# Mirror Symmetry for Dubrovin–Zhang Frobenius Manifolds

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**Abstract**

Frobenius manifolds were formally defined by Boris Dubrovin in the early 1990s [41], and serve as a bridge between a priori very different fields of mathematics such as integrable systems theory, enumerative geometry, singularity theory, and mathematical physics. This thesis concerns, in particular, a specific class of Frobenius manifolds constructed on the orbit space of an extension of the affine Weyl group defined by Dubrovin together with Youjin Zhang in [46]. Here, Landau-Ginzburg superpotentials, or B–model mirrors, are found for these Frobenius structures by considering the characteristic equation for Lax operators of relativistic/periodic Toda chains as proposed by Andrea Brini in [17]. As a bonus, the results open up various applications in topology, integrable hierarchies, and Gromov–Witten theory, making interesting research questions in these areas more accessible. Some such applications are considered in this thesis. The form of the determinant of the Saito metric on discriminant strata is investigated, applications to the combinatorics of Lyashko-Looijenga maps are given, and investigations into the integrable systems theoretic and enumerative geometric applications are commenced.

# Declaration

I confirm that the work contained in this project is my own work unless otherwise stated.

<div align="right">Karoline van Gemst</div>

# Disclaimer

The results of Chapter 5 were obtained jointly with Dr. Andrea Brini and formed parts of the paper *Mirror Symmetry for affine Weyl groups* published in Journal de l'École Polytechnique, Vol. 9, 2022. Thus, while a substantial part of the content of this chapter has been derived, calculated and written down by me under the guidence of Dr. Andrea Brini, some proofs and technical details are largely due to Dr. Brini. This is the case for the proof that the character relations for exceptional Dynkin types are solvable (Definition 5.3.1 – Corollary 5.3.3), and the discussion concerning the failure of the $\mu$-foliation to be transverse to fibres of the universal curve in Section 5.2.

# Acknowledgments

First and foremost, I want to thank my supervisor, Andrea Brini, for his guidance, many (sometimes heated) discussions and for sharing his immense knowledge with me, which I hope to retain a fraction of. Secondly, I will always be indebted to the brilliant Marta Mazzocco who first took me on as a PhD student.

Thirdly, I want to show my appreciation for my parents, who throughout my life have supported me without exception, and my best friend and little sister, Marion, who have asked me countless questions and tried hard to understand what I do. I am also especially grateful to Callan Corbett for his patience and for bringing calmness to my chaos.

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Til Mormor.

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# INTRODUCTION

## 1.1 A tale of three constructions

Frobenius manifolds were first formally defined by B. Dubrovin in 1994, [41], as a coordinate free formulation of the famous Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations of two-dimensional topological field theory. Since then, they have been a key point of confluence of very different mathematical areas such as algebraic geometry, singularity theory, quantum field theory, and the theory of integrable systems. In algebraic geometry, they serve as a model for the quantum cohomology (genus zero Gromov–Witten theory) of smooth projective varieties; in singularity theory, they encode the existence of pencils of flat pairings on the base of the miniversal deformations of hypersurface singularities; in physics, they codify the associativity of the chiral ring of topologically twisted $\mathcal{N} = (2,2)$ supersymmetric field theories in two dimensions; and in the theory of integrable systems, they provide a loop–space formulation of hydrodynamic bihamiltonian integrable hierarchies in 1+1 dimensions.

Even though Frobenius manifolds arise in such different contexts they mainly comprise of three types, or rather three descriptions*, which is closely connected to the concept of *mirror symmetry*.

The three main constructions are as follows.

1. The *quantum cohomology* of, say, a smooth complex variety can be endowed with the structure of a Frobenius manifold.

2. There is also a way to construct a Frobenius manifold from ramified covers of the Riemann sphere, called a Landau–Ginzburg (LG) model.

3. Finally, given a Coxeter group, $G$, acting on its representation space, $V$, one can equip the orbit space $V/G$ with the structure of a Frobenius manifold.

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*A Frobenius manifold can have all three descriptions!

quantum cohomology can be understood as a deformation of the usual cohomology theory. It is intimately connected to enumerative algebraic geometry. In fact it is the genus 0 part of Gromov–Witten theory, which is a modern curve counting theory usually referred to as the *A–model* in mirror symmetry. People working in mirror symmetry are often concerned with finding a mirror to this A–model, called a *B–model*. In the context of Frobenius manifolds, the B–model is a description of the second type, such that the two resulting Frobenius manifolds are isomorphic.

Sometimes we may even have a "triangle" of isomorphisms between all the three main types of Frobenius manifolds;
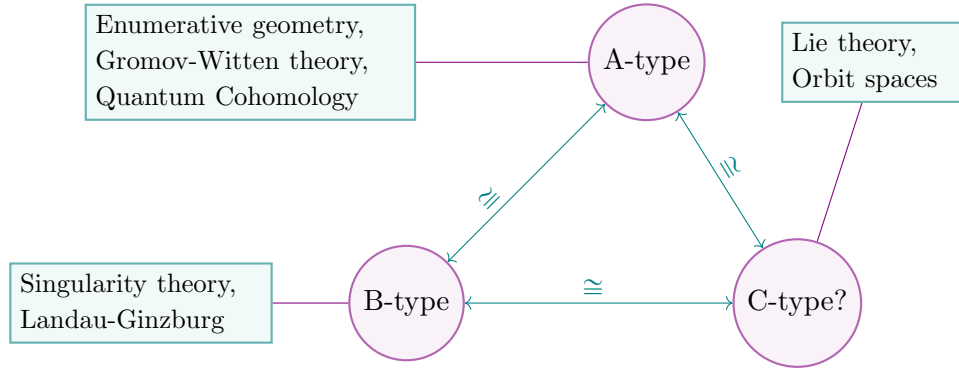


Figure 1.1: Frobenius mirror symmetry.

Mirror symmetry has many different descriptions, some of which are described briefly in Section 5.4. In this thesis, however, mirror symmetry will be synonymous with an isomorphism (as Frobenius manifolds) between any pair of the three types mentioned above (or some generalisation of these).

One of the main reasons researchers search for the B–model in particular is because it is often a lot easier to work with, and makes research questions in the area of Frobenius manifolds (and beyond) far more accessible. Therefore, a typical situation is as follows. Suppose you are given a Frobenius manifold, $N$, of type $M_{\text{Q-coh}}$ or $M_{V/G}$,

**can you find an LG–model such that $M_{\mathbf{LG}} \cong N$?[†]**

It can be extremely difficult to find such a description, and it is certainly not unique, but a small comfort when dealing with Frobenius manifolds is that we at least can rely on one existing[‡].

On top of the physics-inspired $A$– and B–models, an interesting source of Frobenius manifolds is the third construction; Frobenius manifolds from orbit spaces. In [74], Claus Hertling proved that there is a bijection between polynomial solutions of the aforementioned WDVV equations and the

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[†]Isomorphic as Frobenius manifolds.
[‡]Under some mild conditions, as proven by Dubrovin [40].

third construction. In the case when the group $G$ is the Weyl group of a simple Lie algebra of simply laced Dynkin type $\mathcal{R}$, then, famously, the B–side mirror is the miniversal deformation of an isolated singularity of type $\mathcal{R}$.

The main theme of this thesis is Frobenius manifold mirror symmetry and its consequences.

**Remark.** It should be noted, for completion, that Frobenius manifold mirror symmetry is only one formulation of mirror symmetry, among many. What is called mirror symmetry is a more general way of relating symplectic and algebraic geometry. It was discovered by physicists in the early 1990s [22, 23, 73], that one has a choice when constructing string theories. Very roughly, one can do this in two different ways resulting in two very different manifolds. Mirror symmetry in this context means that even though the two manifolds are so different, the physics (the set of observables), is in fact the same. If this is the case the two manifolds are called *mirror* to each other, and each is better suited than the other in various contexts. In particular, mirror symmetry relates the complex structure on one of the manifolds, called the *B–model* to the Kähler structure of the other manifold, the *A–model* [122].

On the other end of the spectrum of abstraction we find the Homological Mirror Symmetry Conjecture [86], which was first posed by Maxim Kontsevich at the 1994 International Congress of Mathematicians in Zürich. This is a way to formulate mirror symmetry in terms of an equivalence of derived categories, and was according to Kontsevich the appropriate way to describe the physics phenomenon in terms of rigorous pure mathematics. More specifically, the conjecture states that given a smooth variety, $X$, the bounded derived category of coherent sheaves on $X$ is equivalent to the derived Fukaya category[§] of its mirror, $Y$:

$$D^b(Coh(X)) \cong DFuk(Y). \tag{1.1.1}$$

Here, the right-hand side is the symplectic geometry perspective, while the left-hand side the algebraic one. On one hand, the physics formulation lends itself well to do explicit computations while not being very rigorous. On the other hand, the homological mirror symmetry formulation is very elegant, but not straight-forward to use. Thus, there have been many constructions, suitable for different circumstances and under a variety of restrictions, sitting in a sense between the two; where we still have mathematical rigour and some level of elegance, but which are more accessible with respect to proofs and computations, compared to homological mirror symmetry. Examples of these include Batyrev–Borisov mirror symmetry (a duality among lattice polytopes realises mirror symmetry for Calabi–Yau hypersurface in Fano toric varieties [6, 11]), Hori–Vafa mirror symmetry (which assigns to a Fano variety or Calabi–Yau hypersurfaces, as gauged linear sigma models, a Landau–Ginzburg model (B–model mirror) [79]), and Berglund–Hübsch–Krawitz

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[§]The Fukaya category of $X$ is an $A^\infty$–category whose objects are Lagrangian submanifolds of $X$ and morphisms are Floer chain groups [114].

mirror symmetry (relating a Landau–Ginzburg model to another, mirror, Landau–Ginzburgh model [9, 87]), and its geometric intepretation by Chiodo and Ruan [30]. There are many overlaps between these constructions and connections to Frobenius manifolds, for instance related by Picard–Fuchs equations satisfied by mirror periods (in particular for Calabi–Yau hypersurfaces in weighted projective spaces see [38]), which in the Frobenius context arise from compatibility of the *deformed* connection (which we shall describe in Chapter 2). The Frobenius manifolds perspective, however, encompasses many of these constructions, as is emphasised by our treatment of (non-toric) Fano orbifolds in Chapter 9.

Thus, while all these formulations and constructions are extremely fascinating in their own right, and the connections with Frobenius manifolds worth investigating, we will not touch on any of these in this thesis. We will define mirror symmetry to be Frobenius manifold mirror symmetry, and make few mentions of the plethora of mirror symmetry constructions available.

## 1.2 A bigger picture

Figure 1.2 shows interconnections in and adjacent to the realm of Frobenius manifolds*, which will be referred to throughout this thesis.
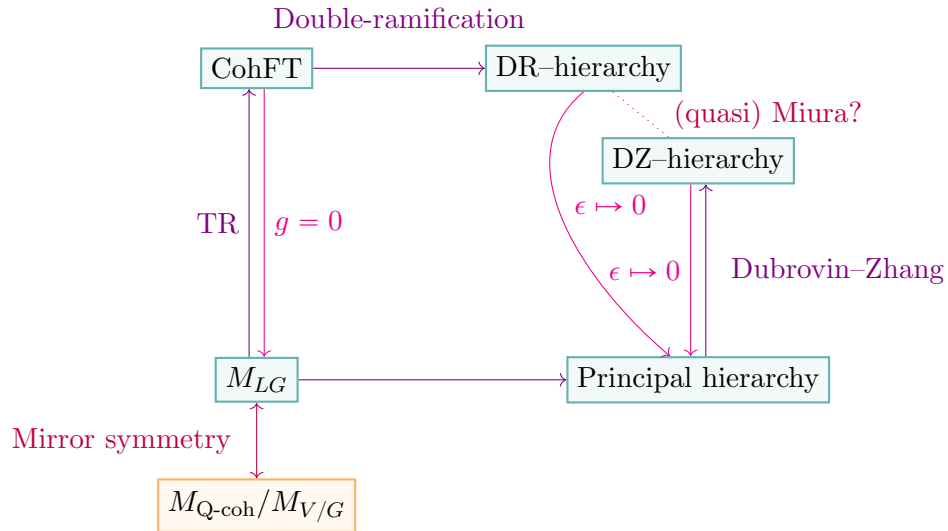


Figure 1.2: The bigger picture: Frobenius manifolds and friends.

The left-most vertical upwards arrow of 1.2 is the (Chekhov–Eynard–Orantin) topological recursion (TR) procedure [52]. This is a universal recursive formalism discovered first in the context of random

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*A semisimplicity condition is necessary and assumed for this diagram.

matrix models, but was found to have connections to a myriad of other fields of mathematics such as string theory, knot theory and enumerative geometry. It is a very general recursive formalism that takes an input, called the spectral curve, and produces an infinite sequence of differential forms, called TR invariants. Special cases of this formalism recovers Mirzakhani's celebrated recursion of hyperbolic volumes of moduli spaces, Hurwitz numbers (counting certain meromorphic functions on complex algebraic curves), and Gromov–Witten invariants (counting parametrised curves in a smooth variety), to mention some. We will describe this procedure more in detail in Part 3 of this thesis, and connect it to our main results.

The bottom horizontal arrow in Figure 1.2 represents the principal hierarchy associated to a Frobenius manifold. This is a dispersionless integrable hierarchy with two compatible Hamiltonian structures, which are induced by a pencil of metrics existing on a Frobenius manifold. An interesting question is to attempt to describe this hierarchy, which is a lot easier if we have an LG–description. The principal hierarchy will be explain more in depth in Chapter 8. The right-most vertical arrow is its dispersive analogue, obtained via the Dubrovin–Zhang (DZ) procedure [47].

The top horizontal arrow in Figure 1.2 represents the double-ramification (DR) hierarchy. This is an integrable hierarchy first introduced by Buryak in [21], starting from a cohomological field theory. Buryak conjectures that this hierarchy is equivalent to the hierarchy obtained by the DZ–procedure (right-most vertical arrow) by a Miura transformation (this conjecture is represented in Figure 1.2 by the small dotted diagonal line in the top corner). One advantage of this conjecture is that constructing the full hierarchy from a Frobenius manifold, going through the bottom-right half of Figure 1.2, is quite technical, and so given a B–model description one could gain access to the hierarchy through the TR procedure instead.

Notice that the small vertical arrow in the bottom left of Figure 1.2 is A–B mirror symmetry, and it is through the B-side, the LG–description, one has access to the left-most vertical arrow, and consequently the DR–hierarchy, as well as making the bottom horizontal arrow easier. This emphasises the value of obtaining such a description of a Frobenius manifold, and the typical research ambition of searching for one. Throughout this thesis, we will also see additional consequences of having a B–model taking the form of applications of our main results; this includes investigations of the Saito metric on discriminant strata and showing that discriminant strata are natural submanifolds in the language of [115, 116] (Chapter 7), an easy derivation of the degrees of the Lyashko–Looijenga mapping ([94, 95]) which in general takes the form of complicated combinatorial calculations (Chapter 6), and the formulation of a *generalised Norbury–Scott* conjecture, through the existence of a mirror symmetry triangle like Figure 1.1 (Chapter 9).

## 1.3 Main results

We give here a more technical summary of the original results obtained throughout this thesis.

**Mirror symmetry for Dubrovin–Zhang Frobenius manifolds [Chapter 5]**

Let $\mathfrak{g}_\mathcal{R}$ be a simple complex Lie algebra associated to an irreducible root system $\mathcal{R}$, and write $\mathfrak{h}_\mathcal{R}$ and $\mathfrak{g}_\mathcal{R}^{(1)}$ for, respectively, its Cartan subalgebra and the associated untwisted affine Lie algebra. A remarkable extension of the construction described as the third type above (or *C–model*[*]) was provided by Dubrovin and Zhang [46], who defined a Frobenius manifold structure, $\mathcal{M}_\mathcal{R}^{\mathrm{DZ}}$, on quotients of $\mathfrak{h}_\mathcal{R} \times \mathbb{C}$ by a suitable semi-direct product $\mathrm{Weyl}(\mathfrak{g}_\mathcal{R}^{(1)}) \ltimes \mathbb{Z}$. They furthermore provided a mirror symmetry construction for Dynkin type A, $\mathfrak{g}_{A_{N-1}} = \mathrm{sl}_N(\mathbb{C})$, in terms of Laurent-polynomial one-dimensional Landau–Ginzburg models, which was later generalised to classical Lie algebras in [44]. A question raised by [44,46] was whether a similar uniform mirror symmetry construction for all Dynkin types could be established, including exceptional Lie algebras.

One main goal of this thesis is to give an affirmative answer to this question. We will present a constructive and explicit Lie-theoretic construction, which provides closed-form expressions for the flat coordinates of the analogue of the Saito–Sekiguchi–Yano metric and for the Frobenius prepotential. The mirror theorem has simultaneous implications for singularity theory, integrable systems, the Gromov–Witten theory of Fano orbicurves, and Seiberg–Witten theory, some of which are explored in this thesis.

Our main result is the following general mirror theorem for Dubrovin–Zhang Frobenius manifolds (see Theorem 5.2.5 for the complete statement, and Tables 1.1, 1.2, for details of the notation employed). Let $\mathcal{H}_{g,\mathsf{m}}$ be the Hurwitz space of isomorphism classes $[\lambda : C_g \to \mathbb{P}^1]$ of covers of the complex line by a genus $g$ curve $C_g$ with ramification profile at infinity described by $\mathsf{m} \in \mathbb{N}_0^{\ell(\mathsf{m})}$, $\ell(\mathsf{m}) \geqslant 1$. Fixing a third kind differential $\phi$ on $C_g$ with simple poles at $\lambda^{-1}([1:0])$ induces, as a particular case of a classical construction of Dubrovin [39,41], a semisimple Frobenius manifold structure $\mathcal{H}_{g,\mathsf{m}}^\phi$ on $\mathcal{H}_{g,\mathsf{m}}$.

**Theorem 1.3.1** (=5.2.5)**.** *For any simple Dynkin type $\mathcal{R}$ there exists a highest weight $\omega$ for the corresponding simple Lie algebra $\mathfrak{g}$, pairs of integers $(g_\omega, \mathsf{n}_\omega)$, an explicit embedding $\iota_\omega : \mathcal{M}_\mathcal{R}^{DZ} \hookrightarrow \mathcal{H}_{g_\omega, \mathsf{n}_\omega}^\phi$, and a choice of third kind differential on the fibres of the universal family $\pi : \mathcal{C}_{g_\omega} \to \mathcal{H}_{g_\omega, \mathsf{n}_\omega}$ such that $\iota_\omega$ is a Frobenius manifold isomorphism onto its image $\mathcal{M}_\omega^{LG} := \iota_\omega(\mathcal{M}_\mathcal{R}^{DZ})$.*

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[*]Note that while we will be referring to the third construction as a C–model throughout this thesis, this is far from standard or established terminology.

In other words, $\iota_\omega$ identifies the Frobenius manifold $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ with a distinguished stratum $\mathcal{M}_\omega^{\mathrm{LG}}$ of a Hurwitz space, which is an affine-linear subspace of $\mathcal{H}_{g_\omega, \mathsf{n}_\omega}^\phi$ in flat coordinates for the latter. The datum of the covering map and third kind differential on $\mathcal{H}_{g_\omega, \mathsf{n}_\omega}$ define a one-dimensional B–model, or Landau-Ginzburg superpotential, for $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ in terms of a family of (trigonometric) meromorphic functions $\mathcal{M}_\omega^{\mathrm{LG}}$, whose Landau–Ginzburg residue formulas determine the Dubrovin–Zhang flat pencil of metrics and the Frobenius product structure on $T\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$.

Theorem 1.3.1 is proved in two main steps. We first associate to $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ a family of spectral curves (a subvariety of a Hurwitz space) given by the characteristic equation for a pencil of group elements $\mathbf{g}(\lambda) \in \mathcal{G} := \exp \mathfrak{g}$ in the irreducible representation $\rho_\omega$. The construction of the family hinges on determining all character relations of the form $\chi_{\wedge^k \rho_\omega} = \mathfrak{p}_k^\omega(\chi_{\rho_1}, \dots, \chi_{\rho_{l_{\mathcal{R}}}})$ in the Weyl character ring of $\mathcal{G}$, where $\rho_i$ is the $i^{\mathrm{th}}$ fundamental representation of $\mathcal{G}$, and $l_{\mathcal{R}}$ is the rank of $\mathcal{R}$. This is done via a combination of Mathematica and SageMath, as described in the proof of Corollary . Different choices of $\omega$ induce different families of spectral curves, and therefore different embeddings $\iota_\omega : \mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}} \hookrightarrow \mathcal{H}_{g_\omega, \mathsf{m}_\omega}$ inside a parent Hurwitz space. Our construction is motivated by a conjectural relation of the almost-dual Frobenius manifold [43] for $\mathcal{R}$ of type ADE with the orbifold quantum cohomology of the associated simple surface singularity, as proposed in [17, 19], which is in turn described by a degeneration of a family of spectral curves for the relativistic Toda chain associated to (a co-extension of) the corresponding affine Poisson–Lie group of type ADE [61, 120]. The one-parameter family of group elements $g(\lambda)$ in our construction is given by the Lax operator for relativistic Toda, where $\lambda$ is the spectral parameter, and the relation to the associated Dubrovin–Zhang Frobenius manifold of type ADE is suggested by analogous results for the simple Lie algebra case due to Lerche–Warner, Ito–Yang, and Dubrovin [43, 80, 92], and then generalising to all Dynkin types.

We prove Theorem 1.3.1 for all $\mathcal{R}$ with dominant weights $\omega$ in a minimal nontrivial Weyl orbit, but we also provide verifications that non-minimal choices of $\omega$ indeed give rise to isomorphic Frobenius manifolds. The target Hurwitz space $\mathcal{H}_{g_\omega, \mathsf{n}_\omega}$ is a space of rational functions ($g_\omega = 0$) for type $\mathcal{R} = A_l$, $B_l$, $C_l$, $D_l$ and $G_2$, and it is a space of meromorphic functions on higher genus curves in the other exceptional types.

## Application: Frobenius prepotentials [Chapter 5]

The original Dubrovin–Zhang construction establishes the existence of a Frobenius manifold structure on $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ by abstractly constructing a flat pencil of metrics $\gamma^* + \lambda\eta^*$ on $T^*\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$, where $\gamma^*$ arises from the extended Killing pairing on $\mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C}$, without reference to an actual system of flat coordinates for $\eta^*$ (the analogue of the Saito–Sekiguchi–Yano metric for finite reflection groups). From Theorem 1.3.1, the metric $\eta^*$ and Frobenius product on the base of the family of spectral

curves can then be computed using Landau–Ginzburg residue formulas for the superpotential: the associativity of the Frobenius product reduces the analysis of the pole structure of the Landau–Ginzburg residues to the sole poles of the superpotential, giving closed-form expressions for the flat coordinates of $\eta^*$ and its prepotential. We then obtain the following

**Theorem 1.3.2** (Theorem 5.2.5 and Examples in 5.4). *For all $\mathcal{R}$, we provide flat coordinates for the Saito metric of the Dubrovin–Zhang pencil and closed-form prepotentials for $\mathcal{M}_{\mathcal{R}}^{DZ}$.*

Our expressions recover results of [44, 46] for classical Lie algebras; the statements for exceptional Dynkin types are new. Theorem 1.3.1 is key to the determination of the prepotential: the Landau–Ginzburg calculation reduces the computation of flat coordinates for $\eta^*$ and a distinguished subset of structure constants to straightforward residue calculations on the spectral curves, from which the entire product structure on the Frobenius manifold can be recovered by using Mathematica to solve a large overdetermined linear system of equations arising from the WDVV equations, as shown in Example 14.

## Application: Lyaskho–Looijenga multiplicities of meromorphic functions [Chapter 6]

The enumeration of isomorphism classes of covers of $S^2$ with prescribed ramification over a point is a classical problem in topology and enumerative combinatorics, going back to Hurwitz's formula for the case in which the covering surface is also a Riemann sphere. The result of the enumeration for a cover of arbitrary geometric genus $g$ and branching profile $\mathsf{n} = (n_0, \ldots, n_m)$ is the Hurwitz number $h_{g,\mathsf{n}}$, whose significance straddles several domains in enumerative combinatorics [71, 72], representation theory of the symmetric group [70], moduli of curves [50], and mathematical physics [15, 24, 35]. It was first noticed by Arnold [3] that when the branching profile has maximal degeneration (i.e. for polynomial maps $f : \mathbb{P}^1 \to \mathbb{P}^1$) this problem is intimately related to considering the topology of the complement of the discriminant for the base of the type $A_l$ miniversal deformation, and in particular to the degree of the Lyashko–Looijenga mapping [94, 95] LL : $\mathbb{C}[\mu] \to \mathbb{C}[\mu]$, which assigns to a polynomial $\lambda(\mu)$ the unordered set of its critical values LL$(\lambda)(\mu) = \prod_{\lambda'(\tilde{z})=0}(\mu - f(\tilde{z}))$. This is a finite polynomial map [3, 94], inducing a stratification of $\mathbb{C}[\mu]$ according to the degeneracy of the critical values of $\lambda$. The computation of the topological degree of this mapping on a given stratum, enumerating the number of polynomials sharing the same critical values counted with multiplicity, can usually be translated into a combinatorial problem enumerating some class of embedded graphs. This connection was used by Looijenga [94] to reprove Cayley's formula for the enumeration of marked trees (corresponding to the codimension zero stratum), and by Arnold [3] to encompass the case of Laurent polynomials (see also [90, 91] for

generalisations to rational functions and discriminant strata). The extension of this combinatorial approach to arbitrary strata at higher genus, involving enumerations of suitable coloured oriented graphs ($k$-constellations), appears unwieldy [110]. However, when $\lambda(\mu)$ is the Landau–Ginzburg superpotential of a semisimple, conformal Frobenius manifold, the graded structure of the latter can be used to determine the Lyashko–Looijenga multiplicity of $\lambda(\mu)$ by a direct application of the quasihomogeneous Bézout theorem [4], with no combinatorics involved. In particular Theorem 1.3.1 has the following immediate consequence.

**Theorem 1.3.3** (Theorem 6.0.2). *For all $\mathcal{R}$ we compute the Lyashko–Looijenga multiplicity of the stratum $\iota_\omega(\mathcal{M}_\mathcal{R}^{DZ}) = \mathcal{M}_\omega^{LG} \subset \mathcal{H}_{g_\omega, \mathsf{n}_\omega}$.*

This includes, in particular, the higher genus Hurwitz spaces appearing for types $\mathcal{R} = E_n$ and $F_4$ (see Table 6.1).

**APPLICATION: Saito determinants on discriminant strata [Chapter 7]**

In [1], the authors consider semisimple Frobenius manifolds embedded as discriminant strata on the Dubrovin–Hertling polynomial Frobenius structures on the orbits of the reflection representation of Coxeter groups. In particular, they use the Landau–Ginzburg mirror superpotentials to establish structural results on the determinant of the restriction of the Saito metrics to arbitrary strata. A specific question asked by [1] is how much of that story can be lifted to the study of the Dubrovin–Zhang Frobenius manifolds on extended affine Weyl group orbits. The Landau–Ginzburg presentation of Theorem 1.3.1 unlocks the power to employ the same successful methodology in the affine setting as well. In Chapter 7 we show that it is indeed the case that one could perform the exact same procedure for the classical affine cases $\mathcal{R} \in \{A_l, B_l, C_l, D_l\}$. We prove the following

**Theorem 1.3.1.** Let $\mathcal{R} \in \{A_l, B_l, C_l, D_l\}$. Then,

$$\det(\eta_D) \propto w_0^{d+1} \prod_{H \in \mathcal{A}_D} l_H^{k_H} \mathcal{F}, \tag{1.3.1}$$

with

$$\mathcal{F} = \prod_{j=1}^{|\mathrm{Sing}_\mu(\lambda(\mu))|} \tilde{l}_j^{\tilde{k}_j}, \tag{1.3.2}$$

where $l_H$, $\tilde{l}_j$ are linear forms in exponentiated linear coordinates, $\mathcal{A}_D$ is the corresponding non-affine hyperplane arrangement, $k_H$, $\tilde{k}_j$, $d + 1 \in \mathbb{N}$, and $\mathrm{Sing}_\mu(\lambda(\mu))$ indicates the set of (finite) poles of $\lambda$.

**Theorem 1.3.2.** Let $H \in \mathcal{A}_D$ and $\beta \in \mathcal{R}$ be such that $\beta_D$ is a non-zero multiple of the linear form $l_H$ (after taking the logarithms of the exponentiated coordinates). Then, the multiplicity of $l_H$ in

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Theorem 7.1.4, $k_H$, is the Coxeter number of the root system $R_{D,\beta}^{(0)}$, as defined in (7.1.2), and $\tilde{k}_j$ is such that $\lambda_{\mathcal{R}} \sim \mathcal{O}(\mu^{-\tilde{k}_j})$, near $\infty_j$.

Theorems 1.3.1, and 1.3.2 were about using the Landau-Ginzburg superpotentials obtained in Section 5, but in terms of linear coordinates on the representation space of the associated Weyl group, which correspond to flat coordinates for the intersection form. This frame is given from the natural coordinates $w$ as in Equations (5.4.10), and (5.4.11), and were in practice computed using Mathematica.

For the exceptional cases $\mathcal{R} \in \{E_6, E_7, E_8, F_4, G_2\}$ we propose analogous conjectures to Theorems 1.3.1–1.3.2, and provide a detailed suggestion for how they may be proved.

## Application: The dispersionless extended type-$\mathcal{R}$ Toda hierarchy [Chapter 8]

The datum of a semisimple conformal Frobenius manifold is equivalent to the existence of a $\tau$-symmetric quasilinear integrable hierarchy, which is bihamiltonian with respect to a Dubrovin–Novikov hydrodynamic Poisson pencil. Having a description of the Frobenius manifold in terms of a closed-form prepotential allows to give an explicit presentation of the hierarchy in terms of an infinite set of commuting 1+1 PDEs in normal coordinates. The loop-space version of Theorem 1.3.2 is then the following

**Theorem 1.3.4** (Theorem 8.2.1). *For all $\mathcal{R}$, we construct a bihamiltonian dispersionless hierarchy on the loop space $\mathcal{LM}_{\mathcal{R}}^{DZ}$ in Hamiltonian form for the canonical Poisson pencil associated to $\mathcal{M}_{\mathcal{R}}^{DZ}$.*

For type $A_n$ this integrable hierarchy is the zero-dispersion limit of Carlet's extended bigraded Toda hierarchy [25], and for type $D_n$ it is the long-wave limit of the Cheng–Milanov extended $D$-type hierarchy [28]. For simply laced $\mathcal{R}$, we expect that the principal hierarchies of Theorem 1.3.4 should coincide with the dispersionless limit of the Hirota integrable hierarchies constructed by Milanov–Shen–Tseng in [102]. The non-simply laced cases are, to the best of our knowledge, new examples of hydrodynamic integrable hierarchies: our construction of the Landau–Ginzburg superpotential is highly suggestive that these should be obtained as symmetry reductions of the hierarchies in [102] by the usual folding procedure of the Dynkin diagram. Aside from laying the foundation for determining the prepotential of $\mathcal{M}_{\mathcal{R}}^{DZ}$, Theorem 1.3.1 also provides a dispersionless Lax formulation for the hierarchy as an explicit reduction of Krichever's genus $g_\omega$, $\ell(\mathsf{n}_\omega)$-pointed universal Whitham hierarchy.

**Application: The orbifold Norbury–Scott conjecture [Chapter 9]**

When $\mathcal{R} = A_1$, the Frobenius manifold $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ famously coincides with the quantum cohomology $\mathrm{QH}(\mathbb{P}^1, \mathbb{C})$ of the complex projective line. In [106], the authors propose a higher genus version of this statement and conjecture that the Chekhov–Eynard–Orantin topological recursion applied to the Landau–Ginzburg superpotential of $\mathbb{P}^1$ computes the $n$-point, genus-$g$ Gromov–Witten invariants of $\mathbb{P}^1$ with descendant insertions of the Kähler class (the "stationary" invariants) in terms of explicit residues on the associated spectral curve (see [49] for a proof). It was shown in [103] and [111], for $\mathcal{R} = A_l$, and $\mathcal{R} = D_l, E_l$, respectively, that the Dubrovin–Zhang Frobenius manifolds $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ are isomorphic to the orbifold quantum cohomology of the Fano orbicurves $\mathscr{C}_{\mathcal{R}}$ given by

$$\mathscr{C}_{\mathcal{R}} \simeq \begin{cases} \mathbb{P}_{\alpha_1, \alpha_2}, & \alpha_1 + \alpha_2 = l + 1, \quad \mathcal{R} = A_l, \\ \mathbb{P}_{2,2,l-2}, & \mathcal{R} = D_l, \\ \mathbb{P}_{2,3,l-3} & \mathcal{R} = E_l. \end{cases} \tag{1.3.3}$$

The construction of the LG superpotentials of Theorem 1.3.1 now associates a family of mirror spectral curves to the quantum cohomology of these orbifolds. As anticipated in [17], it is natural to conjecture that the Norbury–Scott Theorem receives an orbifold generalisation through Theorem 1.3.1, whereby higher genus stationary Gromov–Witten invariants of $\mathscr{C}_{\mathcal{R}}$ can be computed by residue calculus on the respective type $\mathcal{R}$ spectral curve mirrors. The investigation of the correct phrasing for the topological recursion is ongoing, but more details and some work in this direction is described in Chapter 9.

## 1.4   Notation, conventions and breakdown of thesis

This thesis consists of three main parts. Part 1 contains background theory on Frobenius manifolds and mirror symmetry needed for the remainder of the thesis. It is comprised of three Chapters; Chapters 2–4, the first two of which give an introduction to the most important properties of Frobenius manifolds, different ways such spaces may be constructed and connections between them (mirror symmetry). Part 1 ends with a special generalised construction, called the Dubrovin– Zhang construction, which is the C–model for our main mirror theorem, and in a sense a protagonist of this thesis.

The second part consists of three chapters; Chapters 5–7, and is comprised entirely of original results, apart from a motivating introductory section. In Chapter 5, we state and prove the main theorem of this thesis, a mirror theorem for Dubrovin-Zhang Frobenius manifolds. That is, we find B–model descriptions for the C–models introduced at the end of Part 1. In addition, we give

explicit flat coordinates and prepotentials. Chapter 6 includes an immediate and quick topological application of the results of Chapter 5. Finally, Chapter 7 make up the second series of main results, and is a generalisation of the results in [1] to the context of Dubrovin-Zhang manifolds, made approachable from a B–model description, as obtained in Chapter 5.

Part 3 consists of applications of the main results of Chapter 5 as well as some current and future research directions. Firstly, in Chapter 8, we consider integrable hierarchies associated to Dubrovin-Zhang Frobenius manifolds, describing the $G_2$ case in detail (which is another piece of original work). In Chapter 9, after introducing the notions of (Chekhov–Eynard–Orantin) topological recursion and orbifold Gromov–Witten theory, we go on and formalise a conjecture of Norbury-Scott type, first stated in [17], and show some work towards proving this conjecture.

## Notation

We will always be denoting partial derivatives $\frac{\partial}{\partial y^i} = \partial_{y^i}$, and in the case of the Saito-flat frame $\{t_\alpha\}$, we will simply write $\partial_\alpha$. Furthermore, Einstein summation convention* will be assumed, unless otherwise specified. Finally, while it is entirely possible to define many of the concepts of this thesis over any algebraically closed field (or often even non-algebraically closed), we will always be working over the field of complex numbers, $\mathbb{C}$, unless otherwise specified. See Tables 1.1-1.2 for general notation employed throughout this thesis.

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*This is where pairs of repeated indices one in subscript and one in superscript are implicitly summed over.

| Symbol | Meaning |
|---|---|
| $\eta$ | Saito metric |
| $g$ | Intersection form |
| $c$ | The totally symmetric $(0,3)$-tensor |
| $e$ | Identity vector field |
| $E$ | Euler vector field |
| $F$ | Prepotential |
| $d$ | Frobenius manifold charge |
| $d_\alpha$ | Degree of $t_\alpha$ |
| $U$ | Operator induced by $E\cdot$ |
| $t_\alpha$ | Flat coordinates for $\eta$ |
| $x_i$ | Flat coordinates for $g$ |
| $u_i$ | Canonical coordinates/eigenvalues of $U$ |
| $\lambda$ | Landau-Ginzburg superpotential |
| $\phi$ | Primary differential |
| $\mathcal{M}, M$ | Frobenius manifolds, and its underlying complex manifold, respectively |
| $\mathcal{L}_X Y$ | The Lie derivative of $Y$ in the direction of $X$ |
| $\mathbb{P}$ | $\mathbb{CP}^1$ |
| $\star$ | Product of the almost-dual algebra |
| $F_*$ | Almost dual potential associated to prepotential $F$ |
| $\delta_{ij}$ | The Kronoecker delta function |

Table 1.1: General notation.

| Symbol | Meaning |
|---|---|
| $\mathcal{R}$ | An irreducible root system |
| $l_{\mathcal{R}}$ | The rank of $\mathcal{R}$ |
| $\mathfrak{g}_{\mathcal{R}}$ | The complex simple Lie algebra with root system $\mathcal{R}$ |
| $\mathcal{G}_{\mathcal{R}}$ | The simply connected complex simple Lie group $\exp(\mathfrak{g}_{\mathcal{R}})$ |
| $\mathfrak{h}_{\mathcal{R}}$ | The Cartan subalgebra of $\mathfrak{g}_{\mathcal{R}}$ |
| $\mathcal{T}_{\mathcal{R}}$ | The Cartan torus $\exp(\mathfrak{h}_{\mathcal{R}})$ of $\mathfrak{g}_{\mathcal{R}}$ |
| $\mathsf{g}$ | A regular element of $\mathcal{G}_{\mathcal{R}}$ |
| $\mathcal{W}_{\mathcal{R}}/\widehat{\mathcal{W}}_{\mathcal{R}}/\widetilde{\mathcal{W}}_{\mathcal{R}}$ | The Weyl/affine Weyl/extended affine Weyl group of Dynkin type $\mathcal{R}$ |
| $\mathsf{h}$ | A regular element of $\mathfrak{h}_{\mathcal{R}}$ |
| $\{\alpha_1, \ldots, \alpha_{l_{\mathcal{R}}}\}$ | The set of simple roots of $\mathcal{R}$ |
| $\{\omega_1, \ldots, \omega_{l_{\mathcal{R}}}\}$ | The set of fundamental weights of $\mathcal{R}$ |
| $\Lambda_r(\mathcal{R})$ (resp. $\Lambda_r(\mathcal{R})^{\pm}$) | The lattice of roots of $\mathcal{R}$ (resp. the semi-group of positive/negative roots) |
| $\Lambda_w(\mathcal{R})$ (resp. $\Lambda_w(\mathcal{R})^{\pm}$) | The lattice of all weights of $\mathcal{R}$ (resp. the monoid of non-negative/non-positive weights) |
| $\rho_{\omega}$ | The irreducible representation of $\mathcal{G}_{\mathcal{R}}$ with highest weight $\omega$ |
| $\rho_i$ | The $i^{\text{th}}$ fundamental representation of $\mathcal{G}_{\mathcal{R}}$, $i = 1, \ldots, l_{\mathcal{R}}$ |
| $\Gamma(\rho)$ | The weight system of the representation $\rho$ |
| $\chi_{\omega}$ | The formal character of $\rho_{\omega}$ |
| $\chi_i$ | The formal character of $\rho_i$ |
| $[i_1 \ldots i_{l_{\mathcal{R}}}]_{\mathcal{R}}$ (without commas) | Components of a weight in the $\omega$-basis of $\mathcal{R}$ |
| $(x_1, \ldots, x_{l_{\mathcal{R}}})$ | Linear coordinates on $\mathfrak{h}_{\mathcal{R}}$ w.r.t. the coroot basis $\{\alpha_1^*, \ldots, \alpha_{l_{\mathcal{R}}}^*\}$ |
| $(Q_1, \ldots, Q_{l_{\mathcal{R}}})$ | $(\exp(x_1), \ldots, \exp(x_{l_{\mathcal{R}}}))$ |
| $C_{\mathcal{R}}$ | The Cartan matrix of $\mathcal{R}$ |
| $\mathcal{K}_{\mathcal{R}}$ | The symmetrised Cartan matrix of $\mathcal{R}$ |
| $\mathsf{n} \vdash d$ | A padded (i.e. parts are allowed to be zero) partition of $d \in \mathbb{N}$ |
| $|\mathsf{n}|$ (resp. $\ell(\mathsf{n})$) | The length of (resp. the number of parts in) a partition $\mathsf{n}$ |

Table 1.2: General notation.

# Part I

# Background on Frobenius Manifolds

# FROBENIUS MANIFOLDS

We will here define the protagonist of my research area, Frobenius manifolds, and present their most important properties and features.

## 2.1 Basic definitions

Roughly speaking, a Frobenius manifold is a complex manifold with the structure of a *Frobenius algebra* on its tangent space[*].

**Definition 2.1.1.** A *Frobenius algebra* is a pair $((A, \cdot), \eta)$, where $(A, \cdot)$ is a unital associative commutative algebra and $\eta$ a symmetric nondegenerate bilinear form on $A \times A$ such that the *Frobenius property* holds:

$$\eta(X \cdot Y, Z) = \eta(X, Y \cdot Z), \qquad \forall\, X, Y, Z \in A. \tag{2.1.1}$$

A (conformal) Frobenius manifold can be viewed as a family of (graded) Frobenius algebras.

**Definition 2.1.2.** A (holomorphic, conformal) *Frobenius manifold* is a tuple $\mathcal{M} = (M, \cdot, \eta, e, E)$. Here, $M$ is a finite dimensional complex manifold such that at each point $p \in M$, the fibre $T_p M$ of the holomorphic tangent bundle at $p$ has the structure of a Frobenius algebra with multiplication $\cdot$, bilinear form $\eta$, and identity element $e$, varying holomorphically. Additionally, we require the $(0, 2)$-tensor $\eta$ to be flat on $M$, and the following properties to be satisfied:

(i) the unit vector field is horizontal

$$\nabla e = 0, \tag{2.1.2}$$

w.r.t. the Levi-Civita connection $\nabla$ associated to $\eta$;

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[*]Note that this is stricter than what some call a Frobenius algebra, such as in cohomological field theory where neither commutativity nor unitality is required.

(ii) there exists a $(0,3)$-tensor $c \in \Gamma(M, \mathrm{Sym}^3 T^* M)$ such that

$$\nabla_W c(X, Y, Z) \tag{2.1.3}$$

is totally symmetric $\forall\, W, X, Y, Z \in \Gamma(M, TM)$;

(iii) there exists $E \in \Gamma(M, TM)$ such that $\nabla E$ is covariantly constant, and the corresponding 1-parameter group of diffeomorphisms acts by conformal transformations of the metric, and by rescalings of the tangent algebras.

Since the metric[†], $\eta$, is flat, a Frobenius manifold carries a canonical affine equivalence class of charts given by flat frames for $\eta$. We will denote flat coordinates for $\eta$ by $\{t^\alpha\}_{\alpha=1,\cdots,\dim(M)}$. In this chart the objects defined above take the following form.

(i) The total symmetry of $\nabla_W c(X, Y, Z)$ implies, by repeated application of the Poincaré lemma, the existence of a local function $F(t)$ such that

$$c_{\alpha\beta\gamma} = c(\partial_\alpha, \partial_\beta, \partial_\gamma) \equiv \eta(\partial_\alpha \cdot \partial_\beta, \partial_\gamma) = \partial^3_{\alpha\beta\gamma} F.$$

We call $F$ *the* prepotential of the Frobenius algebra.

Note that by (2.1.2) one can always apply coordinate transformations such that $e = \partial_1$.

$$\eta_{\alpha\beta} \equiv \eta(\partial_\alpha, \partial_\beta) = \partial^2_{\alpha\beta} e(F).$$

(ii) The structure coefficients of the Frobenius algebra is determined by $F$ as

$$\partial_\alpha \cdot \partial_\beta = c^\delta_{\alpha\beta} \partial_\delta \equiv \eta^{\delta\gamma} c_{\gamma\alpha\beta} = \eta^{\delta\gamma} \partial^3_{\gamma\alpha\beta} F,$$

where $\eta^{\alpha\beta} \equiv (\eta^{-1})_{\alpha\beta}$.

(iii) Assuming $Q := \nabla E$ is diagonalisable, the Euler vector field of the Frobenius manifold, $E$, induces a grading on $M$, and is linear with coefficients specifying the degrees[‡] of the coordinate functions;

$$E(t) = \sum_{d_\alpha \neq 0} d_\alpha t^\alpha \partial_\alpha + \sum_{d_\alpha = 0} r_\alpha \partial_\alpha.$$

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[†]As is usual in the context of Frobenius manifolds, we shall mean metric synonomously with a symmetric nondegenerate bilinear form, i.e. it is not necessarily positive definite.

[‡]These degrees are the eigenvalues of the grading operator $Q$.

(iv) The Euler vector field realises $F$ as a quasihomogeneous function up to a quadratic polynomial in $t$:

$$\mathcal{L}_E F = E(F) = (3 - d)F + \frac{A_{\alpha,\beta}t^\alpha t^\beta}{2} + B_\alpha t^\alpha + C, \tag{2.1.4}$$

where $\mathcal{L}_E$ denotes the Lie derivative in the direction of $E$, $A_{\alpha,\beta}$, $B_\alpha$, and $C$ are constants and $d \in \mathbb{Z}$ is called the charge of the Frobenius manifold.

(v) Thus,

$$\mathcal{L}_E e = -e, \quad \text{and} \quad \mathcal{L}_E \eta = (2 - d)\eta. \tag{2.1.5}$$

Furthermore, it can be shown, by considering the associativity of the tangent algebra, that the prepotential must satisfy the following system of nonlinear partial differential equations:

$$\frac{\partial^3 F}{\partial t^i \partial t^j \partial t^k} \eta^{kl} \frac{\partial^3 F}{\partial t^l \partial t^m \partial t^n} = j \longleftrightarrow m. \tag{2.1.6}$$

These are the celebrated Witten-Dijkgraaf-Verline-Verline (WDVV) equations, which are protagonists in the story of Frobenius manifolds. In a sense, one of Dubrovin's contribution by defining a Frobenius manifold is to give a geometrisation of the WDVV equations.

Hence, the definition of a Frobenius manifold gives, locally in flat coordinates for $\eta$, a quasi-homogeneous solution of the WDVV equations. Conversely, in order to specify a Frobenius manifold it is sufficient to give a quasihomogeneous solution to the WDVV equations together with a choice of identity vector field[§]. Then on a sufficiently small open subset, all the axioms (i)-(iii) of Definition 2.1.2 may be recovered[¶] [41].

Notice that the prepotential is only defined up to scaling, allowed coordinate changes (linear combination of coordinates of equal degrees), and up to polynomials in $t$ of degree at most two. Such transformations induce an equivalence, or isomorphism, of Frobenius manifolds. Thus one can add to $F$ some quadratic polynomial in $t$ such that (2.1.4) takes the form

$$\mathcal{L}_E \tilde{F}(t) \equiv E(\tilde{F}) = (2 - d)\tilde{F}, \tag{2.1.7}$$

where $\tilde{F}$ gives "the same" Frobenius manifold as $F$. It is up to such transformations a function is said to be *the* prepotential for a Frobenius manifold.

**Example 1.** The simplest Frobenius manifold is the unique (up to equivalence) 1-dimensional Frobenius manifold given by,

$$M = \mathbb{C} = < t >, \quad e = \partial_t, \quad E = t\,\partial_t, \quad F = \frac{t^3}{3!} \implies c_{ttt} = \eta_{tt} = < e, e > = \eta = 1. \tag{2.1.8}$$

This is called the *trivial* Frobenius manifold.

---

[§]Assuming $\det(\partial_e F \neq 0)$.
[¶]Assuming $d_e \neq 0$.

**Example 2** (Singularity Theory). Let $\lambda(q)$ be the miniversal unfolding of a simple singularity of type $A_n$ (c.f. Arnold's classification of simple singularities [2], and Table 3.2). That is,

$$\lambda(q) = q^{n+1} + \sum_{i=0}^{n-1} a_i q^i, \tag{2.1.9}$$

where $a_i \in \mathbb{C}$. Let us build a Frobenius manifold on the parameter space of $\lambda$:

$$M = B^n \subset \mathbb{C}^n = <\{a_i\}_{i=0,\cdots,n-1}>, \tag{2.1.10}$$

where $B^n$ is an $n$-dimensional ball in $\mathbb{C}^n$. The Frobenius structure on $M$ is defined using Saito's theory of primitive forms [112]. Consider the sheaf of Milnor–Jacobi algebras of $\lambda$. This is a rank $n$ free sheaf of $\mathcal{O}_M$-algebras where, at any fixed point $a \in M$, the fibre is given by the local algebra $\frac{\mathbb{C}[q]}{<\lambda'(q,a)>}$, where $\lambda' \equiv \frac{\partial \lambda}{\partial q}$. Furthermore, there exists a Kodaira–Spencer isomorphism, $\kappa$, from the tangent space of $M$ onto the local algebra mapping any vector field $v$ to the class $\mathcal{L}_v(\lambda) \equiv v(\lambda) \mod \lambda'$. This induces a multiplication on $T_a M$;

$$v \bullet w := \kappa^{-1}(v(\lambda) \cdot w(\lambda) \mod \lambda'). \tag{2.1.11}$$

The identity and Euler vector fields are the images of $1$ and $\lambda \mod \lambda'$ under $\kappa$, respectively. Furthermore, the bilinear form $\eta$, and the $c$-tensor are given by the residue formulae[‖]

$$\eta_a(v, w) := \sum_{|\lambda|<\infty} \operatorname{Res}_{d\lambda=0} \frac{v(\lambda dq) w(\lambda dq)}{d\lambda} = \sum_{|\lambda|<\infty} \operatorname{Res}_{d\lambda=0} \frac{v(\lambda) w(\lambda)}{\lambda'} dq, \tag{2.1.12a}$$

and

$$c_a(u, v, w) := -\sum_{|\lambda|<\infty} \operatorname{Res}_{d\lambda=0} \frac{u(\lambda dq) v(\lambda dq) w(\lambda dq)}{dq \, d\lambda} = \sum_{|\lambda|<\infty} \operatorname{Res}_{d\lambda=0} \frac{u(\lambda) v(\lambda) w(\lambda)}{\lambda'} dq, \tag{2.1.12b}$$

where $q$ is kept fixed. The nondegeneracy and flatness properties of (2.1.12a) are highly nontrivial, and were proven in [4], and [113], respectively. Moreover, it was proven in [10] that this construction does indeed define a Frobenius manifold structure on the parameter space $M$. See [37, 41, 42], for proofs and properties of this type of Frobenius manifold. For $n = 1$, we recover the trivial Frobenius manifold of Example 1 (as it is the only 1-dimensional Frobenius manifold, up to equivalence). Thus, Let us explicitly describe the second simplest case in which we have $n = 2$. From (2.1.9) we find that $\lambda = q^3 + a_1 q + a_0$, with

$$e = \partial_{a_0}, \quad \text{and} \quad E = a_0 \partial_{a_0} + \frac{2}{3} a_1 \partial_{a_1}. \tag{2.1.13}$$

For the residue formulae, (2.1.12a), (2.1.12b), we may use the fact that the sum of residues at all poles is zero in order to turn the contour around. In this way we pick up residues at the poles of

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[‖] This is defined through the Grothendieck residue. If interested, see [33] for further details.

$\lambda$ instead. For $\lambda$ as in (2.1.9) there is only one pole lying at $q = \infty$ which in the case of $n = 2$ has multiplicity 3. By using the residue theorem we find

$$c_{ijk} = \operatorname*{Res}_{q=0} \frac{q^{3-i-j-k}}{3} \mathrm{d}q, \quad \text{with } i, j, k \in \{0, 1\}, \tag{2.1.14}$$

$$\implies c_{ijk} = \begin{cases} \frac{1}{3} & \text{if } \{i, j, k\} = \{0, 0, 1\}, \\ -\frac{a_1}{9} & \text{if } \{i, j, k\} = \{1, 1, 1\}, \\ 0 & \text{otherwise.} \end{cases}$$

Hence,

$$\eta = \begin{pmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 0 \end{pmatrix}, \tag{2.1.15}$$

meaning the natural coordinates $\{a_0, a_1\}$ are flat coordinates for $\eta^{**}$. Integrating gives

$$F = \frac{t_1^2 t_2}{6} - \frac{t_2^4}{216}, \tag{2.1.16}$$

after relabelling $a_i = t_{i+1}$.

**Example 3.** Consider now the cohomology of $\mathbb{P}^1$,

$$M = H^*(\mathbb{P}^1, \mathbb{C}) \cong \mathbb{C}^2 = < \{[\mathbb{1}], [\omega^2]\} >, \tag{2.1.17}$$

where $\mathbb{1}$ is the Poincaré dual of the fundamental class, and $\omega$ is the standard Kähler form. Let us identify the tangent space at any point in $H^*(\mathbb{P}^1)$ with the space itself, $e = \partial_1$, and $E = t_1 \partial_1 + 2\partial_2$. The flat metric $\eta$ is identified with the Poincaré pairing, giving

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

There is only one additional nontrivial multiplication, and by identifying $\partial_2 := x$ we find:

$$\partial_2 \cdot \partial_2 = e^{t_2} \partial_1, \quad (\mathcal{A}, \cdot) \cong \mathbb{C}[x]/(x^2 - e^{t_2}) \quad \implies \quad F = \frac{t_1^2 t_2}{2} + e^{t_2}. \tag{2.1.18}$$

**Remark.** Examples 1-3, are closely related to the concept of mirror symmetry as described in the introduction. This will be further explained in Section 3.4.

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$^{**}$This is by far not the case in general. In fact, for $A_n$, with $n \geqslant 3$ this will no longer be true, and a transformation into flat coordinates is necessary.

## 2.2 Semisimplicity and canonical coordinates

**Definition 2.2.1** (Semisimplicity). A Frobenius manifold is *semisimple*, or *massive*, if the set $\text{Discr}(M) := \{p \in M \mid \exists v \in T_p M \text{ with } v \cdot v = 0\}$ has positive complex codimension.

This implies that, if the Frobenius manifold is semisimple, the Frobenius algebra $T_p M$ for a generic point $p \in M$, splits into one dimensional summands. The subset $\text{Discr}(M)$ on which the Frobenius manifold fails to be semisimple is called the *discriminant*.

Let $\mathcal{U}$ be the linear operator induced by multiplying with the Euler vector field:

$$\mathcal{U} : TM \to TM,$$
$$X \mapsto E \cdot X.$$

In flat coordinates for $\eta$ the elements of this operator take the form

$$\mathcal{U}^\alpha_\beta = E^\epsilon c_{\epsilon\beta\rho}\eta^{\rho\alpha}, \tag{2.2.1}$$

where $\alpha$, $\beta$ denotes the row and column, respectively.

A sufficient, but not always necessary, condition for semisimplicity of a Frobenius manifold is that the eigenvalues of the operator $\mathcal{U}$ are pairwise distinct. When this holds the set of eigenvalues of $\mathcal{U}$ make up an important set of coordinates for $M$ called *canonical coordinates*, and can be found as independent solutions of the system of partial differential equations

$$\partial_\gamma u c^\gamma_{\alpha\beta}(t) = \partial_\alpha u \partial_\beta u.$$

Canonical coordinates are determined uniquely up to shifts and permutations of indices.

An advantage of canonical coordinates is that the Frobenius manifold structure takes an especially nice form;

**Proposition 2.2.2.** In canonical coordinates $\{u_i\}$ we have the following.

The vector fields $\{\partial_{u_i}\}$ associated to the canonical coordinates, make up an idempotent basis for the tangent-algebra[*]:

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij}\partial_{u_i}, \tag{2.2.2a}$$

$$< \partial u_i, \partial u_j > = \eta_{ii}\delta_{ij}, \quad \text{with } \eta_{ii} = \partial_{u_i} t_1(u) \tag{2.2.2b}$$

and

$$g^{ij}(u) = u^i \eta_{ii}^{-1} \delta_{ij}, \tag{2.2.2c}$$

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[*]This makes the connection to the usual notion of semisimplicity clear.

with

$$e = \sum_{i=1}^{\dim(M)} \partial_{u_i}, \qquad E = \sum_{i=1}^{\dim(M)} u^i \partial_{u_i}. \tag{2.2.2d}$$

When the Frobenius manifold is of type $M_{\mathrm{LG}}$, the condition that the eigenvalues of $\mathcal{U}$ are pairwise distinct may be phrased as the superpotential having distinct critical values. The critical values form a coordinate system of the parameter space of the superpotential by the Riemann existence theorem and indeed coincides with the canonical coordinates as defined above. Semisimplicity is an important property in the sense that the WDVV equations does not constrain the prepotential much without it. In addition, semisimplicity is needed in order to access the big picture described in the introduction Figure 1.2 (via Givental theory), and to be able to define associated integrable hierarchies to a Frobenius manifold, as will be described further in Chapter 8.

Examples 1-3 are all semisimple;

**Example 4.**

- Example 1.
  Here, using (2.2.1),

  $$\mathcal{U} \equiv \mathcal{U}_1^1 \equiv E^1 c_{11\rho} \eta^{\rho 1} = E^1 c_{111} \eta^{-1} = E^1 \partial_1^3 F \eta^{-1} = t,$$

  $$\implies u_1 \equiv u = t.$$

- Example 2.
  The critical values of $\lambda$ are given by evaluating $\lambda$ at the zeros of $\lambda'$ (the critical points). Here $\lambda(q) = q^3 + a_2 q + a_1$, and so

  $$\lambda' \equiv \partial_q \lambda = 3q^2 + a_2 = 0$$
  $$\implies q = \pm \frac{i\sqrt{a_2}}{\sqrt{3}}$$
  $$\implies u_j = a_1 + (-1)^j \frac{2ia_2^{\frac{3}{2}}}{3\sqrt{3}}, \quad j = 1, 2.$$

  Clearly, $u_1 \neq u_2$ away from the line $a_2 = 0$.

- Example 3.
  Again, using (2.2.1), we find

  $$\mathcal{U} \equiv \left(\mathcal{U}_\beta^\alpha\right) = \begin{pmatrix} t_1 & 2e^{t_2} \\ 2 & t_1 \end{pmatrix}.$$

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$$\implies (t_1 - u)^2 - 4e^{t_2} = 0 \implies u_j = t_1 + (-1)^j 2e^{\frac{t_2}{2}}, \quad j = 1, 2.$$

Hence, $u_1 \neq u_2$ everywhere in $\mathbb{C}_t^2$.

**Remark.** While semisimplicity is important to have for many applications, there are plenty of Frobenius manifolds which are not semisimple, and for the purposes of cohomological field theories, semisimplicity is no necessity. Furthermore, there exist semisimple Frobenius manifolds in which the eigenvalues of $\mathcal{U}$ are not distinct, which is the topic of Giordano Cotti's work (see for example [33]). Thus, a pairwise distinct set of eigenvalues for $\mathcal{U}$ is a sufficient but not necessary condition for semisimplicity. In this thesis, however, all the Frobenius manifolds treated will generically have distinct canonical coordinates, and consequently, will be semisimple.

## 2.3 A flat pencil of metrics, Frobenius almost duality, and the deformed connection

Whenever $E$ is in the group of units, we may define another flat symmetric nondegenerate bilinear form, $g^*$, on the cotangent bundle, called the intersection form;

**Definition 2.3.1** (Intersection form). The *intersection form*, $g^*$, is defined by

$$g^*(\omega_1, \omega_2) := \iota_{E(p)}\eta^*(\omega_1, \omega_2), \quad \omega_1, \omega_2 \in T_p^* M, \tag{2.3.1}$$

where we have used the isomorphism $\eta : T_p M \to T_p^* M$.

Furthermore, $\eta$ and $g$ are compatible in the sense that we have a flat pencil of metrics on the cotangent space $T^* M$:

$$g^* + \lambda \eta^*, \tag{2.3.2}$$

where $(\eta^*) = (\eta)^{-1}$, with $(\eta)$ being the Gram matrix associated to $\eta^*$. That is, for any $\lambda \in \mathbb{C}$, (2.3.2) is a flat metric such that the Christoffel symbols are compatible;

$$\Gamma(\gamma^* + \lambda\eta^*) = \Gamma(\gamma^*) + \lambda\Gamma(\eta^*).$$

Moreover, they induce a flat pencil of connections.

On the tangent bundle we may define the dual of $g^*$, $g$; a flat metric $g$ on $TM$ is determined from $\eta$ and the Euler vector field $E$ by

$$g(E \cdot X, Y) = \eta(X, Y), \tag{2.3.3}$$

which in $\eta$-flat coordinates takes the form

$$g_{\alpha\beta} = E^\gamma c_{\gamma\alpha\beta} = E^\gamma \partial^3_{\gamma\alpha\beta} F. \tag{2.3.4}$$

---

*We will often be lazy and denote both the bilinear form and its associated Gram matrix by $\eta$.

**Example 5.** For the Examples 1-3 above, using (2.3.4), we have

- Example 1:

$$F = \frac{t^3}{3!}, \quad E = t\partial_t$$

$$\implies g \equiv g_{11} = t, \quad g^* = t^{-1};$$

- Example 2:

$$F = \frac{t_1^2 t_2}{6} - \frac{t_2^4}{216}, \quad E = t_1\partial_1 + \frac{2}{3}t_2\partial_2$$

$$\implies g \equiv (g_{\alpha\beta}) = \frac{1}{3}\begin{pmatrix} \frac{2t_2}{3} & t_1 \\ t_1 & -\frac{2t_2}{9} \end{pmatrix}, \quad g^* = \frac{9}{27t_1^2 + 4t_2^2}\begin{pmatrix} 2t_2 & 9t_1 \\ 9t_1 & -6t_2 \end{pmatrix};$$

- Example 3:

$$F = \frac{t_1^2 t_2}{2} + e^{t_2}, \quad E = t_1\partial_1 + 2\partial_2$$

$$\implies g \equiv (g_{\alpha\beta}) = \begin{pmatrix} 2 & t_1 \\ t_1 & 2e^{t_2} \end{pmatrix}, \quad g^* = \frac{1}{t_1^2 - 4e^{t_2}}\begin{pmatrix} -2e^{t_2} & t_1 \\ t_1 & -2 \end{pmatrix}.$$

Note the similarity with $\mathcal{U}$; by comparing the formulae (2.2.1) and (2.3.4), we see they are simply related by raising one index of $g$ using $\eta$. Moreover, we see that $e(g_{\alpha\beta}) = \eta_{\alpha\beta}$ for all these cases. Indeed this is a more general feature, and is actually the way we define $\eta$ in the orbit space construction with $g$ induced by the Killing form.

It is due to the existence of the pencil of metrics (2.3.2) that we may define a dispersionless bihamiltonian integrable hierarchy associated to any Frobenius manifold called the principal hierarchy. This will be described further in Chapter 8.

Since we have a second flat metric, a natural question is whether one can define a Frobenius manifold by letting $g$ take the role of $\eta$. This turns out to be *almost* the case, and is what Dubrovin calls Frobenius almost duality [43].

**Definition 2.3.2** (Almost Frobenius manifold)**.** An *almost Frobenius manifold* is a quintuple $(M, \cdot, \eta, e, E)$, satisfying all the axioms of a Frobenius manifold except the flatness of the identity vector field (2.1.2).

Almost duality assigns to a Frobenius manifold an almost Frobenius manifold.

**Definition 2.3.3** (Almost Duality)**.** Given a Frobenius manifold $\mathcal{M} = (M, \cdot, \eta, e, E)$, define a new multiplication on $TM$ by

$$u \star v := \frac{u \cdot v}{E} \tag{2.3.8}$$

Then almost duality assigns to $\mathcal{M}$ the tuple $(M, \star, g, E, E)$.

**Proposition 2.3.4** (Proposition 3.1 in [43])**.** Given a Frobenius manifold $(M, \cdot, \eta, e, E)$, the tuple $(M, \star, g, E, E)$ is an almost Frobenius manifold.

It is obvious that almost duality does not provide a true Frobenius manifold since the Frobenius manifold axioms only require $\nabla(\nabla E) = 0$, and not $\nabla E = 0$. The fact that the rest of the axioms are satisfied is less trivial. We refer to [43] for a proof of this.

**Example 6** (Theorem 5.2 in [43])**.** The almost Frobenius manifold associated with prepotential

$$F_*(x) = \frac{n+1}{8} \sum_{i<j} (x_i - x_j)^2 \log(x_i - x_j)^2, \tag{2.3.9}$$

and metric

$$g_{ij} = \delta_{ij}$$

is the almost dual to the Frobenius manifold arising from a simple singularity of type $A_n$ as in Example 2 [76].

The concept of almost duality was crucial in the motivation leading up to the main result of [20], and a main result of this thesis (Theorem 5.2.5), as will be described in more detail in Chapter 5.

A very important object of the Frobenius manifold theory is the so-called *deformed connection*[†]. This is a certain deformation of the Levi-Civita connection, and is essential in the classification of Frobenius manifolds as well as defining their associated integrable hierarchies.

Again, let $\mathcal{M}$ be an $n$-dimensional semisimple Frobenius manifold, and, as usual, let $\{t_\alpha : \mathcal{M} \to \mathbb{C}\}_{\alpha=1}^n$ be a flat chart for $\eta$.

Then we define the deformed connection as follows. This is an affine, torsion free connection on $T(\mathcal{M} \times \mathbb{C}^\star)$ such that

$$\begin{aligned}
\tilde{\nabla}_V W &= i_V \mathrm{d}_{\mathcal{M}} W + z V \cdot W, \\
\tilde{\nabla}_{\partial_z} W &= i_{\partial_z} \mathrm{d}_{\mathbb{C}^\star} W - E \cdot W - z^{-1} \mathcal{V} W,
\end{aligned} \tag{2.3.10}$$

where $z \in \mathbb{C}^\star$, $V(z), W(z) \in \Gamma(TM)$, and $\mathcal{V} \in \Gamma(\mathrm{End}(TM))$ is the grading operator defined in flat coordinates as $\mathcal{V}_\beta^\alpha = (1 - n/2)\delta_\beta^\alpha + \partial_\beta E^\alpha$. The flatness of $\tilde{\nabla}$ imply that the gradients of its

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[†]The deformed connection is also sometimes called the *Dubrovin connection*.

flat sections solve the system of differential equations associated to (2.3.10). Furthermore, the compatibility of these equations is equivalent to the Frobenius manifold axioms [41, Lecture 6]. In Chapter 8, we shall see how to define a dispersionless integrable hierarchy of hydrodynamic type using the flat sections of $\widetilde{\nabla}$ (this is the *the principal hierarchy* depicted as the bottom horizontal arrow in Figure 1.2).

$$— 3 —$$

## MAIN CONSTRUCTIONS

### 3.1 Orbit spaces of Coxeter groups

Let $W$ be a Coxeter group, that is, the finite group of linear transformations of a real vector space, $V$, generated by reflections. Irreducible Coxeter groups are classified into $A_l, B_l, D_l, E_6, E_7, E_8, F_4, G_2$, the symmetry groups of the regular icosahedron ($H_3$), regular 600-cell of 4-dimensional space ($H_4$), and the regular planar $p$-gon ($I_2(p)$) [34]. Note that the subscript is the dimension of the vector space $V$ acted upon by $W$, and consequently the number of generators of $W$. The classification is formulated in terms of Coxeter graphs, as shown in Figure 3.1.

In this section, following [41], we give a succinct description of how to define a Frobenius manifold structure on the orbit space of $W$ for all classes in Figure 3.1.

Equip $V$ with an Euclidean structure $< \cdot, \cdot >: V \times V \to \mathbb{R}$ such that the action of $W$ is orthogonal with respect to $< \cdot, \cdot >$. Further, let $\{x_1, \cdots, x_l\}$ be linear coordinates on $V$.

Let $S(V)$ be the symmetric algebra of polynomials in $\{x\}$, and $\Omega(V)$ the $S(V)$-module of differential forms on $V$ with polynomial coefficients. The action of $W$ on $V$ induces an action on $S(V)$ and $\Omega(V)$, and furthermore on the $W$-invariant parts $S(V)^W$, $\Omega(V)^W$. Note that there exists algebraically independent homogeneous polynomials $\{y^i\}$ such that $S(V)^W = < y^1, \cdots, y^l >$, and similarly a basis for $\Omega(V)^{W*}$ is given $\{\mathrm{d}y^{i_1} \wedge \cdots \wedge \mathrm{d}y^{i_k}\}$, for $k = 1, \cdots, l$. The polynomials $\{y^i\}$ are called *basic invariants* and their degrees are uniquely determined by $W$; $d_i \equiv \deg(y^i) = m_i + 1$, where $\{m_i\}$ are the Coxeter exponents[†]. See Table 3.1 for the set of degrees for each Coxeter type $W$. The basic invariants are defined up to an invertible transformation respecting the grading. Now, the action of $W$ is extended to the complexification of $V$, $V \otimes \mathbb{C}$, and the space of orbits of $W$ is defined to be

$$M := (V \otimes \mathbb{C})/W, \tag{3.1.1}$$

---

[*]As a free $S(V)^W$-module.

[†]I.e. the eigenvalues of a Coxeter element (that is the longest irreducible word in the generators of $W$) have the form $e^{\frac{2\pi i(d_j-1)}{h}}$, where $h$ is the order of the Coxeter element, called the Coxeter number of $W$.

| $W$ | Coxeter number | Coxeter diagram |
|-----|----------------|-----------------|
| $A_l$ | $l+1$ | $\alpha_1 - \alpha_2 - \alpha_3 \cdots \alpha_{l-2} - \alpha_{l-1} - \alpha_l$ |
| $B_l, C_l$ | $2l$ | $\alpha_1 - \alpha_2 - \alpha_3 \cdots \alpha_{l-2} - \alpha_{l-1} \overset{4}{-} \alpha_l$ |
| $D_l$ | $2l-2$ | $\alpha_1 - \alpha_2 - \alpha_3 \cdots \alpha_{l-2}$ with branches to $\alpha_{l-1}$ and $\alpha_l$ |
| $E_6$ | $12$ | $\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5$ with $\alpha_6$ attached to $\alpha_3$ |
| $E_7$ | $18$ | $\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6$ with $\alpha_7$ attached to $\alpha_3$ |
| $E_8$ | $30$ | $\alpha_1 - \alpha_2 - \alpha_3 - \alpha_4 - \alpha_5 - \alpha_6 - \alpha_7$ with $\alpha_8$ attached to $\alpha_3$ |
| $F_4$ | $12$ | $\alpha_1 - \alpha_2 \overset{4}{-} \alpha_3 - \alpha_4$ |
| $G_2$ | $6$ | $\alpha_1 \overset{6}{-} \alpha_2$ |
| $H_3$ | $10$ | $\alpha_1 \overset{5}{-} \alpha_2 - \alpha_3$ |
| $H_4$ | $30$ | $\alpha_1 \overset{5}{-} \alpha_2 - \alpha_3 - \alpha_4$ |
| $I_2(p)$ | $p$ | $\alpha_1 \overset{p}{-} \alpha_2$ |

Figure 3.1: Coxeter diagrams and Coxeter numbers.

which has a structure of an affine algebraic variety. The inner product $< \cdot, \cdot >$ can be extended to the complexification of $V$ as a complex quadratic form.

Away from the discriminant, $\Sigma$, consisting of irregular orbits,[‡] the map $V \otimes \mathbb{C} \to M$ is a local diffeomorphism, and so the linear coordinates in $V$ may serve as local coordinates on sufficiently small open subsets of $M\backslash\Sigma$. Furthermore, we may define

$$g^{ij}(y) := (\mathrm{d}y^i, \mathrm{d}y^j)^* := \sum_{a,b=1}^{l} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} (\mathrm{d}x^a, \mathrm{d}x^b)^*, \tag{3.1.2}$$

---

[‡]An orbit is regular if the size of the orbit equals $|W|$.

| $W$ | $\{d_i\}$ |
|:---:|:---:|
| $A_l$ | $d_i = l + 2 - i$ |
| $B_l, C_l$ | $d_i = 2(l - i + 1)$ |
| $D_l$ | $d_i = \begin{cases} 2(l - i) & \text{if } i \leqslant k, \\ 2(l - i + 1) & \text{if } i \geqslant k + 2, \\ l & \text{if } i = k + 1 \end{cases}$ |
| $E_6$ | $\{12, 9, 8, 6, 5, 2\}$ |
| $E_7$ | $\{18, 14, 12, 10, 8, 6, 2\}$ |
| $E_8$ | $\{30, 24, 20, 18, 14, 12, 8, 2\}$ |
| $F_4$ | $\{12, 8, 6, 2\}$ |
| $G_2$ | $\{6, 2\}$ |
| $H_3$ | $\{10, 6, 2\}$ |
| $H_4$ | $\{30, 20, 12, 2\}$ |
| $I_2(p)$ | $\{k, 2\}$ |

Table 3.1: Coxeter groups and associated degrees, as in [41, Lecture 4, Table 1].

where $(\cdot, \cdot)$ denotes the $W$-invariant bilinear form induced from $< \cdot, \cdot >$, and $(\cdot, \cdot)^*$ its contravariant version. Away from $\Sigma$ (3.1.2) is nondegenerate.

We are now ready to state the main theorem of this section:

**Theorem 3.1.1** (Theorem 4.1. in [41])**.** There exists a unique semisimple Frobenius manifold structure (up to equivalence) on $M \backslash \Sigma$ such that

- (3.1.2) is the intersection form on $M$;

- the Euler vector field is given by

$$E = \sum_{i=1}^{l} \frac{1}{h}(d_i y^i \partial_i) = \frac{1}{h} x^a \partial_{x_a},$$

  where $h$ is the Coxeter number of $W$[§];

- the identity vector field is given by $e = \partial_{y^i}$, such that $d_i = h$.

See [41] for a proof. Here the Saito metric, $\eta$, is given by $\mathcal{L}_e g$.

**Example 7** ($A_n$)**.** Consider the group $W_{A_n}$ which acts on $\mathbb{R}^{n+1} = \{x_0, x_1, \cdots, x_n\}$ by permutation of the coordinates $x_i$, such that

$$\sum_{i=0}^{n} x_i = 0. \tag{3.1.3}$$

---

[§]The Coxeter number is the largest of all Coxeter exponents.

The metric is simply given by the restriction of the Euclidean metric on $\mathbb{R}^{n+1}$ to the hyperplane determined by (3.1.3). A homogeneous basis of $S^W(\mathbb{R}^{n+1}|_{(3.1.3)})$ is given by the elementary symmetric polynomials in $\{x_i\}$:

$$a_k = (-1)^{n-k+1} \sum_{0 \leqslant j_1 < \cdots < j_k \leqslant n} \prod_{i=j_1}^{j_k} x_i, \quad k = 1, \cdots, n.$$

Thus, $M \equiv \mathbb{C}^n/A_n$ may be identified with the space of functions

$$\{\lambda(p) = p^{n+1} + \sum_{i=0}^{n-1} b_i p^i | b_0, \cdots, b_{n-1} \in \mathbb{C}\}. \tag{3.1.4}$$

Note that an element of the set of functions (3.1.4) is precisely the form of $\lambda$ in Example 2. Hence, this is an example of B–C mirror symmetry. Furthermore, notice from Example 5 item 2, that the prepotential is polynomial in Saito–flat coordinates $t$. In fact, there is a correspondence between Frobenius manifolds with polynomial prepotential and orbit spaces of Coxeter groups [74].

**Remark.** It turns out that a version of Theorem 3.1.1 can be stated for more general groups $W$ as we will see when considering extended affine Weyl groups in Chapter 4.

## 3.2 Quantum cohomology

In this subsection we will be following [78] Chapter 26, and [83], with [63] serving as the main source for technical details and proofs. $X$ will be a nonsingular complex projective variety, and $H_*(X)$, $H^*(X)$, will be denoting the singular homology, and cohomology over $\mathbb{Q}$, respectively.

### Gromov–Witten theory

Quantum cohomology is the genus 0 part of a modern curve counting theory, called Gromov–Witten theory, which attempts to count the number of curves of genus $g$, degree $d$, passing through $3d - 1$ [*] (generically positioned) points in $X$. Gromov–Witten theory is formulated in terms of the intersection theory over the *moduli space of stable maps*. This space is often denoted by $\overline{\mathcal{M}}_{g,n}(X, \beta)$ where the genus $g$ is the arithmetic genus $g(\mathcal{C}) := h^1(\mathcal{C}, \mathcal{O}_\mathcal{C})$ and $\beta \in H_2(X, \mathbb{Z})$ is a generalisation of the degree. Note that the bar over $\mathcal{M}_{g,n}(X, \beta)$ in $\overline{\mathcal{M}}_{g,n}(X, \beta)$ denotes the Deligne-Mumford compactification in which certain nodal curves have been added, and $n$ is the number of *marks*,

---

[*]This number is exactly what is needed in order to get a finite number to count. For example, there are infinite number of lines passing through one point in the plane, but a unique line passing through two.

which represents the points of intersection. Together with nodes, these make up the set of *special* points. More precisely, every point in the moduli space represents a class

$$[\mathcal{C}; x_1, \cdots, x_n; f], \tag{3.2.1}$$

such that

- $\mathcal{C}$ is a (possibly) nodal genus $g$ curve;

- $x_i$ are distinct nonsingular points on $\mathcal{C}$, for $i = 1, \cdots, n$;

- $f : \mathcal{C} \to X$ is a morphism such that $\beta = f_*[\mathcal{C}]$;

- If $f(\mathcal{C}_0)$ is a point for $\mathcal{C}_0$ an irreducible component of $\mathcal{C}$, then $2g(\mathcal{C}_0) - 2 + n(\mathcal{C}_0) > 0$, with $n(\mathcal{C}_0)$ being the number of special points on $\mathcal{C}_0$.

The equivalence between a pair of stable maps $(\mathcal{C}; x_1, \cdots, x_n; f)$, $(\mathcal{C}'; x_1', \cdots, x_n'; f')$ is given by an isomorphism

$$\tau : \mathcal{C} \to \mathcal{C}',$$

sending $x_i$ to $x_i'$ such that $f' \cdot \tau = f$.

In addition, we have a notion of families, i.e. a geometric characterisation of deformations. Let $B$ be any scheme. Then a family of stable maps parametrised by B is given by the diagram

$$
\sigma_i \left(\begin{array}{c} \mathcal{C} \xrightarrow{\ f\ } X \\ \Big\downarrow \pi \\ B \end{array}\right. \tag{3.2.2}
$$

where $\sigma_i$ is a section of $\pi$ for $i = 1, \cdots, n$ such that $(\pi^{-1}(b); \sigma_1(b), \cdots, \sigma_n(b); f|_{\pi^{-1}(b)})$ is a stable map of degree $\beta$ for $b \in B$.

Furthermore, a smooth, or fine, moduli space $\overline{\mathcal{M}}_{g,n}(X, \beta)$ admits a *universal family* where the base is the moduli space itself, and every other family may be related to the universal family through pull-back.

**Remark.** $\overline{\mathcal{M}}_{0,3}(\mathbb{P}^1, \beta)$ is a smooth moduli space with $\overline{\mathcal{M}}_{0,4}(\mathbb{P}^1, \beta)$ as universal family. More generally;

$$
\tilde{x}_i \left(\begin{array}{c} \overline{\mathcal{M}}_{0,n+1}(X, \beta) \xrightarrow{\ \mathrm{ev}_{n+1}\ } X \\ \Big\downarrow \phi_{n+1} \\ \overline{\mathcal{M}}_{0,n}(X, \beta) \end{array}\right. \tag{3.2.3}
$$

where $\phi_{n+1}$ is the forgetful map given by forgetting mark $x_{n+1}$

$$\phi_{n+1} : \overline{\mathcal{M}}_{0,n+1}(X, \beta) \to \overline{\mathcal{M}}_{0,n}(X, \beta)$$
$$(\mathcal{C}; x_1, \cdots, x_n, x_{n+1}; f) \mapsto (\mathcal{C}; x_1, \cdots, x_n; f), \tag{3.2.4}$$

and $\mathrm{ev}_{n+1}$ is the evaluation map evaluating $f$ at mark $x_{n+1}$

$$\mathrm{ev}_{n+1} : \overline{\mathcal{M}}_{0,n+1}(X, \beta) \to X$$
$$(\mathcal{C}; x_1, \cdots, x_n, x_{n+1}; f) \mapsto f(x_{n+1}), \tag{3.2.5}$$

and stabilising[†] if necessary.

A key property of the moduli space of stable maps is a stratification of the boundary (the subset $\overline{\mathcal{M}}_{g,n}(X, \beta) \backslash \mathcal{M}_{g,n}(X, \beta)$ consisting of maps from singular curves), which admits a recursive structure vital to computations.

It turns out to be too restrictive to expect a smooth moduli space in general due to the presence of nontrivial automorphisms[‡], in which case there exist no universal family. The moduli space can, however, be considered as a Deligne-Mumford (DM) stack/orbifold[§].

A DM–stack is a generalisation of a variety, or scheme, which can be described locally as a quotient of a scheme (which is called the coarse moduli space of the stack) by a finite group. Thus, to each point in the stack there is an associated group which is called the isotropy group of the point.

Furthermore, $\overline{\mathcal{M}}_{g,n}(X, \beta)$ is compact, but not necessarily irreducible or connected. A consequence of this is that we are forced to replace the usual fundamental class with a *virtual* fundamental class which was introduced by Behrend and Fantechi [7], and Li and Tian [93] in the 1990s. The virtual fundamental class is often denoted by $[\overline{\mathcal{M}}_{g,n}(X, \beta)]^{\mathrm{vir}}$, and is a class in $H_{2\mathrm{vdim}}(X, \mathbb{Q})$, where vdim is the virtual, or expected, dimension defined by

$$\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X, \beta)) := \int_\beta c_1(TX) + (\dim(X) - 3)(1 - g) + n,$$

with $c_1(TX)$ being the first Chern class of the tangent bundle of $X$. The virtual fundamental class is often hard to describe, and its construction very technical. We will not go into the details of the virtual fundamental class here. See the aforementioned original papers, or [5][¶] for more details on this. The enumerative invariants of this theory are rational numbers called the Gromov–Witten invariants.

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[†]That is contracting unstable components.
[‡]The stability does, however, ensure that the set of automorphisms is finite.
[§]Note that this is different from what is usually called an orbifold in physics literature.
[¶]See this reference for a more friendly presentation.

**Definition 3.2.1** (GW-invariants)**.** Let $\gamma_1, \cdots, \gamma_n \in H^*(X)$. Define the associated genus $g$ n-point (primary) *Gromov–Witten* invariant by

$$< \gamma_1 \bullet \cdots \bullet \gamma_n >_{g,\beta}^X := \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \overset{n}{\underset{i=1}{\cup}} \mathrm{ev}_i^*(\gamma_i) \tag{3.2.6}$$

where $\mathrm{ev}_i^*$ is the pull-back of the evaluation map at the $i^{\mathrm{th}}$ mark $x_i$ (given some fixed ordering), and we are capping the integrand against the virtual fundamental class.

Furthermore, let $\psi_i$ be the first Chern class of the cotangent line bundle on $\overline{\mathcal{M}}_{g,n}(X,\beta)$ which has as fibre over a point given by $T_{p_i}^*\mathcal{C}$ ‖, and let

$$< \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) >= \int_{[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}} \overset{n}{\underset{i=1}{\cup}} \psi_i^{a_i} \mathrm{ev}_i^*(\gamma_i).$$

Then, some instrumental properties include

1. The string equation:

$$< \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) T_0 >_{g,\beta} = \sum_{i=1}^{n} < \tau_{a_1}(\gamma_1) \cdots \tau_{a_{i-1}}(\gamma_{i-1}) \tau_{a_{i-1}}(\gamma_i) \tau_{a_{i+1}}(\gamma_{i+1}) \cdots \tau_{a_n}(\gamma_n) >_{g,\beta} .$$
$$\tag{3.2.7}$$

2. The dilaton equation:

$$< \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \tau_1(T_0) >_{g,\beta} = (2g - 2 + n) < \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) >_{g,\beta} . \tag{3.2.8}$$

3. The divisor equation:
   Let $\gamma \in H^2(X)$. Then,

$$< \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) \gamma >_{g,\beta} = \left( \int_\beta \gamma \right) < \tau_{a_1}(\gamma_1) \cdots \tau_{a_n}(\gamma_n) >_{g,\beta} . \tag{3.2.9}$$

In Equations (3.2.7)–(3.2.9), $T_0$ indicates the unit in the cohomology ring (i.e. the Poinaré dual of the fundamental class). By restricting to genus zero invariants, one may construct the quantum cohomology ring, which can be given the structure of a Frobenius manifold.

## Quantum cohomology and Frobenius manifolds

There are two notions of quantum cohomology, called *big* and *small*. The small quantum cohomology is to big quantum cohomology like a Frobenius algebra is to a Frobenius manifold. Here, we

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‖More precisely, the line bundle is given by $\sigma_i^* \omega_\pi$, where $\omega_\pi$ is the sheaf of relative differentials.

give a brief description of both of these rings, starting with the latter. Note that in this section $\tau_i$ is a certain class (to be specified) and not as in Equations (3.2.7)-(3.2.9).

The small quantum cohomology can be viewed as a deformation of the usual cohomology. Consider the evenly graded part of $H^*(X)$ with its usual cup-product and bilinear nondegenerate pairing $( , )$;

$$(a, b) := \int a \cup b. \tag{3.2.10}$$

Let $\mathbf{1}_X$ generate $H^0(X)$, then

**Lemma 3.2.2.** $(H^*(X), ( , ), \cup, \mathbf{1}_X)$ is a Frobenius algebra.

For a proof see [109].

We may generalise this construction in the following way. Let us define

$$a_1 \star_\beta a_2 := \mathrm{ev}_{3*}(\mathrm{ev}_1^*(a_1) \cup \mathrm{ev}_2^*(a_2)), \tag{3.2.11}$$

over $\overline{\mathcal{M}}_{0,3}(X, \beta)$. Moreover, we want to be able to keep track of the curve class $\beta$, so let us introduce a formal parameter $q^\beta$ satisfying the exponential relation $q^{\beta_1} q^{\beta_2} = q^{\beta_1 + \beta_2}$. Then we may define a new product

$$a_1 \star a_2 := \sum_\beta a_1 \star_\beta a_2 \, q^\beta. \tag{3.2.12}$$

Now, extend the product (3.2.12) $\mathbb{Q}[[H_2(X, \mathbb{Z})_+]]$-linearly to $H^*(X) \otimes_{\mathbb{Q}} [[H_2(X, \mathbb{Z})_+]]$, and call $H^*(X) \otimes_{\mathbb{Q}} [[H_2(X, \mathbb{Z})_+]]$ with this structure the small quantum cohomology ring and denote it by $QH_s^*(X)$.

**Proposition 3.2.3.** $QH_s^*(X)$ is a Frobenius algebra with the unit $\mathbf{1}_X$ as in Lemma 3.2.2.

For a proof see proof of Proposition 5.2 in [109].

**Example 8** ($QH_s^*(X)$ of $\mathbb{P}^m$ Example 26.5.2 in [77])**.** Let $\tau_i$ be the class of a linear subspace of $\mathbb{P}^m$ with codimension $\mathrm{codim}(\tau_i) = i$, and let $\beta$ be $d$ times the class of a line. Then we have that

$$< \tau_i \cdot \tau_j \cdot \tau_k > = \delta_{i+j+k, m(m+1)d}. \tag{3.2.13}$$

This forces either $d = 0$ or $d = 1$. In the first case we get $i + j + k = m$ and thus the bracket gives 1, and in the second case, we get $i + j + k = 2m + 1$ which again results in the bracket being 1. This means that the algebraic structure looks as follows.

- If $i + j \leqslant m$, then $\tau_i \star \tau_j = \tau_{i+j}$,

- and if $m + 1 \leqslant i + j \leqslant 2m$ we have $\tau_i \star \tau_j = q_1 \tau_{i+j-m-1}$.

Hence,

$$QH_s^*(X) = \mathbb{Q}[H][[q]]/(H^{m+1} - q), \tag{3.2.14}$$

where $H = \tau_1$ is the hyperplane class.

Notice that only 3-point invariants appear in the definition of the small quantum cohomology. Thus, given a description of the small quantum cohomology ring we are not able to recover all GW-invariants. To be able to do this, it is necessary to consider the *big* quantum cohomology.

Let $\{\tau_i\}_{i=0,\cdots,m}$ be a homogeneous basis for $H^*(X, \mathbb{Z})$, with $\tau_0 = 1 \in H^0(X, \mathbb{Z})$, where $\{\tau_1, \cdots, \tau_p\}$ are Kähler classes[**].

Let

$$F(t) = \sum_{n,\beta} \frac{< \gamma^n >_\beta}{n!} = \sum_{n_0+\cdots+n_m \geqslant 3} \sum_\beta < \tau_0^{n_0} \bullet \cdots \bullet \tau_m^{n_m} > \frac{t_0^{n_0}}{n_0!} \cdots \frac{t_m^{n_m}}{n_m!}, \tag{3.2.15}$$

where $\gamma = \sum_i t_i \tau_i$, the first sum is over where $\gamma$ is defined, i.e. $(n, \beta) \neq (0, 0), (1, 0), (2, 0)$, $\sum_i n_i = n$, $< \tau_0^{n_0} \bullet \cdots \bullet \tau_m^{n_m} > \equiv < \tau_0^{n_0} \bullet \cdots \bullet \tau_m^{n_m} >_\beta q^\beta$ and $F$ is considered as a formal power series in $\mathbb{Q}[[t]]$.

Now, differentiating $F$ corresponds to adding terms to the bracket. In particular;

$$c_{ijk} := \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = \sum_{n \geqslant 0} \sum_\beta \frac{1}{n!} < \gamma^n \bullet \tau_i \bullet \tau_j \bullet \tau_k >_\beta . \tag{3.2.16}$$

Using this, we can define a new product

$$\tau_i \star \tau_j := c_{ijk} \eta^{kf} \tau_f, \tag{3.2.17}$$

where $(\eta_{ij})$ is the intersection matrix of $H^*(X)$;

$$\eta_{ij} := \int_X \tau_i \cup \tau_j, \tag{3.2.18}$$

and $(\eta^{ij}) = (\eta_{ij})^{-1}$.

By extending (3.2.17) $Q[[t]]$-linearly to $H^*(X) \otimes_\mathbb{Q} \mathbb{Q}[[t]] \equiv QH^*(X)$, we have the following

**Theorem 3.2.4.** The big quantum cohomology, $QH^*(X)$, is an associative, commutative unital $\mathbb{Q}[[t]]$-algebra.

*Proof.* A full proof of Theorem 3.2.4 may be found in [63] (proof of Theorem 4). The arguments go roughly as follows.

Commutativity.
By (3.2.17) commutativity of $QH^*(X)$ reduces to symmetry of $c_{ijk}$ in all three indices. By (3.2.16) $c_{ijk}$ is a third partial derivative of $F$ and so the result follows.

---

[**]I.e. $\{\tau_i\}_{i=1,\cdots,p}$ is a basis of $H^{1,1}(X, \mathbb{Z})$.

<u>Unity.</u>

The unit of $QH^*(X)$ is as in Lemma 3.2.2. From (3.2.16), using the string equation (3.2.7), it can be shown that

$$c_{0jk} = < \tau_0 \bullet \tau_j \bullet \tau_k >_0 = \int_X \tau_j \cup \tau_k = \eta_{jk} \implies \tau_0 \star \tau_j = \eta_{je}\eta^{ef}\tau_f = \tau_j. \tag{3.2.19}$$

<u>Associativity.</u>

As is common, the hardest part of the proof concerns associativity, and so we shall not say anymore about this apart from that it follows from the splitting property (Lemma 4.18 in [109]), and the equivalence of points on $\mathbb{P}^1$. $\qquad\square$

Moreover, $QH^*(X)$ is a Frobenius manifold where $e = \tau_0$ via the isomorphism induced by sending $\partial_{t^\alpha} \mapsto \tau_\alpha$. The Euler vector field is given by

$$E = c_1(X) + \sum_\alpha \left(1 - \frac{1}{2}\deg(\tau_\alpha)\right) t^\alpha \tau_\alpha, \tag{3.2.20}$$

satisfying

$$\mathcal{L}_E F_0^X = (3 - \dim_{\mathbb{C}}(X))F_0^X + \text{ quadratic polynomial.} \tag{3.2.21}$$

Here, the prepotential $F_0^X \equiv F_0^X(\gamma)$ is given by

$$F_0^X(\gamma) = \overbrace{\frac{1}{6}\int_X \gamma^3}^{\text{classical}} + \overbrace{\sum_{n=0}^{\infty}\sum_\beta \frac{1}{n!}\underbrace{< \gamma \bullet \cdots \bullet \gamma >}_{n}}^{\text{quantum}} n_{0,n,\beta}^X, \tag{3.2.22}$$

and is indeed a solution of the WDVV equations (2.1.6), with $t^0$ only appearing in the classical part. Note here that the only non-zero terms are when $\beta$ is an effective curve class [97], which implies that the a priori infinite sum, (3.2.22) in fact consists of a finite number of terms [63].

**Remark.** The small quantum cohomology is obtained from the big quantum cohomology by restricting the $\star$-product to the parameters $\{\tau_i\}_{i=1,\cdots,p}$ corresponding to the Kähler classes. Thus, one can think of $F$, or $c_{ijk}$, as having two parts, one coming from the small quantum cohomology which encodes three-point invariants, and one carrying the remaining information. That is,

$$F = F_{\text{classical}} + F_{\text{quantum}} \implies c_{ijk} = \overline{c_{ijk}} + \overline{\Gamma_{ijk}}. \tag{3.2.23}$$

For the $c$-tensor, we have

$$\overline{c_{ijk}} = c_{ijk}(t_0, \cdots, t_p, 0, \cdots, 0) = \int_X \tau_i \cup \tau_j \cup \tau_k, \tag{3.2.24a}$$

$$\overline{\Gamma_{ijk}} = \sum_{n\geqslant 0}\frac{1}{n!}\sum_{\beta\neq 0} < \gamma^n \cdot \tau_i \cdot \tau_j \cdot \tau_k >_\beta = \sum_{\beta\neq 0} < \tau_i \cdot \tau_j \cdot \tau_k > q_1^{\int_\beta \tau_1}\cdots q_p^{\int_\beta \tau_p} \tag{3.2.24b}$$

with $\gamma$ as in (3.2.15) and $q_i = e^{t_i}$.

**Remark.** It is often the case that the small quantum cohomology is much easier to describe, and sometimes this is sufficient, depending on what one wants to study. For example if the small quantum cohomology is generically semisimple this is also the case for the big quantum cohomology. However, if a generic small quantum cohomology is non-semisimple, this does not imply that the big quantum cohomology is non-semisimple, as illustrated by Example 8 [36, 64, 101].

**Remark.** Some final remarks on this are in order. Firstly, there is no guarantee that the series (3.2.15) converges. Thus, one can either treat everything formally, as we did here, leading to a *formal* Frobenius manifold, or one can make the assumption that the series does indeed converge on some subdomain and restrict thereafter. Secondly, at the beginning of this subsection, we restricted to the even part of the cohomology $H^*(X)$ in order for the bilinear form to be symmetric. An alternative approach is to promote the structure to a super-manifold. We will not go further into this, see [97] for more details.

## 3.3 Hurwitz Frobenius manifolds

Dubrovin gives in [41] sufficient conditions for when we may obtain Frobenius manifolds of type $M_{LG}$ from a function by taking residues as in Example 2. This is phrased through the theory of *Hurwitz spaces*, and such Frobenius manifolds are called *Hurwitz Frobenius manifolds.*

Hurwitz spaces are moduli spaces parametrising ramified covers of the Riemann sphere. A point in such a space is an equivalence class $[\lambda : \mathcal{C}_g \to \mathbb{P}^1]$, where $\mathcal{C}_g$ is a smooth genus $g$ algebraic curve and $\lambda$ a morphism to the complex projective line realising $\mathcal{C}_g$ as a branched cover of $\mathbb{P}^1$. The equivalence relation is here given by automorphisms of the cover, i.e. a pair of covers $\lambda_1 : \mathcal{C}_g \to \mathbb{P}^1$, $\lambda_2 : \mathcal{C}_g \to \mathbb{P}^1$, are equivalent if and only if there exists a homeomorphism $h : \mathcal{C}_g \to \mathcal{C}_g$ such that $\lambda_1 = \lambda_2 \circ h^*$.

In a Frobenius manifold context, we want to consider Hurwitz spaces with fixed ramification over infinity. Let the preimage of $\infty$ consist of $m+1$ distinct points, denoted by $\infty_i \in \mathcal{C}_g$ for $i = 0, \cdots, m$, and let's denote $\mathcal{C}_g$ by $\mathcal{C}_{g,n}$ to reflect this. Furthermore, let $\lambda$ have degree $n_i + 1$ near $\infty_i$. We denote such a Hurwitz space by $\mathcal{H}_{g;n}$, where $n := (n_0, \cdots, n_m)$ describes the ramification. This is an irreducible quasi-projective complex variety of dimension[†]

$$d_{g;n} := \dim(\mathcal{H}_{g;n}) = 2g + 2m + \sum_{i=0}^m n_i. \tag{3.3.1}$$

---

[*]This can also be phrased in terms of equivalence classes of monodromy representations.
[†]This follows from the Riemann-Hurwitz formula.

We will write $\pi$, $\lambda$ and $\sigma_i$ for the universal family, the universal map, and the sections marking $\infty_i$, respectively, as depicted in the following commutative diagram:

$$
\begin{array}{ccccc}
\mathcal{C}_{g,n} & \hookrightarrow & \mathcal{C} & \xrightarrow{\ \lambda\ } & \mathbb{P}^1 \\
\scriptstyle P_i \Big\uparrow \Big\downarrow & & \scriptstyle \sigma_i \Big\uparrow \Big\downarrow \pi & & \\
[\lambda] & \xhookrightarrow{\ \mathrm{pt}\ } & \mathcal{H} & &
\end{array}
\tag{3.3.2}
$$

We furthermore denote by $\mathrm{d} = \mathrm{d}_\pi$ the relative differential with respect to the universal family and $p_i^{\mathrm{cr}} \in \mathcal{C}_{g,n} \simeq \pi^{-1}([\lambda])$ the critical points $\mathrm{d}\lambda = 0$ of the universal map. As mentioned, by the Riemann existence theorem, the critical values of $\lambda$, $\{u_i\}_{i=1,\cdots,d_{g;n}}$, serve as local coordinates away from the closed subsets in $\mathcal{H}_{g;n}$ in which $u_i = u_j$ for $i \neq j$, whose union is the discriminant. Additionally, there is an action on a Hurwitz space given by the affine subgroup of the $\mathrm{PGL}_2(\mathbb{C})$-action on the target,

$$
(\mathcal{C}, \lambda) \mapsto (\mathcal{C}, a\lambda + b), \qquad u_i \mapsto au_i + b,
\tag{3.3.3}
$$

for $a, b \in \mathbb{C}$, and $i = 1, \cdots, d_{g;n}$.

Let $\mathcal{H}_{g;n}^\phi$ be defined from $\mathcal{H}_{g;n}$ by fixing a meromorphic differential $\phi$. In [41], Dubrovin classifies the types of allowed differentials leading to a Hurwitz Frobenius manifold into five types. For the purposes of this thesis the differential will always be the square of a third-kind meromorphic differential $\phi \in \Omega_{\mathcal{C}}((\lambda))$, that is, it has at most simple poles at the poles of $\lambda$ and vanishing periods. There is an additional compatibility condition between $\phi$ and $\lambda$, stating that the differential must be *admissible*. We will not describe what this means in full generality, as it is quite technical, but we will define it in Chapter 5 to the generality we require in this thesis.

Let us now construct a Frobenius manifold on the cover of $\mathcal{H}_{g;n}^\phi$. We first posit that the coordinate vector fields in the $u$-chart are idempotents of the algebra

$$
\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij} \partial_{u_i},
\tag{3.3.4}
$$

which provides the structure of a semisimple commutative unital algebra with identity and Euler vector field

$$
e = \sum_{i=1}^{d_{g;n}} \partial_{u_i}, \quad E = \sum_{i=1}^{d_{g;n}} u_i \partial_{u_i},
\tag{3.3.5}
$$

arising as the generators of the affine action in (3.3.3). Again, notation is not chosen randomly, the chart $\{u_i\}$, consisting of the critical values of $\lambda$ corresponds indeed to the canonical coordinates for the resulting Frobenius structure.

What remains to be constructed to define a full-fledged Frobenius manifold structure on $H_{g,n}$ is a flat nondegenerate symmetric pairing playing the role of $\eta$, such that the vector fields $e$ and $E$

are, respectively, horizontal and linear under its Levi-Civita connection. This is where the chosen admissible differential comes in. We may define $\eta$ by the residue formula

$$\eta(X,Y) := \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{X(\lambda)Y(\lambda)}{\mathrm{d}\lambda} \phi^2, \tag{3.3.6}$$

for $X,Y \in \Gamma(\mathcal{H}_{g;\mathsf{n}}, T\mathcal{H}_{g;\mathsf{n}})$. Furthermore, combining (3.3.4) and (3.3.6) the 3-tensor $c$ is defined by the LG formula

$$c(X,Y,Z) \equiv \eta(X,Y \cdot Z) = \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{X(\lambda)Y(\lambda)Z(\lambda)}{\mathrm{d}\lambda} \phi^2, \tag{3.3.7}$$

which clearly satisfies the Frobenius property. Moreover, the second flat pairing is obtained by replacing $\lambda$ by $\log \lambda$ in (3.3.6),

$$g(X,Y) = \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{X(\log \lambda)Y(\log \lambda)}{\mathrm{dlog} \lambda} \phi^2. \tag{3.3.8}$$

**Definition 3.3.1** (LG–model). We call the marked meromorphic function $\lambda$, as above, a *Landau-Ginzburg (LG) superpotential* for the Frobenius manifold, and $\phi$ its primary differential. Together $(\lambda, \phi)$ make up the Landau-Ginzburg model of the Frobenius manifold.

In fact, we have already seen an example of a Hurwitz Frobenius manifold;

**Example 9.** Example 2, as described above, where $\phi = \mathrm{d}q$.

**Example 10** (LG–model for $\mathrm{QH}^*(\mathbb{P}^1)$)**.** Example 3 above has the following LG–description:

$$(\lambda, \phi) = \left( \frac{a_0}{\mu}(1 + \mu^2 - a_1\mu), \frac{\mathrm{d}\mu}{\mu} \right).$$

**Remark.** Note that almost duality in this context can easily be seen as replacing $\lambda$ by its logarithm, due to the forms of the residue formulae (3.3.6)-(3.3.8).

A highly nontrivial problem in general is to find flat coordinates for the Saito metric, $\eta$. One benefit of Hurwitz Frobenius manifolds is that such a set of coordinates may be found in a straightforward way, which we will now describe.

Define local coordinates $\kappa_i$ by $\lambda(\mu) = \kappa_i^{(n_i+1)}(\mu)$ near $\mu = \infty_i$. Then we have the following

41

**Theorem 3.3.2** (Theorem 5.1 in [41]). Given an LG–model $(\lambda(p), \phi = dp)$, the following functions on $H_{g_\omega, n_\omega}$ give $\eta$-flat coordinates

$$t_{i;\alpha} = \operatorname*{Res}_{\infty_i} \kappa_i^{-\alpha} p \, d\lambda, \quad i = 0, \cdots, m, \; \alpha = 1, \cdots, n_i; \tag{3.3.9}$$

$$t_j^{\text{ext}} = \text{p.v.} \int_{\infty_0}^{\infty_j} dp, \quad j = 1, \cdots, m; \tag{3.3.10}$$

$$t_k^{\text{Res}} = \operatorname*{Res}_{\infty_k} \lambda \, dp, \quad k = 1, \cdots, m, \tag{3.3.11}$$

where the principal value, p.v., indicates subtraction of the divergent part in $\kappa_i$.

**Remark.** In general, there are more equations in Theorem 3.3.2, than the dimension of the Frobenius manifold, so it is implicit in the theorem that some of these will result in equivalent coordinates. Moreover, in the theorem as it is stated in [41], there are two additional ways to obtain flat coordinates:

$$r_j = \oint_{b_j} dp, \tag{3.3.12}$$

$$s_j = -\frac{1}{2\pi i} \oint_{a_j} \lambda \, dp, \quad j = 1, \cdots, g, \tag{3.3.13}$$

where $\{a_i, b_i\}_{i=1,\cdots,g}$ is a choice of marked symplectic basis of integral one-cycles of the curve $\mathcal{C}$, and $g$ is the genus of $\mathcal{C}$. These, however, will not be necessary for this thesis as they will either give 0 or coincide with some of the above.

## 3.4 Mirror symmetry

We have now seen three very different settings in which Frobenius manifolds arise. While mirror symmetry is a phenomenon whose definition varies widely, in this thesis it will be synonymous with Frobenius manifold mirror symmetry:

**Definition 3.4.1.** *Mirror symmetry* is an equivalence of a pair of Frobenius manifolds

$$\mathcal{M} \cong \mathcal{M}' \tag{3.4.1}$$

where $\mathcal{M}$, $\mathcal{M}'$ are described in two distinct ways out of the three; $A : \mathcal{M}_{\text{Q-coh}}$, $B : \mathcal{M}_{\text{LG}}$, or $C : \mathcal{M}_{V/G}$ as defined in Sections 3.2, 3.3, and 3.1, respectively. The isomorphism between the pair is called the *mirror map*.

Definition 3.4.1 implies that all the objects described throughout Chapter 2 are equivalent.

We have already seen several examples of mirror symmetry. We have seen an instance of B–C mirror symmetry between Example 2, where the superpotential is the miniversal deformation of a simple singularity of type $A_n$, and Example 7 in which the Frobenius manifold is built on the orbit space of the Weyl group associated to the $A_n$ root system.

The attentive reader may have spotted a pattern, which indeed holds true in full generality; for simply laced Dynkin types (i.e. $\mathcal{R} \in \{A_n, D_l, E_r\}$, for $n \geqslant 1, l \geqslant 4, r = 6, 7, 8$), we have mirror symmetry $M_{\mathrm{LG}} \cong M_{V/G}$ where the LG–superpotential is the miniversal deformation of the simple singularity of type $\mathcal{R}$, as shown in Table 3.2, and $G$ is the Weyl group associated to a simple complex Lie algebra of Dynkin type $\mathcal{R}$ [41];

| A | B |
|---|---|
| Frobenius manifold from V/G | Frobenius manifold from $\lambda(q, \underline{a})$ |
| $G = W_\mathcal{R}$ | $\lambda_\mathcal{R}$ as in Table 3.2 with $\phi = \mathrm{d}q$ |

| $\mathcal{R}$ | Singularity | Miniversal deformation ($\lambda_\mathcal{R}$) |
|---|---|---|
| $A_l$ | $q^{l+1}$ | $q^{l+1} + a_{l-1}q^{l-1} + \cdots + a_1 q + a_0$ |
| $D_l$ | $q_1^{l-1} + q_1 q_2^2$ | $q_1^{l-1} + q_1 q_2^2 + a_{l-1}q_1^{l-2} + \cdots + a_1 + a_0 q_2$ |
| $E_6$ | $q_1^4 + q_2^3$ | $q_1^4 + q_2^3 + a_6 q_1^2 q_2 + a_5 q_1 q_2 + a_4 q_1^2 + a_3 q_2 + a_2 q_1 + a_1$ |
| $E_7$ | $q_1^3 q_2 + q_2^3$ | $q_1^3 q_2 + q_2^3 + a_7 q_1^4 + a_6 q_1^3 + a_5 q_1 q_2 + a_4 q_1^2 + a_3 q_2 + a_2 q_1 + a_1$ |
| $E_8$ | $q_1^5 + q_2^3$ | $q_1^5 + q_2^3 + a_8 q_1^3 q_2 + a_7 q_1^2 q_2 + a_6 q_1^3 + a_5 q_1 q_2 + a_4 q_1^2 + a_3 q_2 + a_2 q_1 + a_1$ |

Table 3.2: Miniversal deformations of hypersurface singularities as classified by Arnold.

We have also seen an instance of A–B mirror symmetry, between the Frobenius manifold built on the quantum cohomology of the complex projective line (Example 3), and the LG–model in Example 10;

$$(\lambda, \phi) = \left( \frac{a_0}{\mu}(1 + \mu^2 - a_1 \mu), \frac{\mathrm{d}\mu}{\mu} \right).$$

Additionally, Example 3 is, in a sense, of type $\mathcal{M}_{V/G}$, but now $G$ is the so-called *extended affine* Weyl group of type $A_1$ [46], which will be described in Chapter 4. Furthermore, the LG–superpotential can be shown to arise from a family of curves realised as a characteristic equation of the Lax operator associated to the relativistic Toda hierarchy of type $A_1$. Thus, in this case we do indeed have an instance of a 'triangle' of mirror symmetry between all the three types. In fact, this is the simplest example of one of the main theorems of this thesis, Theorem 5.2.5, published in [20]. As will later be shown, the B–C mirror symmetry of this kind generalises to any $\mathcal{R} \in \{A_l, B_l, C_l, D_l, E_6, E_7, E_8, F_4, G_2\}$. Moreover, in the case of simply laced Dynkin types (that is $\mathcal{R} \in \{A_l, D_l, E_6, E_7, E_8\}$) we have a A–B–C triangle of mirror symmetry where the A–model description is in terms of the quantum cohomology of certain orbifolds. The latter will be further described in Part III, Chapter 9.

**Remark.** It is worth mentioning that different Frobenius manifold axioms come naturally in the different descriptions. For example, in the A–description, semisimplicity is not guaranteed, and highly nontrivial. In fact, there is a well-known conjecture, first formulated by Dubrovin, stating that semisimplicty of the big quantum cohomology of $X$ is equivalent to the existence of a full exceptional collection in the bounded derived category of coherent sheaves on $X^*$. From the B–model description, semisimplicity occurs by construction, flatness of $\eta, e$, and $\nabla E$, on the other hand, are all non-automatic. It is from this perspective weaker notions of Frobenius manifolds are natural, by removing some of these axioms; for instance an almost Frobenius manifold lacks the flatness of the identity vector field. Furthermore, F-manifolds, which lack a flat metric as well, form an active field of research. One way to view the axioms of Frobenius manifolds (and its weaker cousins) is through the notion of algebraic integrability, as we shall now describe.

## 3.5    Algebraic integrability and LG–models from Lie theory

As mentioned in the introduction, it can be extremely difficult to find a LG–description for a given Frobenius manifold. One way to approach doing so is to relate the Frobenius manifold to some integrable system in such a way that an LG–superpotential, $\lambda$, arises as the *spectral parameter* of an associated *spectral curve.*

That is, through the equivalence of completely algebraically integrable systems and special Kähler manifolds* where the associated Liouville tori, or fibration of abelian varieties, are given by (generalised) Jacobians[†] of a family of algebraic curves which are given as the characteristic equation of a Lax operator with a spectral parameter.

$$\mathcal{C}_b = \{\det(L(\lambda, b) - \mu \mathbf{1}) = 0\}, \tag{3.5.1}$$

where $\lambda$ is the spectral parameter, the parameters $b_i$'s are time independent elements of smooth functions in involution, and act as action variables, and we think of (3.5.1) as a family of complex plane curves in variables $(\lambda, \mu) \in \mathbb{P}^2$. This curve together with the notion of almost duality induces the Frobenius manifold structure [75].

If a Frobenius manifold is characterised in this way, we call it *algebraic integrable*. When this holds, however, is generally unknown. It has been conjectured to have a connection with the Hitchin integrable system [75], and while this is still a conjectural relationship, all currently known instances of algebaic integrable Frobenius manifolds fit into this picture. For example, the case of

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*This implicitly uses homological mirror symmetry.

*with a flat Lagrangian lattice on the cotangent space.

[†]More precisely Prym (-Tuyrins) subvarieties.

the miniversal unfolding of a simple singularity was shown in [43] to arise from the open Toda chain[‡] via almost-duality. Furthermore, some cases which are not Hitchin type is shown to not result in a Frobenius manifold, such as Calogero-Moser systems [76]. I will not define all the associated and aforementioned notions here. Rather, I will describe roughly how such a spectral curve is constructed from Lie theory, and briefly show how it works in the case of Example 2. This will be mostly following [17] for the general description, and [76] for $A_n$. If the reader is interested in a more thorough description of the connection with special geometry, please see [75].

**Definitions and initial setup**

For this part, we generalise the presentation of [17].

We will now show how to obtain an integrable system and spectral curve from Lie theory. The starting point in this description is a complex simple Lie algebra $\mathfrak{g}$ of root type $\mathcal{R}$, with Cartan subalgebra $\mathfrak{h}$. Let $\mathcal{G} = \exp \mathfrak{g}$ be the corresponding simply-connected complex Lie group, and let us fix a choice of simple roots $\Pi$. This choice of simple roots induces a splitting of the full root system into positive and negative roots which again determines Borel subgroups $\mathcal{B}^\pm$ intersecting at the maximal torus $\mathcal{T} = \exp(\mathfrak{h})$. Thus, $\mathcal{G}$ is realised as a disjoint union of double cosets in terms of the Weyl group

$$\mathcal{W} : \mathcal{G} = \mathcal{B}^\pm \mathcal{W} \mathcal{B}^\pm = \bigsqcup_{(w_+, w_-) \in \mathcal{W} \times \mathcal{W}} C_{w_+, w_-},$$

where $C_{w_+, w_-} := \mathcal{B}^+ w_+ \mathcal{B}^+ \cap \mathcal{B}^- w_- \mathcal{B}^-$ are the double Bruhat cells of $\mathcal{G}$.

**Kinematics**

Consider the algebraic torus $\mathcal{P} \cong ((\mathbb{C}_x^*)^{l_\mathcal{R}}, (\mathbb{C}_y^*)^{l_\mathcal{R}}, \{\}_\mathcal{G})$ with Poisson bracket $\{x_i, y_j\}_\mathcal{G} = \mathcal{C}_{ij}^\mathfrak{g} x_i y_j$, where $\mathcal{C}^\mathfrak{g}$ is the Cartan matrix[§], making $\mathcal{P}$ symplectic.

Define $\mathcal{P}^{\text{Toda}} := C_{\bar{w}, \bar{w}} / \mathcal{T} \subset \mathcal{G} / \mathcal{T}$, where $\bar{w}$ is the ordered product of the $l_\mathcal{R}$ simple reflections in $\mathcal{W}$ (i.e. a Coxeter element of $\mathcal{W}$). There is an injective morphism $\mathcal{P} \to \mathcal{P}^{\text{Toda}}$ such that $\mathcal{P}^{\text{Toda}}$ inherits the symplectic structure from $\mathcal{G}$ (since $\mathcal{T}$ is a trivial Poisson submanifold) given by the canonical Belanin-Drinfeld-Olive-Turok solution of the classical Yang-Baxter equation ([8, 107]):

$$\{g_1 \overset{\otimes}{,} g_2\}_{PL} = \frac{1}{2}[r, g_1 g_2], \tag{3.5.2}$$

where

$$r = \sum_{i \in \Pi} h_i \otimes h_i + \sum_{\alpha \in \Delta^+} e_\alpha \otimes e_{-\alpha} \in \mathfrak{g} \otimes \mathfrak{g}, \tag{3.5.3}$$

---

[‡]More precisely, from a limit of the periodic Toda chain.
[§]Semisimplicity of the Lie group implies the nondegeneracy of the Poisson bracket.

with $\Delta^+$ denoting the set of positive roots, and $\{h_i \in \mathfrak{h}, e_{\pm i} \in \mathrm{Lie}(\mathcal{B}^\pm) | i \in \Pi\}$ being a Chevalley basis for $\mathfrak{g}$[¶].

Now, consider the Lax map:

$$L : \mathcal{P} \to \mathcal{P}^{\mathrm{Toda}} \qquad (x,y) \mapsto \prod_{i=1}^{l_\mathcal{R}} H_i(x_i) E_i H_i(y_i) E_{-i}, \tag{3.5.4}$$

where $H_i(x_i) = \exp(x_i h_i)$, $E_i(x_i) = \exp(x_i e_i)$, and similarly for $y_i$, are Chevally generators on $\mathcal{G}$. By Fock-Goncharov [60] this map is an algebraic Poisson embedding into an open subset of $\mathcal{P}^{\mathrm{Toda}}$.

**Dynamics**

Let us now define a complete integrable system of involutive Hamiltonians on $\mathcal{G}$, and hence on $\mathcal{P}^{\mathrm{Toda}}$. This is given by Weyl-invariant functions on $\mathcal{T}$ which is a subring of $\mathcal{O}(\mathcal{P}^{\mathrm{Toda}})$ generated by the regular fundamental characters

$$H_i(g) = \chi_{\rho_i}(g), \qquad i = 1, \cdots, l_\mathcal{R}, \tag{3.5.5}$$

where $\rho_i$ is the irreducible representation having the $i^{\mathrm{th}}$ fundamental weight, $\omega_i$, as highest weight.

The Lax map (3.5.4) pulls back this integrable system to $\mathcal{P}$, and fixing a faithful representation $\rho$, the same dynamics on $\mathcal{P}^{\mathrm{Toda}}$ takes the form of isospectral flows:

$$\frac{\partial \rho(L)}{\partial t_i} = \{\rho(L), H_i(L)\}_{PL} = [\rho(L), (P_i(\rho(L)))_+], \tag{3.5.6}$$

where $P_i \in \mathbb{C}[x]$ is the Weyl-invariant Laurent polynomial $\xi_{\omega_i}$ on the maximal torus in terms of power sums of the eigenvalues of the argument, and $()_+$ denotes the projection to the positive Borel $\mathcal{B}^+$.

Now, since the flows (3.5.6) are isospectral, the spectrum of $\rho(L)$ are integrals of motion, and we may define a family of spectral curves being the plane curve in $\mathbb{A}^2$ given by the zero locus of the characteristic equation of $L(\lambda)$ in representation $\rho$. In particular, this is an integral of motion since the determinant is.

**Example 11** ($A_n$). The Lax pair $(L, P)$ for the open Toda chain is given by

$$L = \sum_{i=1}^{\mathrm{rk}(\mathfrak{g})} p_i h_i + e^{(\alpha_i, q)} e_{\alpha_i} + f_{\alpha_i} + \zeta e_{\alpha_0}, \quad P = \sum_{i=1}^{\mathrm{rk}(\mathfrak{q})} f_{\alpha_i}, \tag{3.5.7}$$

where $\{\alpha_i\}$ is a fixed choice of simple roots, with $\alpha_0$ denoting the highest root and $\{h_i, e_{\alpha_i}, f_{\alpha_i}\}$ is the Chevalley basis.

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[¶]A Chevalley basis is chosen such that it satisfies the relations (2.1) in [17].

If one were to naively follow the description above, however, one would obtain a spectral curve which is not quite the correct one. The way to look at this case is to consider it as a limit of the periodic Toda chain. The periodic Toda chain has the spectral curve

$$\mathcal{C}_b := \left\{ \zeta + \frac{y}{\zeta} = \tilde{S}(x, b_1(p, q), \cdots, b_n(p, q)) \right\}, \tag{3.5.8}$$

where $\tilde{S}(x, b_1, \cdots, b_n)$ is the miniversal unfolding of the simple singularity of type $A_n$ with singular point $x = 0$. The spectral curve for the open Toda chain is obtained from (3.5.8) by taking the limit $y \to 0$, which results in the pairs of branch points merging to form singular points.

If one were to treat the spectral curve $\lim_{y \to 0} \mathcal{C}_b$ as an LG–superpotential on the base $B$ spanned by $\{b_i\}$, the identifications $\zeta = \lambda, x = \mu$, and taking $\phi = d\mu$ (and as usual employing almost duality) one would indeed obtain the superpotential in Example 2.

**Remark.** While a multiplication and a compatible metric may be appropriately defined for any integrable system, and the existence of a prepotential is guaranteed by the presence of the special Kähler structure, there are several Frobenius manifold axioms which are far from automatic from the integrable systems point of view. For instance, for the algebra to be associative, we must have a so-called *extra special* Kähler structure (which implies the presence of a potential, $\mathcal{F}$, satisfying WDVV equations). Furthermore, the metric is certainly not flat by default. If these properties hold, however, we call the associated structure a special F-manifold (in the language of Manin [97]). On the other hand, a given Frobenius manifold does not necessarily have a special Kähler structure. Thus, it would be interesting to investigate this further to attempt to describe the precise relationship between the two structures. Some work has been done in this direction, such as [75] where the authors show that the $F$-manifold is indeed canonical from this picture, but the full Frobenius structure is not. See [76] for a friendly overview of the connection between Frobenius manifolds and integrable structures.

This strategy turned out to be instrumental in finding B–models for a class of Frobenius manifolds called Dubrovin-Zhang (DZ) manifolds, which we shall now describe.

— 4 —

# DUBROVIN–ZHANG MANIFOLDS

In Section 3.1, we described the construction of a Frobenius manifold structure on the orbit space of a Coxeter group. Now, we want to restrict ourselves to the subset of these consisting of *Weyl* groups. These are classified in terms of (undirected) Dynkin diagrams which are a subset of the set of diagrams in Figure 3.1; the three infinite families $(A, B, D)$ - classical types, and the five exceptional types $(E_6, E_7, E_8, F_4, G_2 \cong I_2(6))$. In this section, a Dynkin diagram will be synonymous with a directed Dynkin diagram, as they correspond to root systems and simple Lie algebras. These are shown in Figure 4.1.

Now, the construction of Dubrovin and Zhang in [46] consists of taking the affine analogue of $W$, and extend it, and its action, in a certain way[*].

## 4.1   Frobenius manifold construction

We will here give a condensed description of the Dubrovin–Zhang construction of semisimple Frobenius manifold structures on the space of regular orbits of extended affine Weyl groups, which follows closely the account given in [20].

Let $\mathfrak{g}_{\mathcal{R}}$ be a rank-$l_{\mathcal{R}}$ complex simple Lie algebra associated to a root system $\mathcal{R}$, $\mathfrak{h}_{\mathcal{R}}$ the associated Cartan subalgebra, $\dim \mathfrak{h}_{\mathcal{R}} = l_R$, and $\mathcal{W}_{\mathcal{R}}$ the Weyl group. The construction of Dubrovin–Zhang Frobenius manifolds depends on a canonical choice of a marked node in the Dynkin diagram of $\mathcal{R}$, which will be labelled $\bar{k} \in \{1, \ldots, l_{\mathcal{R}}\}$, and we let $\alpha_{\bar{k}}$ and $\omega_{\bar{k}}$ denote the corresponding simple root and fundamental weight, respectively. The canonical node corresponds to the fundamental weight associated to the fundamental representation of smallest dimension. This node is an "attaching" vertex for the external nodes in the diagram, that is, one which if removed splits the corresponding finite Dynkin diagram into disconnected $A$-type pieces. We depict the canonical nodes on the affine Dynkin diagrams in Figure 4.1[*]. The action of $\mathcal{W}_{\mathcal{R}}$ on $\mathfrak{h}_{\mathcal{R}}$ may be lifted to an action of the affine

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[*]The extension is necessary in order to obtain a natural nondegenerate metric.

[*]Note that the choice of canonical node is unique except when $\mathcal{R} = A_l$ where any node may be taken to be

| $\mathcal{R}$ | $\bar{k}$ | Canonically marked affine Dynkin diagram |
|---|---|---|
| $A_l$ | $1, \ldots, l$ |  |
| $B_l$ | $l-1$ |  |
| $C_l$ | $l$ |  |
| $D_l$ | $l-2$ |  |
| $E_6$ | $3$ |  |
| $E_7$ | $3$ |  |
| $E_8$ | $3$ |  |
| $F_4$ | $2$ |  |
| $G_2$ | $2$ |  |

Figure 4.1: Affine Dynkin diagrams with canonical markings, as in [46, Table 1]. The node corresponding to the affine root is marked in black, and the canonically marked node is indicated by ×.

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canonical, resulting in distinct Frobenius manifolds up to the symmetry $\alpha_i \rightarrow \alpha_{l+1-i}$.

Weyl group $\widehat{\mathcal{W}}_{\mathcal{R}} \cong \mathcal{W}_{\mathcal{R}} \ltimes \Lambda_r^\vee(\mathcal{R})$, with $\Lambda_r^\vee(\mathcal{R})$ being the lattice of coroots:

$$\widehat{\mathcal{W}}_{\mathcal{R}} \times \mathfrak{h}_{\mathcal{R}} \mapsto \mathfrak{h}_{\mathcal{R}}, \tag{4.1.1}$$

$$((w, \alpha^\vee), h) \mapsto w(h) + \alpha^\vee, \tag{4.1.2}$$

with

$$\alpha_i^\vee := 2\frac{\alpha_i}{(\alpha_i, \alpha_i)}$$

Then the *extended* affine Weyl group, $\widetilde{\mathcal{W}}_{\mathcal{R}}$, is defined as the semi-direct product $\widetilde{\mathcal{W}}_{\mathcal{R}} := \widehat{\mathcal{W}}_{\mathcal{R}} \ltimes \mathbb{Z}$ acting on $\mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C}$ by

$$\widetilde{\mathcal{W}}_{\mathcal{R}} \times \mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C} \to \mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C}, \tag{4.1.3}$$

$$((w, \alpha^\vee, l), (h, v)) \mapsto (w(h) + \alpha^\vee + l\omega_{\bar{k}}, v - l). \tag{4.1.4}$$

Let $\Sigma_{\mathcal{R}}$ denote the central[†] hyperplane arrangement associated to the root system $\mathcal{R}$, and $\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}} := \mathfrak{h}_{\mathcal{R}} \backslash \Sigma_{\mathcal{R}}$ be the set of regular elements in $\mathfrak{h}_{\mathcal{R}}$. The restriction of (4.1.4) to $\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}} \oplus \mathbb{C}$ is then a free affine action, whose quotient defines the regular orbit space of the extended affine Weyl group of $\mathcal{R}$ with marked node $\bar{k}$ as

$$M_{\mathcal{R}}^{\mathrm{DZ}} := (\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}} \times \mathbb{C})/\widetilde{\mathcal{W}}_{\mathcal{R}} \cong \mathcal{T}_{\mathcal{R}}^{\mathrm{reg}}/\mathcal{W}_{\mathcal{R}} \times \mathbb{C}^* \cong (\mathbb{C}^*)^{l_{\mathcal{R}}+1}, \tag{4.1.5}$$

where $\mathcal{T}_{\mathcal{R}}^{\mathrm{reg}} = \exp(\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}})$ is the image of the set of regular elements of $\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}}$ under the exponential map to the maximal torus $\mathcal{T}_{\mathcal{R}}$. The isomorphism (4.1.5) follows from the Chevalley theorem proven in [46], in which the ring of $\widetilde{\mathcal{W}}_{\mathcal{R}}$-invariant functions is isomorphic to the polynomial ring $\mathbb{C}[y_1, \cdots, y_l, y_{l+1}]$ where $y_i$ is a Fourier polynomial in the $\mathcal{W}$-invariant polynomials in the linear coordinates, $\{x_i\}$ on $V$, with the final coordinate being exponential in $x_{l+1}$, i.e. the coordinate associated to the affine copy.

**Remark.** Let $(x_1, \ldots, x_{l_{\mathcal{R}}})$ be linear coordinates on $\mathfrak{h}_{\mathcal{R}}$ with respect to $\{\alpha_1^\vee, \ldots, \alpha_{l_R}^\vee\}$, the coroot basis, and extend these to linear coordinates $(x_1, \ldots, x_{l_{\mathcal{R}}}; x_{l_{\mathcal{R}}+1})$ on $\mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C}$. Writing $Q_i = \mathrm{e}^{x_i}$, we denote by $\mathcal{I}_{\mathcal{R}} := \mathbb{C}[Q_1^\pm, \ldots, Q_{l_{\mathcal{R}}+1}^\pm]$, the ring of regular functions on the algebraic torus $\mathcal{T}_{\mathcal{R}} \times \mathbb{C}^\star \simeq (\mathbb{C}^\star)^{l_{\mathcal{R}}+1}$. By its definition in (4.1.5), $M_{\mathcal{R}}^{\mathrm{DZ}}$ is a smooth complex manifold homeomorphic to a Zariski open subset of the affine GIT quotient $\mathrm{Spec}\, \mathcal{I}_{\mathcal{R}}^{\widetilde{\mathcal{W}}_{\mathcal{R}}}$. Note that, in [46, Section 1, Definition and Main Lemma], the authors consider a partial compactification of the latter to an affine scheme $\mathrm{Spec}\, \mathcal{A}_{\mathcal{R}}$, where $\mathcal{A}_{\mathcal{R}}$ is a polynomial subring, satisfying suitable boundedness conditions at infinity, of the Laurent polynomial ring $\mathcal{I}_{\mathcal{R}}$. Then $M_{\mathcal{R}}^{\mathrm{DZ}}$ also sits in the underlying affine variety as an open submanifold.

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[†]A central hyperplane arrangement is one in which hyperplanes pass through the origin.

The linear coordinates $(x_1, \ldots, x_{l_{\mathcal{R}}}; x_{l_{\mathcal{R}}+1})$ on $\mathfrak{h}_{\mathcal{R}}^{\mathrm{reg}} \oplus \mathbb{C}$ can serve as local coordinates on the regular orbit space. By orthogonal extension of minus the Cartan–Killing form on $\mathfrak{h}_{\mathcal{R}}$, we define a nondegenerate pairing $\xi$ on $\mathfrak{h}_{\mathcal{R}} \times \mathbb{C}$ by

$$\xi(\partial_{x_i}, \partial_{x_j}) := \begin{cases} -(\mathcal{K}_{\mathcal{R}})_{ij} & \text{if } i, j < l_{\mathcal{R}} + 1, \\ d_{\bar{k}} & \text{if } i = j = l_{\mathcal{R}} + 1, \\ 0 & \text{otherwise,} \end{cases} \qquad (4.1.6)$$

with $x_{l_{\mathcal{R}}+1}$ parametrising linearly the right summand in $\mathfrak{h}_{\mathcal{R}} \oplus \mathbb{C}$. Here, $d_i := \frac{<\omega_i, \omega_{\bar{k}}>}{<\omega_{\bar{k}}, \omega_{\bar{k}}>}$, as shown in Table 4.1, with $< \alpha, \beta >$ being the pairing on $\mathfrak{h}_{\mathcal{R}}^*$ induced by the restriction of the Killing form to the Cartan subalgebra. The quotient map $\aleph : \mathcal{T}_{\mathcal{R}}^{\mathrm{reg}} \times \mathbb{C}^* \to M_{\mathcal{R}}^{\mathrm{DZ}}$ from (4.1.5) defines a principal $\mathcal{W}_{\mathcal{R}}$-bundle on $M_{\mathcal{R}}^{\mathrm{DZ}}$: a section $\tilde{\sigma}_i$, lifts a (sufficiently small) open $U \subset M_{\mathcal{R}}^{\mathrm{DZ}}$ to the $i^{\mathrm{th}}$ sheet of the cover $V_i \in \tilde{\sigma}_i^{-1}(U) \equiv V_1 \bigsqcup \cdots \bigsqcup V_{|\mathcal{W}_{\mathcal{R}}|}$.

| $W$ | $\{d_i\}$ |
|---|---|
| $A_l$ | $d_i = \begin{cases} \dfrac{i(l - \bar{k} + 1)}{l + 1}, & \text{if } i \leqslant \bar{k}, \\ \dfrac{\bar{k}(l - i + 1)}{l + 1}, & \text{if } i > \bar{k} \end{cases}$ |
| $B_l$ | $d_i = \begin{cases} \dfrac{l - 1}{2}, & \text{if } i = l, \\ i, & \text{otherwise} \end{cases}$ |
| $C_l$ | $d_i = i$ |
| $D_l$ | $d_i = \begin{cases} \dfrac{l - 2}{2}, & \text{if } i \in \{l - 1, l\}, \\ i, & \text{otherwise} \end{cases}$ |
| $E_6$ | $\{2, 3, 4, 6, 4, 2\}$ |
| $E_7$ | $\{4, 6, 8, 12, 9, 6, 3\}$ |
| $E_8$ | $\{10, 15, 20, 30, 24, 18, 12, 6\}$ |
| $F_4$ | $\{3, 6, 4, 2\}$ |
| $G_2$ | $\{3, 6\}$ |

Table 4.1: Affine Weyl groups and associated degrees, as in [46, Table 2].

Then we have a reconstruction theorem, analogous to Theorem 3.1.1, [46, Thm 2.1].

**Theorem 4.1.1.** *There exists a unique (up to isomorphism) semisimple Frobenius structure* $\mathcal{M}_{\mathcal{R}}^{DZ} = (M_{\mathcal{R}}^{DZ}, e, E, \eta, \cdot)$ *on the orbit space of* $\widetilde{\mathcal{W}}_{\mathcal{R}}$ *satisfying the following properties in flat coordinates* $(t_1, \ldots, t_{l_{\mathcal{R}}+1})$ *for* $\eta$:

**DZ-I** $e = \partial_{t_{\bar{k}}}$;

**DZ-II** $E = \dfrac{1}{d_{\bar{k}}} \partial_{x_{l_{\mathcal{R}}+1}} = \sum_{j=1}^{l_{\mathcal{R}}} \dfrac{d_j}{d_{\bar{k}}} t_j \partial_{t_j} + \dfrac{1}{d_{\bar{k}}} \partial_{t_{l_{\mathcal{R}}+1}}$;

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**DZ-III** *the intersection form is $\gamma = \tilde{\sigma}_i^* \xi$;*

**DZ-IV** *the prepotential is polynomial in $t_1, \cdots, t_{l_{\mathcal{R}}+1}$, $e^{t_{l_{\mathcal{R}}+1}}$.*

Such Frobenius manifolds will always be of charge $d = 1$, or equivalently, the prepotential will be a degree 2 quasihomogeneous function of its arguments. Furthermore, by construction, the monodromy group of $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$ is isomorphic to $\widetilde{\mathcal{W}}_{\mathcal{R}}$.

## 4.2   Example and B–model outlook

**Example 12** ($A_2$, $k = 1$, Example 2.2 in [46]). By Theorem 4.1.1 and Table 4.1 we have

$$\{d_i\} = \{(\omega_j, \omega_k)/(\omega_k, \omega_k)\} = \{(\omega_j, \omega_1)/(\omega_1, \omega_1)\} = \left\{1, \frac{1}{2}, \frac{3}{2}\right\}, \quad e = \partial_1, \quad E = t_1 \partial_1 + \frac{1}{2} t_2 \partial_2 + \frac{3}{2} \partial_3. \tag{4.2.1}$$

The ring of $\widetilde{W}_{A_2}$-invariant (Fourier) polynomials $\mathcal{A} \cong \mathbb{C}[\tilde{y}]$, is generated by the global coordinates on $\mathcal{M} = \mathrm{Spec}(\mathcal{A})$, $\tilde{y}$, given by

$$\tilde{y}_j = \begin{cases} e^{2\pi i d_j x_3} y_j & j = 1, 2, \\ e^{2\pi i x_3} & j = 3, \end{cases} \tag{4.2.2}$$

where $y_j$ are the $W$-invariant Fourier polynomials given in this case in terms of the elementary symmetric polynomials $s_j(e^{2\pi i z_i})$ for $i = 1, 2, 3$. As in [46], we use local coordinates $\{y^j\}^*$

$$y^j = \begin{cases} 2\pi i x_3 & j = 3, \\ \tilde{y}_j & \text{otherwise.} \end{cases} \tag{4.2.3}$$

Thus, in this case we have

$$y^1 = e^{\frac{4}{3}\pi i x_3} \left(e^{2\pi i x_1} + e^{-2\pi i x_2} + e^{2\pi i (x_2 - x_1)}\right), \tag{4.2.4a}$$

$$y^2 = e^{\frac{2}{3}\pi i x_3} \left(e^{2\pi i x_2} + e^{-2\pi i x_1} + e^{2\pi i (x_1 - x_2)}\right), \tag{4.2.4b}$$

$$y^3 = 2\pi i x_3. \tag{4.2.4c}$$

From the Killing form we have the induced bilinear form;

$$(\,,\,)^{\sim} := \frac{1}{4\pi} \begin{pmatrix} (\omega_i, \omega_j) & 0 \\ 0 & -\frac{1}{d_k} \end{pmatrix} = \frac{1}{12\pi^2} \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -\frac{9}{2} \end{pmatrix}. \tag{4.2.5}$$

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*Note that $x_3$ and consequently $y^3$ is not invariant.

Then,

$$(g^{ij}) \equiv (\mathrm{d}y^i, \mathrm{d}y^j)^{\sim} = \sum_{a,b=1}^{l+1} \frac{\partial y^i}{\partial x^a} \frac{\partial y^j}{\partial x^b} (\mathrm{d}x^a, \mathrm{d}x^b)^{\sim}, \tag{4.2.6}$$

which can be written as

$$(g^{ij}) = \begin{pmatrix} 2t_2 e^{t_3} & 3e^{t_3} & t_1 \\ 3e^{t_3} & 2t_1 - \frac{t_2^2}{2} & \frac{t_2}{2} \\ t_1 & \frac{t_2}{2} & \frac{3}{2} \end{pmatrix}, \tag{4.2.7}$$

where $t_j = y_j$. Since $\eta^{-1} = \mathcal{L}_e g$ and $e = \partial_k = \partial_1$ we find,

$$(\eta^{ij}) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{4.2.8}$$

Using (2.3.4), together with the Euler vector field as given and $\eta$ as in (4.2.8), we may solve for the elements of the $c$-tensor. The nontrivial element of $c$ are given by

$$c_{2,2,2} = -\frac{t_2}{4}, \qquad c_{3,3,3} = e^{\frac{t_3}{2}}, \tag{4.2.9}$$

up to symmetry. Integrating results in the prepotential

$$F = \frac{1}{2}t_1^2 t_3 + \frac{1}{4}t_1 t_2^2 + \frac{1}{2}e^{t_3} - \frac{1}{96}t_2^4. \tag{4.2.10}$$

**Remark.** In Example 12, the flat coordinates coincided with $\{y^j\}$. This is not the case in general. Furthermore, note that in this construction flat coordinates must be derived on a case-by-case basis.

**Remark.** In the case of Example 12, there is an alternative derivation using the reconstruction theorem, Theorem 4.1.1, more directly. That is, one assumes the prepotential to be of the form as in the Theorem, use the action of the Euler vector field (2.1.4), with $d = 1$, to bound the degrees and solve WDVV (which is easily done using a computer). While this is doable in very small rank such as this example, it quickly becomes unfeasible. Therefore, one cannot obtain an explicit prepotential in this construction in general.

Now, as often is the case, we are interested in B–model, or Landau-Ginzburg, descriptions of these Frobenius manifolds. More precisely, we want to describe all DZ-manifolds as Hurwitz Frobenius manifolds, as described in Section 3.3. Note that the Type 1 transformation, as described in Section 3.3, in general changes the charge of a Frobenius manifold, except when $d = 1$, which is precisely the case for Dubrovin–Zhang manifolds!

Dubrovin and Zhang found Landau-Ginzburg descriptions for the DZ-Frobenius manifolds of A-type already in the original paper [46]. They are given in terms of trigonometric polynomials

$$\lambda_{l,k} = e^{ikp} + a_1 e^{i(k-1)p} + \cdots + a_{l+1} e^{i(k-l-1)p}, \quad \text{with } a_{l+1} \neq 0, \tag{4.2.11}$$

where $\phi = \mathrm{d}p$.

Furthermore, superpotentials for the remaining classical cases (i.e. $\mathcal{R} = B_l, C_l, D_l$) were found in [44]. In fact, the authors also managed to remove the restriction of choosing a special canonical Dynkin node. The resulting B–models are given by

$$\lambda_{l,k,m}(P) = \frac{1}{(P^2 - 1)^m} \sum_{j=0}^{l} a_j P^{2(k+m-j)}, \quad \text{with } P = \cos(p), \tag{4.2.12}$$

where $1 \leqslant k, m \leqslant l$, $k + m \leqslant l$, and $\phi = \mathrm{d}p$. Here the canonical cases are given by setting $k = \bar{k}$, and $m = 0$.

The methods utilised in these papers do not, however, lend themselves to finding superpotentials for the exceptional cases. The inspiration towards tackling this task came from dualities in physics.

# Part II

# B–models for DZ–manifolds

# UNIFORM B–MODELS FOR DZ–MANIFOLDS

## 5.1  Integrable systems and spectral curves from affine Lie theory

The aim of this Section is to outline how to construct an integrable system from an *affine* complex Lie algebra, and derive associated spectral curves, analogously to algebraic integrability as described in Section 3.5, where the integrable system considered is the relativistic Toda chain. Through almost duality, this will lead to LG–superpotentials for DZ-manifolds.

Most of the Lie algebraic setup in Section 3.5 carries through to the setting of the Kac-Moody group $\hat{\mathcal{G}} = \exp(\mathfrak{g}^{(1)})$, with $\mathfrak{g}^{(1)} \cong \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}] \oplus \mathbb{C}c$ being the affine Lie algebra corresponding to $\mathcal{R}$. Here we have adjoined the highest (affine) root which we denote by $\alpha_0$. Elements $g \in \hat{\mathcal{G}}$ have the form $M(\lambda)q^{\lambda \partial_\lambda}$, where $\lambda$ is the spectral parameter.

A set of Chevalley generators is then comprised of the Chevalley generators associated to $\mathcal{G}$ lifted to $\hat{\mathcal{G}}$, as well as three additional generators

$$H_0(q) = q^{\lambda \partial_\lambda}, \quad E_0 = \exp{(\lambda e_0)}, \quad E_{\bar{0}} = \exp{(\bar{e}_0 \lambda)}, \tag{5.1.1}$$

arising from having adjoined $\alpha_0$. Here $e_0 \in \mathrm{Lie}(\mathcal{B}^+)$, and $\bar{e}_0 \in \mathrm{Lie}(\mathcal{B}^-)$.

**Kinematics**

Now we consider the torus $(\mathbb{C}^*)^{2(l_\mathcal{R}+1)} \cong (\mathbb{C}_x^*)^{l_\mathcal{R}+1} \times (\mathbb{C}_y^*)^{l_\mathcal{R}+1}$, with exponentiated linear coordinates $(x_0, x_1, \cdots, x_{l_\mathcal{R}}; y_0, y_1, \cdots, y_{l_\mathcal{R}})$ and (log-constant) Poisson bracket

$$\{x_i, y_j\}_{\hat{\mathcal{G}}} = \mathcal{C}_{ij}^{\mathfrak{g}^{(1)}} x_i y_j. \tag{5.1.2}$$

We consider the hypersurface $\hat{\mathcal{P}}$ given by the zero locus of $\prod_{i=0}^{l_\mathcal{R}}(x_i y_i)^{\delta_i} - 1$, with $\delta_i$ being the $i^{\mathrm{th}}$ Dynkin label*. Now, $\hat{\mathcal{P}}$ is not symplectic, because the kernel of the map associated to the Cartan

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*The Dynkin labels of a weight are the coefficients of its expansion in terms of fundamental weights. The Dynkin labels of an irreducible representation are the Dynkin labels of its highest weight.

matrix is one-dimensional. Instead, we get that the regular function on the hypersurface

$$\mathfrak{N} := \prod_{i=0}^{l_{\mathcal{R}}} x_i^{\delta_i} = \prod_{i=0}^{l_{\mathcal{R}}} y_i^{-\delta_i} \tag{5.1.3}$$

is a Casimir of the bracket (5.1.2), and it foliates $\hat{\mathcal{P}}$ symplectically.

As in the non-affine case, we get a double coset decomposition of $\hat{\mathcal{G}}$ and a distinguished cell $C_{\bar{w},\bar{w}}$, where $\bar{w}$ corresponds to the longest cyclically irreducible word in the generators of $\widehat{W}$. Then after projecting to the trivial central (co-) extension

$$\pi : \hat{\mathcal{G}} \to \text{Loop}(\mathcal{G}); \qquad g = M(\lambda)q^{\lambda\partial\lambda} \mapsto M(\lambda), \tag{5.1.4}$$

we get an induced Poisson structure on each of the cells in the decomposition and on their quotients $C_{w_+,w_-}/\text{Ad}\mathcal{T}$. Letting $\hat{\mathcal{P}}^{\text{Toda}} := C_{\bar{w},\bar{w}}/\text{Ad}\mathcal{T}$, we do obtain a Poisson manifold, of dimension $2l_{\mathcal{R}} - 1$.

Consider

$$\hat{L}(\lambda) = \prod \hat{H}_i(x_i)\hat{E}_i\hat{H}_i(y_i)\hat{E}_{-i} = E_0(\lambda/y_0)E_{\bar{0}}(\lambda) \prod H_i(x_i)E_iH_i(y_i)E_{-i}. \tag{5.1.5}$$

Then $\hat{L}$ is a Poisson morphism by similar argument as $L$ in (3.5.4).

## Dynamics

Now we have that the same statements hold as in the non-affine case with the addition of the central Casimir $\mathfrak{N}$. The Lax map $\hat{L}$ pulls back the integrable dynamics to $\hat{\mathcal{P}}$, and after fixing a representation $\rho$ we have isospectral flows

$$\frac{\partial\rho(\hat{L})}{\partial t_i} = \{\hat{\rho}(L), \hat{H}_i(L)\}_{\text{PL}} = [\widehat{\rho(L)}, (P_i(\widehat{\rho(L)}))_+], \tag{5.1.6}$$

with $P_i \in \mathbb{C}[x]$ expresses the $i^{\text{th}}$ fundamental character in terms of eigenvalues of $\rho$.

Since (5.1.6) are isospectral, functions of the spectrum for $\rho(\hat{L})$ are integrals of motion. Similarly to the non-affine case, we want to consider the integrals of motion given by the vanishing locus of the plane curve

$$\det_\rho(\hat{L}(\lambda) - \mu\mathbf{1}), \tag{5.1.7}$$

for $(\lambda, \mu) \in \mathbb{A}^2$.

For reasons of simplicity, we choose to work in the smallest representation possible, that is, the (nontrivial) irreducible representation of lowest dimension. Let $\rho_\omega$ denote this minimal irreducible representation, with $\omega$ being the associated highest weight. Table 5.1 shows $\rho_\omega$ for each of the root types together with its dimension. Note that for any $\mathcal{R}$, $\rho_\omega$ is quasiminiscule. That is, all non-zero

| $\mathcal{R}$ | $\rho_\omega$ | $\dim(\rho_\omega)$ |
|---|---|---|
| $A_l$ | Fundamental/dual | $l+1$ |
| $B_l$ | Fundamental | $2l+1$ |
| $C_l$ | Fundamental | $2l$ |
| $D_l$ | Fundamental | $2l$ |
| $E_6$ | Fundamental/dual | 27 |
| $E_7$ | Fundamental | 56 |
| $E_8$ | Adjoint | 248 |
| $F_4$ | Fundamental | 26 |
| $G_2$ | Fundamental | 7 |

Table 5.1: Minimal nontrivial irreducible representations for simple Lie algebras.

weights in the weight system $\Gamma(\rho_\omega)$ are in the same Weyl-orbit. Furthermore, if $\mathcal{R} \neq B_l, E_8, F_4, G_2$, it is miniscule (i.e. quasiminiscule with no zero weights).

For any $\mathbf{g} \in \mathcal{G}$ (in the representation $\rho_\omega$) we have,

$$\mathcal{Q}_\omega \quad = \quad \det_{\rho_\omega} (\mathbf{g} - \mu \mathbf{1}) = \sum_{k=0}^{\dim \rho_\omega} (-\mu)^{(\dim \rho_\omega - k)} \chi_{\wedge^k \rho_\omega}(\mathbf{g}), \tag{5.1.8}$$

where $\chi_{\wedge^k \rho_\omega}$ denotes the exterior characters of the minimal representation. The second equality follows from the the cofactor expansion of the determinant, via the correspondence between elementary symmetric polynomials, Young diagrams and wedge products [62]. Recall that the representation ring of a simple Lie group is an integral polynomial ring generated by the fundamental representations. Translating this fact to characters we have that the Weyl character ring associated to a simple Lie group is generated by fundamental characters. Thus,

$$\chi_{\wedge^k \rho_\omega}(\mathbf{g}) = \mathfrak{p}_k^\omega(\chi_1, \ldots, \chi_{l_\mathcal{R}}) \in \mathbb{Z}[\chi_1, \ldots, \chi_{l_\mathcal{R}}], \tag{5.1.9}$$

where $\chi_i(\mathbf{g}) := \mathrm{Tr}_{\rho_i}(\mathbf{g})$ denotes the $i^{\mathrm{th}}$ fundamental character. Since $\rho_\omega$ is quasiminuscule, $\mathcal{Q}_\omega$ factorises as

$$\mathcal{Q}_\omega = (1 - \mu)^{\mathbf{z}_0} \mathcal{Q}_\omega^{\mathrm{red}} = (1 - \mu)^{\mathbf{z}_0} \prod_{0 \neq \omega' \in \Gamma(\rho_\omega)} \left( e^{\omega' \cdot h} - \mu \right), \tag{5.1.10}$$

where $\mathbf{z}_0$ is the dimension of the zero weight space of $\rho_\omega$, and $e^h$ with $[e^h] = [\mathbf{g}]$ is a choice of Cartan torus element conjugate to $\mathbf{g}$. In particular, $\mathbf{z}_0 = 0$ and $\mathcal{Q}_\omega^{\mathrm{red}} = \mathcal{Q}_\omega$ for $\mathcal{R} \neq B_l$, $E_8$, $F_4$, or $G_2$.

Let us now define

$$\mathcal{P}_\omega(w_0, \ldots, w_{l_\mathcal{R}+1}; \lambda, \mu) := \mathcal{Q}_\omega^{\mathrm{red}} \left( \chi_i = w_i - \delta_{i\bar{k}} \frac{\lambda}{w_0}; \mu \right), \tag{5.1.11}$$

and consider, as $w := (w_0; w_1, \ldots, w_{l_\mathcal{R}}) \in \mathbb{C}^* \times \mathbb{C}^{l_\mathcal{R}}$ varies, the family of plane algebraic curves in $\mathrm{Spec}\mathbb{C}[\lambda, \mu]$ with fibre at $w$ given by $C_w^{(\omega)} := \mathbb{V}(\mathcal{P}_\omega)$. We compactify and desingularise the

fibres over $w$ by taking the normalisation, $\overline{C_w^{(\omega)}}$, of their closure in $\mathbb{P}^2$. Marking the $\lambda$-projection $(\lambda, \mu) \to \lambda \in \mathbb{P}^1$ and varying $w$ defines a subvariety $M_\omega^{\mathrm{LG}}$ of the Hurwitz space $H_{g_\omega, \mathsf{n}_\omega}$, where $g_\omega = h^{1,0}\big(\overline{C_w^{(\omega)}}\big)$, and $\mathsf{n}_\omega$ records the ramification profile at infinity of the $\lambda$-projection.

## Why the relativistic Toda chain?

The motivation behind considering the relativistic Toda chain in the first place comes from physics, or more precisely the Gopakumar-Ooguri-Vafa (GOV) correspondence [69, 108]. This correspondence describes a duality between two theories in physics; a topological gauge theory (realised as a $U(N)$ Chern-Simons theory on the 3-dimensional sphere [121]), and a topological string theory (the topological A–model on the resolved conifold $\mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1)) \to \mathbb{P}^1$). The duality has profound consequences for mathematics. In fact, it provides a mirror theorem, in a sense, between Gromov–Witten theory (or Donaldson-Thomas theory) and the theory of quantum invariants of knots (Reshetikhin–Turaev–Witten invariants) [16].

In [12], Gaetan Borot and Andrea Brini consider this correspondence as $N$ grows large for 3-dimensional Spherical Seifert manifolds, which are quotients of the 3-dimensional sphere by the free action of a finite isometry group $\Gamma$, which posesses an $ADE$-classification. They relate this to the Toda integrable system as above and explicitly calculate the (affine co-extended) Toda spectral curve for Dynkin type $D$ (as type $A$ was readily available in the literature), and conjecture that for all simply laced cases, one should obtain a family of spectral curves inducing an LG–model for a DZ-manifold associated to a simple laced Dynkin diagram. In particular, for the $D_{p+2}$-case, the resulting curve is given by

$$X\mathcal{P}_{D_{p+2}}^{\mathrm{Toda}}(X,Y) = w_0(X^2+1)(Y-1)^2(Y+1)^2 Y^p + \sum_{i=0}^{2(p+2)} (-1)^i \epsilon_i XY^i, \qquad (5.1.12)$$

where

$$\tilde{\epsilon}_i(X) = \begin{cases} \epsilon_i + w_0\left(X + \dfrac{1}{X}\right) & i = p, p+4, \\[2mm] \epsilon_{p+2} - 2w_0\left(X + \dfrac{1}{X}\right) & i = p+2, \\[2mm] \epsilon_i & \text{otherwise,} \end{cases} \qquad (5.1.13)$$

with $\tilde{\epsilon}_i = \tilde{w}_i$, for $i \leqslant p$, and

$$\tilde{\epsilon}_{p+1} = \tilde{w}_{p+1}\tilde{w}_{p+2} - \begin{cases} \displaystyle\sum_{k=0}^{\frac{p}{2}-1} \tilde{w}_{2k+1} & p \text{ even}, \\[4mm] \displaystyle\sum_{k=0}^{\frac{p}{2}} \tilde{w}_{2k} & p \text{ odd}, \end{cases} \qquad (5.1.14a)$$

$$\tilde{\epsilon}_{p+1} = \tilde{w}_{p+1}^2 + \tilde{w}_{p+2}^2 - 2 \begin{cases} \displaystyle\sum_{k=0}^{\frac{p}{2}} \tilde{w}_{2k} & p \text{ even}, \\[2em] \displaystyle\sum_{k=0}^{\frac{p}{2}} \tilde{w}_{2k+1} & p \text{ odd}. \end{cases} \tag{5.1.14b}$$

We shall see that (5.1.12), after taking the limit $X \to 0$, is a family of spectral curves inducing an LG–model by the assignment $X \leftrightarrow \lambda$, $y \leftrightarrow \mu$ coinciding with (5.1.10) for $\mathcal{R} = D_{p+2}$ (see (5.3.13)).

In [17], Brini completes the description by deriving the remaining case of $E_8$. Furthermore, he proves that taking the limit as the Kähler volume of the base $\mathbb{P}^1$ goes to infinity (which is obtained by the assignment above) is indeed the correct thing to do. Moreover, Brini connects the setup to the $E_8$ relativistic Toda chain and shows that the resulting family of curves does indeed induce a Landau-Ginzburg description for the DZ-manifold associated to the most complicated case of $\mathcal{R} = E_8$. Following this result, there was little doubt that one could obtain analogous results for the remaining simply laced cases $E_6$ and $E_7$ (and the already solved types $A_l, D_l$). The non-simply laced cases, however, do not follow from the physics, but Brini's construction does not rely on simply-lacedness, and it turns out that one can follow the exact same procedure for all Dynkin types.

Let us now describe this recipe and find explicit expressions for families of spectral curves arising from relativistic Toda chains. This will be reduced to finding relations in the associated Weyl character ring. Before we dive into deriving these curves explicitly, we show that $\lambda$, together with the primary differential $\mathrm{dlog}\mu = \frac{\mathrm{d}\mu}{\mu}$, make up an LG–model for *some* Frobenius manifold.

## 5.2 Frobenius manifold structure

In order to prove that $(\lambda, \phi)$ give an LG-model for a Frobenius manifold, we must impose an additional *admissibillity* condition on $(\lambda, \phi)$, as promised in Section 3.3. We must also describe how we lift the Frobenius manifold structure on $H_{g,n}$ to a twisted structure on $\mathcal{C}_{g,n}$ as described in Section 3.3. This is necessary in order to be able to employ the full machinery of Hurwitz Frobenius manifold theory as developed by Dubrovin in [41][*].

**Definition 5.2.1.** A meromorphic function $\mu : \mathcal{C}_{g,n} \to \mathbb{P}^1$ on the universal family is $\lambda$-*admissible* if it satisfies the following properties:

(i) $\mu$ does not factor through $\lambda$, i.e. $\nexists\, g : \mathbb{P}^1 \to \mathbb{P}^1$ s.t. $\mu = g \circ \lambda$;

(ii) $0 \neq \mathrm{d}\mu \in \Omega^1_{\mathcal{C}_{g,n}/H_{g,n}}$;

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[*]This will be important in the proof of Theorem 1.3.1

(iii) $\mathrm{div}(\mu) = \sum_{i=0}^{l} a_i \sigma_i(H_{g,\mathsf{n}})$  for $a_i \in \mathbb{Z}$,

where $\mathrm{div}(\mu)$ denotes the divisor of $\mu$ (sometimes denoted $(\mu)$), $\sigma_i$ is the $i^{\text{th}}$ section as shown in (3.3.2), and $\Omega^1_{\mathcal{C}_{g,n}/H_{g,n}}$ denotes the $\mathcal{C}_{g,n}$-module of differential 1-forms of $\mathcal{C}_{g,n}$ over $H_{g,n}$ defined by the short exact sequence

$$0 \to \pi^*(\Omega^1_{H_{g,n}}) \to \Omega^1_{\mathcal{C}_{g,n}} \to \Omega^1_{\mathcal{C}_{g,n}/H_{g,n}} \to 0, \tag{5.2.1}$$

with $\pi : \mathcal{C}_{g,n} \to H_{g,n}$, denoting the projection. The datum of a $\lambda$-admissible projection $\mu$ allows to define extra structure on the universal curve, $\mathcal{C}$, as follows. We can first of all canonically associate to it a relative one-form given by $\mathrm{d}\log\mu \in \Omega^1_{\mathcal{C}_{g,n}/H_{g,n}}(\infty_0 + \cdots + \infty_m)$ which denotes an exact third-kind relative differential one-form over $H_{g,n}$, which has simple poles at $\infty_i$ with residues $\mathrm{Res}_{\infty_i}\mathrm{d}\log\mu = a_i$. The second ingredient that the $\mu$-projection provides is a notion of a meromorphic Ehresmann connection[†] on $T\mathcal{C}_{g,\mathsf{n}}$, defined in terms of its singular foliation by level sets of $\mu$. Let $p \in \mathcal{C}_{g,\mathsf{n}}$ with $\mathrm{d}\mu(p) \neq 0$, $\mathsf{m} := \mu(p)$, so that the leaf $\mathcal{C}^{(\mathsf{m})}_{g,\mathsf{n}} := \{p' \in \mathcal{C}_{g,\mathsf{n}} | \mu(p') = \mathsf{m}\}$ is locally transverse to the fibres of the universal curve. Let $U$ be a small neighbourhood of $\pi(p) \in H_{g,\mathsf{n}}$. By transversality, there is a canonical local holomorphic section $\sigma_{\mathsf{m}} : U \to \pi^{-1}(U)$ of $\mathcal{C}_{g,\mathsf{n}}$, lifting $U$ to $\pi^{-1}(U) \cap \mathcal{C}^{(\mathsf{m})}_{g,\mathsf{n}}$. Accordingly, holomorphic vector fields $X \in \Gamma(U, TH_{g,\mathsf{n}})$ are lifted to local holomorphic sections $(\sigma_{\mathsf{m}})_*X$ of $T\mathcal{C}_{g,\mathsf{n}}$ which are tangent to the leaves of the foliation. This defines locally around $p$ a holomorphic derivation as

$$\delta^{(\mu)}_X f := L_{(\sigma_{\mathsf{m}})_*X} f.$$

Globally, however, the leaves of the $\mu$-foliation will fail to be transverse to the fibres of the universal curve at the critical locus of $\mu$. Thus the derivation $\delta^{(\mu)}_X$ takes values in the ring of meromorphic functions on $\mathcal{C}_{g,\mathsf{n}}$, with poles on the ramification divisor of $\mu$:

$$\delta^{(\mu)}_X \; : \; H^0(\mathcal{C}_{g,\mathsf{n}}, \mathcal{O}_{\mathcal{C}_{g,n}}) \longrightarrow H^0(\mathcal{C}_{g,\mathsf{n}}, \mathcal{K}_{\mathcal{C}_{g,n}}),$$
$$f \longmapsto (\delta^{(\mu)}_X f)(p) := (L_{(\sigma_{\mu(p)})_*X})f(p). \tag{5.2.2}$$

In more low-brow terms, and in local coordinates $p = (u_1, \ldots, u_{d_{g;n}}; \mu)$ on the complement of $\mathrm{d}\mu = 0$, the derivation $\delta^{(\mu)}_{\partial_{u_i}} f$ is simply the partial derivative taken with respect to $u_i$ whilst keeping $\mu$ constant. The meromorphicity of the derivation near an order-$r$ ramification point $q^{\mathrm{cr}} \in \mathcal{C}_{g,\mathsf{n}}$ with $\mu(q^{\mathrm{cr}}) = \mathsf{m}(u_1, \ldots, u_{d_{g;n}})$, $\mathrm{d}\mu(q^{\mathrm{cr}}) = 0$, is then just expressing that

$$\delta^{(\mu)}_{\partial_{u_i}}(\mu(p) - \mathsf{m})^{1/r} = -r^{-1}(\mu(p) - \mathsf{m})^{(1-r)/r}\partial_{u_i}\mathsf{m}$$

has a pole of order $r - 1$ at $p = q^{\mathrm{cr}}$ as soon as $\partial_{u_i}\mathsf{m}(u) \neq 0$.

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[†]The Ehresmann connection is a connection that makes sense on any smooth fibre bundle.

With these definitions at hand, we can define a Frobenius manifold structure $\mathcal{H}_{g,\mathsf{n}}^{[\mu]} := (H_{g,\mathsf{n}}, \cdot, \eta, e, E)$ on the Hurwitz space $H_{g,\mathsf{n}}$. For later convenience, we will introduce an additional parametric dependence of the Frobenius structure on a normalisation factor $\mathcal{N} \in \mathbb{C}^\star$: this will be immaterial *per se* in the comparison with the Frobenius manifolds of 4.1.1, as such a factor can be scaled away by a Frobenius manifold isomorphism given by a time-$1/(2\mathcal{N})$ flow along the Euler vector field[‡], but it will be helpful in simplifying the notation of the proof of Theorem 5.2.5. In terms of the datum of $(\lambda, \mu, \mathcal{N})$, the metric $\eta$ is defined by the residue formula

$$\eta(X, Y) := -\mathcal{N} \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{\delta_X^{(\mu)} \lambda\, \delta_Y^{(\mu)} \lambda}{\mathrm{d}\lambda} \left( \frac{\mathrm{d}\mu}{\mu} \right)^2, \qquad (5.2.3)$$

for $X, Y \in \Gamma(H_{g,\mathsf{n}}, T H_{g,\mathsf{n}})$. Furthermore, combining (3.3.4) and (5.2.3) the 3-tensor $c(X, Y, Z)$ is defined by

$$c(X, Y, Z) := \eta(X, Y \cdot Z) := -\mathcal{N} \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{\delta_X^{(\mu)} \lambda\, \delta_Y^{(\mu)} \lambda\, \delta_Z^{(\mu)} \lambda}{\mathrm{d}\lambda} \left( \frac{\mathrm{d}\mu}{\mu} \right)^2, \qquad (5.2.4)$$

which clearly satisfies the Frobenius property. Finally, the second flat pairing is, as implied by the discussion in Section 3.3, obtained upon replacing $\lambda$ by $\log \lambda$ in (5.2.3),

$$\gamma(X, Y) := -\mathcal{N} \sum_i \operatorname*{Res}_{p_i^{\mathrm{cr}}} \frac{\delta_X^{(\mu)} \log \lambda\, \delta_Y^{(\mu)} \log \lambda}{\mathrm{d} \log \lambda} \left( \frac{\mathrm{d}\mu}{\mu} \right)^2. \qquad (5.2.5)$$

Then we have the following

**Proposition 5.2.1.** *The residue formulae (5.2.3)–(5.2.5) define a Frobenius manifold structure* $\mathcal{H}_{g,\mathsf{n}}^{[\mu]} := (H_{g,\mathsf{n}}, \cdot, \eta, e, E)$, *which is semisimple outside the discrimimant of* $H_{g,\mathsf{n}}$.

The statement of the Proposition is a direct specialisation of the Main Lemma and the proof of Thm 5.1 in [41, Lecture 5]. Here Dubrovin classifies admissible differentials into fives types; I–V. If $\mu$ is $\lambda$-admissible, the case $\phi := \sqrt{\mathcal{N}}\mathrm{d} \log \mu$ considered here (an exact third kind differential having at most simple poles at the poles of $\lambda$) is readily seen to satisfy the criteria of Type III given in [41, Lecture 5], and therefore leads to an honest Frobenius manifold with LG–model $(\lambda, \sqrt{\mathcal{N}}\mathrm{d} \log \mu)$.

**Remark 5.2.2.** While $\mathcal{H}_{g,n}^{[\mu]}$ is a Frobenius manifold, it is not obvious that the above defines an immersion of $M_\omega^{\mathrm{LG}}$ into $H_{g_\omega, \mathsf{n}_\omega}$. This will follow from the "rectification statement" in Lemma 5.4.1, according to which $M_\omega^{\mathrm{LG}}$ embeds as a dimension-$(l_\mathcal{R} + 1)$ affine hyperplane into $H_{g_\omega, \mathsf{n}_\omega}$ in the respective flat coordinate systems.

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[‡]As the Frobenius manifolds of Theorem 4.1.1 have charge $d = 1$, their prepotentials are quasihomogeneous of degree $3 - d = 2$. Hence, a time-$s$ Euler flow scales them by $2s$.

In the next two Sections we will calculate the character relations (5.1.9), and therefore determine explicitly the polynomials $\mathcal{P}_\omega$. The following Proposition is an anticipated consequence of this direct calculation.

**Proposition 5.2.3.** *Let* $\widetilde{\mathcal{P}}_\omega^{(j)}(w_0, \ldots, w_{l_\mathcal{R}}; \mu) := [\lambda^j]\mathcal{P}_\omega$ *be the* $j^{th}$ *coefficient of* $\mathcal{P}_\omega$, *as defined in (5.1.11), as a function of* $\lambda$, *and let* $j_\omega^{\max} := \deg_\lambda \mathcal{P}_\omega$. *Then* $\widetilde{\mathcal{P}}_\omega^{(j_\omega^{\max})}$ *is a product of a monomial* $\mu^{a_\omega}$ *and cylotomic polynomials* $\Phi_{k_i}(\mu)$

$$\widetilde{\mathcal{P}}_\omega^{(j_\omega^{\max})}(w_0, \ldots, w_{l_\mathcal{R}}; \mu) = \mu^{a_\omega} \prod_{i=1}^{b_\omega} \Phi_{k_i^\omega}(\mu), \tag{5.2.6}$$

*with* $a_\omega \in \mathbb{Z}_{>0}$ *and* $b_\omega, k_i^\omega \in \mathbb{Z}_{\geqslant 0}$. *Moreover,*

$$\widetilde{\mathcal{P}}_\omega^{(j)}(w_0, \ldots, w_{l_\mathcal{R}}; 0) = \delta_{j0}, \tag{5.2.7}$$

$$\lim_{\mu' \to 0} (-\mu')^{\dim \rho_\omega} \widetilde{\mathcal{P}}_\omega^{(j)}(w_0, \ldots, w_{l_\mathcal{R}}; 1/\mu') = \delta_{j0}. \tag{5.2.8}$$

**Remark 5.2.4.** The projection $\lambda : \overline{C_w^{(\omega)}} \to \mathbb{P}^1$ can only possibly have poles at $\mu = \infty$ or at the zeros of $\widetilde{\mathcal{P}}_\omega^{(j_\omega^{\max})}$. The equality in (5.2.6) entails then that $\delta_{\partial_{w_i}}^{(\mu)} \widetilde{\mathcal{P}}_\omega^{(j_\omega^{\max})} = 0$ since the right-hand side is constant in $w$ at fixed $\mu$. In particular, the ramification profile $\mathsf{n}_\omega$ is independent of $w$. The second part of the Proposition implies that the zeros of $\mu$ must occur at points that are poles of $\lambda$, since by (5.2.7) the equation $\mathcal{P}_\omega(\lambda, \mu)|_{\mu=0} = 0$ has no solutions for finite $\lambda$. Likewise, replacing $\mu \to \mu = 1/\mu'$ reveals that poles of $\mu$ are also poles of $\lambda$ by (5.2.8). Therefore, $\mathrm{d}\log\mu$ has at most simple poles at the poles of $\lambda$, showing in particular that $\mu$ satisfies property (iii) in Definition 5.2.1. As properties (i)-(ii) are obvious it is indeed $\lambda$-admissible.

Following the discussion of 3.3, and fixing $\mathcal{N}_\omega \in \mathbb{C}^\star$, we can then define a family of semisimple, Frobenius algebras on $TM_\omega^{\mathrm{LG}}$ via (5.2.3)-(5.2.4):

$$\eta(\partial_{w_i}, \partial_{w_j}) = -\mathcal{N}_\omega \sum_l \operatorname*{Res}_{p_l^{\mathrm{cr}}} \frac{\delta_{\partial_{w_i}}^{(\mu)} \lambda \, \delta_{\partial_{w_j}}^{(\mu)} \lambda}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}\mu}{\mu}\right)^2, \tag{5.2.9}$$

$$\eta(\partial_{w_i}, \partial_{w_j} \cdot \partial_{w_k}) = -\mathcal{N}_\omega \sum_l \operatorname*{Res}_{p_l^{\mathrm{cr}}} \frac{\delta_{\partial_{w_i}}^{(\mu)} \lambda \, \delta_{\partial_{w_j}}^{(\mu)} \lambda \delta_{\partial_{w_k}}^{(\mu)} \lambda}{\mathrm{d}\lambda} \left(\frac{\mathrm{d}\mu}{\mu}\right)^2, \tag{5.2.10}$$

$$\gamma(\partial_{w_i}, \partial_{w_j}) = -\mathcal{N}_\omega \sum_l \operatorname*{Res}_{p_l^{\mathrm{cr}}} \frac{\delta_{\partial_{w_i}}^{(\mu)} \log \lambda \, \delta_{\partial_{w_j}}^{(\mu)} \log \lambda}{\mathrm{d}\log\lambda} \left(\frac{\mathrm{d}\mu}{\mu}\right)^2, \tag{5.2.11}$$

where $\{p_l^{\mathrm{cr}}\}_l$ are the ramification points of $\lambda : \overline{C_w^{(\omega)}} \to \mathbb{P}^1$. This doesn't yet give a Frobenius manifold, or indeed a Frobenius submanifold of $\mathcal{H}_{g_\omega, \mathsf{n}_\omega}^{[\mu]}$, as $\eta$ is not guaranteed to be nondegenerate or flat at this stage. The following statement establishes that this is the case.

**Theorem 5.2.5** (Mirror symmetry for DZ Frobenius manifolds). *The Landau–Ginzburg formulas* (5.2.9)–(5.2.11) *define a semisimple Frobenius submanifold* $\iota_\omega : \mathcal{M}_\omega^{LG} = (M_\omega^{LG}, \eta, e, E, \cdot) \hookrightarrow \mathcal{H}_{g_\omega, \mathsf{n}_\omega}^{[\mu]}$. *In particular,* (5.2.9) *and* (5.2.11) *give flat, nondegenerate metrics on* $TM_\omega^{LG}$, *and the identity and Euler vector fields read*

$$e = w_0^{-1} \partial_{w_{\bar{k}}}, \qquad E = w_0 \partial_{w_0}. \tag{5.2.12}$$

*Furthermore,*

$$\mathcal{M}_\omega^{LG} \simeq \mathcal{M}_{\mathcal{R}}^{DZ}. \tag{5.2.13}$$

The explicit embedding $\iota_\omega : M_\omega^{\mathrm{LG}} \hookrightarrow H_{g_\omega, \mathsf{n}_\omega}$ is described by the relations (5.1.10) in the character ring of $\mathcal{G}$, setting the coefficients associated to the interior of the Newton polytope of $\mathcal{P}_\omega^{\mathrm{red}}(\lambda, \mu)$ to be the polynomials $\mathfrak{p}_k^\omega(w_1, \ldots, w_r)$.

The proof of Theorem 5.2.5 requires two key steps:

1. computing the exterior relations (5.1.10) in the Weyl character ring of $\mathcal{G}$: this was solved for simply laced cases in [12], [18], and we complete the solution here in full generality;

2. proving that the LG formulas (5.2.9)–(5.2.11), combined with the reconstruction theorem 4.1.1, establish the mirror statement of Theorem 5.2.5.

We are now ready to perform the first step and construct explicitly the family of spectral curves. We will then devote Section 5.4 to show that these do indeed induce LG mirror duals to type-$\mathcal{R}$ Dubrovin–Zhang Frobenius manifolds, and a proof of Theorem 5.2.5.

## 5.3 Character relations and explicit superpotentials

### Superpotentials for classical Lie groups

In the following we present the construction of the spectral curve for the classical root systems $\mathcal{R} = A_l$, $B_l$, $C_l$, $D_l$ independently, and show how our construction for a weight $\omega$ corresponding to a minimal-dimensional representation $\rho_\omega$ recovers the mirror results of [44, 46]. We will use the shorthand notation $\varepsilon_i := \chi_{\wedge^i \rho_\omega}$ for the exterior characters of $\rho_\omega$, while $[0 \cdots, 0, 1, 0, \cdots 0]_{\mathcal{R}}$ in subscript or superscript indicates that the associated expression depends on choosing a minimal nontrivial irreducible representation, for Dynkin type $\mathcal{R}$. Here the "1" is placed in position $i$ if the chosen representation is $\rho_{\omega_i}$ in Dynkin type $\mathcal{R}$, and $[0 \ldots 0, 1, 0 \cdots, 0]$ is the Dynkin label associated to the highest weight $\omega_i$. For instance if $\mathcal{R} = G_2$, then the chosen representation is $\rho_{\omega_1}$ which has associated Dynkin label [10].

$\mathcal{R} = A_l$

The Dynkin diagram for affine $A_l$ is shown in Figure 4.1. In this case we can choose any (non-affine) node to be the marked one, since the removal of any node from the corresponding finite Dynkin diagram results in two disconnected (non-affine) $A$-type pieces, with ranks adding up to $l - 1$.

A choice of minimal, nontrivial, irreducible representation $\rho_\omega := \rho_1 = (\mathbf{l+1})$ for $\mathrm{SL}_\mathbb{C}(l+1)$ is the defining $(l+1)$-dimensional representation, the other choice corresponding to its dual representation, $\rho_l = \wedge^l \square$. We then have that $\varepsilon_i = \chi_i$ for $i = 1, \ldots, l$, and $\varepsilon_0 = \varepsilon_{l+1} = 1$. Thus (5.1.8) becomes

$$\mathcal{P}_{[10\ldots0]_{A_l}} = \frac{(-1)^{\bar{k}} \lambda \mu^{\bar{k}}}{w_0} + 1 + (-1)^{l+1} \mu^{l+1} + \sum_{i=1}^{l} (-1)^i w_i \mu^i, \tag{5.3.1}$$

which defines a family of genus 0 curves. Setting (5.3.1) equal to zero and solving for $\lambda$ gives

$$\lambda = \frac{(-1)^{\bar{k}} w_0 (1 + (-1)^{l+1} \mu^{l+1} + \sum_{i=1}^{l} (-1)^i w_i \mu^i)}{\mu^{\bar{k}}}, \tag{5.3.2}$$

which is, for every point in the moduli space, a meromorphic function of $\mu$ with poles at $0$ and $\infty$ of orders $\bar{k}, l + 1 - \bar{k}$, respectively. We see that we have $l + 1$ parameters $w_0, \cdots, w_l$, and so the resulting Frobenius manifold is $l + 1$ dimensional[*]. In particular, it is an $l + 1$ dimensional submanifold of the Hurwitz space $H_{0,\mathsf{n}_\omega}$, with ramification profile $\mathsf{n}_\omega = (\bar{k} - 1, l - \bar{k})$. This Hurwitz space, however, is of dimension $2 + \bar{k} - 1 + l - \bar{k} = l + 1$, and so the DZ-Frobenius manifold associated to $A_l$ is isomorphic to (a full-dimensional ball inside) its associated Hurwitz space. This, as we will see, will not be the case for the other Dynkin types.

$\mathcal{R} = B_l$

For $l > 2$, the minimal, nontrivial, irreducible representation of $\mathrm{Spin}(2l + 1)$ is the defining representation $\rho_1 = (\mathbf{2l+1})$ of the special orthogonal group in $(2l + 1)$-dimensions[†], which is the irreducible representation with highest weight $\omega_1$. In this case the marked node is $\bar{k} = l - 1$, as depicted in 4.1.

For $i < l$, the $i^{\text{th}}$ fundamental representation $\rho_i$ of $B_l$ is the $i^{\text{th}}$ exterior power of $(\mathbf{2l+1})$. For $i = l$, the decomposition of the tensor square of $\rho_l$ leads to

$$\mathfrak{p}_i^{[10\ldots0]_{B_l}} = \begin{cases} \chi_i & \text{if } i < l, \\ \chi_l^2 - \sum_{j=0}^{l-1} \chi_j & \text{if } i = l. \end{cases} \tag{5.3.3}$$

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[*]This is, in fact, the case for all DZ-manifolds associated to a simple Lie algebra of rank $l$.

[†]For $l = 2$, the 4-dimensional spin representation $\rho_2$ is both minimal and minuscule. It is also isomorphic to the vector representation $\rho_1$ of $C_2$, which is included in the discussion of the next section.

Together with the self-duality relation $\mathfrak{p}_i^{[10...0]_{B_l}} = \mathfrak{p}_{2l+1-i}^{[10...0]_{B_l}}$, we get that the curve is the zero locus of

$$\mathcal{P}_{[10...0]_{B_l}} = \frac{(-1)^l(\mu-1)(\mu+1)^2\mu^{l-1}\lambda}{w_0} + \sum_{i=0}^{l}(-1)^i\mu^i(1-\mu^{2(l-i)+1})\varepsilon_i, \tag{5.3.4}$$

with

$$\varepsilon_i = \begin{cases} 1 & \text{if } i = 0, \\ w_i & \text{if } 1 \leqslant i < l, \\ w_l^2 - \sum_{j=0}^{l-1} w_j & \text{if } i = l. \end{cases} \tag{5.3.5}$$

Note that (5.3.4) has a factor of $(\mu-1)$, since $(\mathbf{2l+1})$ has a one-dimensional zero weight space. Setting to zero the reduced characteristic polynomial $\mathcal{P}_{[10...0]_{B_l}}^{\text{red}} = \mathcal{P}_{[10...0]_{B_l}}/(\mu-1)$ gives

$$\lambda = \frac{(-1)^l w_0}{\mu^{l-1}(\mu+1)^2} \sum_{j=0}^{2l} \mu^j \left( \sum_{i=0}^{\min(j,2l-j)}(-1)^i\varepsilon_i \right). \tag{5.3.6}$$

For each point in the moduli space, this is a rational function in $\mu$ with three poles at $0, -1$, and $\infty$ of orders $l-1, 2, l-1$, respectively. Hence, $M_{[10...0]_{B_l}}^{\text{LG}}$ is a sublocus in the $(2l+1)$-dimensional Hurwitz space $H_{0,\mathsf{n}_\omega}$, with $\mathsf{n}_\omega = (l-2,1,l-2)$. The latter carries an involution given by sending $\mu \to 1/\mu$, and $M_{[10...0]_{B_l}}^{\text{LG}}$ is characterised as the $(l+1)$-dimensional stratum that is fixed by the involution.

## $\mathcal{R} = C_l$

The minimal, nontrivial, irreducible representation for $\text{Sp}(2l)$ is the defining representation $\rho_1 = (\mathbf{2l})$ of the rank $2l$ symplectic group. Again, this representation corresponds to the one in which $\omega_1$ is highest weight. The canonical node is the $l^{\text{th}}$ node, as shown in Figure 4.1.

The exterior powers $\wedge^i\rho_1$ are reducible, with only fundamental representations appearing as direct summands in their decomposition, giving the character relations

$$\mathfrak{p}_i^{[10...0]_{C_l}} = \begin{cases} \sum_{j=0}^{\frac{i}{2}} \chi_{2j} & \text{for } i \text{ even,} \\ \sum_{j=0}^{\frac{i-1}{2}} \chi_{2j+1} & \text{for } i \text{ odd.} \end{cases} \tag{5.3.7}$$

From this, and the fact that $\bar{k} = l$ for $C_l$, we see that the characteristic polynomial (5.1.8) is

$$\mathcal{P}_{[10...0]_{C_l}} = \frac{(-1)^l\mu^l\lambda}{w_0} + \sum_{i=0}^{l-1}(-1)^i\varepsilon_i\mu^i(1+\mu^{2(l-i)}) + (-1)^l\varepsilon_l\mu^l, \tag{5.3.8}$$

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with $\varepsilon_{2i} = \sum_{j=0}^{i} \chi_{2j}$, $\varepsilon_{2i+1} = \sum_{j=0}^{i} \chi_{2j+1}$ (and $\varepsilon_0 = 1$ as usual). Setting equal to zero and solving for $\lambda$ gives

$$\lambda = \frac{(-1)^{l-1} w_0 \left( \sum_{i=0}^{l-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(l-i)}) + (-1)^l \varepsilon_l \mu^l \right)}{\mu^l}, \tag{5.3.9}$$

which is a rational function in $\mu$ with two poles at 0 and $\infty$ both of order $l$. Hence, the associated covering Hurwitz space is $H_{0,\mathsf{n}_\omega}$, with $\mathsf{n}_\omega = (l-1, l-1)$, which has dimension $2l$. As before, there is an involution on this Hurwitz space sending $\mu \to 1/\mu$, with $M^{\mathrm{LG}}_{[10\ldots0]_{C_l}}$ being the $(l+1)$-dimensional stratum that is fixed by it.

## $\mathcal{R} = D_l$

For $l \geqslant 4$, the minimal, nontrivial, irreducible representation of $\mathrm{Spin}(2l)$ is the defining vector[‡] representation $\rho_1 = (\mathbf{2l})_v$ of $\mathrm{SO}(2l)$, which corresponds to the irreducible representation with highest weight $\omega_1$. The canonical node is the one with label $l - 2$, as shown in Figure 4.1.

The character relations for $D_l$ were found in [12] to be

$$\mathfrak{p}_i^{[10\ldots0]_{D_l}} = \chi_i, \quad i < l - 1, \tag{5.3.10}$$

$$\mathfrak{p}_{l-1}^{[10\ldots0]_{D_l}} = \chi_{l-1}\chi_l - \begin{cases} \sum_{j=0}^{\frac{l}{2}-2} \chi_{2j+1} & \text{if } l \text{ is even,} \\ \sum_{j=0}^{\frac{l-3}{2}} \chi_{2j} & \text{if } l \text{ is odd,} \end{cases} \tag{5.3.11}$$

$$\mathfrak{p}_l^{[10\ldots0]_{D_l}} = \chi_{l-1}^2 + \chi_l^2 - 2 \begin{cases} \sum_{j=0}^{\frac{l}{2}-1} \chi_{2j} & \text{if } l \text{ is even,} \\ \sum_{j=0}^{\frac{l-3}{2}} \chi_{2j+1} & \text{if } l \text{ is odd,} \end{cases} \tag{5.3.12}$$

and $\mathfrak{p}_i^{[10\ldots0]_{D_l}} = \mathfrak{p}_{2l-i}^{[10\ldots0]_{D_l}}$, so that

$$\mathcal{P}_{[10\ldots0]_{D_l}} = \frac{(-1)^l \mu^{l-2}(\mu^2-1)^2 \lambda}{w_0} + \sum_{i=0}^{l-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(l-i)}) + (-1)^l \mu^l \varepsilon_l, \tag{5.3.13}$$

where as before we denote $\varepsilon_i(w_1, \ldots, w_l) = \mathfrak{p}_i^{[10\ldots0]_{D_l}}(\chi_j = w_j)$, with $\varepsilon_0 = 1$. Setting (5.3.13) equal to zero and solving for $\lambda$ gives

$$\lambda = (-1)^{l-1} \frac{w_0 \left( \sum_{i=0}^{l-1} (-1)^i \varepsilon_i \mu^i (1 + \mu^{2(l-i)}) + (-1)^l \mu^l \varepsilon_l \right)}{\mu^{l-2}(\mu^2-1)^2}, \tag{5.3.14}$$

which, for every point $w$, is a rational function in $\mu$ with four poles at $0$, $\infty$, $1$, $-1$ of orders $l - 2$, $l - 2$, $2$, $2$, respectively. Hence, the parent Hurwitz space is $H_{0;\mathsf{n}_\omega}$, where $\mathsf{n}_\omega = (l - 3, l - 3, 1, 1)$, which has dimension $2l + 2$. Once more this hosts an involution obtained by sending $\mu \to 1/\mu$, identifying $M^{\mathrm{LG}}_{[10\ldots0]_{D_l}}$ as its fixed locus.

---

[‡]For $l = 4$, this can be any of the irreducible 8-dimensional representations, related by triality.

## Comparison with the Dubrovin–Strachan–Zhang–Zuo construction

For the case of $\mathcal{R} = A_l$, an LG–superpotential was already found in the original paper [41], with[§]

$$\lambda = \sum_{j=0}^{k+m} b_j \mu^{m-j}, \tag{5.3.15}$$

where $b_j \in \mathbb{C}$ and $b_0 b_{k+m} \neq 0$. Moreover, in [44], the authors construct a three-integer parameter family of superpotentials of the form[¶]

$$\lambda^{\mathrm{DSZZ}}(l,k,m) = \frac{4^m \mu^m \sum_{j=0}^{l} a_j 2^{-2(-j+k+m)} \left(\frac{\mu+1}{\sqrt{\mu}}\right)^{2(-j+k+m)}}{(\mu-1)^m}. \tag{5.3.16}$$

The key result of [44] is an identification of (5.3.16) with a superpotential for a Dubrovin–Zhang Frobenius manifold of type $B_l$, $C_l$, $D_l$, possibly with a non-canonical choice of marked node in the Dynkin diagram, for suitable choices of $(l,k,m)$. In particular, the mirror theorem for the canonical label $\bar{k}$ is obtained by setting $(l,k,m)$ equal to $(l, l-1, 1)$, $(l, l, 0)$, and $(l, l-2, 1)$, respectively. We shall now show that the results of [41] and [44] coincide with our construction in the previous section.

### $\mathcal{R} = A_l$

By using the fact that $k + m = l + 1$, (5.3.15) becomes

$$\lambda = \frac{\sum_{j=0}^{l+1} b_j \mu^{l+1-j}}{\mu^k}. \tag{5.3.17}$$

Let

$$\hat{\mu} = \mu w_0^{-\frac{1}{l+1-k}}. \tag{5.3.18}$$

As

$$\log(\hat{\mu}) = \log(\mu) - \frac{1}{l+1-k}\log(w_0) \implies \mathrm{dlog}(\hat{\mu}) = \mathrm{dlog}(\mu), \tag{5.3.19}$$

the primary differential remains invariant under the change (5.3.18) and so it is allowed in Frobenius theory. Then we have that (5.3.17) becomes

$$\frac{\sum_{j=0}^{l+1} b_j \hat{\mu}^{l+1-j} w_0^{\frac{-j}{l+1-k}}}{\hat{\mu}^{\bar{k}}}. \tag{5.3.20}$$

---

[§]To relate this to the expression in [41] let $\mu = e^{i\phi}$.
[¶]To relate this to the expression in [44] let $\mu = e^{2i\phi}$.

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Hence, (5.3.17) is equivalent to (5.3.2) by

$$b_j = (-1)^{\bar{k}} \begin{cases} (-1)^{l+1} & \text{if } j = 0, \\ w_0^{\frac{l+1}{l+1-\bar{k}}} & \text{if } j = l+1, \\ (-1)^j w_j w_0^{\frac{j}{l+1-\bar{k}}} & \text{otherwise.} \end{cases} \tag{5.3.21}$$

$\mathcal{R} = B_l$

In the case of $B_l$ we consider (5.3.16) with $k = l - 1$, $m = 1$, which is

$$\lambda^{\text{DSZZ}}(l, l-1, 1) = \frac{4\mu \sum_{j=0}^{l} a_j 2^{-2(l-j)} \left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}}\right)^{2(l-j)}}{(\mu-1)^2}. \tag{5.3.22}$$

Simplifying (5.3.22) gives:

$$\frac{w_0}{(\mu-1)^2} \sum_{j=0}^{l} \frac{a_j 2^{-2(l-j-1)}(\mu+1)^{2(l-j)}}{w_0 \, \mu^{l-j-1}} = \frac{(-1)^l w_0}{(\mu+1)^2 \mu^{l-1}} \sum_{\beta=0}^{2l} (-1)^\beta C_\beta \mu^\beta, \tag{5.3.23}$$

where we have used the binomial theorem and let $\mu \mapsto -\mu$, with

$$C_\beta = \frac{(-1)^l}{w_0} \sum_{j,\alpha|j+\alpha=\beta} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{\alpha} = \frac{(-1)^l}{w_0} \sum_{j=0}^{\beta} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{\beta-j}. \tag{5.3.24}$$

On the other hand, the superpotential constructed from the spectral curve, (5.3.6), is given by

$$\lambda_{B_l} = \frac{(-1)^l w_0}{\mu^{l-1}(\mu+1)^2} \sum_{j=0}^{2l} \mu^j \left( \sum_{i=0}^{\min(j,2l-j)} (-1)^i \varepsilon_i \right), \tag{5.3.25}$$

which we can write as

$$\lambda_{B_l} = \frac{(-1)^l w_0}{\mu^{l-1}(\mu+1)^2} \sum_{j=0}^{2l} b_j \mu^j, \tag{5.3.26}$$

with $b_j = \sum_{i=0}^{\min(j,2l-j)} (-1)^i \varepsilon_i$; note that $b_j = b_{2l-j}$. This means that we want to match up $b_i = (-1)^i C_i$, hence

$$\sum_{j=0}^{\min(i,2l-i)} (-1)^j \varepsilon_j = \frac{(-1)^{l+i}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-j}. \tag{5.3.27}$$

We claim that

$$\varepsilon_i = \frac{(-1)^l}{w_0} \sum_{j=0}^{l} a_j 2^{-2(l-i-1)} \binom{2l - 2j + 1}{i-j}. \tag{5.3.28}$$

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*Proof.* The $i = 0$ case is clear giving $\varepsilon_0 = \dfrac{(-1)^l}{w_0} 2^{-2(l-1)} a_0$ obtained by taking $j = 0$.

So suppose $0 < i \leqslant l$. Then (5.3.27) becomes

$$
\begin{aligned}
\sum_{j=0}^{i} (-1)^j \varepsilon_j &= \frac{(-1)^{l+i}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-j} \\
\implies \varepsilon_i &= \frac{(-1)^l}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-j} + \frac{(-1)^l}{w_0} \sum_{j=0}^{i-1} a_j 2^{-2(l-j-1)} \binom{2(l-j)}{i-1-j} \\
&= \frac{(-1)^l}{w_0} \left( a_i 2^{-2(l-i-1)} + \sum_{j=0}^{i-1} a_j 2^{-2(l-j-1)} \left( \binom{2(l-j)}{i-j} + \binom{2(l-j)}{i-1-j} \right) \right) \quad (5.3.29)
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\binom{2(l-j)}{i-j} + \binom{2(l-j)}{i-1-j} &= \frac{(2(l-j))!}{(i-j)!(2l-i-j)!} + \frac{(2(l-j))!}{(i-j-1)!(2l-i-j+1)!} \\
&= \binom{2l-2j+1}{i-j},
\end{aligned}
$$

which gives the result for $i \leqslant l$. Hence, since $\varepsilon_i = \varepsilon_{2l+1-i}$, we have (5.3.28) $\forall i$.

$\square$

$\mathcal{R} = C_l$

In the case of $C_l$, we consider (5.3.16) with $k = l, m = 0$ which is

$$
\lambda^{\mathrm{DSZZ}}(l, l, 0) = \sum_{j=0}^{l} a_j 2^{-2(l-j)} \left( \sqrt{\mu} + \frac{1}{\sqrt{\mu}} \right)^{2(l-j)}. \tag{5.3.30}
$$

Simplifying (5.3.30) gives:

$$
\frac{(-1)^{l-1} w_0}{\mu^l} \sum_{j=0}^{l} \frac{(-1)^{l-1} a_j 2^{-2}(l-j)(\mu+1)^{2(l-j)}}{w_0 \mu^{-j}} = \frac{(-1)^{l-1} w_0}{\mu^l} \sum_{\beta=0}^{2l} C_\beta \mu^\beta, \tag{5.3.31}
$$

where we again have used the binomial theorem, and with

$$
C_\beta = \frac{(-1)^{l-1}}{w_0} \sum_{j=0}^{\beta} a_j 2^{-2(l-j)} \binom{2(l-j)}{\beta-j}. \tag{5.3.32}
$$

Thus, the equivalence is obtained in the case of $C_l$ by letting

$$
\varepsilon_i \mapsto \frac{(-1)^{l+i-1}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j)} \binom{2(l-j)}{i-j}. \tag{5.3.33}
$$

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$$\mathcal{R} = D_l$$

For $D_l$, we want to consider (5.3.16) with $k = l - 2, m = 1$, which is of the form

$$\lambda^{\text{DSZZ}}(l, l-2, 1) = \frac{4\mu \sum_{j=0}^{l} a_j 2^{-2(l-j-1)} \left(\sqrt{\mu} + \frac{1}{\sqrt{\mu}}\right)^{2(l-j-1)}}{(\mu - 1)^2}. \tag{5.3.34}$$

This is equivalent to

$$\frac{(-1)^{l-1} w_0}{(\mu - 1)^2 \mu^{l-2}(\mu + 1)^2} \sum_{j=0}^{l} \frac{(-1)^{l-1} a_j 2^{-2(l-j-2)}(\mu + 1)^{2(l-j)}}{w_0 \mu^{-j}} = \frac{(-1)^{l-1} w_0}{\mu^{l-2}(\mu^2 - 1)^2} \sum_{\beta=0}^{2l} C_\beta \mu^\beta, \tag{5.3.35}$$

with

$$C_\beta = \frac{(-1)^{l-1}}{w_0} \sum a_j 2^{-2(l-j-2)} \binom{2(l-j)}{\beta - j}, \tag{5.3.36}$$

where, again, the binomial theorem has been used. Hence, the map

$$\varepsilon_i \mapsto \frac{(-1)^{l+i-1}}{w_0} \sum_{j=0}^{i} a_j 2^{-2(l-j-2)} \binom{2(l-j)}{\beta - j} \tag{5.3.37}$$

gives the equivalence.

## Superpotentials for exceptional Lie groups

We present here the construction of the spectral curve for the exceptional types $E_6$, $E_7$, $F_4$, and $G_2$. The $E_8$ case was treated extensively in [17] and [18], and we only give a very brief presentation here.

As for the classical cases, the construction of the superpotential hinges on determining the character relations (5.1.9) for all $k$. Explicitly, for all dominant weights $\varpi = \sum_i \varpi_i \omega_i \in \Lambda_w^+(\mathcal{R})$ we must determine integers $N_\varpi^{(\omega, k)}$ such that

$$\mathfrak{p}_k^\omega = \sum_{\varpi \in \Lambda_w^+(\mathcal{R})} N_\varpi^{(\omega, k)} \prod_{i=1}^{l_\mathcal{R}} \chi_i^{\varpi_i}. \tag{5.3.38}$$

**Definition 5.3.1.** A set of dominant weights $\Pi_\omega \subset \Gamma(\wedge^\bullet \rho_\omega)$ is called *pivotal* for $\omega$ if, $\forall \varpi' \in \Gamma(\wedge^\bullet \rho_\omega)$, $\exists \varpi \in \Pi_\omega$ such that $\varpi' \le \varpi$.

Here $\varpi' \le \varpi$ denotes the canonical partial ordering of weights given by $\varpi' \le \varpi \Leftrightarrow \varpi - \varpi' = \sum_{i=1}^{l_\mathcal{R}} n_i \alpha_i$ with $n_i \ge 0$. Definition 5.3.1 then states that a set $\Pi_\omega$ of dominant weights is pivotal for a representation $\rho_\omega$ if it is contained in the weight system of the exterior algebra of $\rho_\omega$, and all the

other weights in $\Gamma(\wedge^\bullet \rho_\omega)$ are lower, in the partial order, than some element of $\Pi_\omega$.

It will be useful, in the following, to consider pivotal sets that are as small as possible. As for the classical cases, we take $\omega$ to sit in a minimal nontrivial orbit of $\mathcal{W}_\mathcal{R}$, as described in Table 5.2.

**Example 13.** Consider the decomposition of the exterior algebra of $\rho_\omega$ into irreducible representations,

$$\Gamma(\wedge^\bullet \rho_\omega) = \bigoplus_{\omega'} \mathbf{m}_{\wedge^\bullet \rho_\omega}(\omega') \rho_{\omega'}, \tag{5.3.39}$$

and let $\Pi_\omega$ denote the finite set of dominant weights appearing on the right-hand side with non-zero multiplicity. Then $\Pi_\omega$ is pivotal for $\omega$, although not necessarily of minimal cardinality. Suppose that $\omega = \alpha_0$ is the highest root, so that $\rho_\omega = \mathfrak{g}$ is the adjoint representation. Then the dominant weight is given by twice the Weyl vector, $2\mathsf{w} = 2\sum_i \omega_i = \sum_{\alpha>0} \alpha$, and it appears with multiplicity $\mathrm{Mult}_{\wedge^\bullet \mathfrak{g}}(\mathsf{w}) = 1$ in (5.3.39). Moreover, it is higher than any other highest weight in the decomposition of $\wedge^\bullet \rho_\omega$ into irreducibles. In this case, $\Pi_{\alpha_0} = \{\mathsf{w}\}$ is pivotal and of minimal cardinality.

| $\mathcal{R}$ | $\omega$ | $\rho_\omega$ | $\|\mathfrak{J}_\omega\|$ |
|---|---|---|---|
| $E_6$ | [100000] (resp. [000010]) | $\mathbf{27}_{E_6}$ (resp. $\overline{\mathbf{27}}_{E_6}$) | 111 |
| $E_7$ | [0000010] | $\mathbf{56}_{E_7}$ | 907 |
| $E_8$ | [00000010] | $\mathbf{248}_{E_8}$ | 950077 |
| $F_4$ | [0001] | $\mathbf{26}_{F_4}$ | 74 |
| $G_2$ | [10] | $\mathbf{7}_{G_2}$ | 5 |

Table 5.2: Highest weights of minimal representations for exceptional root systems. The last column indicates the cardinality of their sets of admissible exponents 5.3.2.

**Lemma 5.3.1.** *The sets of dominant weights*

$$\Pi_\omega := \begin{cases} \{[010120], [120010], [200200], [110110], [001030], & \omega = [100000]_{E_6}, \\ \quad [030000], [020020], [000041], [000050]\}, & \\ \{[0002022], [0001113], [0100132], [0101041], [1001051], & \\ \quad [1001061], [0011031], [0010070], [0000204], [0110050], & \omega = [0000010]_{E_7}, \\ \quad [0010070], [1100070], [1000090], [0003011], [0020040], & \\ \quad [0004000], [0000105], [00000100], [0000006]\}, & \\ \{[22222222]\}, & \omega = [00000010]_{E_8}, \\ \{[0022]\}, & \omega = [0001]_{F_4}, \\ \{[20]\}, & \omega = [10]_{G_2}, \end{cases} \tag{5.3.40}$$

*are pivotal and of minimal cardinality for $\omega$.*

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For $\mathcal{R} = E_8$, $F_4$ and $G_2$ the weight system of $\rho$ is the set of short roots of $\mathcal{R}$, and the single element of its minimal pivotal set is then the sum of the positive short roots. For $\mathcal{R} = E_6$, $E_7$ the pivotal sets of minimal cardinality in (5.3.40) can be constructed by direct inspection of the weight system.

**Definition 5.3.2.** Let $\Pi_\omega$ be as in Lemma 5.3.1. We call the finite set

$$\mathfrak{I}_\omega := \left\{ \iota \in (\mathbb{Z}_{\geq 0})^{l_\mathcal{R}} \middle| \exists\, \varpi = \sum_k \varpi_k \omega_k \in \Pi_\omega \text{ s.t. } \sum_k (C_\mathcal{R})_{jk}^{-1} (\iota_k - \varpi_k) \in \mathbb{Z}_{\leq 0}, \forall j \right\} \tag{5.3.41}$$

the set of *admissible exponents* of the exterior algebra $\wedge^\bullet \rho_\omega$.

In other words, $\iota$ is admissible if and only if the corresponding dominant weight $\varpi_\iota := \sum_i \iota_i \omega_i \leq \varpi'$ for some weight $\varpi'$ in the minimal pivotal sets of Lemma 5.3.1[||]. We will use the short-hand notation $\varpi_\iota \leq \Pi_\omega$ when this happens. The terminology is justified by the following

**Lemma 5.3.2.** *Let $\iota \in (\mathbb{Z}_{\geq 0})^{l_\mathcal{R}}$, $\iota \notin \mathfrak{I}_\omega$. Then $N_{\varpi_\iota}^{(\omega,k)} = 0$ for all $k$.*

*Proof.* Consider the representation space version of (5.3.38),

$$\wedge^k \rho_\omega = \bigoplus_\iota N_{\varpi_\iota}^{(\omega,k)} \bigotimes_{i=1}^{l_\mathcal{R}} \rho_i^{\iota_i}. \tag{5.3.42}$$

By Definition 5.3.1, we have $\varpi \leq \Pi_\omega$ for all $\varpi \in \Gamma(\wedge^k \rho_\omega)$, and furthermore, by Definition 5.3.2, the $l_\mathcal{R}$-tuple of its coefficients in the basis of fundamental weights is admissible. Equivalently, if $\iota' \in (\mathbb{Z}_{\geq 0})^{l_\mathcal{R}} \setminus \mathfrak{I}_\omega$ is not admissible, then the corresponding dominant weight $\varpi_{\iota'} := \sum_i \iota_i' \omega_i \notin \Gamma(\wedge^k \rho_\omega)$ is not in the exterior algebra. Taking multiplicities of (5.3.42) at $\varpi_{\iota'}$ then gives

$$0 = \mathbf{m}_{\wedge^k \rho_\omega}(\varpi_{\iota'}) = \sum_\iota N_{\varpi_\iota}^{(\omega,k)} \mathbf{m}_{\otimes_i \rho_i^{\iota_i}}(\varpi_{\iota'}). \tag{5.3.43}$$

Further notice that

$$\mathbf{m}_{\otimes_i \rho_i^{\iota_i}}(\varpi_{\iota'}) = \begin{cases} 1, & \iota' = \iota, \\ 0, & \iota' \neq \iota \text{ and } \iota' - \iota \notin (\mathbb{Z}_{\geq 0})^{l_\mathcal{R}}. \end{cases} \tag{5.3.44}$$

Both equalities in (5.3.44) are consequences of the fact that the weights of a tensor product representation are given by the sums of the weights of its factors, $\Gamma(V \otimes W) = \{v + w | v \in \Gamma(V), w \in \Gamma(W)\}$. In particular, the first equality states that the weight space of $\varpi_\iota$ in $\otimes_i \rho_i^{\iota_i}$ is 1-dimensional, corresponding to the unique decomposition of $\varpi_\iota = \sum_i \iota_i \omega_i$ as a sum of highest weights for each factor. The second is the assertion that $\varpi_{\iota'} \notin \Gamma(\otimes_i \rho_i^{\iota_i})$ when $\iota_j' > \iota_j$ for some $j$. As a consequence, (5.3.43) is a linear homogeneous system in the unknowns $N_{\varpi_\iota}^{(\omega,k)}$ with trivial kernel. The integral

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[||]This follows from the fact that inverse of the Cartan matrix for any $\mathcal{R}$ has positive definite entries, and two times the sum of any row (or column) is an integer. See for example [?OnishchikV:2012] for classical types. The equivalence then follows from the definitions of a fundamental weight, the partial ordering $\leq$ and pivotal sets.

matrix $\mathsf{M}$ with coefficients $(\mathsf{M})_{\iota',\iota} := \mathbf{m}_{\otimes_i \rho_i^{\iota_i}}(\varpi_{\iota'})$, in a choice of basis where the column $(\mathsf{M})_{\iota'}$ is to the left of the column $(\mathsf{M})_\iota$ whenever $\iota' - \iota \in (\mathbb{Z}_{\geqslant 0})^{l_{\mathcal{R}}}$, is upper-triangular and with ones on the diagonal. The claim then follows.

$\square$

By Lemma 5.3.2, the sum in the polynomial character decomposition (5.3.38) localises on the set of admissible exponents, whose cardinality $|\mathfrak{I}_\omega|$ is shown in Table 5.2.

**Corollary 5.3.3.** *Fix a bijection* $\sigma : \{1, \ldots, |\mathfrak{I}_\omega|\} \to \mathfrak{I}_\omega$ *inducing a total order on the set of admissible exponents, and let* $\mathsf{N}_\omega \in \mathrm{Mat}_{|\mathfrak{I}_\omega| \times \dim \rho_\omega}(\mathbb{Z})$ *with* $(\mathsf{N}_\omega)_{lk} := N^{(\omega,k)}_{\varpi_{\sigma(l)}}$. *Then, there exist explicit rational matrices* $\mathsf{A}_\omega \in \mathrm{GL}_{|\mathfrak{I}_\omega|}(\mathbb{Q})$, $\mathsf{B}_\omega \in \mathrm{Mat}_{|\mathfrak{I}_\omega| \times \dim \rho_\omega}(\mathbb{Q})$ *such that* $\mathsf{N}_\omega = \mathsf{A}_\omega \mathsf{B}_\omega$. *In particular,* $\mathsf{N}_\omega$ *is explicitly computable for all* $\omega$.

*Proof.* Fix $\mathcal{Q} := \{(Q_1^{(\kappa)}, \ldots, Q_{l_{\mathcal{R}}}^{(\kappa)}) \in \mathbb{Q}^{l_{\mathcal{R}}}\}_{\kappa=1}^{|\mathfrak{I}_\omega|}$ rational points in $\mathcal{T}_{\mathcal{R}}$ in general position, and define

$$(\mathsf{D}_\omega)_{\iota,\kappa} := \prod_{i=1}^{l_{\mathcal{R}}} \chi_i^{\iota_i}(Q_1^{(\kappa)}, \ldots, Q_{l_{\mathcal{R}}}^{(\kappa)}), \tag{5.3.45}$$

$$(\mathsf{B}_\omega)_{\kappa,l} := \chi_{\wedge^l \rho_\omega}(Q_1^{(\kappa)}, \ldots, Q_{l_{\mathcal{R}}}^{(\kappa)}). \tag{5.3.46}$$

Evaluating (5.3.38) at $\mathcal{Q}$ amounts to saying that

$$\mathsf{D}_\omega \mathsf{N}_\omega = \mathsf{B}_\omega. \tag{5.3.47}$$

The identification of the Weyl character ring as an integral polynomial ring generated by the fundamental characters, (5.1.9), assures that $\det \mathsf{D}_\omega \neq 0$ for general $\mathcal{Q}$, and (5.3.47) is a rank-$|\mathfrak{I}_\omega|$ linear problem over the rationals. We then have $\mathsf{A}_\omega = \mathsf{D}_\omega^{-1}$, and furthermore, the integrality of the coefficients in (5.1.9) assures that $\mathsf{N}_\omega \in \mathbb{Z}$. Furthermore, given any such $\mathcal{Q}$, the rational matrices $\mathsf{A}_\omega$, $\mathsf{B}_\omega$, and $\mathsf{N}_\omega$ are computable in an entirely explicit manner, as follows. The fundamental characters on the right-hand side of (5.3.45) are given as

$$\chi_i(Q_1^{(\kappa)}, \ldots, Q_{l_{\mathcal{R}}}^{(\kappa)}) = \sum_{\omega' \in \Gamma(\rho_i)} \prod_{j=1}^{l_{\mathcal{R}}} \left(Q_j^{(\kappa)}\right)^{\omega'_j}, \tag{5.3.48}$$

and their value can be computed from the known expressions of the elements of the fundamental weight systems $\Gamma(\rho_i)$, $i = 1, \ldots, l_{\mathcal{R}}$. The exterior characters in (5.3.46) can similarly be computed from the knowledge of $\Gamma(\rho_\omega)$ alone to evaluate the power sum virtual characters of $\rho_\omega$,

$$\chi_{\rho_\omega}((Q_1^{(\kappa)})^n, \ldots, (Q^{(\kappa)})_{l_{\mathcal{R}}}^n) = \sum_{\omega' \in \Gamma(\rho_\omega)} \prod_{j=1}^{l_{\mathcal{R}}} \left(Q_i^{(\kappa)}\right)^{n\omega'_j}, \tag{5.3.49}$$

from which the exterior characters $\chi_{\wedge^k \rho_\omega}$ can be recovered using the Girard–Newton identities,

$$k\chi_{\wedge^k \rho_\omega}(Q_1^{(\kappa)}, \ldots, Q_{l_\mathcal{R}}^{(\kappa)}) = \sum_{n=1}^{k} (-1)^{n-1} \chi_{\wedge^{k-n} \rho_\omega}(Q_1^{(\kappa)}, \ldots, Q_{l_\mathcal{R}}^{(\kappa)}) \chi_{\rho_\omega}((Q_1^{(\kappa)})^n, \ldots, (Q_{l_\mathcal{R}}^{(\kappa)^n})^n). \quad (5.3.50)$$

The integral linear system (5.3.47) can then be efficiently solved explicitly for $\mathsf{N}_\omega$ using Dixon's $p$-adic lifting algorithm. $\qquad \square$

Using (5.1.8)–(5.1.11), (5.3.38) and the complete calculation of the coefficients $N_{\varpi_\iota}^{(\omega,k)}$ from Lemma 5.3.2 and Corollary 5.3.3, we can construct Landau–Ginzburg superpotentials for all $\mathcal{R}$. In practice, this was achieved for the exceptional cases** by the help of Mathematica in which the code follows the outline above. We now describe the result of the remaining exceptional Dynkin types. Note that in each case, the genus of the associated spectral curve was obtained using the Riemann-Hurwitz formula, as shown more in detail for $\mathcal{R} = F_4$, matching with that of [17].

$\mathcal{R} = E_6$

The Dynkin diagram for the affine $E_6$ root system is given in 4.1, for which the canonical label is $\bar{k} = 3$. In this case there are two nontrivial minimal irreducible representations, the 27-dimensional fundamental representation $\rho_1 = (\mathbf{27})$ with highest weight $\omega_1$ and its dual representation $\rho_5 = (\overline{\mathbf{27}})$ with highest weight $\omega_5$, related by complex conjugation.



Figure 5.1: Newton polygon for the $E_6$-spectral curve. The horizontal axis indicates the order of $\lambda$ and the vertical axis represents the order of $\mu$. Thus, a dot is present at position $(i, j)$ in the diagram if and only if the coefficient of $\lambda^i \mu^j$ in the associated reduced spectral curve, $\mathcal{P}_\omega$, is nonzero.

The character relations (5.3.38) are given explicitly for $\omega = \omega_1$ in Appendix A.1. The resulting family of spectral curves has fibres which are hyperelliptic curves (as the order of $\lambda$ in this case is

---

**Apart from the case of $\mathcal{R} = G_2$ in which the relations were found manually.

2 as seen on the Newton polygon) of genus 5, with Newton polygon as shown in Figure 5.1, and ramification profile over $\infty$

$$
\left( \overbrace{3,6}^{\mu=0}, \overbrace{6,3}^{\mu=\infty}, \overbrace{3}^{\mu=\varepsilon_3^j} \right), \quad j = 0, 1, 2, \tag{5.3.51}
$$

where $\varepsilon_3$ is a primitive third root of unity. This realises $M_{[100000]_{E_6}}^{\mathrm{LG}}$ as a 7-dimensional subvariety of the 42-dimensional Hurwitz space $H_{g_\omega, \mathsf{n}_\omega}$ with $g_\omega = 5$ and $\mathsf{n}_\omega = (5, 5, 2, 2, 2, 2, 2)$, and explicit embedding described by (A.1.1).

## $\mathcal{R} = E_7$

The Dynkin diagram for the affine $E_7$ root system is given in Figure 4.1, in which we see that the canonical label is $\bar{k} = 3$. For this case there is a unique choice of minimal representation, corresponding to the 56-dimensional fundamental representation having highest weight $\omega_6$. Choosing this representation gives character relations which we include in Appendix A.2 for $k = 1, \ldots, 11$. The resulting family of spectral curves has fibres of genus 33, with a degree 3 morphism to $\mathbb{P}^1$ inducing a $3 : 1$ branched cover of the Riemann sphere with ramification profile over $\infty$

$$
\left( \overbrace{12, 6, 4,}^{\mu=0} \overbrace{12, 6, 4,}^{\mu=\infty} \overbrace{2}^{\mu=\pm 1} \overbrace{4}^{\mu=\pm i} \right). \tag{5.3.52}
$$

Hence, $M_\omega^{\mathrm{LG}}$ is an 8-dimensional submanifold in the 130-dimensional Hurwitz space $H_{g_\omega, \mathsf{n}_\omega}$, with $g_\omega = 33$ and $\mathsf{n}_\omega = (11, 5, 3, 11, 5, 3, 1, 1, 3, 3)$. The associated Newton polygon is shown in Figure 5.2.

## $\mathcal{R} = E_8$

As mentioned, this case was thoroughly treated in [17], [18], and we will provide only a brief presentation here. The Dynkin diagram for the affine $E_8$ root system is given in Figure 4.1, and the canonical label is, as for all the $E$-types, $\bar{k} = 3$. In this case, the minimal, nontrivial, irreducible representation is the 248-dimensional adjoint representation. Explicit character relations were given in [17], where their derivation is explained in detail. It is shown in [17] that the resulting curve is of genus 128 and induces a cover of the Riemann sphere with ramification over $\infty$ at $\mu = 0, \infty$, in addition to second, third and fifth roots of unity, with ramification profile given in [17, Eq. (5.34)]. Explicit flat coordinates and prepotential can also be found in [17]. The resulting parent Hurwitz space is of dimension 518.
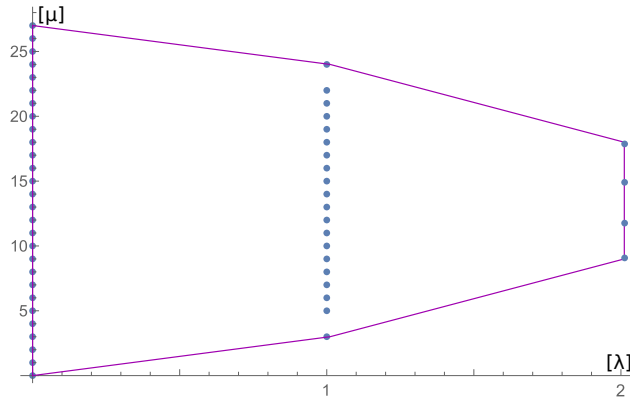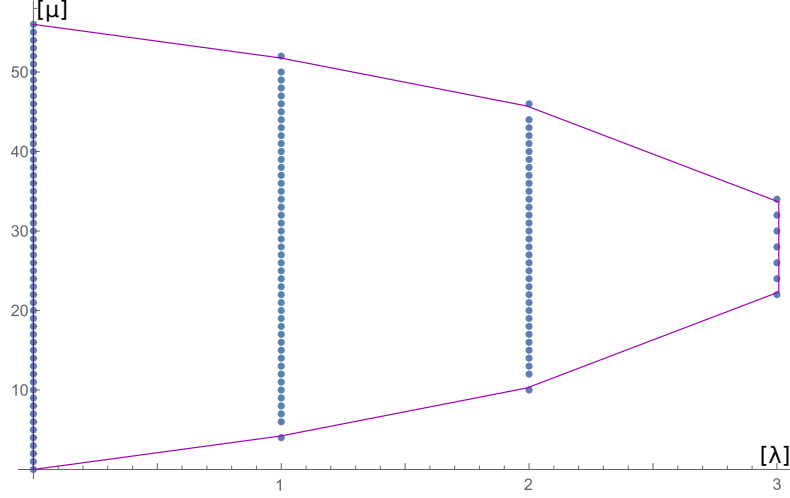
Figure 5.2: Newton polygon for the $E_7$-spectral curve. The horizontal axis indicates the order of $\lambda$ and the vertical axis represents the order of $\mu$. Thus, a dot is present at position $(i, j)$ in the diagram if and only if the coefficient of $\lambda^i \mu^j$ in the associated reduced spectral curve, $\mathcal{P}_\omega$, is nonzero.

$\mathcal{R} = F_4$

The Dynkin diagram for the affine root system of type $F_4$ is shown in 4.1. In this case the canonical
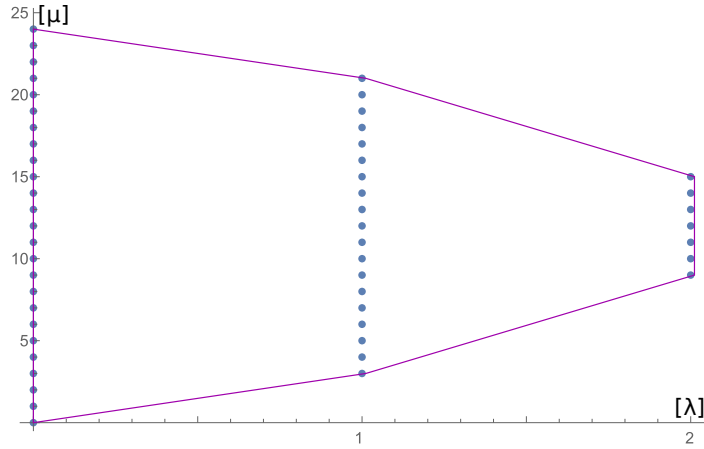


Figure 5.3: Newton polygon for the $F_4$-spectral curve. The horizontal axis indicates the order of $\lambda$ and the vertical axis represents the order of $\mu$. Thus, a dot is present at position $(i, j)$ in the diagram if and only if the coefficient of $\lambda^i \mu^j$ in the associated reduced spectral curve, $\mathcal{P}_\omega$, is nonzero.

node is the one corresponding to the fundamental weight $\omega_2$, $\omega = [0001]_{F_4}$, and $\rho_\omega$ will be the 26-dimensional fundamental representation, i.e. the irreducible representation of highest weight $\omega_4$. The character relations (5.1.9) are then given as follows:

$$\mathfrak{p}_1^{[0001]_{F_4}} = \chi_4,$$

$$\mathfrak{p}_2^{[0001]_{F_4}} = \chi_1 + \chi_3,$$

$$\mathfrak{p}_3^{[0001]_{F_4}} = \chi_2 + \chi_1\chi_4 - \chi_4,$$

$$\mathfrak{p}_4^{[0001]_{F_4}} = \chi_1^2 + \chi_3\chi_1 - \chi_4^2 - \chi_2,$$

$$\mathfrak{p}_5^{[0001]_{F_4}} = \chi_1^2\chi_4 - \chi_4^3 - \chi_1\chi_4 - 2\chi_2\chi_4 + \chi_3\chi_4 + \chi_4 + \chi_3^2 - \chi_2 + \chi_3,$$

$$\mathfrak{p}_6^{[0001]_{F_4}} = \chi_1^3 - \chi_1^2 - \chi_4^2\chi_1 - 3\chi_2\chi_1 + \chi_3\chi_4\chi_1 + \chi_4\chi_1 - \chi_1 - \chi_4^3 + \chi_4^2 - \chi_2 - \chi_2\chi_4 + \chi_3\chi_4 + \chi_4,$$

$$\mathfrak{p}_7^{[0001]_{F_4}} = \chi_1\chi_4^3 - \chi_4^3 - \chi_1\chi_4^2 + \chi_2\chi_4^2 + 2\chi_4^2 - 2\chi_1\chi_4 - 2\chi_2\chi_4 - 3\chi_1\chi_3\chi_4 + 2\chi_3\chi_4$$
$$+ \ \chi_4 + \chi_1^2 - \chi_1 + \chi_1\chi_2 - \chi_2 + \chi_1^2\chi_3 - \chi_1\chi_3 - 2\chi_2\chi_3,$$

$$\mathfrak{p}_8^{[0001]_{F_4}} = \chi_4^2\chi_1^2 - \chi_1^3 - 2\chi_3\chi_1^2 + \chi_2\chi_1 - \chi_3\chi_1 + \chi_2\chi_4\chi_1 - \chi_3\chi_4\chi_1 - \chi_4\chi_1$$
$$+ \ \chi_3\chi_4^3 + \chi_4^3 - 2\chi_3^3 - \chi_4^2 - \chi_2 + 3\chi_2\chi_3 + \chi_3 - 3\chi_3^2\chi_4 + 2\chi_2\chi_4 - 2\chi_3\chi_4 + \chi_4,$$

$$\mathfrak{p}_9^{[0001]_{F_4}} = \chi_4^5 - \chi_1\chi_4^3 - 4\chi_3\chi_4^3 - 2\chi_4^3 + 2\chi_1\chi_4^2 + 4\chi_2\chi_4^2 + \chi_1\chi_3\chi_4^2 + 2\chi_3^2\chi_4 - \chi_2 + \chi_1\chi_2\chi_4 + \chi_2^2$$
$$+ \ 2\chi_1\chi_3\chi_4 + 3\chi_3\chi_4 - 2\chi_1^2 + \chi_2\chi_4 - 2\chi_1\chi_3^2 - 2\chi_1\chi_2 + 2\chi_1\chi_4 - 2\chi_1^2\chi_3 - 2\chi_1\chi_3 - \chi_2\chi_3,$$

$$\mathfrak{p}_{10}^{[0001]_{F_4}} = \chi_4^5 + \chi_1\chi_4^4 - \chi_1\chi_4^3 - 5\chi_3\chi_4^3 - 2\chi_4^3 - 2\chi_1^2\chi_4^2 + 3\chi_2\chi_4^2 - 3\chi_1\chi_3\chi_4^2 - \chi_3\chi_4^2 + 5\chi_3^2\chi_4$$
$$+ \ 3\chi_1\chi_4 + \chi_1\chi_2\chi_4 + \chi_2\chi_4 + 3\chi_1\chi_3\chi_4 + \chi_2\chi_3\chi_4 + 4\chi_3\chi_4$$
$$- \ \chi_4 + \chi_1^3 - \chi_2^2 + \chi_1\chi_3^2 + 3\chi_3^2 - \chi_1\chi_2 + \chi_2 + 2\chi_1^2\chi_3 + 3\chi_1\chi_3 - 3\chi_2\chi_3,$$

$$\mathfrak{p}_{11}^{[0001]_{F_4}} = \chi_1\chi_4^4 + \chi_2\chi_4^3 - \chi_3\chi_4^3 + \chi_4^3 - \chi_1^2\chi_4^2 - 2\chi_1\chi_4^2 - 2\chi_2\chi_4^2 - 5\chi_1\chi_3\chi_4^2 - 2\chi_4^2 + 2\chi_3^2\chi_4$$
$$- \ \chi_1\chi_4\chi_2\chi_4 + \chi_1\chi_3\chi_4 - 3\chi_2\chi_3\chi_4 + \chi_3^3 + 3\chi_1^2$$
$$+ \ \chi_2^2 + 4\chi_1\chi_3^2 + 2\chi_3^2 + \chi_1 + 3\chi_1\chi_2 + 3\chi_1^2\chi_3 + 5\chi_1\chi_3 + 2\chi_2\chi_3 + \chi_3,$$

$$\mathfrak{p}_{12}^{[0001]_{F_4}} = \chi_4^4 - \chi_4^5 - \chi_1\chi_4^4 + 3\chi_1\chi_4^3 + 2\chi_3\chi_4^3 + 2\chi_1^2\chi_4^2 + \chi_3^2\chi_4^2 - \chi_1\chi_4^2 - \chi_2\chi_4^2 + \chi_1\chi_3\chi_4^2$$
$$+ \ \chi_3\chi_4^2 - 5\chi_1\chi_4 - 4\chi_2\chi_4 - 5\chi_1\chi_3\chi_4 - 3\chi_2\chi_3\chi_4 + \chi_3\chi_4 + \chi_4 - \chi_1^3$$
$$+ \ 2\chi_2^2 - \chi_1\chi_3^2 + 3\chi_1\chi_2 - 3\chi_1\chi_3 - 2\chi_2\chi_3,$$

$$\mathfrak{p}_{13}^{[0001]_{F_4}} = 2\chi_4^4 - 2\chi_4^5 - 2\chi_1\chi_4^4 + 4\chi_1\chi_4^3 - 2\chi_2\chi_4^3 + 6\chi_3\chi_4^3 + 2\chi_4^3 + 4\chi_1^2\chi_4^2 + 2\chi_3^2\chi_4^2$$
$$- \ 2\chi_1\chi_4^2 - 2\chi_2\chi_4^2 + 4\chi_1\chi_3\chi_4^2 - 2\chi_4^2 - 2\chi_1^2\chi_4 - 4\chi_3^2\chi_4 - 2\chi_1\chi_4$$
$$+ \ 2\chi_1\chi_2\chi_4 - 4\chi_1\chi_3\chi_4 + 2\chi_2\chi_3\chi_4 - 4\chi_3\chi_4 - 2\chi_3^3 - 4\chi_1^2$$
$$- \ 2\chi_2^2 - 4\chi_1\chi_3^2 - 4\chi_3^2 - 4\chi_1\chi_2 - 4\chi_1^2\chi_3 - 4\chi_1\chi_3 - 2\chi_2\chi_3 + 2, \qquad (5.3.53)$$

with $\mathfrak{p}_{26-i}^{[0001]_{F_4}} = \mathfrak{p}_i^{[0001]_{F_4}}$. Note that the above relations in the character ring follow from those for $\mathcal{R} = E_6$ and $\rho = (\mathbf{27})$ or $\rho = (\overline{\mathbf{27}})$ by folding. In particular $M_{F_4,[0001]_{F_4}}^{\mathrm{LG}}$ sits inside $M_{[100000]_{E_6}}^{\mathrm{LG}}$ as the fixed locus of the involution $w_1 \leftrightarrow w_5$, $w_2 \leftrightarrow w_4$. The generic fibre $\overline{C_w^{\omega_4}}$ is a genus 4 hyperelliptic

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curve with ramification over $\infty$

$$
\left(
\overbrace{3,6}^{\mu=0},\ \overbrace{3,6}^{\mu=\infty},\ \overbrace{3}^{\mu=\varepsilon_3},\ \overbrace{3}^{\mu=\varepsilon_3^2}
\right),
\tag{5.3.54}
$$

where $\varepsilon_3$ is a primitive third root of unity, and the associated Newton polygon is shown in Figure 5.3. The extended affine $F_4$-Frobenius manifold is thus realised as a 5-dimensional submanifold of the 36-dimensional Hurwitz space $H_{g_\omega, \mathsf{n}_\omega}$, with $g_\omega = 4$ and $\mathsf{n}_\omega = (5, 5, 2, 2, 2, 2)$.

## $\mathcal{R} = G_2$

The Dynkin diagram for the affine $G_2$ root system is given in Figure 4.1, and the canonical label is $\bar{k} = 2$. In this case, $\rho_{\omega_1} = (\mathbf{7})$ is the 7-dimensional fundamental representation which is the irreducible representation with highest weight $\omega_1$. We obtain the character relations

$$
\mathfrak{p}_1^{[10]_{G_2}} = \chi_1, \quad \mathfrak{p}_2^{[10]_{G_2}} = \chi_1 + \chi_2, \quad \mathfrak{p}_3^{[10]_{G_2}} = \chi_1^2 - \chi_2,
\tag{5.3.55}
$$

and $\mathfrak{p}_{7-i}^{[10]_{G_2}} = \mathfrak{p}_i^{[10]_{G_2}}$, hence

$$
\mathcal{P}_{[10]_{G_2}}(\lambda, \mu; w_0, w_1, w_2) \equiv \sum_{i=0}^{3}(-1)^i \mathfrak{p}_i^{[10]_{G_2}}\left(w_1, w_2 - \frac{\lambda}{w_0}\right) \mu^i (1 - \mu^{7-2i}),
$$

and solving for $\lambda$

$$
\lambda = \frac{w_0\left(\mu^6 + (1 - w_1)\mu^5 + (1 + w_2)\mu^4 + (1 - w_1^2 + 2w_2)\mu^3 + (1 + w_2)\mu^2 + (1 - w_1)\mu + 1\right)}{\mu^2(\mu + 1)^2}.
\tag{5.3.56}
$$

As was the case for $\mathcal{R} = F_4$, the same superpotential could be obtained from the LG–model of $\omega = [1000]_{D_4}$ by the order three folding of the $D_4$ Dynkin diagram, and $M_{[10]_{G_2}}^{\mathrm{LG}}$ sits inside $M_{[1000]_{D_4}}^{\mathrm{LG}}$ as the fixed locus of the triality action sending $(w_1, w_3, w_4) \to (w_{\epsilon(1)}, w_{\epsilon(3)}, w_{\epsilon(4)})$ with $\epsilon \in S_3$. The outcome is a family of rational functions in $\mu$, with three poles at $\mu = 0, -1, \infty$, all of order two. This means that the resulting Frobenius manifold is a 3-dimensional sublocus in the 7-dimensional Hurwitz space $H_{0, \mathsf{n}_\omega}$, with $\mathsf{n}_\omega = (1, 1, 1)$.

**Remark.** In the computation of the genus of the above spectral curve we have used the Riemann-Hurwitz formula. In the following we show this calculation in the case of $\mathcal{R} = F_4$. Let $X \to \mathbb{P}^1$ be a cover of the projective line of degree $N$ with $X$ being a compact and connected complex curve, then the genus of $X$ is by the Riemann-Hurwitz formula given by

$$
g = 1 - \frac{(N(2 - b) + r))}{2},
\tag{5.3.57}
$$

where $r$ is the number of ramification points and $b$ the number of branch points. When $N = 2$, we must have that $b = r$, and we get

$$g = \frac{b}{2} - 1. \tag{5.3.58}$$

In order to find $b$ (or in this case equivalently $r$), we look at the discriminant of the curve. That is, $b_1^2 - 4b_0b_2$, where $b_j = [\lambda^j]\mathcal{P}$. In this case we obtain

$$\frac{1}{w_0^2}\mu^6(\mu^2 - 1)^2(\mu - 1)^2 f(\mu)^2 g(\mu), \tag{5.3.59}$$

where $f(\mu)$ is a polynomial in $\mu$ of order 12, and $g(\mu)$ is a polynomial in $\mu$ of order 10. Now, $\mu = 1$ is a pole of $\lambda$ of order 2. Furthermore, as the Riemann-Hurwitz formula requires compactness, we must consider the number of solutions of $g(\mu) = 0$, which is 10. This means that the genus of $\mathcal{C}_{F_4}$ is $\frac{10}{2} - 1 = 4$, as claimed.

## 5.4 Mirror symmetry

Having constructed $M_\omega^{\mathrm{LG}}$ for all Dynkin types, we now move on to proving Theorem 5.2.5.

Consider the analogous expressions to Theorem 3.3.2 for our choice of $\phi$ (i.e. $p = \log\mu$):

$$\tau_{i;\alpha} := \mathop{\mathrm{Res}}_{\infty_i} \kappa_i^{-\alpha} \log\mu \, \mathrm{d}\lambda, \quad \alpha = 1, \cdots, n_i, \tag{5.4.1a}$$

$$\tau_j^{\mathrm{ext}} := \mathrm{p.v.} \int_{\infty_0}^{\infty_j} \mathrm{d}\log\mu, \quad j = 1, \cdots, \ell(\mathsf{n}_\omega), \tag{5.4.1b}$$

$$\tau_k^{\mathrm{Res}} := \mathop{\mathrm{Res}}_{\infty_k} \lambda \, \mathrm{d}\log\mu, \quad k = 0, \cdots, \ell(\mathsf{n}_\omega). \tag{5.4.1c}$$

Notice that there are more equations in (5.4.1a)–(5.4.1c) than the dimension $l_\mathcal{R} + 1$. The following Lemma describes how they are related.

**Lemma 5.4.1.** There exist complex numbers $\{\{f_{i,\alpha}^{(j)} \in \mathbb{C}\}_{\alpha=1,\ldots,n_i}\}_{i=1,\ldots,\ell(\mathsf{n}_\omega)}$, $\{q_j \in \mathbb{C}\}_{j=1\cdots,\ell(\mathsf{n}_\omega)}$, $\{r_j \in \mathbb{C}\}_{j=0,\cdots,\ell(\mathsf{n}_\omega)}$, and holomorphic functions $\{t_i : M_\omega^{\mathrm{LG}} \to \mathbb{C}\}_{i=1,\ldots,l_\mathcal{R}+1}$ such that

$$\begin{aligned}
\tau_j^{\mathrm{ext}}\big|_{M_\omega^{\mathrm{LG}}} &= q_j t_{l_\mathcal{R}+1}, \\
\tau_{i;\alpha}\big|_{M_\omega^{\mathrm{LG}}} &= \sum_{r=1}^{l_\mathcal{R}} f_{i,\alpha}^{(r)} t_r, \\
\tau_k^{\mathrm{Res}}\big|_{M_\omega^{\mathrm{LG}}} &= r_k t_{\bar{k}}.
\end{aligned} \tag{5.4.2}$$

Moreover, $(t_1, \ldots, t_{l_\mathcal{R}+1})$ are flat coordinates for the metric (5.4.27) on $\mathcal{M}_\omega^{\mathrm{LG}}$.

*Proof.* Suppose that $\mathcal{R}$ is associated to a genus 0 curve, i.e. any of the root systems $A_l$, $B_l$, $C_l$, $D_l$ or $G_2$. Then, from the discussion of the previous sections, we have that $M_\omega^{\mathrm{LG}} \subset H_{0,\mathsf{n}_\omega}$ is a space of rational functions. The statement of the lemma is then a specialisation of [41, Thm 5.1], which in particular asserts that $\{\tau_{i,\alpha}, \tau_j^{\mathrm{ext}}, \tau_k^{\mathrm{Res}}\}$ make a complete set of flat coordinates for the metric (5.2.9) on a genus zero Hurwitz space; and since $M_\omega^{\mathrm{LG}}$ is a fixed-locus of an involution acting linearly in these coordinates, it is specified by a linear condition of the form (5.4.2). For the remaining four exceptional cases, the linear relations in (5.4.2) and the constancy of the Gram matrix of the metric (5.2.9) in the chart $(t_1, \ldots, t_{l_\mathcal{R}+1})$ follow from a direct residue calculation from (5.4.1a)–(5.4.1c), and from making use of the explicit form of the spectral curve from (5.1.11) for each Dynkin type. We employ Mathematica for large parts of these calculations. Resulting expressions for flat coordinates can be found in Examples 20–23. $\qquad\square$

The following Lemma allows us to notably simplify the explicit computations.

**Lemma 5.4.2.** For a pole of $\lambda$, $\infty_i$, of order $n_i + 1$, it is sufficient to take the Puiseux expansion of the spectral curve to order 0 in $\mu$ to determine Saito flat coordinates via Lemma 5.4.1.

*Proof.* As $\mathrm{d}\log\mu \equiv \dfrac{\mathrm{d}\mu}{\mu}$, it is clear that only the coefficient of $\mu^0$ of the $\lambda$-expansion will contribute to (5.4.1c).

Let us consider (5.4.1a) with $p := \log(\mu)$. We are interested in the residue of

$$\kappa^{-\alpha}\log(\mu)(\kappa)\mathrm{d}\lambda = \kappa^{-\alpha}\log(\mu)(\kappa)\lambda'(\kappa)\mathrm{d}\kappa, \qquad (5.4.3)$$

at $\infty_i$ for $\alpha = 1, \cdots, n_i$. Let $\tilde{\kappa} := \frac{1}{\kappa}$. Then,

$$\mathrm{d}\kappa = \frac{\mathrm{d}\kappa}{\mathrm{d}\tilde{\kappa}}\mathrm{d}\tilde{\kappa} = -\frac{\mathrm{d}\tilde{\kappa}}{\tilde{\kappa}^2}. \qquad (5.4.4)$$

As $\lambda(\kappa) \sim \kappa^{n_i+1}$, we have that $\lambda'(\kappa) \sim (n_i+1)\kappa^{n_i} \implies \lambda'\left(\frac{1}{\tilde{\kappa}}\right) \sim -(n_i+1)\tilde{\kappa}^{-n_i}$. Thus

$$\kappa^{-\alpha}p(\kappa)\mathrm{d}\lambda(\kappa)\mathrm{d}\kappa = -(n_i+1)p\left(\frac{1}{\tilde{\kappa}}\right)\tilde{\kappa}^{\alpha-n_i+1-1}\mathrm{d}\tilde{\kappa}. \qquad (5.4.5)$$

As we want to take the residue, we want to expand $p\left(\frac{1}{\tilde{\kappa}}\right)$ to order $n_i + 1 - \alpha$ in $\tilde{\kappa}$, which implies that we want $p(\kappa)$ to order $\alpha - n_i + 1$ in $\kappa$. Hence, as $p$ is the inverse series of the series expansion of $\lambda$ near $\infty_i$, and $\max(\alpha - n_i - 1) = n_1 - n_1 - 1 = -1$, it is sufficient to expand to order $\mu^{-1}$ to capture the contribution in a.

Finally, by choosing a suitable candidate for $\infty_0$, the lower limit becomes a constant, and so by (5.4.1b), we are simply interested in the constant part of $\frac{1}{\mu(\kappa)}\frac{\mathrm{d}\mu}{\mathrm{d}\kappa}$. Thus, it is sufficient to expand $\mu_\infty(\kappa)$ to order 1 in $\kappa$ to capture the contribution in (5.4.1b). Hence, by the same logic as for a, it is sufficient to consider the series expansion of $\lambda$ near a pole to order $\mu^{-1}$. $\qquad\square$

| $\mathcal{R} = F_4$ | | | |
|---|---|---|---|
| $\lambda$ | $\infty_i$ | $\alpha$ | $\tau_{i,\alpha}$ |
| $\lambda_1$ | $0$ | $1$ | $\frac{1}{6}t_3$ |
| | | $2$ | $-t_1$ |
| | $(-1)^{\frac{2}{3}}$ | $1$ | $-\frac{\nu^5}{2\sqrt{3}}t_3$ |
| | | $2$ | $-\sqrt{3}\,\nu\,t_1$ |
| | $(-1)^{\frac{4}{3}}$ | $1$ | $-\frac{\nu^3}{2\sqrt{3}}t_3$ |
| | | $2$ | $-\sqrt{3}\nu^3 t_1$ |
| $\lambda_2$ | $0$ | $1$ | $0$ |
| | | $2$ | $-\frac{\nu^4}{6}t_3$ |
| | | $3$ | $\frac{3\nu^3}{2}t_2$ |
| | | $4$ | $-\nu^2 t_1$ |
| | | $5$ | $0$ |
| | $(-1)^{\frac{2}{3}}$ | $1$ | $-\frac{\nu^5}{2\sqrt{3}}t_3$ |
| | | $2$ | $-\sqrt{3}\,\nu\,t_1$ |
| | $(-1)^{\frac{4}{3}}$ | $1$ | $-\frac{\nu^3}{2\sqrt{3}}t_3$ |
| | | $2$ | $-\sqrt{3}\,\nu^3\,t_1$ |

Table 5.3: This table shows the result of (5.4.1a) for $\mathcal{R} = F_4$. Here, $\lambda_i$ denotes the $i^{\text{th}}$ branch, i.e. the $i^{\text{th}}$ solution of (5.1.11)$= 0$ for $\lambda$. The second and third columns, respectively, indicate which pole and value of $\alpha$ is associated with the expression shown in the final column, and $\nu := \frac{i+\sqrt{3}}{2}$. The result of $\infty_i = \infty$ is related to the corresponding expression for $\infty_i = 0$ by a minus sign.

For $\mathcal{R} = F_4$, Lemma 5.4.1 is shown in Table 5.3.

**Remark.** The contribution from the pole at $\infty$ is the same as that from $0$. This is due to the symmetry under $\mu \to \frac{1}{\mu}$ (or equivalently, by the realness of the minimal representation for $F_4$ as shown in Table 5.1).

We need an additional Lemma before we give the proof of Theorem 5.2.5.

**Lemma 5.4.1.** *Let $(t_1, \ldots, t_{l_\mathcal{R}+1})$ be flat coordinates for (5.2.9) as in Theorem 3.3.2. Then $t_{l_\mathcal{R}+1}$* $= \log w_0 / d_{\bar{k}}$ *and, for all $i = 1, \ldots, l_\mathcal{R}$,*

$$t_i(w_0, \ldots, w_{l_\mathcal{R}}) \in w_0^{d_i/d_{\bar{k}}} \mathbb{Z}[w_1, \ldots, w_{l_\mathcal{R}}]. \tag{5.4.6}$$

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*Moreover, the change of variables $w \mapsto t(w)$ has a polynomial inverse*

$$w_i(t_1, \ldots, t_{l_{\mathcal{R}}+1}) \in \mathbb{Q}[t_1, \ldots, t_{l_{\mathcal{R}}+1}, e^{t_{l_{\mathcal{R}}+1}}]. \tag{5.4.7}$$

*Proof.* A direct calculation from (5.4.1a)–(5.4.1c), using as above the explicit form of (5.1.11), shows that the flat coordinates $(t_1, \ldots, t_{l_{\mathcal{R}}+1})$ are related (up to normalisation) to $(w_0, \ldots, w_{l_{\mathcal{R}}})$ as

$$
\begin{aligned}
t_i &= w_0^{d_i/d_{\bar{k}}} \mathsf{t}_i(w), \\
t_{l_{\mathcal{R}}+1} &= \frac{\log w_0}{d_{\bar{k}}},
\end{aligned} \tag{5.4.8}
$$

where $\mathsf{t}_i(w)$ are explicit integral polynomials in $(w_1, \ldots, w_{l_{\mathcal{R}}})$. Moreover, it can be verified directly that

$$
\begin{aligned}
\partial_{w_j} \mathsf{t}_i &= 0 \quad \text{if } d_j > d_i, \\
\deg_{w_j} \mathsf{t}_i &= 1 \quad \text{if } d_j = d_i,
\end{aligned} \tag{5.4.9}
$$

implying that the inverse function $t \mapsto w(t)$ is a polynomial in $(t_1, \ldots, t_{l_{\mathcal{R}}}, e^{t_{l_{\mathcal{R}}+1}})$ with rational coefficients. $\qquad \square$

We are now in a position to prove Theorem 5.2.5.

*Proof of Theorem 5.2.5.* Consider the coordinate change

$$w_i(x_1, \ldots, x_{l_{\mathcal{R}}}) = \chi_i(e^{\mathsf{h}}) = \sum_{\omega' \in \Gamma(\rho_i)} \prod_{j=1}^{l_{\mathcal{R}}} e^{\omega'_j x_j}, \quad i = 1, \ldots l_{\mathcal{R}}. \tag{5.4.10}$$

Further, identifying

$$w_0 = e^{c_\omega x_{l_{\mathcal{R}}+1}}, \tag{5.4.11}$$

for $c_\omega \in \mathbb{C}^\star$, extends this to a local analytic isomorphism sending $\mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}} \ni (x_1, \ldots, x_{l_{\mathcal{R}}}; x_{l_{\mathcal{R}}+1}) \mapsto (w_0; w_1, \ldots, w_{l_{\mathcal{R}}}) \in M_\omega^{\mathrm{LG}}$. We shall now prove that this is, in fact, an isomorphism of Frobenius structures upon checking the four defining properties, **DZ-I** through **DZ-IV**, of the reconstruction theorem, Theorem 4.1.1.

**DZ-I and DZ-II.** Define holomorphic vector fields $e, E \in \Gamma(M_\omega^{\mathrm{LG}}, TM_\omega^{\mathrm{LG}})$ as

$$e := \frac{\partial_{w_{\bar{k}}}}{w_0}, \quad E := w_0 \partial_{w_0}. \tag{5.4.12}$$

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Then, the one-parameter group of isomorphisms generated by the horizontal lift $\delta_e^{(\mu)}$ (resp. $\delta_E^{(\mu)}$) to the universal curve acts on the superpotential by translation (resp. conformal transformations) on the superpotential. To see this, note that, by (5.1.11),

$$
\begin{aligned}
w_0 \mapsto a w_0 &\rightsquigarrow \lambda \mapsto a\lambda, \\
w_{\bar{k}} \mapsto w_{\bar{k}} + b/w_0 &\rightsquigarrow \lambda \mapsto \lambda + b.
\end{aligned} \tag{5.4.13}
$$

Since the unit and Euler vector field (3.3.5) of a Hurwitz–Frobenius manifold are characterised as the generators of the affine action (3.3.3) on the superpotential, (5.4.13) identifies $e$ with the unit and $E$ with the Euler vector field of $\mathcal{M}_\omega^{\mathrm{LG}}$. To verify [**DZ-I**] and [**DZ-II**], it then remains to check that the expressions in (5.4.12) coincide with those for the respective vector fields of $\mathcal{M}_\mathcal{R}^{\mathrm{DZ}}$. A simple calculation from Lemma 5.4.1 shows that

$$
e = \frac{1}{w_0} \sum_{i=1}^{l_{\mathcal{R}}+1} \frac{\partial t_i}{\partial w_{\bar{k}}} \partial_{t_i} = \partial_{t_{\bar{k}}}, \tag{5.4.14}
$$

$$
E = w_0 \partial_{w_0} = w_0 \sum_{i=1}^{l_{\mathcal{R}}+1} \frac{\partial t_i}{\partial w_0} \partial_{t_i} = \frac{\partial_{t_{l_{\mathcal{R}}+1}}}{d_{\bar{k}}} + \sum_{i=1}^{l_{\mathcal{R}}} \frac{d_i t_i}{d_{\bar{k}}} \partial_{t_i}, \tag{5.4.15}
$$

thereby matching the expression of the unit and the Euler vector fields in Theorem 4.1.1.

**DZ-III.** Let us consider now the Gram matrix of the intersection pairing on $\mathcal{M}_\omega^{\mathrm{LG}}$ in the $x$-chart. Consider first the argument of the residues in (5.2.11),

$$
\Upsilon_{ij}(p) := \frac{\delta_{\partial_{x_i}}^{(\mu)}\lambda \; \delta_{\partial_{x_j}}^{(\mu)}\lambda}{\lambda \mu^2 \partial_\mu \lambda} \, \mathrm{d}\mu(p), \tag{5.4.16}
$$

so that

$$
\gamma(\partial_{x_i}, \partial_{x_j}) = \sum_l \operatorname*{Res}_{p=p_l^{\mathrm{cr}}} \Upsilon_{ij}(p). \tag{5.4.17}
$$

From (5.4.16), we deduce that the pole structure of $\Upsilon_{ij}(p)$ is as follows.

(i) $\Upsilon_{ij}(p)$ has simple poles at the critical points $\{p_l^{\mathrm{cr}}\}$, for which $\mathrm{d}\lambda(p_i^{\mathrm{cr}}) = 0$;

(ii) $\Upsilon_{ij}(p)$ has at most simple poles at $\lambda(p) = 0$, and only when both $i,j \neq l_\mathcal{R} + 1$. Indeed, from (5.4.11) we have that $\delta_{\partial_{x_{l_{\mathcal{R}}+1}}}^{(\mu)} \lambda = c_\omega \lambda$, thereby cancelling the zeros at the denominator when either $i$ or $j = l_\mathcal{R} + 1$;

(iii) $\Upsilon_{ij}(p)$ has at most simple poles at $\mu(p) = 0$ (for which $\lambda(p) = \infty$, see Remark 5.2.4) and only when $i = j = l_\mathcal{R} + 1$. To see why, notice that locally at a point $p'$ near $\mu(p) = 0$ we have

$$
\lambda(p') = \mathrm{e}^{c_\omega x_{l_{\mathcal{R}}+1}} \mu(p')^{-q_p}(r_p + \mathcal{O}(\mu)), \tag{5.4.18}
$$

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where $q(p) \in \mathbb{Z}_{>0}$ and $r(p) \in \mathbb{C}$. Then, the denominator in (5.4.16) has a leading Puiseux asymptotics in $\mu$ of the form $\mu^{1-2q_p}$, resulting from the combination of the order $q_p$ divergence of $\lambda(\mu)$, the order $q_p + 1$ divergence of $\partial_\mu \lambda(\mu)$, and the double zero of $\mu^2$. For the numerator, we have $\delta_{\partial_{x_i}}^{(\mu)} \lambda \sim \mu^{\delta_{i,l_{\mathcal{R}}+1} - q_p}$, since $r(p)$ is $w$-independent by Proposition 5.2.3, so that

$$\operatorname{ord}_{\mu(p)=0} \Upsilon_{ij} = 1 - \delta_{i,l_{\mathcal{R}}+1} - \delta_{j,l_{\mathcal{R}}+1}; \tag{5.4.19}$$

(iv) Superficially, there may be further poles to be expected at the critical points $\{q_m^{\mathrm{cr}}\}$ of the $\mu$-projection, $\mathrm{d}\mu(q_m^{\mathrm{cr}}) = 0$, that is, where the Ehresmann connection (5.2.2) induced by the $\mu$-foliation is singular and $\delta_{\partial_{x_i}}^{(\mu)} \lambda$ possibly develops a pole. However, these singularities are offset by a vanishing of the same order of $\mathrm{d}\mu/\partial_\mu \lambda$, so that

$$\operatorname{ord}_{q_m^{\mathrm{cr}}} \Upsilon_{ij} = 0. \tag{5.4.20}$$

Based on the above, we turn the contour around and equate the sum of residues at the critical points in (5.2.11) to a much more manageable sum of residues at poles and zeros of $\mu$ and $\lambda$. When $i = j = l_{\mathcal{R}} + 1$, we only have poles at $\mu = 0$, and

$$
\begin{aligned}
\gamma(\partial_{x_{l_{\mathcal{R}}+1}}, \partial_{x_{l_{\mathcal{R}}+1}}) &= \mathcal{N}_\omega \sum_{\mu(p)=0} \operatorname{Res}_{p'=p} \Upsilon_{l_{\mathcal{R}}+1, l_{\mathcal{R}}+1}(p') \\
&= \mathcal{N}_\omega c_\omega^2 \sum_{\mu(p)=0} \operatorname{Res}_{p'=p} \frac{\lambda}{\mu^2 \partial_\mu \lambda} \mathrm{d}\mu(p') \\
&= \mathcal{N}_\omega c_\omega^2 \sum_{\mu(p)=0} \frac{1}{\operatorname{ord}_p \lambda} = -\sum_{i:\mu(\infty_i)=0} \frac{\mathcal{N}_\omega c_\omega^2}{(\mathsf{n}_\omega)_i + 1}. 
\end{aligned}
\tag{5.4.21}
$$

Thus, setting

$$c_\omega = \sqrt{d_{\bar{k}}} \left( -\mathcal{N}_\omega \sum_{i:\mu(\infty_i)=0} \frac{1}{(\mathsf{n}_\omega)_i + 1} \right)^{-1/2}, \tag{5.4.22}$$

gives $\gamma(\partial_{x_{l_{\mathcal{R}}+1}}, \partial_{x_{l_{\mathcal{R}}+1}}) = d_{\bar{k}}$.

Suppose now that $j = l_{\mathcal{R}} + 1$, $i < l_{\mathcal{R}} + 1$. Our analysis above shows that $\Upsilon_{ij}$ is regular outside the critical locus of $\lambda$, and therefore

$$\gamma(\partial_{x_i}, \partial_{x_{l_{\mathcal{R}}+1}}) = \gamma(\partial_{x_{l_{\mathcal{R}}+1}}, \partial_{x_i}) = \delta_{i,l_{\mathcal{R}}+1} d_{\bar{k}}. \tag{5.4.23}$$

Finally, let us look at $i, j < l_{\mathcal{R}} + 1$. In this case, outside $\{p_l^{\mathrm{cr}}\}$, $\Upsilon_{ij}$ has only simple poles at the zeros of $\lambda$, i.e. when

$$
\begin{aligned}
0 = \mathcal{P}_\omega(0, \mu) &= \mathcal{Q}_\omega^{\mathrm{red}}(\mu) = \prod_{0 \neq \omega' \in \Gamma(\rho_\omega)} (\mu - \mathrm{e}^{\omega'(x)}) \\
\iff \quad \mu &= \mathrm{e}^{\omega'(x)}, \quad \omega' \in \Gamma(\rho_\omega),
\end{aligned}
\tag{5.4.24}
$$

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where we've used the shorthand notation $\omega'(x) := \sum_{n=1}^{l_{\mathcal{R}}} \omega'_n x_n$. The evaluation of the residue at $(\lambda, \mu) = (0, e^{\omega'(x)})$ gives

$$
\operatorname*{Res}_{\mu = e^{\omega'(x)}} \Upsilon_{ij} = \left. \frac{\delta_{\partial_{x_i}}^{(\mu)} \lambda \, \delta_{\partial_{x_i}}^{(\mu)} \lambda}{\mu^2 (\partial_\mu \lambda)^2} \right|_{\mu = e^{\omega'(x)}} = \left. \frac{\partial_{x_i} \mathcal{Q}_\omega^{\mathrm{red}} \partial_{x_j} \mathcal{Q}_\omega^{\mathrm{red}}}{\mu^2 (\partial_\mu \mathcal{Q}_\omega^{\mathrm{red}})^2} \right|_{\mu = e^{\omega'(x)}} \tag{5.4.25}
$$

$$
= \frac{e^{2\omega'(x)} \omega'_i \omega'_j \prod_{\omega'', \omega''' \neq \omega'} \left( e^{\omega'(x)} - e^{\omega''(x)} \right) \left( e^{\omega'(x)} - e^{\omega'''(x)} \right)}{e^{2\omega'(x)} \prod_{\omega'', \omega''' \neq \omega'} \left( e^{\omega'(x)} - e^{\omega''(x)} \right) \left( e^{\omega'(x)} - e^{\omega'''(x)} \right)} \tag{5.4.26}
$$

$$
= \omega'_i \omega'_j, \tag{5.4.27}
$$

where the second equality makes use of the implicit function theorem. Summing over $\omega'$ gives

$$
\begin{aligned}
\gamma(\partial_{x_i}, \partial_{x_j}) &= \mathcal{N}_\omega \sum_{\omega' \in \Gamma(\rho_\omega)} \omega'_i \omega'_j = \mathcal{N}_\omega \operatorname{Tr}(\rho_\omega(h_i) \rho_\omega(h_j)) \\
&= \mathcal{N}_\omega C_2(\rho_\omega)(\mathcal{K}_{\mathcal{R}})_{ij} \\
&= \mathcal{N}_\omega \frac{(\omega, \omega + 2\mathsf{w}) \dim_{\mathbb{C}} \rho_\omega}{\dim_{\mathbb{C}} \mathfrak{g}} (\mathcal{K}_{\mathcal{R}})_{ij},
\end{aligned} \tag{5.4.28}
$$

where $\mathsf{w}$ is the Weyl vector.

In (5.4.28) the first equality is immediate. The second reflects the fact that the resulting trace gives a nondegenerate Ad-invariant bilinear form on $\mathfrak{h}_{\mathcal{R}}^\star$, hence proportional to the Killing pairing since $\mathfrak{g}$ is simple, and the final step identifies the proportionality factor with the corresponding quadratic Casimir eigenvalue for the representation $\rho_\omega$ in the appropriate normalisation (see e.g. [62, Lect. 25.1]). Picking $\mathcal{N}_\omega := -\frac{\dim_{\mathbb{C}} \mathfrak{g}}{(\omega, \omega + 2\mathsf{w}) \dim_{\mathbb{C}} \rho_\omega}$ concludes the identification of (5.2.11) with the intersection pairing in Theorem 4.1.1.

**DZ-IV** The final missing piece to invoke the reconstruction theorem, Theorem 4.1.1, is to prove the quasi polynomiality of the prepotential of $\mathcal{M}_\omega^{\mathrm{LG}}$. Consider the change-of-variables[*]

$$
v_i := \begin{cases} \log w_0 & \text{if } i = 0, \\ w_i & \text{otherwise}, \end{cases} \tag{5.4.29}
$$

and define

$$
\Xi_{ijk}(p) := -\mathcal{N}_\omega \frac{\delta_{\partial_{v_i}}^{(\mu)} \lambda \; \delta_{\partial_{v_j}}^{(\mu)} \lambda \; \delta_{\partial_{v_k}}^{(\mu)} \lambda}{\mu^2 \partial_\mu \lambda} \mathrm{d}\mu(p). \tag{5.4.30}
$$

---

[*]This might look like an unnecessary piece of trivial additional notation at this stage and it just amounts to replacing $w_0$-derivatives of the superpotential with logarithmic $w_0$-derivatives. The reader will forgive us as this will help making the discussion of polynomiality completely manifest at the end of the argument. Indeed, while components $c(\partial_{w_i}, \partial_{w_j}, \partial_{w_k})$ of the $c$-tensor in the $w$-chart are Laurent monomials (with coefficients in $\mathbb{Q}[w_1, \ldots, w_{l_{\mathcal{R}}}]$) in $w_0$ with exponent $2 - \delta_{i0} - \delta_{j0} - \delta_{k0}$, and in particular have negative power for $i = j = k = 0$, the components $c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k})$ of the $c$-tensor in the $v$-chart will always be elements of $e^{2v_0} \mathbb{Q}[v_1, \ldots, v_{l_{\mathcal{R}}}]$ (see (5.4.34)).

Then the symmetric $(0,3)$ tensor $(5.2.10)$ in the $v$-chart reads

$$c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k}) = \sum_l \operatorname*{Res}_{p=p_l^{\mathrm{cr}}} \Xi_{ijk}(p). \tag{5.4.31}$$

The same analysis as the one we carried out to verify [**DZ-III**] reveals that $\Xi_{ijk}$ has poles at the critical points (i) and the poles (iii) of $\lambda$. Additionally, it has poles at the critical points (iv) of the $\mu$-projection, due to the singularities of the derivation $\delta_X^{(\mu)}$ in $(5.2.2)$. As noted in $(5.4.20)$, in the case of the intersection pairing, the poles of $\delta_{\partial_{x_i}}^{(\mu)}\lambda$ and $\partial_\mu\lambda$ in $(5.4.16)$ were cancelling out between the numerator and denominator in the Puiseux expansion of $\Upsilon_{ij}(p)$ near $q_m^{\mathrm{cr}}$. On the other hand, the additional factor containing $\delta_{\partial_{w_k}}^{(\mu)}\lambda$ in the numerator of $(5.2.10)$ may give rise to a pole with non-vanishing residue for $\Xi_{ijk}(p)$ at $q_m^{\mathrm{cr}}$.

To compute $(5.4.31)$ we determine individually the residues of $\Xi_{ijk}(p)$ using the known expression of $\mathcal{P}_\omega(\lambda, \mu)$ from $(5.1.11)$. By Proposition $5.2.3$, the $\mu$-coordinate of the poles of $\lambda$ have simple $v$-independent expressions, which are either $\mu = 0$, $\infty$, or a root of unity (see $(5.2.8)$). The calculation of the corresponding residues is straightforward, and we find

$$\operatorname*{Res}_{p=\infty_r} \Xi_{ijk}(p) \in \mathrm{e}^{2v_0}\mathbb{Q}[v_1, \ldots, v_{l_{\mathcal{R}}}], \quad r = 1, \ldots, \ell(\mathsf{n}_\omega). \tag{5.4.32}$$

On the other hand, the $\mu$-coordinates of the critical points $\{p_l^{\mathrm{cr}}\}$ of $\lambda$ (resp. $\{q_m^{\mathrm{cr}}\}$ of $\mu$) are given by the roots of $\operatorname{Discr}_\mu(\mathcal{P}_\omega)(\mu)$ (resp. $\operatorname{Discr}_\lambda(\mathcal{P}_\omega)(\mu)$). When $\deg_\lambda \mathcal{P}_\omega > 1$, these are both high degree polynomials in $\mu$ with $v$-dependent coefficients, and therefore $\mu(p_l^{\mathrm{cr}})$ and $\mu(q_m^{\mathrm{cr}})$ are given by complicated (algebraic) hypergeometric functions of $(\mathrm{e}^{v_0}, v_1, \ldots, v_{l_{\mathcal{R}}+1})$. Since $\Xi_{ijk}$ has in general non-vanishing residues at $p = q_m^{\mathrm{cr}}$, turning the contour around in the sum in $(5.4.31)$ will pick up some intricate hypergeometric contributions from these points, making it difficult to provide a manifest proof of the polynomiality of their sum. One exception however is when $i = 0$, since $\delta_{\partial_{v_0}}^{(\mu)}\lambda = \lambda$. In this case, the same count of the order of divergence as in $(5.4.20)$ shows that

$$\operatorname{ord}_{q_m^{\mathrm{cr}}} \Xi_{0jk} = 0 \implies \operatorname*{Res}_{p=q_m^{\mathrm{cr}}} \Xi_{0jk} = 0. \tag{5.4.33}$$

Here, the residue theorem implies that, closing the contour around the complement of $\{p_l^{\mathrm{cr}}\}$, $(5.4.31)$ equates to a sum of residues coming only from the poles of $\lambda$,

$$c(\partial_{v_0}, \partial_{v_j}, \partial_{v_k}) = -\sum_r \operatorname*{Res}_{p=\infty_r} \Xi_{0jk}(p) \in \mathrm{e}^{2v_0}\mathbb{Q}[v_1, \ldots, v_{l_{\mathcal{R}}}], \tag{5.4.34}$$

where we used $(5.4.32)$. In the same vein, we also obtain

$$\eta(\partial_{v_j}, \partial_{v_k}) = -\sum_r \operatorname*{Res}_{p=\infty_r} \frac{1}{\lambda} \Xi_{0jk}(p) \in \mathrm{e}^{v_0}\mathbb{Q}[v_1, \ldots, v_{l_{\mathcal{R}}}]. \tag{5.4.35}$$

To compute the remaining components $\tilde{c}_{ijk}(v) := c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k})$, we use the associativity of the Frobenius product in the $v$-chart:

$$\sum_{k,l} \left( \tilde{c}_{ijk} \tilde{\eta}^{kl} \tilde{c}_{0lm} - \tilde{c}_{imk} \tilde{\eta}^{kl} \tilde{c}_{0lj} \right) = 0, \qquad (5.4.36)$$

where $\tilde{\eta}^{kl} \in e^{-v_0} \mathbb{Q}(w_1, \ldots, w_{l_{\mathcal{R}}})$ is the inverse of the Gram matrix of (5.4.35). The set of equations (5.4.36) give an a priori overconstrained inhomogeneous linear system for the unknowns $\tilde{c}_{ijk}(v)$ with all $i, j, k \neq 0$. It is not at all obvious, in principle, that a unique solution of (5.4.36) exists; and even so, that such a solution is a polynomial (as opposed to rational) function of $(e^{v_0}, v_1, \ldots, v_{l_{\mathcal{R}}})$. That this is the case, however, can be shown from a direct calculation from (5.4.34)–(5.4.35) (by Mathematica), which gives

$$c(\partial_{v_i}, \partial_{v_j}, \partial_{v_k}) \in e^{2v_0} \mathbb{Q}[v_1, \ldots, v_{l_{\mathcal{R}}}]. \qquad (5.4.37)$$

Using Lemma 5.4.1 together with (5.4.29), we finally deduce that

$$c(\partial_{t_i}, \partial_{t_j}, \partial_{t_k}) = \frac{\partial F_{\mathcal{R}}}{\partial t_i \partial t_j \partial t_k} \in \mathbb{Q}[t_1, \ldots, t_{l_{\mathcal{R}}+1}, e^{t_{l_{\mathcal{R}}+1}}]. \qquad (5.4.38)$$

The claim now follows upon invoking Theorem 4.1.1.

$\square$

**Remark.** The invariants $w_i$ and the basic invariants in [46], $\tilde{y}_j$, do not coincide precisely; they are related by a linear triangular rational transformation. The invariants $\{\tilde{y}_j\}$ are defined by

$$\tilde{y}_j(t) = e^{2\pi i d_j t_{l_{\mathcal{R}}+1}} \sum_{w \in \mathcal{W}} e^{2\pi i (w(\omega_i, t))}, \qquad (5.4.39)$$

and

$$w_i = \sum_j \frac{\mathbf{m}_{\rho_{\omega_i}}(\omega_j)}{|\mathcal{W}_{\omega_j}|} \tilde{y}_j(t) + \mathbf{m}_{\rho_{\omega_i}}(0), \qquad (5.4.40)$$

where $\mathbf{m}_\rho(\omega)$ is the multiplicity of the weight $\omega$ in the weight system, $\Gamma(\rho)$, of the representation $\rho$, and $\mathcal{W}_\omega$ is the Weyl orbit of $\omega$.

**Remark 5.4.2.** The explicit form of (5.1.11) was crucially used in the computation of (5.4.34)–(5.4.35), as well as in determining the flat coordinates in Lemma 5.4.1. Despite the disadvantage of having to carry out a separate analysis of each of the seven Dynkin series, an immediate bonus of this method is that closed-form polynomial expressions for the flat coordinates and, from (5.4.38), the prepotentials are rather powerfully produced from straightforward residue computations. We will illustrate this in detail in the following.

## Examples

The construction of closed-form flat frames for $\eta$ and prepotentials in all Dynkin types does not follow directly from the original Dubrovin–Zhang construction of the flat pencil[†]. One of the advantages of the mirror formulation in Theorem 5.2.5 is that these can now be easily computed using the Landau–Ginzburg formalism and Mathematica.

### Classical root systems

**Example 14** ($A_3$, $k = 1$ - detailed example). From (5.3.2) we have that

$$\lambda = -\frac{w_0}{\mu} \left( \mu^4 - w_3\mu^3 + w_2\mu^2 - w_1\mu + 1 \right), \tag{5.4.41}$$

which has poles at $\mu = 0, \infty$ of order $1, 3$ respectively.

### Step 1: Find flat coordinates.

We want to compute $\eta$-flat coordinates using Theorem 3.3.2.

Note that in general, for $\lambda$ having poles at $0, \infty, a$ of orders $k, m, r$, respectively, we have

$$\log(\mu_0) = -\frac{1}{k}\log(\kappa) + t_0^0 + t_0^1\kappa^{-1/k} + \cdots + t_0^k\kappa^{-1} + \mathcal{O}(\kappa^{-\frac{k+1}{k}}), \tag{5.4.42a}$$

$$\log(\mu_\infty) = \frac{1}{m}\log(\kappa) + t_\infty^1\kappa^{-\frac{1}{m}} + \cdots + t_\infty^{m-1}\kappa^{-\frac{m-1}{m}} + \mathcal{O}(\kappa^{-1}), \tag{5.4.42b}$$

$$\log(\mu_a) = t_a^0 + t_a^1\kappa^{-\frac{1}{r}} + \cdots + t_a^r\kappa^{-1} + \mathcal{O}(\kappa^{-\frac{r+1}{r}}). \tag{5.4.42c}$$

**a)** Since the pole at $\mu = 0$ is of order 1, this will not contribute to part a, due to $\alpha = 0$. Let us therefore focus on the pole at $\mu = \infty$. The series expansion of $\lambda$ near $\mu = \infty$ is of the form

$$\lambda \underset{\mu\to\infty}{\sim} -\mu^3 w_0 + \mu^2 w_0 w_3 - \mu w_0 w_2 \left( -w_0 w_1 + \mathcal{O}\left( \left(\frac{1}{\mu}\right)^0 \right) \right) \equiv \lambda_\infty, \tag{5.4.43}$$

giving

$$\kappa = (\lambda_\infty)^{-\frac{1}{3}} \overset{\text{inverse series}}{\Longrightarrow} \mu_\infty = \frac{1}{\kappa(-w_0)^{\frac{1}{3}}} + \frac{w_3}{3} + \frac{\kappa(-w_0)^{\frac{1}{3}}}{9}(w_3^2 - 3w_2) + \mathcal{O}(\kappa^2). \tag{5.4.44}$$

---

[†]A priori one could try to construct a quasihomogeneous polynomial ansatz for the prepotential, impose the associativity of the Frobenius product, and solve for the coefficients. This typically results in a large system of non-linear equations, owing to the non-linearity of the WDVV equations, whose solution is already unviable e.g. for $\mathcal{R} = E_n$. The proof of Theorem 5.2.5 shows that the Landau–Ginzburg formulas reduce this to an explicit calculation of residues and a relatively small-rank, *linear* problem.

Thus,

$$\log(\mu_\infty) = \log\left(\frac{1}{\kappa(-w_0)^{\frac{1}{3}}}\right) + \frac{\kappa(-w_0)^{\frac{1}{3}}w_3}{3} + \frac{\kappa^2(-w_0)^{\frac{2}{3}}}{18}\left(w_3^2 - 6w_2\right) + \mathcal{O}(\kappa^3). \tag{5.4.45}$$

Removing the divergent part $\log\kappa$ and taking residues gives

$$\alpha = 2: \qquad \left[\kappa^{-1}\right]\left(\frac{3 \text{ p.v. } (\log(\mu_\infty))}{\kappa^2}\right) = (-1)^{\frac{1}{3}}w_0^{\frac{1}{2}}w_3 \equiv (-1)^{\frac{1}{3}}t_1, \tag{5.4.46a}$$

$$\alpha = 1: \qquad \left[\kappa^{-1}\right]\left(\frac{3 \text{ p.v. } (\log(\mu_\infty))}{\kappa^3}\right) = \frac{(-1)^{\frac{2}{3}}}{6}w_0^{\frac{2}{3}}(w_3^2 - 6w_2) \equiv \frac{(-1)^{\frac{2}{3}}}{6}t_2. \tag{5.4.46b}$$

**b)** By the discussion in the proof of Lemma 5.4.2, we need to take the logarithm of the $[\kappa^1]$ coefficient of $\mu_\infty(\kappa)$, if we choose $\infty_0 = 0$. From (5.4.44), we see that (up to proportionality) this is

$$t_0 = \frac{4}{3}\log w_0, \tag{5.4.47}$$

where the constant factor has been chosen for later convenience.

**c)** The final coordinate (as can easily be read off from (5.4.43) as the constant part in $\mu$) is given by

$$-w_0 w_1 \equiv -t_3. \tag{5.4.48}$$

**Remark.** Note that while the pole at $\mu = 0$ does contribute to part c, its contribution is simply negative the contribution from $\mu = \infty$ (due to the fact that residues of all poles of a meromorphic differential form on a compact Riemann surface sum to zero). In fact, as stated in Lemma 5.4.1, we always get a single coordinate from this. In particular, we have that the residue at any pole other than $0, \infty$ vanishes.

This gives us the set of flat coordinates

$$t_0 = \frac{4}{3}\log w_0, \quad t_1 = w_0^{\frac{1}{2}}w_3, \quad t_2 = w_0^{\frac{2}{3}}(w_3^2 - 6w_2), \quad t_3 = w_0 w_1, \tag{5.4.49}$$

with the inverse (mirror) map[‡]

$$w(t) = \{w_0 = e^{\frac{3t_0}{4}}, w_1 = e^{-\frac{3t_0}{4}}t_3, w_2 = \frac{e^{-\frac{t_0}{2}}}{6}(t_1^2 - t_2), w_3 = e^{-\frac{t_0}{4}}t_1\}. \tag{5.4.50}$$

We can immediately see that we have the correct degrees (and indeed Euler vector field) by comparing (5.4.15) and with the Coxeter exponents of the type $A_3$ Weyl group (or the known values for the degrees as in [46]).

---

[‡]Here we have, without loss of generality, made a choice of branch.

**Step 2: Derive $\eta$ and $\tilde{c}_{0ab}$.**

Now, writing $\lambda$ in flat coordinates gives

$$\lambda = \left(\mu^3 + \frac{1}{\mu^3}\right)e^{\frac{3t_0}{4}} - \mu^2 t_1 e^{\frac{t_0}{2}} + \frac{\mu}{6}e^{\frac{t_0}{4}}\left(t_1^2 - t_2\right) - t_3, \tag{5.4.51}$$

and by taking residues as usual (with $e = \partial_3$) we obtain

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{18} & 0 \\ 0 & -\frac{1}{18} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{5.4.52}$$

Following the discussion in the proof of Theorem 5.2.5, we also need $\tilde{c}_{0ab}(w)$, which in this case are given by

$$\tilde{c}_{000} = \frac{2(-4w_3^6 - 135w_2^2 w_3^2 + 162w_3^2 + 108w_2^3 + 4374w_1^2 + 81w_1(6w_2w_3 - w_3^3) + 6w_2(7w_3^4 + 1080))}{2187 w_0},$$

$$\tilde{c}_{001} = \frac{2}{81}(-w_3^3 + 6w_2w_3 + 108w_1), \quad \tilde{c}_{002} = \frac{1}{243}(4w_3^4 - 27w_2w_3^2 + 54w_1w_3 + 36w_2^2 + 432),$$

$$\tilde{c}_{003} = \frac{4}{729}(27w_1(3w_2 - w_3^2) + w_3(-2w_3^4 + 15w_2w_3^2 - 27w_2^2 + 54)), \quad \tilde{c}_{011} = \frac{4w_0}{3}, \quad \tilde{c}_{012} = \frac{w_0 w_3}{9},$$

$$\tilde{c}_{013} = \frac{2}{27}(w_0(3w_2 - w_3^2)) = \tilde{c}_{022}, \quad c_{023} = \frac{w_0}{81}(4w_3^3 - 15w_2w_3 + 27w_1),$$

$$\tilde{c}_{033} = -\frac{2w_0}{243}(4w_3^4 - 18w_2w_3^2 + 27w_1w_3 + 9w_2^2 - 54).$$

**Step 3: Maximal rank.**

We consider the WDVV equations in $w$-coordinates;

$$\tilde{c}_{abc}\tilde{\eta}^{cd}\tilde{c}_{def} = b \longleftrightarrow e, \tag{5.4.54}$$

as a linear system of equations in the unknowns $c_{abc}$ with $a, b, c \neq 0$.

By choosing generic coordinates for $w$ it is shown that there exists a unique solution. For instance, by choosing $(w_0, w_1, w_2, w_3) = (5, 13, 23, 37)$ we obtain

$$c_{111} = c_{112} = c_{113} = c_{122} = 0, \quad c_{123} = c_{222} = \frac{25}{3}, \quad c_{133} = c_{223} = -\frac{1850}{9},$$

$$c_{233} = \frac{135175}{27}, \quad c_{333} = -\frac{9875300}{81}.$$

**Step 4: Making an ansatz for $F$ to solve for its coefficients.**

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We know from Theorem 4.1.1 that the prepotential must be a polynomial in $t_1, t_2, t_3, e^{t_0}, t_0$. Using this fact together with (2.1.4), we can restrict the prepotential to have a maximum of 8 terms. Letting $\gamma_{ijklm}$ denote the coefficient of the term of form $t_1^i t_2^j t_3^k e^{lt_0} t_0^m$ we have the following (potentially) non-zero coefficients:

$$\gamma_{20001}, \ \gamma_{11100}, \ \gamma_{03000}, \ \gamma_{02200}, \ \gamma_{01400}, \ \gamma_{01010}, \ \gamma_{00600}, \ \gamma_{00210}. \tag{5.4.56}$$

Differentiating $F_{\text{ansatz}} \equiv \sum \gamma_{ijklm} t_1^i t_2^j t_3^k e^{lt_0} t_0^m$, using the mirror map (5.4.50) to find the c-tensor in curved $w$-coordinates ($\tilde{c}$), equating the relevant ones with (5.4.53) and solving WDVV with the coefficients (5.4.56) as unknown, we obtain a unique solution (up to equivalence) for the prepotenial in which the coefficients are given by

$$\gamma_{20001} = \frac{1}{2}, \quad \gamma_{11100} = -\frac{1}{18}, \quad \gamma_{03000} = \gamma_{01400} = -\frac{1}{3888}, \quad \gamma_{02200} = -\frac{1}{1296},$$

$$\gamma_{01010} = -\gamma_{00210} = -\frac{1}{6}, \quad \gamma_{00600} = -\frac{1}{19440}.$$

It can be easily verified that the resulting prepotential is indeed a solution to the WDVV system of equations, making it the correct one by the reconstruction theorem, Theorem 4.1.1. This can also be verified by comparison with the expression in [46, Example 2.5]. Note that in this case all the coefficients in the ansatz turned out to be non-zero. This is the case in general.

The remaining examples are obtained in the exact same way, apart from the additional need for taking Puiseux expansions near $\lambda$-poles for curves of non-zero genus. We therefore omit the details and present flat coordinates and prepotentials only.

**Example 15** ($\mathcal{R} = B_3$)**.** From (5.3.6) we have superpotential

$$\lambda_{B_3} = \frac{w_0(\mu^6 + 1 - (\mu^5 + \mu)(w_1 - 1) + (\mu^4 + \mu^2)(-w_1 + w_2 + 1) + \mu^3(-w_3^2 + 2w_2 + 2))}{\mu^2(\mu + 1)^2}, \tag{5.4.58}$$

which gives flat coordinates

$$t_4 = \frac{\log w_0}{2}, \quad t_1 = w_0^{\frac{1}{2}}(w_1 + 1), \quad t_2 = w_0^{\frac{1}{2}} w_3, \quad t_3 = w_0(w_1 + w_2 + 2). \tag{5.4.59}$$

We obtain the prepotential for $B_3$ to be

$$F_{B_3} = t_4 t_3^2 + \frac{1}{2} t_1^2 t_3 + t_2^2 t_3 - \frac{1}{48} t_1^4 - \frac{1}{24} t_2^4 + 2t_1 t_2^2 e^{t_4} + t_1^2 e^{2t_4} + 2t_2^2 e^{2t_4} + \frac{1}{2} e^{4t_4}, \tag{5.4.60}$$

which is seen to be equivalent to the free energy in Example 2.7 in [46] by $F \mapsto \frac{F}{2}$ and $t_2 \leftrightarrow t_3$.

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**Example 16** $(\mathcal{R} = B_4)$. For $B_4$, (5.3.6) is

$$\lambda_{B_4} = \frac{w_0}{\mu^3(\mu+1)^2}\Big(\mu^8 + 1 - (\mu^7 + \mu)(w_1 - 1) + (\mu^6 + \mu^2)(-w_1 + w_2 + 1)$$
$$- (\mu^5 + \mu^3)(w_1 - w_2 + w_3 - 1) + \mu^4(w_4^2 - 2w_1 - 2w_3)\Big), \tag{5.4.61}$$

which gives flat coordinates

$$t_5 = \frac{\log w_0}{3}, \quad t_1 = w_0^{\frac{1}{3}}(w_1 + 1), \quad t_2 = w_0^{\frac{1}{2}}w_4, \quad t_3 = w_0^{\frac{2}{3}}(4w_1 - w_1^2 + 6w_2 + 11),$$

$$t_4 = w_0(2w_1 + w_2 + w_3 + 2).$$

In this case, the resulting prepotential is given by

$$F_{B_4} = \frac{1}{1944}t_1^4 t_3 - \frac{1}{9720}t_1^6 - \frac{1}{648}t_1^2 t_3^2 + \frac{1}{9}t_1 t_3 t_4 - \frac{1}{24}t_2^4 + t_2^2 t_4 + \frac{1}{1944}t_3^3 + t_5 t_4^2 + \frac{1}{3}t_1^2 t_2^2 e^{t_5}$$
$$+ \frac{1}{3}t_2^2 t_3 e^{t_5} + \frac{1}{36}t_1^4 e^{2t_5} + \frac{1}{18}t_1^2 t_3 e^{2t_5} + 2t_1 t_2^2 e^{2t_5} + \frac{1}{36}t_3^2 e^{2t_5} + 2t_2^2 e^{3t_5} + \frac{1}{2}t_1^2 e^{4t_5} + \frac{1}{3}e^{6t_5}. \tag{5.4.63}$$

**Example 17** $(\mathcal{R} = C_3)$. Here, the superpotential is given by

$$\lambda_{C_3} = \frac{w_0(\mu^6 + 1 - (\mu^5 + \mu)w_1 + (\mu^4 + \mu^2)(w_2 + 1) - \mu^3(w_1 + w_3))}{\mu^3}, \tag{5.4.64}$$

which leads to the following flat coordinates:

$$t_4 = \frac{\log w_0}{3}, \quad t_1 = w_0^{\frac{1}{3}}w_1, \quad t_2 = w_0^{\frac{2}{3}}(w_1^2 - 6(w_2 + 1)), \quad t_3 = w_0(w_1 + w_3). \tag{5.4.65}$$

In this case, the prepotential is

$$F_{C_3} = t_4 t_3^2 - \frac{1}{9}t_1 t_2 t_3 - \frac{1}{9720}t_1^6 - \frac{1}{1944}t_1^4 t_2 - \frac{1}{648}t_1^2 t_2^2 - \frac{1}{1944}t_2^3 + \frac{1}{36}t_1^4 e^{2t_4} - \frac{1}{18}t_1^2 t_2 e^{2t_4}$$
$$+ \frac{1}{36}t_2^2 e^{2t_4} + \frac{1}{2}t_1^2 e^{4t_4} + \frac{1}{3}e^{6t_4}, \tag{5.4.66}$$

which is the same as the one found in Example 2.8 in [46] after letting $F \mapsto \frac{F}{2}$, and $t_2 \mapsto -6t_2$.

**Example 18** $(\mathcal{R} = D_4)$. For $l = 4$, (5.3.14) becomes

$$\lambda_{D_4} = \frac{w_0(\mu^8 - w_1(\mu^7 + \mu) + w_2(\mu^6 + \mu^2) + (w_1 - w_3 w_4)(\mu^5 + \mu^3) + 1)}{\mu^2(\mu^2 - 1)^2}, \tag{5.4.67}$$

which has poles at $\mu = 0, \infty, 1, -1$, all of order 2. The resulting flat coordinates are

$$t_5 = \frac{\log w_0}{2}, \quad t_1 = w_0^{\frac{1}{2}}w_1, \quad t_2 = w_0^{\frac{1}{2}}(w_3 + w_4), \quad t_3 = w_0^{\frac{1}{2}}(w_3 - w_4), \quad t_4 = w_0(w_2 + 2), \tag{5.4.68}$$

which leads to the prepotential

$$F_{D_4} = t_5 t_4^2 + \frac{1}{4} t_2^2 t_4 + \frac{1}{4} t_3^2 t_4 - \frac{1}{48} t_1^4 + \frac{1}{2} t_1^2 t_4 - \frac{1}{384} t_2^4 - \frac{1}{64} t_2^2 t_3^2 - \frac{1}{384} t_3^4 + \frac{1}{2} t_1 t_2^2 \mathrm{e}^{t_5} - \frac{1}{2} t_1 t_3^2 \mathrm{e}^{t_5} + t_1^2 \mathrm{e}^{2t_5}$$
$$+ \frac{1}{2} t_2^2 \mathrm{e}^{2t_5} + \frac{1}{2} t_3^2 \mathrm{e}^{2t_5} + \frac{1}{2} \mathrm{e}^{4t_5}.$$

$$(5.4.69)$$

**Example 19** ($\mathcal{R} = D_5$). For $l = 5$, (5.3.14) is given by

$$\lambda_{D_5} = \frac{w_0}{\mu^3(\mu^2 - 1)^2} \Big( \mu^{10} - w_1(\mu^9 + \mu) + w_2(\mu^8 + \mu^2) - w_3(\mu^7 + \mu^3) + (w_2 - w_4 w_5 + 1)(\mu^6 + \mu^4)$$
$$- (w_4^2 + w_5^2 - 2w_1 - 2w_3)\mu^5 + 1 \Big),$$

$$(5.4.70)$$

which has poles at $\mu = 0, \infty, 1, -1$, of orders $3, 3, 2, 2$, respectively. We can compute the flat coordinates

$$t_6 = \frac{\log w_0}{3}, \quad t_1 = w_0^{\frac{1}{3}} w_1, \quad t_2 = w_0^{\frac{1}{2}}(w_4 - w_5), \quad t_3 = w_0^{\frac{1}{2}}(w_4 + w_5),$$
$$t_4 = w_0^{\frac{2}{3}}(w_1^2 - 6(w_2 + 2)), \quad t_5 = w_0(2w_1 + w_3), \quad\quad (5.4.71)$$

which give the prepotential

$$F_{D_5} = t_6 t_5^2 + \frac{1}{4} t_2^2 t_5 + \frac{1}{4} t_3^2 t_5 - \frac{1}{9720} t_1^6 - \frac{1}{1944} t_1^4 t_4 - \frac{1}{648} t_1^2 t_4^2 - \frac{1}{9} t_1 t_4 t_5 - \frac{1}{384} t_2^4 - \frac{1}{64} t_2^2 t_3^2 - \frac{1}{384} t_3^4$$
$$- \frac{1}{1944} t_4^3 - \frac{1}{12} t_1^2 t_2^2 \mathrm{e}^{t_6} + \frac{1}{12} t_1^2 t_3^2 \mathrm{e}^{t_6} + \frac{1}{12} t_2^2 t_4 \mathrm{e}^{t_6} - \frac{1}{12} t_3^2 t_4 \mathrm{e}^{t_6} + \frac{1}{36} t_1^4 \mathrm{e}^{2t_6} - \frac{1}{18} t_1^2 t_4 \mathrm{e}^{2t_6}$$
$$+ \frac{1}{2} t_1 t_2^2 \mathrm{e}^{2t_6} + \frac{1}{2} t_1 t_3^2 \mathrm{e}^{2t_6} + \frac{1}{36} t_4^2 \mathrm{e}^{2t_6} - \frac{1}{2} t_2^2 \mathrm{e}^{3t_6} + \frac{1}{2} t_3^2 \mathrm{e}^{3t_6} + \frac{1}{2} t_1^2 \mathrm{e}^{4t_6} + \frac{1}{3} \mathrm{e}^{6t_6}.$$

$$(5.4.72)$$

**Exceptional root systems**

**Example 20** ($\mathcal{R} = E_6$). For the exceptional cases, we find expressions for $\lambda$ near any ramification point by Puiseux expansions. By doing so, and calculating (5.4.1a)–(5.4.1c), we obtain the following flat coordinates for $\mathcal{R} = E_6$

$$t_7 = \frac{\log(w_0)}{6}, \quad t_1 = w_0^{\frac{1}{3}} w_1, \quad t_2 = w_0^{\frac{1}{3}} w_5, \quad t_3 = w_0^{\frac{1}{2}}(w_6 + 2), \quad t_4 = w_0^{\frac{2}{3}}(-w_5^2 + 12w_1 + 6w_4),$$
$$t_5 = w_0^{\frac{2}{3}}(w_1^2 - 6w_2 - 12w_5), \quad t_6 = w_0(2w_1 w_5 + w_3 + 3w_6 + 3). \quad\quad (5.4.73)$$

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In this case, we find that the prepotential is given by

$$
\begin{aligned}
F_{E_6} = & -\frac{1}{19440}t_1^6 + \frac{1}{72}e^{2t_7}t_2t_1^4 - \frac{1}{3888}t_5t_1^4 + \frac{1}{6}e^{6t_7}t_1^3 + \frac{1}{6}e^{3t_7}t_3t_1^3 + \frac{5}{18}e^{4t_7}t_2^2t_1^2 - \frac{1}{1296}t_5^2t_1^2 \\
& + \frac{1}{36}e^{t_7}t_2^2t_3t_1^2 + \frac{1}{36}e^{4t_7}t_4t_1^2 + \frac{1}{36}e^{t_7}t_3t_4t_1^2 - \frac{1}{36}e^{2t_7}t_2t_5t_1^2 + \frac{1}{72}e^{2t_7}t_2^4t_1 + \frac{1}{2}e^{2t_7}t_2t_3^2t_1 \\
& + \frac{1}{72}e^{2t_7}t_4^2t_1 + \frac{1}{2}e^{8t_7}t_2t_1 + e^{5t_7}t_2t_3t_1 + \frac{1}{36}e^{2t_7}t_2^2t_4t_1 - \frac{1}{6}e^{3t_7}t_3t_5t_1 - \frac{1}{18}t_5t_6t_1 + \frac{1}{12}e^{12t_7} \\
& - \frac{1}{19440}t_2^6 - \frac{1}{96}t_3^4 + \frac{1}{6}e^{6t_7}t_2^3 + \frac{1}{3}e^{3t_7}t_3^3 + \frac{1}{3888}t_4^3 - \frac{1}{3888}t_5^3 + \frac{1}{2}e^{6t_7}t_3^2 - \frac{1}{1296}t_2^2t_4^2 \\
& + \frac{1}{72}e^{2t_7}t_2t_5^2 + \frac{1}{2}t_7t_6^2 + \frac{1}{6}e^{3t_7}t_2^3t_3 + \frac{1}{3888}t_2^4t_4 + \frac{1}{6}e^{3t_7}t_2t_3t_4 - \frac{1}{36}e^{4t_7}t_2^2t_5 - \frac{1}{36}e^{t_7}t_2^2t_3t_5 \\
& - \frac{1}{36}e^{4t_7}t_4t_5 - \frac{1}{36}e^{t_7}t_3t_4t_5 + \frac{1}{4}t_3^2t_6 + \frac{1}{18}t_2t_4t_6.
\end{aligned}
$$

$$\text{(5.4.74)}$$

**Example 21** ($\mathcal{R} = E_7$). Similar to the $E_6$-case, by taking Puiseux expansions of the spectral curve, we obtain the following flat coordinates

$$
t_8 = \frac{\log(w_0)}{12}, \quad t_1 = w_0^{\frac{1}{4}}w_6, \quad t_2 = w_0^{\frac{1}{3}}(w_1 + 2), \quad t_3 = w_0^{\frac{1}{2}}(2w_6 + w_7),
$$

$$
t_4 = w_0^{\frac{1}{2}}(-w_6^2 + 8w_1 + 4w_5 + 12), \quad t_5 = w_0^{\frac{2}{3}}(-w_1^2 + 26w_1 + 6w_2 + 12w_5 + 26),
$$

$$
t_6 = w_0^{\frac{3}{4}}(5w_6^3 + 24(6w_1 - w_5 + 21)w_6 + 96w_4 + 288w_7),
$$

$$
t_7 = w_0(3w_1^2 + 2w_1(w_5 + 8) + 3w_6^2 + 3w_6w_7 + 2w_2 + w_3 + 7w_5 + 14). \qquad \text{(5.4.75)}
$$

The prepotential for $E_7$ takes the form

$$
\begin{aligned}
F_{E_7} = & -\frac{1}{4128768}t_1^8 + \frac{1}{18432}49e^{6t_8}t_1^6 + \frac{1}{18432}e^{2t_8}t_2t_1^6 + \frac{1}{294912}t_4t_1^6 + \frac{1}{384}e^{3t_8}t_3t_1^5 - \frac{1}{2949120}t_6t_1^5 \\
& + \frac{19}{192}e^{12t_8}t_1^4 + \frac{5}{288}e^{4t_8}t_2^2t_1^4 - \frac{1}{49152}t_4^2t_1^4 + \frac{1}{8}e^{8t_8}t_2t_1^4 + \frac{1}{1536}13e^{6t_8}t_4t_1^4 + \frac{1}{1536}e^{2t_8}t_2t_4t_1^4 \\
& + \frac{1}{576}e^{4t_8}t_5t_1^4 + \frac{1}{4}e^{9t_8}t_3t_1^3 + \frac{1}{576}e^{t_8}t_2^2t_3t_1^3 + \frac{25}{96}e^{5t_8}t_2t_3t_1^3 + \frac{7}{384}e^{3t_8}t_3t_4t_1^3 + \frac{1}{576}e^{t_8}t_3t_5t_1^3 \\
& + \frac{1}{9216}e^{6t_8}t_6t_1^3 + \frac{1}{9216}e^{2t_8}t_2t_6t_1^3 + \frac{1}{147456}t_4t_6t_1^3 + \frac{1}{6}e^{18t_8}t_1^2 + \frac{1}{288}e^{2t_8}t_2^4t_1^2 + \frac{5}{24}e^{6t_8}t_2^3t_1^2 \\
& + \frac{1}{24576}t_4^3t_1^2 + \frac{5}{8}e^{10t_8}t_2^2t_1^2 + \frac{5}{8}e^{6t_8}t_3^2t_1^2 + \frac{1}{8}e^{2t_8}t_2t_3^2t_1^2 + \frac{5}{512}e^{6t_8}t_4^2t_1^2 + \frac{1}{512}e^{2t_8}t_2t_4^2t_1^2 \\
& + \frac{1}{288}e^{2t_8}t_5^2t_1^2 - \frac{1}{589824}t_6^2t_1^2 + \frac{1}{2}e^{14t_8}t_2t_1^2 + \frac{1}{32}e^{12t_8}t_4t_1^2 + \frac{1}{24}e^{4t_8}t_2^2t_4t_1^2 + \frac{1}{8}e^{8t_8}t_2t_4t_1^2 \\
& + \frac{1}{144}e^{2t_8}t_2^2t_5t_1^2 + \frac{1}{24}e^{6t_8}t_2t_5t_1^2 + \frac{1}{96}e^{4t_8}t_4t_5t_1^2 + \frac{1}{384}e^{3t_8}t_3t_6t_1^2 + \frac{1}{3}e^{3t_8}t_3^3t_1 + \frac{1}{64}e^{3t_8}t_3t_4^2t_1 \\
& + \frac{1}{6}e^{3t_8}t_2^3t_3t_1 + \frac{7}{6}e^{7t_8}t_2^2t_3t_1 + e^{11t_8}t_2t_3t_1 + \frac{1}{4}e^{9t_8}t_3t_4t_1 + \frac{1}{96}e^{t_8}t_2^2t_3t_4t_1 + \frac{5}{16}e^{5t_8}t_2t_3t_4t_1 \\
& + \frac{1}{6}e^{7t_8}t_3t_5t_1 + \frac{1}{6}e^{3t_8}t_2t_3t_5t_1 + \frac{1}{96}e^{t_8}t_3t_4t_5t_1 + \frac{1}{576}e^{4t_8}t_2^2t_6t_1 - \frac{1}{49152}t_4^2t_6t_1 + \frac{1}{1536}e^{6t_8}t_4t_6t_1 \\
& + \frac{1}{1536}e^{2t_8}t_2t_4t_6t_1 + \frac{1}{576}e^{4t_8}t_5t_6t_1 + \frac{1}{384}t_6t_7t_1 + \frac{1}{24}e^{24t_8} - \frac{1}{19440}t_2^6 + \frac{1}{72}e^{4t_8}t_2^5 + \frac{5}{36}e^{8t_8}t_2^4 \\
& - \frac{1}{96}t_3^4 - \frac{1}{49152}t_4^4 + \frac{1}{6}e^{12t_8}t_2^3 + \frac{1}{384}e^{6t_8}t_4^3 + \frac{1}{3888}t_5^3 + \frac{1}{4}e^{16t_8}t_2^2 + \frac{1}{2}e^{12t_8}t_3^2 + \frac{2}{3}e^{4t_8}t_2^2t_3^2 \\
& + e^{8t_8}t_2t_3^2 + \frac{1}{64}e^{12t_8}t_4^2 + \frac{1}{64}e^{4t_8}t_2^2t_4^2 + \frac{1}{72}e^{8t_8}t_5^2 - \frac{1}{1296}t_2^2t_5^2 + \frac{1}{72}e^{4t_8}t_2t_5^2 + \frac{1}{288}e^{2t_8}t_4t_5^2 \\
& + \frac{1}{18432}e^{6t_8}t_6^2 + \frac{1}{18432}e^{2t_8}t_2t_6^2 + \frac{1}{294912}t_4t_6^2 + \frac{1}{2}t_8t_7^2 + \frac{1}{288}e^{2t_8}t_2^4t_4 + \frac{1}{24}e^{6t_8}t_2^3t_4 \\
& + \frac{1}{8}e^{10t_8}t_2^2t_4 + \frac{1}{8}e^{6t_8}t_3^2t_4 + \frac{1}{8}e^{2t_8}t_2t_3^2t_4 + \frac{1}{3888}t_2^4t_5 + \frac{1}{36}e^{4t_8}t_2^3t_5 + \frac{1}{36}e^{8t_8}t_2^2t_5 + \frac{1}{6}e^{4t_8}t_3^2t_5 \\
& + \frac{1}{144}e^{2t_8}t_2^2t_4t_5 + \frac{1}{24}e^{6t_8}t_2t_4t_5 + \frac{1}{576}e^{t_8}t_2^2t_3t_6 + \frac{1}{96}e^{5t_8}t_2t_3t_6 + \frac{1}{384}e^{3t_8}t_3t_4t_6 + \frac{1}{576}e^{t_8}t_3t_5t_6 \\
& + \frac{1}{4}t_3^2t_7 + \frac{1}{128}t_4^2t_7 + \frac{1}{18}t_2t_5t_7.
\end{aligned}
$$

$$(5.4.76)$$

**Example 22** ($\mathcal{R} = F_4$). In this case we obtain flat coordinates

$$
t_5 = \frac{1}{6}\log(w_0), \quad t_1 = w_0^{\frac{1}{3}}(w_4 + 1), \quad t_2 = w_0^{\frac{1}{2}}(w_4 + w_1 + 2),
$$

$$
t_3 = w_0^{\frac{2}{3}}(-w_4^2 + 16w_4 + 6w_3 + 6w_1 + 11), \quad t_4 = w_0(2w_4^2 + 6w_4 + w_4w_1 + w_3 + w_2 + 4w_1 + 5).
$$

$$(5.4.77)$$

The resulting prepotential is

$$
\begin{aligned}
F_{F_4} = & -\frac{1}{9720}t_1^6 + \frac{1}{36}e^{2t_5}t_1^5 + \frac{5}{18}e^{4t_5}t_1^4 + \frac{1}{36}e^{t_5}t_2t_1^4 + \frac{1}{432}t_3t_1^4 + \frac{1}{3}e^{6t_5}t_1^3 + \frac{1}{3}e^{3t_5}t_2t_1^3 \\
& + \frac{1}{4}e^{2t_5}t_3t_1^3 + \frac{1}{2}e^{8t_5}t_1^2 + \frac{1}{2}e^{2t_5}t_2^2t_1^2 - \frac{1}{32}t_3^2t_1^2 + e^{5t_5}t_2t_1^2 + \frac{1}{4}e^{4t_5}t_3t_1^2 + \frac{1}{4}e^{t_5}t_2t_3t_1^2 \\
& + \frac{9}{16}e^{2t_5}t_3^2t_1 + \frac{3}{2}e^{3t_5}t_2t_3t_1 + \frac{1}{2}t_3t_4t_1 + \frac{1}{12}e^{12t_5} - \frac{1}{96}t_2^4 + \frac{1}{3}e^{3t_5}t_2^3 + \frac{3t_3^3}{64} + \frac{1}{2}e^{6t_5}t_2^2 \\
& + \frac{9}{16}e^{4t_5}t_3^2 + \frac{9}{16}e^{t_5}t_2t_3^2 + \frac{1}{2}t_5t_4^2 + \frac{1}{4}t_2^2t_4.
\end{aligned}
\tag{5.4.78}
$$

**Example 23** ($\mathcal{R} = G_2$). For $G_2$, we obtain flat coordinates

$$
t_3 = \frac{\log(w_0)}{6}, \quad t_1 = w_0^{\frac{1}{2}}(w_1 + 1), \quad t_2 = w_0(2w_1 + w_2 + 2).
\tag{5.4.79}
$$

In this case, the prepotential takes the form

$$
F_{G_2} = \frac{1}{2}t_2^2t_3 + \frac{1}{4}t_2t_1^2 - \frac{1}{96}t_1^4 + \frac{1}{3}t_1^3e^{3t_3} + \frac{1}{2}t_1^2e^{6t_3} + \frac{1}{12}e^{12t_3},
\tag{5.4.80}
$$

which matches exactly the expression found in [46] Example 2.4.

## Non-minimal irreducible representations

It is argued in [17], based on the isomorphism of Toda flows on Prym–Tyurin varieties associated to different representations, [99, 100], that the Frobenius manifold obtained from the construction is independent of the choice of highest weight $\omega$.

Let us verify this explicitly for $\mathcal{R} = G_2$. In this case picking $\omega = \omega_2$ gives the nontrivial irreducible representation of $G_2$ of smallest dimension after the fundamental, which is the 14-dimensional adjoint representation $\rho_{\omega_2} = \mathfrak{g}_2$. By the same method of the previous section we obtain

$$
\begin{aligned}
\mathfrak{p}_0^{[01]_{G_2}} &= 1, \\
\mathfrak{p}_1^{[01]_{G_2}} &= \chi_2, \\
\mathfrak{p}_2^{[01]_{G_2}} &= \chi_1^3 - \chi_1^2 - \chi_1(2\chi_2 + 1), \\
\mathfrak{p}_3^{[01]_{G_2}} &= \chi_1^4 - \chi_1^3 - \chi_1^2(3\chi_2 + 1) + \chi_1 + 2\chi_2^2 + \chi_2, \\
\mathfrak{p}_4^{[01]_{G_2}} &= \chi_1^3(\chi_2 - 1) - \chi_1^2(\chi_2 - 1) - \chi_1(2\chi_2^2 - \chi_2 - 1) - \chi_2^2 + \chi_2, \\
\mathfrak{p}_5^{[01]_{G_2}} &= \chi_1^5 - 2\chi_1^4 - 5\chi_1^3\chi_2 + \chi_1^2(3\chi_2 + 2) + \chi_1(6\chi_2^2 + 5\chi_2 - 1) + \chi_2^3 + 2\chi_2^2, \\
\mathfrak{p}_6^{[01]_{G_2}} &= \chi_1^4 - 3\chi_1^3\chi_2 + \chi_1^2(\chi_2^2 - \chi_2 - 2) + \chi_1\chi_2(4\chi_2 + 3) + 2\chi_2^2 + \chi_2, \\
\mathfrak{p}_7^{[01]_{G_2}} &= 4\chi_1^4 + 2\chi_1^3(3\chi_2 + 1) + 2\chi_1^2(\chi_2^2 - 2\chi_2 - 3) \\
& \quad - 2\chi_1\chi_2(4\chi_2 + 3) - 2\chi_2^3 - 6\chi_2^2 + 2 - 2\chi_1^5,
\end{aligned}
\tag{5.4.81}
$$

and $\mathfrak{p}_i^{[01]_{G_2}} = \mathfrak{p}_{14-i}^{[01]_{G_2}}$ by reality of the adjoint representation. The characteristic polynomial (5.1.8) then factorises as

$$\mathcal{P}_{[01]_{G_2}}^{\text{red}} = (\mu - 1)^2 \mathcal{P}_{G_2, \text{ short}} \mathcal{P}_{G_2, \text{ long}}, \tag{5.4.82}$$

with the three factors corresponding to the three irreducible Weyl orbits of the adjoint representation associated to the zero, short, and long roots of $G_2$:

$$
\begin{aligned}
\mathcal{P}_{G_2, \text{ short}} &= \mathcal{P}_{[10]_{G_2}}^{\text{red}}, \\
\mathcal{P}_{G_2, \text{ long}} &= \mu^6 + \mu^5 (w_1 - w_2 + 1) + \mu^4 \left(w_1^3 - 3w_2 w_1 - w_1 - 2w_2 + 1\right) \\
&+ \mu^3 \left(2w_1^3 - w_1^2 - 4w_2 w_1 - 2w_1 - w_2^2 - 4w_2 + 1\right) \\
&+ \mu^2 \left(w_1^3 - 3w_2 w_1 - w_1 - 2w_2 + 1\right) + \mu (w_1 - w_2 + 1) + 1.
\end{aligned}
\tag{5.4.83}
$$

For any $w$, the curve $\overline{C}_w^{(1)} = \overline{\{\mathcal{P}_{G_2, \text{ long}} = 0\}}$ is a $\mathbb{P}^1$, and the $\lambda$-projection has ramification profile

$$\left( \overbrace{1,2}^{\mu=0}, \overbrace{1,2}^{\mu=\infty} \right). \tag{5.4.84}$$

The embedding $\iota_{[01]_{G_2}} : \mathcal{M}_{[01]_{G_2}}^{\text{LG}} \hookrightarrow \mathcal{H}_{0;(0,1,0,1)}$ gives the same flat coordinates (5.4.79) as for the case $\omega = [10]_{G_2}$, and up to scaling the prepotential coincides with the prepotential (5.4.80), as expected.

## 5.5    Open problem: non-canonical DZ-Manifolds

In [17], it was proposed that the family of Frobenius algebras obtained in (5.1.11) through the shift of $w_j \to w_j + \delta_{ij} \frac{\lambda}{w_0}$ for any $i$ should give the Frobenius structure corresponding to Dynkin node $\alpha_i$. Theorem 5.2.5 shows that this is the case for $i = \bar{k}$, and this proposal is consistent with the analysis of the generalised type-A mirrors of [46]. It turns out, however, that the conjecture is false in this form at the stated level of generality. Let us consider the case $\mathcal{R} = G_2$ where $\bar{k} = 2$. We see that shifting $w_1$ instead of $w_2$ in (5.3.55) yields

$$
\begin{aligned}
\mathcal{P}_{[10]_{G_2}, i=1} &= \left(\frac{\lambda}{w_0}\right)^2 \mu^3 + \frac{\lambda}{w_0}(-\mu^5 - 2w_1\mu^3 - \mu - 2) + \mu^6 + (\mu^5 + \mu)(1 - w_1) \\
&+ (\mu^4 + \mu^2)(w_2 + 1) - \mu^3(w_1^2 - 2w_2 - 1) + 1.
\end{aligned}
\tag{5.5.1}
$$

Computing the metric in curved $w$-coordinates, $\eta = \sum_{ij} \tilde{\eta}_{ij} dw^i dw^j$, gives

$$
\tilde{\eta}_{ij} = \begin{pmatrix} \dfrac{8w_1 + 1}{4w_0} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{5.5.2}
$$

which is clearly a singular matrix. Hence, (5.5.1) cannot define a Frobenius manifold structure, and the conjecture cannot hold. Three different efforts were made in an attempt to remedy this failure.

## Attempt 1: A more general shift

Firstly we attempted to generalise the shift in $w_1$, keeping closely to the original conjecture.

Consider the shift

$$\mathcal{P}(\mu; w_1, w_2) \mapsto \mathcal{P}\left(\mu; aw_1 + bw_2 + \frac{\lambda}{w_0}, w_2\right), \tag{5.5.3}$$

for some constants $a, b \in \mathbb{C}$. By setting (5.5.3) equal to zero and solving for $\lambda$ we find

$$\lambda = -\frac{u_0}{\mu^2}\left(\mu^4 + \mu^2(au_1 + bu_2 + 1) - (\mu + 1\sqrt{\mu^6 + 2\mu^5 - \mu^4 + 4\mu^3(u_2 + 1) - \mu^2 + 2\mu + 1})\right). \tag{5.5.4}$$

We see that we still have $\mu \mapsto 1/\mu$ symmetry, however the poles are now 0 and $\infty$ (both of order 2).

The metric (in curved $w$-coordinates) is given by

$$\eta = \begin{pmatrix} \dfrac{4(aw_1 + w_2)}{w_0} & 2a & 2b \\ 2a & 0 & 0 \\ 2b & 0 & 0 \end{pmatrix}, \tag{5.5.5}$$

which might look slightly better, but is still a singular matrix for any choice of $a, b$. Hence, this method does not solve the issue.

## Attempt 2: Multiplying the canonical superpotential by a pole factor

The reason for considering this approach is that this seems to be what happens in [46], and [44], for $A_l$, and the remaining classical cases respectively.

This attempt consists of manipulating $\lambda$ by multiplying it by a product of integer powers of $(\mu - p_j)$ where $p_j$ is a (finite) pole of $\lambda_C$. That is, we want to let

$$\lambda \mapsto \mu^m(\mu + 1)^n \lambda, \quad m, n \in \{-1, 0, 1\}. \tag{5.5.6}$$

Note that as long as $|m|, |n| \leqslant 1$, we will always have poles at $0, \infty, -1$, however the orders change by $-m, -n$ for 0 and $-1$ respectively.

The resulting metrics turn out to all be nondegenerate, which improves the situation somewhat. Furthermore, of these only $(m, n) = (0, -1)$ gives a flat metric. Given that flat metrics are rare, this is a hopeful sign. Let us investigate this case further.

The new superpotential in the case of $(m, n) = (0, -1)$ is given by

$$\tilde{\lambda} = \frac{1}{(\mu + 1)} \lambda. \tag{5.5.7}$$

The metric is now

$$\eta = \begin{pmatrix} \dfrac{25u_1^2 + 140u_1 + 72u_2 + 52}{72u_0} & \dfrac{25u_1 + 52}{36} & \dfrac{1}{2} \\ \dfrac{25u_1 + 52}{36} & \dfrac{25u_0}{18} & 0 \\ \dfrac{1}{2} & 0 & 0 \end{pmatrix}. \tag{5.5.8}$$

This is a nonsingular matrix for generic points in the moduli space, and the Riemann tensor

$$R^\rho_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma} \tag{5.5.9}$$

is identically 0. Here, $\Gamma^a_{bc}$ are the Christoffel symbols of the second kind;

$$\Gamma^d_{ab} = \eta^{dc} \Gamma_{cab}, \tag{5.5.10}$$

with

$$\Gamma_{cab} = \frac{1}{2} \left( \partial_b \eta_{ca} + \partial_a \eta_{cb} - \partial_c \eta_{ab} \right). \tag{5.5.11}$$

The inverse matrix to $\eta$ is given by:

$$\eta^{-1} = \begin{pmatrix} 0 & 0 & 2 \\ 0 & \dfrac{18}{25u_0} & -\dfrac{25u_1 + 52}{25u_0} \\ 2 & -\dfrac{25u_1 + 52}{25u_0} & -\dfrac{2(25u_1 + 50u_2 - 39)}{25u_0} \end{pmatrix}. \tag{5.5.12}$$

Furthermore, we know that flat coordinates for $\eta$ are solutions $t(u)$ of the system

$$\partial^2_{ij} t = \Gamma^k_{ij} \partial_k t, \tag{5.5.13}$$

with non-zero Christoffel symbols $\Gamma^k_{ij}$ given by:

$$\Gamma^1_{11} = -\frac{1}{u_0} = -\Gamma^3_{13}, \quad \Gamma^2_{11} = \frac{25u_1 + 34}{100u_0^2}, \quad \Gamma^2_{12} = \frac{1}{2u_0} = \Gamma^3_{12}, \quad \Gamma^3_{11} = \frac{75u_1 + 100u_2 - 26}{100u_0^2}.$$

Solving this system we could find

$$t_0 = \log(u_0), \quad t_1 = u_0 \left( u_1 + u_2 + \frac{2}{25} \right), \quad t_2 = u_0^{\frac{1}{2}} \left( u_1 + \frac{34}{25} \right). \tag{5.5.14}$$

In these coordinates, the metric takes the form

$$\eta = \begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & \dfrac{25}{18} \end{pmatrix}, \tag{5.5.15}$$

which verifies that $t$ is indeed a flat frame for $\eta$.

Furthermore, $\widetilde{\lambda}$ becomes

$$\widetilde{\lambda}(\mu; t) = -\frac{e^{t_0}}{625\mu^2(\mu+1)^3}\left(625\mu^6 + \left(625e^{-\frac{t_0}{2}}t_2 + 1475\right)\mu^5 + \left(625t_1e^{-t_0} + 1425e + 625e^{-\frac{t_0}{2}}r_2\right)\mu^4\right.$$

$$\tag{5.5.16}$$

$$+ \left((1250t_1 - 625t_2^2)e^{-t_0} - 450e^{-\frac{t_0}{2}}t_2 + 1069\right)\mu^3 + \left(625e^{-t_0}t_1 + 1425 + 625e^{-\frac{t_0}{2}}t_2\right)\mu^2$$

$$\tag{5.5.17}$$

$$+ \left(625e^{-\frac{t_0}{2}}t_2 + 1475\right)\mu + 625\Big).$$

$$\tag{5.5.18}$$

Notice that the differential of $\widetilde{\lambda}(\mu; t)$ with respect to $t_1$ is not a constant (it is $-\frac{1}{\mu+1}$). Hence we will not get that $c_{1ij} = \eta_i j$, and consequently, that $\partial_1$ will not be identity in the tangent algebra. Regardless of this we can find the components of the $c$-tensor in the usual way by taking residues. Since $\partial_1$ cannot be considered a unit, the only hope left is that the identity is actually a combination of $t$-coordinates. Finding such a unit vector field, however, turned out to be not straight-forward. Furthermore, even if a unit exists, it may not be covariantly constant, which suggests that we have obtained a structure weaker than that of a Frobenius manifold, perhaps an almost Frobenius manifold.

Note that we have only considered multiplying by powers of size at most 1. While it is possible to consider more general powers, we decided to not pursue this direction as it seemed to only complicate the structure even further.

## Attempt 3: Changing the primary differential

So far we have attempted to make changes to the superpotential. We could, however, try to change the primary differential instead while keeping the superpotential unchanged. The motivation behind this attempt is that making such a change results in new solutions to the WDVV equations.

The idea of this attempt is to let the primary differential be

$$p = \alpha \log(\mu) + \beta \log(\mu + 1), \tag{5.5.19}$$

where $(\alpha, \beta) \in \mathbb{C}$ (not both zero). More complicated changes to the differential would most likely result in the $g$-metric no longer being Weyl-invariant.

The precise form of (5.5.19) expresses the positions of the poles of $\lambda$. In fact, the overall scaling of $\phi$ is not important, as multiplying $\phi$ by some constant $\alpha$ will result in the metric and the flat coordinates being multiplied by $\alpha$ as well. Hence, it is only important how $\beta$ relates to $\alpha$.

For generic $(\alpha, \beta)$ we obtain generically (in $w$) a nondegenerate metric with components

$$\eta_{00} = \frac{\alpha(\alpha + \beta)(3w_1^2 + 22w_1 + 8w_2 + 19) + \beta^2(w_1^2 + 32w_1 + 8w_2 + 47)}{4w_0}, \qquad (5.5.20)$$

$$\eta_{01} = \eta_{10} = \frac{\alpha(\alpha + \beta)(3w_1^2 + 10w_1 + 7) + \beta^2(w_1^2 + 18w_1 + 2w_2 + 21)}{2(w_1 + 1)}, \qquad (5.5.21)$$

$$\eta_{02} = \eta_{20} = \alpha^2 + \alpha\beta + \beta^2, \qquad \eta_{11} = w_0\left(3\alpha^2 + 3\alpha\beta + \frac{\beta^2(w_1 + 9)}{w_1 + 1}\right), \qquad (5.5.22)$$

$$\eta_{12} = \eta_{21} = \frac{\beta^2 w_0}{w_1 + 1}, \qquad \eta_{22} = 0. \qquad (5.5.23)$$

By calculating the curvature, we see that $\beta(\beta + 2\alpha)$ is a factor of every non-zero component of the Riemann tensor. Since setting $\beta = 0$ recovers the canonical case, we are forced to let $\beta = -2\alpha$, and by the above discussion we can choose $(\alpha, \beta) = (1, -2)$. This means that we are considering the transformation

$$\mathrm{dlog}(\mu) \equiv \frac{\mathrm{d}\mu}{\mu} \mapsto \frac{(1 - \mu)}{\mu(1 + \mu)}\mathrm{d}\mu, \qquad (5.5.24)$$

with $\lambda'\mu \mapsto \lambda'\left(\frac{(1 - \mu)}{\mu(1 + \mu)}\right)^{-1}$. Furthermore, we can find flat coordinates

$$t_0 = \log(u_0^{\frac{3}{2}}(u_1 + 1)^2), \quad t_1 = u_0(4u_1 + u_2 + 6), \quad t_2 = \sqrt{u_0}(u_1 + 5). \qquad (5.5.25)$$

In these coordinates the metric takes the simple form

$$\eta = \begin{pmatrix} 0 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (5.5.26)$$

For this case the components of the $c$-tensor even seem to have reasonable form, and there is hope that this structure does indeed correspond to a Frobenius manifold. The difficulty here arises due to $t_0$ having quite a complicated form, or rather the inverse transformation is complicated enough so that it makes it difficult to write the $c$–tensor in flat coordinates in order to find a prepotential. One way around this could be to make some additional assumptions for the form of the prepotential in order to make an ansatz which could be verified. For the classical cases, [44], non-canonical prepotentials could have terms rational in $t_l$. While this could be an avenue for the future, it is very likely that it is not the manifold we are looking for. Chances are, we have changed the structure too much. More precisely, we are expecting the intersection form to be invariant under node-change (up to change in flat coordinates), but here we have changed the intersection form in a way that is not compatible with this.

We have seen that the conjecture in its most general form does not hold. Furthermore, opening for more general shifts as well as allowed changes in the primary differential, i.e. scalings and

translations of $\mathrm{d}\log\mu$ which are type I (Legendre) transformations [41], under which the intersection form is invariant, does not resolve the nondegeneracy, and neither does attempting to change the pole structure manually akin to that of (5.3.16). There is a further possible solution viewed from the DZ–perspective. By making appropriate assumptions about the prepotential, one could choose degrees, identity and Euler vector fields, as we would expect, and solve the WDVV equations. While feasible in theory for $G_2$, this is not a possible avenue for the other exceptional Dynkin types due to computational complexity.

In summary, the problem of finding Frobenius manifolds associated to non-canonical nodes for exceptional groups is still open, and indeed even the existence of non-canonical Frobenius manifold structures for DZ-manifolds of exceptional types is, at present, unknown. We hope to investigate this problem further at a later stage.

In the following Sections we will look at some applications and new directions of research arising from the main theorem, Theorem 5.2.5.

# Part III

# Applications & Ongoing Research

$$— 6 —$$

# APPLICATION I:
# LYASHKO-LOOIJENGA MAPS

This Section describes an application of Theorem 5.2.5 to the world of topology.

The semisimple locus of a generically semisimple, $n$-dimensional Frobenius manifold $\mathcal{M}$ is topologically a covering of finite multiplicity over a quotient by $S_n$ of the complement of their discriminant, with covering map

$$
\begin{array}{rcl}
\mathrm{LL} : \mathcal{M} & \longrightarrow & (\mathbb{C}^n \backslash \mathrm{Discr}_{\mathcal{M}})/S_n \\
t & \longmapsto & \{e_1(u(t)), \ldots, e_n(u(t))\}
\end{array}
\tag{6.0.1}
$$

assigning to $t \in \mathcal{M}$ the unorderet set of its canonical coordinates in the form of their elementary symmetric polynomials $e_i(u_1, \ldots, u_n)$. When $\mathcal{M}$ has a Landau–Ginzburg description as a holomorphic family of meromorphic functions, the application LL is the classical Lyashko–Looijenga (LL) mapping, sending a meromorphic function to the unordered set of its critical values.

As anticipated in 1.3, a direct corollary of Theorem 5.2.5 is the computation of the degree of the LL map of $\mathcal{M}_\omega^{\mathrm{LG}} \simeq \mathcal{M}_{\mathcal{R}}^{\mathrm{DZ}}$. The calculation of the LL-degree can be tackled combinatorially through the enumeration of certain embedded graphs [89, Section 1.3.2], which unfortunately proves to be intractable for a general stratum of a Hurwitz space of arbitrary genus and ramification profile. In the case of $\deg \mathrm{LL}(\mathcal{M}_\omega^{\mathrm{LG}})$, however, its realisation as a conformal Frobenius manifold allows to bypass the problem altogether by the use of the quasihomogeneous Bézout theorem. To this aim, note that

$$
\det(z - (E(t)\cdot)) = \prod_{i=1}^{l_{\mathcal{R}}+1} (z - u_i(t)) = \sum_{j=0}^{l_{\mathcal{R}}+1} (-1)^j e_j(u_1(t), \ldots, u_{l_{\mathcal{R}}+1}(t)) z^{l_{\mathcal{R}}+1-j}.
\tag{6.0.2}
$$

Setting $\mathcal{Q} := \mathrm{e}^{t_{l_{\mathcal{R}}+1}}$, it follows from Theorems 4.1.1, 5.2.5 that the LL-map (6.0.1) is polynomial in $(t_1, \ldots, t_{l_{\mathcal{R}}+1}, \mathcal{Q})$ since both the product and the Euler vector field are in (6.0.2). Moreover, it follows from Definition (3.3.4) of the quantum product in canonical coordinates that the canonical idempotents $\partial_{u_i}$ have weight $-1$ under the Euler scaling, meaning that $e_i(u)$ is quasihomogeneous

of degree $i$. We can then avail ourselves of the graded generalisation of Bézout's theorem (see e.g. [89, Theorem 3.3]).

**Theorem 6.0.1.** *Let $F : \mathbb{A}^n \to \mathbb{A}^n$ be a finite morphism induced by a quasihomogeneous map $F_* : \mathbb{C}[y_1, \ldots y_n] \to \mathbb{C}[x_1, \ldots, x_n]$ with degrees $p_i$ (resp. $q_i$) for $y_i$ (resp. $x_i$) for $i = 1, \ldots, n$. Then*

$$\deg F = \prod_{i=1}^{n} \frac{p_i}{q_i}. \tag{6.0.3}$$

An immediate consequence of Theorems 4.1.1, 5.2.5, 6.0.1 is the following

**Corollary 6.0.2.** *The degree of the LL-map of the Hurwitz stratum $\mathcal{M}_\omega^{LG}$ is*

$$\deg \mathrm{LL}(\mathcal{M}_\omega^{LG}) = \frac{(l_\mathcal{R} + 1)!(\omega_{\bar{k}}, \omega_{\bar{k}})^{l_\mathcal{R}}}{\prod_{j=1}^{l_\mathcal{R}}(\omega_j, \omega_{\bar{k}})}. \tag{6.0.4}$$

Corollary 6.0.2 is clear by the following. Note that the degrees of the canonical coordinates $\{u_i\}$, i.e. the critical values, are all one[*], while the degrees of the flat coordinates $\{t_i\}$ are given by $\deg(t_i) = \frac{(\omega_i, \omega_{\bar{k}})}{(\omega_{\bar{k}}, \omega_{\bar{k}})}$ (by Theorem 4.1.1 part II). Furthermore, the DZ-invariant $y_i$ is an elementary symmetric polynomial in the canonical coordinates of degree $i$. Thus, by considering the pull back of the composition $\mathbb{C}[t_1, \cdots, t_{l+1}] \to \mathbb{C}[y_1, \cdots, y_{l+1}] \to \mathbb{C}[u_1, \cdots, u_{l+1}]$, Corollary 6.0.2 is obtained by Theorem 6.0.1. We collect in Table 6.1 the calculation of the degrees for the minimal choices of weight $\omega$. Our expectation that $\mathcal{M}_\omega^{\mathrm{LG}} \simeq \mathcal{M}_\mathcal{R}^{\mathrm{DZ}}$ for any of the infinitely many choices of dominant weight $\omega \in \Lambda_w^+(\mathcal{R})$ implies that the same formula (6.0.4) would hold for the Hurwitz strata associated to those non-minimal choices by the construction of 5.3.

For type $A_l$, Corollary 6.0.2 recovers Arnold's formula for the LL-degree of the space of complex trigonometric polynomials, as was already shown in [46]. The fifth column indicates how $\iota_\omega(\mathcal{M}_\mathcal{R}^{\mathrm{DZ}})$ sits as a stratum inside the parent Hurwitz space $\mathcal{H}_{g_\omega, n_\omega}$, either through explicit character relations in $\mathrm{Rep}(\mathcal{G}_\mathcal{R})$ or as a fixed locus of an automorphism of the Hurwitz space induced by the folding of the Dynkin diagram.

---

[*]This can be seen from (2.2.2).

| $\mathcal{R}$ | $g_\omega$ | $\mathsf{n}_\omega$ | $d_{g_\omega;\mathsf{n}_\omega}$ | $\iota_\omega(\mathcal{M}_\mathcal{R}^{\mathrm{DZ}})$ | $\deg(\mathrm{LL})$ |
|---|---|---|---|---|---|
| $A_l$ | $0$ | $(\bar{k}-1, l-\bar{k})$ | $l+1$ | $\mathcal{H}_{g_\omega,\mathsf{n}_\omega}$ | $\dfrac{(l-\bar{k})!(l+1)^l}{(l-\bar{k}+1)^{\bar{k}}\bar{k}^{l-\bar{k}}\prod_{j=\bar{k}+1}^{l}(l-j+1)}$ |
| $B_l$ | $0$ | $(l-2, l-2, 1)$ | $2l+1$ | $\left(\mathcal{H}_{g_\omega,\mathsf{n}_\omega}\right)^{\mu_2}$ | $2(l+1)l(l-1)^{l-1}$ |
| $C_l$ | $0$ | $(l-1, l-1)$ | $2l$ | $\left(\mathcal{H}_{g_\omega,\mathsf{n}_\omega}\right)^{\mu_2}$ | $(l+1)l^l$ |
| $D_l$ | $0$ | $(l-3, l-3, 1, 1)$ | $2l+2$ | $\left(\mathcal{H}_{g_\omega,\mathsf{n}_\omega}\right)^{\mu_2}$ | $4(l+1)l(l-1)(l-2)^{l-2}$ |
| $E_6$ | $5$ | $(5, 5, 2, 2, 2, 2, 2)$ | $42$ | (A.1.1) | $2^3 \cdot 3^6 \cdot 5 \cdot 7$ |
| $E_7$ | $33$ | $(11, 5, 3, 11, 5,$ $3, 1, 1, 3, 3)$ | $130$ | (A.2.1) | $2^{12} \cdot 3^3 \cdot 5 \cdot 7$ |
| $E_8$ | $128$ | $(29, 29, 14, 14, 14,$ $14, 14, 14, 9, 9,$ $9, 9, 5, 5, 4,$ $4, 4, 4, 4, 4,$ $2, 2, 0, 0)$ | $518$ | [17, 18] | $2^4 \cdot 3^5 \cdot 5^5 \cdot 7$ |
| $F_4$ | $4$ | $(5, 5, 2, 2, 2, 2)$ | $36$ | (5.3.53); $\left(\mathcal{M}^{\mathrm{LG}}_{[100000]_{E_6}}\right)^{\mu_2}$ | $2^3 \cdot 3^3 \cdot 5$ |
| $G_2$ | $0$ | $(1, 1, 1)$ | $7$ | (5.3.55); $\left(\mathcal{M}^{\mathrm{LG}}_{[1000]_{D_4}}\right)^{S_3}$ | $12$ |

Table 6.1: Lyashko–Looijenga degrees for all Dynkin types. Here, the superscripts in the fifth column, $\mu_2$, and $S_3$ indicate that the Frobenius manifold lives inside a larger associated Hurwitz (sub)space as a fixed locus of the action of an involution, or the symmetric group on a set of three elements, respectively, as described in Section 5.3.

# APPLICATION II:
# SAITO DISCRIMINANT STRATA

In this chapter, we study certain submanifolds of DZ-manifolds given by discriminant strata. In particular, we investigate the form of the determinant of the Saito metric restricted to discriminant strata, and prove an associated structural theorem. The result is the analogy of [1] to affine Weyl groups. We also prove that discriminant strata are natural submanifolds as anticipated in [115].

## 7.1 Saito discriminant strata for non-affine Weyl groups

In [1] the authors treat Frobenius manifolds arising as orbit spaces of (non-extended, non-affine) Weyl groups of simple complex Lie algebras, and investigate the form of the restriction of the Saito metric to a discriminant stratum $D$, $\eta_D{}^*$. In order to describe their results, we require some notation.

Let $S = \{\beta_j\}_{j=1,\cdots,k}$ be a fixed collection of linearly independent roots in $\mathcal{R}$ determining a discriminant stratum $D = D_S = \cap_{j=1}^{k}\Pi_{\beta_j}$, where $\Pi_{\beta_j} := \{x \in V | (\beta_j, x) = 0\}$ is the mirror to the hyperplane defined by $\beta_j$. Define the root system $\mathcal{R}_D$ by the intersection $\mathcal{R} \cap \langle S \rangle$. It has an orthogonal decomposition

$$\mathcal{R}_D = \bigsqcup_{i=1}^{l} \mathcal{R}_D^{(i)}, \tag{7.1.1}$$

where $\mathcal{R}_D^{(i)}$ is irreducible[†] for each $i$, and $l \in \mathbb{N}$. Let $\mathcal{I}(\mathcal{A})$ denote the defining polynomial for the Coxeter arrangement, $\mathcal{A}$, corresponding to $W$. Furthermore, let $\mathcal{A}^D$, $\mathcal{A}_D$, and $\mathcal{A}_{\Pi_\gamma}$ denote the Coxeter arrangement corresponding to $\mathcal{R}_D$, the restriction of $\mathcal{A}$ to $D$, and the restriction of $\mathcal{A}^D$ to the hyperplane $\Pi_\gamma$ determined by some $\gamma \in \mathcal{R}_D$, respectively.

---

[*]More precisely, this is the pull back $\iota^*\eta$ along the immersion $\iota : D \hookrightarrow M_\mathcal{R}$, for $M_\mathcal{R}$ being the underlying complex manifold of the Frobenius manifold corresponding to the Weyl orbit space associated to $\mathcal{R}$.

[†]A root system is irreducible if and only if the associated Dynkin diagram is connected.

| Symbol | Meaning |
|---|---|
| $\mathcal{R}$ | An irreducible root system |
| $l_{\mathcal{R}}$ | The rank of $\mathcal{R}$ |
| $\mathfrak{g}_{\mathcal{R}}$ | The complex simple Lie algebra with root system $\mathcal{R}$ |
| $\mathcal{G}_{\mathcal{R}}$ | The simply connected complex simple Lie group $\exp(\mathfrak{g}_{\mathcal{R}})$ |
| $\mathfrak{h}_{\mathcal{R}}$ | The Cartan subalgebra of $\mathfrak{g}_{\mathcal{R}}$ |
| $\mathcal{T}_{\mathcal{R}}$ | The Cartan torus $\exp(\mathfrak{h}_{\mathcal{R}})$ of $\mathfrak{g}_{\mathcal{R}}$ |
| g | A regular element of $\mathcal{G}_{\mathcal{R}}$ |
| $\mathcal{W}_{\mathcal{R}}/\widehat{\mathcal{W}}_{\mathcal{R}}/\widetilde{\mathcal{W}}_{\mathcal{R}}$ | The Weyl/affine Weyl/extended affine Weyl group of Dynkin type $\mathcal{R}$ |
| h | A regular element of $\mathfrak{h}_{\mathcal{R}}$ |
| $\{\alpha_1, \ldots, \alpha_{l_{\mathcal{R}}}\}$ | The set of simple roots of $\mathcal{R}$ |
| $\{\omega_1, \ldots, \omega_{l_{\mathcal{R}}}\}$ | The set of fundamental weights of $\mathcal{R}$ |
| $\Lambda_r(\mathcal{R})$ (resp. $\Lambda_r(\mathcal{R})^{\pm}$) | The lattice of roots of $\mathcal{R}$ (resp. the semi-group of positive/negative roots) |
| $\Lambda_w(\mathcal{R})$ (resp. $\Lambda_w(\mathcal{R})^{\pm}$) | The lattice of all weights of $\mathcal{R}$ (resp. the monoid of non-negative/non-positive weights) |
| $\rho_{\omega}$ | The irreducible representation of $\mathcal{G}_{\mathcal{R}}$ with highest weight $\omega$ |
| $\rho_i$ | The $i^{\text{th}}$ fundamental representation of $\mathcal{G}_{\mathcal{R}}$, $i = 1, \ldots, l_{\mathcal{R}}$ |
| $\Gamma(\rho)$ | The weight system of the representation $\rho$ |
| $\chi_{\omega}$ | The formal character of $\rho_{\omega}$ |
| $\chi_i$ | The formal character of $\rho_i$ |
| $[i_1 \ldots i_{l_{\mathcal{R}}}]_{\mathcal{R}}$ (without commas) | Components of a weight in the $\omega$-basis of $\mathcal{R}$ |
| $(x_1, \ldots, x_{l_{\mathcal{R}}})$ | Linear coordinates on $\mathfrak{h}_{\mathcal{R}}$ w.r.t. the coroot basis $\{\alpha_1^*, \ldots, \alpha_{l_{\mathcal{R}}}^*\}$ |
| $(Q_1, \ldots, Q_{l_{\mathcal{R}}})$ | $(\exp(x_1), \ldots, \exp(x_{l_{\mathcal{R}}}))$ |
| $C_{\mathcal{R}}$ | The Cartan matrix of $\mathcal{R}$ |
| $\mathcal{K}_{\mathcal{R}}$ | The symmetrised Cartan matrix of $\mathcal{R}$ |
| $\mathsf{n} \vdash d$ | A padded (i.e. parts are allowed to be zero) partition of $d \in \mathbb{N}$ |
| $|\mathsf{n}|$ (resp. $\ell(\mathsf{n})$) | The length of (resp. the number of parts in) a partition $\mathsf{n}$ |

Table 7.1: Notation for Chapter 7.

Finally, defining, for any $\beta \in \mathcal{R} \backslash \mathcal{R}_D$, $\mathcal{R}_{D,\beta} := <\mathcal{R}_D, \beta> \cap \mathcal{R}$, which has the decomposition

$$\mathcal{R}_{D,\beta} = \bigsqcup_{i=0}^{p} \mathcal{R}_{D,\beta}^{(i)}, \tag{7.1.2}$$

with $\beta \in \mathcal{R}_{D,\beta}^{(0)}$.

In [1] the authors prove, using the B–model for the classical cases and theory of hyperplane arrangements for the exceptional cases, the following theorem.

**Theorem 7.1.1** (Theorem 1.3 in [1])**.** Let

$$\mathcal{I}_i := \mathcal{I}(\mathcal{A}_{\Pi_{\gamma_i}} \backslash \mathcal{A}_{\Pi_{\gamma_i}}^D),$$

with $\gamma_i \in \mathcal{R}_D^{(i)\ddagger}$. Then

$$\det(\eta_D) \propto \mathcal{I}(\mathcal{A} \backslash \mathcal{A}^D)^m \prod_{i=1}^{l} \mathcal{I}_i^{r_i}, \tag{7.1.3}$$

on $D$, where $m = 2 - \sum_{i=1}^{l} r_i$ with $r_i = \mathrm{rk}(\mathcal{R}_D^{(i)})$ and $\propto$ represents proportionality up to a constant (complex number).

Theorem 7.1.1 is proven in two steps by showing that it is equivalent to the following statements.

**Theorem 7.1.2** (Theorem 2.7 in [1]). Let $H \in \mathcal{A}_D$, and $l_H \in D^*$ be a covector such that $H = \{x \in D | l_H(x) = 0\}$. Then the determinant of the Saito metric is proportional to a product of linear forms

$$\det(\eta_D) \propto \prod_{H \in \mathcal{A}_D} l_H^{k_H}, \tag{7.1.4}$$

where $k_H \in \mathbb{N}$.

**Theorem 7.1.3** (Theorem 2.8 in [1]). Let $H \in \mathcal{A}_D$, and $\beta \in \mathcal{R}$ be such that $\beta_D$ is a non-zero multiple of $l_H$. Then, $k_H$ in Theorem 7.1.2 equals the Coxeter number of $\mathcal{R}_{D,\beta}^{(0)}$.

For the classical cases, the authors prove Theorem 7.1.2 by manipulating associated superpotentials in flat coordinates for the intersection form to find an expression for the determinant in canonical coordinates. They then translate this expression to a "nice" frame via a Jacobian expressible in terms of derivatives of $\lambda$ restricted to the stratum evaluated at critical points, in which (7.1.4) is obvious.

For exceptional Dynkin types, the authors of [1] prove Theorem 7.1.1 by a different path as the authors did not have access to LG–models. They combine two approaches; they used the theory of hyperplane arrangements which lead implicitly to the result in high and low codimension, and then explicit calculations did so for the remaining cases (for $E_l$ type strata).

The present chapter serves as an attempt to obtain analogous results for Dubrovin-Zhang manifolds. In summary, we prove the following

**Theorem 7.1.4.** Let $\mathcal{R} \in \{A_l, B_l, C_l, D_l\}$. Then,

$$\det(\eta_D) \propto w_0^{d+1} \prod_{H \in \mathcal{A}_D} l_H^{k_H} \mathcal{F}, \tag{7.1.5}$$

with

$$\mathcal{F} = \prod_{j=1}^{|\mathrm{Sing}_\mu(\lambda(\mu))|} \tilde{l}_j^{\tilde{k}_j}, \tag{7.1.6}$$

---

‡It can be shown that $\mathcal{I}_i|_D$ is independent of the choice $\gamma_i$.

where $l_H, \tilde{l}_j$ are linear forms in exponentiated linear coordinates, $\mathcal{A}_D$ is the corresponding non-affine hyperplane arrangement, $k_H, \tilde{k}_j, d+1 \in \mathbb{N}$, and $\mathrm{Sing}_\mu(\lambda(\mu))$ indicates the set of (finite) poles of $\lambda$.

**Theorem 7.1.5.** Let $H \in \mathcal{A}_D$ and $\beta \in \mathcal{R}$ be such that $\beta_D$ is a non-zero multiple of the linear form $l_H$ (after taking the logarithms of the exponentiated coordinates). Then, the multiplicity of $l_H$ in Theorem 7.1.4, $k_H$, is the Coxeter number of the root system $R_{D,\beta}^{(0)}$, as defined in (7.1.2), and $\tilde{k}_j$ is such that $\lambda_\mathcal{R} \sim \mathcal{O}(\mu^{-\tilde{k}_j})$, near $\infty_j$.

Similarly to [1], we must employ different strategies for classical and exceptional Dynkin types. For the former, we will perform derivations very similar to that of [1]. For the latter, we do have access to an explicit LG–description, but are still forced to employ a different method due to the complexity of the associated spectral curves. While the method is different for the classical cases, it will also differ from that of [1]. The idea here is to make use of the phenomenon that the dual of discriminant strata embed into subalgebras of the corresponding simple Lie algebras. Then we may perform residue calculations in coordinates representing fundamental characters in the subalgebra to obtain the result. For the classical cases, we will prove Theorems 7.1.4–7.1.5, while for the exceptional we shall present a conjecture together with a detailed proposal for a method of proof.

First, however, we must make some comments related to almost duality on $D$.

## 7.2 Almost duality for DZ-discriminant strata

Recall from Chapter 2 that almost duality (Definition 2.3.3) associates to a Frobenius manifold $(M, \cdot, \eta, e, E)$, an almost Frobenius manifold (Definition 2.3.2) using the multiplication, $\star$, (2.3.8), arising from the second nondegenerate bracket $g$, (2.3.1), defined whenever the Euler vector field is a unit in the tangent algebra. That is,

$$g(E \cdot X, Y) = \eta(X, Y), \quad u \star v = E^{-1} \cdot u \cdot v. \tag{7.2.1}$$

We will here show that a restricted version of this holds on discriminant strata as well, as the authors of [1] proved in the non-affine case.

**Lemma 7.2.1.** Let $x$ be $g$-flat coordinates, and let $x_0 \in D$. Then, $e^{-1}(x)$ is well defined as $x$ approaches $x_0$, and $e^{-1}(x_0) \in T_{x_0}D$.

*Proof.* Consider the components of $e^{-1}$ with respect to $g$. By [1],

$$\left(e^{-1}\right)_j = \eta\left(E, \partial_{x_j}\right), \tag{7.2.2}$$

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and from Theorem 4.1.1 we have that

$$E = \sum_{\alpha=1}^{l_{\mathcal{R}}} \frac{d_\alpha}{d_{\bar{k}}} t_\alpha \partial_\alpha + \frac{1}{d_{\bar{k}}} \partial_{l_{\mathcal{R}}+1}. \tag{7.2.3}$$

Then (7.2.2) becomes

$$\left(e^{-1}\right)_j = \sum_{\alpha=1}^{l_{\mathcal{R}}} \frac{d_\alpha}{d_{\bar{k}}} t_\alpha \eta \left(\partial_\alpha, \partial_{x_j}\right) + \frac{1}{d_{\bar{k}}} \eta \left(\partial_{l_{\mathcal{R}}}, \partial_{x_j}\right) \tag{7.2.4}$$

$$= \sum_{\alpha=1}^{l_{\mathcal{R}}} \frac{d_\alpha}{d_{\bar{k}}} t_\alpha \partial_{x_j}(t_\beta) \eta_{\alpha\beta} + \frac{1}{d_{\bar{k}}} \eta_{l_{\mathcal{R}}+1,j} \tag{7.2.5}$$

$$= \sum_{\alpha=1}^{l_{\mathcal{R}}} \frac{d_\alpha}{d_{\bar{k}}} t^\alpha \partial_{x_j}(t_\beta) + \frac{1}{d_{\bar{k}}} \partial_{x_j}(t_1), \tag{7.2.6}$$

as, by [46], $t$ can be scaled to give $\eta = (\delta_{\alpha+\beta,l_{\mathcal{R}}+2})$. This establishes the first part of the lemma. Now, consider $\gamma \in R_D$. Then $\partial_\gamma$ is orthogonal to any $D$, and by (7.2.6)

$$\lim_{x \mapsto x_0} \left(e^{-1}(x), \partial_\gamma\right) = 0, \tag{7.2.7}$$

since $\partial_\gamma t^{l_{\mathcal{R}}+2-\alpha}\big|_D = 0$. Hence, $e^{-1}(x_0) \in T_{x_0}D$. $\qquad\square$

In [59], it was shown that the $\star$-multiplication is well-defined on $D$ for $u, v \in T_{x_0}D$, and that the resulting vector field $u \star v|_{x_0} \in T_{x_0}D$, i.e. the restricted star multiplication is closed in $TD$. Furthermore, it was proved in [1] that $u \cdot v \equiv e^{-1} \star u \star v$. Thus, Lemma 7.2.1 implies the following

**Proposition 7.2.2.** If $u, v \in T_{x_0}D$ with $x_0 \in D\backslash\Sigma_D$, then $u \cdot v \in T_{x_0}D$, with $u \cdot v \equiv e^{-1} \star u \star v$.

In order words, also the multiplication $\cdot$ is well-defined and closed on $D$, and the relationship with $\star$ descends to $D$. This implies that discriminant strata $D$ are indeed natural submanifolds as defined (and anticipated) in [115], which allow us to define the Euler vector field on $D^*$.

As the $x$-frame constitute flat coordinates for the intersection form, we must have that the determinant of the restricted metric $g_D$ in coordinates which are linear combinations of $x_i$, is also constant. Thus, as we have established the analogous relation (7.2.1) on $D$, we have the following

**Corollary 7.2.2.1.** The determinant of the Euler vector field on $D$ is proportional to the determinant of the Saito-metric restricted to $D$;

$$\det(E_D) \propto \det(\eta_D). \tag{7.2.8}$$

In anticipation of forthcoming results, we state the following

---

$^*$This follows from the fact that any natural submanifold of an $\mathcal{F}$–manifold $(M, \cdot, E, g)$ is an $\mathcal{F}$–manifold [115].

**Lemma 7.2.3.** Let $\mathcal{R} \in \{A_l, B_l, C_l, D_l\}$. Then, $\eta_D$ is nondegenerate on $D \backslash \Sigma_D$.

Lemma 7.2.3 is established through the explicit proof of Theorem 7.1.4 in the upcoming section.

**Conjecture 7.2.3.1.** $\eta_D$ is nondegenerate on $D \backslash \Sigma_D$, for any $\mathcal{R}$.

For the exceptional Dynkin types (and in general), we expect the statement to follow from the embedding of the dual of $D$ into subalgebras of the associated Lie algebra to $\mathcal{R}$, and LG–formulae on $D$. It turns out that in coordinates representing fundamental character on the subalgebra, $\det(\eta)$ is a non-zero polynomial. Thus the nondegeneracy should follow from the nonvanishing of the Jacobian of the coordinate transformation from fundamental characters and linear coordinates on the representation space of the Weyl group [46].

## 7.3 Classical Dynkin Types

In this section, we prove Theorem 7.1.4 for extended affine Weyl groups of classical type $\mathcal{R} \in \{A_l, B_l, C_l, D_l\}$, case-by-case. In order to do so, we write the superpotentials found in Section 5.3 in terms of linear coordinates $x$ on $V$, which coincides with flat coordinates for the intersection form.

This is done as follows. Recall that superpotentials are written in terms of $w$-coordinates, which correspond to fundamental characters. To write fundamental characters explicitly in terms of $x$-coordinates, one uses the Weyl character formula, or equivalently,

$$
w_k = \begin{cases} \sum_{\omega'} \mathfrak{m}' e^{x \cdot \omega'} & \text{if } k \neq 0, \\ e^{c_\omega x_{l+1}} & \text{if } k = 0, \end{cases} \tag{7.3.1}
$$

where $c_\omega \in \mathbb{C}^*$, as given in (5.4.22), and $\omega'$ occurs in the weight system of $\rho_k$ with multiplicity $\mathfrak{m}'$.

**Example 24** ($A_3$)**.** The weight systems are (in the Dynkin-basis) given by

$$
\Gamma(\rho_1) = \{(1,0,0), (-1,1,0), (0,-1,1), (0,0,-1)\}, \tag{7.3.2a}
$$

$$
\Gamma(\rho_2) = \{\pm(0,1,0), \pm(1,-1,1), \pm(-1,0,1)\}, \tag{7.3.2b}
$$

$$
\Gamma(\rho_3) = \{(0,0,1), (0,1,-1), (1,-1,0), (-1,0,0)\}. \tag{7.3.2c}
$$

In this case, all weights have multiplicity one, and so we find

$$
w_1 = e^{x_1} + e^{x_2 - x_1} + e^{-x_3} + e^{x_3 - x_2}, \tag{7.3.3a}
$$

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$$w_2 = e^{-x_2} + e^{x_2} + e^{x_1-x_3} + e^{-x_1+x_2-x_3} + e^{x_3-x_1} + e^{x_1-x_2+x_3}, \tag{7.3.3b}$$

$$w_3 = e^{-x_1} + e^{x_1-x_2} + e^{x_2-x_3} + e^{x_3}. \tag{7.3.3c}$$

Thus, by (5.3.2),

$$\lambda(x) = \frac{w_0(\mu - e^{x_3})(\mu - e^{-x_1})(\mu - e^{x_1-x_2})(\mu - e^{x_2-x_3})}{\mu}. \tag{7.3.4}$$

In the following we denote the restricted superpotential $\lambda|_D := \lambda_D$ and the restricted Saito metric $\eta_D := \eta|_D$, for some discriminant stratum $D$, with

$$\eta_D(\zeta_1, \zeta_2) = \sum_{q_s \in \mathrm{Crit}(\lambda_D)} \mathrm{Res}_{\mu=q_s} \frac{\zeta_1(\lambda_D)\zeta_2(\lambda_D)}{\mu^2 \lambda'_D(\mu)} \mathrm{d}\mu, \tag{7.3.5a}$$

$$c_D(\zeta_1, \zeta_2, \zeta_3) \equiv \eta_D(\zeta_1 \star \zeta_2, \zeta_3) = \sum_{q_s \in \mathrm{Crit}(\lambda_D)} \mathrm{Res}_{\mu=q_s} \frac{\zeta_1(\lambda_D)\zeta_2(\lambda_D)\zeta_3(\lambda_D)}{\mu^2 \lambda'_D(\mu)} \mathrm{d}\mu, \tag{7.3.5b}$$

for $\zeta_1, \zeta_2, \zeta_3 \in T_p D$, $p \in D\backslash\Sigma_D$, and as usual $\lambda'_D(\mu) \equiv \frac{\mathrm{d}\lambda_D(\mu)}{\mu}$.

## $\mathcal{R} = A_l$

**LG–superpotentials**

As suggested by the above, we want to write the LG-superpotentials obtained in Section 5.3 in terms of the linear coordinates $x$ on $V \oplus \mathbb{C}^*$ using (7.3.1). Doing so for type $A$ of rank $l$ we get[*].

$$\lambda(\mu; x) = \frac{(-1)^{k+l} w_0 \prod_{i=1}^{l+1}(\mu - a_i)}{\mu^k}, \tag{7.3.6}$$

where

$$a_i = \begin{cases} e^{-x_1}, & i = 1, \\ e^{x_l}, & i = l+1, \\ e^{x_{i-1}-x_i}, & \text{otherwise.} \end{cases}$$

That $\lambda$ has this form in the $x$-frame can also be seen directly from the spectral curve, as zeros of $\lambda$ are the zeros of $[\lambda^0]\mathcal{P}_{A_l}$, which are precisely $a_i$ as in (7.3). As in [46], we normalise the superpotential by bringing $w_0$ into the product, and redefining $\mu$ and $\{a_i\}$ accordingly. Then (7.3.6) becomes

$$\lambda(\hat{\mu}; x) = \frac{(-1)^{k+l} \prod_{i=1}^{l+1}(\hat{\mu} - \hat{a}_i)}{\hat{\mu}^k}, \tag{7.3.7}$$

---

[*]This can be done by hand using the correspondence between the natural coordinates $\{w_i\}$ and the DZ-invariants $\{\tilde{y}_j\}$ as given in (5.4.40), and the fact that $y_j = S_W(e^{2\pi i(\omega_j, x)})$ which is related to $\tilde{y}_j$ as in (4.2.8). For any specific rank however, it is much more easily done using the Mathematica package LieArt [58].

where $\hat{a}_i = a_i e^{\alpha x_0}$, $\forall\, i = 1, \cdots, l+1$, and similarly for $\mu$, with $\alpha \in \mathbb{C}^*$ being some constant[†]. In the following we will drop $\hat{\ }$ for readability.

**Discriminant Strata for type $A$**

The discriminant, $\Sigma$, consists of a union of hyperplanes, on which the $\eta$-pairing is nondegenerate. From the Landau-Ginzburg perspective, it is the set

$$\Sigma = \{\lambda \mid \lambda(q) = 0 \text{ for some critical point } q\}.$$

Clearly, $\lambda(q) = 0$ if and only if $q = a_i$ for some $i$. Suppose $q = a_m$ for $m \in \{1, \cdots, l+1\}$. Then,

$$\lambda'(\mu)\Big|_{\mu=q} = \overbrace{\frac{-k}{\mu}\lambda(\mu)\Big|_{\mu=a_m}}^{=0} + \sum_{j=1}^{l+1} \frac{\lambda(\mu)}{(\mu - a_j)}\Big|_{\mu=a_m} = 0$$

$$\implies \prod_{\substack{i=1 \\ i \neq m}}^{l+1} (a_m - a_i) = 0.$$

From this it is clear that in order to consider the discriminant, we must have $a_i = a_j$ for at least one pair of distinct labels $\{i, j\}$.

Thus, we can describe any discriminant stratum $D$ by the following equations,

$$a_1 = \cdots = a_{m_1} = \xi_1$$

$$a_{m_1+1} = \cdots = a_{m_1+m_2} = \xi_2$$

$$\vdots$$

$$a_{\sum_{j=1}^d m_j + 1} = \cdots = a_{\sum_{j=1}^{d+1} m_j} = \xi_{d+1},$$

where $\sum_{j=0}^d m_j = l+1$.

The corresponding superpotential is given by

$$\lambda_D(\mu) = \frac{(-1)^{l+k}}{\mu^k} \prod_{i=1}^{d+1} (\mu - \xi_i)^{m_i}. \tag{7.3.8}$$

**Lemma 7.3.1.**

$$\lambda'_D(\mu) = \frac{(-1)^{l+k}(l+1-k)}{\mu^{k+1}} \prod_{i=1}^{d+1} (\mu - \xi_i)^{m_i-1} \prod_{j=0}^{d} (\mu - q_j), \tag{7.3.9}$$

for some $q_j \in \mathbb{C}$.

---

[†]Notice that this can always be achieved by letting $\hat{a}_i = a_i e^{\frac{c_\omega x_0}{l+1-k}}$, and $\hat{\mu} = \mu e^{-\frac{c_\omega x_0}{l+1-k}}$, similarly to the scaling in Section 5.3, with $c_\omega$ as in (5.4.22) for $\mathcal{N}_\omega = 1$.

*Proof.* Differentiating (7.3.8) with respect to $\mu$:

$$
\lambda_D'(\mu) = \frac{(-1)^{l+k}}{\mu^k} \left( \frac{-k}{\mu} \prod_{i=1}^{d+1}(\mu - \xi_i)^{m_i} + \sum_{j=1}^{d+1} m_j(\mu - \xi_j)^{m_j-1} \prod_{\substack{i=1 \\ i \neq j}}^{d+1}(\mu - \xi_i)^{m_i} \right)
$$

$$
= \frac{(-1)^{l+k}}{\mu^{k+1}} \prod_{i=1}^{d+1}(\mu - \xi_i)^{m_i-1} f(\mu; \xi),
$$

where $f(\mu; \xi)$ is a polynomial in $\mu$ of order $d+1$ with $[\mu^{d+1}] = -k+\sum_{j=1}^{d+1} m_j = l+1-k$. Factorising $f(\mu; \xi)$ in terms of its roots $\{q_i\}_{i=0}^{d}$ gives the statement. $\qquad\square$

**Lemma 7.3.2.**

$$
\lambda_D''(q_r) = \frac{(-1)^{l+k}(l+1-k)}{q_r^{k+1}} \prod_{i=1}^{d+1}(q_r - \xi_i)^{m_i-1} \prod_{\substack{j=0 \\ j \neq r}}^{d}(q_r - q_j). \tag{7.3.10}
$$

*Proof.* Differentiating (7.3.9) with respect to $\mu$:

$$
\lambda_D''(\mu)\big|_{\mu=q_r} = (-1)^{l+k}(l+1-k)\frac{\partial}{\partial\mu}\left(\frac{\prod_{i=1}^{d+1}(\mu - \xi_i)^{m_i-1}}{\mu^{k+1}}\right) \overbrace{\prod_{j=0}^{d}(\mu - q_j)}^{=0}\Bigg|_{\mu=q_r}
$$

$$
+ \frac{\prod_{i=1}^{d+1}(\mu - \xi_i)^{m_i-1}}{\mu^{k+1}} \sum_{s=0}^{d}\prod_{\substack{j=0 \\ j \neq s}}^{d}(\mu - q_j)\Bigg|_{\mu=q_r}
$$

$$
= \frac{(-1)^{l+k}(l+1-k)}{\mu^{k+1}} \prod_{i=1}^{d+1}(\mu - \xi_i)^{m_i-1} \prod_{\substack{j=0 \\ j \neq r}}^{d}(\mu - q_j)\Bigg|_{\mu=q_r}.
$$

$\qquad\square$

**Lemma 7.3.3.** Let $u_i$ be the critical values of $\lambda$; $u_i := \lambda_D(q_i)$. Then

$$
\partial_{u_i}\lambda_D(\mu)\big|_{\mu=q_j} = \delta_{ij}. \tag{7.3.11}
$$

*Proof.* Proof is identical to that of [1]. $\qquad\square$

**Proposition 7.3.4.**

$$
\partial_{u_r}\lambda_D(\mu) = \frac{\mu}{q_r(\mu - q_r)}\frac{\lambda_D'(\mu)}{\lambda_D''(q_r)}. \tag{7.3.12}
$$

*Proof.* Differentiating (7.3.8) with respect to $u_r$:

$$\partial_{u_r}\lambda_D(\mu) = \frac{(-1)^{l+k}}{\mu^k}\sum_{j=1}^{d+1}(-m_j)\frac{\partial\xi_j}{\partial u_r}(\mu-\xi_j)^{m_j-1}\prod_{\substack{i=1\\i\neq j}}^{d+1}(\mu-\xi_i)^{m_i}$$

$$= \frac{(-1)^{l+k}}{\mu^k}\prod_{i=1}^{d+1}(\mu-\xi_i)^{m_i-1}F(\mu;r).$$

Here $F(\mu;r)$ is a polynomial in $\mu$ of degree $d$, and so is fully determined by its values on $\{q_i\}$ by Lagrange interpolation. From Lemma 7.3.3

$$F(q_i;r) = \frac{\delta_{ri}(-1)^{l+k}q_i^k}{\prod_{j=1}^{d+1}(q_i-\xi_j)^{m_j-1}}.$$

Hence,

$$F(\mu;r) = \sum_{s=0}^{d}F_s(\mu;r) = F_r(\mu;r),$$

where

$$F_s(\mu;r) = F(q_s;r)\prod_{\substack{i=0\\i\neq s}}^{d}\frac{(\mu-q_i)}{(q_r-q_i)},$$

which gives

$$\partial_{u_r}\lambda_D(\mu) = \frac{(-1)^{l+k}}{\mu^k}\prod_{i=1}^{d+1}(\mu-\xi_i)^{m_i-1}\cdot\frac{(-1)^{l+k}q_r^k}{\prod_{j=1}^{d+1}(q_r-\xi_j)^{m_j-1}}\prod_{\substack{i=0\\i\neq r}}^{d}\frac{(\mu-q_i)}{(q_r-q_i)}.$$

Finally, using Lemmas 7.3.1 and 7.3.2 to calculate $\frac{\mu\lambda'_D(\mu)}{q_r(\mu-q_r)\lambda''_D(q_r)}$ gives the statement. $\qquad\square$

**Proposition 7.3.5.**

$$\partial_{u_r}(\xi_a) = \frac{\xi_a}{q_r(q_r-\xi_a)\lambda''_D(q_r)}.$$

*Proof.* Differentiating (7.3.8) with respect to $u_r$, setting equal to (7.3.12), multiplying both sides by $\frac{1}{(\mu-\xi_a)^{m_a-1}}$, and evaluating at $\mu=\xi_a$ gives

$$\frac{\xi_a}{q_r(\xi_a-q_r)\lambda''_D(q_r)}\frac{\lambda'_D(\mu)}{(\mu-\xi_a)^{m_a-1}}\bigg|_{\mu=\xi_a} = \sum_{j=1}^{d+1}\frac{\partial\xi_j}{\partial u_r}\frac{m_j\lambda_D(\mu)}{(\mu-\xi_j)(\mu-\xi_a)^{m_a-1}}\bigg|_{\mu=\xi_a}.$$

Since

$$\frac{\lambda_D(\mu)}{(\mu-\xi_a)^{m_a}}\bigg|_{\mu=\xi_a} = \frac{\lambda'_D(\mu)}{m_a(\mu-\xi_a)^{m_a-1}}\bigg|_{\mu=\xi_a},$$

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we do indeed find

$$-\frac{\partial \xi_j}{\partial u_r} = \frac{\xi_a}{q_r(\xi_a - q_r)\lambda_D''(q_r)}.$$

□

**Lemma 7.3.6.**

$$\eta_D(\partial_{u_i}, \partial_{u_j}) = \frac{(-1)^{k+1}\partial_{ij}}{q_i^2 \lambda_D''(q_i)}, \tag{7.3.13}$$

$$\partial_{u_i} \star \partial_{u_j} = \delta_{ij}\partial_{u_i}. \tag{7.3.14}$$

*Proof.* Using (7.3.5a) together with Lemma 7.3.4, we have

$$\eta_D(\partial_{u_i}, \partial_{u_j}) = (-1)^{k+1} \sum_{s=0}^d \operatorname*{Res}_{\mu=q_s} \frac{\partial_{u_i}(\lambda_D(\mu))\partial_{u_j}(\lambda_D(\mu))}{\mu^2 \lambda_D'(\mu)} d\mu$$

$$= \frac{(-1)^{k+1}}{q_i q_j \lambda_D''(q_i)\lambda_D''(q_j)} \sum_{s=0}^d \operatorname*{Res}_{\mu=q_s} \frac{\lambda_D'(\mu)}{(\mu - q_i)(\mu - q_j)} d\mu.$$

Due to the form in (7.3.9), it is clear that the residues will yield zero unless $i = j$. Letting $i = j$ gives

$$\frac{(-1)^{k+1}}{q_i^2 \lambda_D''(q_i)} \left( \overbrace{\frac{(-1)^{l+k}(l+1-k)\prod_{j=1}^{d+1}(q_i - \xi_j)^{m_j-1}\prod_{\substack{j=0\\j\neq i}}^d(q_i - q_j)}{q_i^{k+1}}}^{= \lambda_D''(q_i)} \right).$$

For the second part of the proposition, we consider $c_D(\partial_{u_i}, \partial_{u_j}, \partial_{u_k})$. It follows from Lemma 7.3.4 that the residues are zero unless $i = j = k$, for which we have

$$c_D(\partial_{u_i}, \partial_{u_i}, \partial_{u_i}) = (-1)^{k+1} \sum_{s=0}^d \operatorname*{Res}_{\mu=q_s} \frac{(\partial_{u_i}\lambda_D)^3}{\mu^2 \lambda_D'(\mu)} d\mu = \frac{(-1)^{k+1}}{\lambda_D''(q_i)^3 q_i^3} \sum_{s=0}^d \operatorname*{Res}_{\mu=q_s} \frac{\mu \lambda_d'(\mu)^2}{(\mu - q_i)^3} d\mu. \tag{7.3.15}$$

By Lemma 7.3.1 we find

$$c_D(\partial_{u_i}, \partial_{u_i}, \partial_{u_i}) = \frac{(-1)^{k+1}(l+1-k)^2}{\lambda_D''(q_i)^3 q_i^{2k+4}} \sum_{s=0}^d \operatorname*{Res}_{\mu=q_s} \frac{\prod_{j=1}^{d+1}(\mu - \xi_j)^{2(m_j-1)}\prod_{\substack{j=0\\j\neq i}}^d(\mu - q_j)^2}{(\mu - q_i)} d\mu$$

$$= \frac{(-1)^{k+1}(l+1-k)^2}{\lambda_D''(q_i)^3 q_i^{2k+4}} \prod_{j=1}^{d+1}(q_i - \xi_j)^{2(m_j-1)}\prod_{\substack{j=0\\j\neq i}}^d(q_i - q_j)^2 = \frac{(-1)^{k+1}}{q_i^2 \lambda_D''(q_i)},$$

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where we have used Lemma 7.3.2 in the final equality. Hence,

$$c_D(\partial_{u_i}, \partial_{u_j}, \partial_{u_k}) = \frac{\delta_{ij}\delta_{jk}(-1)^{k+1}}{q_i^2 \lambda_D''(q_i)}, \tag{7.3.16}$$

which implies the statement by the nondegeneracy of $\eta_D$ and Proposition 7.2.2. $\qquad\square$

Before we set out to calculate the determinant of the Saito metric in $\xi$-coordinates, we prove some Lemmas which will be useful when doing so.

**Lemma 7.3.7.**

$$\prod_{i=0}^{d} q_i = -\frac{k}{(l+1-k)} \prod_{a=1}^{d+1} \xi_a. \tag{7.3.17}$$

*Proof.* Differentiating (7.3.8) with respect to $\mu$ and setting equal to (7.3.9) gives

$$\frac{(-1)^{l+k}}{\mu^{k+1}} \left( -k \prod_{a=1}^{d+1} (\mu - \xi_a)^{m_a} + \mu \frac{\partial}{\partial \mu} \left( \prod_{a=1}^{d+1} (\mu - \xi_a)^{m_a} \right) \right)$$

$$= \frac{(-1)^{l+k}(l+1-k)}{\mu^{k+1}} \prod_{i=1}^{d+1} (\mu - \xi_i)^{m_i - 1} \prod_{j=0}^{d} (\mu - q_j)$$

$$\implies \quad -k \prod_{a=1}^{d+1} (\mu - \xi_a) + \mu \frac{\partial}{\partial \mu} \left( \prod_{a=1}^{d+1} (\mu - \xi_a)^{m_a} \right) = (l+1-k) \prod_{j=0}^{d} (\mu - q_j).$$

Comparing constant terms gives

$$-k \prod_{a=1}^{d+1} (-\xi_a) = (l+1-k) \prod_{j=0}^{d} (-q_j),$$

and rearranging gives the statement. $\qquad\square$

**Lemma 7.3.8.**

$$\prod_{i=0}^{d} (\xi_a - q_i) = \frac{\xi_a m_a}{(l+1-k)} \prod_{\substack{i=1 \\ i \neq a}}^{d+1} (\xi_a - \xi_i). \tag{7.3.18}$$

*Proof.* From Lemma 7.3.1 we have that

$$\prod_{i=0}^{d} (\xi_a - q_i) = \frac{(-1)^{l+k} \mu^{k+1} \lambda_D'(\mu)}{(l+1-k) \prod_{i=1}^{d+1} (\mu - \xi_i)^{m_i - 1}} \Bigg|_{\mu = \xi_a}$$

$$= \frac{(-1)^{l+k} \xi_a^{k+1}}{(l+1-k)} \frac{m_a}{\prod_{\substack{i=1 \\ i \neq a}}^{d+1} (\mu - \xi_i)^{m_i - 1}} \frac{\lambda_D(\mu)}{(\mu - \xi_i)^{m_a}} \Bigg|_{\mu = \xi_a}.$$

Inserting (7.3.8), evaluating and using the fact that

$$\frac{\lambda_D(\mu)}{(\mu - \xi_a)^{m_a}}\bigg|_{\mu = \xi_a} = \frac{\lambda'_D(\mu)}{m_a(\mu - \xi_a)^{m_a-1}}\bigg|_{\mu = \xi_a},$$

gives the statement. □

**Lemma 7.3.9.**

$$z := \frac{\prod_{i=0}^d \lambda''_D(q_i)}{\prod_{\substack{i=0 \\ i<j}}^d (q_i^2 - q_j^2)^2} = C \cdot \frac{\prod_{\substack{i=0,\cdots,d \\ j=1,\cdots,d+1}} (q_i - \xi_j)^{m_j - 1}}{\prod_{a=1}^{d+1} \xi_a^{k+1}}, \tag{7.3.19}$$

where $C = \frac{(l+1-k)^{d+k+2}(-1)^{(l+k)d+\frac{d(d+1)}{2}+l+1}}{k^{k+1}}$.

*Proof.* From Lemma 7.3.2, we have

$$\frac{\prod_{i=0}^d \lambda''_D(q_i)}{\prod_{\substack{i=0 \\ i<j}}^d (q_i^2 - q_j^2)^2} = \prod_{i=0}^d \left( (-1)^{l+k}(l+1-k) \right) \frac{\prod_{i,j}(q_i - \xi_j)^{m_j - 1}}{\prod_{i=0}^d q_i^{k+1}} \frac{\prod_i \prod_{\substack{j \\ j \neq i}}(q_i - q_j)}{\prod_{\substack{i<j}} (q_i^2 - q_j^2)^2}$$

$$= (-1)^{(l+k)(d+1)}(l+1-k)^{d+1} \frac{\prod_{i,j}(q_i - \xi_j)^{m_j - 1}}{\left( -\frac{k}{(l+1-k)} \prod_{a=1}^{d+1} \xi_a \right)^{k+1}} \cdot (-1)^{\frac{d(d+1)}{2}}$$

$$= \frac{(l+1-k)^{d+k+2}(-1)^{(l+k)d+\frac{d(d+1)}{2}+(l+1)}}{k^{k+1}} \frac{\prod_{i,j}(q_i - \xi_j)^{m_j - 1}}{\prod_{a=1}^{d+1} \xi_a^{k+1}},$$

where the second equality is obtained using Lemma 7.3.7. □

Recall that in the beginning of this section, we dropped "hats" on $a_i$, and consequently $\xi_i$, where these are related by $\hat{a}_i = a_i e^{\frac{c_\omega x_0}{l+1-k}}$, with $\hat{\xi}_i = \hat{a}_i|_D$. We are now ready to state and prove the main result of this section.

**Theorem 7.3.10.**

$$\det(\eta_D(\hat{\xi})) = \kappa \cdot \prod_{a=1}^{d+1} \hat{\xi}_a^{m_a - (k+2)} \prod_{1 \leqslant a < b \leqslant d+1} (\hat{\xi}_a - \hat{\xi}_b)^{m_a + m_b}, \tag{7.3.20}$$

and

$$\det(\eta_D(\xi)) = \kappa \cdot e^{c_\omega x_0(d+1)} \prod_{a=1}^{d+1} \xi_a^{-(k+2)} \prod_{1 \leqslant a < b \leqslant d+1} (\xi_a - \xi_b)^{m_a + m_b}, \tag{7.3.21}$$

where $\kappa = \frac{(-1)^{k+1+\sum_{i=1}^d i m_i}}{(l+1-k)^{l-k} k^{k+1}} \prod_{a=0}^d m_a^{m_a + 1}$.

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*Proof.* Using Lemma 7.3.6,

$$\det(\eta_D(\xi)) = (\det(B))^{-2}\det(\eta_D(u))$$

$$= (\det(B))^{-2}\frac{(-1)^{(k+1)(d+1)}}{\prod_{i=0}^{d} q_i^2 \prod_{i=0}^{d} \lambda_D''(q_i)},$$

where $B$ is the $(d+1)$-dimensional Jacobian matrix $\left(\frac{\partial \xi_a}{\partial u_b}\right)_{\substack{b=0,\cdots,d \\ a=1,\cdots,d+1}}$. From Lemma 7.3.5;

$$\det(B) = \frac{\prod_{a=1}^{d+1} \xi_a}{\prod_{r=0}^{d} q_r \prod_{r=0}^{d} \lambda_D''(q_r)} \cdot \det(A),$$

where $A$ is the $d+1 \times d+1$ matrix with entries $\frac{1}{q_r-\xi_a}$, for $r = 0,\cdots,d$, and $a = 1,\cdots,d+1$. $\det(A)$ is given by the Cauchy determinant

$$(-1)^{\frac{d(d+1)}{2}}\frac{\prod_{\substack{i=1 \\ i<j}}^{d+1}(\xi_i - \xi_j) \prod_{\substack{i=0 \\ i<j}}^{d}(q_i - q_j)}{\prod_{\substack{i=0,\cdots,d \\ a=1,\cdots,d+1}}(q_i - \xi_a)}.$$

Hence,

$$\det(\eta_D(\xi)) = (-1)^{(k+1)(d+1)}\frac{\prod_{i,a}(q_i - \xi_a)^2}{\prod_a \xi_a^2 \prod_{\substack{i \\ i<j}}(\xi_i - \xi_j)^2} \cdot z$$

$$= \frac{C(-1)^{(d+1)(k+1)}\prod_{i,a}(q_i - \xi_a)^{m_a+1}}{\prod_a \xi_a^{k+3}\prod_{\substack{i \\ i<j}}(\xi_i - \xi_j)^2}$$

$$= \frac{C(-1)^{(d+1)(k+1)+(d+1)(d+l+1)}\prod_{a=1}^{d+1} m_a^{m_a+1}}{(l+1-k)^{d+l+2}k^{k+1}} \frac{\prod_{a=1}^{d+1} \xi_a^{m_a-k-2}\prod_{a=1}^{d+1}\prod_{\substack{i=1 \\ i\neq a}}^{d+1}(\xi_a - \xi_i)^{m_a+1}}{\prod_{1\leqslant a<b\leqslant d+1}(\xi_a - \xi_b)^2}$$

$$= \kappa \cdot \prod_{a=1}^{d+1} \xi_a^{m_a-k-2}\prod_{1\leqslant a<b\leqslant d+1}(\xi_a - \xi_b)^{m_a+m_b},$$

where $\kappa = \dfrac{(l+1-k)^{d+k+2}\prod_{a=1}^{d+1} m_a^{m_a+1}}{k^{k+1}(l+1-k)^{d+l+2}}(-1)^\gamma$, with

$$\gamma = (d+1)(k+1+(d+1)(d+l+1)+\frac{d(d+1)}{2}+(l+k)(d+1)+k+1+\frac{d(d+1)}{2}+\sum_{i=1}^{d} im_i) = k+1+\sum_{i=1}^{d} im_i.$$

Here $z$ is as defined in Lemma 7.3.9, the third equality is obtained by using Lemma 7.3.8, and in the final equality we have used the fact that

$$\prod_{a=1}^{d+1}\prod_{\substack{b=1 \\ b\neq a}}^{d+1}(\xi_a - \xi_b)^{m_a+1} = (-1)^{\sum_{i=1}^{d} im_i - \frac{d(d+1)}{2}}\prod_{1\leqslant a<b\leqslant d+1}(\xi_a - \xi_b)^{m_a+m_b+2}. \tag{7.3.22}$$

Thus, recalling that we removed "hats" over $\xi_i$ in the very beginning of this section, gives the first part of Theorem 7.3.10.

Now, since

$$\hat{\xi}_a = \xi_a e^{\frac{c_\omega x_0}{l+1-k}} \implies \frac{\partial \hat{\xi}_a}{\partial \xi_b} = \delta_{ab} e^{\frac{c_\omega x_0}{l+1-k}}, \tag{7.3.23}$$

we have that the Jacobian of the coordinate transformation $\hat{\xi} \mapsto \xi$ is

$$J := \left( \partial_{\xi_a} \hat{\xi}_b \right) = e^{\frac{c_\omega x_0 (d+1)}{l+1-k}}. \tag{7.3.24}$$

This gives,

$$\det(\eta_D) = \kappa \, e^{\frac{2c_\omega x_0 (d+1)}{l+1-k}} \prod_{a=1}^{d+1} e^{\frac{c_\omega x_0 (d+1)}{l+1-k}(m_a - (k+2))} \prod_{1 \leqslant a < b \leqslant d+1} e^{\frac{c_\omega x_0 (d+1)}{l+1-k}(m_a + m_b)}$$
$$\cdot \prod_{a=1}^{d+1} \xi_a^{m_a - (k+2)} \prod_{1 \leqslant a < b \leqslant d+1} (\xi_a - \xi_b)^{m_a + m_b}.$$

Notice that

$$\sum_{1 \leqslant a < b \leqslant d+1} = \sum_{b=2}^{d+1}(m_1 + m_b) + \sum_{b=3}^{d+1}(m_2 + m_b) + \cdots + m_d + m_{d+1}$$
$$= dm_1 + \sum_{b=2}^{d+1} m_b + (d-1)m_2 + \sum_{b=3}^{d+1} m_b + \cdots + m_d + m_{d+1}$$
$$= \sum (d-a)m_a \sum_{b=a}^{d+1} m_b = d \sum_{a=1}^{d+1} m_a = d(l+1),$$

and so

$$2(d+1) + \sum_{a=1}^{d+1} m_a - (d+1)(k+2) \sum_{1 \leqslant a < b \leqslant d+1}(m_a + m_b) = l+1-k-dk+d(l+1) = (d+1)(l+1-k), \tag{7.3.25}$$

which gives

$$e^{\frac{c_\omega x_0}{l+1-k}(2(d+1) + \sum_{a=1}^{d+1} m_a - (d+1)(k+2) \sum_{1 \leqslant a < b \leqslant d+1}(m_a + m_b))} = e^{c_\omega x_0(d+1)}. \tag{7.3.26}$$

The second part of the theorem is then obtained by the fact that $\prod_{a=1}^{d+1} \xi_a^{m_a} = 1$.

$\square$

*Proof of Theorems 7.1.4 and 7.1.5 for $\mathcal{R} = A_l$.* From Theorem 7.3.10, we have

$$\det(\eta_D(\hat{\xi})) = \kappa \cdot \prod_{a=1}^{d+1} \hat{\xi}_a^{m_a - (k+2)} \prod_{1 \leqslant a < b \leqslant d+1} (\hat{\xi}_a - \hat{\xi}_b)^{m_a + m_b}. \tag{7.3.27}$$

We also know that

$$\partial_{\xi_a}\hat{\xi}_b = \delta_{ab}e^{\frac{c_\omega x_0}{l+1-k}}, \tag{7.3.28}$$

and that, for $i, j \neq 0$,

$$\partial_{x_j}(a_i) = \begin{cases} -a_i & \text{if } i = j, i \neq l+1, \\ a_i & \text{if } i = j+1, i \neq 1, \\ 0 & \text{otherwise}. \end{cases} \tag{7.3.29}$$

This implies that in order to find the Jacobian between $\hat{\xi}$- and $x$-coordinates, we must compute the determinant of a $(d+1) \times (d+1)$ matrix, $M$, with entries of the form

$$M_{ij} = \begin{cases} \dfrac{c_\omega}{l+1-k}\hat{\xi}_i & \text{if } j = 0, \\ \alpha_{ij}\hat{\xi}_j & \text{otherwise}, \end{cases} \tag{7.3.30}$$

where $\alpha_{ij} = \partial_{x_j}(\xi_i)/\xi_i \in \mathbb{Z}$. Thus,

$$\det(M) = \frac{c_\omega}{l+1-k}\prod_{i=1}^{d+1}\hat{\xi}_i \cdot \det(P), \tag{7.3.31}$$

where $P$ is some constant integral nondegenerate matrix[‡]. Let $\det(P) = \beta \in \mathbb{Z}$, then

$$\det(\eta_D) = \det(\eta_D)(\hat{\xi}) \cdot \frac{\beta^2 c_\omega^2}{(l+1-k)^2}\prod_{i=1}^{d+1}\hat{\xi}_i^2 = \tilde{\kappa}\cdot\prod_{a=1}^{d+1}\hat{\xi}_a^{\overbrace{m_a - k - 2 + 2}^{=-k}}\prod_{1\leqslant a < b \leqslant d+1}(\hat{\xi}_a - \hat{\xi}_b)^{m_a+m_b}, \tag{7.3.32}$$

where $\tilde{\kappa} = \kappa \cdot \frac{\beta^2 c_\omega^2}{(l+1-k)^2}$. Letting $\hat{\xi} = \hat{\xi}(\xi)$, we get

$$\det(\eta_D) = \tilde{\kappa}\cdot e^{c_\omega x_0(d+1)}\prod_{a=1}^{d+1}\xi_a^{-k}\prod_{1\leqslant a<b\leqslant d+1}(\xi_a - \xi_b)^{m_a+m_b}, \tag{7.3.33}$$

where we have used that $\sum_{1\leqslant a<b\leqslant d+1}(m_a + m_b) = d(l+1)$, and $\prod_{a=1}^{d+1}\xi_a^{m_a} = 1$, as in the proof of Theorem 7.3.10. Hence, Theorem 7.1.4 holds for $\mathcal{R} = A_l$.

Furthermore, notice that the second set of products in (7.3.33) is exactly the non-affine A-type result of [1], Theorem 7.1.2. Therefore, the first part of Theorem 7.1.5 holds automatically from Theorems 7.3.10 and 7.1.3. The second part of Theorem 7.1.5 holds as well as the superpotential, (7.3.8), has a single pole for finite $\mu$ located at 0, of order $k$. □

**Example 25** ($A_3^{k=1}$). Let $\mathcal{R} = A_3$ with $k = 1$. The associated superpotential is given by (7.3.6):

$$\lambda_{A_3}^{k=1} = \frac{w_0(\mu - e^{-x_1})(\mu - e^{x_1-x_2})(\mu - e^{x_2-x_3})(\mu - e^{x_3})}{\mu}, \tag{7.3.34}$$

---

[‡]The nondegeneracy of $P$ can be seen from (7.3.29) and keeping in mind the restrictions of $a_i$ to $D$.

with $w_0 = e^{cx_0}$. Let us consider the discriminant stratum obtained by letting $x_3 \mapsto -x_1$, that is, $a_3 \mapsto a_1$ in the notation of (7.3.6). Thus,

$$\lambda_D = \frac{e^{cx_0}(\mu - e^{-x_1})^2(\mu - e^{x_1-x_2})(\mu - e^{x_1+x_2})}{\mu}. \tag{7.3.35}$$

Using (7.3.5a), and taking the determinant we obtain:

$$\det(\eta_D(x)) = -\frac{128}{81}c^2(e^{2x_2}-1)^2(e^{2x_1+x_2}-1)^3(e^{2x_1-x_2}-1)^3 e^{3cx_0-5x_1-2x_2}. \tag{7.3.36}$$

The discriminant strata corresponds to $m_i = 1 + \delta_{i1}$, and $d \equiv \dim(D) - 1 = 2$. Hence, from Theorem 7.3.10 and the proof of Theorem 7.1.4, we have

$$\det(\eta_D) = \tilde{\kappa} \cdot e^{3cx_0}(\xi_1\xi_2\xi_3)^{-1}(\xi_1 - \xi_2)^3(\xi_1 - \xi_3)^3(\xi_2 - \xi_3)^2, \tag{7.3.37}$$

with

$$\tilde{\kappa} = \frac{\beta^2 c^2}{(l+1-k)^2} \cdot \frac{(-1)^{k+1+\sum_{i=1}^{d+1} im_i}}{(l+1-k)^{l-k}} \prod_{a=1}^{d} \frac{m_a^{m_a+1}}{k^{k+1}} = \frac{\beta^2 c^2}{3^2} \cdot \frac{(-1)^{1+1+2+2+3}}{(3+1-1)^{3-1}} \frac{2^3 \cdot 1^2 \cdot 1^2}{1^2} = -\beta^2 c^2 \frac{2^3}{3^4}.$$

Recall that $\beta$ is the determinant of the constant matrix $P = (\alpha_{ij}) \equiv \partial_{x_j}(\xi_i)/\xi_i$ from the proof of Theorem 7.1.4, which in this case is of the form

$$P = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{pmatrix} \implies \det(P) \equiv \beta = 4, \tag{7.3.38}$$

giving $\tilde{\kappa} = -c^2 \frac{2^7}{3^4}$. Writing (7.3.37) in terms of $\{x_i\}_{i=0,1,2}$ then gives

$$
\begin{aligned}
\det(\eta_D) &= -c^2 \frac{2^7}{3^4} e^{3cx_0} \cdot e^{-x_1}(e^{-x_1} - e^{x_1-x_2})^3(e^{-x_1} - e^{x_1+x_2})^3(e^{x_1-x_2} - e^{x_1+x_2})^2 \\
&= -c^2 \frac{2^7}{3^4} e^{3cx_0-5x_1-2x_2}(e^{2x_2}-1)^2(e^{2x_1-x_2}-1)^3(e^{2x_1+x_2}-1)^3,
\end{aligned}
$$

which precisely matches (7.3.36).

### $\mathcal{R} \in \{B_l, C_l, D_l\}$

In this section, we prove Theorem 7.1.4 for the remaining classical Dynkin types; $B_l, C_l, D_l, G_2$, including non-canonical marking.

**Superpotentials**

As for $\mathcal{R} = A_l$, we want to write the LG-superpotentials for $\mathcal{R} = B_l, C_l, D_l$ in terms of linear coordinates $x$, or in other words in flat coordinates for the second metric, $g$, using (7.3.1). Doing so the superpotentials take the form

$$\lambda(\mu) = \frac{(-1)^{k+1}w_0}{\mu^k(\mu+1)^{k_1}(\mu-1)^{k_2}} \prod_{j=1}^{l}(\mu - a_j)(\mu - a_j^{-1}), \tag{7.3.39}$$

where $w_0 = e^{c_\omega x_0}$, $(k, k_1, k_2) = (l-1, 2, 0), (l, 0, 0), (l-2, 2, 2)$, for $B_l, C_l, D_l$, respectively. Again, we can see that this form is correct simply by factorising $[\lambda^0]\mathcal{P}$ (and checking the pole orders and positions).

Furthermore,

$$a_i = \begin{cases} e^{x_1}, & i = 1 \\ e^{x_i - x_{i-1}}, & i = 2, \cdots, \widehat{k+1}, \cdots, l \\ e^{2x_l - x_{l-1}}, & \mathcal{R} = B, i = l \\ e^{x_l + x_{l-1} - x_{l-2}}, & \mathcal{R} = D, i = l-1. \end{cases}$$

In [44], we have that BCD superpotentials (including non-canonical) are given by,

$$\lambda(P) = \frac{a_0}{(P^2-1)^m P^{2n}} \prod_{j=1}^{l}(P^2 - p_j^2). \tag{7.3.40}$$

The equivalence between (7.3.40) and (7.3.39) is given by $\mu = e^{2i\phi}$ with $P = \cos\phi$. Consequently, $p_j^2 = \frac{a_j + a_j^{-1} + 2}{4}$, $l = k + m + n$, $k_1 = 2n$, $k_2 = 2m$, and $a_0 = (-1)^m 4^k \tilde{a}_0$, with $\tilde{a}_0 \equiv (-1)^{k+1}w_0$:

$$\lambda(P) = \frac{(-1)^m a_0}{(\sin^2(\phi))^m(\cos(\phi))^{2n}} \prod_{j=1}^{l}\left(\cos^2(\phi) - p_j^2\right)$$

$$= \frac{2^{2(n+m)}(-1)^m a_0}{e^{-(m+n)2i\phi}(e^{2i\phi}+1)^{2n}(e^{2i\phi}-1)^{2m}} \prod_{j=1}^{l}(4e^{2i\phi})^{-1}(e^{2i\phi} - a_j)(e^{2i\phi} - a_j^{-1})$$

$$= \frac{(-1)^m 2^{2(n+m-l)}a_0}{e^{2i\phi(l-n-m)}(e^{2i\phi}+1)^{2n}(e^{2i\phi}-1)^{2m}} \prod_{j=1}^{l}(e^{2i\phi} - a_j)(e^{2i\phi} - a_j^{-1}).$$

**Discriminant Strata**

Again, the discriminant is given by the set

$$\Sigma = \{\lambda | \lambda(q) = 0 \text{ for some critical point } q\}.$$

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From (7.3.39) it is easy to see that the zeros of $\lambda(\mu)$ are given by $\mu = a_i^\epsilon$ for some $a \in \{1, \cdots, l\}$ with $\epsilon \in \{1, -1\}$. Furthermore,

$$
\lambda'(P)\Big|_{P^2 \to p_a^2} = a_0 \overbrace{\frac{\mathrm{d}}{\mathrm{d}p}\left(\frac{1}{(P^2-1)^m P^{2n}}\right) \prod_{j=1}^{l}(P^2 - p_j^2)}^{=0}\Bigg|_{P^2 \to p_a^2}
$$
$$
+ \frac{a_0}{(P^2-1)^m P^{2n}} \sum_{j=1}^{l} 2P \prod_{\substack{i=1 \\ i \neq j}}^{l}(P^2 - p_j^2)\Bigg|_{P^2 \to p_a^2}.
$$

For this to equal zero we must have that $p_a = p_i$ for at least one $i \in \{1, \cdots, l\}\backslash\{a\}$, unless $n = 0$, in which case we can also have $p_a = 0$.

Therefore, similarly to the $A$-case, we get that a discriminant strata $D$ is determined by the following equations;

$$
p_1 = \cdots = p_{m_0} = 0
$$
$$
\epsilon_1 p_{m_0+1} = \cdots = \epsilon_{m_1} p_{m_0+m_1} = \xi_1
$$
$$
\epsilon_{m_1+1} p_{m_0+m_1+1} = \cdots = \epsilon_{m_1+m_2} p_{m_0+m_1+m_2} = \xi_2
$$
$$
\vdots
$$
$$
\epsilon_{\sum_{j=1}^{d-1} m_j+1} p_{\sum_{j=0}^{d-1} m_j+1} = \cdots = \epsilon_{\sum_{j=1}^{d} m_j} p_{\sum_{j=0}^{d} m_j} = \xi_d,
$$

where $\epsilon_i \in \{\pm 1\}$, for all $i$, $m_0 \in \mathbb{N} \cup \{0\}$, with $m_0 = 0$ whenever $n \neq 0$, $m_j \in \mathbb{N}$ for all $j \in \{1, \cdots, d\}$, and $\sum_{j=1}^{d} m_j = l - m_0$. Note that the dimension of $D$ is $d+1$. In fact, we will use $\{\xi_i\}_{i=1,\cdots,d} \cup \{a_0\}$ as a basis for $D$. For simplicity we shall omit writing $\epsilon_i$.

Hence,

$$
\lambda_D(P) = \frac{a_0}{(P^2-1)^m P^{2(n-m_0)}} \prod_{i=1}^{d}(P^2 - \xi_i^2)^{m_i}, \tag{7.3.41}
$$

is the restriction of $\lambda(P)$ to the stratum $D$.

**Lemma 7.3.11.**

$$
\lambda_D'(P) = \frac{2ka_0}{(P^2-1)^{m+1} P^{2(n-m_0)+1}} \prod_{i=1}^{d}(P^2 - \xi_i^2)^{m_i-1} \prod_{i=0}^{d}(P^2 - q_i^2), \tag{7.3.42}
$$

where $q_i \in \mathbb{C}$.

*Proof.*

$$\lambda'_D(P) = \frac{a_0}{(P^2 - 1)^{m+1} P^{2(n-m_0)+1}} \prod_{i=1}^{d} (P^2 - \xi_i^2)^{m_i - 1}$$

$$\cdot \underbrace{\left( (-2mP^2 - 2(n - m_0)(P^2 - 1)) \prod_{i=1}^{d} (P^2 - \xi_i^2) + 2P^2 (P^2 - 1) \sum_{j=1}^{d} m_j \prod_{\substack{i=1 \\ i \neq j}}^{d} (P^2 - \xi_i^2) \right)}_{G(P)}.$$

Notice that $G(P)$ is an even polynomial in $P$ of degree $2(d+1)$, with $[P^{2(d+1)}] = 2(-m - n + m_0 + \sum_{j=1}^{d} m_j) = 2(-m - n + m_0 + l - m_0) = 2k$.

$$\lambda'_D(P) = \frac{2ka_0}{(P^2 - 1)^{m+1} P^{2(n-m_0)+1}} \prod_{i=1}^{d} (P^2 - \xi_i^2)^{m_i - 1} \prod_{i=0}^{d} (P^2 - q_i^2),$$

where $q_i, i = 0, \cdots, d$ are the critical points of $\lambda_D$. $\qquad\square$

**Lemma 7.3.12.**

$$\lambda''_D(q_r) = \frac{4ka_0}{q_r^{2(n-m_0)} (q_r^2 - 1)^{m+1}} \prod_{i=1}^{d} (q_r^2 - \xi_i^2)^{m_i - 1} \prod_{\substack{i=0 \\ i \neq r}}^{d} (q_r^2 - q_i^2). \tag{7.3.43}$$

*Proof.* Using Lemma 7.3.11, we have

$$\lambda''_D(q_r) = \frac{\partial}{\partial P} \left( \lambda'_D(P) \right) \Big|_{P \to q_r} = 2ka_0 \frac{\partial}{\partial P} \left( \frac{\prod_{i=1}^{d} (P^2 - \xi_i^2)^{m_i - 1}}{(P^2 - 1)^{m+1} P^{2(n-m_0)+1}} \right) \overbrace{\prod_{i=0}^{d} (P^2 - q_i^2)}^{= 0} \Big|_{P \to q_r}$$

$$+ \frac{2ka_0 \prod_{i=1}^{d} (P^2 - \xi_i^2)^{m_i - 1}}{(P^2 - 1)^{m+1} P^{2(n-m_0)+1}} \sum_{j=0}^{d} 2P \prod_{\substack{i=0 \\ i \neq j}}^{d} (P^2 - q_i^2) \Big|_{P \to q_r}$$

$$= \frac{4ka_0 \prod_{i=1}^{d} (P^2 - \xi_i^2)^{m_i - 1}}{(P^2 - 1)^{m+1} P^{2(n-m_0)}} \prod_{\substack{i=0 \\ i \neq r}}^{d} (P^2 - q_i^2) \Big|_{P \to q_r},$$

hence, evaluation gives the lemma. $\qquad\square$

As usual, we have the following

**Lemma 7.3.13.**

$$\partial_{u_i} \lambda_D(P) \Big|_{P = q_j} = \delta_{ij}. \tag{7.3.44}$$

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The proof of this Lemma is identical to the proof of Lemma 4.2 in [1].

Furthermore,

**Proposition 7.3.14.**

$$\partial_{u_r} \lambda_D(P) = \frac{2P(P^2-1)}{(q_r^2-1)(P^2-q_r^2)} \frac{\lambda_D'(P)}{\lambda_D''(q_r)}. \tag{7.3.45}$$

*Proof.* Differentiating (7.3.41), we find

$$\partial_{u_r}(\lambda_D(P)) = \frac{a_0 \prod_{i=1}^d (P^2-\xi_i^2)^{m_i-1}}{P^{2(n-m_0)}(P^2-1)^m} \left( \frac{\partial_{u_r}(a_0)}{a_0} \prod_{i=1}^d (P^2-\xi_i^2) + \sum_{j=1}^d m_j(-2\xi_j \partial_{u_r}(\xi_j)) \prod_{\substack{i=1 \\ i\neq j}}^d (P^2-\xi_i^2) \right)$$

$$= \frac{a_0 \prod_{i=1}^d (P^2-\xi_i^2)^{m_i-1}}{P^{2(n-m_0)}(P^2-1)^m} F(P;r).$$

Note that $F(P;r)$ is an even polynomial in $P$ of degree $2d$, and thus is fully determined by its values on the critical points of $\lambda_D$, $\{q_i\}$, through the Lagrange interpolation formula[§].

From Lemma 7.3.13, we have that

$$F(q_i;r) = \frac{\delta_{ri} q_i^{2(n-m_0)} (q_i^2-1)^m}{a_0 \prod_{j=1}^d (q_i^2-\xi_j^2)^{m_j-1}}.$$

Then, $F(P;r) = F_r(P;r)$, with

$$F_s(P;r) = F(q_s;r) \prod_{\substack{i=0 \\ i\neq s}}^d \frac{P^2-q_i^2}{q_r^2-q_i^2}.$$

Hence,

$$\partial_{u_r}(\lambda_D(P)) = \frac{\prod_{i=1}^d (P^2-\xi_i^2)^{m_i-1}}{P^{2(n-m_0)}(P^2-1)^m} \frac{q_r^{2(n-m_0)}(q_r^2-1)^m}{\prod_{j=1}^d (q_r^2-\xi_j^2)^{m_j-1}} \prod_{\substack{i=0 \\ i\neq r}}^d \frac{P^2-q_i^2}{q_r^2-q_i^2}. \tag{7.3.46}$$

Furthermore, using Lemmas 7.3.11 and 7.3.12,

$$\frac{2P}{(P^2-q_r^2)} \frac{\lambda_D'(P)}{\lambda_D''(q_r)} = \frac{2P}{(P^2-q_r^2)} \frac{2ka_0 \prod_{i=1}^d (P^2-\xi_i^2)^{m_i-1}}{(P^2-1)^{m+1} P^{2(n-m_0)+1}} \prod_{\substack{i=0 \\ i\neq r}}^d (P^2-q_i^2)$$

$$\cdot \frac{q_r^{2(n-m_0)}(q_r^2-1)^{m+1}}{4ka_0 \prod_{i=1}^d (q_r^2-\xi_i^2)^{m_i-1}} \left( \prod_{\substack{i=0 \\ i\neq r}}^d (q_r^2-q_i^2) \right)^{-1}.$$

---

Simplification of the above expression gives precisely (7.3.46).

$\square$

Now, let us use the results obtained thus far to describe the change of coordinates between the $\xi$-frame and canonical coordinates.

**Proposition 7.3.15.** Let $\xi_0 \equiv a_0$. Then,

$$\partial_{u_r}(\xi_a) = \begin{cases} \dfrac{2\xi_a}{(q_r^2 - \xi_a^2)} \dfrac{(\xi_a^2 - 1)}{(q_r^2 - 1)\lambda_D''(q_r)}, & a \neq 0, \\[4mm] a_0 \left( \dfrac{\delta_{r0}}{\lambda_D(q_0)} + \dfrac{4}{(q_r^2 - 1)\lambda_D''(q_r)} \left( \sum\limits_{j=1}^{d} \dfrac{m_j \xi_j^2(\xi_a^2 - 1)}{(q_0^2 - \xi_j^2)(q_r^2 - \xi_j^2)} \right) \right), & a = 0. \end{cases} \tag{7.3.47}$$

*Proof.*   • <u>$a \neq 0$</u>:

$$\partial_{u_r}(\lambda_D(\phi)) = \lambda_D(P)\frac{\partial_{u_r}(a_0)}{a_0} - 2\sum_{j=1}^{d} m_j \xi_j \partial_{u_r}(\xi_j)\frac{\lambda_D(P)}{(P^2 - \xi_j^2)}.$$

Setting the above expression equal to (7.3.45), dividing both sides by $(P^2 - \xi_a^2)^{m_a - 1}$ and letting $P \to \xi_a$ gives:

$$\overbrace{\frac{\lambda_D(P)}{(P^2 - \xi_a^2)^{m_a - 1}} \frac{\partial_{u_r}(a_0)}{a_0}\bigg|_{P \to \xi_a}}^{= 0} - 2\sum_{j=1}^{d} m_j \xi_j \partial_{u_r}(\xi_j)\frac{\lambda_D(P)}{(P^2 - \xi_j^2)(P^2 - \xi_a^2)^{m_a - 1}}\bigg|_{P \to \xi_a}$$

$$= \frac{2P(P^2 - 1)}{(P^2 - q_r^2)(q_r^2 - 1)\lambda_D''(q_r)} \frac{\lambda_D'(P)}{(P^2 - \xi_a^2)^{m_a - 1}}\bigg|_{P \to \xi_a},$$

and since

$$\frac{\lambda_D(P)}{(P^2 - \xi_a^2)^{m_a}}\bigg|_{P \to \xi_a} = \frac{\lambda_D'(P)}{(P^2 - \xi_a^2)^{m_a - 1}}\bigg|_{P \to \xi_a} \cdot \frac{1}{2m_a \xi_a},$$

we find

$$\partial_{u_r}(\xi_a) = -\frac{2\xi_a}{(\xi_a^2 - q_r^2)} \frac{(\xi_a^2 - 1)}{(q_r^2 - 1)\lambda_D''(q_r)}.$$

• <u>$a = 0$</u>:

Similarly,

$$\begin{aligned} \frac{\partial_{u_r}(\lambda_D(\phi))}{\lambda_D(P)} &= \frac{\partial_{u_r}(a_0)}{a_0} - 2\sum_{j=1}^{d} \frac{m_j \xi_j \partial_{u_r}(\xi_j)}{P^2 - \xi_j^2} \\ &= \frac{\partial_{u_r}(a_0)}{a_0} - 2\sum_{j=1}^{d} \frac{m_j \xi_j}{(P^2 - \xi_j^2)} \frac{2\xi_j(\xi_j^2 - 1)}{(\xi_j^2 - q_r^2)} \frac{1}{(q_r^2 - 1)\lambda_D''(q_r)}, \end{aligned}$$

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where we have used the already proven expression for $a \neq 0$.

Letting $P \to q_0$, and using Lemma 7.3.13 gives

$$\frac{\delta_{r0}}{u_0} = \frac{\partial_{u_r}(a_0)}{a_0} + \frac{4}{(q_r^2 - 1)\lambda_D''(q_r)} \sum_{j=1}^{d} \frac{m_j \xi_j^2 (\xi_j^2 - 1)}{(q_0^2 - \xi_j^2)(q_r^2 - \xi_j^2)}, \tag{7.3.48}$$

and by rearranging we obtain the statement.

$\square$

Let us now find the Saito metric in $\{u_i\}$-coordinates, as well as showing that these are indeed orthonormal in the tangent space, i.e. they act as canonical coordinates.

**Lemma 7.3.16.**
$$\eta_D(\partial_{u_i}, \partial_{u_j}) = \frac{(-1)^k \delta_{ij}}{\lambda_D''(q_i)} \frac{2}{(q_i^2 - 1)}, \tag{7.3.49}$$

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij} \partial_{u_j}. \tag{7.3.50}$$

*Proof.* The restriction of the $\eta$-metric to the stratum $D$ is defined as

$$\eta_D(\partial_{u_i}, \partial_{u_j}) := (-1)^k \sum_{s=0}^{d} \operatorname*{Res}_{p=q_s} \frac{\partial_{u_i}(\lambda_D(P)) \partial_{u_j}(\lambda_D(P))}{(P^2 - 1)\lambda_D'(P)} dP. \tag{7.3.51}$$

By Lemma 7.3.14, this becomes

$$\frac{4(-1)^k}{(q_i^2 - 1)(q_j^2 - 1)\lambda_D''(q_i)\lambda_D''(q_j)} \sum_{s=0}^{d} \operatorname*{Res}_{P=q_s} \frac{P^2(P^2 - 1)^2 \lambda_D'(P)}{(P^2 - q_i^2)(P^2 - q_j^2)(P^2 - 1)} dP.$$

From Lemma 7.3.11, it is clear that the residues will be zero unless $i = j$, in which case we find

$$\frac{4(-1)^k q_i}{\lambda_D''(q_i)^2} \frac{2k a_0}{(q_i^2 - 1)^{m+2} q_i^{2(n-m_0)+1}} \prod_{j=1}^{d} (q_i^2 - \xi_j^2)^{m_j - 1} \prod_{\substack{j=0 \\ j \neq i}}^{d} (q_i^2 - q_j^2) = \frac{(-1)^k}{\lambda_D''(q_i)} \frac{2}{(q_i^2 - 1)}.$$

In order to prove (7.3.50) we consider the $c$-tensor, (7.3.5b) (in $P$), in canonical coordinates. Again, by Lemma 7.3.14, we have that the residues vanish unless $i = j = k$. Thus, the non-zero elements are given by

$$c_D(\partial_{u_i}, \partial_{u_i}, \partial_{u_i}) = (-1)^k \sum_{s=0}^{d} \operatorname*{Res}_{P=q_s} \frac{(\partial_{u_i} \lambda_D)^3}{\lambda_D'(P)(P^2 - 1)} dP = \frac{(-1)^k}{\lambda_D''(q_i)^3} \sum_{s=0}^{d} \operatorname*{Res}_{P=q_s} \frac{8P^3 \lambda_D'(P)^2}{(P^2 - q_i^2)^3 (P^2 - 1)} dP.$$

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By Lemma 7.3.11, we find

$$c_D(\partial_{u_i}, \partial_{u_i}, \partial_{u_i}) = \frac{8(-1)^k}{(q_i^2-1)^3\lambda_D''(q_i)^3}\sum_{s=0}^{d}\operatorname*{Res}_{P=q_s}\frac{P(P^2-1)^3(2ka_0)^2}{(P^2-q_i^2)(P^2-1)^{2m+3}P^{4(n-m_0)}}$$

$$\cdot\prod_{j=1}^{d}(P^2-\xi_i^2)^{2m_i-2}\prod_{\substack{j=0\\j\neq i}}^{d}(P^2-q_j^2)^2 dP$$

$$= \frac{2(-1)^k}{\lambda_D''(q_i)^3}\frac{1}{(q_i^2-1)}\frac{(4ka_0)^2}{(q_i^2-1)^{2m+2}q_i^{4(n-m_0)}}\prod_{j=1}^{d}(q_i^2-\xi_j^2)^{2m_i-2}\prod_{\substack{j=0\\j\neq i}}^{d}(q_i^2-q_j^2)^2$$

$$= \frac{(-1)^k}{\lambda_D''(q_i)}\frac{2}{(q_i^2-1)}.$$

This (and comparing to (7.3.49)) implies the statement by the nondegeneracy of $\eta_D$ and Proposition 7.2.2. $\qquad\square$

Now, before we try to calculate the determinant of $\eta_D$, we will obtain some lemmas which will prove very useful in the subsequent manipulations.

**Lemma 7.3.17.**

$$\prod_{i=0}^{d}q_i^2 = \frac{(m_0-n)}{k}\prod_{a=1}^{d}\xi_a^2. \tag{7.3.52}$$

*Proof.*

$$\lambda_D'(P) = a_0\frac{d}{dP}\left(\frac{1}{(P^2-1)^mP^{2(n-m_0)}}\right)\prod_{a=1}^{d}(P^2-\xi_a^2)^{m_a}$$

$$+\frac{a_0}{(P^2-1)^mP^{2(n-m_0)}}\frac{d}{dP}\left(\prod_{a=1}^{d}(P^2-\xi_a^2)^{m_a}\right)$$

$$= \frac{a_0}{(P^2-1)^{m+1}P^{2(n-m_0)+1}}\prod_{a=1}^{d}(P^2-\xi_a^2)^{m_a-1}((-2mP^2-2(n-m_0)(P^2-1))\prod_{a=1}^{d}(P^2-\xi_a^2)+$$

$$+P(P^2-1)\frac{d}{dP}(\prod_{a=1}^{d}(P^2-\xi_a^2)^{m_a}))$$

$$= \frac{2ka_0}{(P^2-1)^{m+1}P^{2(n-m_0)+1}}\prod_{i=1}^{d}(P^2-\xi_i^2)^{m_i-1}\prod_{i=0}^{d}(P^2-q_i^2).$$

Simplifying and comparing the constant term gives:

$$\prod_{a=1}^{d}(-\xi_a^2)^{m_a-1}2(n-m_0)\prod_{a=1}^{d}(-\xi_a^2) = 2k\prod_{i=1}^{d}(-\xi_i^2)^{m_i-1}\prod_{i=0}^{d}(-q_i^2). \tag{7.3.53}$$

$$\square$$

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**Lemma 7.3.18.**

$$\prod_{i=0}^{d}(\xi_a^2 - q_i^2) = \frac{\xi_a^2 m_a(\xi_a^2 - 1)}{k}\prod_{\substack{i=1\\i\neq a}}^{d}(\xi_a^2 - \xi_i^2).\tag{7.3.54}$$

*Proof.* From Lemma 7.3.11, we have

$$\begin{aligned}
\prod_{i=0}^{d}(\xi_a^2 - q_i^2) &= \frac{(P^2-1)^{m+1}P^{2(n-m_0)+1}}{2ka_0}\frac{\lambda_D'(P)}{\prod_{i=1}^{d}(P^2-\xi_i^2)^{m_i-1}}\bigg|_{P\to\xi_a}\\
&= \frac{(\xi_a^2-1)^{m+1}\xi_a^{2(n-m_0)+1}}{2ka_0}\frac{2\xi_a m_a\lambda_D(P)}{(P^2-\xi_a^2)^{m_a}\prod_{\substack{i=1\\i\neq a}}^{d}(P^2-\xi_i^2)^{m_i-1}}\bigg|_{P\to\xi_a},
\end{aligned}$$

where we have used L'Hospital's rule in reverse to obtain the second equality. Inputting (7.3.41) gives the lemma. $\square$

**Lemma 7.3.19.**

$$\prod_{i=0}^{d}(q_i^2 - 1) = \frac{m}{k}\prod_{a=1}^{d}(\xi_a^2 - 1).\tag{7.3.55}$$

*Proof.* This proof starts the same way as the proof of Lemma 7.3.17, only now we let $P \to 1$ after the differentiation, instead of 0. This gives

$$\lambda_D'(P)\big|_{P\to 1} = (-2m)\prod_{a=1}^{d}(1-\xi_a^2) = 2(l-m-n)\prod_{i=0}^{d}(1-q_i^2).$$

$\square$

**Lemma 7.3.20.**

$$z := \frac{\prod_{i=0}^{d}\lambda_D''(q_i)}{\prod_{\substack{i=0\\i<j}}^{d}(q_i^2-q_j^2)^2} = C\cdot a_0^{d+1}\prod_{a=1}^{d}\xi_a^{2(m_a-1-n+m_0)}\prod_{a=1}^{d}(\xi_a^2-1)^{m_a-m-2}\prod_{a=1}^{d}\prod_{\substack{i=1\\i\neq a}}^{d}(\xi_a^2-\xi_i^2)^{m_a-1},\tag{7.3.56}$$

where $C = \dfrac{4^{d+1}k^{2(d+1)}(-1)^{\frac{d(d+1)}{2}+(d+1)l-n-m_0}\prod_{a=1}^{d}m_a^{m_a-1}}{(n-m_0)^{n-m_0}m^{m+1}k^k}.$

*Proof.* Using Lemmas 7.3.12, 7.3.17, 7.3.19 and 7.3.19;

$$z = \prod_{r=0}^{d} \left( \frac{4ka_0}{q_r^{2(n-m_0)}(q_r^2-1)^{m+1}} \prod_{i=1}^{d}(q_r^2-\xi_i^2)^{m_i-1} \prod_{\substack{i=0 \\ i \neq r}}^{d}(q_r^2-q_i^2) \right) \left( \prod_{\substack{i=0 \\ i<j}}^{d}(q_i^2-q_j^2)^2 \right)^{-1}$$

$$= \frac{\prod_{r=0}^{d}(4ka_0)}{\left( \frac{(m_0-n)\prod_{a=1}^{d}\xi_a^2}{k} \right)^{n-m_0} \left( \frac{m\prod_{a=1}^{d}(\xi_a^2-1)}{k} \right)^{m+1}} (-1)^{\frac{d(d+1)}{2}} \prod_{r=0}^{d}\prod_{i=1}^{d}(q_r^2-\xi_i^2)^{m_i-1}$$

$$= \frac{4^{d+1}(-1)^{\frac{d(d+1)^2}{2}+l-n+d(d+1)+d\sum_{j=1}^{d}m_j}k^{2(d+1)}}{m^{m+1}(n-m_0)^{n-m_0}k^k} \prod_{a=1}^{d}m_a^{m_a-1}$$

$$\cdot a_0^{d+1} \prod_{a=1}^{d}\xi_a^{2(m_a-1-n+m_0)} \prod_{a=1}^{d}(\xi_a^2-1)^{m_a-m-2} \prod_{a=1}^{d}\prod_{\substack{i=1 \\ i \neq a}}^{d}(\xi_a^2-\xi_i^2)^{m_a-1}.$$

Note that if $m = 0$ or $m_0 = n = 0$, then the resulting expression for $z$ is simply the above expression multiplied by $\left( \frac{m\prod(\xi_a^2-1)}{k} \right)^{m+1}$, or $\left( \frac{(m-n_0)\prod\xi_a^2}{k} \right)^{n-m_0}$, respectively. □

We are now ready to prove Theorem 7.1.4 for $\mathcal{R} = B_l, C_l, D_l$, i.e. to complete the proof of this theorem for classical Dynkin types.

**Theorem 7.3.21.**

$$\det(\eta(x)) = \rho\alpha a_0^{d+1} \cdot \prod_{a=1}^{d}\xi_a^{2(m_a+m_0-n)}(\xi_a^2-1)^{m_a-m} \prod_{1 \leqslant a < b \leqslant d}(\xi_a^2-\xi_b^2)^{m_a+m_b}, \tag{7.3.57}$$

where $\rho = \dfrac{a_0^{d+1}(-1)^{-d^2+(d+1)(l+k)-n-m_0+\sum_j jm_{j+1}}\prod_{a=1}^{d}m_a^{m_a+1}}{2^{d+3}k^{k-1}m^m(n-m_0)^{n-m_0}}$, $\alpha = c^2\frac{\partial\hat{\xi}_d}{\partial x_d}$, and if $m = 0$ or $n = m_0 = 0$, one simply removes the corresponding singular factor.

*Proof.*

$$\det(\eta(\xi)) = (\det(B))^{-2}\det(\eta(u)), \tag{7.3.58}$$

where $B = (\partial_{u_r}(\xi_a))$. Then, $\det(B) = \sum_{r=0}^{d}(-1)^r B_{r0}\det(J_r)$, where $J_r$ is the $(d \times d)$ matrix $(\partial_{u_i}\xi_a)$, with $a = 1, \cdots, d$, and $i = 0, \cdots, \hat{r}, \cdots, d$. Hence, by Proposition 7.3.15, $\det(B)$ can be expressed as

$$\frac{\det(B)}{a_0} = \frac{\det(J_0)}{\lambda_D(q_0)} + \sum_{r=0}^{d}\frac{4(-1)^r}{\lambda_D''(q_r)(q_r^2-1)} \left( \sum_{j=1}^{d}\frac{m_j\xi_j^2(\xi_j^2-1)}{(q_0^2-\xi_j^2)(q_r^2-\xi_j^2)} \right) \det(J_r),$$

where

$$\det(J_r) = \frac{(-1)^d 2^d \prod_{a=1}^{d}\xi_a(\xi_a^2-1)}{\prod_{\substack{i=0 \\ i \neq r}}^{d}\lambda_D''(q_i)(q_i^2-1)} \cdot \det(A),$$

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and $A$ is the $d \times d$ matrix with entries $\dfrac{1}{\xi_a^2 - q_i^2}$, where $a = 1, \cdots, d$, $i = 0, \cdots, \hat{r}, \cdots, d$. The determinant of $A$ is the Cauchy determinant

$$\frac{(-1)^{\frac{d(d-1)}{2}} \prod_{\substack{s=1 \\ s<j}}^{d}(\xi_s^2 - \xi_j^2) \prod_{\substack{s=0 \\ s<j,\, s,j\neq r}}^{d}(q_s^2 - q_j^2)}{\prod_{\substack{a=1,j=0 \\ j\neq r}}^{d}(\xi_a^2 - q_j^2)}.$$

Combining, we find

$$\frac{\det(B)}{a_0} - \frac{\det(J_0)}{\lambda_D(q_0)} = (-1)^{\frac{d(d+1)}{2}} 2^{d+2} \prod_{a=1}^{d} \xi_a(\xi_a^2 - 1) \prod_{i<j}(\xi_i^2 - \xi_j^2) \sum_{r=0}^{d}$$

$$\cdot \frac{(-1)^r \prod_{\substack{s<j \\ s\neq r}}(q_s^2 - q_j^2)}{\prod_{i=0}^{d}(q_i^2 - 1)\lambda_D''(q_i) \prod_{a=1}^{d} \prod_{\substack{i=0 \\ i\neq r}}^{d}(\xi_a^2 - q_i^2)} \sum_{j=1}^{d} \frac{m_j \xi_j(\xi_j^2 - 1)}{(\xi_j^2 - q_0^2)(\xi_j^2 - q_r^2)}$$

$$= \frac{(-1)^{\frac{d(d+1)}{2}} 2^{d+2} \prod_{a=1}^{d} \xi_a(\xi_a^2 - 1) \prod_{i<j}(\xi_i^2 - \xi_j^2) \sum_{j=1}^{d} m_j \xi_j(\xi_j^2 - 1)}{\prod_{i=0}^{d}(q_i^2 - 1)\lambda_D''(q_i)}$$

$$\cdot \frac{\prod_{j=0}^{d} \prod_{s<j}(q_s^2 - q_j^2)}{\prod_{a=1}^{d} \prod_{i=0}^{d}(\xi_a^2 - q_i^2)} \sum_{r=0}^{d} \frac{\prod_{a=1}^{d}(\xi_a^2 - q_r^2)}{(\xi_j^2 - q_0^2)(\xi_j^2 - q_r^2) \prod_{i\neq r}(q_r^2 - q_i^2)}.$$

Let $z = P^2$, then

$$\frac{\lambda(z)}{z(z-1)\frac{d\lambda}{dz}} = \frac{\prod_{j=1}^{d}(z - \xi_j^2)}{k \prod_{i=0}^{d}(z - q_i^2)}.$$

Hence,

$$\sum_{r=0}^{d} \frac{\prod_{a=1}^{d}(q_r^2 - \xi_a^2)}{(q_0^2 - \xi_j^2)(z - \xi_j^2) \prod_{i\neq r}(q_r^2 - q_i^2)} = k \sum_{r=0}^{d} \frac{(z - q_r^2)\lambda(z)}{(q_0^2 - \xi_j^2)(z - \xi_j^2)z(z-1)\frac{d\lambda}{dz}}\Bigg|_{z=q_r}$$

$$= k \left( \sum_{r=1}^{d} \operatorname*{Res}_{z\to q_r} \frac{\lambda(z)}{(q_0^2 - \xi_j^2)(z - \xi_j^2)z(z-1)\frac{d\lambda}{dz}} dz + \right.$$

$$\left. + \operatorname*{Res}_{z\to q_0} \frac{\lambda(z)}{(z - \xi_j^2)(z - \xi_j^2)z(z-1)\frac{d\lambda}{dz}} dz \right)$$

$$= -k \left( \operatorname*{Res}_{z\to\infty} + \operatorname*{Res}_{z\to\xi_j^2} \right) \left( \frac{\lambda(z)dz}{(z - \xi_j^2)z(z-1)\frac{d\lambda}{dz}} \left( \frac{1}{(q_0^2 - \xi_j^2)} + \frac{1}{(z - \xi_j^2)} \right) \right)$$

$$= -k \operatorname*{Res}_{z\to\xi_j^2} \frac{\lambda(z)}{(z - \xi_j^2)^2 z(z-1)\frac{\lambda}{dz}} dz$$

$$= -\frac{k}{m_j \xi_j^2(\xi_j^2 - 1)}.$$

Using this,

$$\frac{\det(B)}{a_0} = \frac{\det(J_0)}{\lambda_D(q_0)} + \frac{(-1)^{\frac{d(d+1)}{2}}2^{d+2}\prod_{a=1}^d \xi_a(\xi_a^2-1)\prod_{i<j}(\xi_i^2-\xi_j^2)\prod_{\substack{i<j\\i\neq r}}(q_i^2-q_j^2)}{\prod_{i\neq r}(q_i^2-1)\lambda_D''(q_i)\prod_{a=1}^d\prod_{i\neq r}(\xi_a^2-q_i^2)}$$

Furthermore,

$$\frac{\det(J_0)}{\lambda_D(q_0)} = \frac{(-1)^{\frac{d(d+1)}{2}})2^{d+2}k\prod_{a=1}^d \xi_a(\xi_a^2-1)\prod_{\substack{s=1\\s<j}}^d(\xi_s^2-\xi_j^2)\prod_{\substack{s=0\\s<j}}^d(q_s^2-q_j^2)}{\prod_{r=0}^d \lambda_D''(q_r)(q_r^2-1)\prod_{\substack{a=1\\j=0}}^d(\xi_a^2-q_j^2)},$$

which gives,

$$\det(B) = \gamma \cdot \frac{\prod_{a=1}^d \xi_a(\xi_a^2-1)\prod_{\substack{s=1\\s<j}}^d(\xi_s^2-\xi_j^2)\prod_{\substack{s=0\\s<j}}^d(q_s^2-q_j^2)}{\prod_{r=0}^d \lambda_D''(q_r)(q_r^2-1)\prod_{a=1}^d\prod_{j=0}^d(\xi_a^2-q_j^2)},$$

where

$$\gamma = 2^{d+3}(-1)^{\frac{d(d+1)}{2}}ka_0.$$

Hence, by Lemma 7.3.20,

$$\det(\eta(\xi)) = (\det(B))^{-2}\frac{(-1)^{k(d+1)}2^{d+1}}{\prod_{i=0}^d \lambda_D''(q_i)(q_i^2-1)}$$

$$= \gamma^{-2}(-1)^{k(d+1)}2^{d+1} \cdot \frac{\prod_{a=1}^d\prod_{j=0}^d(\xi_a^2-q_j^2)^2}{\prod_{a=1}^d(\xi_a(\xi_a^2-1))^2\prod_{\substack{s=1\\s<j}}^d(\xi_s^2-\xi_j^2)^2}\frac{\prod_{r=0}^d \lambda_D''(q_r)(q_r^2-1)}{\prod_{\substack{s=0\\s<j}}^d(q_s^2-q_j^2)^2}$$

$$= \gamma^{-2}(-1)^{k(d+1)}2^{d+1} \cdot \frac{\prod_{r=0}^d(q_r^2-1)\prod_{a=1}^d\prod_{j=0}^d(\xi_a^2-q_j^2)^2}{\prod_{a=1}^d(\xi_a(\xi_a^2-1))^2\prod_{\substack{s=1\\s<j}}^d(\xi_s^2-\xi_j^2)^2} \cdot z$$

$$= \rho \cdot \frac{\prod_{a,j}(\xi_a^2-q_j^2)^2\prod_{a=1}^d \xi_a^{2(m_j-n+m_0-1)}(\xi_a^2-1)^{m_j-m-2}\prod\prod(\xi_j^2-\xi_i^2)^{m_j-1}\prod_{i=0}^d(q_i^2-1)}{\prod_{a=1}^d \xi_a^2(\xi_a^2-1)^2\prod_{i<j}(\xi_i^2-\xi_j^2)^2}$$

$$= \rho \cdot \frac{\prod_{a=1}^d \xi_a^{2(m_j-2-n+m_0)}(\xi_a^2-1)^{m_j-m-4}\prod_a\prod_{i\neq a}(\xi_a^2-\xi_b^2)^{m_a-1}}{\prod_{i<j}(\xi_i^2-\xi_j^2)^2}$$

$$\cdot \left(\frac{\xi_a^2(\xi_a^2-1)m_a\prod_{i\neq a}(\xi_a^2-\xi_i^2)}{k}\right)^2 \cdot \left(\frac{m}{k}\prod_{a=1}^d(\xi_a^2-1)\right)$$

$$= \rho \cdot \prod_{a=1}^d \xi_a^{2(m_j-n+m_0)}(\xi_a^2-1)^{m_j-m-1}\prod_{a<b}(\xi_a^2-\xi_b^2)^{m_a+m_b},$$

where

$$\rho = \frac{Ca_0^{d-1}2^{d+1}(-1)^{k(d+1)}}{\gamma^2}\frac{m\prod m_a^2}{k^{2d+1}}(-1)^{-\frac{d(d-1)}{2}+\sum_{j=1}^{d-1}jm_{j+1}}$$

$$= \frac{a_0^{d+1}2^{d-3}(-1)^{-d^2+(d+1)(l+k)-n-m_0+\sum_j jm_{j+1}}\prod_{a=1}^d m_a^{m_a+1}}{k^{k-1}m^m(n-m_0)^{n-m_0}}.$$

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Here we have used Lemmas 7.3.17–7.3.19, and the analogous fact to (7.3.22).

Now, as $p_j^2 - \dfrac{a_j + a_j^{-1} + 2}{4}$, we have that $\xi_j = \pm \dfrac{(\hat{\xi}_j + \hat{\xi}_j^{-1} + 2)^{\frac{1}{2}}}{2}$ for $j \neq 0$ and

$$
\frac{\partial \xi_j}{\partial x_i} = 
\begin{cases}
c\, a_0\, \delta_{i0} & j = 0, \\[2ex]
\pm \dfrac{1}{4} \dfrac{1}{(\hat{\xi}_j + \hat{\xi}_j^{-1} + 2)^{\frac{1}{2}}} \cdot \dfrac{\hat{\xi}_j^2 - 1}{\hat{\xi}_j^2} \cdot \dfrac{\partial \hat{\xi}_j}{\partial x_i} & \text{otherwise .}
\end{cases}
$$

Furthermore, from (7.3) we see that $(\partial_{x_i} \hat{\xi}_j)$ is a lower triangular matrix with $\beta_j \hat{\xi}_j$ at the $j^{\text{th}}$ diagonal entry, where

$$
\beta_j = 
\begin{cases}
\dfrac{\partial \hat{\xi}_d}{\partial x_d} & j = d, \\[2ex]
\hat{\xi}_j & \text{otherwise .}
\end{cases}
$$

Then, since

$$
\xi_j^2 - 1 = \frac{\hat{\xi}_j + \hat{\xi}_j^{-1} + 2}{4} - 1 = \frac{(\hat{\xi}_j - 1)^2}{4\hat{\xi}_j},
$$

we get that

$$
\frac{\partial \xi_j}{\partial x_i} = \frac{\beta_j}{2\xi_j}(\xi_j(\xi_j^2 - 1)^{\frac{1}{2}}).
$$

Hence,

$$
\det(\eta(\hat{\xi})) = \frac{a_0^2}{2^{2d}} \det(\eta(\xi)) \cdot \alpha(\xi_a^2 - 1),
$$

with $\alpha = \frac{\partial \hat{\xi}_d}{\partial x_d} c^2$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

*Proof of Theorems 7.1.4 and 7.1.5 for $\mathcal{R} = B_l, C_l, D_l$.* From Theorem 7.3.21, we already have that Theorem 7.1.4 holds. Furthermore, as for the A-case, we see the non-affine contribution arising as a factor. In [1], the authors obtain (Theorem 4.14)

$$
\det(\eta_D(\xi)) \propto \prod_{i=1}^{d} \xi_i^{2(m_i + m_0)} \prod_{i \leqslant i < j \leqslant d} (\xi_i^2 - \xi_j^2)^{m_i + m_j}. \tag{7.3.59}
$$

Thus, in order to complete the proof of Theorems 7.1.4 and 7.1.5 for classical Dynkin types, we must explain the origin of the multiplicities of the additional factors. By comparing the result to (7.3.41), we see indeed that they match precisely with the pole locations and orders of the appropriate superpotential. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

**Remark.** As this method works for meromorphic superpotentials (and not just holomorphic as in [1]), it is possible to perform a similar derivation for other superpotentials found in literature. For example, it should be straight-forward to prove an analogous result for the generalised extensions introduced in [96, 124].

## 7.4 Exceptional Dynkin types

**Discriminant strata for implicit superpotentials**

Let the spectral curve associated to $\mathcal{R}$ be given by

$$\mathcal{C}_\mathcal{R} : \mathcal{P}(\mu, \lambda; x) = \sum_{i=0}^{M} b_i \lambda^i = 0, \qquad (7.4.1)$$

where $M < \infty$ and $b_i$ are polynomials of $\mu, x$. By setting $\lambda = 0$, we see that the zeros of $\lambda$ coincide with the zeros of $b_0$. Moreover, $b_0$ may be factorised in $x$-coordinates as

$$b_0 = w_0 \frac{\prod_{i=0}^{\dim(\rho)}(\mu - a_i)}{\prod_{i=1}^{N}(\mu - \infty_i)^{n_i+1}}, \qquad (7.4.2)$$

for some $3 \leqslant N \leqslant 8$, and $a_i = e^{f_i}$, with $f_i$ being integral linear polynomial functions of $x$. Thus, the zeros of $\lambda$ are given by $\{a_i\}$. By the implicit function theorem we have

$$\partial_\mu \lambda = -\frac{\partial_\mu \mathcal{P}}{\partial_\lambda \mathcal{P}}, \qquad (7.4.3)$$

which means that the critical points of $\lambda$, i.e. the zeros of $\lambda'(\mu)$, are the zeros of $\partial_\mu \mathcal{P}$. Hence,

$$\partial_\mu \mathcal{P}\big|_{\lambda=0} = \partial_\mu b_0 \big|_{\mu=a_j} = \frac{w_0}{\prod_{b=1}^{N}(\mu - \infty_b)^{n_i+1}} \sum_i \prod_{i \neq j}(a_j - a_i) = \frac{w_0}{\prod_{b=1}^{N}(a_j - \infty_b)^{n_b+1}} \prod_{i \neq j}(a_j - a_i). \qquad (7.4.4)$$

Thus, what we saw for the classical cases makes up a more general fact; the discriminant strata are determined by the zeros of $b_0$, and $\lambda_D$ is still obtained by setting $a_i = a_j$ for some distinct pair $(i, j)$.

**Discriminant strata as duals to parabolic subalgebras**

While we do have a B–model description for the exceptional cases, and the discriminant strata may be determined similarly to the classical cases, the above method, and even its overall philosophy, proves unfeasible for exceptional Dynkin types. The reasons for this are the following. Firstly, in the polynomial defining the spectral curve, $\lambda$ appears now of order greater that one[*]. This results in multiple branches and non-rational expressions for $\lambda$, which complicates attempting the above method. More importantly, an expression for $\lambda_D''(q_i)$, which was the crux of the classical calculations, will be highly complicated, and not likely to be useful. Therefore, we are forced to

---

[*]When $\mathcal{R} = G_2$ we still have a genus zero spectral curve and $\lambda$ appearing of order one, so it could in theory be possible to perform a similar calculation for this case. It is not obvious, however, how to do this since the step consisting of using Lagrange interpolation is not automatically available due to the form of $\lambda_D$.

find an alternate route to the result. Instead, we want to take advantage of the simple form that the spectral curves take in $w$-coordinates. We want to write

$$\det(\eta(\xi)) = \left( \det \left( \frac{\partial \tilde{w}_i}{\partial \xi_j} \right) \right)^2 \det(\eta(\tilde{w})), \qquad (7.4.5)$$

where $\tilde{w}$ is the induced independent $w$-coordinates on the discriminant. It turns out that this is actually computable, and it relies on the fact that discriminant strata are unions of hyperplanes in the root space corresponding to Weyl walls [46]. Therefore, as these are determined by roots, the dual, $D^*$, of a discriminant, $D$, may be realised by the removal of the associated node(s) in the appropriate Dynkin diagram.

Recall that $a_i = e^{f_i(x)}$, meaning a discriminant strata is given by the relation $f_i = f_j$ for some $i \neq j$. In Example 24 we saw that the functions $f_i$ were determined by the weight system $\Gamma(\rho)$ (due to the weight system being a union of Weyl orbits).

This means that

$$f_i - f_j = \begin{cases} 0, & \text{if } a_i = a_j^{-1} \\ \alpha_k, & \text{otherwise,} \end{cases} \qquad (7.4.6)$$

for some $\alpha_k \in \mathcal{R}$, and that every $\alpha_k \in \mathcal{R}$ may be written as a difference $f_i - f_j$.

Hence, setting $a_i = a_j$ for $i \neq j$ is equivalent to setting $\alpha_k = 0$ for some $k \in \{1, \cdots, l_{\mathcal{R}}\}$, which again is equivalent to removing a node in the associated Dynkin diagram. In other words, we may characterise a discriminant strata, $D$, by the (potentially disconnected) Dynkin diagram found by removing the nodes corresponding to the relations determining $D$. While this is a general fact, it is quite easy to confirm it in the exceptional cases explicitly, which are the ones we shall employ this reasoning to. Let us do so for $\mathcal{R} = G_2$.

**Example 26** ($G_2$). The root system for $G_2$ is given by the vectors (in the $\alpha$-basis):

$$\pm\{(1,0), (0,1), (3,-1), (1,-1), (3,2), (2,-1), (0,0)\}. \qquad (7.4.7)$$

Furthermore, the functions $f_i$ appearing in the exponentials of the zeros of $b_0$ for $G_2$ are

$$\pm\{x_1, x_2 - x_1, x_2 - 2x_1\}. \qquad (7.4.8)$$

Setting $(a, b) \equiv ax_1 + bx_2$, it is easy to see that every difference of a pair in (7.4.8) gives an element in (7.4.7), and, conversely, that every element in (7.4.7) can be written as the difference of a pair of expressions in (7.4.8).

Thus, we may characterise all the discriminant strata for any given $\mathcal{R}$, by their dual root systems. See Figure 7.1 for the case of $\mathcal{R} = F_4$.
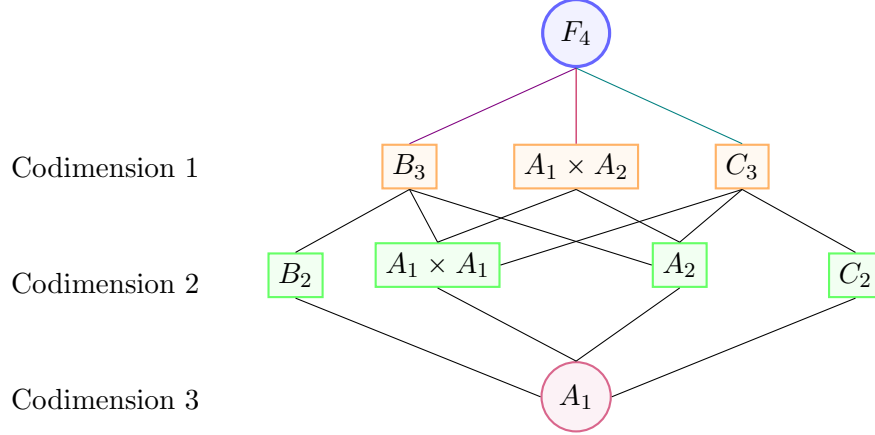
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Figure 7.1: $F_4$ discriminant strata. The purple, red, and teal lines represent removing $\alpha_4$, $\alpha_2$ or $\alpha_3$, $\alpha_1$, respectively from the Dynkin diagram associated to $F_4$ as shown in Figure 4.1.

More importantly, this gives us a way to compute natural curved coordinates $\tilde{w}$ on discriminant strata. Since,

$$w_i = \mathrm{Tr}_{\rho_i}\left(e^{h(x)}\right), \tag{7.4.9}$$

where $h(x) = \sum_{i=1}^l x_i h_i \in \mathrm{Cartan}(\mathfrak{g})$, with $\{h_i\}$ being Chevalley generators for $\mathcal{G}$, we get the corresponding expression on $D$

$$w_i = \mathrm{Tr}_{\rho_i}\left(e^{h(x)}\right)\Bigg|_{x\in D} = \sum_j n_{ij}\mathrm{Tr}_{\tilde{R}_j}e^{\tilde{h}(x)}, \tag{7.4.10}$$

where $\tilde{h}(x) \in \mathrm{Cartan}(\mathrm{Lie}(\mathcal{G}_D))$ with $\mathcal{G}_D$ being a Lie subgroup of $\mathcal{G}$, and $n_{ij} \in \mathbb{Z}_{\geqslant 0}$ the coefficients of the decomposition

$$\rho_i = \oplus_j n_{ij}\tilde{R}_j, \tag{7.4.11}$$

for $\tilde{R}_j \in \mathrm{Rep}(\mathcal{G}_D)$. Thus,

$$w_i = \sum_{ij} n_{ij}\tilde{Q}_j(\tilde{w}), \tag{7.4.12}$$

where $\tilde{Q}_j$ is polynomial in $\tilde{w}$ which are the $\dim(D)$ independent natural coordinates on $D$ given by

$$\tilde{w}_i = \mathrm{Tr}_{\tilde{\rho}_i}\left(e^{\tilde{h}(x)}\right). \tag{7.4.13}$$

This makes it possible to find restricted $w-$coordinates $\tilde{w}$, and $w(\tilde{w})$, which lets us employ the method of turning around the contour to compute $\eta_D(\tilde{w})$ as in Chapter 5. Furthermore, the Jacobian $\left(\dfrac{\partial \tilde{w}_i}{\partial \xi_j}\right)$ is already known since we now know $w(\tilde{w})$, $w(x)$, and $\xi(x)$. Let us see how this works through an easy example.

**Example 27** ($A_1 \subset G_2$)**.** One way of decomposing the fundamental weights of $G_2$ into $A_1$ gives

$$w_1(\tilde{w}) = 3 + 2\tilde{w}_1, \quad w_2(\tilde{w}) = \tilde{w}_1^2 + 4\tilde{w}_1 + 2. \tag{7.4.14}$$

Note that the dimension of $\rho_1$ in $A_1$ is 2, and so $3 + 2 \cdot 2 = 7$, and $2^2 + 4 \cdot 2 + 2 = 14$, which is exactly the dimension of $\rho_1$ and $\rho_2$ in $G_2$, respectively. Such a dimension test may serve as a sanity check in general. We call the corresponding discriminant strata $D_1$. Letting $w_0 = \tilde{w}_0$, (5.3.56) becomes

$$\lambda_{D_1}(\tilde{w}) = -\frac{(\mu - 1)^2 \tilde{w}_0 (\mu^2 - \mu\tilde{w}_1 + 1)^2}{\mu^2(\mu + 1)^2}. \tag{7.4.15}$$

Note that the positions and orders of the poles of $\lambda_{D_1}$ and $\lambda$ are unchanged. This will always be the case, as the leading order coefficient of $\lambda$ near any pole is independent of $w_1, \cdots, w_{l_{\mathcal{R}}}$ for any $\mathcal{R}$. Furthermore, notice that $\mu \mapsto \frac{1}{\mu}$ symmetry is maintained and that $\lambda_D$ looks like two copies of non–reduced $A_1$ with the addition of the extra pole structure arising from $G_2$.

Turning the contour around and evaluating the residues at the poles of $\lambda_{D_1}$, we find that

$$\eta_{D_1} = \begin{pmatrix} -1\dfrac{5\tilde{w}_1^2 + 28\tilde{w}_1 + 32}{\tilde{w}_0} & 8\tilde{w}_1 + 20 \\ 8\tilde{w}_1 + 20 & 12\tilde{w}_0 \end{pmatrix} \implies \det(\eta_{D_1}(\tilde{w})) = -4(\tilde{w}_1 - 2)^2. \tag{7.4.16}$$

On the other hand, by writing fundamental characters of $G_2$ in $x$-coordinates, (5.3.56) becomes

$$\lambda(x) = -\frac{e^{cx_0}(\mu - e^{x_1})(\mu - e^{-x_1})(\mu - e^{x_2 - x_1})(\mu - e^{x_1 - x_2})(\mu - e^{2x_1 - x_2})(\mu - e^{x_2 - 2x_1})}{\mu^2(\mu + 1)^2}. \tag{7.4.17}$$

Furthermore, (7.4.15) in $x$-coordinates is given by

$$-\frac{e^{cx_0}(\mu - 1)^2(\mu - e^{x_1})^2(\mu - e^{-x_1})}{\mu^2(\mu + 1)^2}. \tag{7.4.18}$$

Thus, we see that we obtain (7.4.18) from (7.4.17) by letting

$$x_2 \mapsto x_1, \quad x_2 \mapsto 2x_1, \quad \text{or } x_1 \mapsto 0. \tag{7.4.19}$$

This makes sense from the perspective of the root system. A natural choice of simple roots are (in the $\alpha$-basis) given by

$$\{(1, 0), (0, 1)\}, \tag{7.4.20}$$

and so we see that the discriminant stratum considered is characterised by removing the first node in the Dynkin diagram. The remaining relations are given by

$$x_2 \mapsto 0, \quad x_2 \mapsto 3x_1, \quad x_2 \mapsto \frac{3x_1}{2}, \tag{7.4.21}$$

and corresponds to removing the second Dynkin node. Finding $\det(\eta(x))$ from (7.4.18) directly in $x$ gives

$$-4c^2 e^{2cx_0 - 4x_1}(e^{x_1} - 1)^4(e^{2x_1} - 1)^2. \tag{7.4.22}$$

Now,

$$\tilde{w}_1(x) = e^x + e^{-x}, \quad \text{and} \quad \tilde{w}_0 = e^{cx_0}. \tag{7.4.23}$$

Thus, the Jacobian matrix becomes

$$J \equiv \left(\frac{\partial \tilde{w}_i}{\partial x_j}\right) = \begin{pmatrix} ce^{cx_0} & 0 \\ 0 & e^{x_1 - e^{-x_1}} \end{pmatrix} \quad \Longrightarrow \quad \det(J)^2 = c^2 e^{2cx_0 - 2x_1}(e^{2x_1} - 1)^2. \tag{7.4.24}$$

Hence,

$$\det(\eta_D(x)) \equiv \det(\eta_D)(\tilde{\omega})\det(J)^2 = -4e^{-2x_1}(e^{x_1-1})^4 \cdot c^2 e^{2cx_0 - 2x_1}(e^{2x_1} - 1)^2, \tag{7.4.25}$$

which is precisely (7.4.22).

Finally, Let $\xi = e^{x_1}$. Then, as $e^{x_1} + e^{-x_1} = \dfrac{\xi^2 + 1}{\xi}$, (7.4.22) is given by

$$-\frac{4c^2\xi_0(\xi^2 - 1)^2(\xi - 1)^4}{\xi^2} = -\frac{4c^2\xi_0(\xi^2 - 1)^4(\xi - 1)^2}{\xi^2(\xi + 1)^2}, \tag{7.4.26}$$

meaning Theorem 7.1.4 holds for $\mathcal{R} = G_2$ and $D_1$.

Moreover, notice that the denominator in the final expression is now in the correct form for Theorem 7.1.5. The result of employing the method of [1] is shown in Table 7.2. Consequently, we find that Theorem 7.1.5 also holds for $\mathcal{R} = G_2$ and $D_1$.

| $\beta_D$ | $R_{D,\beta}$ | $\mathcal{R}_{D,\beta}^{(0)}$ | $h(\mathcal{R}_{D,\beta}^{(0)})$ |
|---|---|---|---|
| $2x_1$ | $B_2$ | $B_2$ | 4 |
| $x_1$ | $A_1 \times A_1$ | $A_1$ | 2 |

Table 7.2: $\mathcal{R}_{D_1,\beta}$ for $\mathcal{R} = G_2$.

**Remark.** Now we may sense the reason for the non-affine contributions appearing among the linear form decomposition of the determinant, as these factors arise from the Jacobian, $\left(\frac{\partial \tilde{w}_i}{\partial x_j}\right)$ in the subalgebra, which sees nothing of the affine extension or of the spectral curve itself.

While it is possible to find expressions for $w$ in terms of characters associated to subalgebras for $G_2$, $F_4$, $E_6$ and $E_7$, as in Example 27, $\mathcal{R} = E_8$ proves a challenge. More specifically, when attempting to consider the subalgebra associated to $\mathcal{R} = E_7$ in the Lie algebra associated to $\mathcal{R} = E_8$ (that is we remove node $\alpha_7$ in Figure 3.1), it is possible to decompose the fundamental characters associated to nodes $1, 2, 5, 6, 7, 8$ (in the labelling of Figure 4.1) by using a combination of Mathematica and SageMath. However, $w_3$ and $w_4$ seem unfeasible. It is therefore necessary to provide the following workaround.

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1. Firstly, the Weyl character ring $\mathbb{Z}[\chi_i]$ can be shown to be isomorphic to the ring $\mathbb{Z}[\{\chi_{\wedge^k \rho_7}\}_{k=1}^5, \{\chi_{\wedge^k \rho_1}\}_{k=1}^2, \chi_8]$ (where nodes 1, 7, 8 are the leaves of the Dynkin diagram as in Figure 3.1). This means that it is possible to decompose $\chi_j$ for all $j$ into expressions only involving $\chi_{\wedge^k \rho_i}$, for $i \in \{1, 7, 8\}$, and consequently also $\chi_3$, and $\chi_4$. Indeed, the resulting relations are given by

$$\chi_3 = \chi_{\wedge^5 \rho_7} - \chi_{\wedge \rho_7} \chi_{\wedge^3 \rho_7} + 2\chi_{\wedge \rho_7} \chi_{\wedge^2 \rho_7} - 2\chi_{\rho_7}^2 + \chi_{\rho_7}, \tag{7.4.27a}$$

$$\chi_4 = \chi_{\wedge^4 \rho_7} - \chi_{\rho_7} \chi_{\wedge^2 \rho_7} + \chi_{\wedge^2 \rho_7} + \chi_{\rho_7}^2 - \chi_{\rho_7}. \tag{7.4.27b}$$

Notice that these only involve characters of wedge sums of $\rho_7$, which simplifies the process significantly.

2. Now one can decompose the direct sums using the Künneth formula for exterior powers;

**Theorem 7.4.1.** Let $R$ be a ring and let $M$, $N$ be $R$-modules. Then, for $k \geqslant 0$ there exists an $R$-module isomorphism

$$\wedge^k(M \oplus N) \cong \bigoplus_{i=0}^k \left( \wedge^i(M) \otimes_R \wedge^{k-i}(N) \right). \tag{7.4.28}$$

As $\rho_7$ can be decomposed into polynomials of fundamental characters in $E_7$, using Mathematica, we are simply left with wedge products of these on the subalgebra associated to $\mathcal{R} = E_7$. Thus, the remaining step is to decompose wedge products of fundamental representations in the subalgebra in terms of fundamental representations of the subalgebra. This brings us to the next step, which must be performed for most $\mathcal{R}$, not just the two difficult nodes in $E_8$. Nevertheless, we explain it in the context of $E_7 \hookrightarrow E_8$.

3. Decompose $\chi_{\wedge^k \rho_i}$ into polynomials in $\chi_{\rho_i}$ for $i = 1, \cdots, 7$ in $E_7$. This may be done by taking a "sampling" of sufficiently many points on the Cartan torus, giving a linear system of equations in the coefficients of an ansatz polynomial. The sampling size is necessarily finite. As a sanity check, one can then test if the result is correct by comparing the dimensions. For instance, if $\chi_{\rho_i}$ on $E_8$ decomposes into $\sum_{j=1}^7 \alpha_j \chi_{\rho_j}$ for $\alpha \in \mathbb{Z}$ (after performing the necessary wedge decomposition on $E_7$), we should have that $\dim \rho_i = \sum_{j=1}^7 \alpha_j \dim(\rho_j)$. This is straight-forward to do in Mathematica.

By knowing $w_i(\tilde{w})$, where $\tilde{w}_i$ is the $i^{\text{th}}$ fundamental character of the appropriate subalgebra, we can find the determinant of the Saito metric as follows. Simply input $w(\tilde{w})$ into the spectral curve. Then, we can perform residue calculations by LG–formulae after turning the contour around in the usual way to compute $\det(\eta(\tilde{w}))$. Note that as the leading coefficient in the expansion of $\lambda$ near any pole only depends on $w_0 = \tilde{w}_0$, the orders of the poles do not change. Then, the result is found by taking the Jacobian (which is known to have the appropriate behaviour by [1]), and factorise into appropriate $\xi$-coordinates as indicated by the factorisation of $b_0$.

Since Theorems 7.1.4–7.1.5 are shown to be true for $\mathcal{R} = A_l, B_l, C_l, D_l, G_2$ and the corresponding results in [1] include all exceptional cases, it is tempting to conjecture a factorisation of an analogous form for all $\mathcal{R}$.

**Conjecture 7.4.1.1.** Let $\mathcal{R}$ be as in Figure 4.1. Then,

$$\det(\eta_D) \propto w_0^{d+1} \prod_{H \in \mathcal{A}_D} l_H^{k_H} \mathcal{F}, \qquad (7.4.29)$$

with

$$\mathcal{F} = \prod_{j=1}^{|\mathrm{Sing}_\mu(\lambda(\mu))|} \tilde{l}_j^{\tilde{k}_j}, \qquad (7.4.30)$$

where $l_H, \tilde{l}_j$ are linear forms in exponentiated linear coordinates, $\mathcal{A}_D$ is the corresponding non-affine hyperplane arrangement, $k_H, \tilde{k}_j, d+1 \in \mathbb{N}$, and $\mathrm{Sing}_\mu(\lambda(\mu))$ indicates the set of (finite) poles of $\lambda$.

**Conjecture 7.4.1.2.** Let $H \in \mathcal{A}_D$, and let $\beta \in \mathcal{R}$ be such that $\beta_D$ is a non-zero multiple of the linear form $(l_H)$ (after taking the logarithms of the exponentiated coordinates). Then, the multiplicity of $l_H$ in Theorem 7.1.4, $k_H$, is the Coxeter number of the root system $R_{D,\beta}^{(0)}$, as defined in (7.1.2), $d = \dim(D) - 1$, and $\tilde{k}_j$ is such that $\lambda_\mathcal{R} \sim \mathcal{O}(\mu^{-\tilde{k}_j})$, near $\infty_j$.

We aim to prove Conjectures 7.4.1.1–7.4.1.2 in the near future.

**Remark.** While the second method must be performed on a case-by-case basis, it is beneficial when considering non-minimal representations as this usually results in a higher genus curve, and higher orders of $\lambda$ even if one can solve for $\lambda$ in the minimal representation.

**Remark.** While much of the Frobenius structure remains intact under the projection, the resulting submanifold is not in general a Frobenius manifold. In our context, in fact, none of them are as $\eta$ has generically non-zero curvature in all cases. This is consistent with the statements in [115], that a natural submanifold of a Frobenius manifold is a Frobenius manifold if and only if it is a pure caustic.

**Remark.** There are a couple of natural outstanding tasks in the context of this thesis. One is to relate the critical locus of $\lambda$, and $\lambda_D$ to the dimension of the associated Frobenius manifold, or discriminant stratum $D$, respectively, as the critical values of a Landau-Ginzburg superpotential correspond to canonical coordinates. Secondly, it would make sense to relate the discriminant strata to mirrors of the affine Weyl group. We hope to accomplish both these tasks shortly.

**Remark.** When the submanifold is Frobenius, we know that it encodes certain contracted Gromov–Witten invariants of the ambient manifold (see for example [116]). An interesting question is

whether the submanifolds consisting of discriminant strata also can be related to enumerative geometry.

# APPLICATION III:
# DUBROVIN-ZHANG INTEGRABLE HIERARCHIES

In this section we focus predominantly on the bottom horizontal arrow in Figure 1.2. That is, the principal hierarchy associated to a semisimple Frobenius manifold. We also mention briefly its dispersive analogue (the right-most vertical arrow in Figure 1.2).

In the context of Frobenius manifolds, we are interested in systems of hydrodynamic type. That is, systems defined by equations of the form

$$r_t^i = V_j^i(r)r_X^j, \quad i = 1, \cdots, N. \tag{8.0.1}$$

If the system is diagonalisable, we can write (8.0.1) as

$$u_t^i = v^i(u)u_X^i, \tag{8.0.2}$$

and we call $\{u^i\}$ Riemann invariants, and $\{v^i\}$ characteristic velocities (here we assume $v^i \neq v^j$ for $i \neq j$).

Further, the system is Tsarev *integrable*[*] if

$$\partial_{u_k}\left(\frac{\partial_{u_j} v^i}{v^i - v^j}\right) = \partial_{u_j}\left(\frac{\partial_{u_k} v^i}{v^i - v^k}\right), \tag{8.0.3}$$

for $i, j, k$ pairwise distinct. If a system is (Tsarev) integrable, then the general solution of the system depends on $N$ functions of a single variable.

**Theorem 8.0.1** (Sevennec)**.** A (Tsarev) integrable system can be written as a system of conservation laws

$$r_t^i = \partial_X f^i(r). \tag{8.0.4}$$

Conversely, if a system of conservation laws admits Riemann invariants, then its characteristic velocities satisfy the integrability conditions (8.0.3).

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[*]Alternative terminology is *semi-Hamiltonian* or *rich*.

## 8.1 The principal hierarchy of a Frobenius manifold

Here, we recall the general theory of principal hierarchies associated to semisimple Frobenius manifolds as formulated by Dubrovin in [40].

The principal hierarchy is bihamiltonian as we have two compatible Hamiltonian structures (induced by the two compatible metrics $\eta$ and $g$). It is of hydrodynamic type and is diagonalisable with the canonical coordinates as Riemann invariants. Now, however, instead of a finite system, we have an infinite quasilinear system of partial differential equations. We call $m$ the level of the hierarchy and we have a system as (8.0.1) for every level, with $N = n$, the dimension of the Frobenius manifold.

### Hamiltonian structure

Recall from Chapter 2 that we have a deformed connection $\widetilde{\nabla}$ on any (semisimple) Frobenius manifold, whose flatness implied a system of differential equations solved by (the gradients of) its flat sections. A basis of flat sections for $\widetilde{\nabla}$, $\sum_\alpha W_\beta^\alpha \partial_\alpha$, can be taken to have the form $W_\alpha^\beta = \sum_\nu \eta^{\alpha\nu} \partial_\beta h_\nu(\mathfrak{u}, z) z^{\mathcal{V}} z^R$ for some constant matrix $R$ (determined by the monodromy data of the Frobenius manifold; see [42, Lecture 2]) and $h(\mathfrak{u}, z) \in \Gamma(\mathcal{O}_\mathcal{M})[[z]]$.

We now build an integrable hierarchy on the formal loop space of $\mathcal{M}$, $\mathcal{L}\mathcal{M} := \{\mathfrak{u} : S^1 \to \mathcal{M}\}$, where an element $\mathfrak{u} \in \mathcal{L}\mathcal{M}$ is an $n$-tuple $\mathfrak{u} = (\mathfrak{u}^1, \ldots, \mathfrak{u}^n)$ with $\mathfrak{u}_i \in \mathbb{C}[[X, X^{-1}]]$ a formal Laurent series in a periodic coordinate $X \in S^1$.

The Taylor coefficients $h_{\alpha,m}(\mathfrak{u}) := [z^{m+1}]h_\alpha$ define Hamiltonian densities for which the corresponding local Hamiltonians

$$H_{\alpha,m}[\mathfrak{u}] := \int_{S^1} h_{\alpha,m}((\mathfrak{u}(X))\mathrm{d}X, \tag{8.1.1}$$

are in involution

$$\{H_{\alpha,m}, H_{\beta,n}\}_{[\lambda]} = 0, \tag{8.1.2}$$

with respect to any element (that is for any $\lambda$) of the pencil of (hydrodynamic) Poisson brackets:

$$\{\mathfrak{u}^\alpha(X), \mathfrak{u}^\beta(Y)\}_{[\lambda]} = (\gamma^{\alpha\beta} + \lambda\eta^{\alpha\beta})\delta'(X - Y) + \sum_\delta \Gamma_\delta^{\alpha\beta}(\mathfrak{u})\partial_X \mathfrak{u}^\delta \; \delta(X - Y), \tag{8.1.3}$$

where $\Gamma_\delta^{\alpha\beta}$ denotes the Christoffel symbol of the Levi-Civita connection of $\gamma^*$ in the flat coordinate chart $t$ for $\eta^*$. The corresponding involutive Hamiltonian flows

$$\partial_{T_{\alpha,m}}\mathfrak{u}^\beta := \{\mathfrak{u}^\beta, H_{\alpha,m}\}_{[\lambda]} = \sum_{\delta\epsilon}\left[(\gamma^{\beta\delta} + \lambda\eta^{\beta\delta})\partial_{\mathfrak{u}^\delta \mathfrak{u}^\epsilon}^2 h_{\alpha,m}(\mathfrak{u})\mathfrak{u}_X^\epsilon + \Gamma_\epsilon^{\beta\delta}\partial_{\mathfrak{u}^\delta}h_{\alpha,m}(\mathfrak{u})\partial_X\mathfrak{u}^\epsilon\right], \tag{8.1.4}$$

for $\alpha = 1, \ldots, n$ and $m = 0, \ldots, \infty$ define an integrable hierarchy of quasilinear PDEs on $\mathcal{L}\mathcal{M}$, called the *principal hierarchy* of $\mathcal{M}$; the dependent variables $\mathfrak{u}^\alpha = \mathfrak{u}^\alpha(X, T)$ are called the normal

coordinates of the hierarchy. This hierarchy moreover satisfies the $\tau$-symmetry condition

$$\partial_{T_{\mu,m}} h_{\nu,n}((\mathfrak{u}(X,T)) = \partial_{T_{\nu,n}} h_{\mu,m}(\mathfrak{u}(X,T)) = \frac{\partial^3 \log \tau(X,T)}{\partial X \partial T_{\mu,m} \partial T_{\nu,n}}, \tag{8.1.5}$$

for some function $\tau(X;(T_{\mu,m})_{\mu,m})$. In particular, $\mathfrak{u}^\alpha(X,T) = \partial^2_{X,T_{\alpha,0}} \log \tau = \partial^2_{X,T_{\alpha,0}} F$. In general, however, this system is very difficult to solve. The process of finding solutions simplifies significantly if starting from a B–model, as we will see in the next section.

**Example 28** (KdV)**.** The KdV hierarchy is determined by the following relations

$$u_{t_i} = \begin{cases} u_x & i = 0, \\[2mm] u u_x & i = 1, \\[2mm] \dfrac{u^2}{2!} u_x & i = 2, \\[2mm] \quad \vdots \end{cases} \tag{8.1.6}$$

**Remark.** There is a general construction by Dubrovin and Zhang to promote the principal hierarchy to have dispersion. This is a deformation of the principal hierarchy, introducing a small dispersion parameter $\epsilon$, and in the dispersionless limit $\epsilon \mapsto 0$ one recovers the principal hierarchy.

**Example 29** (KdV)**.** The KdV hierarchy with dispersion is determined by the following relations

$$u_{t_i} = \begin{cases} u_x & i = 0, \\[3mm] u u_x + \dfrac{\epsilon^2}{12} u_{xxx} & i = 1, \\[3mm] \dfrac{u^2}{2!} u_x + \dfrac{\epsilon^2}{12} \left( u u_{xxx} + 2 u_x u_{xx} \right) + \dfrac{\epsilon^4}{240} u^{(5)} & i = 2, \\[3mm] \quad \vdots \end{cases} \tag{8.1.7}$$

with $u_{t_k} \in \mathbb{C}[\epsilon, u, u_x, u_{xx}, \cdots, u^{(2k+1)}]$, for general $k$. It is easy to see that as $\epsilon \mapsto 0$, we recover Example 28.

We shall not go further into this construction here as it is past the scope of this thesis. If interested, see [47] for more details.

## 8.2    Principal hierarchy from the B–model

A notable consequence of the determination of the prepotential and the superpotential of a Frobenius manifold is a straight-forward route to describe its principal hierarchy, i.e. having such a mirror

theorem helps us solve the problem of finding flat sections of the deformed connection and gives us a presentation of the principal hierarchy in normal form.

An immediate adaptation of [41, Proposition 6.3] gives the following

**Proposition 8.2.1.** *With conventions as in Section 5.4, let $\widetilde{h}_{i,\alpha}(\tau, z)$, $\widetilde{h}_j{}^{ext}(\tau, z)$, $\widetilde{h}_k{}^{res}(\tau, z)$ be the flat coordinates for the deformed connection* (2.3.10) *on $\mathcal{M}_{\mathcal{R}}^{DZ} \times \mathbb{C}^{\star}$ normalised such that $\widetilde{h}_{i,\alpha} = \tau_{i,\alpha} + \mathcal{O}(z)$, $\widetilde{h}_j{}^{ext} = \tau_j{}^{ext} + \mathcal{O}(z)$, $\widetilde{h}_k{}^{res} = \tau_k{}^{res} + \mathcal{O}(z)$. Then,*

$$\widetilde{h}_{i,\alpha}(\tau, z) = -\frac{n_i + 1}{\alpha} \operatorname*{Res}_{\infty_i} \kappa_i^{\alpha} \,_1F_1\Big(1, 1 + \frac{\alpha}{n_i + 1}; z\lambda(\mu)\Big)\frac{\mathrm{d}\mu}{\mu}, \tag{8.2.1}$$

$$\widetilde{h}_j{}^{ext}(\tau, z) = \text{p.v.}\int_{\infty_0}^{\infty_j} \mathrm{e}^{z\lambda}\frac{\mathrm{d}\mu}{\mu}, \tag{8.2.2}$$

$$\widetilde{h}_k{}^{res}(\tau, z) = \operatorname*{Res}_{\infty_i}\frac{\mathrm{e}^{z\lambda} - 1}{z}\frac{\mathrm{d}\mu}{\mu}, \tag{8.2.3}$$

*where $_1F_1(a, b; x) = \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!}$ is Kummer's confluent hypergeometric function, and $(a)_n := \Gamma(a + n)/\Gamma(a)$.*

Theorem 5.2.5 also provides a dispersionless Lax–Sato description of (8.1.4) and (8.2.1)–(8.2.3) as a specific reduction of the universal genus-$g_\omega$ Whitham hierarchy with $\ell(\mathsf{n}_\omega)$ punctures [40,88]. Let $\widetilde{C}_w$ be the universal covering of $\overline{C_w^{\omega,\overline{k}}}\backslash\{\infty_i\}_i$, the fibre at $w$ of the Landau–Ginzburg family of 5.2.5, viewed as an analytic variety. Following [41], we consider second and third kind differentials $\Omega_{i,\alpha}$, $\Omega_j^{\text{ext}}$, $\Omega_k^{\text{res}}$ defined on $\widetilde{C}_w$ such that

$$\Omega_{i,\alpha;m} = -\frac{1}{n_i + 1}\left[\left(\frac{\alpha}{n_i + 1}\right)_{m+1}\right]^{-1} \mathrm{d}\lambda^{\alpha/(n_i+1)+m} + \text{regular},$$

$$\Omega_{j;m}^{\text{ext}} = \begin{cases} -\dfrac{\mathrm{d}\psi_m(\lambda)}{n_j+1} + \text{regular} & \text{near } \infty_i, \\[2mm] \dfrac{\mathrm{d}\psi_m(\lambda)}{n_0+1} + \text{regular} & \text{near } \infty_0, \end{cases}$$

$$\Omega_{i;m}^{\text{res}} = -\mathrm{d}\left(\frac{\lambda^{m+1}}{(m+1)!}\right) + \text{regular}, \tag{8.2.4}$$

where $\psi_m(\lambda) := \lambda^m/m!(\log \lambda - H_m)$, and $H_m$ is the $m^{\text{th}}$ harmonic number. Then, (8.1.4) and (8.2.1)–(8.2.3) are equivalent to the dispersionless Lax system

$$\partial_{T_{(i,\alpha);m}}\lambda = \{\lambda, q_{i,\alpha;m}\}_{\text{LS}}, \quad \partial_{T_{i;m}^{\text{ext}}}\lambda = \{\lambda, q_{j;m}^{\text{ext}}\}_{\text{LS}}, \quad \partial_{T_{i;m}^{\text{res}}}\lambda = \{\lambda, q_{i;m}^{\text{res}}\}_{\text{LS}}, \tag{8.2.5}$$

where

$$q_{i,\alpha;m}(\mu) := \int^{\mu}\Omega_{i,\alpha;m}, \qquad q_{j;m}^{\text{ext}}(\mu) := \int^{\mu}\Omega_{j;m}^{\text{ext}}, \qquad q_{j;m}^{\text{res}}(\mu) := \int^{\mu}\Omega_{j;m}^{\text{res}}, \tag{8.2.6}$$

and

$$\{f(\mu, X), g(\mu, X)\}_{\text{LS}} := \mu\left(\partial_\mu f \partial_X g - \partial_X f \partial_\mu g\right). \tag{8.2.7}$$

Having a closed-form superpotential for $\mathcal{M}_{\mathcal{R}}^{\text{DZ}}$ from Theorem 5.2.5 in particular provides explicit expressions for the Lax–Sato and Hamiltonian densities.

## 8.3 The type-$\mathcal{R}$ extended Toda hierarchy

We call the bihamiltonian integrable hierarchy defined by (8.1.4) and (8.2.1)–(8.2.3) in the case of a DZ-Frobenius manifold of Dynkin type $\mathcal{R}$ the *dispersionless extended $\mathcal{R}$-type Toda hierarchy*. For $\mathcal{R} = A_n$, this coincides with the dispersionless limit of the bi-graded Toda hierarchy of [25]. The adjective "extended" refers to the Hamiltonian flows generated by $H_j^{\text{ext}}[\mathfrak{u}]$, which are higher order versions of the space translation. We refer to the Hamiltonian flows generated by $H_{i,\alpha}[\mathfrak{u}]$ and $H_k^{\text{res}}[\mathfrak{u}]$ as the *stationary flows* of the hierarchy.

**Example 30** ($\mathcal{R} = G_2$). Let us consider for example $\mathcal{R} = G_2$. We construct explicitly the whole tower of Hamiltonian densities for the stationary flows of the type-$G_2$ dispersionless Toda hierarchy. In principle, these can be computed (up to a triangular linear transformation in the flow variables $T^{\alpha,m}$) by imposing the recursion relation, coming from the first line of (2.3.10),

$$\partial_{t_\alpha t_\beta}^2 h_{\gamma,m} = \sum_\delta c_{\alpha\beta}^\delta \partial_{t_\delta} h_{\gamma,m-1}, \tag{8.3.1}$$

with $\gamma = 1, 2, 3$, $m \geqslant 0$, $h_{\gamma,0} = \sum_\delta \eta_{\gamma,\delta} t_\delta$, and $c_{\alpha\beta}^\gamma$ are the structure constants of the quantum product determined by the prepotential (5.4.80). While the recursion (8.3.1) is ostensibly very hard to solve directly, the combination of Theorem 5.2.5 and Proposition 8.2.1 allows to give closed forms for the stationary Hamiltonians $H_{\gamma,m}$, $\gamma = 1, 3$, parametrically in $m$. This is most easily achieved in flat coordinates $(x_0, x_1, x_2)$ for the second metric $\gamma$, and in Hamiltonian form for the corresponding Poisson bracket. From (5.3.56) we have

$$\lambda(\mu) = \frac{w_0}{\mu^2(\mu+1)^2} \prod_{i=1}^6 (\mu - a_i(x)), \tag{8.3.2}$$

with

$$a_1 = \mathrm{e}^{x_1}, \quad a_2 = \mathrm{e}^{-x_1+x_2}, \quad a_3 = \mathrm{e}^{2x_1-x_2}, \quad a_4 = \mathrm{e}^{-x_1}, \quad a_5 = \mathrm{e}^{x_1-x_2}, \quad a_6 = \mathrm{e}^{x_2-2x_1}. \tag{8.3.3}$$

Labelling the punctures at $\mu = 0, -1$ and $\infty$ as $\infty_0$, $\infty_1$ and $\infty_2$ respectively we have:

$$\begin{aligned}
\widetilde{h}_{0;m}^{\text{res}} &= -\widetilde{h}_{2;m}^{\text{res}} = h_{3,m}, & \widetilde{h}_{1;m}^{\text{res}} &= 0, \\
\widetilde{h}_{0,1/2;m} &= \widetilde{h}_{2,1/2;m} = h_{1,m}, & \widetilde{h}_{1,1/2;m} &= -2\widetilde{h}_{0,1/2;m}.
\end{aligned} \tag{8.3.4}$$

From (8.2.1) and (8.2.3) we then get that

$$
\begin{aligned}
h_{1,m} &= \sum_{j=0}^{2m+1} \sum_{\substack{k_1,\ldots,k_6=0,\ldots,2m+1 \\ \sum_i k_i = 2m+1-j}} \frac{(2m+1)_j \mathrm{e}^{(m+1/2)x_0}}{j!(\frac{3}{2})_m} \prod_{i=1}^{6} \frac{a_i(x)^{m+1/2-k_i}\,(m-k_i+3/2)_{k_i}}{k_i!}, \\
h_{3,m} &= \sum_{j=0}^{2m} \sum_{\substack{k_1,\ldots,k_6=0,\ldots,m \\ \sum_i k_i = 2m-j}} \frac{(2m)_j \mathrm{e}^{mx_0}}{j!m!} \prod_{i=1}^{6} \frac{a_i(x)^{m-k_i}\,(m-k_i+1)_{k_i}}{k_i!},
\end{aligned}
$$

$$(8.3.5)$$

where $(a)_m = \Gamma(a+m)/\Gamma(m) = a(a+1)\cdots(a+m-1)$ is the Pochhammer symbol. In these coordinates, the Gram matrix of the second metric $\gamma$ and its inverse read

$$
(\gamma) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 12 & -6 \\ 0 & -6 & 4 \end{pmatrix}, \qquad (\gamma^{-1}) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{2} \\ 0 & \frac{1}{2} & 1 \end{pmatrix}.
$$

$$(8.3.6)$$

Let $x_i = f_i(t_1,t_2,t_3)$, $i=0,1,2$ be the change-of-variables expressing the $x$-coordinates in the flat coordinate chart $(t_1,t_2,t_3)$ for the first metric $\eta$, and define accordingly $\mathfrak{w}^i = f_i(\mathfrak{u}^1,\mathfrak{u}^2,\mathfrak{u}^3)$ for the corresponding dependent variables for the principal hierarchy. In these coordinates, the second Poisson bracket takes the form (recall that $\gamma^{ij} := (\gamma^{-1})_{ij}$)

$$
\{\mathfrak{w}^i(X), \mathfrak{w}^j(Y)\}_0 = \gamma^{ij}\delta'(X-Y)
$$

$$(8.3.7)$$

and the stationary flows are given by

$$
\frac{\partial \mathfrak{w}^i}{\partial T_{j,m}} = \{\mathfrak{w}^i, H_{j,m}\}_0 = \sum_{k=0,1,2} \gamma^{ik}\partial_{\mathfrak{w}^k} h_{j,m}(\mathfrak{w})\partial_X \mathfrak{w}^k, \quad j=1,3.
$$

$$(8.3.8)$$

**Remark.** As mentioned in the introductory result summary, there are some expectations with regards to the integrable hierarchies associated to DZ-manifolds, for simply laced Dynkin types. It is expected that these will be found by taking the dispersionless limit of the Hirota integrable hierarchies constructed in [102] via the relation to 1-dimensional Deligne-Mumford stacks as described in Section 9.2. For non-simply laced Dynkin types, we expect them to be new. In these cases it would be of interest to explicitly describe the hierarchies further, and show whether they may be obtained from symmetry reductions of the Hirota hierarchies.

**Remark 8.3.1.** In [118], a deformation scheme for the genus zero universal Whitham hierarchy is introduced in terms of a Moyal-type quantisation of the dispersionless Lax–Sato formalism. It would be intriguing to apply these ideas to the cases when $\mathcal{M}_\omega^{\mathrm{LG}}$ embeds into a genus $g_\omega = 0$ Hurwitz space, and verify that the resulting dispersionful deformation of the Principal Hierarchy is compatible with the hierarchy obtained by the quantisation of the underlying semisimple CohFT. We hope to attempt this in the future.

# APPLICATION IV:
# AN ORBICURVE NORBURY-SCOTT CONJECTURE

We will now discuss a conjecture arising naturally from the results of Chapter 5. It concerns a recursive procedure by the name of Topological Recursion, and is represented by the left-most vertical arrow in Figure 1.2.

## 9.1  Topological recursion

Topological Recursion (TR) is a universal recursive procedure in which one inputs a *spectral curve*, and obtains a sequence of symmetric differential forms as output. It was first discovered in the context of random matrix theory, and formally defined in [52]. It turns out that there is a surprising connection between this formalism and various problems in enumerative geometry. For instance, Mirzakhani's recursion of hyperbolic volumes [104] was proven in [54] by the use of random matrix models and geometrically in [123] to be a special case of the procedure. This is also the case for Gromov–Witten invariants of toric Calabi-Yau threefolds (the BKMP conjecture [14,98]) as proven in [53], and in [57] for Calabi-Yau orbifolds, as well as Hurwitz numbers (as conjectured in [15] and proven in [51]). The specific connection to enumerative geometry, for instance for what class of spectral curve one should expect to obtain geometric invariants, is still largely unknown and very much an active area of research. Nevertheless, it is hard to understate its importance as the formalism provides a straight-forward way to obtain enumerative invariants in algebraic geometry which can be extremely difficult to compute using the standard methods, as exemplified above.

The recursion has an initial input called a spectral curve which is not quite the same as what we called a spectral curve in the previous parts of this thesis, as it contains some additional data. A spectral curve in this context, which we will call a TR spectral curve, is defined as follows.

**Definition 9.1.1** (TR Spectral Curve). A *TR spectral curve*, $\mathcal{S}$, is a tuple $(\Sigma, x, y, \omega_{0,2})$, where

- $\Sigma$ is a Riemann surface[*];

- $x$ is a meromorphic function providing a ramified cover of the Riemann sphere with simple ramification points (zeros of $\mathrm{d}x$);

- $y$ is a meromorphic function such that the zeros of $\mathrm{d}y$ are not zeros of $\mathrm{d}x$;

- $\omega_{0,2}$ is a symmetric bidifferential on $\Sigma \times \Sigma$ with a double pole on the diagonal with bi-residue 1.

**Definition 9.1.2.** (Topological Recursion) Let $\omega_{0,1} := y\mathrm{d}x$. Then, the topological recursion procedure produces from $\mathcal{S}$ multi-differentials $\omega_{g,n}$ on $\Sigma^n$ by recursion on $2g - 2 + n > 0$ by

$$\omega_{g,n}(z_1,\cdots,z_n) := \sum_{i=1}^{r}\operatorname*{Res}_{z=a_i}\mathcal{K}(z_1,z)\Big(\omega_{g-1,n+2}(z,\sigma(z),z_2,\cdots,z_n)$$

$$+ \sum_{\substack{g_1+g_2=g \\ I_1\sqcup I_2=\{2,\cdots,n\}}}' \omega_{g_1,|I_1|+1}(z,z_{I_1})\omega_{g_2,|I_2|+1}(\sigma(z),z_{I_2})\Big), \tag{9.1.1}$$

where $\{a_i\}$ are the ramification points of $x$, $\sigma$ is the involution swapping a pair of sheets in the cover, the *dash* above the sum indicates the omittance of the case $(g_i,|I_i|+1) = (0,1)$, and $\mathcal{K}$ is the recursion kernel defined by

$$\mathcal{K}(z_1,z) := \frac{1}{2}\frac{\int_{w=\sigma(z)}^{z}\omega_{0,2}(z_1,w)}{(y(z)-y(\sigma(z))\mathrm{d}x(z)}. \tag{9.1.2}$$

It is a nontrivial fact that the multidifferential $\omega_{g,n}$ is symmetric in all its arguments (despite the seemingly special role of $z_1$ in (9.1.1)). In addition, its only poles are at the ramification points $\{a_i\}$.

The TR invariants $\omega_{g,n}$ also satisfy the dilaton equation; let $2g - 2 + n > 0$, then

$$\sum_{i=1}^{r}\operatorname*{Res}_{z=a_i}\omega_{g,n+1}(z_1,\cdots,z_n,z_{n+1})\,\Phi(z_n) = (2-2g-n)\,\omega_{g,n}(z_1,\cdots,z_n), \tag{9.1.3}$$

where $\Phi$ is such that $\mathrm{d}\Phi = \omega_{0,1} \equiv y\mathrm{d}x$.

**Remark.** The reason for the name topological recursion is that one can pictorially represent $\omega_{g,n}$ and the formula (9.1.1) in terms of surfaces of genus $g$ and with $n$ boundaries. Thus $2g - 2 + n$ becomes the (negative of the) Euler characteristic of the surface.

---

[*]In TR theory, this Riemann surface need not be connected or compact. To use Givental–Teleman theory to link it to Frobenius manifolds, however, we do require compactness, and so we shall assume this property implicitly.

As (9.1.1) is quite complicated, we provide here an example calculation to get a sense of how the recursion works.

**Example 31** $((g, n) = (0, 3))$**.** Note that in this case, $g - 1$ is negative, and we are only left with the second sum in (9.1.1). Moreover, the second sum can only be split such that $g_1 = 0 = g_2$, and $I_1 = \{2\}$, or $I_2 = \{3\}$, up to relabelling. This gives

$$\omega_{0,3}(z_1, z_2, z_3) = \sum_{i=1}^{r} \operatorname*{Res}_{z=a_i} \mathcal{K}(z_1, z) \left(\omega_{0,2}(z, z_2)\omega_{0,2}(\sigma(z), z_3) + \omega_{0,2}(z, z_3)\omega_{0,2}(\sigma(z), z_2)\right), \quad (9.1.4)$$

which is now fully described by the input data.

**Example 32** $((g, n) = (1, 1))$**.** In this case, the second sum in (9.1.1) vanishes as we are not allowing for $I_i = \varnothing$ and $g = 0$ at the same time. Hence,

$$\omega_{1,1}(z_1) = \sum_{i=1}^{r} \operatorname*{Res}_{z=a_i} \mathcal{K}(z_1, z) \, \omega_{0,2}(z, \sigma(z)), \quad (9.1.5)$$

which again only contains data from the TR spectral curve.

In general, to obtain the $n$-form $\omega_{g,n}$, one needs the multidifferentials $\omega_{g',n'}$ with $2g - 2 + n > 2g' - 2 + n'$, although it can be seen from (9.1.1) that for $\omega_{0,n}$, only invariants with $g = 0$ are required as the first sum vanishes.

**Example 33** (Mirzakhani's recursion)**.** Consider the TR spectral curve given by[†]

$$(\Sigma, \, x(z), \, y(z), \omega_{0,2}(z_1, z_2)) \; = \; \left(\mathbb{P}^1, \, \frac{z^2}{2}, \, \frac{\sin(2\pi z)}{2\pi}, \, \frac{\mathrm{d}z_1, \mathrm{d}z_2}{(z_1 - z_2)^2}\right). \quad (9.1.6)$$

We see that $\sigma(z) = -z$, as this keeps $x$ invariant, and there is a single ramification point at $z = 0$. The recursion kernel is

$$\mathcal{K}(z_1, z) = \frac{1}{2} \frac{\int_{w=-z}^{z} \frac{\mathrm{d}z_1 \mathrm{d}w}{(z_1-w)^2}}{\left(\frac{\sin(2\pi z)}{2\pi} - \frac{\sin(-2\pi z)}{2\pi}\right) \frac{\mathrm{d}x}{\mathrm{d}z} \mathrm{d}z} = \frac{2\pi}{2} \frac{\frac{2z\mathrm{d}z_1}{z_1^2-z^2}}{2\sin(2\pi z)z\mathrm{d}z} = \frac{1}{(z_1^2 - z^2)} \frac{\pi}{\sin(2\pi z)} \frac{\mathrm{d}z_1}{\mathrm{d}z}, \quad (9.1.7)$$

which by Example 32 gives

$$\omega_{1,1}(z_1) = \operatorname*{Res}_{z=0} \frac{1}{(z^2 - z_1^2)} \frac{\pi}{\sin(2\pi z)} \frac{\mathrm{d}z_1}{\mathrm{d}z} \frac{\mathrm{d}z \, \mathrm{d}z}{(2z)^2} = \frac{1}{8z_1^2} \left(\frac{1}{z_1^2} + \frac{4\pi^2}{6}\right) \mathrm{d}z_1. \quad (9.1.8)$$

On the other hand, the hyperbolic volume of the moduli space of genus 1 bordered Riemann surface with one geodesic of length $L$, is given by (Table 1 in [104][‡])

$$\mathcal{V}_{1,1} = \frac{L^2 + 4\pi^2}{48}. \quad (9.1.9)$$

---

[†]Technically speaking, $y(z)$ is here not defined at $z = \infty \in \mathbb{P}^1$. This is not a problem, however, as $\omega_{0,1} \equiv y\mathrm{d}x$ is.

[‡]In Table 1 in [104] the volume appears as twice this value. This is due to Mirzakhani using the imaginary part of the Weil-Petersson pairing as opposed to the Weil-Petersson Kähler form given by half this.

Taking the Laplace transform

$$\int_0^\infty \mathcal{V}_{1,1}(L)e^{-z_1 L}L\mathrm{d}L = \frac{1}{8z_1^4} + \frac{\pi^2}{12z_1^2},\tag{9.1.10}$$

we obtain precisely the functional part of (9.1.8).

**Remark.** In Example 33, we saw that the $\omega_{1,1}$ was related to the associated hyperbolic volume through the Laplace transform. This seems to be a general feature, and in a sense, the Laplace transform is here taking the role of the mirror map. We will see the Laplace transform appearing again when considering Frobenius manifolds in this context.

**Remark.** The $\omega_{0,2}$ as in Example 33, is often called the Bergmann Kernel. It is the unique bidifferential on the Riemann sphere, with a pole of order two on the diagonal with vanishing residue (up to holomorphic terms)[§]. In mathematics, this choice has historically been the correct one when considering TR on genus-zero curves, however, note that the Topological Recursion procedure also allows additional poles away from the diagonal, and more general forms have been found in physics literature. This, as we will see, will be crucial when treating DZ–manifolds.

**Remark.** What is described here may be called *global* topological recursion. There is also a local version (which is often how TR is usually defined) in which one only requires small disks near the ramification points, and the involution $\sigma$ is replaced by a set of local deck-transformations $\{\sigma_i\}$, where $\sigma_i$ is defined near the ramification point $a_i$. However, for Frobenius manifolds in which we have only simple ramification points, which we will be concerned with, these two notions of TR coincide. There are also several generalisations of TR. Examples of this include the Bouchard–Eynard topological recursion [13], in which ramification points of $x$ need not be simple, and the newly introduced *refined* topological recursion, currently valid for genus 0 spectral curves [82], which takes as input a *refined* spectral curve and allows for fractional $g$, to name a few.

## Connection with Frobenius manifolds

The connection between Frobenius manifolds is ultimately due to the results of Alexander Givental. In Givental theory [66–68], one associates to a semisimple Frobenius manifold a formal Gromov–Witten potential of the form

$$\hat{S}_t^{-1}\hat{\Psi}_t\hat{R}_t\hat{\Delta}_t Z_{\mathrm{KdV}}^{\times_r},\tag{9.1.11}$$

where $t$ is a semisimple point in the manifold, $r$ is its dimension, $S_t, R_t$ are $r \times r$ matrix series', $\Psi_t$ and $\Delta_t$ are $r \times r$ matrices, and $Z_{\mathrm{KdV}}$ is the Kontsevich-Witten $\tau$ function of the KdV integrable

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[§]For higher genus curve one must also normalise along a fix choice of symplectic basis.

hierarchy, with the ˆ indicating that a certain quantisation procedure has been performed. For instance let $R = \sum_{k=0}^{\infty} R_k z^k := e^{\sum_{k=1}^{\infty} r_k z^k}$, we have that

$$\hat{R} := e^{\sum_{k=1}^{\infty}(-1)^k r_k z^k}. \tag{9.1.12}$$

All objects of (9.1.11) are defined using the Frobenius structure. $S$ is called a calibration of the Frobenius manifold, and the matrix $\Psi$ represents the transformation matrix between canonical and flat coordinates. To describe $R$, consider the system

$$\nabla_z S = 0. \tag{9.1.13}$$

There exists a solution in a neighbourhood of a semisimple point $t$, $\Psi_t R_t(z) e^{\frac{U}{z}}$ where $R = \mathbb{1} + \sum_{l=1}^{\infty} R_k z^k$, satisfying $R_t^*(-z) R_t(z) = \mathbb{1}$ and the recursion relation

$$\Psi^{-1} d(\Psi R_{k-1}) = [dU, R_k], \tag{9.1.14}$$

with $U := \Psi^{-1} \mathcal{U} \Psi$ being the matrix associated to the multiplication by the Euler vector field, in canonical coordinates. This solution is uniquely determined by the homogeneity condition

$$\left( z \partial_z + \sum u^i \partial_{u^i} \right) R_t(z) = 0. \tag{9.1.15}$$

Finally, $\Delta$ is related to the entries of $\eta$ in the canonical basis: $\Delta_i^{-1} = \eta \left( \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i} \right)$. See Example 34 for how these are calculated. It was proved in [119] that the formal potential (9.1.11) does indeed coincide with the actual Gromov–Witten potential, to all genera, when the associated Frobenius manifold is constructed through quantum cohomology.

**Example 34.** For the case $\mathcal{R} = A_2$ it is possible to derive the $R$-matrices to any finite order (computer time permitting) in the formal variable by Frobenius manifold theory and some help from Mathematica from the recursion relations (9.1.14). This goes as follows. By (9.1.14) in order to consider $R$, we must obtain the matrix $\Psi$, which is defined by

$$\partial_\alpha = \sum_{i=1}^{\infty} \psi_{i\alpha} f_i, \tag{9.1.16}$$

where $\partial_\alpha$ is the vector, in the Saito-flat basis, with 1 in position $\alpha$, and zeros otherwise, and $f_i$ is an idempotent basis obtained by normalising the eigenvectors of $\mathcal{U}$ [41].

The Frobenius structure is given by

$$F = \frac{t_2^2 t_0}{2} + \frac{t_2 t_1^2}{4} - \frac{t_1^4}{96} + t_1 e^{t_0}, \quad e = \partial_2, \quad E = \frac{3}{2}\partial_0 + \frac{1}{2}t_1\partial_1 + t_2\partial_2. \tag{9.1.17}$$

Hence

$$\eta = \begin{pmatrix} 0 & 0 & 1 \\ 0 & \frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \implies \quad \eta^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \tag{9.1.18}$$

Recall that the matrix corresponding to multiplication by the Euler vector field is defined by

$$\mathcal{U}^\alpha_\beta := E^\epsilon c^\alpha_{\beta\epsilon} \equiv E^\epsilon c_{\epsilon\beta\rho}\eta^{\rho\alpha}, \tag{9.1.19}$$

where $\alpha$ denotes the row and $\beta$ denotes the column.

Thus, we get

$$\mathcal{U} \equiv (\mathcal{U}^\alpha_\beta) = \begin{pmatrix} t_2 & \frac{t_1}{4} & \frac{3}{2} \\ 3e^{t_0} & t_2 - \frac{t_1^2}{4} & \frac{t_1}{2} \\ 2t_1 e^{t_0} & \frac{3e^{t_0}}{2} & t_2 \end{pmatrix}, \tag{9.1.20}$$

which has characteristic polynomial

$$-u^3 + u^2\left(3t_2 - \frac{1}{4}t_1^2\right) - u\left(3t_2^2 - \frac{1}{2}t_2 t_1^2 - \frac{9}{2}t_1 e^{t_0}\right) + t_2^3 - \frac{1}{4}t_2^2 t_1^2 - \frac{9}{2}t_2 t_1 e^{t_0} + t_1^3 e^{t_0} + \frac{27}{4}e^{2t_0}. \tag{9.1.21}$$

At the origin this becomes $\frac{27}{4} - u^3$ which shows that the eigenvalues evaluated at the origin are pairwise distinct, and thus it is indeed a semisimple point on the Frobenius manifold.

Let us set $t_1 = t_2 = 0$, but keep $t_0$. Then we have

$$\mathcal{U} = \begin{pmatrix} 0 & 0 & \frac{3}{2} \\ 3e^{t_0} & 0 & 0 \\ 0 & \frac{3e^{t_0}}{2} & 0 \end{pmatrix}, \quad u_j = \frac{3}{2^{\frac{2}{3}}}(-1)^{\frac{4j}{3}}, \quad \text{for } j = 1, 2, 3. \tag{9.1.22}$$

The corresponding eigenvectors are given by

$$v_j = \begin{pmatrix} \frac{1}{2^{\frac{1}{3}}}e^{-\frac{2t_0}{3}}(-1)^{-\frac{2j}{3}} \\ 2^{\frac{1}{3}}e^{-\frac{-t_0}{3}}(-1)^{\frac{2j}{3}} \\ 1 \end{pmatrix}, \quad \text{for } j = 1, 2, 3. \tag{9.1.23}$$

However, we are interested in obtaining an idempotent basis, $\{f_i\}_{i=1,2,3}$, which means we need to normalise

$$f_j := \frac{v_i}{\sqrt{<v_j, v_j>}} = v_j \cdot \left(-\frac{2^{\frac{1}{6}}e^{\frac{t_0}{3}}}{3^{\frac{1}{2}}(-1)^{\frac{2j-1}{3}}}\right) = \begin{pmatrix} \frac{1}{2^{\frac{1}{6}}3^{\frac{1}{2}}}e^{-\frac{t_0}{3}}(-1)^{-\frac{2(2j+1)}{3}} \\ -\frac{2^{\frac{1}{2}}}{3^{\frac{1}{2}}}(-1)^{\frac{1}{3}} \\ -\frac{2^{\frac{1}{6}}}{3^{\frac{1}{2}}}e^{\frac{t_0}{3}}(-1)^{\frac{1-2j}{3}} \end{pmatrix}. \tag{9.1.24}$$

We are now ready to calculate the transition matrix $\Psi$. Using (9.1.16) we get

$$\Psi = \frac{1}{\sqrt{3}}\left(2^{\frac{1}{6}}(-1)^{\frac{\gamma-2}{3}}e^{\frac{t_0}{3}} \quad 2^{-\frac{1}{2}}(-1)^\gamma \quad 2^{-\frac{1}{6}}(-1)^{\frac{2-\gamma}{3}}e^{-\frac{t_0}{3}}\right)_{\gamma=1,2,3}. \tag{9.1.25}$$

It can be easily checked that $\Psi$ satisfies $\Psi\mathcal{U}\Psi^{-1} = U$, where $U$ is the diagonal matrix with $U_{ii} = u_i$ and $\Psi^T\Psi = \eta$, as is required from theory, and serve as a sanity check. In addition, we see that at $t_0 = 0$ we get

$$\Psi(0) = \frac{1}{\sqrt{3}}\left(2^{\frac{1}{6}}(-1)^{\frac{\gamma-2}{3}} \quad 2^{-\frac{1}{2}}(-1)^\gamma \quad 2^{-\frac{1}{6}}(-1)^{\frac{2-\gamma}{3}}\right)_{\gamma=1,2,3}. \tag{9.1.26}$$

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Now, setting $e^{\frac{2t_0}{3}} = z^{-1}$, and using the recursion (9.1.14), we may obtain a truncated solution for $R$ to any fixed order in $z$. See Appendix B for the result up to $[z^9]$. It can be easily checked that (9.1.15) is indeed satisfied.

**Remark.** It is a highly nontrivial fact that the expression (9.1.11) is independent of the chosen semisimple point $t$.

In [49], the authors, using a graphical approach, provide a local identification of Givental theory and the theory of Topological Recursion. This goes roughly as follows. Let $x$, $y$, $B$ be as in Definition 9.1.1, and let the number of ramification points of $x$ be $n$. As all ramification points are simple, we can define local coordinates $w_i$ on a neighbourhood $U_i$ of the ramification point $a_i$ by

$$x\big|_{U_i} = -\frac{w_i^2}{2} + x(a_i). \tag{9.1.27}$$

Then the identification goes as follows.

**Theorem 9.1.3** (Theorem 4.1 in [49] rephrased following [48])**.**

$$\Delta_i^{-\frac{1}{2}} = \frac{\mathrm{d}y}{\mathrm{d}w_i}(0); \tag{9.1.28a}$$

$$R^{-1}(\zeta^{-1})_i^j = -\frac{1}{2\pi\zeta} \int_{-\infty}^{\infty} \frac{B(w_i, w_j)}{\mathrm{d}w_i}\bigg|_{w_i=0} \cdot e^{x(w_j)-x(a_j)\zeta}; \tag{9.1.28b}$$

$$\sum_{k=1}^{n} \left(R^{-1}(\zeta^{-1})\right)_k^i \Delta_k^{-\frac{1}{2}} = \frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y(w_i)e^{x(w_j)-x(a_j)\zeta}. \tag{9.1.28c}$$

Additionally, $B$ satisfies the Laplace relation

$$\frac{\sqrt{\zeta_1\zeta_2}}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(w_i, w_j)e^{x(w_j)-x(c_j)\zeta_1+x(w_j)-x(c_j)\zeta_2} = \frac{\sum + R^{-1}(\zeta_1^{-1})_k^i R^{-1}(\zeta_2)_k^j}{\zeta_1^{-1} + \zeta_2^{-1}}. \tag{9.1.29}$$

Moreover, in [48], the authors relate the identification in Theorem 9.1.3 to an LG–description of a Frobenius manifold, as in the following

**Theorem 9.1.4** (Theorem 6.1 in [48])**.** Consider a Frobenius manifold with an associated LG–model $(\lambda, \mathrm{d}p)$ induced by a Riemann surface $\mathcal{C}$ of genus $g$ such that there is precisely one critical point in each singular fibre $\lambda : \mathcal{C} \to \mathbb{C}$. Fix a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ of $H_1(\mathcal{C}, \mathbb{Z})$. Then, the following identification ((somethingsomething);

$$x \longleftrightarrow \lambda, \quad y \longleftrightarrow p, \quad \omega_{0,2} \longleftrightarrow B(p_1, p_2), \tag{9.1.30}$$

where $B(p_1, p_2)$ is the unique bidifferential on $\mathcal{C}$ with a double pole on the diagonal of vanishing residue with the normalization

$$\oint_{p_1 \in \mathcal{A}_i} B(p_1, p_2) = 0, \quad \forall\, i = 1, \cdots, g. \tag{9.1.31}$$

There are, however, some caveats to Theorem 9.1.4. Firstly, it is assumed throughout [48] that the charge for a Frobenius manifold, $d$ is different from one. Secondly, the authors impose a compatibility condition for the pair $(p, B(p_1, p_2))$; that the 1-form

$$\mathrm{d}\left(\frac{\mathrm{d}y}{\mathrm{d}x}(z)\right) + \sum_{i=1}^{n} \operatorname*{Res}_{z'=a_i} \frac{\mathrm{d}y}{\mathrm{d}x}(z') B(z, z'), \tag{9.1.32}$$

is invariant under each local involution $\sigma_i$[¶].

While the second assumption may hold for DZ-manifolds, the first ensures that we cannot apply Theorem 9.1.4 directly to DZ-manifolds.

Theorems 9.1.3, 9.1.4 together with (9.1.11), imply that the TR–invariants (under a suitable limiting procedure) should match with the GW–invariants of the associated space (if it exists). Despite the fact that Theorem 9.1.4 is not directly applicable to DZ-manifolds, the implication still holds for the simplest case of DZ-manifolds in which $\mathcal{R} = A_1$. This is called the Norbury–Scott Conjecture.

Let the spectral curve $\mathcal{S}$ be defined by

$$\mathcal{S} = \begin{cases} \Sigma = \mathbb{P}^1; \\ x = z + \dfrac{1}{z}; \\ y = \log z; \\ \omega_{0,2} = B(z_1, z_2) \equiv \dfrac{\mathrm{d}z_1 \mathrm{d}z_2}{(z_1 - z_2)^2}. \end{cases} \tag{9.1.33}$$

The connected stationary[‖] Gromov–Witten invariants of $\mathbb{P}^1$ are given by

$$\left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_{g,d} = \int_{[\overline{\mathcal{M}}_n^g(\mathbb{P}^1, d)]^{\mathrm{vir}}} \prod_{i=1}^{n} \psi_i^{b_i} \mathrm{ev}_i^*(\omega) \tag{9.1.34}$$

where notation is as in Section 3.2.

Assembling these into a generating function gives

$$\Omega_n^g(x_1, \cdots, x_n) = \sum_{\mathbf{b}} \left\langle \prod_{i=1}^{n} \tau_{b_i}(\omega) \right\rangle_{g,d} \cdot \prod_{i=1}^{n} (b_i + 1)! \, x_i^{-b_i - 2} \mathrm{d}x_i. \tag{9.1.35}$$

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[¶]An equivalent condition is that the 1-form is given as a pull-back $x^*\omega$.

[‖]The stationary invariants include intersections with $\psi$-classes.

In [106] Paul Norbury and Nick Scott conjectured the following.

**Conjecture 9.1.4.1** (Norbury-Scott)**.** For $2g - 2 + n > 0$, the generating function of the Gromov–Witten invariants of $\mathbb{P}^1$ gives an analytic expansion of the TR-invariants of $\mathcal{S}$ around a branch of $\{x_i = \infty\}$:

$$\omega_{g,n} \sim \Omega_{g,n}(x_1, \cdots, x_n). \tag{9.1.36}$$

**Remark.** Note that the unstable terms $(g, n) = (0, 1), (0, 2)$, are not analytic at $x \to \infty$, and so in order to perform the expansion, we must remove this singularity, which is done by

$$\omega_{0,1} - \log(x)\,\mathrm{d}x \sim \Omega_{0,1}(x), \quad \omega_{0,2} - \frac{\mathrm{d}x_1 \mathrm{d}x_2}{(x_1 - x_2)^2} \sim \Omega_{0,2}(x_1, x_2). \tag{9.1.37}$$

The conjecture was proven in genus 0 and 1 in the original paper by Norbury and Scott [106], by finding explicit relations between the coefficients of the generating function arising from TR and the Gromov–Witten invariants, in a highly nontrivial fashion. Their methods, however, did not lend themselves to a general proof. The proof was completed in [49], using the identification of Theorem 9.1.3, deriving the objects arising from the Frobenius structure explicitly and making a direct comparison. Most importantly the authors managed to find an explicit closed formula for the $R$–matrix from TR–theory matching the one explicitly found from Frobenius manifold theory. This proved the Conjecture on the level of cohomological field theories. Then they promoted the proof to Gromov–Witten theory by equating the contribution from the $S$-matrices in (9.1.11).

In order to formulate an analogous conjecture for the rest of the DZ-manifolds associated to simply laced Dynkin diagrams, we must first connect them to the quantum cohomology of some space. As mentioned in the introduction, these spaces will be 1-dimensional orbifolds.

## 9.2 Quantum cohomology of DM–stacks and a mirror symmetry triangle

There is a generalisation of Gromov–Witten theory to orbifolds, as described in Section 3.2. The theory describing the quantum cohomology of such spaces is called Chen-Ruan orbifold cohomology [27]. We shall now describe this theory to the generality needed, mainly following [81].

**Definition 9.2.1** (Orbifold/Stack)**.** A *Deligne-Mumford* (DM) *stack*, or *orbifold*[*], $\mathcal{X}$, is a coarse moduli space[†], with underlying topological space $|\mathcal{X}|$, such for any point $x \in |\mathcal{X}|$ a neighbourhood of $x$, $U_x$, is isomorphic to $\mathbb{C}^n/G_x$ for some $n$ and a finite, possibly trivial, group $G_x$.

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[*]Note that this is not quite the same as what physicists call an orbifold.
[†]Recall that this means that there remains a finite number of automorphisms.

We denote the Chen-Ruan cohomology ring of a stack $\mathcal{X}$ by $H^*_{\text{CR}}(\mathcal{X})$. As vector spaces we have,

$$H^*_{\text{CR}}(\mathcal{X}) \cong H^*_{\text{dR}}(\mathcal{IX}, \mathbb{C}), \tag{9.2.1}$$

where the subscript dR on the right-hand side denotes the de-Rham cohomology, and $\mathcal{IX}$ is the *inertia* stack associated to $\mathcal{X}$ which can be viewed as follows.

Each element in $\mathcal{IX}$ may be characterised as a pair $(x, g) \in \mathcal{X} \times G_x$, where $G_x$ is the isotropy group at $x$. Thus the inertia stack has several connected components, one which arises from the identity element;

$$\{(x, e_G)\} \cong \mathcal{X}. \tag{9.2.2}$$

The other components are called *twisted sectors*.

While $H^*_{CR}(\mathcal{X})$ is isomorphic to the cohomology of the inertia stack as vector spaces, the cup product and graded structure differ. The grading is given by the concept of *age* which implies a degree shift by a rational number which is constant on each component of the inertia stack. See Example 35 for how it works in a simple case. For a precise definition of the cup product and age see [27].

We will be concerned with $\mathcal{X} = \mathbb{P}^1$ which is thoroughly treated in [81]. Furthermore, these orbifolds will be of the type *effective*, that is the isotropy group of a generic point of the orbifold is trivial. More precisely, we will consider DM stacks with 2 or 3 points $p_i$ with nontrivial isotropy group, called *orbifold points*. The orders of the group associated to $p_i$, $a_i$ is called a weight, and we have a restriction of these weights given by the Euler characteristic needing to be positive[‡]:

$$\chi_{\text{orb}} := \sum_i \frac{1}{a_i} > 0, \tag{9.2.3}$$

which is necessary in order for the associated semisimple Frobenius manifold to be semisimple.

**Example 35** ($\mathbb{P}^1_{r,s}$)**.** Let $\mathbb{P}^1_{r,s}$ denote the DM–stack with coarse moduli space $|\mathbb{P}^1_{r,s}| = \mathbb{P}^1$, with two orbifold points $p_1, p_2$ which can, without loss of generality, be placed at 0, and $\infty$, respectively by the automorphism group of $\mathbb{P}^1$. We have an action of $\mathbb{Z}_r$ ($\mathbb{Z}_s$) at 0 ($\infty$) given by multiplication by roots of unity. Let us calculate $H^*_{\text{CR}}(\mathbb{P}^1_{r,s})$. From the discussion above, we know that the inertia stack has $r + s - 1$ components: $\mathcal{I}\mathbb{P}^1_{r,s}(1)$, $\mathcal{I}\mathbb{P}^1_{r,s}(x_1)$, and $\mathcal{I}\mathbb{P}^1_{r,s}(x_2)$, where $x_1 = k_1/r$ for $k_1 = 1, \cdots r - 1$, and $x_2 = k_2/s$ for $k_2 = 1, \cdots, s - 1$. By (9.2.2), the untwisted sector $\mathcal{I}\mathbb{P}^1_{r,s}(1)$ is simply $\mathbb{P}^1_{r,s}$ which is topologically equivalent to $\mathbb{P}^1$. Thus, $H^*(\mathbb{P}^1_{r,s}, \mathbb{C}) = <\mathbf{1}, \omega>$, where $<\mathbf{1}> = H^0(\mathbb{P}^1)$, and $<\omega> = H^2(\mathbb{P}^1)$. Furthermore, we have $r + s - 2$ components arising from the $r - 1$ twisted sectors over 0, and the $s - 1$ twisted sectors over $\infty$ corresponding to the nontrivial elements of $\mathbb{Z}_r$ and $Z_s$, respectively. Each twisted sector is a copy of $\mathcal{B}\mathbb{Z}_r$ ($\mathcal{B}\mathbb{Z}_s$) where $\mathcal{B}G$ denotes the classifying space

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[‡]Equivalently, $\mathcal{X}$ is a *Fano* orbifold.

of $G$. Hence, $H^*(\mathcal{I}\mathcal{C}_{r,s}(x_1)) = < 0_{k_1} >$, and $H^*(\mathcal{I}z, \mathbb{P}^1_{r,s}(x_2)) = < \infty_{k_2} >$, with $0_{k_1}, \infty_{k_2}$ being the multiplicative identity of $\mathbb{Z}_r$, $\mathbb{Z}_s$, respectively. Since $\mathbb{Z}_r$ $(\mathbb{Z}_s)$ acts on $T\mathbb{P}^1_{r,s}$ by $e^{\frac{2\pi k_1}{r}}$ $(e^{\frac{2\pi k_1}{s}})$, we have that $0_{k_1} \in H^{\frac{2k_1}{r}}_{\mathrm{CR}}(\mathbb{P}^1_{r,s})$, $(\infty_{k_2} \in H^{\frac{2k_1}{s}}_{\mathrm{CR}}(\mathbb{P}^1_{r,s}))$.

**Remark.** For an orbicurve to be toric, we must have precisely two orbifold points, and so whenever there exist three such points, we have non-toric orbifolds.

**Remark.** Note that when the weights $r, s$ are pairwise coprime, we have an equivalence with the weighted projective line $\mathbb{P}_{r,s} \cong \mathbb{P}[r, s]$. For non-coprime weights on the other hand, the two spaces are distinct which can be seen by considering the relation $\mathbb{P}^1[\gamma b, \gamma c] \cong \mathbb{P}^1[b, c]$ for weighted projective spaces, which clearly does not hold for Example 35.

In this section mirror symmetry is understood as an isomorphism of Frobenius manifolds arising from quantum cohomology and orbit spaces (i.e. $A - C$ mirror symmetry). However, we will see that such a mirror symmetry implies a triangle $A - B - C$ mirror symmetry by Theorem 5.2.5.

Paolo Rossi proves in [111], using techniques from Symplectic Field theory, that in the simply laced cases the DZ-manifolds are indeed isomorphic, as Frobenius manifolds, to the quantum cohomology of Fano $\mathbb{P}^1$ orbifolds. More specifically, he considers a Frobenius manifold structure on the space of tripolynomials, $M_{p,q,r}$, consisting of polynomials of the form

$$F(x,y,z) = -xyz + P(x) + P_2(y) + P_3(z), \tag{9.2.4}$$

of degrees $p, q, r$ in $x, y, z$, respectively such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$, and constructs a structure of Frobenius manifold on them. He then proves the following

**Theorem 9.2.2.** Let $M_{p,q,r}$ be the space of tripolynomials, $\mathcal{M}$ the corresponding Frobenius manifold structure, and $\mathbb{P}^1_{p,q,r}$ be the orbicurve with three orbifold points of weights $p, q, r$, respectively. Then we have the following isomorphisms of Frobenius manifolds:

$$\mathcal{M}_{p,q,r} \cong \mathcal{M}\left(QH^*\left(\mathbb{P}^1_{p,q,r}\right)\right). \tag{9.2.5}$$

Theorem 9.2.2 is proved by comparing the associated ring structures, and explicitly writing the relations in the quantum product.

**Remark.** In the case of $r = 1$, the above theorem was already proven in [103]. Furthermore, $\mathcal{M}_{p,q,1}$ is trivially equivalent, as Frobenius manifolds, to $\mathcal{M}^{\mathrm{DZ}}_{A_l}$, for $P + q = l + 2$ as proved by Dubrovin and Zhang [46], and [103] by relating the Frobenius manifold structure to the extended bigraded Toda hierarchy [25].

Rossi finds that the only cases in which the prepotential is polynomial in $t_1, \cdots, t, e^t$ are precisely those with $(p, q, r) = (k, l - k + 1, 1), (2, 2, l - 2)$, or $(2, 3, l - 3)$, where in the final case we must have $l \in \{6, 7, 8\}$. Note that $k$ in the first case does indeed correspond to the choice of canonical node, $\bar{k}$, for $A_l$. The remaining isomorphisms then hold by the reconstruction theorem, Theorem 4.1.1. Hence, for simply laced Dynkin types, we have a *triangle* of mirror symmetry.

As we have now connected (the simply laced) DZ–manifolds to quantum cohomology, we are ready to state a conjecture akin to that of Norbury-Scott, Conjecture 9.1.4.1.

## 9.3 Generalised conjecture and preliminary work towards a proof

Unfortunately, the Frobenius manifolds obtained from extended affine Weyl groups do not satisfy the criteria to make usage of Theorem 9.1.4, as we have $d = 1$, which is highlighted by the authors considering $\mathbb{P}^1$ in the appendix of [48]. In fact, it can be easily seen that the Bergmann kernel cannot be the correct two-point function. For instance, in the case $\mathcal{R} = A_2$ this gives an $\omega_{0,3} = 0$. Thus we must add some holomorphically nontrivial terms to the Bergmann kernel. This arises from the fact that we are now in the equivariant setting.

An additional consequence of the GOV correspondence and the results of [12, 17], is precisely a conjectural form of the input data for running topological recursion on the spectral curves computed in Chapter 5, and it turns out that only the two-point function needs to be altered. That is, we want to take $(\Sigma, x, y) = (\mathcal{C}_\mathcal{R}, \lambda_\mathcal{R}, \log(\mu))$, as in Chapter 5, while the two-point function should correspond to the *symmetrised Bergmann Kernel*, in the language of [17], which is given by

$$\omega_{0,2}(z_1, z_2) := \mathcal{P}_\mathfrak{g}^* E_\Psi(z_1, z_2), \tag{9.3.1}$$

where

$$\mathcal{P}_\mathfrak{g} := \sum_{\omega \in \mathcal{W}/\mathcal{W}_{\alpha_0}} < \omega^{-1}(\alpha_0), \alpha_0 > \omega, \tag{9.3.2}$$

in the notation of Chapter 5. Here $\Psi$ is a sub-lattice of the first integral homology of the spectral curve satisfying some technical conditions and, $E$ is a primitive of the Bergmann Kernel (see [17] for the details).

**Remark.** We expect to evaluate $\lambda$ at a suitable semisimple point, to mimic the Norbury-Scott Conjecture. For $\mathcal{R} = A_l$ we propose this point to be given by $t$ being the origin giving $(w_0, \underline{w}) = (1, \underline{0})$, which is equivalent to taking the standard Kähler structure. By the previous discussion, the result should be independent of the point chosen.

We are now ready to state the main Conjecture.

**Conjecture 9.3.0.1** (Orbicurve Norbury–Scott). Let $\mathcal{S} = (\mathcal{C}_\mathcal{R}, \lambda_\mathcal{R}, \log(\mu), \omega_{0,2})$, where $\mathcal{C}_\mathcal{R}$ is the spectral curve as in Chapter 5 associated to root system $\mathcal{R}$, $\lambda_\mathcal{R}$ is the corresponding LG–super-potential and $\omega_{0,2}$ as in (9.3.1). Then, the Topological Recursion procedure applied to $\mathcal{S}$ recovers the (orbifold) Gromov–Witten theory of $\mathcal{X}_\mathcal{R}$, in the limit $\mu \to \infty$ by

$$\omega_{g,n} \sim \Omega_n^g(x), \tag{9.3.3}$$

where $\Omega_n^g(x)$ is the generating function of the associated (orbifold, stationary) Gromov–Witten invariants, and

$$\mathcal{X}_\mathcal{R} = \begin{cases} \mathbb{P}_{\alpha_1,\alpha_2}, & \mathcal{R} = A_l, \quad \text{with } \alpha_1 + \alpha_2 = l + 1, \\ \mathbb{P}_{2,2,l-2}, & \mathcal{R} = D_l, \\ \mathbb{P}_{2,3,l-3} & \mathcal{R} = E_l. \end{cases} \tag{9.3.4}$$

**Remark.** Note that, while $y \equiv \log(\mu)$ is not meromorphic, it is sufficient to consider

$$y \sim \sum_{k=1}^{6g-6+2n} \frac{(1 - \mu^2)^k}{-2k}, \tag{9.3.5}$$

to obtain TR–invariants up to $\omega_{g,n}$ [106].

Let us investigate Conjecture 9.3.0.1 for $\mathcal{R} = A_l$.

## $\mathcal{R} = A_l$

We consider the TR spectral curve

$$\mathcal{S} = \begin{cases} \Sigma = \mathbb{P}^1, \\ x = \lambda_{l,k}(\mu; 1, \underline{0}) = \dfrac{(-1)^k(1 + (-1)^{l+1}\mu^{l+1})}{\mu^k}, \\ y = p \equiv \log(\mu), \\ \omega_{0,2}(\mu_1, \mu_2) = \displaystyle\sum_{k \in \mathbb{Z}_l} (-1)^{\frac{2k}{l}} \dfrac{d\mu_1\, d\mu_2}{(\mu_1 - (-1)^{\frac{2k}{l}}\mu_2)^2}, \end{cases} \tag{9.3.6}$$

where the superpotential has been scaled by a factor of $(-1)^{l-k}$ for later convenience[*].

We can write $\omega_{0,2}$ as

$$\omega_{0,2}(\mu_1, \mu_2) = \sum_{k \in \mathbb{Z}_l} B_S(\mu_1, \xi_k\mu_2) = B_S(\mu_1, \mu_2) + \sum_{k \in \mathbb{Z}_n \setminus \{0\}} B_S(\mu_1, \xi_k\mu_2), \tag{9.3.7}$$

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[*]Such a scaling has no effect on the Frobenius structure.

where $B_S(\mu_1,\mu_2)$ is the Bergmann Kernel, and $\xi_k$ is the $k^{\text{th}}$ root of unity. It is easy to see that in the case of $A_1$, in which $\mathbb{Z}_l\backslash\{0\}$ is empty, (9.3.7) reduces to the Bergmann Kernel, which is proven to be the correct choice in [49]. Note that (9.3.7) clearly satisfies the axioms of Topological Recursion as

$$B_k(\mu_2,\mu_1) = \mathrm{d}\mu_1\mathrm{d}\mu_2\left(\frac{e^{\frac{2\pi k}{l}}}{(\mu_2 - e^{\frac{2\pi k}{l}}\mu_1)^2} + \frac{e^{-\frac{2\pi k}{l}}}{(\mu_2 - e^{-\frac{2\pi k}{l}}\mu_1)^2}\right) \tag{9.3.8}$$

$$= \mathrm{d}\mu_1\mathrm{d}\mu_2\left(\frac{e^{\frac{2pik}{l}}}{e^{\frac{4\pi k}{l}}(e^{-\frac{2\pi k}{l}}\mu_2 - \mu_1)^2} + \frac{e^{-\frac{2\pi k}{l}}}{e^{-\frac{4\pi k}{l}}(e^{\frac{2\pi k}{l}} - \mu_1)^2}\right) = B_k(\mu_1,\mu_2), \tag{9.3.9}$$

where

$$B_k(\mu_1,\mu_2) = B_S(\mu_1, e^{\frac{2\pi k}{n}}\mu_2) + B_S(\mu_1, e^{-\frac{2\pi k}{n}}\mu_2), \quad \omega_{0,2}(z_1,z_2) = \sum_{k=1}^{\lceil\frac{l-1}{2}\rceil} B_k(z_1,z_2). \tag{9.3.10}$$

By Theorem 9.1.3, the two-point function $\omega_{0,2}$ and the $R$–matrix are intimately connected. Thus, it makes sense to consider the $R$-matrix more in depth for $\mathcal{R} = A_l$. Note that it is usually a highly nontrivial problem to obtain the matrices occurring in $R$ explicitly.

The $R$–matrix for $A_l$

We consider the origin, $(w_0,\underline{w}) = (1,\underline{0})$, at which $\lambda = \frac{(-1)^l}{\mu^{\bar{k}}} - \mu^{l+1-\bar{k}}$. Near a ramification point $a_j$ of $\lambda$, as all ramification points are simple, we may choose a local coordinate $\zeta_j$ such that $\lambda = -\frac{\zeta_j^2}{2} + \lambda(a_j)$, with $\lambda(a_j) = u_j$. Then, from Theorem 9.1.3, we have

$$\xi_i(\mu) := \int^\mu \left.\frac{\omega_{0,2}(\zeta_i,\cdot)}{\mathrm{d}\zeta_i}\right|_{\zeta_i=0} = \sum_{k\in\mathbb{Z}_l}(-1)^{\frac{2k}{l}}\int^\mu \frac{\mathrm{d}\mu_1(\zeta_i)\mathrm{d}\mu_2}{(\mu_1(\zeta_i) - (-1)^{\frac{2k}{l}}\mu_2)^2} \tag{9.3.11}$$

$$= (-1)^{\frac{2k}{l}}\int^\mu \frac{\mathrm{d}\mu_1(\zeta_i)}{\mathrm{d}\zeta_i}\mathrm{d}\mu_2\left(\frac{1}{(\mu_1(\zeta_i) - (-1)^{\frac{2k}{l}}\mu_2)^2}\right)\Bigg|_{\zeta_i=0}. \tag{9.3.12}$$

Letting $\mu(\zeta_i) = a_i + \beta_i\Delta_i\zeta_i + \mathcal{O}(\zeta_i^2)$, we get

$$\xi_i(\mu) = \beta_i\Delta_i\sum(-1)^{\frac{2k}{l}}\int^\mu \frac{\mathrm{d}\mu_2}{(a_i - (-1)^{\frac{2k}{l}}\mu_2)^2} = \beta_i\Delta_i\sum_{k\in\mathbb{Z}_l}\frac{1}{a_i - (-1)^{\frac{2k}{l}}\mu}, \tag{9.3.13}$$

giving

$$(R^{-1})_i^j(u) = -\frac{\beta_i\Delta_i e^{\lambda(a_j)}}{\sqrt{2\pi u}}\sum_{k\in\mathbb{Z}_l}\int_{-\infty}^\infty \frac{e^{-\frac{\zeta_j^2}{2u}}}{\left(a_i - (-1)^{\frac{2k}{l}}\mu(\zeta_j)\right)}\mathrm{d}\zeta_j. \tag{9.3.14}$$

In $\mu$ (9.3.14) becomes

$$(R^{-1})_i^j(u) = -\frac{\beta_i \Delta_i e^{-\frac{\lambda(a_j)}{u}}}{\sqrt{2\pi u}} \sum_{k \in \mathbb{Z}_l} \int_{\Gamma_j} \left( \frac{\frac{(-1)^{l+1}\bar{k}}{\mu^{\bar{k}+1}} - (l+1-\bar{k})\mu^{l-\bar{k}}}{((-1)^{\frac{2k}{l}}\mu - a_i)} \right) e^{\frac{1}{u}\left(\frac{(-1)^l}{\mu^{\bar{k}}} - \mu^{l+1-\bar{k}}\right)} d\mu, \quad (9.3.15)$$

where $\Gamma_j$ is a Hänkel, or keyhole type, contour enclosing $0$ and $-\infty$.

## Attempt 1: Explicitly equating invariants

In [31, 32], the authors defined a framework for computing Gromov–Witten invariants for a class of complete intersections in toric DM–stacks. As the orbifolds associated to $\mathcal{R} = A_l$ are indeed included in this framework, one could explicitly compute GW–invariants, and try to match these up with coefficients of the expansion of TR–invariants in the vein of [105, 106]. While this could be feasible in low genus, and would give evidence for the validity of Conjecture 9.3.0.1, it is highly unlikely to give a general proof as the level of difficulty increases exponentially with the genus. Furthermore, as the simply laced cases with $\mathcal{R} \neq A_l$ are not toric, this method would, at best, provide proof for the $A$-case only. Therefore, it would be better to use the framework as a check after the fact.

## Attempt 2: Explicit coordinates

In [49], the authors equate the $R$–matrix for $\mathcal{R} = A_1$ from TR with the expression constructed from the Frobenius manifold structure, by finding a closed form expression. This is done by inverting to find and insert $\mu(\zeta)$ explicitly. While this is doable for $l = 1$, it is highly unlikely to work in general. Already for low rank, the inversion requires the use of Lagrange inversion theorem leading to complicated expressions which seem to make general rank expressions unattainable. We are therefore encouraged to pursue other directions.

## Attempt 3: Method of Steepest Descent

A common method for analysing integrals of exponential form is the method of steepest descent. The aim of the method of steepest descent is to approximate an integral of the form

$$\int_\gamma f(z) e^{\alpha g(z)}, \quad (9.3.16)$$

by evaluating its asymptotic as $\alpha \mapsto \infty$. The method requires $f(z)$, $g(z)$ to (locally) be complex analytic functions in order to deform the contour $\gamma \to \gamma'$ such that the imaginary part of $g(z)$ is constant along the new contour, making it possible to approximate the remaining integral by Laplace's method.

Consider (9.3.14). As the contour goes along the real line, the imaginary part of the function in the exponent of the integrand is indeed constant, and so we are already on the required steepest descent integration contour. However, whenever $i = j$, and $k = 0$ we have a pole at the $\mu = a_j \Leftrightarrow \zeta_j = 0$, which necessarily lies on the steepest descent path, making the integral ill-defined. This severely complicates using this method to obtain the $R$-matrix, even order-by-order.

## Attempt 4: Relating to ODEs

A different approach, inspired by [26, 29], consists of relating the integrals to an integral representation of a solution of an ordinary differential equation (at least asymptotically, which is sufficient for the purposes of the $R$-matrix). More precisely, suppose we have an integral of the form

$$\psi(x) = \int_\gamma e^{V(t) + tp(x)} \mathrm{d}t, \qquad (9.3.17)$$

such that $V'(t) \in \mathbb{C}[t, t^{-1}]^\dagger$, and $\gamma$ is some domain. Then the associated differential equation is given by

$$V'\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)\psi = p(x)\psi. \qquad (9.3.18)$$

Thus, the analysis of the asymptotics of exponential integrals, is reduced to the analysis of the asymptotics of a collection of ODEs.

Explicitly, we have the following recipe

1. integrate by parts to eliminate the poles in the pre-exponential factor (while in $\mu$);

2. expand into a finite number of terms forming a basis of solutions, whose size should be equal to the number of ramification points of $\lambda$;

3. find an associated ODE, as above, for each term.

**Remark.** Note that in our case, we have some additional facts to our disposal;

- if the minimal nontrivial irreducible representation $\rho$ as in Chapter 5 is real, we have $\lambda(\mu) = \lambda\left(\frac{1}{\mu}\right)$,

- for type $A$, the Frobenius manifold associated to $k$ is equivalent to the one associated to $l + 1 - k$ for any rank $l$,

- changing $\lambda \mapsto \alpha\lambda$ for any constant $\alpha$ results in an equivalent Frobenius manifold.

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$^\dagger$Note that this allows for simple logarithmic terms in $V(t)$.

These facts could aid in the analysis of the $R$-matrix integral.

**Example 36** ($\mathcal{R} = A_1$)**.** In this case we have

$$\lambda = -\frac{w_0(\mu^2 + 1 - w_1\mu)}{\mu} \xrightarrow{(w_0,\underline{w})=(1,\underline{0})} -\mu - \frac{1}{\mu}, \tag{9.3.19}$$

which matches that of [49]. Then, by (9.3.15):

$$\left(R^{-1}\right)_i^j = -\frac{\beta_i\Delta_i e^{-\frac{\lambda(a_j)}{u}}}{\sqrt{2\pi u}} \int_{\Gamma_j} \frac{1 - \frac{1}{\mu^2}}{a_i - \mu} e^{\frac{\lambda}{u}} \mathrm{d}\mu, \tag{9.3.20}$$

with $a_i = \pm 1$. Note that by inputting $a_i$, we obtain two terms proportional to

$$I_1 := \int_{\Gamma_j} \frac{1}{\mu} e^{\frac{\lambda}{u}} \mathrm{d}\mu, \quad I_2 = \int_{\Gamma_j} \frac{1}{\mu^2} e^{\frac{\lambda}{u}} \mathrm{d}\mu. \tag{9.3.21}$$

Let us consider the recipe above. We have already integrated by parts to eliminate the poles, i.e. step 1, as well as expanded into terms whose number equals the number of ramification points. Thus, we want to manipulate the two integrals $I_1$, $I_2$ to be of the appropriate form (9.3.17). Letting $u \mapsto \frac{1}{u}$, and $\tilde{\mu} = -\frac{\mu}{u}$, we get

$$\frac{I_1}{u} = -\int_{\Gamma_j} e^{-(\tilde{\mu}u + \frac{1}{\tilde{\mu}}) - \log(\mu)} \mathrm{d}\tilde{\mu}, \quad \frac{I_2}{u} = -\int_{\Gamma_j} e^{-(\tilde{\mu}u + \frac{1}{\tilde{\mu}}) - 2\log(\mu)} \mathrm{d}\tilde{\mu}, \tag{9.3.22}$$

which are both of the form (9.3.17), with

$$V_1(t) = -\log(t) - \frac{1}{t}, \implies V_1'(t) = -\frac{1}{t} + \frac{1}{t^2}, \tag{9.3.23a}$$

$$V_1(t) = -2\log(t) - \frac{1}{t}, \implies V_1'(t) = -\frac{2}{t} + \frac{1}{t^2}, \tag{9.3.23b}$$

and $p_1(x) = p_2(x) = x$. Now, if we further let $u \mapsto -\frac{u^2}{4}$, we find

$$I_1 \sim \frac{1}{2\pi i} \int_{-\infty}^{0+} e^{\tilde{\mu} - \frac{u^2}{4\tilde{\mu}}} \frac{\mathrm{d}\tilde{\mu}}{t}, \quad I_2 \sim \frac{u}{2\pi i} \int_{-\infty}^{0+} e^{\tilde{\mu} - \frac{u^2}{4\tilde{\mu}}} \frac{\mathrm{d}\tilde{\mu}}{t^2}, \tag{9.3.24}$$

which is an integral representation of the Bessel function of the first kind , i.e. an integral representation of Bessel's differential equation [55, Equation 10.9.19], given by

$$\Omega_\nu(z) = \frac{(\frac{z}{2})^\nu}{2\pi i} \int_{-\infty}^{0+} e^{t - \frac{z^2}{4t}} \frac{\mathrm{d}t}{t^{\nu+1}}, \tag{9.3.25}$$

for $z = u, t = \tilde{\mu}$ and with $\nu = 0, 1$ for $I_1$, $I_2$, respectively, for which the asymptotic behaviour is well known.

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The next step would be to consider the asymptotic analysis of the ODE associated to the case $\mathcal{R} = A_2$ obtained by the recipe, and make a comparison to the $R$-matrix arising from the Frobenius structure in this case as found by Example 34. Thereafter, it would be natural to attempt to find an integral representation of generalised Bessel type, from the $R$-matrix from TR-theory for arbitrary rank $l$, and perhaps show that it respects the recursion relation and the homogeneity condition (9.1.14), (9.1.15). This would provide a proof on the level of cohomological field theories. We see this as one of the more promising directions towards a proof of Conjecture 9.3.0.1.

**Remark.** An additional possible validation of Conjecture 9.3.0.1, comes from the notion of the $G$–function. The $G$–function is a function on a semisimple Frobenius manifold solving the Getzler system of equations [65], [45]:

$$\sum_{1 \leqslant \alpha_1, \alpha_2, \alpha_3, \alpha_4 \leqslant \dim} z_{\alpha_1} z_{\alpha_2} z_{\alpha_3} z_{\alpha_4} \Delta_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = 0, \tag{9.3.26}$$

where

$$\Delta_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} = 3 c^\mu_{\alpha_1 \alpha_2} c^\nu_{\alpha_3 \alpha_4} \frac{\partial^2 G}{\partial t^\mu \partial t^\nu} - 4 c^\mu_{\alpha_1 \alpha_2} c^\nu_{\alpha_3 \mu} \frac{\partial^2 G}{\partial t^{\alpha_4} \partial t^\nu} - c^\mu_{\alpha_1 \alpha_2} c^\nu_{\alpha_3 \alpha_4 \mu} \frac{\partial G}{\partial t^\nu} + 2 c^\mu_{\alpha_1 \alpha_2 \alpha_3} c^\nu_{\alpha_4 \mu} \frac{\partial G}{\partial t^\nu} +$$

$$+ \frac{1}{6} c^\mu_{\alpha_1 \alpha_2 \alpha_3} c^\nu_{\alpha_4 \mu \nu} + \frac{1}{24} c^\mu_{\alpha_1 \alpha_2 \alpha_3 \alpha_4} c^\nu_{\mu \nu} - \frac{1}{4} c^\mu_{\alpha_1 \alpha_2 \nu} c^\nu_{\alpha_3 \alpha_4 \mu},$$

with $c_{i_1, \cdots, i_m} := \frac{\partial^m F(t)}{\partial t^{i_1}, \cdots, \partial t^{i_m}}$.

Furthermore, the $G$–function encodes the data of the first order deformation of the associated principal hierarchy (the first step of the right-most vertical arrow in Figure 1.2) as

$$G = \log\left(\frac{\tau_I}{\Psi^{\frac{1}{24}}}\right) \implies \mathcal{F}_1(t, t_X) = \frac{1}{24} \log(\det(c_{\alpha\beta\gamma}(t) t_X^\gamma)) + G(t), \tag{9.3.27}$$

with $\tau_I$ being the isomonodromic Tau–function, and $\mathcal{F}_1$ the free energy.

Thus, as the results of Chapter 5 give explicit prepotentials, it is possible to explicitly solve (9.3.26) and describe genus one GW-invariants by comparing to the result obtained by utilising the framework described in Method 1 above. More generally, the methods of [84, 85], would lend themselves to the verification of the two-point function in the TR-spectral data. As a bonus, the $G$–function together with a description of the DZ-manifold caustic (for $\mathcal{R} \neq A_l$), would finally settle a conjecture stated by Ian Strachan in [117]. We hope to find $G$–functions associated to DZ-manifolds in the near future.

**Remark.** In [56], the authors considered the equivariant version of the spectral curve associated to $\mathcal{R} = A_1$ (that is including parameters instead of fixing at a semisimple point) and proved results which in the non-equivariant limit, reduced to the Norbury–Scott Conjecture. They also considered another limit, the large radius limit, at which their results provide a proof of the Bouchard–Marino

conjecture of simple Hurwitz numbers [15]. Thus, it would be natural and interesting to formulate a similar conjecture and consider the analogous limit for simply laced root types other than $A_1$.

# CONCLUSION

In this thesis we have derived a mirror theorem for Dubrovin-Zhang Frobenius manifolds taking a uniform form for all Dynkin types, in which its proof is reduced to quite straight-forward complex analysis and Lie theory. These results are important for a number of reasons. Firstly, it is nice to finally settle a problem which has stood open since the late 1990s, and to do so in a uniform way. Secondly, via the relation with orbicurves, these Frobenius manifolds provide explicit examples of a class of non-toric DM–stacks, which are certainly not of abundance in literature. Finally, the results open up many fascinating avenues of research, some of which have been successfully dealt with in this thesis. We have considered four applications belonging to quite different areas of mathematics; topological degrees of LL-maps, Saito discriminant strata, integrable hierarchies, and the Topological Recursion procedure. Some of these have been, or are near to be, completed (the first two), and some have been commenced (the last two). We hope to continue research in these directions in the near future. There are many more applications and adjacent research questions which have not been dealt with in this thesis. These include calculating $G$-functions associated to DZ-manifolds, the Bouchard–Marino conjecture (both of which were briefly mentioned in the final chapter), Buryak's conjecture which relates the Dubrovin-Zhang hierarchy to the double-ramification hierarchy constructed using moduli spaces of curves (this conjecture is represented by the small diagonal dotted line in the top right corner of Figure 1.2), and integrable hierarchies associated to discriminant strata, to name a few.

We hope the reader has enjoyed this window into the world of Frobenius manifolds, and its large variety of applications. We expect the results of this thesis will lead to many exciting publications in the future.

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# Part IV

# Appendices

# Character relations for $E_6$ and $E_7$

## A.1  $\mathcal{R} = E_6$

The character relations for $\rho = \rho_{\omega_1}$ and $k = 1, \ldots, 13$ are:

$$\mathfrak{p}_0^{[100000]_{E_6}} = 1,$$

$$\mathfrak{p}_1^{[100000]_{E_6}} = \chi_1,$$

$$\mathfrak{p}_2^{[100000]_{E_6}} = \chi_2,$$

$$\mathfrak{p}_3^{[100000]_{E_6}} = \chi_3,$$

$$\mathfrak{p}_4^{[100000]_{E_6}} = -\chi_5^2 - \chi_2\chi_5 + \chi_1 + \chi_4 + \chi_4\chi_6,$$

$$\mathfrak{p}_5^{[100000]_{E_6}} = \chi_1^2 - 2\chi_5^2\chi_1 + 2\chi_4\chi_1 + \chi_4^2 + \chi_5\chi_6^2 + \chi_2 - 2\chi_3\chi_5 + \chi_5 - \chi_2\chi_6 - \chi_5\chi_6,$$

$$\begin{aligned}
\mathfrak{p}_6^{[100000]_{E_6}} = {} & \chi_1\chi_5 - \chi_5^3 - \chi_2\chi_5^2 + 2\chi_4\chi_5 - 2\chi_1\chi_6\chi_5 + \chi_4\chi_6\chi_5 + \chi_6^3 \\
& + 2\chi_1\chi_2 - 2\chi_3 + \chi_2\chi_4 - 3\chi_3\chi_6,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p}_7^{[100000]_{E_6}} = {} & 2\chi_2^2 + \chi_5\chi_2 - 2\chi_5\chi_6\chi_2 + \chi_3\chi_5^2 + \chi_1\chi_6^2 + \chi_4\chi_6^2 - 3\chi_1\chi_3 \\
& - 2\chi_3\chi_4 + \chi_4 - \chi_5^2\chi_6 - \chi_1\chi_6 + \chi_4\chi_6,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p}_8^{[100000]_{E_6}} = {} & \chi_2\chi_5^3 - \chi_1\chi_6\chi_5^2 + \chi_6^2\chi_5 + \chi_1\chi_2\chi_5 - 2\chi_3\chi_5 - 3\chi_2\chi_4\chi_5 + \chi_3\chi_6\chi_5 \\
& - 2\chi_6\chi_5 + \chi_5 - \chi_1^2 - \chi_2\chi_6^2 + \chi_2\chi_3 + \chi_1\chi_4 + \chi_1^2\chi_6 - \chi_2\chi_6 + 2\chi_1\chi_4\chi_6,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p}_9^{[100000]_{E_6}} = {} & \chi_1\chi_5^4 - \chi_6\chi_5^3 + \chi_2\chi_5^2 - 4\chi_1\chi_4\chi_5^2 + \chi_2\chi_6\chi_5^2 - \chi_2^2\chi_5 - \chi_1\chi_6^2\chi_5 - 4\chi_1\chi_5 \\
& + \chi_4\chi_5 + 3\chi_4\chi_6\chi_5 + \chi_6^3 + \chi_3^2 + 2\chi_1\chi_4^2 + \chi_1\chi_2 - 6\chi_3 + 4\chi_1^2\chi_4 - 4\chi_2\chi_4 \\
& + 4\chi_1\chi_3\chi_5 - 2\chi_2\chi_4\chi_6 - 3\chi_3\chi_6 + 3,
\end{aligned}$$

$$\begin{aligned}
\mathfrak{p}_{10}^{[100000]_{E_6}} = {} & \chi_5^5 - 5\chi_4\chi_5^3 + \chi_1\chi_6\chi_5^3 - \chi_6^2\chi_5^2 - \chi_1\chi_2\chi_5^2 + 5\chi_3\chi_5^2 - \chi_5^2 - 2\chi_1^2\chi_5 + 5\chi_4^2\chi_5 + \chi_2\chi_5 \\
& + \chi_2\chi_3\chi_5 + 4\chi_1\chi_4\chi_5 + \chi_1^2\chi_6\chi_5 - 2\chi_2\chi_6\chi_5 - 3\chi_1\chi_4\chi_6\chi_5 + \chi_1\chi_6^2 + 2\chi_4\chi_6^2 + 2\chi_1 \\
& + \chi_1^2\chi_2 - 5\chi_1\chi_3 + 2\chi_1\chi_2\chi_4 - 5\chi_3\chi_4 - \chi_2^2\chi_6 - 2\chi_1\chi_6 + \chi_1\chi_3\chi_6 - \chi_4\chi_6,
\end{aligned}$$

$$\mathfrak{p}_{11}^{[100000]_{E_6}} = \chi_6\chi_5^4 - \chi_2\chi_5^3 + \chi_1\chi_3\chi_5^2 - \chi_4\chi_5^2 + 2\chi_1\chi_6\chi_5^2 - 4\chi_4\chi_6\chi_5^2 - 2\chi_6^2\chi_5 - \chi_1\chi_2\chi_5 + \chi_2\chi_6$$
$$+ \quad 3\chi_3\chi_5 + 3\chi_2\chi_4\chi_5 - 2\chi_1\chi_2\chi_6\chi_5 + 3\chi_3\chi_6\chi_5 - \chi_6\chi_5 + \chi_1\chi_2^2 + 2\chi_4^2 + 2\chi_4^2\chi_6$$
$$+ \quad 2\chi_1^2\chi_6^2 - 2\chi_2\chi_6^2 + 2\chi_2 - 3\chi_1^2\chi_3 + \chi_2\chi_3 + \chi_2^2\chi_4 + \chi_1\chi_4 - 2\chi_1\chi_3\chi_4 - 2\chi_1^2\chi_6,$$

$$\mathfrak{p}_{12}^{[100000]_{E_6}} = 2\chi_6\chi_1^3 + \chi_5^2\chi_1^2 - \chi_4\chi_1^2 - 2\chi_2\chi_5\chi_1^2 - \chi_5^2\chi_6\chi_1^2 + \chi_4\chi_6\chi_1^2 + 3\chi_5\chi_6^2\chi_1 + 2\chi_2\chi_1 + 6\chi_3\chi_6$$
$$- \quad 3\chi_2\chi_3\chi_1 + \chi_2\chi_4\chi_5\chi_1 + \chi_5\chi_1 - 5\chi_2\chi_6\chi_1 - 2\chi_5\chi_6\chi_1 + \chi_2^3 + \chi_3\chi_5^3 - 3\chi_4\chi_5\chi_6 - \chi_1^3$$
$$- \quad \chi_5^3 - 2\chi_6^3 + 3\chi_3^2 - \chi_3 + \chi_2\chi_4 + 3\chi_2^2\chi_5 - 3\chi_3\chi_4\chi_5 + 2\chi_4\chi_5 + \chi_5^3\chi_6 - \chi_2\chi_5^2\chi_6,$$

$$\mathfrak{p}_{13}^{[100000]_{E_6}} = \chi_1^4 - 2\chi_5^2\chi_1^3 + 2\chi_4\chi_1^3 + \chi_4^2\chi_1^2 - 3\chi_2\chi_1^2 - 2\chi_3\chi_5\chi_1^2 - \chi_5\chi_1^2 - \chi_2\chi_6\chi_1^2 + 4\chi_5\chi_6\chi_1^2$$
$$+ \quad 2\chi_5^3\chi_1 + 2\chi_2\chi_5^2\chi_1 - 2\chi_6^2\chi_1 + \chi_3\chi_1 - 4\chi_2\chi_4\chi_1 + \chi_2^2\chi_5\chi_1 - 4\chi_4\chi_5\chi_1 - \chi_5^3\chi_6\chi_1$$
$$+ \quad 3\chi_3\chi_6\chi_1 + \chi_4\chi_5\chi_6\chi_1 - \chi_6\chi_1 + 2\chi_1 + \chi_2^2 - 2\chi_2\chi_4^2 + \chi_3\chi_5^2 + \chi_2\chi_4\chi_5^2$$
$$+ \quad \chi_5^2\chi_6^2 - 3\chi_3\chi_4 + 2\chi_4 + \chi_2\chi_5 - \chi_2\chi_3\chi_5 + \chi_2^2\chi_6 - 2\chi_5^2\chi_6 - 2\chi_2\chi_5\chi_6. \qquad \text{(A.1.1)}$$

## A.2 $\mathcal{R} = E_7$

We include here the character relations for $\rho = \rho_{\omega_6}$ and $k = 1, \ldots, 11$; note that by reality, we have $\mathfrak{p}_{56-k}^{\rho} = \mathfrak{p}_{k}^{\rho}$. The full set of character relations for $k$ up to 28 is available upon request.

$$\mathfrak{p}_0^{[0000010]_{E_7}} = 1,$$
$$\mathfrak{p}_1^{[0000010]_{E_7}} = \chi_6,$$
$$\mathfrak{p}_2^{[0000010]_{E_7}} = \chi_5 + 1,$$
$$\mathfrak{p}_3^{[0000010]_{E_7}} = \chi_4 + \chi_6,$$
$$\mathfrak{p}_4^{[0000010]_{E_7}} = \chi_3 + \chi_5 + 1,$$
$$\mathfrak{p}_5^{[0000010]_{E_7}} = -(\chi_1 - 1)\chi_4 + \left(-\chi_1^2 + \chi_1 + \chi_2 + \chi_5 + 1\right)\chi_6 + \chi_2\chi_7,$$
$$\mathfrak{p}_6^{[0000010]_{E_7}} = -2\chi_1^3 + (1 - 2\chi_5)\chi_1^2 + \left(\chi_6^2 - \chi_7\chi_6 + \chi_7^2 + 4\chi_2 - 2\chi_3 + 2\chi_5 + 2\right)\chi_1 + \chi_2^2 + \chi_5^2$$
$$- \quad \chi_3 + 2\chi_5 + 2\chi_2(\chi_5 + 1) + \chi_4\chi_6 - \chi_4\chi_7 - \chi_6\chi_7 + 1,$$
$$\mathfrak{p}_7^{[0000010]_{E_7}} = \chi_4\left(-\chi_1^2 + \chi_1 + \chi_2 + 2\chi_5 + 2\right) + (-\chi_1^3 + (2\chi_2 + \chi_5 + 3)\chi_1 + 2\chi_2 - 2\chi_3 + \chi_5$$
$$+ \quad 1)\chi_6 + \chi_7\left(-2\chi_1^2 + (\chi_2 - 2\chi_5 + 1)\chi_1 + \chi_7^2 + 3\chi_2 - 3\chi_3\right),$$
$$\mathfrak{p}_8^{[0000010]_{E_7}} = (2\chi_6^2 + \chi_4\chi_6 - 2\chi_7\chi_6 + \chi_7^2 + 4\chi_2 - 2\chi_1^3 - 4\chi_3 + 2\chi_5 - 2\chi_4\chi_7)\chi_1$$
$$+ \quad \chi_2^2 + 2\chi_4^2 + \chi_6^2 + \chi_5\chi_7^2 + \chi_7^2 - 2\chi_3 - 3\chi_3\chi_5 + 3\chi_4\chi_6 + \chi_4\chi_7 - \chi_5\chi_6\chi_7 + \chi_6\chi_7$$
$$+ \quad \chi_2(\chi_6^2 + \chi_7\chi_6 + \chi_7^2 - 2\chi_3 + 2\chi_5) + (\chi_3 - 2\chi_5 - \chi_6\chi_7 + 2)\chi_1^2,$$

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$$\mathfrak{p}_9^{[0000010]_{E_7}} = (\chi_1 + 2)\chi_6^3 - 2\chi_1\chi_7\chi_6^2 + \left(-\chi_1^3 + \chi_1^2 + \left(\chi_7^2 + 2\chi_2 - 2\chi_3 - 2\chi_5\right)\chi_1 - \chi_5^2\right.$$
$$- \; 3\chi_3 - 2\chi_5 + \chi_2(\chi_5 + 1) - 1)\chi_6 + (-(\chi_5 + 2)\chi_1^2 + (\chi_2 + \chi_3 + \chi_5$$
$$+ \; 2)\chi_1 + \chi_5^2 - \chi_3 + 3\chi_5 + 2\chi_2(\chi_5 + 1) + 2)\chi_7 + \chi_4(\chi_1^3 - \chi_1^2 + (-3\chi_2 + \chi_5$$
$$+ \; 2)\chi_1 - \chi_7^2 - 2\chi_2 + \chi_3 + 3\chi_5 - \chi_6\chi_7 + 4),$$

$$\mathfrak{p}_{10}^{[0000010]_{E_7}} = \chi_5\chi_1^4 - \chi_6\chi_7\chi_1^3 + \left(-\chi_6^2 - 2\chi_7\chi_6 + \chi_7^2 - (4\chi_2 + 1)\chi_5 + \chi_4(\chi_6 + \chi_7)\right)\chi_1^2$$
$$- \; (\chi_4^2 + 3\chi_7\chi_4 - 4\chi_5^2 - \chi_2\chi_6^2 - 2\chi_6^2 + \chi_7^2 - 3\chi_2\chi_6\chi_7 - 3\chi_6\chi_7 + \chi_5\left(4\chi_6^2 + \chi_7^2 - 2\right))\chi_1$$
$$+ \; \chi_6\chi_7^3 + \chi_3^2 + 4\chi_2\chi_5^2 + 2\chi_5^2 + \chi_2\chi_6^2 - 5\chi_5\chi_6^2 - 5\chi_6^2 - \chi_2\chi_7^2 - \chi_5\chi_7^2 + 3\chi_6^4$$
$$+ \; 5\chi_5 - 4\chi_2\chi_4\chi_6 + 3\chi_4\chi_6 + \chi_4\chi_5\chi_6 - 2\chi_2\chi_4\chi_7 + \chi_4\chi_7 + 2\chi_2\chi_6\chi_7 + 2\chi_5\chi_6\chi_7$$
$$+ \; 2\chi_2\chi_5 + \chi_6\chi_7 + \chi_3\left(-6\chi_6^2 - 3\chi_7\chi_6 + (4\chi_1 + 5)\chi_5 + 4\right) - \chi_7^2 + 2\chi_2^2\chi_5 + 3,$$

$$\mathfrak{p}_{11}^{[0000010]_{E_7}} = \left(-\chi_1^2 + 4\chi_1 + 2\chi_5\right)\chi_6^3 + (\chi_1 - \chi_2 - 2(\chi_5 + 1))\chi_7\chi_6^2 + (\chi_1^5 - 5(\chi_2 + 1)\chi_1^3$$
$$+ \; \left(-\chi_7^2 + \chi_2 + 5\chi_3 - 4\chi_5 + 3\right)\chi_1^2 + (5\chi_2^2 + 4(\chi_5 + 2)\chi_2 - 2\chi_5^2$$
$$- \; 8\chi_3 - 1)\chi_1 + 3\chi_2^2 - \chi_5^2 + \chi_5\chi_7^2 + \chi_7^2 - 2\chi_3 - 5\chi_3\chi_5$$
$$- \; \chi_5 + \chi_2\left(2\chi_7^2 - 5\chi_3 + 7\chi_5 + 2\right) + 1)\chi_6 - \chi_4^2\chi_7 + \chi_4(2\chi_1^3 - (\chi_5 - 1)\chi_1^2$$
$$+ \; \left(\chi_6^2 - 2\chi_7\chi_6 - 6\chi_2 + \chi_3 + 4\chi_5 - 3\right)\chi_1 + \chi_5^2 - \chi_6^2 - \chi_7^2 + 5\chi_3$$
$$+ \; 3\chi_5 + \chi_2(2\chi_5 - 7) + 2\chi_6\chi_7 + 3) + \chi_7(-\chi_1^4 + \chi_5\chi_1^3 + (4\chi_2 - 2\chi_5 + 3)\chi_1^2 + (\chi_5^2$$
$$+ \; \chi_5 + \chi_7^2 - 4\chi_3 - \chi_2(3\chi_5 + 1) - 3)\chi_1 - 2\chi_2^2 + \chi_5^2 + 3\chi_3 + \chi_3\chi_5 + 3\chi_5 - \chi_2(\chi_5 + 2)$$
$$+ \; 2).$$

$$(\text{A.2.1})$$

# R-matrices for affine $A_2$

Note that in the notation of Chapter 9, $x = \frac{1}{z}$.

$[x^{-1}]:$
$$\begin{pmatrix} \dfrac{1-i\sqrt{3}}{36\sqrt[3]{2}x} & -\dfrac{i\left(\sqrt{3}-i\right)}{18\sqrt[3]{2}x} & -\dfrac{1}{9\sqrt[3]{2}x} \\[2ex] -\dfrac{i\left(\sqrt{3}-i\right)}{18\sqrt[3]{2}x} & -\dfrac{1}{18\sqrt[3]{2}x} & \dfrac{i\left(\sqrt{3}+i\right)}{18\sqrt[3]{2}x} \\[2ex] -\dfrac{1}{9\sqrt[3]{2}x} & \dfrac{i\left(\sqrt{3}+i\right)}{18\sqrt[3]{2}x} & \dfrac{1+i\sqrt{3}}{36\sqrt[3]{2}x} \end{pmatrix},$$

$[x^{-2}]:$
$$\begin{pmatrix} \dfrac{1+i\sqrt{3}}{432\,2^{2/3}x^2} & \dfrac{-7-3i\sqrt{3}}{324\,2^{2/3}x^2} & \dfrac{1+5i\sqrt{3}}{324\,2^{2/3}x^2} \\[2ex] \dfrac{4+i\sqrt{3}}{162\,2^{2/3}x^2} & -\dfrac{1}{216\,2^{2/3}x^2} & \dfrac{4-i\sqrt{3}}{162\,2^{2/3}x^2} \\[2ex] \dfrac{1-5i\sqrt{3}}{324\,2^{2/3}x^2} & \dfrac{-7+3i\sqrt{3}}{324\,2^{2/3}x^2} & \dfrac{1-i\sqrt{3}}{432\,2^{2/3}x^2} \end{pmatrix},$$

$[x^{-3}]:$
$$\begin{pmatrix} -\dfrac{5}{69984x^3} & -\dfrac{205+5i\sqrt{3}}{23328x^3} & \dfrac{5\left(41-i\sqrt{3}\right)}{23328x^3} \\[2ex] \dfrac{5i\left(\sqrt{3}+41i\right)}{23328x^3} & -\dfrac{5}{69984x^3} & -\dfrac{205+5i\sqrt{3}}{23328x^3} \\[2ex] \dfrac{5\left(41+i\sqrt{3}\right)}{23328x^3} & \dfrac{5i\left(\sqrt{3}+41i\right)}{23328x^3} & -\dfrac{5}{69984x^3} \end{pmatrix},$$

$[x^{-4}]:$
$$\begin{pmatrix} \dfrac{4025i\left(\sqrt{3}+i\right)}{10077696\sqrt[3]{2}x^4} & \dfrac{175i\left(23\sqrt{3}+61i\right)}{1259712\sqrt[3]{2}x^4} & \dfrac{175\left(2-21i\sqrt{3}\right)}{629856\sqrt[3]{2}x^4} \\[2ex] \dfrac{175\left(65-19i\sqrt{3}\right)}{1259712\sqrt[3]{2}x^4} & \dfrac{4025}{5038848\sqrt[3]{2}x^4} & \dfrac{175\left(65+19i\sqrt{3}\right)}{1259712\sqrt[3]{2}x^4} \\[2ex] \dfrac{175\left(2+21i\sqrt{3}\right)}{629856\sqrt[3]{2}x^4} & \dfrac{175\left(-61-23i\sqrt{3}\right)}{1259712\sqrt[3]{2}x^4} & -\dfrac{4025\left(1+i\sqrt{3}\right)}{10077696\sqrt[3]{2}x^4} \end{pmatrix},$$

$[x^{-5}]:$
$$\begin{pmatrix} -\dfrac{17395\left(1+i\sqrt{3}\right)}{20155392\,2^{2/3}x^5} & \dfrac{1225i\left(639\sqrt{3}+529i\right)}{90699264\,2^{2/3}x^5} & \dfrac{1225\left(-1223-55i\sqrt{3}\right)}{90699264\,2^{2/3}x^5} \\[2ex] \dfrac{1225i\left(292\sqrt{3}+347i\right)}{45349632\,2^{2/3}x^5} & \dfrac{17395}{10077696\,2^{2/3}x^5} & \dfrac{1225\left(-347-292i\sqrt{3}\right)}{45349632\,2^{2/3}x^5} \\[2ex] \dfrac{1225i\left(55\sqrt{3}+1223i\right)}{90699264\,2^{2/3}x^5} & \dfrac{1225\left(-529-639i\sqrt{3}\right)}{90699264\,2^{2/3}x^5} & \dfrac{17395i\left(\sqrt{3}+i\right)}{20155392\,2^{2/3}x^5} \end{pmatrix},$$

$[x^{-6}]:$
$$\begin{pmatrix} \dfrac{31237745}{19591041024x^6} & \dfrac{2695\left(811+11591i\sqrt{3}\right)}{3265173504x^6} & \dfrac{2695i\left(11591\sqrt{3}+811i\right)}{3265173504x^6} \\[2ex] \dfrac{2695\left(811-11591i\sqrt{3}\right)}{3265173504x^6} & \dfrac{31237745}{19591041024x^6} & \dfrac{2695\left(811+11591i\sqrt{3}\right)}{3265173504x^6} \\[2ex] -\dfrac{2695i\left(11591\sqrt{3}-811i\right)}{3265173504x^6} & \dfrac{2695\left(811-11591i\sqrt{3}\right)}{3265173504x^6} & \dfrac{31237745}{19591041024x^6} \end{pmatrix},$$

$$[x^{-7}]: \begin{pmatrix} \dfrac{961865905i\left(\sqrt{3}+i\right)}{705277476864\sqrt[3]{2}x^7} & \dfrac{385385\left(17159+17471i\sqrt{3}\right)}{352638738432\sqrt[3]{2}x^7} & \dfrac{385385\left(17393-78i\sqrt{3}\right)}{176319369216\sqrt[3]{2}x^7} \\[3mm] \dfrac{385385\left(17627+17315i\sqrt{3}\right)}{352638738432\sqrt[3]{2}x^7} & \dfrac{961865905}{352638738432\sqrt[3]{2}x^7} & \dfrac{385385\left(17627-17315i\sqrt{3}\right)}{352638738432\sqrt[3]{2}x^7} \\[3mm] \dfrac{385385\left(17393+78i\sqrt{3}\right)}{176319369216\sqrt[3]{2}x^7} & \dfrac{385385\left(17159-17471i\sqrt{3}\right)}{352638738432\sqrt[3]{2}x^7} & -\dfrac{961865905\left(1+i\sqrt{3}\right)}{705277476864\sqrt[3]{2}x^7} \end{pmatrix},$$

$$[x^{-8}]: \begin{pmatrix} -\dfrac{63122484425\left(1+i\sqrt{3}\right)}{33853318889472\;2^{2/3}x^8} & \dfrac{3578575\left(148873+52917i\sqrt{3}\right)}{6347497291776\;2^{2/3}x^8} & -\dfrac{3578575i\left(100895\sqrt{3}-4939i\right)}{6347497291776\;2^{2/3}x^8} \\[3mm] -\dfrac{3578575i\left(23989\sqrt{3}-76906i\right)}{3173748645888\;2^{2/3}x^8} & \dfrac{63122484425}{16926659444736\;2^{2/3}x^8} & \dfrac{3578575i\left(23989\sqrt{3}+76906i\right)}{3173748645888\;2^{2/3}x^8} \\[3mm] \dfrac{3578575i\left(100895\sqrt{3}+4939i\right)}{6347497291776\;2^{2/3}x^8} & \dfrac{3578575\left(148873-52917i\sqrt{3}\right)}{6347497291776\;2^{2/3}x^8} & \dfrac{63122484425i\left(\sqrt{3}+i\right)}{33853318889472\;2^{2/3}x^8} \end{pmatrix},$$

$$[x^{-9}]: \begin{pmatrix} \dfrac{1588239577925}{16452712980283392x^9} & \dfrac{60835775\left(4410203+26107i\sqrt{3}\right)}{1828079220031488x^9} & \dfrac{60835775i\left(26107\sqrt{3}+4410203i\right)}{1828079220031488x^9} \\[3mm] \dfrac{60835775\left(4410203-26107i\sqrt{3}\right)}{1828079220031488x^9} & \dfrac{1588239577925}{16452712980283392x^9} & \dfrac{60835775\left(4410203+26107i\sqrt{3}\right)}{1828079220031488x^9} \\[3mm] -\dfrac{60835775i\left(26107\sqrt{3}-4410203i\right)}{1828079220031488x^9} & \dfrac{60835775\left(4410203-26107i\sqrt{3}\right)}{1828079220031488x^9} & \dfrac{1588239577925}{16452712980283392x^9} \end{pmatrix}.$$