RESEARCH ARTICLE

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Exterior and interior metrics with quadrupole moment

Abstract We present the Ernst potential and the line element of an exact solution of Einstein's vacuum field equations that contains as arbitrary parameters the total mass, the angular momentum, and the quadrupole moment of a rotating mass distribution. We show that in the limiting case of slowly rotating and slightly deformed configuration, there exists a coordinate transformation that relates the exact solution with the approximate Hartle solution. It is shown that this approximate solution can be smoothly matched with an interior perfect fluid solution with physically reasonable properties. This opens the possibility of considering the quadrupole moment as an additional physical degree of freedom that could be used to search for a realistic exact solution, representing both the interior and exterior gravitational field generated by a self-gravitating axisymmetric distribution of mass of perfect fluid in stationary rotation.

Keywords Quadrupole, Exterior solutions, Interior solutions

1 Introduction

Astrophysical compact objects are in general not spherically symmetric rotating mass distributions. To describe the corresponding gravitational field one can assume axial symmetry, with an axis of symmetry that coincides with the axis of rotation. Then, the deviation from spherical symmetry can be described by means of axisymmetric multipole moments.

In general relativity, vacuum solutions with multipole moments have been known for a long time. In fact, if we restrict ourselves to solutions with only

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monopole moment, Birkoff's theorem guarantees that the only solution is the Schwarzschild metric. Moreover, if rotation is also taken into account, the black hole uniqueness theorems state that the only asymptotically flat solution with a regular horizon is the Kerr metric [1]. The next interesting multipole is the quadrupole. In this case, the uniqueness theorems do not apply and it is possible to find a large number of different vacuum solutions with the same quadrupole. Differences appear only at the level of higher multipoles. The first static solution with an arbitrary quadrupole was found by Weyl [1], using cylindrical coordinates. Later on, Erez and Rosen (ER) [2] discovered a solution with arbitrary quadrupole in prolate spheroidal coordinates which are more convenient for the investigation of multipole solutions. Zipoy [3] and Voorhees [4] found a simple transformation which allows to generate static solutions from a given one. In particular, the Zipoy–Voorhees (ZV) transformation can be used to generate the simplest solution with quadrupole, starting from the Schwarzschild metric. Stationary solutions represent an additional challenge. In fact, the first physically relevant rotating solution was found only in 1963 by Kerr [5]. The Ernst representation of stationary axisymmetric fields was an important achievement that allowed to search for the symmetries of the field equations upon which modern solution generating techniques are based.

In this work, we will limit ourselves to the study of a particular stationary axisymmetric vacuum solution which was derived by Quevedo and Mashhoon (QM) in [6; 7] as a generalization of the ER metric [2]. The QM solution contains in general an infinite number of gravitational and electromagnetic multipole moments. Here, however, we neglect the electromagnetic field and focus on the contribution of the gravitational quadrupole only. The main goal of the present work is to show that the QM solution can be used to describe the exterior gravitational field of rotating compact objects. First, we will see that the set of independent and arbitrary parameters entering the metric determines the total mass, angular momentum and mass quadrupole moment of the source. Although higher multipole moments are present, they all can be expressed in terms of the independent lower multipoles. This result is obtained by using the invariant definition of relativistic multipole moments proposed by Geroch and Hansen [8; 9; 10]. The spacetime turns out to be asymptotically flat and and free of singularities outside a region which can be "covered" by an interior perfect fluid solution. This last property, however, is shown only in the limiting case of a slightly deformed body with uniform and slow rotation.

There exists in the literature a reasonable number of interior spherically symmetric solutions which can be matched with the exterior Schwarzschild metric. Nevertheless, a major problem of classical general relativity consists in finding a physically reasonable interior solution for the exterior Kerr metric. Although it is possible to match numerically the Kerr solution with the interior field of an infinitely tiny rotating disk of dust [11], such a hypothetical system does not seem to be of relevance to describe astrophysical compact objects. It is now widely believed that the Kerr solution is not appropriate to describe the exterior field of rapidly rotating compact objects. Indeed, the Kerr metric takes into account the total mass and the angular momentum of the body. However, the moment of inertia is an additional characteristic of any realistic body which should be considered in order to correctly describe the gravitational field. As a consequence, the multipole

moments of the field created by a rapidly rotating compact object are different from the multipole moments of the Kerr metric. For this reason a solution with arbitrary sets of multipole moments, such as the QM solution, can be used to describe the exterior field of arbitrarily rotating mass distributions.

To completely characterize the spacetime it is necessary to find an exact solution with a set of interior multipole moments. Due to the generality of the exterior solution, one can expect that the corresponding interior solution can be derived by postulating an infinite series of inner multipoles with arbitrary functions which can be matched one by one with the exterior solution. To see if this procedure is realizable, in this work we analyze the special case of a deformed rotating body with quadrupole moment only. Despite this simplification, the problem of finding interior solutions is at present still out of reach due, in part, to the complexity of Einstein's equations with a realistic model for the inner configuration. We therefore limit ourselves here to the study of approximate perfect fluid solutions.

In fact, in the case of slowly rotating compact objects it is possible to find approximate interior solutions with physically meaningful energy-momentum tensors and state equations. Because of its physical importance, in this work we will study the Hartle–Thorne (HT) [12; 13] interior solution which can be coupled to an approximate exterior metric. Hereafter this solution will be denoted as the HT solution. One of the most important characteristics of this family of solutions is that the corresponding equation of state has been constructed using realistic models for the internal structure of relativistic stars. Semi-analytical and numerical generalizations of the HT metrics with more sophisticated equations of state have been proposed by different authors. A comprehensive review of these solutions is given in [14]. In all these cases, however, it is assumed that the multipole moments (quadrupole and octupole) are relatively small and that the rotation is slow. We will find the explicit coordinate transformation that transforms an approximation of the QM solution into the exterior HT solution and can be matched with an approximate solution. In this manner, we show that the QM solution satisfies the main physical conditions to be matched with an interior solution.

2 Exterior solution

It is well known that the main physical information about an axisymmetric stationary vacuum gravitational field can be encoded in the complex Ernst potential [15; 16]. For a special case of the QM solution the Ernst potential reads

$$E = e^{-2\psi} \frac{x - 1 - (x+1)\lambda\mu + i[y(\lambda + \mu) + \lambda - \mu]}{x + 1 - (x-1)\lambda\mu + i[y(\lambda + \mu) - \lambda + \mu]}, \tag{1}$$

where the function ψ can be written in terms of the Legendre polynomials P and Legendre functions Q as

$$\psi = (1-\delta)Q_0 - \delta q P_2 Q_2, \quad P_2 = \frac{1}{2}(3y^2 - 1), \tag{2}$$

$$Q_0 = \frac{1}{2}\ln\frac{x+1}{x-1}, \ Q_1 = \frac{x}{2}\ln\frac{x+1}{x-1} - x, \ Q_2 = \frac{1}{4}(3x^2 - 1)\ln\frac{x+1}{x-1} - \frac{3}{2}x. \tag{3}$$

Here δ and q are arbitrary real constants. Moreover, λ and μ are functions of x and y

$$\lambda = \alpha(x^2-1)^{1-\delta}(x+y)^{2\delta-2}e^{2\delta q\beta_-}, \quad \mu = \alpha(x^2-1)^{1-\delta}(x+y)^{2\delta-2}e^{2\delta q\beta_+}, \quad (4)$$

with

$$\beta_\pm = \frac{1}{2}\ln\frac{(x\pm y)^2}{x^2-1} + \frac{3}{2}(1-y^2\mp xy) + \frac{3}{4}\left[x(1-y^2)\mp y(x^2-1)\right]\ln\frac{x-1}{x+1}. \quad (5)$$

The constant α can be represented in terms of the additional constants a and m as

$$\alpha = \frac{\sigma-m}{a}, \quad \sigma = \sqrt{m^2-a^2}. \quad (6)$$

The solution is asymptotically flat and possesses the following independent parameters: m, a, δ, and q. In the limiting case $\alpha = 0$, $a = 0$, $q = 0$ and $\delta = 1$, the only independent parameter is m and the Ernst potential (1) determines the Schwarzschild spacetime. Moreover, for $\alpha = a = 0$ and $q = 0$ we obtain the Ernst potential of the ZV static solution which is characterized by the parameters m and δ. Furthermore, for $\alpha = a = 0$ and $\delta = 1$, the resulting solution coincides with the ER static spacetime [2]. The Kerr metric is also contained as a special case for $q = 0$ and $\delta = 1$.

The physical significance of the parameters entering the potential (1) can be established in an invariant manner by calculating the relativistic Geroch–Hansen multipole moments. We use here the procedure formulated in [17] which allows us to derive the gravitoelectric M_n as well as the gravitomagnetic J_n multipole moments. A lengthly but straightforward calculation yields

$$M_{2k+1} = J_{2k} = 0, \quad k = 0,1,2,... \quad (7)$$
$$M_0 = m+\sigma(\delta-1) \quad (8)$$
$$M_2 = \frac{2}{15}\sigma^3\delta q - \frac{1}{3}\sigma^3(\delta^3-3\delta^2-4\delta+6) - m\sigma^2\delta(\delta-2) - 3m^2\sigma(\delta-1) - m^3, \quad (9)$$
$$J_1 = ma+2a\sigma(\delta-1), \quad (10)$$
$$J_3 = \frac{4}{15}a\sigma^3\delta q - a\left[\frac{2}{3}\sigma^3(\delta^3-3\delta^2-\delta+3)\right.$$
$$\left. + m\sigma^2(3\delta^2-6\delta+2)+4m^2\sigma(\delta-1)+m^3\right]. \quad (11)$$

The even gravitomagnetic and the odd gravitoelectric multipoles vanish identically because the solution possesses and additional reflection symmetry with respect to the hyperplane $y = 0$. Higher odd gravitomagnetic and even gravitoelectric multipoles can be shown to be linearly dependent since they are completely determined in terms of the parameters m, a, q and δ. From the above expressions we see that the ZV [3; 4] parameter changes the value of the total mass M_0 as well as the angular momentum J_1 of the source. The mass quadrupole M_2 can be interpreted as a nonlinear superposition of the quadrupoles corresponding to the ZV, ER and Kerr spacetimes. Notice that in the limiting static case of the ZV metric the only

non-vanishing parameters are $m = \sigma$ and δ so that all gravitomagnetic multipoles vanish and we obtain $M_0 = m\delta$ and $M_2 = \delta m^3(1 - \delta^2)/3$ for the leading gravitoelectric multipoles. This means that the ZV solution represents the field of a static deformed body. To our knowledge, this is the simplest generalization of the Schwarschild solution which includes a quadrupole moment.

In order to completely describe the geometric and physical properties of the spacetime, it is convenient to calculate the explicit form of the metric. In fact, the Ernst potential is defined as

$$E = f + i\Omega, \quad \text{with} \quad \sigma(x^2 - 1)\Omega_x = f^2\omega_y, \ \sigma(1 - y^2)\Omega_y = -f^2\omega_x, \quad (12)$$

where f and ω are the main metric functions which determine the line element in prolate spheroidal coordinates (t, x, y, φ):

$$ds^2 = f(dt - \omega d\varphi)^2$$
$$- \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2 - 1} + \frac{dy^2}{1 - y^2}\right) + (x^2 - 1)(1 - y^2)d\varphi^2\right]. \quad (13)$$

The metric function γ can be calculated by quadratures once f and ω are known. The calculation of the metric functions results in

$$f = \frac{R}{L}e^{-2q\delta P_2 Q_2}, \quad (14)$$

$$\omega = -2a - 2\sigma\frac{M}{R}e^{2q\delta P_2 Q_2}, \quad (15)$$

$$e^{2\gamma} = \frac{1}{4}\left(1 + \frac{m}{\sigma}\right)^2 \frac{R}{(x^2 - 1)^\delta}e^{2\delta^2\hat{\gamma}}, \quad (16)$$

where

$$R = a_+ a_- + b_+ b_-, \quad L = a_+^2 + b_+^2, \quad (17)$$

$$M = (x + 1)^{\delta - 1}\left[x(1 - y^2)(\lambda + \eta)a_+ + y(x^2 - 1)(1 - \lambda\eta)b_+\right], \quad (18)$$

$$\hat{\gamma} = \frac{1}{2}(1 + q)^2 \ln\frac{x^2 - 1}{x^2 - y^2} + 2q(1 - P_2)Q_1 + q^2(1 - P_2)[(1 + P_2)(Q_1^2 - Q_2^2)$$
$$\quad (19)$$
$$+ \frac{1}{2}(x^2 - 1)(2Q_2^2 - 3xQ_1 Q_2 + 3Q_0 Q_2 - Q_2')]. \quad (20)$$

Furthermore

$$a_\pm = (x \pm 1)^{\delta - 1}[x(1 - \lambda\eta) \pm (1 + \lambda\eta)], \quad (21)$$

$$b_\pm = (x \pm 1)^{\delta - 1}[y(\lambda + \eta) \mp (\lambda - \eta)]. \quad (22)$$

It is easy to show that this solution is asymptotically flat and free of singularities outside the sphere $x = 1$ which represents a naked singularity. In fact, a numerical investigation of the Kretschmann invariant shows that the hypersurface $x = 1$ is singular, independently of the value of the parameters m, $\delta \neq 1$, a, and q. Only in the limiting case $q = 0$ and $\delta = 1$, the hypersurface $x = 1$ coincides with the exterior horizon of the Kerr spacetime. This situation is illustrated in Fig.

Fig. 1 Structure of the hypersurface $x = 1$ in the spacetime of a rotating mass with arbitrary quadrupole parameter. The plot (**a**) illustrates the case of vanishing quadrupole, $q = 0$, with $\delta = 1$. The hypersurface $x = 1$ corresponds to an event horizon (*dashed line*). The horizon covers the ring singularity which is caused by the rotation of the body. The case (**b**) is for $\delta = 2$ and $q = 10$. The horizon $x = 1$ becomes singular due to the presence of the quadrupole moment so that the spacetime is characterized by a naked singularity (*solid line*). By varying the value of the quadrupole parameter q it is possible to generate additional singularities (**c**) which are always inside the outer naked singularity situated at $x = 1$

1 which shows that a naked singularity is always present when the quadrupole parameter q is non zero or when $\delta \neq 1$. Moreover, the symmetry axis $y = \pm 1$ is free of singularities (except at $x = 1$), indicating that the condition of elementary flatness is satisfied [1].

All the properties mentioned above seem to indicate that the solution can be used to describe the exterior field of a deformed, rotating mass distribution. From the point of view of general relativity, however, to describe the entire manifold of a rotating body it is necessary to find an interior solution which takes into account the inner properties of the body and that can be matched to the exterior vacuum solution. However, rather few exact stationary solutions that involve a matter distribution in rotation are to be found in the literature. In particular, the interior solution for the rotating Kerr solution is still unknown. In fact, the quest for a realistic exact solution, representing both the interior and exterior gravitational field generated by a self-gravitating axisymmetric distribution of mass of perfect fluid in stationary rotation is considered a major problem in general relativity. We believe that the inclusion of a quadrupole in the exterior and in the interior solutions adds a new physical degree of freedom that could be used to search for realistic interior solutions. To see if this is true, we will use a special limit of the above solution with quadrupole moment and will show that in fact it can be matched with a realistic approximate interior solution.

3 The field of a slowly rotating and slightly deformed body

In this section we calculate the limiting case where the deviation from spherical symmetry is small and the body is slowly rotating. It is convenient to introduce the new spatial coordinates r and θ by means of [18]

$$x = \frac{r - m}{\sigma}, \quad y = \cos\theta, \tag{23}$$

and to choose the ZV parameter as $\delta = 1 + sq$, where s is a real constant. Then, expanding the metric (13) to first order in the quadrupole parameter q and to second

order in the rotation parameter a, we obtain

$$f = 1 - \frac{2m}{r} + \frac{2a^2 m \cos^2\theta}{r^3} + q(1+s)\left(1 - \frac{2m}{r}\right)\ln\left(1 - \frac{2m}{r}\right)$$

$$+ 3q\left(\frac{r}{2m} - 1\right)\left[\left(1 - \frac{m}{r}\right)(3\cos^2\theta - 1)\right.$$

$$\left. + \left\{\left(\frac{r}{2m} - 1\right)(3\cos^2\theta - 1) - \frac{m}{r}\sin^2\theta\right\}\ln\left(1 - \frac{2m}{r}\right)\right], \qquad (24)$$

$$\omega = \frac{2amr\sin^2\theta}{r - 2m}, \qquad (25)$$

$$\gamma = \frac{1}{2}\ln\frac{r(r-2m)}{(r-m)^2 - m^2\cos^2\theta} + \frac{a^2}{2}\left[\frac{m^2\cos^2\theta\sin^2\theta}{r(r-2m)((r-m)^2 - m^2\cos^2\theta)}\right]$$

$$+ q(1+s)\ln\frac{r(r-2m)}{(r-m)^2 - m^2\cos^2\theta}$$

$$- 3q\left[1 + \frac{1}{2}\left(\frac{r}{m} - 1\right)\ln\left(1 - \frac{2m}{r}\right)\right]\sin^2\theta. \qquad (26)$$

Furthermore, we introduce coordinates R and Θ by means of

$$r = R + \mathcal{M}q + \frac{3}{2}\mathcal{M}q\sin^2\Theta\left[\frac{R}{\mathcal{M}} - 1 + \frac{R^2}{2\mathcal{M}^2}\left(1 - \frac{2\mathcal{M}}{R}\right)\ln\left(1 - \frac{2\mathcal{M}}{R}\right)\right]$$

$$- \frac{a^2}{2R}\left[\left(1 + \frac{2\mathcal{M}}{R}\right)\left(1 - \frac{\mathcal{M}}{R}\right) - \cos^2\Theta\left(1 - \frac{2\mathcal{M}}{R}\right)\left(1 + \frac{3\mathcal{M}}{R}\right)\right], \qquad (27)$$

$$\theta = \Theta - \sin\Theta\cos\Theta\left\{\frac{3}{2}q\left[2 + \left(\frac{R}{\mathcal{M}} - 1\right)\ln\left(1 - \frac{2\mathcal{M}}{R}\right)\right] + \frac{a^2}{2R}\left(1 + \frac{2\mathcal{M}}{R}\right)\right\}, \qquad (28)$$

where

$$\mathcal{M} = m(1-q), \quad J = -ma, \quad Q = \frac{J^2}{m} - \frac{4}{5}m^3 q. \qquad (29)$$

A straightforward computation shows that the metric (24)–(26) can be written as

$$ds^2 = \left(1 - \frac{2\mathcal{M}}{R}\right)\left[1 + 2k_1 P_2(\cos\Theta) + 2\left(1 - \frac{2\mathcal{M}}{R}\right)^{-1}\frac{J^2}{R^4}(2\cos^2\Theta - 1)\right]dt^2$$

$$- \left(1 - \frac{2\mathcal{M}}{R}\right)^{-1}\left[1 - 2k_2 P_2(\cos\Theta) - 2\left(1 - \frac{2\mathcal{M}}{R}\right)^{-1}\frac{J^2}{R^4}\right]dR^2$$

$$- R^2[1 - 2k_3 P_2(\cos\Theta)](d\Theta^2 + \sin^2\Theta d\phi^2) + 4\frac{J}{R}\sin^2\Theta dt d\phi \qquad (30)$$

where

$$k_1 = \frac{J^2}{\mathcal{M}R^3}\left(1 + \frac{\mathcal{M}}{R}\right) + \frac{5}{8}\frac{Q - J^2/\mathcal{M}}{\mathcal{M}^3}Q_2^2\left(\frac{R}{\mathcal{M}} - 1\right),$$

$$k_2 = k_1 - \frac{6J^2}{R^4},$$

$$k_3 = k_1 + \frac{J^2}{R^4} - \frac{5}{4}\frac{Q - J^2/\mathcal{M}}{\mathcal{M}^2 R}\left(1 - \frac{2\mathcal{M}}{R}\right)^{-1/2}Q_2^1\left(\frac{R}{\mathcal{M}} - 1\right),$$

and we choose for the free parameter s the particular value $s = -1$ for the sake of simplicity. Here Q_l^m are the associated Legendre functions of the second kind

$$Q_2^1(x) = (x^2 - 1)^{1/2}\left[\frac{3x^2 - 2}{x^2 - 1} - \frac{3}{2}x\ln\frac{x+1}{x-1}\right],$$

$$Q_2^2(x) = \frac{3}{2}(x^2 - 1)\ln\frac{x+1}{x-1} + \frac{5x - 3x^3}{x^2 - 1}. \tag{31}$$

The line element (30) was first derived in an equivalent representation by Hartle [12] and Hartle and Thorne [13]. Consequently, our results show that the general solution (13)–(22) contains as a special case the HT metric which describes the exterior gravitational field of any slowly and rigidly rotating deformed body.

In the case of ordinary stars, such as the Sun, the smallness of the parameters $\mathcal{M}_{Sun}/\mathcal{R}_{Sun} \approx 10^{-6}$, $J_{Sun}/\mathcal{R}_{Sun}^2 \approx 10^{-12}$, $Q_{Sun}/\mathcal{R}_{Sun}^3 \approx 10^{-12}$, allows us to simplify the metric (30) to obtain [19]

$$ds^2 = \left[1 - \frac{2\mathcal{M}}{R} + \frac{2Q}{R^3}P_2(\cos\Theta)\right]dt^2 + \frac{4J}{R}\sin^2\Theta dt d\phi$$

$$- \left[1 + \frac{2\mathcal{M}}{R} - \frac{2Q}{R^3}P_2(\cos\Theta)\right]dR^2$$

$$- \left[1 - \frac{2Q}{R^3}P_2(\cos\Theta)\right]R^2(d\Theta^2 + \sin^2\Theta d\phi^2). \tag{32}$$

The accuracy of this metric is of one part in 10^{12}. Consequently, it describes the gravitational field for a wide range of compact objects, and only in the case of very dense ($\mathcal{M} \sim \mathcal{R}$) or very rapidly rapidly rotating ($J \sim \mathcal{R}^2$) objects large discrepancies will appear.

4 The interior solution

If a compact object is rotating slowly, the calculation of its equilibrium properties reduces drastically because it can be considered as a linear perturbation of an already-known non-rotating configuration. A very good approximation of the internal structure of the body is delivered by a perfect fluid model satisfying a one-parameter equation of state, $\mathcal{P} = \mathcal{P}(\mathcal{E})$, where $\mathcal{P}$ is the pressure and $\mathcal{E}$ is the density of total mass–energy. A further simplification follows from the assumption that the configuration is symmetric with respect to an arbitrary axis which can be

taken as the rotation axis. Furthermore, the rotating object should be invariant with respect to reflections about a plane perpendicular to the axis of rotation, i.e, about the equatorial plane. As for the rotation, it turns out that configurations which minimize the total mass–energy (e.g., all stable configurations) must rotate uniformly [20]. Uniform rotation facilitates the study of the internal properties of the body, especially if we restrict ourselves to the case of slow rotation. That is, we assume that angular velocities Ω are small enough so that the fractional changes in pressure, energy density and gravitational field due to the rotation are all less than unity, i.e. $\Omega^2 \ll \left(\frac{c}{\mathscr{R}}\right)^2 \frac{G\mathscr{M}}{c^2\mathscr{R}}$ where $\mathscr{M}$ is the mass and $\mathscr{R}$ is the radius of the non-rotating configuration. The above condition is equivalent to the physical requirement $\Omega \ll c/\mathscr{R}$. The metric functions must be found by solving Einstein's equations

$$R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R = 8\pi T_\mu^\nu, \tag{33}$$

where the stress-energy tensor is that of a perfect fluid

$$T_\mu^\nu = (\mathscr{E} + \mathscr{P})u^\nu u_\mu - \mathscr{P}\delta_\mu^\nu. \tag{34}$$

The 4-velocity which satisfies the normalization condition $u^\mu u_\mu = 1$ is

$$u^R = u^\Theta = 0, \ \ u^\phi = \Omega u^t, \ \ u^t = \left(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi}\right)^{-1/2}, \tag{35}$$

where the angular velocity Ω is a constant throughout the fluid.

The explicit integration of the inner Einstein's equations depends on the particular choice of the density function $\mathscr{E}(R)$, where R is the radial coordinate, and on the equation of state $\mathscr{P} = \mathscr{P}(\mathscr{E})$. For the sake of simplicity, we consider here only the simplest case with $\mathscr{E} =$const and let the pressure $\mathscr{P}$ be determined by the field equations. Then, the total mass of the non-rotating body is $\mathscr{M} = 4\pi\mathscr{E}\mathscr{R}^3/3$, where $\mathscr{R}$ is the radius of the body. Under these conditions, the resulting line element can be written as

$$ds^2 = (1 + 2\Phi)\,dt^2 - \left[1 + 2R\frac{d\Phi_0(R)}{dR} + \Phi_2(R)P_2(\cos\Theta)\right]dR^2$$
$$-R^2\left[1 + 2\Phi_2(R)P_2(\cos\Theta)\right]\left[d\Theta^2 + \sin^2\Theta(d\phi - \tilde{\omega}dt)^2\right], \tag{36}$$

where

$$\Phi = \Phi_0(R) + \Phi_2(R)P_2(\cos\Theta), \tag{37}$$

is the interior Newtonian potential. Here Φ_0 is the interior Newtonian potential for the non-rotating configuration and $\Phi_2(R)$ is the perturbation due to the rotation. Moreover,

$$\tilde{\omega} = \Omega - \bar{\omega} \tag{38}$$

is the angular velocity of the local inertial frame, where $\Omega =$ const is the angular velocity of the fluid and $\bar{\omega} = \bar{\omega}(R)$ is the angular velocity of the fluid relative to

the inertial frame. With a suitable choice of the integration constant, the interior unperturbed Newtonian potential can be expressed as

$$\Phi_0(R) = -2\pi\rho\left(\mathscr{R}^2 - \frac{R^2}{3}\right),$$ (39)

where $\mathscr{R}$ is the radius of the non-rotating configuration. The function $\Phi_2(R)$ satisfies the following equation

$$\frac{d\Phi_2(R)}{dR} = \Phi_2(R)\left(\frac{4\pi R^2\rho}{\mathscr{M}} - \frac{2}{R}\right) - \frac{2\chi}{\mathscr{M}} + \frac{4\pi}{3\mathscr{M}}\rho\Omega^2 R^4,$$ (40)

where χ is defined by

$$\frac{d\chi}{dR} = -\frac{2\mathscr{M}}{R^2}\Phi_2(R) + \frac{8\pi}{3}\rho\Omega^2 R^3,$$ (41)

For the angular velocity relative to the local inertial frame $\bar{\omega}$ one obtains the differential equation

$$\frac{1}{R^4}\frac{d}{dR}\left(R^4 j\frac{d\bar{\omega}}{dR}\right) + \frac{4}{R}\frac{dj}{dR}\bar{\omega} = 0, \quad j(R) = \left[1 + \frac{4\pi}{3}\rho R^2\right]^{1/2},$$ (42)

which is related to the equilibrium condition of a rotating self-gravitating body, namely, the condition that there exists a balance between pressure forces, gravitational forces and centrifugal forces. The solution for $\bar{\omega}(R)$ must be regular at the origin and outside the body it must take the form $\bar{\omega}(R) = \Omega - 2J/R^3$, where J is the total angular momentum of the star. This condition guarantees a smooth matching of the corresponding metric functions on the surface of the mass distribution.

For the integration of the field equations it is convenient to consider also an additional differential equation which follows from the conservation law $T^{\mu\nu}_{;\nu} = 0$ and involves the pressure and its partial derivatives. Even in the simple case of $\mathscr{E} = $ const, the resulting expression is rather cumbersome. A detailed analysis of this equation will be presented elsewhere [19]. The integration of the system of partial differential equations (40)–(42), together with the differential equation for the pressure, cannot be carried out analytically. Nevertheless, it is possible to find numerical solutions not only in the simple case $\mathscr{E} = $ const, but also in the case of more realistic density functions [13].

It is easy to show analytically that the interior solution (36) can be matched smoothly on the surface $R = \mathscr{R}$ with the exterior solution (30) in the special case of an unperturbed configuration with $\Phi_2 = 0$ and Φ_0 as given in Eq. (39). In the general case of a rotating deformed body, the matching can be performed only numerically. In fact, the solutions of the field equations for the inner distribution of mass are calculated using the matching conditions as boundary conditions. In this manner, it is possible to find realistic solutions that describe the interior and exterior gravitational field of slowly rotating and slightly deformed mass distributions.

5 Conclusions

In this work we studied an exact solution of Einstein's vacuum field equations which represents the exterior gravitational field of a stationary axisymmetric rotating mass distribution. It contains four free parameters that determine the total mass, angular momentum, and quadrupole momentum of the body. It was shown that the solution is asymptotically flat and is free of singularities outside a region that can be covered by an interior solution. In fact, we show that the presence of a gravitoelectric quadrupole changes drastically the geometric properties of the spacetime. Independently of the value of the quadrupole, there always exists a singular surface which is not surrounded by an event horizon. This implies that the gravitoelectric quadrupole moment of the source can always be associated with the presence of naked singularities. However, the naked singularities are situated very close to the origin of coordinates so that a physically reasonable interior solution could be used to cover them.

The main goal of this work was to take the first step to prove that the exact exterior QM solution can be matched in general with an interior solution. To this end we calculated the limiting case of an approximate solution for a slowly rotating and slightly deformed source. This means that the solution is expanded to first order in the quadrupole parameter and to second order in the rotation parameter. We show that a particular choice of the ZV parameter allows us to find explicitly a coordinate transformation that transforms the solution into the approximate exterior Hartle metric. Assuming that the inner configuration can be described by a perfect fluid which rotates uniformly with respect to an axis that coincides with the axis of symmetry and that the source is symmetric with respect to reflections about the equatorial plane, the set of differential equations which follow from Eintein's interior equations was derived and investigated. A particular analytic solution was presented that can be matched with the approximate exterior solution in the case of an unperturbed non-rotating perfect fluid. To consider the general case of a slowly rotating and slightly deformed perfect fluid it is necessary to perform numerical calculations to integrate the corresponding set of differential equations. Our numerical results show that it is possible to obtain numerical solutions with realist physical properties and to match them with the exterior approximate solution. The inclusion of the quadrupole parameter in the interior and exterior metrics is equivalent to adding a new degree of freedom which facilitates to search for interior solutions.

We conclude that the particular QM solution presented in this work can be used to describe the gravitational field of a rotating body with arbitrary quadrupole moment. In this work, it was shown that in the case of a slightly deformed body with uniform and slow rotation, there exist a set of solutions of Einstein solutions which can be used to completely describe the spacetime. The case of an arbitrarily rotating perfect fluid with arbitrary quadrupole parameter will be investigated in a future work.

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