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On quasimap invariants of moduli spaces of Higgs bundles

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Abstract

We compute odd-degree genus 1 quasimap and Gromov–Witten invariants of moduli spaces of Higgs SL_2-bundles on a curve of genus $g \geqslant 2$. We also compute certain invariants for all prime ranks. This proves some parts of the author's conjectures on quasimap invariants of moduli spaces of Higgs bundles. More generally, our methods provide a computation scheme for genus 1 quasimap and Gromov–Witten invariants in the case when degrees of maps are coprime to the rank. This requires an analysis of the localisation formula for certain Quot schemes parametrising higher-rank quotients on an elliptic curve. Invariants for degrees that are not coprime to the rank exhibit a very different structure for a reason that we explain.

Contents

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1. Introduction

1.1 Quasimaps

Let $M(\mathsf{d})$ be a moduli space of semistable Higgs SL_2-bundles of degree d on a curve C of genus $g \geqslant 2$. In this work, we consider quasimaps from a fixed elliptic curve E to $M(\mathsf{d})$. These are maps from E to the stack of all Higgs sheaves mapping generically to $M(\mathsf{d})$.

The (reduced) expected dimension of a moduli space of quasimaps up to translations by E is 0, hence by [Nes23], it produces an invariant

$$\mathsf{QM}^{\bullet}_{\mathsf{d},\mathsf{w}} \in \mathbb{Q},$$

where $\mathsf{w} \in \mathbb{Z}$ is the degree of quasimaps. The Picard rank of $M(\mathsf{d})$ is 1, and the degree is taken with respect to the ample generator of $\mathrm{Pic}(M(\mathsf{d}))$. Assuming $\mathsf{d} = 1$ or, equivalently, that d is odd, we determine these invariants for odd degrees w. Let

$$\mathsf{U}(q) := \log\left(\prod_{k \geqslant 0}(1-q^k)\right).$$

Theorem 1.1. *We have*

$$\sum_{\text{odd } \mathsf{w}} \mathsf{QM}^{\bullet}_{1,\mathsf{w}} q^{\mathsf{w}} = (2-2g)2^{2g-1}\left(\mathsf{U}(q) - \mathsf{U}(-q)\right).$$

By [Nes23, Corollary 10.12], this also determines genus 1 Gromov–Witten invariant $\mathsf{GW}^{\bullet}_{1,\mathsf{w}}$ of $M(1)$ since

$$\mathsf{QM}^{\bullet}_{1,\mathsf{w}} = \mathsf{GW}^{\bullet}_{1,\mathsf{w}}$$

if w is odd. The invariant $\mathsf{GW}^{\bullet}_{1,\mathsf{w}}$ is defined analogously but via the moduli space of stable maps from E. Moreover, by Corollary 3.7, these quasimap invariants determine certain Vafa–Witten invariants with insertions on the product $C \times E$.

1.2 Degree 0 Higgs bundles

In [Nes23], a notion of extended degree[1] $(\mathsf{w}, \mathsf{a}) \in \mathbb{Z} \oplus \mathbb{Z}_2$ of quasimaps to $M(\mathsf{d})$ was defined (see Definition 2.5) as were the associated invariants for an arbitrary d,

$$\mathsf{QM}^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}} \in \mathbb{Q}.$$

If $\mathsf{d} = 1$, the extended degree is determined by the parity of w; that is, $(\mathsf{w}, \mathsf{a}) = (\mathsf{w}, [\mathsf{w}]_2)$, where $[\mathsf{w}]_2 := \mathsf{w} \bmod 2$. In particular, for odd w we have

$$\mathsf{QM}^{1,\bullet}_{1,\mathsf{w}} = \mathsf{QM}^{\bullet}_{1,\mathsf{w}}.$$

In this article, we also determine the quasimap invariants $\mathsf{QM}^{1,\bullet}_{0,\mathsf{w}}$ associated to a moduli space of degree 0 Higgs SL_2-bundles $M(0)$.

Theorem 1.2. *We have*

$$\sum_{\mathsf{w}} \mathsf{QM}^{1,\bullet}_{0,\mathsf{w}} q^{\mathsf{w}} = (2-2g)2^{2g-1}\left(\mathsf{U}(q) + \mathsf{U}(-q)\right).$$

As it was argued in [Nes23], these invariants can be seen as Gromov–Witten–type invariants of the stack $M(0)$. Moreover, we compute certain invariants for $\mathsf{w} = 0$ in Corollary 3.9. Theorems 1.1 and 1.2, with Corollary 3.9, confirm parts of [Nes23, Conjecture A, B], thus providing evidence for [Nes23, Conjecture A, B, C].

We want to draw the reader's attention to a peculiar coincidence: Genus 1 positive-degree Gromov–Witten invariants of an elliptic curve E are given by the following generating series:

$$-\mathsf{U}(q) = -\log\left(\prod_{k\geqslant 0}(1-q^k)\right),$$

as is shown in [Dij95]. Our methods make this coincidence in some sense less surprising,[2] because in fact, the roles of C and E can be exchanged, allowing us to treat invariants $\mathsf{QM}^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}}$ in terms of other invariants that are expressible via E alone. We now explain how this is done (see also Remark 4.15).

1.3 Methods

The correspondence between quasimap invariants of $M(\mathsf{d})$ and Vafa–Witten invariants of $C \times E$, discussed in [Nes23], is essential. Our argument uses a combination of

- wall-crossing for Vafa–Witten invariants and
- quasimap wall-crossing of [Nes21].

The wall-crossing for Vafa–Witten invariants is conjectured to be trivial for a complex surface S with $p_g(S) > 0$ or, equivalently, with $b_2^+(S) > 1$. We will sketch an argument for its triviality by assembling various results from the existing literature. However, these results usually assume that $b_1(S) = 0$, mainly in order to simplify the exposition. In our case, S is a product of two non-rational curves; hence, $b_1(S) \neq 0$. We therefore make the assumption that existing results extend to the case of S with $b_1(S) \neq 0$. See Section 2.1 for more details.

[1]The extended degree aims to capture the presence of torsion classes in the cohomology of moduli spaces of PGL_r-bundles. It is essential for the formulation of enumerative mirror symmetry.

[2]However, we do not claim that we can fully explain this coincidence, so we urge the reader to treat this sentence as mostly rhetorical.

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$$\boxed{\#^{\mathrm{vir}}\{E \dashrightarrow M(\mathsf{d})\} \approx \#^{\mathrm{vir}}\{C \dashrightarrow M'(\mathsf{a})\} \approx \#^{\mathrm{vir}}\{G \twoheadrightarrow Q \text{ on } E\}}$$

Figure 1. Summary.

Changing stability on $C \times E$ from the one that has a high degree on E to the one that has a high degree on C corresponds to passing from quasimaps $E \dashrightarrow M(\mathsf{d})$ to quasimaps $C \dashrightarrow M'(\mathsf{a})$, where $M'(\mathsf{a})$ is a moduli space of Higgs SL_2-bundles on E. Since the wall-crossing for Vafa–Witten invariants is trivial, this gives rise to an equivalence of associated invariants: Corollaries 2.12 and 3.7.

In this way, invariants $\mathsf{QM}^{1,\bullet}_{\mathsf{d},\mathsf{w}}$ correspond to quasimap invariants of $M'(1)$. This makes computation more accessible because $M'(1)$ is just a point, as there is a unique stable Higgs SL_2-bundle of degree 1 on E. Hence, the corresponding quasimap invariants can be effectively computed by the quasimap wall-crossing: They will be equal to the wall-crossing invariants of the quasimap wall-crossing formula, which are just Euler characteristics of certain Quot schemes on E (which are quotiented by the action of E). A summary of the preceding discussion is depicted in Figure 1.

Complications arise due to the fact that in reality, one needs to consider quasisections of C to $M'(\mathsf{a})$ instead of quasimaps. This is a reason we did not put signs of equality in Figure 1 (another reason is that we have to find the quotient using E). Moduli spaces of quasisections and quasimaps are essentially isomorphic in this case, but the obstruction theories are not. This is the main source of technicalities in our calculations: The cosection of the obstruction theory of quasisections takes values in the canonical bundle of the curve C, as is explained in Section 4.4. Nevertheless, we obtain the same vanishing results as in the case of an obstruction theory with a standard surjective cosection, Theorem 4.10. We refer to Section 2.3 for more on quasisections in our context. Quasisections are treated in greater detail for more general fibrations in [LW23].

1.4 Higher rank

Almost everything presented in this article applies to an arbitrary rank r, except for the following two results. Firstly, Claim 2.3 is stated for a prime rank r because of Thomas' vanishing result [Tho20, Corollary 5.30]. Secondly, the analysis of the wall-crossing Quot schemes in Section 5 is done only for $\mathsf{r} = 2$.

The case of $\mathsf{r} > 2$ requires Quot schemes to parametrise quotients of higher rank on E. One can always deform to a sum of line bundle and use torus-localisations. However, since a sum of line bundles is not stable and since we consider higher-rank quotients, the resulting Quot schemes have non-trivial obstruction theories. This slightly obscures localisation formulas; hence, it will be addressed elsewhere. To this end, we conjecture an expression for higher-rank invariants in Conjecture 4.17 and provide a basic check of the conjecture, Proposition 4.16.

1.5 Even degrees

There is a good reason we cannot compute invariants for even degrees w (or, more generally, for degrees coprime to the rank) using the same methods. This case corresponds to the moduli space of degree 0 Higgs SL_2-bundles $M'(0)$ on E. The space $M'(0)$ is no longer a point. In fact, all Higgs bundles of degree 0 on E are strictly semistable and are given by direct sums of degree

0 line bundles. As such, the moduli space $M'(0)$ is not complicated, but since it is a stack, its moduli spaces of quasimaps are not easily accessible.

Quasimap invariants of even degrees are in some sense more appealing. For example, if $\mathsf{w}=0$, then the corresponding invariants give Euler characteristics of moduli spaces of Higgs bundles. If $\mathsf{w}>0$, then by [Nes23, Section 7.2], they determine quasimap invariants of the gerbe given by the class α. Moreover, K-theoretic invariants should give more refined topological invariants. In particular, one could potentially compute these topological invariants via the moduli space of degree 0 Higgs bundles on E by using Vafa–Witten wall-crossing, once quasimaps to moduli space of degree 0 Higgs bundles on E is better understood.

1.6 Notation and conventions

We denote the torus that scales Higgs fields by $\mathbb{C}^*_t$, while the torus that scales $\mathbb{P}^1$ (with weight 1 at $0\in\mathbb{P}^1$) by $\mathbb{C}^*_z$. We also denote

$$\mathbf{t} := \text{weight 1 representation of } \mathbb{C}^*_t \text{on } \mathbb{C};$$

$$\mathbf{z} := \text{weight 1 representation of } \mathbb{C}^*_z \text{ on } \mathbb{C},$$

such that $t := e_{\mathbb{C}_t}(\mathbf{t})$ and $z := e_{\mathbb{C}_z}(\mathbf{z})$ are the associated classes in the equivariant cohomology of a point.

Moduli spaces of Higgs sheaves are not proper; hence, we will always use the virtual localisation to define invariants. In order to make the notation less complicated, we denote

$$\int_{[M]^{\mathrm{vir}}} \cdots := \int_{[M^{\mathbb{C}^*_t}]^{\mathrm{vir}}} \frac{\cdots}{e(N^{\mathrm{vir}})}.$$

Finally, we will frequently use the fact that an obstruction theory of some space M descends to the quotient $[M/G]$. That this is indeed true can be seen either by taking quotients in the category of derived stacks, since our group actions preserve the naturally defined derived enhancements, or by viewing the descent of an obstruction theory of M to $[M/G]$ as an obstruction theory of $[M/G]$ relative to $[\mathrm{pt}/G]$, which requires certain compatibility of the corresponding moduli problems, which also holds in our case.

2. Vafa–Witten and quasimap invariants

2.1 Preliminaries

Throughout the present article, we fix a rank $\mathsf{r}\geq 2$. Only in the very end of Section 4 will we restrict to $\mathsf{r}=2$. We need $\mathsf{r}=2$ for the analysis of Quot schemes in Section 5.

Let C and C' be smooth, non-rational projective curves, and let

$$L_\delta := \mathcal{O}_C(1)\boxtimes\mathcal{O}_{C'}(\delta), \quad \delta\in\mathbb{Q}_{>0}$$

be an ample $\mathbb{Q}$-line bundle on the product $C\times C'$. For the extremal values of δ, we introduce the following notation:

$$\begin{aligned}\delta &= + \text{ if } \delta\gg 1;\\ \delta &= - \text{ if } \delta\ll 1.\end{aligned} \tag{1}$$

Throughout this article, we will be using the identification

$$H^2(C\times C',\mathbb{Z})\cong\mathbb{Z}\oplus H^1(C,\mathbb{Z})\otimes H^1(C',\mathbb{Z})\oplus\mathbb{Z},$$

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as provided by the Kunneth's decomposition theorem. We define

$$\Gamma_C := \mathrm{Jac}(C)[\mathsf{r}]$$

to be a group of r-torsion lines bundles on C.

Definition 2.1. Let d and a be integers such that $0 \leqslant \mathsf{d}, \mathsf{a} < \mathsf{r}$. We define

$$M^\delta(C \times C', \mathsf{d}, \mathsf{a}, \mathsf{w})$$

to be a moduli space of Higgs sheaves (F, ϕ), with a fixed determinant and traceless Higgs field $\phi \in \mathrm{Hom}(F, F \otimes \omega_{C\times C'})$ on $C \times C'$, which are Gieseker-stable with respect to L_δ. The class of F is given as follows:

$$\begin{aligned} &\mathrm{rk}(F) = \mathsf{r}; \\ &\mathrm{c}_1(F) = (\mathsf{d}, 0, \mathsf{a}); \\ &\Delta(F) := \mathrm{c}_1(F)^2 - 2\mathrm{r}\mathrm{ch}_2(F) = 2\mathsf{w}. \end{aligned}$$

Throughout this article, we assume

$$\gcd(\mathsf{r}, \mathsf{d}, \mathsf{a}) = 1,$$

which implies that there are no strictly semistable Higgs sheaves.

Remark 2.2. The assumption on the middle component of $c_1(F)$ being 0 is not restrictive because if C and C' are chosen in the same way that, as the Jacobian of one curve is not an isogenous component of another, then $H^1(C) \otimes H^1(C')$ does not contain algebraic classes. The curves C' and C can always be deformed to such a setup. By the deformation invariance of Vafa–Witten invariants, we can therefore assume that the middle component is zero.

2.2 Vafa–Witten wall-crossing

Conjecturally, Vafa–Witten invariants are independent of stabilities for surfaces with $p_g(S) > 1$. Let us present some evidences of this: By [MM21], (physically derived) formulas for Vafa–Witten invariants for a surface S with $b_1(S) = 0$ and $p_g(S) > 1$ are independent of stabilities (see also the discussion in [TT20, Section 1.6]). By [DPS98], the same holds for Donaldson invariants for a surface with $b_1(S) \neq 0$ and $p_g(S) > 1$. It is therefore reasonable to expect that Vafa–Witten invariants (with even insertions) for a surface with $b_1(S) \neq 0$ and $p_g(S) > 1$ are also independent of stabilities. Products of non-rational curves are among such surfaces. We now derive a proof for this claim.

Claim 2.3. Assume $p_g(S) > 0$. If r is prime and there are no strictly semistable sheaves, then Vafa–Witten invariants with even μ-insertions are independent of stability.

Derivation of proof: Vafa–Witten invariants consist of instanton and monopole contributions. The instanton contributions are integrals on moduli spaces of stable sheaves on the surface (descendent Donaldson invariants). On the other hand, the monopole contributions are given by integrals on moduli spaces of flags of sheaves. See [TT20] for more details.

The claim can therefore be proven by assembling the following results from the literature:

- Mochizuki's universal expressions for descendent invariants on moduli spaces of sheaves on surfaces [Moc09]. To express instanton contributions via descendent invariants, we use Göttsche–Kool's expressions of the virtual equivariant Euler class in terms of descendent invariants [CY22];

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- Thomas' double-cosection argument, which shows that only vertical components contribute to the monopole branch [Tho20, Corollary 5.30];
- Laarakker's expressions of vertical contributions in terms of integrals on nested Hilbert schemes [Laa20, Theorem A] (see also [GSY20, Theorem 3] for the rank 2 case). Laarakker's analysis extends to invariants with insertions.

It can be readily checked that Thomas' and Laarakker's results are independent of the assumption on $b_1(S)$. On the other hand, Mochizuki's result is more involved.

More conceptually, the independence of stability for Vafa–Witten invariants should be studied within the framework of Joyce's wall-crossing [Joy21]. □

Definition 2.4. Following [TT20], we define Vafa–Witten invariants associated to a moduli space $M^\delta(C\times C', \mathsf{d}, \mathsf{a}, \mathsf{w})$ by the $\mathbb{C}^*_t$-localisation,

$$\mathsf{VW}^{\mathsf{a}}_{\mathsf{d},\mathsf{w}}(C\times C') := \int_{[M^\delta(C\times C', \mathsf{d},\mathsf{a},\mathsf{w})^{\mathbb{C}^*_t}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})} \in \mathbb{Q}.$$

By Claim 2.3, they are independent of δ. For short, we will write

$$\int_{[M^\delta(C\times C', \mathsf{d}, \mathsf{a},\mathsf{w})]^{\mathrm{vir}}} 1 := \int_{[M^\delta(C\times C', \mathsf{d},\mathsf{a},\mathsf{w})^{\mathbb{C}^*_t}]^{\mathrm{vir}}} \frac{1}{e(N^{\mathrm{vir}})},$$

the same notation of which applies to all integrals that require $\mathbb{C}^*_t$-localisations.

2.3 Quasisection invariants

The importance of quasisections was already observed in [Oko19, Section 7]. Here we apply them in the context of a relative moduli space of Higgs bundles.

Let $K_{C\times C'}$ be the total space of the canonical bundle $\omega_{C\times C'}$ on $C\times C'$. The variety $K_{C\times C'}$ admits projections both to C and to C':

$$\pi_C : K_{C\times C'} \to C, \quad \pi_{C'} : K_{C\times C'} \to C'.$$

We will consider various moduli spaces (that is, Quot schemes and moduli spaces of Higgs sheaves) relative to these projections.

Definition 2.5. We define $\mathfrak{M}^{\mathrm{rel}}_C(\mathsf{d}) \to C'$ to be a relative moduli space of 1-dimensional compactly supported sheaves on $\pi_{C'} : K_{C\times C'} \to C'$ whose associated Higgs sheaves are of rank r, degree d and with a fixed determinant and a traceless Higgs field. We refer to this moduli space as a relative moduli space of Higgs sheaves. By $M^{\mathrm{rel}}_C(\mathsf{d}) \to C'$, we denote its semistable locus.

Let us denote

$$\mathsf{v} = (\mathsf{r}, \mathsf{d}) \in H^{\mathrm{ev}}(C, \mathbb{Z}).$$

As in the absolute case, we have a determinant-line-bundle map

$$\lambda : H^{\mathrm{ev}}(C,\mathbb{Z}) \to \mathrm{Pic}(\mathfrak{M}^{\mathrm{rel}}_C(\mathsf{d})),$$

such that a class $u \in H^{\mathrm{ev}}(C,\mathbb{Z})$ that satisfies

$$\chi(\mathsf{v}\cdot u) = \int_C \mathsf{v}\cdot u \cdot \mathrm{td}_C = \mathsf{r}\cdot u_2 + \mathsf{d}\cdot u_1 + \mathsf{r}\cdot u_1(1-g) = 1$$

gives a trivilisation of the $\mathbb{C}^*$-gerbe

$$\mathfrak{M}^{\mathrm{rel}}_C(\mathsf{d}) \to \mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},C}(\mathsf{d}) := \mathfrak{M}^{\mathrm{rel}}_C(\mathsf{d}) /\!\!/ \mathbb{C}^*$$

or, in other words, a universal family on $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},C}(\mathsf{d})$. For a class $u \in H^{\mathrm{ev}}(C,\mathbb{Z})$ such that $\chi(\mathrm{v}\cdot u) = 0$, the line bundle $\lambda(u)$ descends to $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},C}(\mathsf{d})$.

We define the theta line bundle $\Theta \in \mathrm{Pic}\Big(\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},C}(\mathsf{d})\Big)$ as follows:

$$\theta = (-\mathsf{r}, \mathsf{d} - \mathsf{r}(g-1));$$
$$\Theta = \lambda(\theta).$$

We also define the class of SL-trivialisations of the universal family of $\mathfrak{M}^{\mathrm{rel}}_{C}(\mathsf{d})$:

$$\alpha \in H^2\Big(\mathfrak{M}^{\mathrm{rel}}_{C}(\mathsf{d}), \mathbb{Z}_{\mathsf{r}}\Big).$$

Equivalently, α is the Kunneth component of the first Chern class of the universal family modulo r. The class α is the gerbe class of [HT03]. The classes Θ and α will be used to define degrees of quasisections.

Remark 2.6. If $\gcd(\mathsf{r},\mathsf{d})$=1, then α is a multiple of Θ modulo r. However, this is not the case otherwise. It is useful to keep α for notational purposes, however, because even in the case when $\gcd(\mathsf{r},\mathsf{d})$=1, invariants behave very differently depending on the degree with respect to α.

Definition 2.7. A *quasisection* of $M^{\mathrm{rel}}_{C}(\mathsf{d})$ is a section of the projection $p_{C'} : \mathfrak{M}^{\mathrm{rel}}_{rg,C}(\mathsf{d}) \to C'$,

$$f : C' \to \mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},C}(\mathsf{d}), \quad p_{C'} \circ f = \mathrm{id}_{C'},$$

which maps generically to $M^{\mathrm{rel}}_{C}(\mathsf{d})$. A quasisection is of degree $(\mathsf{w},\mathsf{a}) \in \mathbb{Z}\oplus\mathbb{Z}_{\mathsf{r}} := \mathbb{Z}\oplus\mathbb{Z}/\mathsf{r}\mathbb{Z}$, if

$$\deg(f^*\Theta) = \mathsf{w}, \quad f^*\alpha = \mathrm{a}.$$

We denote the moduli space of quasisections of $M^{\mathrm{rel}}_{C}(\mathsf{d})$ of degree (w,a) by $Q\big(M^{\mathrm{rel}}_{C}(\mathsf{d}),\mathsf{a},\mathsf{w}\big)$.

The moduli spaces $Q\big(M^{\mathrm{rel}}_{C}(\mathsf{d}),\mathsf{a},\mathsf{w}\big)$ inherit $\mathbb{C}^*_t$-actions from $\mathfrak{M}^{\mathrm{rel}}_{C}(\mathsf{d})$. The properness of quasisections and the existence of a perfect obstruction theory is proven in the same way as in [Nes21, Nes23]; see also [LW23].

Definition 2.8. If $\gcd(\mathsf{r},\mathsf{d}) = 1$, we define

$$\mathsf{QM}^{\mathsf{a}}_{\mathsf{d},\mathsf{w}}(C) = \int_{[Q(M^{\mathrm{rel}}_{C}(\mathsf{d}),\mathsf{a},\mathsf{w})]^{\mathrm{vir}}} 1 \in \mathbb{Q}$$

to be quasisection invariants associated to a moduli space $Q\big(M^{\mathrm{rel}}_{C}(\mathsf{d}),\mathsf{a},\mathsf{w}\big)$.

Note that by the definition of a relative moduli of sheaves, a section

$$f : C' \to \mathfrak{M}^{\mathrm{rel}}_{C}(\mathsf{d})$$

is given by a sheaf on

$$K_{C\times C'} \times_{C'} C' = K_{C\times C'}.$$

Hence by [Nes23, Proposition 5.10], a moduli space of L_+-stable Higgs sheaves on $C\times C'$ is naturally a $\Gamma_{C'}$-torsor over the moduli space of quasisections of $M^{\mathrm{rel}}_{C}(\mathsf{d})$. On the other hand, the moduli space of L_--stable Higgs sheaves on $C\times C'$ is naturally a Γ_C-torsor over the moduli space of quasisections of $M^{\mathrm{rel}}_{C'}(\mathsf{d})$. Moreover, the corresponding obstruction theories match. This is summarised in the following proposition:

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Proposition 2.9. *If* $\gcd(\mathsf{r},\mathsf{d})=1$, *we have*

$$Q\Big(M^{\mathrm{rel}}_{C'}(\mathsf{d}),\mathsf{a},\mathsf{w}\Big)\cong\big[M^{-}(C\times C',\mathsf{d},\mathsf{a},\mathsf{w})\Gamma_C\big]$$
$$Q\Big(M^{\mathrm{rel}}_{C}(\mathsf{d}),\mathsf{a},\mathsf{w}\Big)\cong\big[M^{+}(C\times C',\mathsf{d},\mathsf{a},\mathsf{w})/\Gamma_{C'}\big]\,,$$

such that the naturally defined obstruction theories on both sides match.

Proof. Similar to [Nes23, Proposition 5.10], see also [LW23]. □

We use Proposition 2.9 as a justification for the following definition of invariants in the case of $\gcd(\mathrm{r},\mathrm{d})\neq 1$:

Definition 2.10. If $\gcd(\mathsf{r},\mathsf{d})\neq 1$, we define

$$\mathsf{QM}^{\mathsf{a}}_{\mathsf{d},\mathsf{w}}(C):=\int_{[M^{+}(C\times C',\mathsf{d},\mathsf{a},\mathsf{w})/\Gamma_{C'}]^{\mathrm{vir}}}1\in\mathbb{Q}.$$

If $\gcd(\mathsf{r},\mathsf{a})\neq 1$, we define

$$\mathsf{QM}^{\mathsf{d}}_{\mathsf{a},\mathsf{w}}(C'):=\int_{[M^{-}(C\times C',\mathsf{d},\mathsf{a},\mathsf{w})/\Gamma_{C}]^{\mathrm{vir}}}1\in\mathbb{Q}.$$

Remark 2.11. Proposition 2.9 implicitly depends on the choice of the universal family on the rigidified stack $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},C}$. We return to this point in Section 3.2 for an elliptic curve.

Using Claim 2.3 and Proposition 2.9, we obtain a curious correspondence between quasisection invariants of $M^{\mathrm{rel}}_{C}(\mathsf{d})$ and $M^{\mathrm{rel}}_{C'}(\mathsf{a})$.

Corollary 2.12. *If* r *is prime, we have*

$$\mathsf{r}^{2g(C')}\mathsf{QM}^{\mathsf{a}}_{\mathsf{d},\mathsf{w}}(C)=\mathsf{VW}^{\mathsf{a}}_{\mathsf{d},\mathsf{w}}(C\times C')=\mathsf{r}^{2g(C)}\mathsf{QM}^{\mathsf{d}}_{\mathsf{a},\mathsf{w}}(C')\,.$$

3. Genus 1 invariants

3.1 Group actions

For the rest of this article we assume that C' is an elliptic curve,

$$C'=E.$$

Since $\pi_E:K_{C\times E}\to E$ is a trivial fibration, the moduli space of quasisections to $M^{\mathrm{rel}}_{C}(\mathsf{d})$ is canonically isomorphic to a moduli space of quasimaps from E to an absolute moduli space of Higgs bundles $M_C(\mathsf{d})$ on C:

$$Q\Big(M^{\mathrm{rel}}_{C}(\mathsf{d}),\mathsf{a},\mathsf{w}\Big)\cong Q_E(M_C(\mathsf{d}),\mathsf{a},\mathsf{w})\,.$$

In fact, our primary interest is in quasimaps up to translations of E; that is, in the quotient

$$[Q_E(M(\mathsf{d}),\mathsf{a},\mathsf{w})/E]\,,$$

where E acts on $Q_E(M(\mathsf{d}),\mathsf{a},\mathsf{w})$ by precomposition with a translation. A similar action exists on the level of moduli spaces $M^{\delta}(C\times E,\mathsf{d},\mathsf{a},\mathsf{w})$, which we now explain.

The group

$$E\times\mathrm{Jac}(E)$$

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naturally acts on sheaves. Here, E acts by pulling back a sheaf with respect to a translation τ_p by a point $p \in E$, while $\mathrm{Jac}(E)$ acts by tensoring a sheaf with a line bundle L. These operations commute. Overall,

$$F \mapsto \tau_p^* F \otimes L.$$

Let

$$\Phi(\mathcal{L}) \subset E \times \mathrm{Jac}(E)$$

be the subgroup that fixes the determinant line bundle $\mathcal{L}$ of sheaves in a moduli space $M^\delta(C \times E, \mathsf{d}, \mathsf{a}, \mathsf{w})$. We define

$$\Phi_{\mathsf{a}} = (\mathrm{id}, \mathrm{r})^{-1} \Phi(\mathcal{L})\,. \tag{2}$$

The group Φ_{a} preserves rank r sheaves with determinant $\mathcal{L}$. The action of Φ_{a} on sheaves therefore restricts to an action on $M^\delta(C \times E, \mathsf{d}, \mathsf{a}, \mathsf{w})$.

By our assumption on the classes in Definition 2.1, the line bundle $\mathcal{L}$ is of the form $L \boxtimes L'$. Hence, $\Phi(\mathcal{L})$ and therefore Φ_{a} depend only on the degree of a. The group Φ_{a} also acts on $\mathfrak{M}_E^{\mathrm{rel}}(\mathsf{a})$. In the case of

$$Q\Big(M_E^{\mathrm{rel}}(\mathsf{a}), \mathsf{d}, \mathsf{w}\Big) \cong \big[M^-(C \times E, \mathsf{d}, \mathsf{a}, \mathsf{w})\,/\Gamma_C\big]\,,$$

the action of Φ_{a} on $Q\big(M_E^{\mathrm{rel}}(\mathsf{a}), \mathsf{d}, \mathsf{w}\big)$ can be seen as identification of maps by the automorphisms of the target. The importance of this action is due to the next two lemmas:

Lemma 3.1. *There is a canonical identification*

$$[Q_E(M_C(\mathsf{d}), \mathsf{a}, \mathsf{w})\,/E] \cong \big[M^+(C \times E, \mathsf{d}, \mathsf{a}, \mathsf{w})\,/\Phi_{\mathsf{a}}\big]$$

such that the naturally defined obstruction theories on both sides match.

Proof. There exists a natural map

$$M^+(C \times E, \mathsf{d}, \mathsf{a}, \mathsf{w}) \to Q_E(M_C(\mathsf{d}), \mathsf{a}, \mathsf{w}) \tag{3}$$

that is a Γ_E-torsor. There also exists a natural projection

$$\Phi_{\mathsf{a}} \to E \tag{4}$$

that is also a Γ_E-torsor. The map (3) is equivariant with respect to (4) and the corresponding actions of Φ_{a} and E on the source and the target. It is not difficult to check that we obtain the desired identification after taking quotients. The rest follows from the same arguments as in [Nes23, Section 5.5]. □

The action of Φ_{a} can be exchanged for an insertion. We are interested in μ-insertions, which are defined as follows:

$$\mu : H^*(C \times E, \mathbb{Q}) \to H^*_{\mathbb{C}^*}\left(M^\delta(C \times E, \mathsf{d}, \mathsf{a}, \mathsf{w})\,, \mathbb{Q}\right);$$
$$\beta \mapsto \pi_{M*}\left(\Delta(\mathcal{F})/2\mathrm{r} \cdot \pi^*_{X \times E}\beta\right),$$

where $\mathcal{F}$ is the universal sheaf on $M^\delta(C \times E, \mathsf{d}, \mathsf{a}, \mathsf{w})$. Consider now the class

$$B_{\mathsf{w}} := \frac{\mathbb{1} \boxtimes [\mathrm{pt}]}{\mathrm{rw}} \in H^*(C \times E, \mathbb{Q})\,.$$

Lemma 3.2. *We have*

$$\mathsf{VW}^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}}(C\times E):=\int_{[M^{\delta}(C\times E,\mathsf{d},\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}1=\int_{[M^{\delta}(C\times E,\mathsf{d},\mathsf{a},\mathsf{w})]^{\mathrm{vir}}}\mu(B_{\mathsf{w}}).$$

Proof. Similar to [Nes23, Proposition 5.26], □

by Lemma 3.2 and Claim 2.3, invariants associated to a moduli space $\left[M^{\delta}(C\times E,\mathsf{d},\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}\right]$ are independent of δ.

Definition 3.3. If $\gcd(\mathsf{r},\mathsf{d})=1$, we define

$$\mathsf{QM}^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}}(C)=\int_{[Q_E(M(\mathsf{d}),\mathsf{a},\mathsf{w})/E]^{\mathrm{vir}}}1\in\mathbb{Q}t$$

to be invariants associated with quotient moduli spaces $[Q_E(M(\mathsf{d}),\mathsf{a},\mathsf{w})\,/E]$. If $\gcd(\mathsf{r},\mathsf{a})=1$, we also define

$$\mathsf{QM}^{\mathsf{d},\bullet}_{\mathsf{a},\mathsf{w}}(E)=\int_{[Q(M^{\mathrm{rel}}_E(\mathsf{d}),\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}1\in\mathbb{Q}t$$

to be invariants associated with the quotient moduli space $\left[Q\left(M^{\mathrm{rel}}_E(\mathsf{a}),\mathsf{d},\mathsf{w}\right)/\Phi_{\mathsf{a}}\right]$.

We use Lemma 3.1 as a justification for the following definition of invariants in the case of $\gcd(\mathsf{r},\mathsf{d})\neq 1$ and $\gcd(\mathsf{r},\mathsf{a})$.

Definition 3.4. If $\gcd(\mathsf{r},\mathsf{d})\neq 1$, we define

$$\mathsf{QM}^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}}(C)=\int_{[M^{+}(C\times E,\mathsf{d},\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}1\in\mathbb{Q}t.$$

If $\gcd(\mathsf{r},\mathsf{a})\neq 1$, we define

$$\mathsf{QM}^{\mathsf{d},\bullet}_{\mathsf{a},\mathsf{w}}(E)=\int_{[M^{-}(C\times E,\mathsf{d},\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}1\in\mathbb{Q}t.$$

Remark 3.5. Note the presence of the equivariant parameter t. This is due to the existence of $\mathbb{C}^*_t$-equivariant cosections, which map to a line bundle of $\mathbb{C}^*_t$-weight 1 and are constructed in [Nes23, Proposition 3.15]. We can divide by t, thereby obtaining $\mathbb{Q}$-valued invariants. This also corresponds to reducing the obstruction theory. However, the reduction of the obstruction theory is not necessary, as the cosection is equivariant and therefore does not lead to the vanishing of the virtual fundamental class; rather, it becomes a multiple of the equivariant parameter.

Remark 3.6. In the case of invariants up to translation by E, the role of the E-action is exchanged after passing from quasimaps of $M_C(\mathsf{d})$ to quasisections of $M_E(\mathsf{a})$. For $[Q_E(M(\mathsf{d}),\mathsf{a},\mathsf{w})\,/E]$, taking the quotient is an identification of maps by translations of the source curve E. On the other hand, for $\left[Q\left(M^{\mathrm{rel}}_E(\mathsf{a}),\mathsf{d},\mathsf{w}\right)/\Phi_{\mathsf{a}}\right]$, taking quotient can be seen as identification of quasisections by automorphisms of the target $M^{\mathrm{rel}}_E(\mathsf{d})$.

Using Claim 2.3 and Lemma 3.2, we obtain the following result:

Corollary 3.7. *If* r *is prime, we have*

$$\mathsf{QM}^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}}(C)=\mathsf{VW}^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}}(C\times E)=\mathsf{r}^{2g(C)}\mathsf{QM}^{\mathsf{d},\bullet}_{\mathsf{a},\mathsf{w}}(E).$$

Remark 3.8. Note that unlike in Corollary 2.12, we do not have the factor $\mathsf{r}^{2g(E)}$ on the left-hand side. This is because the group Φ_{a} contains Γ_E.

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3.2 Moduli spaces of Higgs sheaves and sheaves

From now on, we will assume that $\gcd(\mathsf{r}, \mathsf{a}) = 1$, unless stated otherwise. A moduli space of rank r and degree a stable Higgs GL_{r}-bundles on E, denoted by $M_E^{\mathrm{GL}}(\mathsf{a})$, is isomorphic to K_E via the determinant-trace map,

$$(\det, \mathrm{tr}) : M_E^{\mathrm{GL}}(\mathsf{a}) \xrightarrow{\cong} K_E.$$

Hence, a moduli space of stable Higgs sheaves on E with a fixed determinant and a traceless Higgs field is a point,

$$M_E(\mathsf{a}) = \{(G, 0)\} = \mathrm{pt},$$

where G is the unique stable sheaf with the given determinant. This is also holds relatively for the projection $\pi_C : K_{C\times E} \to C$,

$$M_E^{\mathrm{rel}}(\mathsf{a}) = \{(G, 0)\} \times C = C.$$

Using Corollary 2.12, we obtain an immediate consequence for quasimaps of degree $\mathsf{w} = 0$, which confirms a part of [Nes23, Conjecture B].

Corollary 3.9. *If* r *is prime and* $\mathsf{a} \neq 0$*, then*

$$\mathsf{QM}^{\mathsf{a}}_{0,0}(C) = \mathsf{r}^{2g-2}.$$

Let us now consider quasimaps of degree $\mathsf{w} \neq 0$. Since the unique Higgs sheaf in $M_E^{\mathrm{rel}}(\mathsf{a})$ has a zero Higgs field, a quasisection to $M_E^{\mathrm{rel}}(\mathsf{a})$ must factor through the moduli stack of Higgs sheaves with zero Higgs fields. The latter is just a moduli stack of sheaves on E,

$$\mathfrak{N}_E^{\mathrm{rel}}(\mathsf{a}) \hookrightarrow \mathfrak{M}_E^{\mathrm{rel}}(\mathsf{a}). \tag{5}$$

Since $\mathfrak{N}_E^{\mathrm{rel}}(\mathsf{a})$ is a relative moduli space of sheaves associated to a trivial fibration $C \times E \to C$, it trivialises canonically,

$$\mathfrak{N}_E^{\mathrm{rel}}(\mathsf{a}) = \mathfrak{N}_E(\mathsf{a}) \times C. \tag{6}$$

The same applies for rigidified stacks.

The obstruction theory of $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$ is constructed as follows: Let

$$\mathcal{G} \in \mathrm{Coh}\Big(K_{C\times E} \times_C \mathfrak{M}_E^{\mathrm{rel}}(\mathsf{a})\Big) \quad \text{and} \quad \pi : K_{C\times E} \times_C \mathfrak{M}_E^{\mathrm{rel}}(\mathsf{a}) \to \mathfrak{M}_E^{\mathrm{rel}}(\mathsf{a})$$

be the universal C-relative 1-dimensional sheaf and the canonical projection. The complex $R\mathcal{H}om_\pi(\mathcal{G}, \mathcal{G})$ descends to $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$. The obstruction theory for Higgs sheaves on a surface is constructed in [TT20, Section 6], and the construction applies to Higgs sheaves on a curve. The spectral-curve construction identifies C-relative 1-dimensional sheaves on $K_{C\times E}$ with C-relative Higgs sheaves on $C \times E$ with Higgs fields valued in $\omega_{C\times E}$. Hence, the C-relative obstruction theory of $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$ is given by the complex

$$\mathbb{T}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})} = R\mathcal{H}om_\pi(\mathcal{G}, \mathcal{G})_0\ [1]\,,$$

where $R\mathcal{H}om_\pi(\mathcal{G}, \mathcal{G})_0$ is defined to be the cone

$$\mathrm{Cone}\big(R\mathcal{H}om_\pi(\mathcal{G}, \mathcal{G}) \to \big(H^{*-1}(\omega_E) \oplus H^*(\mathcal{O}_E)\big) \otimes \omega_C\big)\,[-1]\,.$$

Note that $\mathbb{T}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})}$ does not restrict to the virtual tangent complex of the stack $\mathfrak{N}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$.

3.3 Chern characters

For the purposes of wall-crossing, one needs to make a choice for a universal family on the rigidified stack $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$. As explained in [Nes23, Section 3], this amounts to choosing $\mathsf{u} \in H^{\mathrm{ev}}(E,\mathbb{Z})$ such that $\chi(\mathsf{u}\cdot\mathsf{v})=1$. For a choice of such class $\mathsf{u}=(\mathsf{u}_1,\mathsf{u}_2)$, the sheaf F on $C\times E$ associated to a quasisection $f : C \to \mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$ of degree w has the following Chern character,

$$\mathrm{ch}(F) = (\mathsf{v}, \check{\mathsf{w}}) \in H^{\mathrm{ev}}(E,\mathbb{Z}) \oplus H^{\mathrm{ev}}(E,\mathbb{Z})(-1),$$

where $\check{\mathsf{w}}$ is defined by the following system of equations:

$$\begin{aligned} \check{\mathsf{w}}_1\cdot\mathsf{u}_2 + \check{\mathsf{w}}_2\cdot\mathsf{u}_1 &= 0, \\ \check{\mathsf{w}}_1\cdot\mathsf{a} - \check{\mathsf{w}}_2\cdot\mathsf{r} &= \mathsf{w}. \end{aligned} \tag{7}$$

For example, if $(\mathsf{r},\mathsf{a})=(\mathrm{r},1)$, then $\mathsf{u}=(1,0)$ clearly satisfies

$$\chi(\mathsf{v}\cdot\mathsf{u})=1.$$

Using (7), we deduce that in this case,

$$\check{\mathsf{w}}=(\mathsf{w},0).$$

4. Wall-crossing

4.1 ϵ-stable quasisections

We will use Corollary 3.7 to compute genus 1 quasimap invariants of moduli spaces of Higgs bundles on C. If $\gcd(\mathsf{r},\mathsf{a})=1$, then

$$M_E(\mathsf{a}) = \{(G,0)\};$$

hence, there are no sections of nonzero degree, and there is a unique section of degree zero. The quasimap wall-crossing for $\mathsf{w}>0$ is therefore particularly simple here, as it gives equality of invariants associated to $\epsilon=0^+$ and to the wall-crossing invariants. However, there are two complications:

- the action of Φ_{a} on $\mathfrak{M}^{\mathrm{rel}}_E(\mathsf{a})$ and
- C-relative setup,

which obscure otherwise-simple computations.

Let us start with defining ϵ-stable quasisections. From now on, we simplify the notation in the following way:

$$\begin{aligned} Q(\mathsf{a},\mathsf{w}) &:= Q\left(M^{\mathrm{rel}}_E(\mathsf{a}),\mathsf{w}\right), \\ Q(\mathsf{a},\mathsf{w})^{\bullet} &:= \left[Q\left(M^{\mathrm{rel}}_E(\mathsf{a}),\mathsf{w}\right)\Phi_{\mathsf{a}}\right]; \end{aligned}$$

the same applies to other related spaces.

Definition 4.1. A *marked bubbling* of C is a pair $(C',\mathbf{p},\iota)$, where $(C',\mathbf{p})$ is a connected, marked nodal curve of genus equal to $g(C)$ and where

$$\iota : C \hookrightarrow C'$$

is a closed immersion. In other words, C' is an isotrivial, semistable degeneration of C.

Definition 4.2. Given $\epsilon \in \mathbb{Q}_{>0}$, we define $Q^{\epsilon}_{k}(\mathsf{a},\mathsf{w})$ to be the moduli space of quasimaps of degree w,

$$f\colon (C',\mathbf{p}) \to \mathfrak{N}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a}) = \mathfrak{N}_{\mathrm{rg},E}(\mathsf{a}) \times C,$$

such that

- $f_{\mathfrak{N}(\mathsf{a})}\colon (C',\mathbf{p}) \to \mathfrak{N}_{\mathrm{rg},E}(\mathsf{a})$ is ϵ-stable [Nes23, Definition 3.5];
- $(C',\mathbf{p})$ is a marked bubbling of C with k markings;
- $[f_C \circ \iota\colon C \to C] = \mathrm{id}_C$.

Since quasisections to $\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$ factor through $\mathfrak{N}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$, a moduli space $Q^{\epsilon}_{k}(\mathsf{a},\mathsf{w})$ should be viewed as a moduli space of ϵ-stable quasisections to $\mathfrak{M}^{\mathrm{rel}}_{E}(\mathsf{a})$. The fact that these moduli spaces are proper follows from the arguments of [Nes21, Nes23]. Recall the embedding (5), which also holds for rigidified stacks,

$$\mathfrak{N}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a}) \hookrightarrow \mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a});$$

we thus endow $Q^{\epsilon}_{k}(\mathsf{a},\mathsf{w})$ with the obstruction theory given by the complex

$$\mathbb{T}^{\mathrm{vir}}_{Q^{\epsilon}_{k}(\mathsf{a},\mathsf{w})} := \pi_* f^* \mathbb{T}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})}. \tag{8}$$

Its perfectness is proven in the same vein as in [Nes21, Nes23].

Let us indicate what moduli spaces $Q^{\epsilon}_{k}(\mathsf{a},\mathsf{w})$ are for the extremal values of ϵ. Using the same notation as in (1), if $\epsilon = -$, we get

$$Q^{-}_{0}(\mathsf{a},\mathsf{w}) \cong Q(\mathsf{a},\mathsf{w}).$$

If $\epsilon = +$, then

$$Q^{+}_{0}(\mathsf{a},\mathsf{w}) = \mathrm{pt} \quad \text{if } \mathsf{w} = 0;$$
$$Q^{+}_{0}(\mathsf{a},\mathsf{w}) = \emptyset \quad \text{if } \mathsf{w} \neq 0.$$

We now discuss the wall-crossing between invariants associated to different values of ϵ.

Definition 4.3. Let $GQ(\mathsf{a},\mathsf{w})$ be a moduli space of prestable quasimaps of degree w,

$$f\colon \mathbb{P}^1 \to \mathfrak{N}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a}) = \mathfrak{N}_{\mathrm{rg},E}(\mathsf{a}) \times C,$$

such that $\infty \in \mathbb{P}^1$ is mapped to the stable locus. Consider a $\mathbb{C}^*_z$-action on the source $\mathbb{P}^1$ with weight 1 at $0 \in \mathbb{P}^1$. It thus induces a $\mathbb{C}^*_z$-action on $GQ(\mathsf{a},\mathsf{w})$. We define

$$W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \subset GQ(\mathsf{a},\mathsf{w})$$

to be the $\mathbb{C}^*_z$-fixed locus.

As in the case of $Q^{\epsilon}_{k}(\mathsf{a},\mathsf{w})$, we endow $GQ(\mathsf{a},\mathsf{w})$ with the obstruction theory given by the complex

$$\mathbb{T}^{\mathrm{vir}}_{GQ(\mathsf{a},\mathsf{w})} := \pi_* f^* \mathbb{T}^{\mathrm{vir}}_{\mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})},$$

using the embedding $\mathfrak{N}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a}) \hookrightarrow \mathfrak{M}^{\mathrm{rel}}_{\mathrm{rg},E}(\mathsf{a})$. The space $W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ inherits the obstruction theory defined by the fixed part of the obstruction theory of $GQ(\mathsf{a},\mathsf{w})$, as well as the virtual normal bundle N^{vir} defined by the moving part of the obstruction theory.

4.2 Moduli spaces of flags

As before, there exists a canonical identification of moduli spaces

$$W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) = W(\mathsf{a},\mathsf{w}) \times C, \tag{9}$$

where $W(\mathsf{a},\mathsf{w})$ is the analogous space defined via quasimaps to $\mathfrak{N}_E(\mathsf{a})$. By [Obe21, Section 4], the space $W(\mathsf{a},\mathsf{w})$ admits a description in terms of moduli spaces of flags, which we now recall. In what follows, by G we denote the unique stable sheaf of degree a with a fixed determinant supported on the zero section in K_E. We also define

$$\mathbf{z} := \text{weight 1 representation of } \mathbb{C}^*_z \text{ on } \mathbb{C}.$$

Let

$$\mathrm{Fl}(\mathsf{a}) = \{F_1 \subseteq F_2 \subseteq \ldots \subset F_{r-1} \subseteq F_r = G\}$$

be a moduli space of flags such that consecutive terms are allowed to be equal. To each $F_\bullet$ and a choice of an integer $k \in \mathbb{Z}$, we can associated a $\mathbb{C}^*$-equivariant, torsion-free sheaf $\mathcal{F}$ on $K_E \times \mathbb{A}^1$,

$$\mathcal{F} = F_1\mathbf{z}^{k+1} \oplus F_2\mathbf{z}^{k+2} \oplus \ldots F_{r-1}\mathbf{z}^{k+r-1} \oplus G\mathbf{z}^{k+r} \oplus G\mathbf{z}^{k+r+1} \ldots \tag{10}$$

In fact, for torsion-free sheaves G, such association is an equivalence between $\mathbb{C}^*$-fixed torsion-free sheaves on $K_E \times \mathbb{A}^1$ and weighted flags (up to a choice of k). Moreover, each $\mathbb{C}^*$-fixed sheaf in $W(\mathsf{a},\mathsf{w})$ is canonically $\mathbb{C}^*$-equivariant. Let us denote by

$$\mathrm{Fl}(\mathsf{a},\mathsf{w}) \subset \mathrm{Fl}(\mathsf{a})$$

the locus of flags that correspond to sheaves in $W(\mathsf{a},\mathsf{w})$. By construction, we have

$$W(\mathsf{a},\mathsf{w}) \cong \mathrm{Fl}(\mathsf{a},\mathsf{w}). \tag{11}$$

Analogously, let $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ be the relative moduli space of flags of $(\pi_E^*G)_{|C\times E}$ on the relative surface $\pi_C : K_{C\times E} \to C$. Viewing quotients of G as quotients of a sheaf on E, on K_E or C-relatively on $K_{C\times E}$ is equivalent. Hence, identification (11) also holds relatively:

$$W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \cong \mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \cong \mathrm{Fl}(\mathsf{a},\mathsf{w}) \times C. \tag{12}$$

Let

$$\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \subset \mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$$

be the connected component corresponding to Quot schemes; that is, flags with $r = 2$. By

$$\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})^c \subset \mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}),$$

we denote its complement. We define

$$\mathcal{Q}^{\mathrm{rel}} := \mathcal{F}_2/\mathcal{F}_1 \quad \text{and} \quad \mathcal{K}^{\mathrm{rel}} := \mathcal{F}_1$$

to be the universal quotient and the universal kernel of $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$, respectively. Quot schemes of stable sheaves on smooth curves are smooth; hence, so are $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ by (12).

4.3 Obstruction theory of flags

By [Obe21, Section 4], obstruction theories of moduli spaces $W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ and $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ also agree. We now describe the obstruction theory of $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ and the associated virtual normal bundle N^{vir}. Let

$$\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_r = G$$

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be the universal flag on $K_{C\times E}\times_C \mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$, and let

$$\pi : K_{C\times E}\times_C \mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \to \mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$$

be the natural projection.

Theorem 4.4 *The C-relative obstruction theory of $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ is given by the complex*

$$\mathbb{T}^{\mathrm{vir}}_{\mathrm{Fl}(\mathsf{a},\mathsf{w})^{\mathrm{rel}}} = \mathrm{Cone}\left(\bigoplus_{i=1}^{i=r-1} R\mathcal{H}om_\pi(\mathcal{F}_i,\mathcal{F}_i) \to \bigoplus_{i=1}^{i=r-1} R\mathcal{H}om_\pi(\mathcal{F}_i,\mathcal{F}_{i+1})\right).$$

The K-class of N^{vir} is

$$\begin{aligned} N^{\mathrm{vir}} = &-\sum_{i\geqslant 1}\sum_{k\geqslant 1} R\mathcal{H}om_\pi(\mathcal{F}_{i+k}/\mathcal{F}_{i+k-1},\mathcal{F}_i)\mathbf{z}^k \\ &+\sum_{i\geqslant 1}\sum_{k\geqslant 1} R\mathcal{H}om_\pi(\mathcal{F}_{i+k+1}/\mathcal{F}_{i+k},\mathcal{F}_i)^\vee\mathbf{z}^{-k}. \end{aligned}$$

Proof. See [Obe21, Section 4]. Note the sign difference in z-weights; this occurs because $0\in\mathbb{P}^1$ has weight z in contrast with [Obe21], where its weight is equal to $-z$. □

The next lemmas will be useful for the analysis presented in Section 4.6 and in Section 5.

Lemma 4.5. *We have the following identity in the K-group,*

$$R\mathcal{H}om_\pi(\mathcal{F}_{j+1}/\mathcal{F}_j,\mathcal{F}_i) = \mathsf{K}(1-\omega_C\mathbf{t}),$$

for some K-class K.

Proof. Let us denote $\mathcal{A} := \mathcal{F}_{j+1}/\mathcal{F}_j$ and $\mathcal{B} := \mathcal{F}_i$. Both $\mathcal{A}$ and $\mathcal{B}$ are scheme-theoretically supported on the zero section $C\times E\subset K_{C\times E}$; they can therefore be extended to the entire $K_{C\times E}$ by pulling them back by the projection $K_{C\times E}\to C\times E$. We denote these extensions by $\bar{\mathcal{A}}$ and $\bar{\mathcal{B}}$.

Consider now the sequence on $K_{C\times E}$,

$$0\to \mathcal{O}(-C\times E)\to \mathcal{O}_{K_{C\times E}}\to \mathcal{O}_{C\times E}\to 0.$$

We tensor it with $\bar{\mathcal{A}}$,

$$0\to \bar{\mathcal{A}}(-C\times E)\to \bar{\mathcal{A}}\to \mathcal{A}\to 0,$$

and then we apply $R\mathcal{H}om_\pi(-,\mathcal{B})$ to obtain the distinguished triangle

$$R\mathcal{H}om_\pi(\mathcal{A},\mathcal{B})\to R\mathcal{H}om_\pi\left(\bar{\mathcal{A}},\mathcal{B}\right)\to R\mathcal{H}om_\pi\left(\bar{\mathcal{A}}(-C\times E),\mathcal{B}\right)\to. \tag{13}$$

There is a natural $\mathbb{C}^*_t$-equivariant identification

$$\mathcal{O}_{K_{C\times E}}(-C\times E)_{|C\times E}\cong \omega^\vee_{C\times E}\mathbf{t}^{-1}\cong \omega_C^\vee\mathbf{t}^{-1}, \tag{14}$$

which gives us that

$$R\mathcal{H}om_\pi\left(\bar{\mathcal{A}}(-C\times E),\mathcal{B}\right)\cong R\mathcal{H}om_\pi(\bar{\mathcal{A}},\mathcal{B})\boxtimes\omega_C\mathbf{t}.$$

Passing to the K-group, the distinguished triangle (13) therefore gives us that

$$R\mathcal{H}om_\pi(\mathcal{A},\mathcal{B}) = R\mathcal{H}om_\pi\left(\bar{\mathcal{A}},\mathcal{B}\right)(1-\omega_C\mathbf{t}).$$

This proves the claim. □

Lemma 4.6. *With respect to the identification* (12), *the obstruction bundle of a Quot scheme* $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ *leads to the following expression:*

$$\mathrm{Ob}_{\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})} \cong T_{\mathrm{Quot}(\mathsf{a},\mathsf{w})} \boxtimes \omega_C \mathbf{t}.$$

Proof. Assume $r=2$; then

$$\mathbb{T}^{\mathrm{vir}}_{\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})} = R\mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}).$$

Using the same distinguished triangle (13) and passing to the associated long exact sequence, we obtain

$$\begin{aligned} 0 &\to \mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}(-C\times E), \mathcal{Q}^{\mathrm{rel}}) \to \\ &\to \mathcal{E}xt^1_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{E}xt^1_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{E}xt^1_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}(-C\times E), \mathcal{Q}^{\mathrm{rel}}) \to \dots. \end{aligned}$$

Since Q is scheme-theoretically supported on the zero section, the map $\mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}(-C\times E), \mathcal{Q}^{\mathrm{rel}})$ is zero; hence, the map

$$\mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}})$$

is an isomorphism. This also implies that

$$\mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}(-C\times E), \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{E}xt^1_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}})$$

is injective. Since we are considering quotients of the stable sheaf $\mathrm{ext}^2(K,Q)=0$, hence

$$\mathrm{ext}^0(K,Q) - \mathrm{ext}^1(K,Q) = \mathrm{ch}(K^\vee)\cdot \mathrm{ch}(Q) = 0,$$

we conclude that $\mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}(-C\times E), \mathcal{Q}^{\mathrm{rel}}) \to \mathcal{E}xt^1_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}})$ is in fact an isomorphism. Using the $\mathbb{C}^*_t$-equivariant identification (14), we obtain that

$$\begin{aligned} \mathcal{E}xt^1_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) &\cong \mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}(-C\times E), \mathcal{Q}^{\mathrm{rel}}) \\ &\cong \mathcal{H}om_\pi(\bar{\mathcal{K}}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \boxtimes \omega_C\mathbf{t} \cong \mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}}) \boxtimes \omega_C\mathbf{t}. \end{aligned}$$

The sheaf $\mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}}, \mathcal{Q}^{\mathrm{rel}})$ is the C-relative tangent bundle of $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$, and due to the decomposition (12), its C-relative tangent bundle is exactly the pull-back of the tangent bundle of $\mathrm{Quot}(\mathsf{a},\mathsf{w})$. □

4.4 Cosections

Cosections of the obstruction theory of $\mathrm{Fl}(\mathsf{a},\mathsf{w})^{\mathrm{rel}}$ are constructed in exactly the same way as in [PT16, Section 5.4] (see also [Obe21, Section 4] and [Nes21, Section 10.2]). However, since we work relative to C, the cosections map is not to a trivial line bundle but to ω_C. For example, this can already be seen in Lemma 4.6. This is because the relative canonical sheaf of

$$K_{C\times E} \to C$$

is the pull-back of $\omega_C^\vee$. By the argument from [Nes21, Section 10.2], which uses Serre's duality, we therefore get a C-relative cosection to $\omega_C^{\oplus 2}$ instead of the trivial bundle $\mathcal{O}^{\oplus 2}$ (here, we use the identification (9)):

$$\sigma = (\sigma_1, \sigma_2) : h^1\left(T^{\mathrm{vir}}_{\mathrm{Fl}(\mathsf{a},\mathsf{w})^{\mathrm{rel}}}\right) \to \omega_C^{\oplus 2}\mathbf{t},$$

where

$$\mathbf{t} := \text{weight 1 representation of } \mathbb{C}^*_t \text{ on } \mathbb{C}.$$

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Remark 4.7. By (12), the absolute obstruction theory of $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ is a direct sum of the relative obstruction theory of $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ and T_C. Hence, the cosection constructed above extends to a cosection of the absolute obstruction theory. However, since we are working $\mathbb{C}_t^*$-equivariantly, we do not need to reduce our obstruction theory, as the cosections will manifest themselves only in terms of equivariant parameters in the expressions of virtual fundamental cycles.

As in [Nes21, Proposition 10.6], we have the following result:

Proposition 4.8. *The cosection σ is surjective on $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})^c$ in $W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$. On $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$, only the component σ_1 is surjective.*

Proof. Similar to [PT16, Proposition 12]. □

By the description of the obstruction theory of $W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ in terms of flags from [Obe21, Section 4], we can compute its virtual dimension. Indeed, for any two sheaves F_1 and F_2 supported on the zero section of K_E, we have

$$\sum_i (-1)^i \mathrm{ext}^i(F_1,F_2) = \mathrm{ch}(F_1^\vee)\cdot \mathrm{ch}(F_2) = 0, \tag{15}$$

and the C-relative virtual dimension of $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ is therefore 0. Hence the absolute virtual dimension of $\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ is 1.

Both the virtual normal bundle and the cosections are Φ_{a}-equivariant by the construction; hence, they descend to the quotients

$$[\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}].$$

This, in conjunction with Proposition 4.8, implies that the virtual fundamental cycles of quotients $\left[\mathrm{Fl}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}\right]$, when restricted to Quot schemes and to their complements, are of the following corollary:

Corollary 4.9. *We have*

$$\begin{aligned}
&[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}} = \mathsf{B}\boxtimes[\mathrm{pt}] \in H_0(W(\mathsf{a},\mathsf{w})\times C,\mathbb{Q})\,[t],\\
&[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})^c/\Phi_{\mathsf{a}}]^{\mathrm{vir}} = t\mathsf{B}'\boxtimes[\mathrm{pt}] \in H_2(W(\mathsf{a},\mathsf{w})\times C,\mathbb{Q})\,[t],
\end{aligned}$$

4.5 Master space

Let $\epsilon_0\in\mathbb{Q}_{>0}$ be a wall of ϵ-stabilities for quasisections, and let ϵ^+ and ϵ^- be the values close to the wall ϵ_0 from the right-hand side and from the left-hand side, respectively. Consider the master space $MQ^{\epsilon_0}(\mathsf{a},\mathsf{w})$ for the wall-crossing around the wall ϵ_0; we refer to [Zho22, Section 4] for the construction of the master space. Let $\mathsf{w}_0 = 1/\epsilon_0$. By construction, there is a $\mathbb{C}_z^*$-action on $MQ^{\epsilon_0}(\mathsf{a},\mathsf{w})$. In what follows, we use the identification

$$M_E^{\mathrm{rel}}(\mathsf{a}) = \{(G,0)\}\times C = C.$$

Following the terminology of [Zho22], we define $\tilde{Q}^{\epsilon^+}(\mathsf{a},\mathsf{w})$ to be the pull-back of $Q_k^{\epsilon^+}(\mathsf{a},\mathsf{w})$ to the moduli space of entangled semistable degenerations $\tilde{\mathfrak{M}}_{C,k,\mathsf{w}}$, which are constructed in [Zho22, Section 2.2] as a blow-up of the moduli space of weighted semistable degenerations $\mathfrak{M}_{C,k,\mathsf{w}}$,

$$\tilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w}) := Q_k^{\epsilon^+}(\mathsf{a},\mathsf{w})\times_{\mathfrak{M}_{C,k,\mathsf{w}}}\tilde{\mathfrak{M}}_{C,k,\mathsf{w}}.$$

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We also define $\tilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w})'$ to be a k-root stack of $\tilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w})$ associated to the calibration bundle $\mathbb{M}$ of $\tilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w})$, defined in [Zho22, Section 2.8]. By the analysis of [Zho22, Section 6], the $\mathbb{C}_z^*$-fixed locus of $MQ^{\epsilon_0}(\mathsf{a},\mathsf{w})$ then has the following form,

$$MQ^{\epsilon_0}(\mathsf{a},\mathsf{w})^{\mathbb{C}_z^*} = \widetilde{Q}^{\epsilon^+}(\mathsf{a},\mathsf{w}) \cup Q^{\epsilon^-}(\mathsf{a},\mathsf{w}) \cup \coprod_k \left(\widetilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w}_1)' \times_{C^k} W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}_0)^k\right), \tag{16}$$

such that $\mathsf{w} = \mathsf{w}_1 + k\mathsf{w}_0$.

The group Φ_{a} acts on the master space $MQ^{\epsilon_0}(\mathsf{a},\mathsf{w})$. Since the action of Φ_{a} and $\mathbb{C}_z^*$ on $MQ^{\epsilon_0}(\mathsf{a},\mathsf{w})$ commute, the operations of taking the quotient by Φ_{a} and taking the $\mathbb{C}_z^*$-fixed locus also commute. We therefore obtain

$$\left[MQ^{\epsilon_0}(\mathsf{a},\mathsf{w})^{\mathbb{C}_z^*}/\Phi_{\mathsf{a}}\right] = \widetilde{Q}^{\epsilon^+}(\mathsf{a},\mathsf{w})^\bullet \cup Q^{\epsilon^-}(\mathsf{a},\mathsf{w})^\bullet \cup \coprod_k \left(\left[\widetilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w}_1)' \times_{C^k} W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}_0)^k/\Phi_{\mathsf{a}}\right]\right), \tag{17}$$

such that the action on the wall-crossing components (the components on the right-hand side in the expression above) is given by the diagonal action of Φ_{a}. By [Zho22, Section 6], the wall-crossing formula is obtained by taking residues of the localisation formula associated with (17). Let $\mathcal{N}^{\mathrm{vir}}$ be the virtual normal bundle of wall-crossing components. The wall-crossing invariants are therefore given by the following residues:

$$\mathrm{Res}_{z=0}\left(\frac{\left[\tilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w}_1)' \times_{C^k} W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}_0)^k/\Phi_{\mathsf{a}}\right]^{\mathrm{vir}}}{e_{\mathbb{C}_{z,t}^*}(\mathcal{N}^{\mathrm{vir}})}\right).$$

We now show that most of the wall-crossing invariants essentially vanish by the second-cosection argument, except that our cosections are twisted, as explained in Section 4.4, which forces us to work a bit harder to obtain the vanishing.

Theorem 4.10. *If* $\epsilon_0 = 1/\mathsf{w}$, *then*

$$\deg\left[Q^{\epsilon^+}(\mathsf{a},\mathsf{w})^\bullet\right]^{\mathrm{vir}} - \deg[Q^{\epsilon^-}(\mathsf{a},\mathsf{w})^\bullet]^{\mathrm{vir}} = \deg\,\mathrm{Res}_{z=0}\left(\frac{[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}{e_{\mathbb{C}_{z,t}^*}(\mathcal{N}^{\mathrm{vir}})}\right).$$

Otherwise,

$$\deg\left[Q^{\epsilon^+}(\mathsf{a},\mathsf{w})^\bullet\right]^{\mathrm{vir}} = \deg[Q^{\epsilon^-}(\mathsf{a},\mathsf{w})^\bullet]^{\mathrm{vir}}.$$

4.6 Proof of Theorem 4.10

By [Zho22, Section 6], we have to analyse the wall-crossing components in the decomposition (21); see also [Nes21, Section 6], [Nes23, Section 10] and [LW23].

Assuming that $k \geqslant 2$ or $\mathsf{w}_1 \neq 0$, then the space

$$X := \left[\tilde{Q}_k^{\epsilon^+}(\mathsf{a},\mathsf{w}_1)' \times_{C^k} W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}_0)^k/\Phi_{\mathsf{a}}\right]$$

has an additional action of Φ_{a} coming from any of the components of the product. To distinguish it from the diagonal action of Φ_{a}, we denote it by Φ'_{a}. The obstruction theory of the quotient $[X/\Phi'_{\mathsf{a}}]$ is compatible with the obstruction theory of X; hence,

$$\pi^*[X/\Phi'_{\mathsf{a}}]^{\mathrm{vir}} = [X]^{\mathrm{vir}}$$

for the quotient map $\pi: X \to [X/\Phi'_{\mathsf{a}}]$. Moreover, the virtual normal bundles from [Zho22, Section 6] are Φ'_{a}-equivariant; hence, they descend to the quotient $[X/\Phi'_{\mathsf{a}}]$. Overall, we obtain that the wall-crossing class is a pull-back of some class A from the quotient $[X/\Phi'_{\mathsf{a}}]$,

$$\frac{\left[\tilde{Q}^{\epsilon^+}_k(\mathsf{a},\mathsf{w}_1)' \times_{C^k} W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}_0)^k/\Phi_{\mathsf{a}}\right]}{e_{\mathbb{C}^*_{z,t}}(\mathcal{N}^{\mathrm{vir}})} = \pi^*\mathsf{A};$$

that its degree is therefore 0 and that it does not contribute to the wall-crossing formula,

$$\deg \mathrm{Res}_{z=0}\left(\frac{\left[\tilde{Q}^{\epsilon^+}_k(\mathsf{a},\mathsf{w}_1)' \times_{C^k} W^{\mathrm{rel}}(\mathsf{a},\mathsf{w}_0)^k/\Phi_{\mathsf{a}}\right]}{e_{\mathbb{C}^*_{z,t}}(\mathcal{N}^{\mathrm{vir}})}\right) = 0.$$

It remains for us to determine the contribution of terms

$$\left[\tilde{Q}^{+}_1(\mathsf{a},0)' \times_C W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}\right] = \left[Q^{+}_1(\mathsf{a},0) \times_C W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}\right].$$

Since $Q^+_1(\mathsf{a},0) = C$, we obtain that

$$\left[Q^{+}_1(\mathsf{a},0) \times_C W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}\right] = [W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}].$$

We will now show that the complement of $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \subset W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ does not contribute. Ideally, one would say that this statement follows from the double cosection argument. However, in this case, the cosections are twisted due to the relative setup; hence, one has to do a little bit of additional work. By [Zho22, Lemma 6.5.6] and the dimension constraint, degrees of the following residues are equal,

$$\deg \mathrm{Res}_{z=0}\left(\frac{[W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})}\right) = \deg \mathrm{Res}_{z=0}\left(\frac{[W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*_{z,t}}(\mathcal{N}^{\mathrm{vir}})}\right), \tag{18}$$

where N^{vir} is the normal bundle of $W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ inside $GQ(\mathsf{a},\mathsf{w})$, whose expression is given in Theorem 4.4.

We argue that $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})^c \subset W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ does not contribute because the quantity (18) is a multiple of t^2. As taking quotient by Φ_{a} can be exchanged with taking an insertion, it is enough to show that (18) is a multiple of t^2 before taking quotient. By Corollary 4.6, we know that $[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})^c]^{\mathrm{vir}}$ is a multiple of t, and we therefore have to show that the residue of $e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})^{-1}$ is a multiple of t, too. By Theorem 4.4, the class $e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})^{-1}$ leads to the following expression:

$$e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})^{-1} = \prod_{i\geqslant 1, k\geqslant 1} \frac{e_{\mathbb{C}^*_{z,t}}(R\mathcal{H}om_\pi(\mathcal{F}_{i+k}/\mathcal{F}_{i+k-1},\mathcal{F}_i)\mathbf{z}^k)}{e_{\mathbb{C}^*_{z,t}}(R\mathcal{H}om_\pi(\mathcal{F}_{i+k+1}/\mathcal{F}_{i+k},\mathcal{F}_i)^\vee\mathbf{z}^{-k})}. \tag{19}$$

According to (15), the rank of $R\mathcal{H}om_\pi(\mathcal{F}_{i+k}/\mathcal{F}_{i+k-1},\mathcal{F}_i)$ is 0; hence,

$$\begin{aligned} e_{\mathbb{C}^*_{z,t}}(R\mathcal{H}om_\pi(\mathcal{F}_{i+k}/\mathcal{F}_{i+k-1},\mathcal{F}_i)\otimes \mathbf{z}^k) &= \sum_{j\geqslant 0}(kz)^{-j}\mathrm{c}_j(R\mathcal{H}om_\pi(\mathcal{F}_{i+k}/\mathcal{F}_{i+k-1},\mathcal{F}_i)) \\ &= 1-(kz)^{-1}\mathrm{c}_1(R\mathcal{H}om_\pi(\mathcal{F}_{i+k+1}/\mathcal{F}_{i+k},\mathcal{F}_i)) + O(z^{-2}), \end{aligned}$$

the same applies to the denominator of (19). We therefore obtain that

$$\begin{aligned} e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})^{-1} = 1 &- \sum_{i,k}(kz)^{-1}\mathrm{c}_1(R\mathcal{H}om_\pi(\mathcal{F}_{i+k}/\mathcal{F}_{i+k-1},F_i)) \\ &- \sum_{i,k}(kz)^{-1}\mathrm{c}_1(R\mathcal{H}om_\pi(\mathcal{F}_{i+k+1}/\mathcal{F}_{i+k},\mathcal{F}_i))^\vee + O(z^{-2}). \end{aligned} \tag{20}$$

By Lemma 4.5, we obtain that

$$\mathrm{c}_1(R\mathcal{H}om_\pi(\mathcal{F}_j/\mathcal{F}_{j-1},\mathcal{F}_i) = \mathsf{A}t + \mathsf{A}\cdot \mathrm{c}_1(\omega_C) \in H^2\left(\mathrm{Quot}^c,\mathbb{Q}\right),$$

for some class A of cohomological degree 0. Using Corollary 4.6 and (20), we conclude that

$$\mathrm{Res}_{z=0}\left(\frac{[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})^c]^{\mathrm{vir}}}{e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})}\right) = \mathsf{A}'t^2 \in H_2(\mathrm{Quot}^c,\mathbb{Q})\,[t],$$

for some class A' of homological degree 2. Taking the degree of the class above, we obtain 0. This finishes the proof of Theorem 4.10.

4.7 Contributions from Quot schemes

We now have to determine the contributions of $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \subset W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$. Firstly, by [Obe21, Section 4], the component $\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) \subset W^{\mathrm{rel}}(\mathsf{a},\mathsf{w})$ is composed of the following Quot schemes

$$\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w}) = \coprod_{\mathsf{m}|\mathsf{w}} \mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},u_{\mathsf{m}}), \tag{21}$$

and the classes u_{m} are defined as follows:

$$u_{\mathsf{m}} := h_{\mathsf{m}}\mathsf{v} - \frac{\check{\mathsf{w}}}{\mathsf{m}},$$

where $\check{\mathsf{w}}$ is given by (7) and h_{m} is the unique integer such that

$$h_{\mathsf{m}}\mathsf{r} - \frac{\check{\mathsf{w}}_1}{\mathsf{m}} \in [0,\mathsf{r}-1].$$

We therefore obtain the following proposition:

Proposition 4.11. *We have*

$$[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}} = \sum_{\mathsf{m}|\mathsf{w}} [\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},u_{\mathsf{m}})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}.$$

Proof. See [Obe21, Section 4]. □

Let us analyse N^{vir} over each component $\left[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},u_{\mathrm{m}})/\Phi_{\mathsf{a}}\right]$. In what follows, we use the notation from Section 5. By [Obe21, Section 4.4], the equivariant Euler class of the virtual normal bundle N^{vir} can be expressed as follows:

$$\begin{aligned} e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})^{-1} &= e_{\mathbb{C}^*_{z,t}}(R\mathcal{H}om_\pi(\mathcal{Q}^{\mathrm{rel}},\mathcal{K}^{\mathrm{rel}})\mathbf{z}^{\mathsf{m}}) \\ &= e_{\mathbb{C}^*_{z,t}}(R\mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}},\mathcal{Q}^{\mathrm{rel}})^\vee\mathbf{t}\mathbf{z}^{\mathsf{m}}) \\ &= \sum_{k\in\mathbb{Z}} (\mathsf{m}z)^{-k}\mathrm{c}_k(R\mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}},\mathcal{Q}^{\mathrm{rel}})^\vee\mathbf{t}), \end{aligned}$$

where $\mathcal{K}^{\mathrm{rel}}$ and $\mathcal{Q}^{\mathrm{rel}}$ are as in Section 4.2. We are interested in the residue of $e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})^{-1}$,

$$\begin{aligned} \mathrm{Res}_{z=0}(e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})^{-1}) &= \mathsf{m}^{-1}\mathrm{c}_1(R\mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}},\mathcal{Q}^{\mathrm{rel}})^\vee\mathbf{t}) \\ &= \mathsf{m}^{-1}\mathrm{rk}(\mathcal{H}om_\pi(\mathcal{K}^{\mathrm{rel}},\mathcal{Q}^{\mathrm{rel}})^\vee)(\mathrm{c}_1(\omega_C)+t) \\ &= \mathsf{m}^{-1}\dim(\mathrm{Quot}(\mathsf{a},u_{\mathsf{m}}))\left(\mathrm{c}_1(\omega_C)+t\right). \end{aligned}$$

Using Corollary 4.9, the total residue then takes the following form:

$$\operatorname{Res}_{z=0}\left(\frac{[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},u_{\mathsf{m}})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*_{z,t}}(N^{\mathrm{vir}})}\right)=\mathsf{m}^{-1}\dim(\mathrm{Quot}(\mathsf{a},u_{\mathsf{m}}))\,[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},u_{\mathsf{m}})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}t.$$

Assume that $(\mathsf{r},\mathsf{a})=(2,1)$; using the analysis from Corollaries 5.3 and 5.4, we get

$$\deg\ \operatorname{Res}_{z=0}\left(\frac{[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},u_{\mathsf{m}})/\Phi_{\mathsf{a}}]^{\mathrm{vir}}}{e_{\mathbb{C}^*}(N^{\mathrm{vir}})}\right)=(2g-2)\mathsf{m}^{-1}t.$$

Now, applying Theorem 4.10 repeatedly and using the fact that $Q^{+}(\mathsf{a},\mathsf{w})$ is empty for $\mathsf{w}\neq 0$, we obtain the following result (for how d and w are related, see (7)):

Theorem 4.12. *If* $(\mathsf{r},\mathsf{a})=(2,1)$, *then*

$$\mathsf{QM}(E)^{\mathsf{d},\bullet}_{1,\mathsf{w}}=\begin{cases}(2g-2)\sum_{\mathsf{m}|\mathsf{w}}\mathsf{m}^{-1}t, & \text{if } \mathsf{w}=\mathsf{d} \bmod 2\\ 0, & \text{otherwise.}\end{cases}$$

Using Corollary 3.7, we obtain the desired quasimap invariants.

Theorem 4.13. *If* $(\mathsf{r},\mathsf{a})=(2,1)$, *then*

$$\mathsf{QM}(C)^{1,\bullet}_{\mathsf{d},\mathsf{w}}=\begin{cases}(2g-2)2^{2g}\sum_{\mathsf{m}|\mathsf{w}}\mathsf{m}^{-1}t, & \text{if } \mathsf{w}=\mathsf{d} \bmod 2\\ 0, & \text{otherwise.}\end{cases}$$

This gives us Theorems 1.1 and 1.2 (after passing to reduced invariants; that is, after dividing by t).

Remark 4.14. Since $M_E(1)=\{(G,0)\}$, invariants $\mathsf{QM}(C)^{1,\bullet}_{\mathsf{d},\mathsf{w}}$ have only instanton contributions; that is, on C they correspond to invariants of moduli spaces of stable bundles $T^*N_C(\mathsf{d})\subset M_C(\mathsf{d})$. On the other hand, the even-degree invariants are completely monopole; that is, they correspond to invariants of the complement of $N_C(\mathsf{d})$ in the nilpotent cone. This was expected from [MM21]; see also [Nes23, Remark 7.2].

Remark 4.15. Let us now comment on the fact that genus 1 Gromov–Witten invariants of E have very similar expressions, as was mentioned in the introduction. If $\gcd(\mathsf{r},\mathsf{d})=1$, then a moduli space of stable sheaves on E is naturally isomorphic to E via the determinant map. Hence, Gromov–Witten theory of E is equivalent to one of its moduli spaces of sheaves. Here, we study a kind of twisted Gromov–Witten theory of moduli spaces of sheaves on E. Hence, (posteriori) it is not so unexpected that we get similar answers. Perhaps this phenomenon can be made even precise.

4.8 Higher rank

By (7), if we assume that all divisors m of w are congruent to 0 or a modulo r, then

$$u_{\mathsf{m}}=(0,\mathsf{k})\ \text{ or }\ (\mathsf{r}-1,\mathsf{k})\,;$$

therefore, using the arguments of the previous section, the analysis of Section 5.2 is enough to conclude the following:

Proposition 4.16. *If* r *is prime,* $\mathsf{a}\neq 0$ *and all divisors* m *of* w *satisfy*

$$\mathsf{m}=0 \text{ or } \mathsf{a} \bmod \mathsf{r},$$

then

$$\mathsf{QM}(C)^{\mathsf{a},\bullet}_{\mathsf{d},\mathsf{w}} = \begin{cases} (2g-2)\mathsf{r}^{2g}\sum_{\mathsf{m}|\mathsf{w}} \mathsf{m}^{-1}t, & \text{if } \mathsf{w}=\mathsf{d}\cdot\mathsf{a} \bmod \mathsf{r} \\ 0, & \text{otherwise.}\end{cases}$$

This agrees with [Nes23, Conjecture E]. Our methods involving Quot schemes lead to an obvious conjectural extension of Proposition 4.16:

Conjecture 4.17. *If* $\gcd(\mathsf{r},\mathsf{a})=1$, *then*

$$\mathsf{QM}(C)^{\mathsf{d},\bullet}_{\mathsf{a},\mathsf{w}} = \begin{cases} (2g-2)\mathsf{r}^{2g}\sum_{\mathsf{m}|\mathsf{w}} \mathsf{m}^{-1}t, & \text{if } \mathsf{w}=\mathsf{d}\cdot\mathsf{a} \bmod \mathsf{r} \\ 0, & \text{otherwise.}\end{cases}$$

5. Quot schemes

5.1 Group actions on Quot schemes

The group Φ_{a} acts naturally on Quot schemes $\mathrm{Quot}(\mathsf{a},\mathsf{w})$. The stabilisers of the action are finite, as long as $\mathsf{w}\neq 0$. The quotient stack $[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]$ is therefore a Deligne–Mumford stack. Taking the quotient respects the identification (12),

$$[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}] = [\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]\times C.$$

The obstruction theory of $\left[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}\right]$ is a descent of the obstruction theory of $\left[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})\right]$. By Lemma 4.6, the C-relative obstruction bundle of the quotient is therefore given by the descent of $T_{\mathrm{Quot}(\mathsf{a},\mathsf{w})}\boxtimes\omega_C\mathbf{t}$. More precisely, let

$$q:\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})\to[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]$$

be the quotient map; then we have the Φ_{a}-equivariant identification

$$q^*\mathrm{Ob}_{[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\cong\mathrm{Ob}_{\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})},$$

such that $\mathrm{Ob}_{\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})}$ is Φ_{a}-equivariantly isomorphic to $T_{\mathrm{Quot}(\mathsf{a},\mathsf{w})}\boxtimes\omega_C\mathbf{t}$. We will denote the descent of $T_{\mathrm{Quot}(\mathsf{a},\mathsf{w})}$ to $[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]$ by the same symbol. Hence, by Lemma 4.6, we obtain the following corollary:

Corollary 5.1. *There is a natural identification on* $\left[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}\right]$,

$$\mathrm{Ob}_{[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\cong T_{\mathrm{Quot}(\mathsf{a},\mathsf{w})}\boxtimes\omega_C\mathbf{t}.$$

We are now ready to determine the virtual degree.

Proposition 5.2. *For any* r *and* a, *we have*

$$\deg[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}} = (2g-2)e\big(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\big)\,.$$

Proof. Considering the map

$$[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]\to[\mathrm{pt}/\Phi_{\mathsf{a}}],$$

the associated sequence of tangent complexes takes the following form:

$$0 \to T_{\Phi_{\mathsf{a}}} \to T_{\mathrm{Quot}(\mathsf{a},\mathsf{w})} \to T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]} \to 0.$$

Since $T_{\Phi_{\mathsf{a}}}$ is trivial, we obtain that

$$e\left(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})]}\right) = 0,$$
$$\mathrm{c}_{\mathsf{rk}-1}\left(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})]}\right) = e\left(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\right).$$

Using Corollary 5.1, we therefore obtain

$$\begin{aligned}[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}} &= e_{\mathbb{C}_t^*}\left(\mathrm{Ob}_{[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\right) \\ &= e\left(T_{\mathrm{Quot}(\mathsf{a},\mathsf{w})}\right) + \mathrm{c}_{\mathrm{rk}-1}\left(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})]}\right)\cdot(\mathrm{c}_1(\omega_C)-t)+\ldots \\ &= e\left(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\right)\cdot(\mathrm{c}_1(\omega_C)-t)+\ldots.\end{aligned}$$

Taking the degree, we arrive at the statement of the proposition,

$$\deg[\mathrm{Quot}^{\mathrm{rel}}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]^{\mathrm{vir}} = (2g-2)e\left(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\right).$$

□

We now have to compute $e\left(T_{[\mathrm{Quot}(\mathsf{a},\mathsf{w})/\Phi_{\mathsf{a}}]}\right)$. The analysis might be split, depending on the Chern character of quotients in the decomposition (21).

5.2 Relevant Quot schemes

For this section we assume that $\mathsf{a} \neq 0$. Consider firstly the class

$$u = (\mathsf{r}-1, \mathsf{k}).$$

Let

$$\mathsf{dim} := \dim(\mathrm{Quot}(\mathsf{a},u)) = \mathsf{r}\cdot(\mathsf{k}-\mathsf{a})+\mathsf{a};$$

then $\mathrm{Quot}(\mathsf{a},u)$ is a $\mathbb{P}^{\mathsf{dim}-1}$-bundle over $\mathrm{Jac}(E)$ given by the natural projection

$$\mathrm{Quot}(\mathsf{a},u) \to \mathrm{Pic}(E)$$
$$[K \hookrightarrow G \twoheadrightarrow Q] \mapsto K.$$

A fiber

$$\mathbb{P}^{\mathsf{dim}-1} \hookrightarrow \mathrm{Quot}(\mathsf{a},u)$$

is a slice of the Φ_{a}-action on $\mathrm{Quot}(\mathsf{a},u)$. In other words, let $\Gamma_{\mathsf{k}} \subset \Phi_{\mathsf{a}}$ be a finite subgroup that fixes $\mathbb{P}^{\mathsf{dim}-1}$, then we have the following diagram:

$$\begin{array}{ccc}\mathbb{P}^{\mathsf{dim}-1} & \hookrightarrow & \mathrm{Quot}(\mathsf{a},u) \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_2} \\ \left[\mathbb{P}^{\mathsf{dim}-1}/\Gamma_{\mathsf{k}}\right] & \xrightarrow{\cong} & [\mathrm{Quot}(\mathsf{a},u)/\Phi_{\mathsf{a}}]\end{array}$$

The diagram above gives us that

$$e\left(T_{[\mathrm{Quot}(\mathsf{a},u)/\Phi_{\mathsf{a}}]}\right) = \frac{\chi\left(\mathbb{P}^{\mathsf{dim}-1}\right)}{|\Gamma_{\mathsf{k}}|} = \frac{\mathsf{dim}}{|\Gamma_{\mathsf{k}}|}. \tag{22}$$

It therefore remains to determine Γ_{k}. The subgroup $\Gamma_{\mathsf{k}} \subset \Phi_{\mathsf{a}}$ that fixes the slice is exactly the subgroup that fixes a line bundle $\mathcal{O}(\mathsf{a}-\mathsf{k})$ of degree $\mathsf{a}-\mathsf{k}$. A translation of $\mathcal{O}(\mathsf{a}-\mathsf{k})$ by τ_p that is associated to a point $p \in E$ can be described as follows:

$$\tau_p^* \mathcal{O}(\mathsf{a}-\mathsf{k}) = \mathcal{O}(\mathsf{a}-\mathsf{k}) \otimes L_p^{\mathsf{a}-\mathsf{k}},$$

where L_p is line bundle corresponding to p under the natural identification

$$\begin{aligned} E &\xrightarrow{\cong} \operatorname{Jac}(E) \\ p &\mapsto \mathcal{O}(0_E) \otimes \mathcal{O}(p)^{-1} =: L_p. \end{aligned}$$

By the definition of Φ_{a} from (2), determining the stabiliser of $\mathcal{O}(\mathsf{a}-\mathsf{k})$ in Φ_{a} therefore amounts to finding pairs

$$\left(p, L_p^{\frac{\mathsf{a}}{\mathsf{r}}}\right) \in E \times \operatorname{Jac}(E)$$

such that

$$L_p^{\mathsf{a}-\mathsf{k}} \otimes L_p^{-\frac{\mathsf{a}}{\mathsf{r}}} = \mathcal{O}_E. \tag{23}$$

Raising the expression to the power of r, we conclude that $L_p^{\mathsf{dim}} = \mathcal{O}_E$. As a group, E is isomorphic to $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$; hence, with respect to the identification $E \cong \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$, we obtain that

$$L_p = \left(\frac{n_1}{\mathsf{dim}}, \frac{n_2}{\mathsf{dim}}\right) \in \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z},$$

and that r-roots of $a_i = \frac{n_i}{\mathsf{dim}}$ are given by the following elements:

$$a_i^{\frac{1}{\mathsf{r}}} = \frac{n_i + h\mathsf{dim}}{\mathsf{rdim}}, \quad h \in \{0, \dots, \mathsf{r}-1\}.$$

However, only one satisfies the equation

$$\frac{(\mathsf{a}-\mathsf{k})n_i}{\mathsf{dim}} - \frac{\mathsf{a}(n_i + h\mathsf{dim})}{\mathsf{rdim}} = 0 \in \frac{\mathbb{R}}{\mathbb{Z}},$$

more specifically, h is uniquely defined by the following equation:

$$\frac{(n_i + h\mathsf{a})\,\mathsf{dim}}{\mathsf{rdim}} = 0 \in \mathbb{R}/\mathbb{Z}.$$

We therefore conclude that for all $p \in E$, there exists a unique root $L_p^{\frac{\mathsf{a}}{\mathsf{r}}}$ that satisfies the equation (23) and that it must be dim-torsion; hence p is also dim-torsion. We therefore obtain that

$$\Gamma_{\mathsf{k}} = E[\mathsf{dim}] \cong \mathbb{Z}_{\mathsf{dim}}^{\oplus 2},$$

or, in particular, that

$$|\Gamma_{\mathsf{k}}| = \mathsf{dim}^2.$$

Proposition 5.2 and (22) give us the following:

Corollary 5.3. *If* $u = (\mathsf{r}-1, \mathsf{k})$, *then*

$$\deg([\operatorname{Quot}(\mathsf{a}, u)^{\mathrm{rel}}/\Phi_{\mathsf{a}}]^{\mathrm{vir}}) = (2g-2)\dim(\operatorname{Quot}(\mathsf{a}, u))^{-1}.$$

Consider now the class

$$u = (0, \mathsf{k}).$$

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In this case,

$$\dim(\mathrm{Quot}(\mathsf{a},u)) = \mathsf{r}\cdot\mathsf{k},$$

and $\mathrm{Quot}(\mathsf{a},u)$ admits the natural projection

$$\begin{aligned}\mathrm{Quot}(\mathsf{a},u) &\to \mathrm{Pic}(E)\\ [G \twoheadrightarrow Q] &\mapsto \det(Q),\end{aligned}$$

which provides a slice of the action of Φ_{a} on $\mathrm{Quot}(\mathsf{a},u)$. More specifically, let

$$\mathrm{Quot}(\mathsf{a},u)_0 \hookrightarrow \mathrm{Quot}(\mathsf{a},u)$$

be the fibre of the projection, and let $\Gamma_{\mathsf{k}} \subset \Phi_{\mathsf{a}}$ be the stabiliser of $\mathrm{Quot}(\mathrm{a},u)_0$. We obtain that

$$e\big(T_{[\mathrm{Quot}(\mathsf{a},u)/\Phi_{\mathsf{a}}]}\big) = \frac{\chi(\mathrm{Quot}(\mathsf{a},u)_0)}{|\Gamma_{\mathsf{k}}|}. \tag{24}$$

The subgroup $\Gamma_{\mathsf{k}} \in \Phi_{\mathsf{a}}$ that fixes the slice is exactly the subgroup that fixes the determinant of a 0-dimension sheaf of degree k; this means that it consists of pairs

$$\left(p, L_p^{\frac{\mathsf{a}}{\mathsf{r}}}\right) \in E \times \mathrm{Jac}(E),$$

such that

$$L_p^{\mathsf{k}} = \mathcal{O}_E,$$

hence,

$$|\Gamma_{\mathsf{k}}| = \mathsf{r}^2\cdot\mathsf{k}^2.$$

Let us now determine the Euler characteristics of $\mathrm{Quot}(\mathsf{a},u)_0$. Firstly, any vector bundle on a curve can be deformed to a direct sum of line bundles. By the deformation invariance of the (virtual) Euler characteristics, we can assume that $G = \oplus_{i=1}^{i=\mathsf{r}} L_i$. In this case, $\mathrm{Quot}(\mathsf{a},u)_0$ admits a torus-action of $T = \prod_{i=1}^{i=\mathsf{r}}\mathbb{C}^*$ acting by scaling line bundles. The associated fixed locus has the following description:

$$\mathrm{Quot}(\mathsf{a},u)_0^T = \coprod_{u_1+\cdots+u_{\mathsf{r}}=u}\left(\prod_{i=1}^{i=\mathsf{r}}\mathrm{Quot}(L_i,u_i)\right)_0,$$

we refer to [MR22, Section 3] for more details in case of usual Quot schemes, which extend in a straightforward manner to our slices. Let us now analyse Euler characteristics of the components in the decomposition above. Firstly, if at least two class u_k and u_j are nonzero, then $\left(\prod_i \mathrm{Quot}(L_i,u_i)\right)_0$ admits an extra fixed-point-free action of E; therefore,

$$\chi\left(\left(\prod_{i=1}^{i=\mathsf{r}}\mathrm{Quot}(L_i,u_i)\right)_0\right) = 0, \qquad \text{if } u_k \neq 0 \text{ and } u_j \neq 0.$$

If only one class u_j is nonzero, then

$$\left(\prod_{i=1}^{i=\mathsf{r}}\mathrm{Quot}(L_i,u_i)\right)_0 = \mathrm{Quot}(L_j,u_j)_0 = \mathbb{P}^{\mathsf{k}-1}.$$

We therefore obtain

$$\chi\left(\left(\prod_{i=1}^{i=\mathsf{r}}\mathrm{Quot}(L_i,u_i)\right)_0\right) = \mathsf{k}, \qquad \text{if only one } u_j \neq 0.$$

Overall,

$$\chi(\mathrm{Quot}(\mathsf{a},u)_0) = e(\mathrm{Quot}(\mathsf{a},u)_0^T) = \mathsf{r}\cdot\mathsf{k}. \tag{25}$$

Combining Proposition 5.2, (24) and (25), we obtain the following:

Corollary 5.4. *If* $u = (0,\mathsf{k})$, *then*

$$\deg([\mathrm{Quot}(\mathsf{a},u)^{\mathrm{rel}}/\Phi_{\mathsf{a}}]^{\mathrm{vir}}) = (2g-2)\dim(\mathrm{Quot}(\mathsf{a},u))^{-1}.$$

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Conflicts of Interest

None.

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