

where the $\hat{\Phi}_i$ can be arbitrary combinations of $\hat{\Psi}$, $\hat{\varepsilon}$, $\hat{Q}_B\hat{\Psi}$ and $\hat{Q}_B\hat{\varepsilon}$.

The proof of these properties goes through as before, with small modifications due to the presence of CP factors:

- Properties (7.1.5) and (7.1.6) follow directly from the properties of the BRST charge.

- To prove properties (7.1.7) and (7.1.8), one should bear in mind that the GSO+ field is Grassmann odd while fields in the GSO− sector are Grassmann even and that one picks up signs when Pauli matrices are interchanged. The Chan-Paton factor assignments ensure that the string field $\hat{\Psi}$ is odd with respect to $\hat{Q}_B$, while the gauge parameter $\hat{\varepsilon}$ is even with respect to $\hat{Q}_B$. This explains the signs in (7.1.7) and (7.1.8).

- We presently outline the proof of the cyclicity property (7.1.9) for the case of zero momentum fields (this is the case relevant for the fields entering in the calculation of the tachyon potential).

  It is easy to see that all the fields in the GSO+ sector are tensored with either I or σ_3, and all the fields in the GSO− sector with either σ_1 or $i\sigma_2$. At zero momentum, the fields in the GSO+ sector have integer conformal weight, and the fields in the GSO− sector have half-integer conformal weight. We compute

$$\begin{aligned}
\langle\langle\, \widehat{\Phi}_1 \cdots \widehat{\Phi}_n \,\rangle\rangle &= \mathrm{Tr}\left\langle \widehat{Y} f_1^n \circ \widehat{\Phi}_1 \widehat{Z} \cdots \widehat{Z} f_{n-1}^n \circ \widehat{\Phi}_{n-1} \widehat{Z} f_n^n \circ \widehat{\Phi}_n \right\rangle\\
&= \mathrm{Tr}\left\langle \widehat{Y} f_2^n \circ \widehat{\Phi}_1 \widehat{Z} \cdots \widehat{Z} f_n^n \circ \widehat{\Phi}_{n-1} \widehat{Z}\, R \circ f_1^n \circ \widehat{\Phi}_n \right\rangle\,,
\end{aligned}$$

  where R is the rotation over an angle of 2π. Next we use $R \circ f_1^n \circ \hat{\Phi}_n = \pm f_1^n \circ \hat{\Phi}_n$, with a plus sign if the field has integer weight (GSO+ sector), and a minus sign if the field has half-integer weight (GSO− sector). We also use the cyclicity of the trace to move the field $\hat{\Phi}_n$ in front. Although we are manipulating Grassmann objects, we do not get an additional minus sign, due to the fact that the total amplitude is Grassmann odd.

$$\langle\langle\, \widehat{\Phi}_1 \cdots \widehat{\Phi}_n \,\rangle\rangle = \pm\mathrm{Tr}\left\langle \widehat{Y} f_1^n \circ \widehat{\Phi}_n f_2^n \circ \widehat{\Phi}_1 \widehat{Z} \cdots \widehat{Z} f_n^n \circ \widehat{\Phi}_{n-1} \widehat{Z} \right\rangle$$

$$\begin{aligned}
&= \pm\mathrm{Tr}\left\langle \hat{Y}\hat{Z}\circ f_1^n\circ\hat{\Phi}_n f_2^n\circ\hat{\Phi}_1\hat{Z}\cdots\hat{Z}f_n^n\circ\hat{\Phi}_{n-1}\right\rangle \\
&= \mathrm{Tr}\left\langle \hat{Y}f_1^n\circ\hat{\Phi}_n\hat{Z}f_2^n\circ\hat{\Phi}_1\hat{Z}\cdots\hat{Z}f_n^n\circ\hat{\Phi}_{n-1}\right\rangle \\
&= \langle\langle\, \hat{\Phi}_n\hat{\Phi}_1\cdots\hat{\Phi}_{n-1}\,\rangle\rangle\,. \qquad (7.1.10)
\end{aligned}$$

In the next to last line we have used that the picture changing operator $\hat{Z}$ commutes with the GSO+ fields and anticommutes with the GSO− fields, cancelling the minus sign in front of the amplitude. ∎

Making use of the properties (7.1.5-7.1.9), the (formal) proof of gauge invariance goes through as in the GSO+ sector.

An important question is whether the gauge-invariant extension to the GSO− sector we have constructed is unique. Writing down a general cubic action including GSO+ fields Ψ_+ as well as GSO− fields Ψ_- and general gauge transformations for these fields, one finds after tedious computation a unique nontrivial[2] solution for the action and gauge transformations (modulo rescalings of the fields and gauge parameters):

$$\begin{aligned}
-g^2S &= \frac{1}{2}\langle\langle\, \Psi_+Q_B\Psi_+\,\rangle\rangle + \frac{1}{3}\langle\langle\, \Psi_+\Psi_+\Psi_+\,\rangle\rangle + \frac{1}{2}\langle\langle\, \Psi_-Q_B\Psi_-\,\rangle\rangle - \langle\langle\, \Psi_+\Psi_-\Psi_-\,\rangle\rangle \\
\delta\Psi_+ &= Q_B\varepsilon_+ + [\Psi_+,\varepsilon_+] + \{\Psi_-,\varepsilon_-\} \\
\delta\Psi_- &= Q_B\varepsilon_- + \{\Psi_+,\varepsilon_-\} + [\Psi_-,\varepsilon_+] \qquad (7.1.11)
\end{aligned}$$

where the correlator of unhatted fields was defined in (3.2.4). When performing this calculation, one needs to use the extension of the cyclicity property (3.2.7) to the GSO− sector:

$$\langle\langle\, \Phi_1\cdots\Phi_n\Psi_\pm\,\rangle\rangle = \pm\langle\langle\, \Psi_\pm\Phi_1\cdots\Phi_n\,\rangle\rangle\,. \qquad (7.1.12)$$

The result (7.1.11) is precisely what one gets when working out (7.1.2) in terms of GSO+ and GSO− components of the fields.

We can now also come back to our choice of branch (7.1.3) in the definition of the conformal transformations of the fields. Had we chosen the other branch, the property (7.1.12) would have changed to $\langle\langle\, \Phi_1\cdots\Phi_n\Psi_\pm\,\rangle\rangle = \langle\langle\, \Psi_\pm\Phi_1\cdots\Phi_n\,\rangle\rangle$. A similar calculation then yields the result that, with this cyclicity property, no nontrivial gauge-invariant extension to the GSO− sector exists. We conclude that equations (7.1.2) define the unique, gauge-invariant extension of Witten's cubic theory to the GSO− sector.

[2] A 'trivial' solution would be to add just a kinetic term $\langle\langle\, \Psi_-Q_B\Psi_-\,\rangle\rangle$ for the GSO− field. We discard this possibility because it doesn't reproduce the necessary coupling between GSO+ and GSO− fields.

7.1.2 The fields up to level 2

As was the case for the bosonic string, we can restrict the fields in the calculation of the tachyon potential to a suitable subspace $\mathcal{H}_1$. An argument similar to the one made in section 5.2 shows that it is a consistent truncation of the string field theory equations to restrict the string field to lie in a subspace $\mathcal{H}_1$ formed by acting only with modes of the stress-energy tensor, the supercurrent and the ghost fields $b,\ c,\ \eta,\ \xi,\ \phi$. Again, the resulting tachyon potential will be universal since it is insensitive to the precise details of the matter theory as long as it furnishes a representation of the $(1,1)$ superconformal algebra with central charge 15. Also, the calculation of the tachyon potential for the the non-BPS D-brane extends to the brane-antibrane system as well: the only thing that changes is the overall normalisation of the potential [51].

We fix the gauge freedom (7.1.4) by imposing the Feynman-Siegel gauge $b_0\hat{\Psi} = 0$ on fields with non-zero conformal weight. The justification of this gauge choice follows the one in the bosonic case (see section 5.4). In the bosonic theory, we were able to further restrict the number of contributing states by making use of a $\mathbf{Z}_2$ twist invariance of the action. We have not been able to find such a symmetry for this particular action. We will, however, encounter twist symmetry in the other two superstring field theory actions we will consider and we will comment on the reasons for its absence in Witten's theory at that point.

Taking all this together we get the list of contributing fields up to level 2 shown in table 7.1. The level of a field is defined here to be the conformal weight shifted by 1/2, so that the tachyon is a level 0 field. We use the notation $|q>$ for the state corresponding to the operator $:e^{q\phi}:$.

The reality condition (3.2.3) determines whether the coefficients of these fields are real or imaginary. The action of BPZ conjugation on various oscillator modes follows from the general formula (2.3.30), while Hermitean conjugation is defined to act as:

$$\begin{aligned}
\mathrm{hc}(\alpha_n^\mu) &= \alpha_{-n}^\mu \\
\mathrm{hc}(\psi_n^\mu) &= \psi_{-n}^\mu \\
\mathrm{hc}(b_n) &= b_{-n} \\
\mathrm{hc}(c_n) &= c_{-n} \\
\mathrm{hc}(\beta_n^\mu) &= -\beta_{-n}^\mu \\
\mathrm{hc}(\gamma_n^\mu) &= \gamma_{-n}^\mu .
\end{aligned} \tag{7.1.13}$$

Level	GSO	state	vertex operator
0	$-$	$c_1\|-1>$	$T = ce^{-\phi}$
1/2	$+$	$\xi_{-1}c_1c_0\|-2>$	$R = \partial\xi c\partial c\ e^{-2\phi}$
1	$-$	$c_1\phi_{-1}\|-1>$	$S = c\partial\phi\ e^{-\phi}$
3/2	$+$	$2c_1c_{-1}\xi_{-1}\|-2>$ $\eta_{-1}\|0>$ $c_1G^m_{-3/2}\|-1>$	$A = c\partial^2 c\partial\xi\ e^{-2\phi}$ $E = \eta$ $F = cG^m\ e^{-\phi}$
2	$-$	$c_1\left[(\phi_{-1})^2 - \phi_{-2}\right]\|-1>$ $c_1\phi_{-2}\|-1>$ $c_1L^m_{-2}\|-1>$ $2c_{-1}\|-1>$ $\xi_{-1}\eta_{-1}c_1\|-1>$	$K = c\ \partial^2\left(e^{-\phi}\right)$ $L = c\ \partial^2\phi\ e^{-\phi}$ $M = cT^m\ e^{-\phi}$ $N = \partial^2 c\ e^{-\phi}$ $P = \partial\xi\eta c\ e^{-\phi}$

Table 7.1: The fields up to level contributing to the tachyon potential. The field R, which contains the zero mode c_0, is included because the Feynman-Siegel gauge can only be imposed on fields with nonzero conformal weight (see section 5.4). The level 0,1 and 2 fields should be tensored with σ_1 and the level 1/2 and 3/2 fields with I. We list the conformal transformations of these fields in appendix A.1.

These definitions are chosen in order to meet two criteria: they are consistent with the commutation relations (2.3.25, 2.3.33, 2.5.4, 2.5.13) and they guarantee that the BRST operator Q_B (2.5.17) is Hermitean. The latter fact can be used to prove the reality of the quadratic term in the action. We further define the the ground state $|-1>$ of the superghost system to be invariant under the action of BPZ conjugation followed by Hermitean conjugation:

$$\text{hc}^{-1} \circ \text{bpz}(|-1>) \equiv |-1> .$$

These rules enable us to work out the reality condition (3.2.3). We presently give some examples.

Consider first the tachyon field $|T>= c_1|-1>$. Using the definitions (7.1.13), one finds that

$$\text{hc}^{-1} \circ \text{bpz}(|T>) = |T>$$

so the tachyon comes with a real coefficient. Next we consider the state R. In terms of (β, γ) modes, it can be written as $R = c_1c_0\beta_{-1/2}|-1>$. From (7.1.13) we find[3]

$$\begin{aligned} \mathrm{hc}^{-1}\circ \mathrm{bpz}(|R>) &= \beta_{-1/2}c_0c_1|-1> \\ &= -|R> . \end{aligned}$$

Hence the state R comes with an imaginary coefficient. Similarly, one finds that the S field comes with an imaginary coefficient, while the coefficients of the fields at levels $3/2$ and levels 2 should be real.

Summarising, we can write the tachyon string field containing states up to level two as:

$$\widehat{T} = t\hat{T} + ir\hat{R} + is\hat{S} + a\hat{A} + e\hat{E} + f\hat{F} + k\hat{K} + l\hat{L} + m\widehat{M} + n\hat{N} + p\hat{P}. \quad (7.1.14)$$

7.1.3 The tachyon potential

In [115], we calculated the level $(2,4)$ tachyon potential in the level truncation method which we will now present. The computation was done in the following way: the conformal transformations of the fields, which we include for completeness in appendix A, were calculated by hand. A nontrivial check on the computation of these conformal transformations was performed by evaluating the action in two coordinate systems related by a global conformal transformation: the results should not (and did not) change. Note that one does not need to evaluate the operators $Q_B\cdot\mathcal{V}_\Psi$ and their conformal transformations explicitly, since:

$$\begin{aligned} f\circ Q_B\cdot\mathcal{V}_\Psi(z) &= Q_B\cdot(f\circ\mathcal{V}_\Psi(z)) \\ &= \mathrm{Res}_{z_1\to f(z)}j_B(z_1)f\circ\mathcal{V}_\Psi(z) \end{aligned}$$

so it suffices to compute correlators with an insertion of the BRST current j_B and then take the residue.

The computation of the necessary CFT correlators between the transformed fields was done with a computer program in Mathematica, written in collaboration with Pieter-Jan De Smet. The program applies Wick's theorem in a rather straightforward manner. The main technical subtleties arose from the implementation of the normal ordering prescription, the ghost zero modes and the treatment of the exponential operators $:e^{q\phi}:$.

[3] One should recall from section 2.3.5 that hc reverses the order of the oscillator modes while bpz doesn't.

Denoting, as before, the level (m, n) approximation to the tachyon potential by $f_{(m,n)}$, we have obtained the following results:

$$
\begin{aligned}
f_{(0,0)} &= -\frac{t^2}{2} \\
f_{(1/2,1)} &= f_{(0,0)} - \frac{9}{2}\, r\, t^2 + 2\, r^2 \\
f_{(1,2)} &= f_{(1/2,1)} - 8\, r\, s\, + \frac{1}{2} s^2 \\
f_{(3/2,3)} &= f_{(1,2)} + 10\, a\, t^2 - 2\, e\, t^2 + 20\, f\, t^2\, - \frac{16}{3\sqrt{3}}\, e\, r^2 - \frac{10}{3}\, r\, s^2 \\
&\quad + \frac{64}{3}\, a\, s\, t + \frac{80}{3}\, f\, s\, t\, + 4\, a\, e + 10\, f^2\,) \\
f_{(2,4)} &= f_{(3/2,3)} + \frac{64}{3\sqrt{3}}\, a\, e\, r - \frac{32}{9\sqrt{3}}\, e^2\, r + \frac{320}{9\sqrt{3}}\, e\, f\, r - \frac{320}{9\sqrt{3}}\, f^2\, r \\
&\quad + 8\, k\, r\, s - \frac{160}{27}\, l\, r\, s + \frac{100}{9}\, m\, r\, s - \frac{80}{9}\, n\, r\, s - \frac{152}{27}\, p\, r\, s \\
&\quad - \frac{460}{27}\, a\, k\, t + \frac{28}{27}\, e\, k\, t - \frac{280}{27}\, f\, k\, t - \frac{128}{27}\, a\, l\, t - \frac{64}{27}\, e\, l\, t \\
&\quad - \frac{250}{9}\, a\, m\, t + \frac{50}{9}\, e\, m\, t - \frac{220}{3}\, f\, m\, t + \frac{104}{9}\, a\, n\, t \\
&\quad - \frac{40}{9}\, e\, n\, t + \frac{880}{27}\, f\, n\, t - \frac{20}{9}\, a\, p\, t + \frac{4}{9}\, e\, p\, t + \frac{200}{27}\, f\, p\, t \\
&\quad + 6\, k^2 - 6\, k\, l + 3\, l^2 + \frac{45}{4}\, m^2 - 6\, n^2 - \frac{3}{2}\, p^2.
\end{aligned} \tag{7.1.15}
$$

At level 1 and 2 the fields r and s can be integrated out exactly to give the following effective potentials:

$$
\begin{aligned}
f_{(0,0)}(t) &= -\frac{t^2}{2}, \qquad f_{(1/2,1)}(t) = -\frac{t^2}{2} - \frac{81\, t^4}{32}, \\
f_{(1,2)}(t) &= -\frac{t^2}{2} + \frac{81\, t^4}{2\,(4 - 64\, t^2)^2} - \frac{648\, t^6}{(4 - 64\, t^2)^2} - \frac{81\, t^4}{4\,(4 - 64\, t^2)}.
\end{aligned}
$$

The main difference with the bosonic potential is that the level 0 contribution has no minimum. We see that the inclusion of these higher modes does not alter this fact, on the contrary, the slope of the potential becomes even steeper and a singularity is encountered at level 2, see figure 7.1. The singularity in the tachyon potential encountered here might be contrasted with the singularities in the tachyon potential of open bosonic string theory encountered in section 5.6. In the case at hand, the potential diverges at the singular

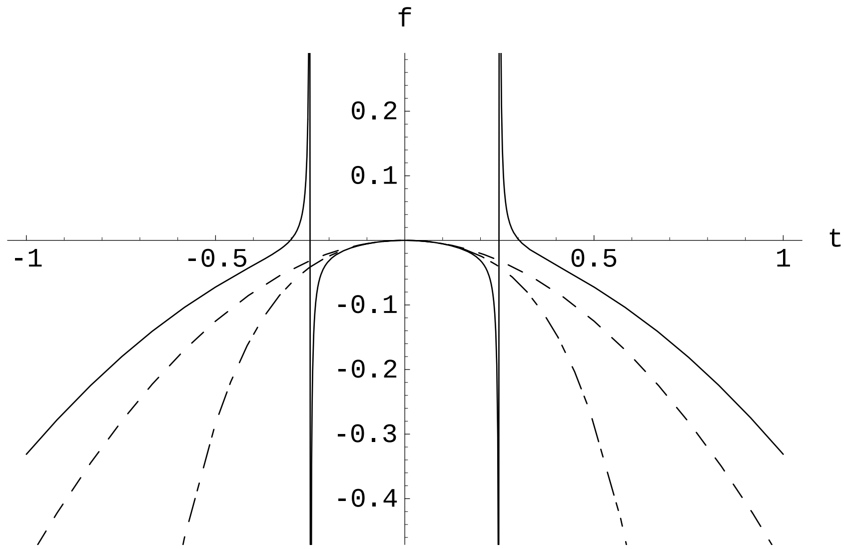


Figure 7.1: The tachyon potential $V(t)$ at level 0 (dotted line), level 1 (dashed line) and level 2 (full line).

point. Moreover, it doesn't have the interpretation as a point where different branches of the effective potential connect since, at level 2, the equations for the fields that are integrated out are still linear.

Integrating out the fields numerically for the higher levels, one finds that the more fields are included, the steeper the slope of the potential becomes. This behaviour was anticipated in the conclusions of [51]. In conclusion, the evidence presented here strongly suggests that Witten's cubic superstring field theory does not support Sen's conjecture.

7.2 The tachyon potential in modified cubic theories

7.2.1 Non-BPS D-branes in modified cubic theories

The construction of a gauge-invariant extension of the modified cubic theories to the GSO$-$ sector proceeds exactly as in the previous section. One assigns CP factors to the fields and operations defining the cubic action:

$$\begin{aligned}\hat{\Psi} &= \Psi_+ \otimes I + \Psi_- \otimes \sigma_1\\ \hat{\varepsilon} &= \varepsilon_+ \otimes \sigma_3 + \varepsilon_- \otimes i\sigma_2\,,\\ \hat{Q}_B &= Q_B \otimes \sigma_3\,,\\ \hat{\oint} &= \mathrm{Tr}\int\,,\\ \hat{\star} &= \star \otimes \sigma_3.\end{aligned} \tag{7.2.1}$$

The action takes the form

$$\begin{aligned}g^2 S[\hat{\Psi}] &= -\frac{1}{2}\hat{\oint}\hat{\Psi}\,\hat{\star}\,\hat{Q}_B\hat{\Psi} - \frac{1}{3}\hat{\oint}\hat{\Psi}\,\hat{\star}\,\hat{\Psi}\,\hat{\star}\,\hat{\Psi}\\ &= -\frac{1}{2}\langle\langle\,\hat{\Psi}\,\hat{Q}_B\hat{\Psi}\,\rangle\rangle - \frac{1}{3}\langle\langle\,\hat{\Psi}\hat{\Psi}\hat{\Psi}\,\rangle\rangle\,.\end{aligned}$$

where the double brackets should now be interpreted as

$$\begin{aligned}\langle\langle\,\hat{\Phi}_1\hat{\Phi}_2\cdots\hat{\Phi}_{n-1}\hat{\Phi}_n\,\rangle\rangle &= \frac{1}{2}\mathrm{Tr}\Big\langle \hat{Y}_{-2}(0) f_1^n\circ\hat{\Phi}_1(0)\sigma_3 f_2^n\circ\hat{\Phi}_2(0)\,\times\\ &\quad\times\,\sigma_3\cdots\sigma_3 f_{n-1}^n\circ\hat{\Phi}_{n-1}(0)\sigma_3 f_n^n\circ\hat{\Phi}_n(0)\Big\rangle\,.\end{aligned}$$

We have defined

$$\hat{Y}_{-2} = Y_{-2}\otimes I \tag{7.2.2}$$

The conformal transformations are defined as in (7.1.3) and the action is invariant under gauge transformations of the form (3.2.11). Again one can show that the action and gauge invariance is unique modulo a rescaling of the fields [117].

7.2.2 The tachyon potential

We now turn to the study of the tachyon potential in the modified cubic theories. We impose the Feynman-Siegel gauge $b_0\Psi = 0$ on states with nonzero conformal weight and make a consistent truncation to a subspace $\mathcal{H}_1$ formed by acting only with modes of the stress-energy tensor, the supercurrent and the ghost fields $b,\ c,\ \eta,\ \xi,\ \phi$.

We have seen in section 3.2.2 that there are two proposed modifications to Witten's cubic theory corresponding to different choices of the double-step picture changing operator Y_{-2}. The first choice (3.2.8)

$$Y_{-2} = \frac{1}{3}e^{-2\phi} + \frac{1}{15}\partial\xi c G^m e^{-3\phi}, \tag{7.2.3}$$

is not suited for the extension to the GSO− sector for the following reason. The pure tachyon in the 0-picture is given by the operator $T = te^{\phi}\eta$. Substituting this field into the action, we see that the kinetic term for the tachyon vanishes due to ϕ-charge conservation! Hence this theory is unable to reproduce the correct couplings of the tachyon field. The reason can be traced to the fact that the tachyon belongs to the kernel of the operator Y_{-2}. The fact that fields such as the tachyon have a vanishing kinetic term theory is precisely what makes the propagator in this theory ill-defined, the latter fact being the main source of criticism for this model [45].

Next, we consider the second choice for Y_{-2} (3.2.9)

$$Y_{-2} = Y(0)Y(\infty).$$

With this choice, the tachyon kinetic term has the correct form. The action so obtained possesses a $\mathbf{Z}_2$ twist symmetry when restricted to the fields in $\mathcal{H}_1$. Under this symmetry, the fields in the GSO+ sector carry charge $(-1)^{h+1}$ and the GSO− sector fields carry charge $(-1)^{h+\frac{1}{2}}$. In the calculation of the tachyon potential, we can consistently truncate the string field to be twist even. The proof of twist invariance of the action can be found in [117] and relies on the same properties (5.4.2) that were used to prove twist invariance in the bosonic theory. A crucial element in the proof is the fact that that the Y_{-2} insertion is separately invariant under the conformal transformation $\tilde{I}(z) = 1/z$. This hints at the reason why Witten's action does not possess this twist invariance (if it did, terms like the rt^2 term in 7.1.15 would be absent): there, the cubic term contains an $Z(0)$ insertion which is *not* invariant under $\tilde{I}(z)$. The same remark applies to the theory with the other choice (7.2.3) for Y_{-2} considered in the last paragraph and twist invariance is absent there as well.

We can now undertake the calculation of the tachyon potential in this theory. We define the level of a field to be $L_0 + 1$ so that the field of lowest weight has level zero. In table 7.2 we list the contributing fields up to level 5/2.

It is important to note that in this theory the lowest level field, denoted by U, is not the pure tachyon but an auxiliary field. The pure tachyon contribution T comes into play at level $\frac{1}{2}$. For completeness, we list the conformal transformations of these fields in A.2.

Level	GSO	state
0	+	$U = c$
1/2	-	$T = e^{\phi}\eta$
2	+	$V_1 = \partial^2 c,\ V_2 = cT^m,\ V_3 = cT_{\eta\xi}$ $V_4 = cT_\phi,\ V_5 = c\partial^2\phi,\ V_6 = e^{\phi}\eta G^m, V_7 = b\eta\partial\eta e^{2\phi}$
5/2	-	$W_1 = c\partial^2 c\partial\xi e^{-\phi},\ W_2 = bc\partial\eta e^{\phi},\ W_3 = c\partial\phi G^m$ $W_4 = bc\partial(\eta e^{\phi}),\ W_5 = c\partial G^m,\ W_6 = \eta\partial^2(e^{\phi}),\ W_7 = \partial\eta\partial(e^{\phi})$ $W_8 = \eta(\partial\phi)^2 e^{\phi},\ W_9 = \partial^2\eta e^{\phi},\ W_{10} = \eta e^{\phi}T^m,\ W_{11} = \partial bc\eta e^{\phi}$

Table 7.2: States contributing to the level 5/2 tachyon string field.

We have computed the tachyon potential in the level (5/2, 5) approximation. The level 5/2 string field is given by[4]:

$$\widehat{\mathcal{T}} = u\hat{U} + \frac{t}{2}\hat{T} + \sum_{i=1}^{7} v_i\hat{V}_i + \sum_{j=1}^{11} w_j\widehat{W}_j.$$

We denote the universal tachyon potential, whose conjectured value at its minimum is -1, by $f(\hat{\mathcal{T}})$. This is proportional to the action $S[\hat{\mathcal{T}}]$ [117]:

$$f(\widehat{\mathcal{T}}) = -\frac{\pi^2}{2}S[\widehat{\mathcal{T}}].$$

We denote the level (m,n) approximation to this function by $f_{(m,n)}$.

At level (1/2, 1) we have [117]

$$f_{(1/2,1)} = -\frac{\pi^2}{2}\Big(\frac{t^2}{4} + \frac{9\,t^2\,u}{16} + u^2\Big).$$

This result is encouraging: when solving for the u field in terms of t one finds an effective potential for t which has a local maximum at $t = 0$ and two global minima at $t = \pm 1.257$. The value at the global minima is already 97.5% of the conjectured exact value.

At level (2, 4), we have found the following result:

$$f_{(2,4)} = f_{(1/2,1)} - \frac{\pi^2}{2}\Big(\frac{9\,t^2\,v_1}{8} + 4\,u\,v_1 + 4\,{v_1}^2 - \frac{25\,t^2\,v_2}{32} + \frac{15\,{v_2}^2}{2} - \frac{9\,t^2\,v_3}{16}$$

[4]It can be shown that the reality condition on the string field restricts the coefficients in this expansion to be real.

$$
\begin{aligned}
&-2\,u\,v_3 + 8\,v_1\,v_3 + {v_3}^2 - \frac{59\,t^2\,v_4}{32} - 8\,u\,v_4 - 32\,v_1\,v_4 - 16\,v_3\,v_4 \\
&+\frac{77\,{v_4}^2}{2} + \frac{43\,t^2\,v_5}{24} + 8\,u\,v_5 + 24\,v_1\,v_5 + 4\,v_3\,v_5 - 52\,v_4\,v_5 \\
&+22\,{v_5}^2 + 30\,v_2\,v_6 - 30\,v_4\,v_6 + 20\,v_5\,v_6 + 10\,{v_6}^2 + \frac{160\,u\,{v_6}^2}{9\,\sqrt{3}} \\
&+2\,u\,v_7 + 2\,\sqrt{3}\,u^2\,v_7 + 8\,v_1\,v_7 + \frac{280\,u\,v_1\,v_7}{9\,\sqrt{3}} - 15\,v_2\,v_7 \\
&-\frac{50\,u\,v_2\,v_7}{3\,\sqrt{3}} + 4\,v_3\,v_7 + \frac{20\,u\,v_3\,v_7}{9\,\sqrt{3}} - 5\,v_4\,v_7 - \frac{386\,u\,v_4\,v_7}{9\,\sqrt{3}} + 4\,v_5\,v_7 \\
&+\frac{344\,u\,v_5\,v_7}{9\,\sqrt{3}}\Big).
\end{aligned}
$$

The level $(2,4)$ approximation to the tachyon potential was also studied in [117]. However, we disagree with that study on several points.

First of all, our field V_7 seems to be overlooked in that paper[5].

Also, it was proposed in [117] that one should discard the fields v_2 and v_6 in order to obtain a potential with more pleasing properties. We strongly object to this procedure since putting v_2 and v_6 to zero is not consistent with the equations of motion, nor is their contribution to the potential at this level negligible.

We now discuss the structure of the level $(2,4)$ effective tachyon potential keeping all the fields. Already at this level, the potential has a complicated branch structure. Making use of a numerical algorithm and using the values of the fields at the minimum of the level $(1/2,1)$ potential as starting values, we find a branch of the potential which has the expected behaviour (see figure 7.2). The two minima now occur at $t \approx \pm 1.219$ and the value of the potential is $f_{2,4} \approx -1.083$, which exceeds the conjectured exact value by about 8%. In analysing the other branches of the potential using numerical methods, we have not found a branch with a better behaviour. Hence the level $(2,4)$ potential does not yield the expected convergence towards the conjectured value. However, before drawing any definitive conclusion, we deemed it important to study the tachyon potential including the states at the next level as well.

Including the 11 states at level $\frac{5}{2}$, we obtained the following result for the

[5] We are indebted to Pieter-Jan De Smet for pointing this out to us.

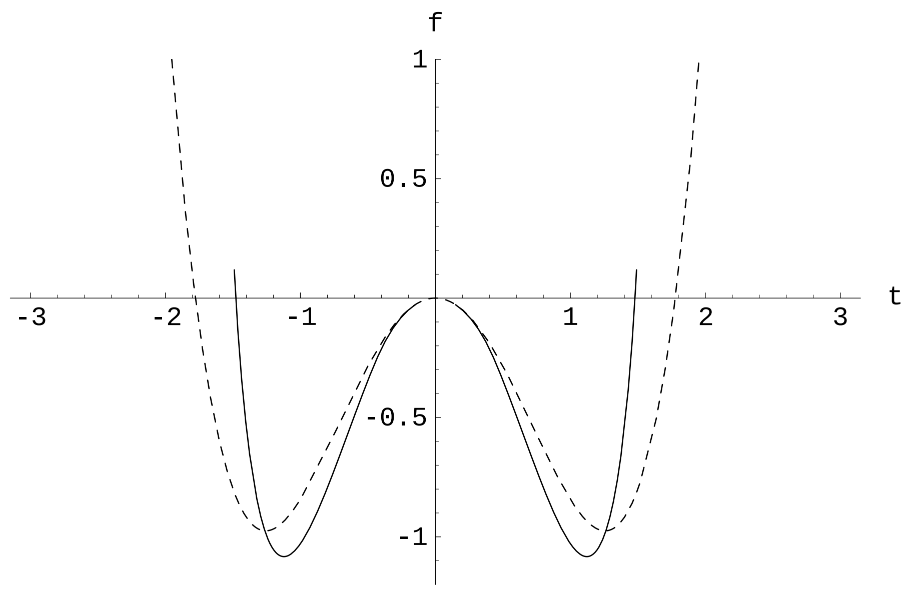


Figure 7.2: The level $(2,4)$ tachyon potential (full line) as compared to the level $(1/2,1)$ potential (dotted line). The $(2,4)$ branch ends at $t \approx \pm 1.49$.

$(5/2,5)$ potential:

$$
\begin{aligned}
f_{(5/2,5)} = f_{(2,4)} - \frac{\pi^2}{2}\Big(& \frac{256\,t\,v_7\,w_1}{81} + \frac{25\,t\,v_1\,w_{10}}{4} - \frac{4435\,t\,v_2\,w_{10}}{432} - \frac{25\,t\,v_3\,w_{10}}{8} \\
& - \frac{1475\,t\,v_4\,w_{10}}{144} + \frac{1075\,t\,v_5\,w_{10}}{108} + \frac{15\,{w_{10}}^2}{2} + \frac{4435\,u\,{w_{10}}^2}{432} + \frac{169\,t\,v_1\,w_{11}}{54} \\
& - \frac{275\,t\,v_2\,w_{11}}{72} - \frac{11\,t\,v_3\,w_{11}}{4} - \frac{649\,t\,v_4\,w_{11}}{72} + \frac{473\,t\,v_5\,w_{11}}{54} + 12\,w_1\,w_{11} \\
& + \frac{275\,u\,w_{10}\,w_{11}}{36} + {w_{11}}^2 + \frac{121\,u\,{w_{11}}^2}{36} + \frac{172\,t\,v_1\,w_2}{81} - \frac{25\,t\,v_2\,w_2}{27} \\
& + \frac{74\,t\,v_3\,w_2}{81} - \frac{59\,t\,v_4\,w_2}{27} + \frac{172\,t\,v_5\,w_2}{81} + 15\,w_{10}\,w_2 + \frac{50\,u\,w_{10}\,w_2}{27} \\
& + 7\,w_{11}\,w_2 + \frac{44\,u\,w_{11}\,w_2}{27} + \frac{16\,u\,{w_2}^2}{9} - 30\,w_{10}\,w_3 + 50\,{w_3}^2 \\
& + \frac{86\,t\,v_1\,w_4}{27} - \frac{25\,t\,v_2\,w_4}{18} + \frac{47\,t\,v_3\,w_4}{81} - \frac{467\,t\,v_4\,w_4}{162} + \frac{226\,t\,v_5\,w_4}{81} \\
& + 15\,w_{10}\,w_4 + \frac{25\,u\,w_{10}\,w_4}{9} + 7\,w_{11}\,w_4 + \frac{22\,u\,w_{11}\,w_4}{9} + \frac{80\,u\,w_2\,w_4}{27} \\
& + \frac{4\,u\,{w_4}^2}{9} + \frac{320\,t\,v_6\,w_5}{27} - 15\,w_{10}\,w_5 + 30\,{w_5}^2 - \frac{101\,t\,v_1\,w_6}{6} \\
& + \frac{2525\,t\,v_2\,w_6}{216} + \frac{101\,t\,v_3\,w_6}{12} + \frac{719\,t\,v_4\,w_6}{24} - \frac{511\,t\,v_5\,w_6}{18} - 72\,w_1\,w_6 \\
& - \frac{2525\,u\,w_{10}\,w_6}{108} - 8\,w_{11}\,w_6 - \frac{1111\,u\,w_{11}\,w_6}{54} - 6\,w_2\,w_6 - \frac{404\,u\,w_2\,w_6}{81} \\
& - 80\,w_3\,w_6 - 6\,w_4\,w_6 - \frac{734\,u\,w_4\,w_6}{81} - 120\,w_5\,w_6 + 20\,{w_6}^2 \\
& + \frac{1361\,u\,{w_6}^2}{36} - \frac{4\,t\,v_1\,w_7}{3} + \frac{25\,t\,v_2\,w_7}{27} - \frac{74\,t\,v_3\,w_7}{81} + \frac{113\,t\,v_4\,w_7}{81} \\
& - \frac{4\,t\,v_5\,w_7}{3} + 48\,w_1\,w_7 - \frac{50\,u\,w_{10}\,w_7}{27} - 8\,w_{11}\,w_7 - \frac{44\,u\,w_{11}\,w_7}{27} \\
& + \frac{32\,u\,w_2\,w_7}{27} + 40\,w_3\,w_7 + 3\,w_4\,w_7 + \frac{16\,u\,w_4\,w_7}{81} + 60\,w_5\,w_7 \\
& + 16\,w_6\,w_7 + \frac{220\,u\,w_6\,w_7}{27} - 4\,{w_7}^2 - \frac{80\,u\,{w_7}^2}{27} - \frac{3\,t\,v_1\,w_8}{2} \\
& + \frac{25\,t\,v_2\,w_8}{24} + \frac{3\,t\,v_3\,w_8}{4} + \frac{59\,t\,v_4\,w_8}{24} - \frac{259\,t\,v_5\,w_8}{162} - 48\,w_1\,w_8 \\
& - \frac{25\,u\,w_{10}\,w_8}{12} - \frac{11\,u\,w_{11}\,w_8}{6} - 2\,w_2\,w_8 - \frac{4\,u\,w_2\,w_8}{9} - 60\,w_3\,w_8 \\
& - 2\,w_4\,w_8 - \frac{118\,u\,w_4\,w_8}{81} - 60\,w_5\,w_8 + 4\,w_6\,w_8 + \frac{1933\,u\,w_6\,w_8}{162}
\end{aligned}
$$

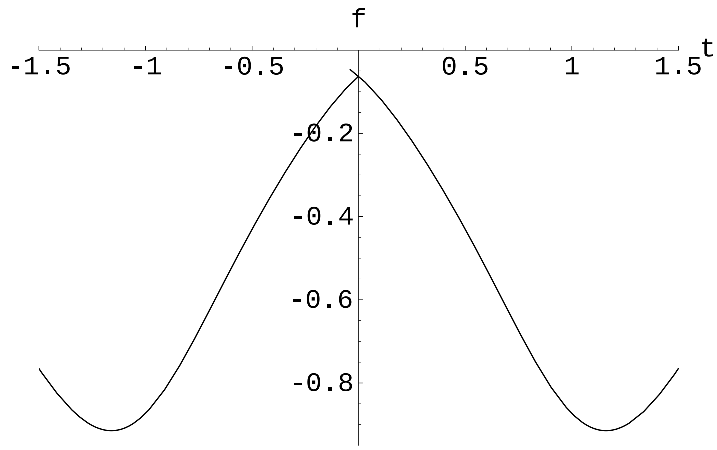


Figure 7.3: The branch of the $(5/2, 5)$ tachyon potential discussed in the text. Here, the two minima belong to different branches of the potential, who come to an end close to $t = 0$. These branches are not connected to the unstable vacuum, contrary to the expectations from Sen's conjecture.

$$
\begin{aligned}
&+4\,w_7\,w_8 + \frac{164\,u\,w_7\,w_8}{81} + 2\,w_8{}^2 + \frac{283\,u\,w_8{}^2}{108} + 17\,t\,v_1\,w_9 \\
&-\frac{425\,t\,v_2\,w_9}{36} - \frac{203\,t\,v_3\,w_9}{54} - \frac{1003\,t\,v_4\,w_9}{36} + \frac{731\,t\,v_5\,w_9}{27} \\
&-24\,w_1\,w_9 + \frac{425\,u\,w_{10}\,w_9}{18} + 24\,w_{11}\,w_9 + \frac{187\,u\,w_{11}\,w_9}{9} - 6\,w_2\,w_9 \\
&+\frac{136\,u\,w_2\,w_9}{27} - 6\,w_4\,w_9 + \frac{68\,u\,w_4\,w_9}{9} - 48\,w_6\,w_9 - \frac{1717\,u\,w_6\,w_9}{27} \\
&-12\,w_7\,w_9 - \frac{136\,u\,w_7\,w_9}{27} - \frac{17\,u\,w_8\,w_9}{3} + 36\,w_9{}^2 + \frac{289\,u\,w_9{}^2}{9}\Big).
\end{aligned}
$$

We have found that the behaviour of the potential at this level conflicts with the expectations for tachyon condensation. Taking again the minima from the previous level as starting values, we again find two minima of the effective potential, at $t \approx \pm 1.160$, where the potential takes the value ≈ -0.915. First of all, we notice that the value at the minimum has increased with respect to the value at the previous level. However, closer scrutiny reveals that these minima in fact lay on different branches, crossing each other at $t = 0$ where they both take the value ≈ -0.063 (see figure 7.3). Due to the fact that these two minima lay on branches that are not connected to the unstable vacuum, the potential at this level no longer satisfies the basic requirements for tachyon condensation. Furthermore, in analysing the other branches of the potential found by numerical methods, we have not found any branch which does display the expected behaviour. Therefore, we tentatively conclude that the behaviour of the tachyon potential in modified cubic string field theory conflicts with the predictions from Sen's conjecture.

7.3 Tachyon condensation in Berkovits' theory

We now present our results concerning tachyon condensation in Berkovits' superstring field theory. The results in this section were obtained in our paper [118].

7.3.1 Non-BPS D-branes in Berkovits' theory

We first extend Berkovits' action to the GSO− sector as required for the description of a non-BPS D-brane. This is done once more by tensoring the string fields with the appropriate CP factors:

$$\widehat{\Psi} = \Psi_+ \otimes I + \Psi_- \otimes \sigma_1$$

One further associates CP factors with the BRST charge and the η_0 field:

$$\widehat{Q}_B = Q_B \otimes \sigma_3$$
$$\widehat{\eta}_0 = \eta_0 \otimes \sigma_3.$$

In contrast to the previously discussed theories, we don't need to extend the $\star$ operation to include multiplication by a Pauli-matrix in order to preserve gauge invariance. The string field theory action for the non-BPS D-brane takes the following form:

$$\begin{aligned} S[\widehat{\Psi}] &= \frac{1}{4g^2}\Big\langle\Big\langle (e^{-\widehat{\Psi}}\widehat{Q}_B e^{\widehat{\Psi}})(e^{-\widehat{\Psi}}\widehat{\eta}_0 e^{\widehat{\Psi}}) \\ &- \int_0^1 dt\,(e^{-t\widehat{\Psi}}\partial_t e^{t\widehat{\Psi}})\{(e^{-t\widehat{\Psi}}\widehat{Q}_B e^{t\widehat{\Psi}}),(e^{-t\widehat{\Psi}}\widehat{\eta}_0 e^{t\widehat{\Psi}})\}\Big\rangle\Big\rangle. \end{aligned} \quad (7.3.1)$$

Here the double brackets mean the following:

$$\langle\langle\, \widehat{\Psi}_1 \cdots \widehat{\Psi}_n \,\rangle\rangle = \mathrm{Tr}\Big\langle f_1^{(n)} \circ \widehat{\mathcal{V}}_{\Psi_1}(0) \cdots f_n^{(n)} \circ \widehat{\mathcal{V}}_{\Psi_n}(0)\Big\rangle\,.$$

The action (7.3.1) is invariant under the gauge transformations

$$\delta e^{\widehat{\Psi}} = \left(\widehat{Q}_B\widehat{\Omega}\right) e^{\widehat{\Psi}} + e^{\widehat{\Psi}}\left(\widehat{\eta}_0\widehat{\Omega}'\right),$$

where $\widehat{\Omega}$ and $\widehat{\Omega}'$ are independent gauge parameters. The proof is similar to the one in the GSO+ sector and relies on the properties (compare 3.2.16)

$$\{\widehat{Q}_B, \widehat{\eta}_0\} = 0, \qquad \widehat{Q}_B^2 = \widehat{\eta}_0^2 = 0 \tag{7.3.2}$$

$$\begin{aligned}
\langle\langle\, \widehat{Q}_B(...)\,\rangle\rangle &= \langle\langle\, \widehat{\eta}_0(...)\,\rangle\rangle = 0 \\
\langle\langle\, \dots \widehat{Q}_B(\widehat{\Psi}_1\widehat{\Psi}_2)\dots\,\rangle\rangle &= \langle\langle\, \dots (\widehat{Q}_B\widehat{\Psi}_1)\widehat{\Psi}_2 + \widehat{\Psi}_1(\widehat{Q}_B\widehat{\Psi}_2)\dots\,\rangle\rangle \\
\langle\langle\, \dots \widehat{\eta}_0(\widehat{\Psi}_1\widehat{\Psi}_2)\dots\,\rangle\rangle &= \langle\langle\, \dots (\widehat{\eta}_0\widehat{\Psi}_1)\widehat{\Psi}_2 + \widehat{\Psi}_1(\widehat{\eta}_0\widehat{\Psi}_2)\dots\,\rangle\rangle \\
\langle\langle\, \widehat{\Phi}_1\dots\widehat{\Phi}_{n-1}\widehat{\Psi}\,\rangle\rangle &= \langle\langle\, \widehat{\Psi}\widehat{\Phi}_1\dots\widehat{\Phi}_{n-1}\,\rangle\rangle \\
\langle\langle\, \widehat{\Phi}_1\dots\widehat{\Phi}_{n-1}\widehat{Q}_B\widehat{\Psi}\,\rangle\rangle &= -\langle\langle\, \widehat{Q}_B\widehat{\Psi}\widehat{\Phi}_1\dots\widehat{\Phi}_{n-1}\,\rangle\rangle \\
\langle\langle\, \widehat{\Phi}_1\dots\widehat{\Phi}_{n-1}\widehat{\eta}_0\widehat{\Psi}\,\rangle\rangle &= -\langle\langle\, \widehat{\eta}_0\widehat{\Psi}\widehat{\Phi}_1\dots\widehat{\Phi}_{n-1}\,\rangle\rangle
\end{aligned} \tag{7.3.3}$$

The gauge invariance can be fixed[6] by imposing

$$b_0|\widehat{\Psi}>= 0 \quad \text{and} \quad \xi_0|\widehat{\Psi}>= 0\,. \tag{7.3.4}$$

In the calculation of the tachyon potential, we can restrict the string field to lie in a subspace $\mathcal{H}_1$ formed by acting only with modes of the stress-energy tensor, the supercurrent and the ghost fields b, c, η, ξ, ϕ, since the other excitations can be consistently put to zero. Furthermore, when restricted to fields lying in $\mathcal{H}_1$ the action has a $\mathbf{Z}_2$ twist invariance under which the fields in the GSO+ sector carry charge $(-)^{h+1}$ and the fields in the GSO− sector carry charge $(-)^{h+1/2}$ (h is the conformal weight). The proof of this property, which can be found in appendix B of [51], relies on the properties (5.4.2) of the conformal transformations $f_k^{(n)}$. In the computation of the tachyon potential we can therefore further restrict ourselves to the twist even fields.[7]

The non-polynomial action (7.3.1) should be formally expanded in the string field $\widehat{\Psi}$, and each term should be accompanied by the appropriate conformal transformations. However, because we will only compute the interactions between a finite number of fields, it is easy to see that one does not need all the terms in the action. The conformal field theory correlators in the action (7.3.1) are nonvanishing only if the total $(b,\ c)$ number is 3, the $(\eta,\ \xi)$

[6] This is a reachable gauge choice for states with $L_0 \neq 0$ but we have not been able to prove that it fixes the gauge freedom completely.

[7] This restriction projects out the only state with $L_0 = 0$, namely $\xi\partial\xi\ c\partial c\ e^{-2\phi}$.

number is 1 and the total ϕ-charge is -2. In the following we will need only the terms in the action involving up to 6 string fields.

Making use of the properties (7.3.3)), the action to this order can be written as [51]

$$
\begin{aligned}
S = \frac{1}{2g^2} \langle\langle\, & \frac{1}{2}(\hat{Q}_B\hat{\Psi})(\hat{\eta}_0\hat{\Psi}) + \frac{1}{3}(\hat{Q}_B\hat{\Psi})\hat{\Psi}(\hat{\eta}_0\hat{\Psi}) \\
& + \frac{1}{12}(\hat{Q}_B\hat{\Psi})\left(\hat{\Psi}^2(\hat{\eta}_0\hat{\Psi}) - \hat{\Psi}(\hat{\eta}_0\hat{\Psi})\hat{\Psi}\right) + \frac{1}{60}(\hat{Q}_B\hat{\Psi})\left(\hat{\Psi}^3(\hat{\eta}_0\hat{\Psi}) - 3\hat{\Psi}^2(\hat{\eta}_0\hat{\Psi})\hat{\Psi}\right) + \\
& + \frac{1}{360}(\hat{Q}_B\hat{\Psi})\left(\hat{\Psi}^4(\hat{\eta}_0\hat{\Psi}) - 4\hat{\Psi}^3(\hat{\eta}_0\hat{\Psi})\hat{\Psi} + 3\hat{\Psi}^2(\hat{\eta}_0\hat{\Psi})\hat{\Psi}^2\right) \rangle\rangle \,.
\end{aligned}
$$

7.3.2 The fields up to level 2

Taking all this together we get the list of contributing fields up to level 2 (table 7.3). The level of a field is here defined to be the conformal weight shifted by 1/2. In this way the tachyon is a level 0 field. We use the notation $|q>$ for the state corresponding with the operator $:e^{q\phi}:$. The level 0 and level 2 fields should be tensored with σ_1 and the level 3/2 fields with I. We list the conformal transformations of these fields in A.3.

Level	state	vertex operator
0	$\xi_0 c_1 \vert -1>$	$T = \xi c e^{-\phi}$
3/2	$2c_1 c_{-1}\xi_0\xi_{-1}\vert -2>$	$A = c\partial^2 c\xi\partial\xi\ e^{-2\phi}$
	$\xi_0\eta_{-1}\vert 0>$	$E = \xi\eta$
	$\xi_0 c_1 G^m_{-3/2}\vert -1>$	$F = \xi c G^m\ e^{-\phi}$
2	$\xi_0 c_1 \left[(\phi_{-1})^2 - \phi_{-2}\right]\vert -1>$	$K = \xi c\ \partial^2\left(e^{-\phi}\right)$
	$\xi_0 c_1 \phi_{-2}\vert -1>$	$L = \xi c\ \partial^2\phi\ e^{-\phi}$
	$\xi_0 c_1 L^m_{-2}\vert -1>$	$M = \xi c T^m\ e^{-\phi}$
	$2\xi_0 c_{-1}\vert -1>$	$N = \xi\partial^2 c\ e^{-\phi}$
	$\xi_0\xi_{-1}\eta_{-1}c_1\vert -1>$	$P = \xi\partial\xi\eta c\ e^{-\phi}$

Table 7.3: The contributing states up to level two and the corresponding vertex operators.

The resulting level 2 tachyon string field is then given by

$$
\hat{\mathcal{T}} = t\hat{T} + a\hat{A} + e\hat{E} + f\hat{F} + k\hat{K} + l\hat{L} + m\hat{M} + n\hat{N} + p\hat{P}. \tag{7.3.5}
$$

7.3.3 The tachyon potential

In [118], we have calculated the tachyon potential in the level $(2,4)$ approximation using the Mathematica program that was also used for the calculations in sections 7.1 and 7.2. An extra check on the calculations was performed by calculating some of the correlators on the upper half plane instead of the disc. As before, the universal tachyon potential $f(\widehat{\mathcal{T}})$ is equal to the action $S[\widehat{\mathcal{T}}]$ divided by the brane tension. The latter was calculated in [51] and is equal to $1/2\pi^2 g^2$, so we have

$$f(\widehat{\mathcal{T}}) = -2\pi^2 S[\widehat{\mathcal{T}}].$$

We now give the result for the tachyon potential with coefficients evaluated numerically up to 6 significant digits:

$$\begin{aligned}
f_{(0,0)} &= 2\pi^2\Big(- 0.25t^2 + 0.5t^4\Big) \\
f_{(3/2,3)} &= f_{(0,0)} + 2\pi^2\Big(- at^2 - 0.25et^2 - 0.518729et^4 \\
&\quad 2ae + 5f^2 + 4.96405aet^2 - 0.66544e^2t^2 \\
&\quad +5.47589eft^2 + 5.82107f^2t^2 + 0.277778e^2t^4\Big) \\
f_{(2,4)} &= f_{(3/2,2)} + 2\pi^2\Big(- 3.03704akt - 7.11111alt + 2.77778amt \\
&\quad -1.62963ant - 1.55556apt + 0.12963ekt - 0.296296elt \\
&\quad +0.694444emt - 1.2963ent + 0.944444ept - 11.8519flt \\
&\quad -8.88889fmt - 2.96296fpt - 2.87299ekt^3 - 1.94348elt^3 \\
&\quad +4.35732emt^3 - -4.77364ent^3 + 0.605194ept^3 \\
&\quad +3k^2 - 3kl + 1.5l^2 + 5.625m^2 - 3n^2 - 0.75p^2 + 10.3958k^2t^2 + \\
&\quad +0.791667klt^2 - 1.875kmt^2 + 5.54167knt^2 - 1.4375kpt^2 + \\
&\quad +6.70833l^2t^2 - 10.3125lmt^2 + 11.9167lnt^2 - 0.875lpt^2 + \\
&\quad +14.7656m^2t^2 - 15.9375mnt^2 - 1.40625mpt^2 + \\
&\quad +5.83333n^2t^2 - 0.5npt^2 - 1.5p^2t^2\Big).
\end{aligned}$$

The level $(0,0)$ and $(3/2,3)$ contributions to the potential were already calculated in [116, 51] and our results agree with the ones displayed there[8].

At the lowest level, the potential $f_{(0,0)}$ has two minima at $t_0 = \pm 0.5$, where the potential takes the value $f_{(0,0)} = -0.61685$. Hence this level already accounts for about 62% of the conjectured exact value $f = -1$.

[8]Note that our field F is defined with a different sign from the one in [51].

At the next level, the two extrema of the potential $f_{(3/2,3)}$ occur at

$$t_0 = \pm 0.588823, \ a_0 = 0.056363, \ e_0 = 0.093175, \ f_0 = 0.012603$$

where the potential takes the value $f_{(3/2,3)} = -0.854458$ and accounts for 85% of the conjectured exact value.

In the level $(2,4)$ approximation, the potential has extrema at

$$\begin{array}{lllllllll} t_0 & = & \pm 0.602101 & & & & & & \\ a_0 & = & 0.052193\,, & e_0 & = & 0.043037\,, & f_0 & = & -0.013816\,, \\ k_0 & = & \mp 0.010190\,, & l_0 & = & \mp 0.045043\,, & m_0 & = & \pm 0.032213\,, \\ n_0 & = & \pm 0.047311\,, & p_0 & = & \pm 0.021291\,. & & & \end{array}$$

At these extrema

$$f_{(2,4)} = -0.891287$$

so we see that in the level $(2,4)$ approximation we obtain 89% of the conjectured exact value.

One can also obtain the effective potential $f(t)$ for the tachyon by solving for the other field in terms of t. To this order at least, this does not lead to different branches of the potential since the fields we integrate out appear at most quadratically in the potential. The result in the different approximations is shown in figure 7.4. The extrema of the full potential correspond to minima of the effective potential.

In conclusion, the results presented in this section show that the version of superstring field theory proposed by Berkovits yields sensible results for the tachyon potential in good agreement with the behaviour predicted by Sen's conjecture. It would be nice, however, to establish agreement beyond a shadow of a doubt, as has been done for the bosonic string, by including more fields. The next contribution to the potential comes at level 7/2, which contains 23 fields. To perform calculations involving such a large number of fields, a new calculational approach is probably required, perhaps some extension of the Neumann coefficient method which has proven fruitful in the bosonic analysis.

7.4 Conclusions

In this chapter, we have analysed the tachyon potential in the three different versions of superstring field theory proposed in the literature. The analysis

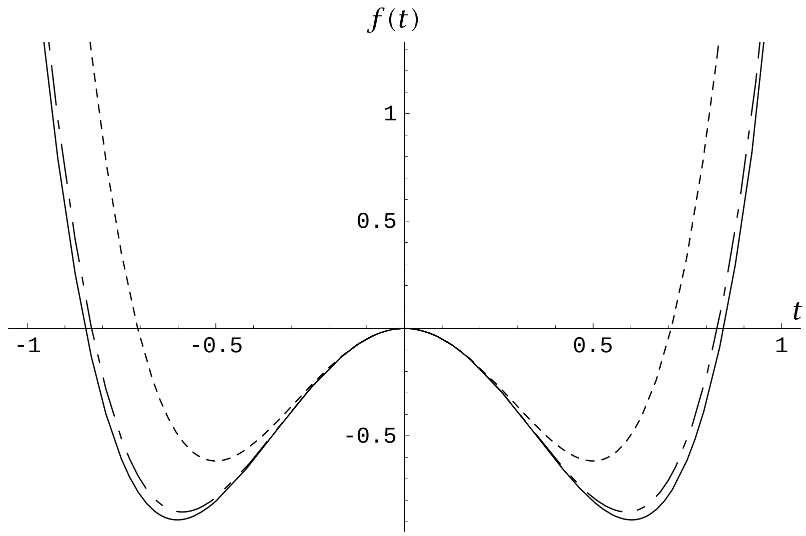


Figure 7.4: The tachyon potential $f(t)$ in the level $(0,0)$ (dotted line), level $(3/2,3)$ (dashed line) and level $(2,4)$ (full line) approximations.

was done in the level truncation scheme. As mentioned in the introduction to this chapter, evidence for the validity of Sen's conjecture for the superstring has been provided by arguments using a well-established duality symmetry (T-duality), and it is a more or less common belief that a sensible string field theory should produce results in agreement with the conjecture.

Since there exist three inequivalent proposals for open string field theory, we see that Sen's conjecture provides an opportunity to put these different theories to the test in a reasonably straightforward classical computation. Our analysis showed that the behaviour of the tachyon potential in Witten's theory and in the modified cubic string field theories conflicts with the expectations from Sen's conjecture. On the other hand, our results for the tachyon potential in Berkovits' theory showed good agreement with Sen's conjecture.

In retrospect, the fact that we found agreement with the predictions of Sen's conjecture only in Berkovits' formulation is perhaps not so surprising. Indeed, the two other theories had received some criticism in the literature. In the case of Witten's proposal, it has been known for some time that the theory suffers from contact term divergences which spoil the required gauge invariance [42]. The modified cubic theories, which don't suffer from these divergences, have been criticised for a different reason [45]: the presence of the double-step picture changing operator leads to a kinetic operator that is not invertible even after gauge-fixing, so that one runs into trouble when trying to extract an off-shell propagator for the theory (recall that, in the other string field theories, it was possible to fix a gauge in which the kinetic operator reduces to L_0, the natural propagator in conformal field theories encountered in section 2.3.3). So far, Berkovits' theory is the only candidate which is gauge-invariant and leads to a sensible off-shell propagator.

Appendix A

Conformal transformations of the fields

A.1 Witten's action

Here we list the conformal transformations of the fields necessary for the computation of the level $(2,4)$ approximation to the tachyon potential in section in Witten's theory. To shorten the notation we denote $w = f(z)$.

$$
\begin{aligned}
f \circ T(z) &= (f'(z))^{-1/2} T(w)\,, \\
f \circ R(z) &= R(w)\,, \\
f \circ S(z) &= (f'(z))^{1/2} S(w) - \frac{1}{2}\frac{f''(z)}{f'(z)}(f'(z))^{-1/2} c e^{-\phi}(w)\,, \\
f \circ A(z) &= f'(z) A(w) - \frac{f''(z)}{f'(z)} c\partial c\, \partial\xi\, e^{-2\phi}(w)\,, \\
f \circ E(z) &= f'(z) E(w)\,, \\
f \circ F(z) &= f'(z) F(w)\,, \\
f \circ K(z) &= (f'(z))^{3/2} K(w) + 2\frac{f''(z)}{f'(z)}(f'(z))^{1/2} c\partial\left(e^{-\phi}\right)(w) + \\
&\quad + \left[\frac{1}{2}\frac{f'''}{f'} - \frac{1}{4}\left(\frac{f''}{f'}\right)^2\right](f'(z))^{-1/2} c e^{-\phi}(w)\,, \\
f \circ L(z) &= (f'(z))^{3/2} L(w) + \frac{f''(z)}{f'(z)}(f'(z))^{1/2} c\partial\phi\, e^{-\phi}(w) +
\end{aligned}
$$

$$
\begin{aligned}
& + \left[\frac{3}{4} \left(\frac{f''}{f'} \right)^2 - \frac{2}{3} \frac{f'''}{f'} \right] (f'(z))^{-1/2} c e^{-\phi}(w) \,, \\
f \circ M(z) &= (f'(z))^{3/2} M(w) + \frac{15}{12} \left[\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-1/2} c e^{-\phi}(w) \,, \\
f \circ N(z) &= (f'(z))^{3/2} N(w) - \frac{f''(z)}{f'(z)} (f'(z))^{1/2} \partial c \, e^{-\phi}(w) + \\
& + \left[2 \left(\frac{f''}{f'} \right)^2 - \frac{f'''}{f'} \right] (f'(z))^{-1/2} c \, e^{-\phi}(w) \,, \\
f \circ P(z) &= (f'(z))^{3/2} P(w) + \\
& + \left[\frac{1}{4} \left(\frac{f''}{f'} \right)^2 - \frac{1}{6} \frac{f'''}{f'} \right] (f'(z))^{-1/2} c e^{-\phi}(w) \,.
\end{aligned}
$$

A.2 Modified cubic string field theory

Here we list the conformal transformations of the fields necessary for the computation of the level $(5/2, 5)$ approximation to the tachyon potential in modified cubic string field theory ($w \equiv f(z)$).

$$
\begin{aligned}
f \circ U(z) &= (f'(z))^{-1} U(w) \\
f \circ T(z) &= (f'(z))^{-1/2} T(w) \\
f \circ V_1(z) &= f'(z) V_1(w) - \frac{f''(z)}{f'(z)} \partial c(w) \\
& - \left[\frac{f'''}{f'} - 2 \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-1} c(w) \\
f \circ V_2(z) &= f'(z) V_2(w) + \frac{15}{12} \left[\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-1} c(w) \\
f \circ V_3(z) &= f'(z) V_3(w) - \frac{1}{6} \left[\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-1} c(w) \\
f \circ V_4(z) &= f'(z) V_4(w) + \frac{13}{12} \left[\frac{f'''}{f'} - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-1} c(w) \\
f \circ V_5(z) &= f'(z) V_5(w) + \frac{f''(z)}{f'(z)} c \partial \phi(w)
\end{aligned}
$$

$$
\begin{aligned}
& - \left[\frac{f'''}{f'} - \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-1} c(w) \\
f \circ V_6(z) &= f'(z) V_6(w) \\
f \circ V_7(z) &= f'(z) V_7(w) \\
f \circ W_1(z) &= (f'(z))^{3/2} W_1(w) - \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} c\partial c \partial \xi e^{-\phi} \\
f \circ W_2(z) &= (f'(z))^{3/2} W_2(w) + \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} \left(bc\eta e^{\phi}(w) + \frac{3}{2} \partial \eta e^{\phi}(w) \right) \\
& + \frac{3}{2} \left(\frac{f''}{f'} \right)^2 (f'(z))^{-\frac{1}{2}} \eta e^{\phi}(w) \\
f \circ W_3(z) &= (f'(z))^{3/2} W_3(w) - \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} c G^m(w) \\
f \circ W_4(z) &= (f'(z))^{3/2} W_4(w) - \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} \Big(\frac{1}{2} bc\eta e^{\phi}(w) \\
& - \frac{3}{2} \partial(\eta e^{\phi})(w) \Big) - \frac{3}{4} \left(\frac{f''}{f'} \right)^2 (f'(z))^{-\frac{1}{2}} \eta e^{\phi}(w) \\
f \circ W_5(z) &= (f'(z))^{3/2} W_5(w) + \frac{3}{2} \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} c G^m(w) \\
f \circ W_6(z) &= (f'(z))^{3/2} W_6(w) - 2 \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} \eta \partial(e^{\phi})(w) \\
& - \left[\frac{3}{2} \frac{f'''}{f'} - \frac{15}{4} \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-\frac{1}{2}} \eta e^{\phi}(w) \\
f \circ W_7(z) &= (f'(z))^{3/2} W_7(w) - \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} \left(\frac{3}{2} \partial \eta e^{\phi}(w) - \eta \partial(e^{\phi})(w) \right) \\
& - \frac{3}{2} \left(\frac{f''}{f'} \right)^2 (f'(z))^{-\frac{1}{2}} \eta e^{\phi}(w) \\
f \circ W_8(z) &= (f'(z))^{3/2} W_8(w) - 3 \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} \eta \partial(e\phi)(w) \\
& - \left[\frac{1}{6} \frac{f'''}{f'} - \frac{5}{2} \left(\frac{f''}{f'} \right)^2 \right] (f'(z))^{-\frac{1}{2}} \eta e^{f}(w) \\
f \circ W_9(z) &= (f'(z))^{3/2} W_9(w) + 3 \frac{f''(z)}{f'(z)} (f'(z))^{\frac{1}{2}} \partial \eta e^{\phi}(w) \\
& + \frac{f'''}{f'} (f'(z))^{-\frac{1}{2}} \eta e^{\phi}(w)
\end{aligned}
$$

$$
\begin{aligned}
f \circ W_{10}(z) &= (f'(z))^{3/2} W_{10}(w) + \frac{15}{12}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right](f'(z))^{-\frac{1}{2}} \eta e^{f}(w) \\
f \circ W_{11}(z) &= (f'(z))^{3/2} W_{11}(w) + 2\frac{f''(z)}{f'(z)}(f'(z))^{\frac{1}{2}} bc\eta e^{\phi} \\
&\quad + \left[\frac{5}{6}\frac{f'''}{f'} + \frac{1}{4}\left(\frac{f''}{f'}\right)^2\right](f'(z))^{-\frac{1}{2}} \eta e^{\phi}(w)
\end{aligned}
$$

A.3 Berkovits' theory

Here we list the conformal transformations of the fields necessary for the computation of the level $(2,4)$ approximation to the tachyon potential in Berkovits' theory ($w \equiv f(z)$).

$$
\begin{aligned}
f \circ T(z) &= (f'(z))^{-1/2} T(w) \\
f \circ A(z) &= f'(z) A(w) - \frac{f''(z)}{f'(z)} c\partial c\ \xi\partial\xi\ e^{-2\phi}(w) \\
f \circ E(z) &= f'(z) E(w) - \frac{f''(z)}{2f'(z)} \\
f \circ F(z) &= f'(z) F(w) \\
f \circ K(z) &= (f'(z))^{3/2} K(w) + 2\frac{f''(z)}{f'(z)}(f'(z))^{1/2} \xi c\partial\left(e^{-\phi}\right)(w) + \\
&\quad + \left[\frac{1}{2}\frac{f'''}{f'} - \frac{1}{4}\left(\frac{f''}{f'}\right)^2\right](f'(z))^{-1/2} \xi c e^{-\phi}(w) \\
f \circ L(z) &= (f'(z))^{3/2} L(w) + \frac{f''(z)}{f'(z)}(f'(z))^{1/2} \xi c\partial\phi\ e^{-\phi}(w) + \\
&\quad + \left[\frac{3}{4}\left(\frac{f''}{f'}\right)^2 - \frac{2}{3}\frac{f'''}{f'}\right](f'(z))^{-1/2} \xi c e^{-\phi}(w) \\
f \circ M(z) &= (f'(z))^{3/2} M(w) + \frac{15}{12}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right](f'(z))^{-1/2} \xi c e^{-\phi}(w) \\
f \circ N(z) &= (f'(z))^{3/2} N(w) - \frac{f''(z)}{f'(z)}(f'(z))^{1/2} \xi\partial c\ e^{-\phi}(w) + \\
&\quad + \left[2\left(\frac{f''}{f'}\right)^2 - \frac{f'''}{f'}\right](f'(z))^{-1/2} \xi c\ e^{-\phi}(w)
\end{aligned}
$$

$$
\begin{aligned}
f \circ P(z) \quad = \quad & (f'(z))^{3/2} P(w) + \frac{1}{2}\frac{f''(z)}{f'(z)}(f'(z))^{1/2}\partial\xi\, c e^{-\phi}(w) + \\
& + \left[\frac{1}{4}\left(\frac{f''}{f'}\right)^2 - \frac{1}{6}\frac{f'''}{f'}\right](f'(z))^{-1/2}\xi c e^{-\phi}(w).
\end{aligned}
$$

Bijlage B

Samenvatting

B.1 Situering van het onderzoek

Vooraleer we een overzicht geven van de resultaten die naar voor gebracht worden in deze thesis, willen we aangeven waar ons onderzoek zich situeert in het breder kader van de snaartheorie.

B.1.1 Snaartheorie

Unificatie

Een van de belangrijkste concepten in de ontwikkeling van de theoretische natuurkunde is het streven naar *unificatie*. De geünificeerde beschrijving van verscheiden fysische fenomenen in een enkele theorie is in vele gevallen niet alleen esthetisch aantrekkelijk gebleken, vaak heeft ze ook geleid tot het verwerven van diepe inzichten in de grondslagen van de natuurkunde. Zo lag Maxwells unificatie van de elektrische en magnetische krachten in de theorie van het elektromagnetisme aan de basis van Einsteins formulering van de speciale relativiteitstheorie in het begin van de vorige eeuw.

Dit streven naar unificatie vond zijn voorlopig eindpunt in de jaren ’70 in de formulering van het *Standaardmodel* van elementaire deeltjes en interacties. Dit model beschrijft de geobserveerde elementaire deeltjes en de manier waarop zij interageren tengevolge van elektromagnetische wisselwerkingen en zwakke en sterke kernkrachten. Bovendien biedt het model een microscopische beschrijving in overeenstemming met de wetten van de kwantummechanica en is het ondertussen op talloze manieren experimenteel geverifieerd.

Het grote obstakel in het verderzetten van het ambitieuze project van de unificatie van alle gekende deeltjes en interacties was het incorporeren van de zwaartekracht. Deze kracht kreeg een zeer elegante beschrijving in de vorm van Einsteins *algemene relativiteitstheorie*, die ook experimenteel bevestigd werd. Deze theorie is echter een klassieke theorie, die slechts betrouwbaar is voor de beschrijving van fenomenen op macroscopische schaal. Fundamentele problemen treden op wanneer men tracht de algemene relativiteitstheorie in overeenstemming te brengen met de microscopische wetten van de kwantummechanica: algemene relativiteitstheorie is een niet-renormaliseerbare theorie, hetgeen erop neerkomt dat het toepassen van kwantummechanische stoornisrekening op deze theorie leidt tot onoverkomelijke oneindigheden in verstrooiingsamplitudes.

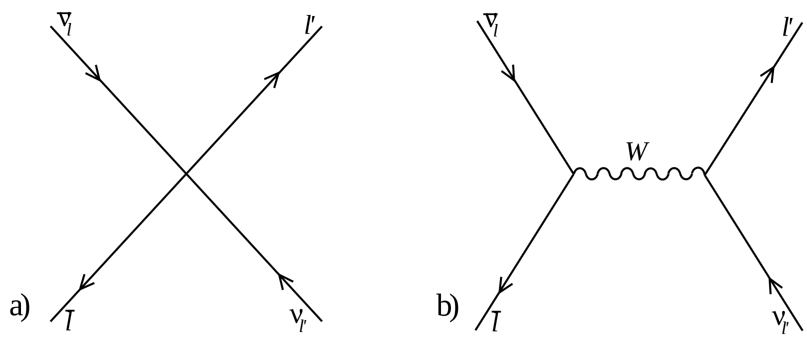
Niet-renormaliseerbaarheid en effectieve theorieën

Laten we het probleem van de niet-renormaliseerbaarheid even illustreren aan de hand van een ander voorbeeld, de vier-fermion theorie van de zwakke interacties. In deze theorie worden zwakke interacties beschreven als interacties van vier fermionen in eenzelfde punt van de ruimte-tijd zoals geïllustreerd in figuur B.1(a). De sterkte van deze interactie is vervat in een koppelingsconstante G_F die de dimensies[1] heeft van $[\text{energie}]^{-2}$. Dit betekent dat, in een proces bij karakteristieke energie E, de effectieve dimensieloze koppeling evenredig is met $G_F E^2$. Deze koppeling wordt willekeurig groot bij hoge energieën en leidt tot oneindigheden in verstrooiingsamplitudes die, in tegenstelling tot de oneindigheden die voorkomen in renormaliseerbare theorieën, niet kunnen geabsorbeerd worden in herdefinitie van de fysische parameters van de theorie[2].

We hebben reeds vermeld dat het Standaardmodel wel een goede kwantummechanische beschrijving geeft van de zwakke interacties, en het loont de moeite om even stil te staan bij de manier waarop het probleem van de niet-renormaliseerbaarheid van het oorspronkelijke vier-fermion model hier een oplossing kreeg. In figuur B.1(b) hebben we hetzelfde proces als in fi-

[1] We werken hier in eenheden waarin $c = \hbar = 1$.

[2] Juister gezegd, het wegwerken van de oneindigheden in een niet-renormaliseerbare theorie zou de introductie van *oneindig* veel experimenteel te bepalen fysische parameters vergen, hetgeen uiteraard nefast is voor de voorspellende kracht van de theorie.



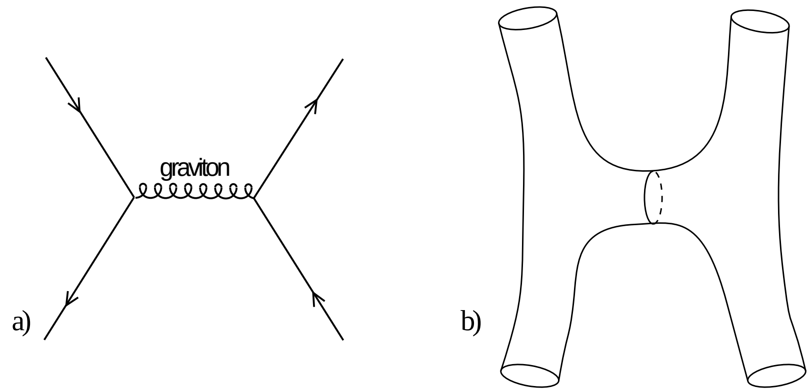
Figuur B.1: (a) Een vier fermion-interactie in de oorspronkelijke vier-fermi beschrijving. (b) Hetzelfde proces in de Standaard Model-beschrijving: de interactie wordt 'uitgesmeerd' door de uitwisseling van een W-boson.

guur B.1(a) voorgesteld, maar dan in de Standaard model beschrijving. We zien dat, wanneer men het proces op voldoende kleine schaal bekijkt, de vier-fermion interactie blijkt te ontstaan tengevolge van de uitwisseling van een nieuw deeltje, het W-boson. Op die manier is de interactie 'uitgespreid' en zijn de hoge-energie (kleine afstand) divergenties van het vier-fermion model afwezig. De oorspronkelijke vier-fermion beschrijving geeft echter wel een goede benadering bij voldoende lage energieën. Men zegt dat de vier-fermion theorie een *effectieve beschrijving* is.

Dit voorbeeld illustreert de historisch gegroeide interpretatie van niet-renormaliseerbare theorieën: niet-renormaliseerbaarheid wijst erop dat de theorie in kwestie slechts een effectieve beschrijving levert van een meer fundamentele theorie. Deze laatste theorie bevat nieuwe vrijheidsgraden (zoals het W-boson in ons voorbeeld) die ervoor zorgen dat de interacties worden uitgesmeerd en het probleem van de hoge-energie divergenties oplossen.

Snaartheorie: een fundamentele beschrijving van de zwaartekracht

Keren we nu terug naar de algemene relativiteitstheorie. Ook hier heeft de koppelingsconstante, de konstante van Newton G_N, de dimensie van $[\text{energie}]^{-2}$. Dezelfde problemen van niet-renormaliseerbaarheid treden ook hier op, en men verwacht dan ook te maken te hebben met een effectieve



Figuur B.2: (a) Een verstrooiingsamplitude met uitwisseling van een graviton. (b) Hetzelfde proces in snaartheorie: het vervangen van deeltjes door snaren heeft tot gevolg dat de interactie wordt ‘uitgesmeerd’.

beschrijving van een meer fundamentele theorie. De zoektocht naar een fundamentele beschrijving van de zwaartekracht in termen van een ‘traditionele’ deeltjestheorie heeft evenwel niets opgeleverd. Het leek er dan ook op dat zo’n beschrijving een radicaal nieuwe aanpak zou vereisen. Zulk een radicaal nieuw idee, en voorlopig nog steeds het enige gekende idee dat de zojuist beschreven problemen oplost, kwam, nu ongeveer dertig jaar geleden, in de belangstelling in de vorm van *snaartheorie*.

In de snaartheorie zijn de fundamentele objecten geen deeltjes, maar submicroscopisch kleine, trillende, snaren. Van op voldoende grote afstand bekeken zien deze trillende snaartjes eruit als deeltjes, en elke trillingswijze van de snaar komt overeen met een deeltje met bepaalde fysische kenmerken. Op die manier kunnen vele theorieën van elementaire deeltjes gezien worden als effectieve beschrijvingen van een onderliggende theorie van snaren. Het mooie is nu dat één van deze trillingswijzen van de snaar kan geïdentificeerd worden als het deeltje dat verantwoordelijk is voor het overbrengen van de zwaartekracht, het *graviton*. Snaartheorie is dus een kandidaat voor de meer fundamentele theorie waarvan de algemene relativiteitstheorie een effectieve beschrijving geeft (zie figuur B.2), en het is inderdaad gebleken dat snaartheorie voldoet aan het voornaamste criterium dat men aan zo’n fundamentele theorie stelt, namelijk eindigheid van de stoornisrekening.

Snaartheorie en unificatie

Naast het incorporeren van gravitatie bevat snaartheorie nog een aantal andere ingrediënten die in de loop van de twintigste eeuw zijn voorgesteld in de zoektocht naar een geünificeerde beschrijving van de natuur. We pikken er hier slechts de meest prominente uit.

- Naast de hoger vermelde graviton-excitatie bevat de snaartheorie nog vele andere excitaties, en sommige hiervan dragen de juiste kenmerken om geïdentificeerd te worden als de elementaire deeltjes en dragers van krachten uit het Standaardmodel. In snaarmodellen (zie bv. [1]) is de symmetriegroep van het Standaardmodel typisch ingebed in een grotere symmetriegroep. Hierdoor verenigt snaartheorie gravitatie met het idee van de *Grote Geünificeerde Theorieën (GUT's)* voorgesteld in de jaren '70.

- Snaartheorie realiseert ook het concept van *extra ruimtelijke dimensies*. Inderdaad, de gekende consistente snaartheorieën vereisen maar liefst 9 ruimtelijke dimensies. Dit idee is niet in strijd met experimentele waarnemingen zolang 6 van deze dimensies maar voldoende klein of 'opgerold' zijn, en werd reeds in de twintiger jaren voorgesteld door Kaluza en Klein als een mogelijk mechanisme voor de unificatie van zwaartekracht en elektromagnetisme [4]. Traditioneel werd vermoed dat deze extra dimensies zich zouden uitstrekken over een grootte van de orde van de Planck-lengte, $l_p = G_N^{-1/2} = 1.6 \times 10^{-33}$ cm. De laatste jaren wordt ook terdege rekening gehouden met de mogelijkheid van grotere extra dimensies, tot zelfs van de grootte orde van 1 mm [5]!

- Een derde belangrijk ingrediënt in snaartheorie is *supersymmetrie*, een symmetrie tussen bosonische en fermionische vrijheidsgraden, die deel uitmaakt van alle gekende consistente snaartheorieën. Ook supersymmetrie was reeds voorgesteld als een onderdeel van een mogelijke uitbreiding van het Standaardmodel (zie bv. [6] voor een recent overzicht), onder meer omdat het een natuurlijke verklaring geeft voor de hiërarchie van de grootte-orden van fysische parameters in het Standaardmodel.

B.1.2 Waarom snaarveldentheorie?

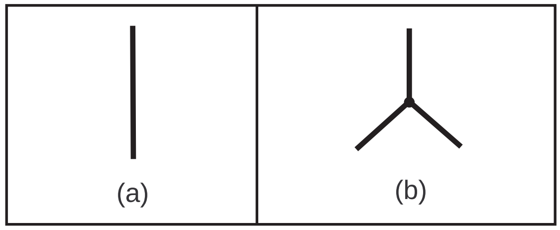
In de vorige paragraaf zagen we dat snaartheorie, in de vorm waarin zij conventioneel wordt geformuleerd, een formalisme geeft waarin verstrooiingsprocessen, onder andere tussen gravitonen, perturbatief kunnen berekend worden. We duiden deze conventionele en zeer uitvoerig bestudeerde formulering dan ook aan met de term *perturbatieve snarentheorie*. Voor een aantal toepassingen, waarvan een specifiek voorbeeld uitvoerig aan bod komt in deze thesis, is het echter noodzakelijk gebruik te maken van een meer uitgebreid formalisme, dat van de *snaarveldentheorie*. Om een idee te krijgen van de bestaansredenen van de snaarveldentheorie dienen we even stil te staan bij de beperkingen inherent in de formulering van de perturbatieve snaartheorie. Weerom loont het de moeite de situatie te vergelijken met deze in de elementaire deeltjesfysica.

Veldentheorie versus S-matrix theorie

Het natuurlijke kader voor de beschrijving van interacties tussen relativistische deeltjes is de *kwantumveldentheorie*, waarin deeltjes beschreven worden als fluctuaties van een *kwantumveld*. De kwantumveldentheorie beschrijft de interacties tussen deeltjes (waarbij ook deeltjes kunnen worden gecreëerd of geannihileerd) op een manier die in overeenstemming is met de vereisten van de kwantummechanica en de speciale relativiteitstheorie. In het bijzonder leidt kwantumveldentheorie tot een perturbatieve methode voor het berekenen van overgangsamplitudes tussen initiële en finale toestanden die elk een vast aantal deeltjes bevatten. Zulke overgangsamplitudes noemt men *S-matrix*[3] elementen en deze vormen de basis voor de berekening van meetbare grootheden zoals vervaltijden en werkzame doorsneden.

S-matrix elementen worden hier benaderd door een som van bijdragen, waarbij elke afzonderlijke contributie kan worden voorgesteld door een *Feynman diagram* (enkele voorbeelden van Feynman diagrammen zagen we reeds in de figuren B.1, B.2). Deze diagrammen zijn opgebouwd volgens een set van *Feynman regels* die men op een eenvoudige manier kan aflezen uit de actie voor het kwantumveld. We geven een eenvoudig voorbeeld: de theorie van

[3] De fysische beperkingen op de S-matrix, zoals Lorentz-invariantie, unitariteit en causaliteit zijn zelfs zodanig restrictief dat ze op een natuurlijke manier leiden tot het invoeren van kwantumvelden [7].



Figuur B.3: Feynman diagrammen in scalaire ϕ^3 theorie worden opgebouwd met behulp van (a) een propagator en (b) een drie-punts interactie vertex.

een scalair veld ϕ met de volgende actie:

$$S[\phi] = \int d^4x[\frac{1}{2}\phi(\partial_\mu\partial^\mu - m^2)\phi + \frac{g}{3!}\phi^3].$$

In termen van Feynman diagrammen geeft de eerste term aanleiding tot de *propagator* (figuur 1.3(a)), die vrije propagatie van het deeltje voorstelt, terwijl de tweede term aanleiding geeft tot interacties waarbij twee deeltjes annihileren en een derde deeltje wordt gecreëerd. Deze interactie wordt dan voorgesteld door een drie-punts *interactie vertex* (figuur 1.3(b)).

In de perturbatieve snaartheorie is de situatie in zekere zin omgekeerd: hier beschikt men over een perturbatieve expansie van de S-matrix, maar is het niet a priori duidelijk of deze expansie ook een veldentheoretische oorsprong heeft. De doelstelling van de snaarveldentheorie is dan ook het introduceren van *snaarvelden*, waarvan de fluctuaties overeenkomen met snaren, en het opstellen van een *actie* voor deze velden die, via een bijhorende set van Feynman regels, de perturbatieve S-matrix expansie van de snaartheorie genereert.

De voordelen van een veldentheorie-beschrijving

Een eerste vraag die men zich dient te stellen is uiteraard: is zo'n aanpak wel echt noodzakelijk voor de beschrijving van snaren? Indien immers alle fysisch relevante vragen zouden kunnen beantwoord worden met behulp van perturbatieve snaartheorie, zou het invoeren van snaarveldentheorie geen noodzaak zijn maar hoogstens kunnen leiden tot een alternatieve beschrijving van dezelfde fenomenen.

In de theorie van deeltjes zijn er vele toepassingen te vinden waarvoor de perturbatieve S-matrix expansie te kort schiet, maar die wel adequaat kunnen beschreven worden in de veldentheorie. We geven een paar voorbeelden.

- Allereerst is deze aanpak, zoals de naam reeds suggereert, perturbatief van aard in de zin dat er een expansie gemaakt wordt in machten van een koppelingsconstante, zoals de parameter g in het voorbeeld (1.2.1), die klein verondersteld wordt. Sterker nog, in het algemeen (en dit geldt waarschijnlijk ook voor de perturbatieve snaartheorie) is deze reeks niet convergent, doch eerder een asymptotische ontwikkeling voor kleine g. Zo'n asymptotische reeks kan weliswaar een zeer goede benadering geven voor voldoende kleine waarden van g, maar kan, voor grotere waarden van g, belangrijke aspecten van de fysica van het model missen. Dit is bijvoorbeeld het geval voor de beschrijving van *solitonische objecten*, zoals monopolen in ijktheorieën, die typisch een massa hebben evenredig met $1/g$. Een ander voorbeeld zijn de zogenaamde *instanton correcties* op verstrooiingsamplitudes, die typisch van de orde $e^{-1/g}$ zijn.

- Voor de beschrijving van processen in aanwezigheid van achtergrondvelden en voor het berekenen van kwantum correcties op de klassieke actie dient men te beschikken over meer algemene amplitudes dan degene die vervat zijn in de elementen van de S-matrix: deze worden *off-shell* amplitudes genoemd.

- De S-matrix aanpak schiet ook tekort in de beschrijving van *collectieve fenomenen* waarbij grote aantallen deeltjes betrokken zijn. Een voorbeeld van zo'n fenomeen, dat een cruciale rol speelt in de formulering van het Standaardmodel, is het Brout-Englert-Higgs-Kibble-effect, waarbij er een condensaat van scalaire deeltjes wordt gevormd dat onder meer verantwoordelijk is voor het genereren van massa's van deeltjes in het Standaard Model.

De ambitie van snaarveldentheorie is om een kader te bieden waarin gelijkaardige toepassingen in de snaartheorie kunnen worden behandeld. Merkwaardig genoeg is het hier mogelijk gebleken een aantal niet-perturbatieve effecten te beschrijven enkel gebruik makend van technieken uit de perturbatieve snaartheorie. Het is namelijk gebleken dat, in vele gevallen, een snaartheorie bij hoge waarde van de koppeling g equivalent ('duaal') is met een andere snaartheorie bij een kleine koppeling g' evenredig[4] met $1/g$. Op deze manier is het mogelijk het gedrag van een snaartheorie bij sterke koppeling te bestuderen aan de hand van perturbatieve berekeningen in de duale theorie.

Ofschoon snaarveldentheorie weinig of geen rol heeft gespeeld bij de ontdekking van dit soort dualiteitsrelaties, is het toch een noodzakelijk hulpmiddel gebleken in een aantal toepassingen. Eén van deze toepassingen, tachyon condensatie, vormt het hoofdonderwerp van deze thesis.

Wittens snaarveldentheorie

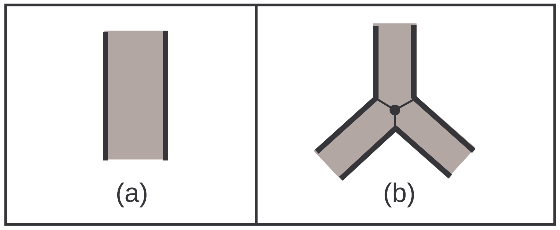
De hoop en ambitie van de snaarveldentheorie is dus om een beschrijving te kunnen geven van aspecten van snaartheorie waarvoor de perturbatieve aanpak tekort schiet. De tot nu toe meest succesvolle snaarveldentheorie werd voorgesteld door Witten in 1986 en beschrijft de interacties tussen bosonische open snaren.

De actie ziet eruit als volgt:

$$S[\Psi] = \oint [\frac{1}{2}\Psi \star Q\Psi + \frac{g}{3}\Psi \star \Psi \star \Psi]. \qquad \text{(B.1.1)}$$

Zonder hier in te gaan op de precieze betekenis van alle symbolen (dit gebeurt in hoofdstuk 3), wensen we toch kort de fysische betekenis van de twee termen in deze actie te bespreken. Wanneer we de actie vergelijken met de actie voor het scalaire veld, zien we een aantal gelijkenissen: het scalaire veld ϕ is vervangen door het snaarveld Ψ, de kinetische energie operator door een nog mysterieuze operator Q, de operaties van vermenigvuldiging en integratie zijn vervangen door analoge operaties $\star$ en $\oint$ op snaarvelden. De betekenis van de twee termen in (B.1.1) is dan ook vergelijkbaar met met deze van de twee termen in (1.2.1): de eerste term beschrijft vrije propagatie van snaren (figuur B.4(a)) terwijl de tweede term snaarinteracties beschrijft waarbij twee snaren aaneengehecht worden ter vorming van een nieuwe snaar (figuur B.4(b)).

[4]Dit type dualiteit staat bekend onder de naam *S-dualiteit*. Een ander soort dualiteitsrelaties aanwezig in de snaartheorie zijn de zogenaamde *T-dualiteiten*, die we bestudeerden in ons artikel [76].



Figuur B.4: Feynman diagrammen in open snaartheorie worden opgebouwd met behulp van (a) een propagator en (b) een drie-snaar interactie vertex.

B.1.3 Onstabiele objecten in snaartheorie

D-branen

Van groot belang in de recente ontwikkelingen in de snaartheorie is de vaststelling dat gesloten snaartheorie meer is dan een theorie van snaren alleen: ze bevat ook andere objecten, die *D-branen* worden genoemd. D-branen zijn objecten die zich kunnen uitstrekken over meerdere ruimtelijke dimensies: zo spreekt men van het D-deeltje, de D-snaar, het D-membraan enzoverder.

Ofschoon D-branen uitgebreide objecten zijn die deel uitmaken van gesloten snaartheorie, spelen open snaren een belangrijke rol in de beschrijving ervan. Een D-braan kan immers gezien worden als een hyperoppervlak waarop open snaren kunnen eindigen. Deze open snaren komen overeen met fluctuaties van het D-braan.

De D-branen in gesloten supersnaartheorieën hebben een belangrijke fysische eigenschap: zij dragen een bepaald soort lading, die Ramond-Ramond lading genoemd wordt. Deze eigenschap hangt nauw samen met supersymmetrie: het feit dat D-branen deze lading dragen identificeert hen als objecten in wier aanwezigheid de theorie invariant blijft onder een aantal supersymmetrieën. Zulke objecten worden ook BPS-objecten genoemd. Een belangrijk gevolg van de BPS-eigenschap is dat D-branen *stabiel* zijn en niet kunnen vervallen naar lichtere objecten.

Tachyonen en instabiliteiten in veldentheorie

Vooraleer we overgaan tot de bespreking van onstabiele branen in snaartheorie, zeggen we eerst iets over instabiliteiten in kwantumveldentheorie.

In de kwantumveldentheorie worden instabiliteiten van het systeem gesignaleerd door de aanwezigheid van *tachyonen* in het perturbatieve spectrum van de theorie. Met de term tachyon bedoelen we een fluctuatie van het veld die, indien men zou insisteren op een deeltjesinterpretatie, een deeltje met een imaginaire rustmassa (en bijgevolg, een snelheid groter dan de lichtsnelheid) zou beschrijven. Een kleine verstoring van zo'n systeem geeft aanleiding tot een vervalproces naar een stabiele configuratie. Dit proces wordt *tachyon condensatie* genoemd omdat, in de stabiele toestand, er zich een condensaat van scalaire deeltjes heeft gevormd. De aanwezigheid van dit condensaat heeft belangrijke gevolgen voor de fysica van het model; zo beïnvloedt het de massa's van de andere deeltjes in het model. Dit mechanisme speelt een zeer belangrijke rol in het Standaardmodel, waar het verantwoordelijk is voor het genereren van de massa's van materiedeeltjes en van deze van de dragers van de zwakke kernkracht.

Onstabiele branen en Sens conjectuur

Naast de zojuist besproken stabiele branen, bestaan er in de snaartheorie ook *onstabiele branen*, die de BPS-eigenschap niet bezitten. Hun bestaan werd voor het eerst aangetoond door Sen.

Net als in veldentheorie wordt de instabiliteit van deze objecten ook hier gesignaleerd door de aanwezigheid van een tachyon in het fluctuatiespectrum. Ook hier zal het systeem vervallen naar een stabiele configuratie en zal er tachyon condensatie plaatsvinden. Een belangrijke vraag betreft hier de aard van het eindstadium van dit proces.

In deze context heeft Sen, op basis van een aantal argumenten gebaseerd op string dualiteiten, een hypothese voorgesteld die de motivatie vormt voor het onderzoek gepresenteerd in deze thesis. Volgens *Sens conjectuur* is het eindproduct van tachyon condensatie op onstabiele branen niets anders dan de grondtoestand van gesloten snaartheorie, of anders gezegd: het onstabiele braan vervalt naar de lege ruimte onder tachyon condensatie.

Indien de hypothese waar is, moet alle energie die vervat zit in de massa van het onstabiele braan opgebruikt worden in het proces van tachyon condensatie. Dit kan alleen maar als de potentiaal voor het tachyon aan volgende voorwaarde voldoet: het verschil tussen de waarde van de potentiaal op zijn maximum (dat overeenkomt met het onstabiele braan) en de waarde op het minimum (dat de stabiele eindtoestand voorstelt) moet precies gelijk zijn aan de spanning van het onstabiele braan.

Het berekenen van de tachyon potentiaal vormt dus een concrete test van Sens conjectuur. Bovendien is dit een berekening waarvoor de perturbatieve snaartheorie tekort schiet en die dus thuishoort in het formalisme van de snaarveldentheorie. We zien hier een ideale gelegenheid om snaarveldentheorie te testen in een concrete toepassing. Inderdaad, voor de snaarveldentheorie werd gebruikt om Sens conjectuur te testen, was het een wijdverspreide opvatting dat stringveldentheorie gefaald had omdat ze nog geen resultaten had opgeleverd die niet konden verkregen worden met behulp van de perturbatieve snaartheorie. We zijn dan ook van mening dat de studie van de tachyon potentiaal in snaarveldentheorie, waar ons onderzoek deel van uitmaakt, deze visie heeft ontkracht.

B.2 Overzicht van de thesis

Hoofdstuk 2: perturbatieve snaartheorie

In hoofdstuk 2 geven we een overzicht van de perturbatieve snaartheorie. Zo'n kort overzicht is uiteraard verre van volledig, en de hier gegeven selectie van topics is gemaakt met twee doelstellingen voor ogen.

Enerzijds willen we voor de geïnteresseerde niet-expert een aantal basisresultaten in de perturbatieve snaartheorie vermelden, waarvan we er reeds enkele aanstipten in de vorige paragraaf. In sectie 2.1 bespreken we de Polyakov actie, die aan de basis ligt van de perturbatieve snaartheorie. In sectie 2.2 geven we aan hoe Feynmans padintegraalformalisme leidt tot de perturbatieve expansie van de S-matrix in snaartheorie, en bespreken we hoe het fixen van de lokale symmetrieën in snaartheorie aanleiding geeft tot de introductie van 'spoken'. Sectie 2.4 behandelt het fysische spectrum van de bosonische snaar in het formalisme van de BRST-kwantisatie. Hier stellen we vast dat het fysische spectrum van de gesloten snaar onder meer een graviton bevat. In sectie 2.5 bespreken we de uitbreiding naar supersnarentheorie. Het fysische spectrum van de supersnaar wordt behandeld in paragraaf 2.5.5. Tenslotte geven we in paragraaf 2.5.6 een overzicht van de gekende consistente supersnaartheorieëen.

Anderzijds willen we van de gelegenheid gebruik maken om een aantal technische aspecten uit te werken die nodig zijn voor een grondig begrip van de technische kanten van hoofdstukken 3-7. In deze categorie valt paragraaf 2.3 waarin we tweedimensionale conforme veldentheorieën bespreken, die kunnen gezien worden als de bouwstenen van de snaartheorie. In latere hoofdstukken zullen we regelmatig gebruik maken van volgende resultaten: het bepalen van het gedrag van lokale operatoren onder conforme afbeeldingen (paragraaf 2.3.2), het verband tussen Fock-ruimte toestanden en lokale operatoren (paragraaf 2.3.3), en de inwendige producten in de Fock-ruimte van conforme veldentheorieën (paragraaf 2.3.5). In paragrafen 2.5.4 en 2.5.5 komen enkele technische details aan bod in verband met de spoken in supersnaartheorie. Deze spelen een belangrijke rol in sectie 3.2 en hoofdstuk 7.

Hoofdstuk 3: snaarveldentheorie

In dit hoofdstuk voeren we het formalisme van de snaarveldentheorie in. We introduceren het formalisme in de oorspronkelijke formulering van Witten, die het nauwst aansluit bij het intuïtieve beeld dat snaarinteracties hun oorsprong vinden in het opsplitsen en aaneenhechten van snaren, en maken dan de overstap naar de equivalente formulering in termen van correlatoren in conforme veldentheorie. Deze laatste formulering zullen we uitvoerig gebruiken in de concrete berekeningen van hoofdstukken 5 en 7.

In paragraaf 3.1.1 introduceren we snaarvelden en overlopen we de verschillende voorstellingen van deze objecten die gangbaar zijn in de literatuur. In paragrafen 3.1.2 en 3.1.3 gaan we in op de structuur en de betekenis van de symbolen in Wittens actie (B.1.1) voor open snaarveldentheorie. In paragraaf 3.1.4 leiden we de equivalente formulering af in termen van correlatoren in conforme veldentheorie. Onze afleiding, die gebruik maakt van de padintegraal voorstelling van het stringveld, is nog niet eerder in de literatuur verschenen. In 3.1.6 illustreren we het voorgaande aan de hand van een eenvoudig voorbeeld. Ofschoon we in deze thesis enkel klassieke aspecten van de snaarveldenheorie bestuderen, staan we in paragraaf 3.1.7 toch even stil bij de kwantisatie van de theorie, die de gekende S-matrix expansie van perturbatieve snaartheorie reproduceert.

Sectie 3.2 is gewijd aan de veldentheorie beschrijving van open supersnaren. In de literatuur zijn er drie verschillende concrete voorstellen gedaan voor zo'n beschrijving, die we een voor een overlopen. Het eerste voorstel werd gedaan door Witten en is terug te vinden in paragraaf 3.2.1. Het is een vrij directe veralgemening van de bosonische snaarveldentheorie. Deze theorie bleek echter niet vrij van problemen te zijn, en in een poging om deze problemen op te lossen, werd in de literatuur een licht gewijzigde theorie voorgesteld die besproken wordt in paragraaf 3.2.2. Een derde actie voor supersnaar veldentheorie werd voorgesteld door Berkovits. Deze theorie, die we bespreken in paragraaf 3.2.3, heeft een drastisch gewijzigde structuur ten opzichte van de twee andere voorstellen.

Tenslotte staan we in sectie 3.3 even stil bij de onvolmaaktheden en open problemen in de snaarveldentheorie anno 2001.

Hoofdstuk 4: tachyon condensatie

In hoofdstuk 4 geven we de motivatie voor ons onderzoek inzake tachyon condensatie in snaartheorie en plaatsen we het in een ruimer kader.

In sectie 4.1 geven we voorbeelden van tachyon condensatie in veldentheorie waar zij een belangrijk ingrediënt vormt van het Standaardmodel. In secties 4.2 en 4.3 bespreken we het bestaan van BPS D-branen in snaartheorie en hun rol in de ontdekking van dualiteitsrelaties tussen op het eerste gezicht verschillende snaartheorieën. In sectie 4.4.2 behandelen we Sens constructie van onstabiele branen in snaartheorie, en stellen we vast dat de instabiliteit ook hier gesignaleerd wordt door de aanwezigheid van een tachyon in het spectrum van fluctuaties van het braan. In secties 4.4.3 en 4.4.4 formuleren we Sens conjectuur in verband met de potentiaal van het tachyon en geven we aan waarom de verificatie van deze conjectuur thuishoort in het formalisme van de snaarveldentheorie.

Hoofdstuk 5: tachyon condensatie in bosonische snaartheorie

In dit hoofdstuk bespreken we het fenomeen van tachyon condensatie in bosonische snaartheorie. Ofschoon het grootste deel van de resultaten in dit hoofdstuk te vinden is in de literatuur, proberen we hier toch enkele losse eindjes aan elkaar te knopen. Tevens is het bosonische model een ideale gelegenheid om een aantal basisprincipes te illustreren in een relatief eenvoudige context, die we dan in hoofdstuk 7 gemakkelijk kunnen vergalgemenen naar het technisch meer veeleisende kader van de supersnaren.

In secties 5.1 en 5.2 geven we aan welke componenten van het snaarveld betrokken zijn bij tachyon condensatie en tonen we aan dat de tachyon potentiaal een universele functie is die niet afhangt van specifieke details van het model. In sectie 5.3 bespreken we de benaderingsmethode die we gebruiken in alle berekeningen in deze thesis. Deze methode is bekend als de level truncatie methode en blijkt in de praktijk een reeks van opeenvolgende benaderingen te geven die snel convergeert naar het exacte resultaat. Secties 5.5 en 5.6 behandelen dan de berekening van de tachyon potentiaal in de zogenaamde level $(4, 8)$ benadering. Het resultaat is in zeer goede overeenstemming met de voorspellingen van Sens conjectuur. Tenslotte bespreken we in 5.7 een interessante hypothese omtrent de fysica van het model na tachyon condensatie. Er wordt geopperd dat het model hier een beschrijving zou kunnen geven van gesloten snaren en hiermee het oude idee van 'gesloten snaren uit open snaren' concreet zou kunnen realiseren. Dit laatste idee, ofschoon zeer speculatief, vormt ons inziens één van de meestbelovende richtingen voor verder onderzoek in het domein.

Hoofdstuk 6: vereenvoudigd model voor tachyon condensatie in bosonische snaartheorie

In dit hoofdstuk willen we een belangrijk aspect van de berekening van de tachyon potentiaal, namelijk de level truncatie methode, op een iets steviger basis plaatsen. Deze methode is immers grotendeels 'experimenteel' in de zin dat zij in de praktijk goed blijkt te convergeren maar dat hier weinig a-priori argumenten voor bestaan. Daarom bekijken we in dit hoofdstuk deze methode in een vereenvoudigd model dat geïnspireerd is op het volledige probleem in snaarveldentheorie.

In secties 6.1 en 6.2 stellen we het model voor, geven we aan hoe het gerelateerd is tot bosonische snaarveldentheorie, en leiden we de vergelijkingen voor het condensaat af. In sectie 6.3 construeren we de exacte oplossing voor een specifieke waarde van de parameters van het model. In sectie 6.4 maken we de vergelijking met de resultaten bekomen in de level truncatie methode. We kunnen hier aantonen dat de level truncatie methode, in functie van het level, op exponentiële wijze convergeert naar het exacte resultaat. Dit gedrag stemt overeen met de 'experimentele' bevindingen in de volledige snaarveldentheorie.

Hoofdstuk 7: tachyon condensatie in supersnaartheorie

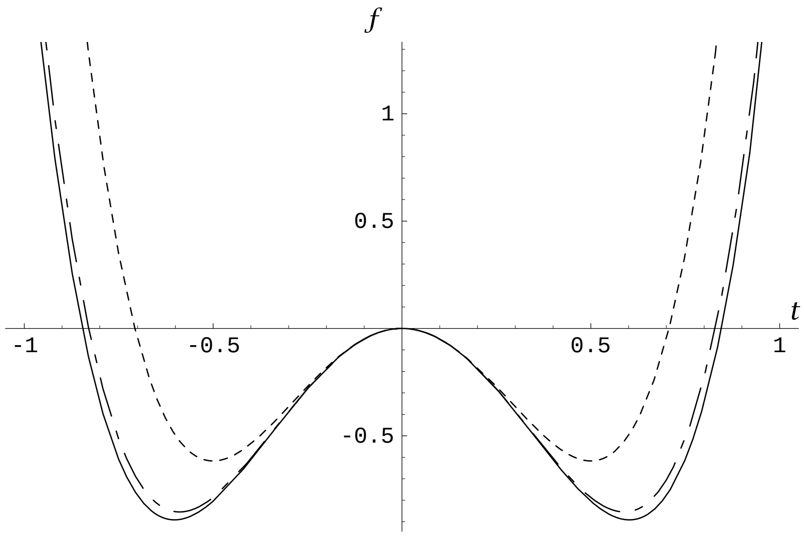
In hoofdstuk 7 presenteren we onze resultaten omtrent de studie van de tachyon potentiaal in supersnaartheorie. Zoals we reeds vermeldden, bestaan er in de literatuur drie verschillende voorstellen voor de veldentheorie van open supersnaren; in dit hoofdstuk bestuderen we de tachyon potentiaal in elk van deze theorieën. De analyse gebeurt, net zoals voor de bosonische snaar, met de methode van level truncatie.

De studie van tachyon potentiaal in supersnaartheorie moet in een ander licht gezien worden dan de gelijkaardige studie voor de bosonische snaar: daar waar, voor de bosonische snaar, de snaarveldentheorie beschrijving buiten kijf stond en Sens conjectuur onzeker was, is in de supersnaartheorie Sens conjectuur beter geargumenteerd terwijl de correcte veldentheorie beschrijving onzeker is. Daarom dient de berekening van de tachyon potentiaal hier eerder beschouwd te worden als een test om te zien of de veldentheorie beschrijving de verwachte resultaten kan reproduceren.

In sectie 7.1 bespreken we de tachyon potentiaal in Wittens supersnaar veldentheorie. Deze resultaten zijn verschenen in ons artikel [115]. We bespreken eerst hoe de theorie dient uitgebreid te worden om de fluctuaties van een onstabiel braan te beschrijven, en gaan dan over tot de concrete berekening van de tachyon potentiaal in de level $(2,4)$ benadering. We stellen vast dat de tachyon potentiaal in de eerste benadering geen minimum vertoont, en dat dit het geval blijft wanneer men de invloed van hogere levels in rekening brengt. De potentiaal vertoont dan zelfs een singulier gedrag. We besluiten dan ook dat deze theorie resultaten oplevert die in strijd zijn met het door Sens conjectuur voorspelde gedrag.

In sectie 7.2 bestuderen we het gedrag van de tachyon potentiaal in de gewijzigde kubische snaarveldentheorie. In een eerste benadering vertoont de potentiaal een minimum, en de waarde van de potentiaal in dit minimum bedraagt 97% van de voorspelde exacte waarde. Wanneer we ook een volgend level in rekening nemen, stellen we vast dat dat de waarde in het minimum de voorspelde waarde overschrijdt: zij bedraagt hier 108% van de verwachte waarde. Om uitsluitsel te geven, bestuderen we de potentiaal ook in de volgende benadering, namelijk op level $(5/2,5)$. Hier vinden we een gedrag dat manifest in tegenspraak is met Sens conjectuur. We zien ons dus genoodzaakt het optimisme van sommige auteurs omtrent het gedrag van de tachyon potentiaal in deze theorie enigszins te temperen en concluderen dat er ook hier geen overeenstemming is met Sens conjectuur.

Tenslotte richten we in sectie 7.3 onze aandacht op de tachyon potentiaal in de theorie die werd voorgesteld door Berkovits. De resultaten, die ook te vinden zijn in ons artikel [118], blijken hier wel in goede overeenstemming te



Figuur B.5: De tachyon potentiaal f in Berkovits' snaarveldentheorie in drie opeenvolgende benaderingen in de level truncatiemethode. De door Sen voorspelde waarde van f in de minima is $f = -1$.

zijn met de voorspellingen van Sens conjectuur: in de eerste benadering vindt men een minimum waar de waarde van de potentiaal 62% van de voorspelde exacte waarde bedraagt, op level $(3/2, 3)$ vindt men 85% van de voorspelde waarde, en de level $(2, 4)$ benadering levert 89% van de voorspelde waarde (zie figuur B.5).

B.3 Samenvatting van de resultaten

De belangrijkste nieuwe resultaten die in deze thesis naar voor worden gebracht, zijn:

- De afleiding van snaarveldentheorie in de conforme veldentheorie representatie (sectie 3.1.4), waarbij we gebruik maken van de padintegraal voorstelling van het snaarveld.
- De studie van de level truncatie methode in een vereenvoudigd model (hoofdstuk 6), waar we kunnen aantonen dat deze methode een algo-

ritme levert dat, als functie van het level, exponentieel convergeert naar het exacte resultaat.

- De studie van de tachyon potentiaal op level $(2, 4)$ in Wittens supersnaar veldentheorie, waar we vaststellen dat het gedrag van de potentiaal niet overeenstemt met de voorspellingen van Sens conjectuur.
- De studie van de tachyon potentiaal op level $(5/2, 5)$ in de gewijzigde kubische snaarveldentheorie, waar we eveneens vaststellen dat het gedrag van de potentiaal niet overeenstemt met de voorspellingen van Sens conjectuur.
- De studie van de tachyon potentiaal op level $(2, 4)$ in de snaarveldentheorie van Berkovits, waar we kunnen vaststellen dat het gedrag van de potentiaal in goede overeenkomst is met de voorspellingen van Sens conjectuur.

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