Particle Accelerators
1983 Vol. 14 pp. 91–108
0031-2460/83/1401/0091$18.50/0

THEORY OF LONGITUDINAL BUNCHED-BEAM INSTABILITIES BASED ON THE FOKKER-PLANCK EQUATION

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(Received February 1, 1983)

The Fokker-Planck equation for synchrotron motion is solved by expansion in terms of generalized Laguerre polynomials. The convergence of the expansion is confirmed by computer calculation to be very rapid. The Robinson instability occurs for sharply peaked impedance. Calculation shows that a simplified approach based on the Vlasov equation can be taken when combined with an approximate stability criterion that the growth rate should be less than $|m|/\tau$, where m is the azimuthal mode number and τ is the radiation damping time for synchrotron oscillations. Fast instability occurs when coupling between different azimuthal modes is taken into account. The threshold of this instability calculated by the Fokker-Planck equation is almost equal to that obtained by the Vlasov equation. Thus radiation effects are shown to play little role in fast instability.

1. INTRODUCTION

Many works on bunched-beam instabilities are based on Sacherer's general theories.[1,2] Sacherer used the Vlasov equation expressed in polar coordinates in phase space and derived integral equations known as Sacherer's integral equations. First,[1] he dealt with a case where beam intensity is relatively low and different azimuthal modes, such as dipole mode, quadrupole mode, etc., are well separated. In this theory, the coupled-bunch instabilities are explained. A single bunch is shown to be stable for a broadband impedance and only for a sharply peaked impedance, such as a cavity impedance, does a single bunch become unstable (Robinson effect.)[3]

Later[2] he extended his theory to the case where the beam intensity is high and two different azimuthal modes couple to give rise to fast instability. This theory explains microwave instabilities in proton beams and anomalous bunch lengthening in electron beams.

Sacherer's theories are mainly for proton beams, but the same formalism is widely applied to explain electron-beam instabilities. In general, electrons emit synchrotron radiation and the beam dynamics should be dealt with by use of the Fokker-Planck equation[4] rather than the Vlasov equation. However, in many previous works, the growth rates of the coupled-bunch instabilities and the Robinson instabilities are calculated by use of Sacherer's formalism, which is based on the Vlasov equation, and a stability criterion that the growth rate should be smaller than the radiation damping rate is used. In explaining a fast single-bunch instability, Sacherer's theory is also widely used, since the growth time of the fast instability is much shorter than the radiation damping time. The radiation effect is therefore expected not to play an important role.

In this spirit, Chao and Gareyte[5] derived from the Vlasov equation a simple scaling law for the anomalous bunch lengthening in SPEAR II which explained the qualitative features of the experimental data. Suzuki, *et al.*, [6] applied Sacherer's formalism using a Gaussian beam model to the anomalous bunch lengthening in SPEAR II. They found that the theory explained the qualitative features of the data well, but that the theory disagreed quantitatively with the experimental data by a factor of about three to four.

On the other hand, several workers[7–10] used the Fokker-Planck equation to explain anomalous bunch lengthening. In these treatments, however, the unique features of the Fokker-Planck equation approach, which are different from the results obtained by the Vlasov-equation approach, are not always clear. Here we develop a new formalism for solving the Fokker-Planck equation. The Fokker-Planck equation is first expressed in action-angle variables. In this way an integro-differential equation is obtained. This is parallel to Sacherer's formalism for solving the Vlasov equation. The azimuthal modes and radial modes are thus introduced naturally. Then the Fokker-Planck equation is solved by expansion in terms of generalized Laguerre polynomials. This method is similar to that of Besnier,[11] who used it to solve Sacherer's integral equations. The present method is also somewhat similar to that of Renieri,[8] who used an expansion in terms of Hermite polynomials to solve the Fokker-Planck equation. However, he used Cartesian coordinates and the present method is more suitable to introduce azimuthal and radial modes. Moreover, in the limit where the radiation damping time becomes infinite and the diffusion constant becomes zero, the present formalism becomes identical to Sacherer's formalism. Thus the similarities and differences between the Fokker-Planck and the Vlasov approaches will become more apparent.

In Section 2, we derive the Fokker-Planck equation for synchrotron motion and write it in terms of action-angle variables. In Section 3, we develop a formalism for solving the Fokker-Planck equation. The formalism is applied in Section 4 to a sharply peaked impedance and the validity of the usual stability criterion for the Vlasov-equation treatment, that the growth rate should be smaller than the radiation damping rate, is studied carefully. In Section 5, the formalism is applied to anomalous bunch lengthening. Conclusions are given in Section 6.

2. THE FOKKER-PLANCK EQUATION IN ACTION-ANGLE VARIABLES

We first derive the Fokker-Planck equation for synchrotron motion in order to establish notation. The derivation is based on the paper of Chandrasekhar.[4] The equations of motion for synchrotron oscillations are

$$\frac{d\theta}{dt} = k_0\epsilon \tag{1}$$

$$\frac{d\epsilon}{dt} = -\frac{\omega_s{}^2}{k_0}\theta - \frac{2}{\tau}\epsilon + \frac{eV(\theta, t)}{T} + g(t) \tag{2}$$

where

$$k_0 = \frac{\alpha\omega_0}{E} \tag{3}$$

In these equations, the notation is as follows:

θ: angular position with respect to the synchronous particle
t: time
ϵ: energy error
ω_s: incoherent synchrotron angular frequency
τ: radiation damping time
$eV(\theta, t)$: energy gain per turn due to self-fields
T: revolution period $= 2\pi/\omega_0$
$g(t)$: stochastic function denoting the effect of quantum emission of synchrotron radiation
α: momentum compaction factor
ω_0: revolution angular frequency
E: beam energy.

In Eqs. (1) and (2), a more suitable independent variable is the azimuthal angle around the ring circumference instead of time, as explained in Ref. 14, but the difference is neglected here because it is quite small. From Eqs. (1) and (2), the changes $\Delta\theta$ and $\Delta\epsilon$ in time Δt are given by

$$\Delta\theta = k_0 \epsilon \Delta t \tag{4}$$

$$\Delta\epsilon = -\frac{\omega_s^2}{k_0}\theta\Delta t - \frac{2}{\tau}\epsilon\Delta t + \frac{eV(\theta, t)}{T}\Delta t + B(\Delta t), \tag{5}$$

where

$$B(\Delta t) = \int_t^{t+\Delta t} g(t)dt. \tag{6}$$

The distribution $w(B(\Delta t))$ of the stochastic quantity $B(\Delta t)$ is given by[4]

$$w(B(\Delta t)) = \frac{1}{(4\pi D\Delta t)^{1/2}} \exp(-|B(\Delta t)|^2/4D\Delta t), \tag{7}$$

where D is the diffusion constant.

Now we introduce a particle distribution function $\psi(\theta, \epsilon, t)$ in phase space. It satisfies the integral equation[4]

$$\psi(\theta, \epsilon, t + \Delta t) = \int\int \psi(\theta - \Delta\theta, \epsilon - \Delta\epsilon, t) w(\theta - \Delta\theta, \epsilon - \Delta\epsilon; \Delta\theta, \Delta\epsilon) d(\Delta\epsilon) d(\Delta\theta) \tag{8}$$

Since

$$w(\theta, \epsilon; \Delta\theta, \Delta\epsilon) = w(\theta, \epsilon; \Delta\epsilon)\delta(\Delta\theta - k_0\epsilon\Delta t), \tag{9}$$

Eq. (8) is rewritten as

$$\psi(\theta + k_0\epsilon\Delta t, \epsilon, t + \Delta t) = \int \psi(\theta, \epsilon - \Delta\epsilon, t) w(\theta, \epsilon - \Delta\epsilon; \Delta\epsilon) d(\Delta\epsilon). \tag{10}$$

Expanding Eq. (10) into Taylor series in Δt and $\Delta\epsilon$, we obtain the Fokker-Planck equation

$$\frac{\partial\psi}{\partial t} + k_0\epsilon\frac{\partial\psi}{\partial\theta} + \left\{-\frac{\omega_s^2}{k_0}\theta + \frac{eV(\theta, t)}{T}\right\}\frac{\partial\psi}{\partial\epsilon} = \frac{2}{\tau}\frac{\partial(\epsilon\psi)}{\partial\epsilon} + D\frac{\partial^2\psi}{\partial\epsilon^2}. \tag{11}$$

Here we have used the form

$$w(\theta, \epsilon - \Delta\epsilon; \Delta\epsilon) = \frac{1}{(4\pi D\Delta t)^{1/2}}\exp\left[-\left(\Delta\epsilon + \frac{\omega_s^2}{k_0}\theta\Delta t + \frac{2}{\tau}\epsilon\Delta t - \frac{eV}{T}\Delta t\right)^2 \bigg/ 4D\Delta t\right], \tag{12}$$

which is obtained from Eqs. (5) and (7). When $\tau \to \infty$ and $D \to 0$, the Fokker-Planck Eq. (11) reduces to the Vlasov equation.

We introduce polar coordinates (r, ϕ) in phase space that are defined by

$$\theta = r\cos\phi \tag{13}$$

$$\epsilon = -\frac{\omega_s}{k_0} r\sin\phi. \tag{14}$$

Then the Fokker-Planck equation is written as

$$\frac{\partial\psi}{\partial t} + \omega_s\frac{\partial\psi}{\partial\phi} - \frac{k_0}{\omega_s}\frac{eV(\theta, t)}{T}\sin\phi\frac{d\psi_0}{dr} = \frac{2}{\tau}\left\{\psi + r\sin^2\phi\frac{\partial\psi}{\partial r} + \frac{\sin 2\phi}{2}\frac{\partial\psi}{\partial\phi}\right\} + D\left(\frac{k_0}{\omega_s}\right)^2 \times\left\{\sin^2\phi\frac{\partial^2\psi}{\partial r^2} - \frac{\sin 2\phi}{r^2}\frac{\partial\psi}{\partial\phi} + \frac{\sin 2\phi}{r}\frac{\partial^2\psi}{\partial r\,\partial\phi} + \frac{\cos^2\phi}{r}\frac{\partial\psi}{\partial r} + \frac{\cos^2\phi}{r^2}\frac{\partial^2\psi}{\partial\phi^2}\right\}, \tag{15}$$

where ψ_0 is a stationary distribution function which is a function of r only. Only terms linear in ψ are retained in Eq. (15).

We first calculate the stationary distribution function $\psi_0(r)$ which is independent of ϕ and t. We neglect the self-force term eV. Then Eq. (15) reduces to

$$\frac{2}{\tau}\left\{\psi_0 + r\sin^2\phi\frac{d\psi_0}{dr}\right\} + D\left(\frac{k_0}{\omega_s}\right)^2\left\{\sin^2\phi\frac{d^2\psi_0}{dr^2} + \frac{\cos^2\phi}{r}\frac{d\psi_0}{dr}\right\} = 0. \tag{16}$$

We average Eq. (16) over ϕ and obtain

$$\frac{2}{\tau}\left\{\psi_0 + \frac{r}{2}\frac{d\psi_0}{dr}\right\} + \frac{1}{2}D\left(\frac{k_0}{\omega_s}\right)^2\left\{\frac{d^2\psi_0}{dr^2} + \frac{1}{r}\frac{d\psi_0}{dr}\right\} = 0. \tag{17}$$

Eq. (17) can be easily solved to give

$$\psi_0 = C\exp\left[-\frac{r^2}{\tau D}\left(\frac{\omega_s}{k_0}\right)^2\right], \tag{18}$$

where C is a normalization constant. The rms bunch length σ_θ in units of θ is then given by

$$\sigma_\theta{}^2 = \frac{\tau D}{2}\left(\frac{k_0}{\omega_s}\right)^2. \tag{19}$$

We normalize the distribution function ψ_0 by

$$\int \psi_0(r)\,d\theta\,d\epsilon = Ne, \tag{20}$$

where N is the number of electrons in a bunch.

Then the normalization constant C in Eq. (18) is given by

$$C = \frac{Ne\alpha}{2\pi\nu_s E\sigma_\theta{}^2}, \tag{21}$$

where $\nu_s = \omega_s/\omega_0$ is the synchrotron tune number. When the self-force term eV is included in Eq. (17), the stationary distribution is different from the Gaussian distribution given by Eq. (18).[8,12] We neglect in this paper such potential-well distortion effects. This approximation is the one usually used in the Vlasov-equation treatment.

We now decompose the distribution function ψ into a stationary part ψ_0 and a time-dependent perturbed part ψ_1 as

$$\psi = \psi_0 + \psi_1 \exp(-i\Omega t) \tag{22}$$

$$\psi_1(r, \phi) = \sum_{n=-\infty}^{\infty} R_n(r)\exp(in\phi), \tag{23}$$

where Ω is a coherent frequency. Instability occurs if the imaginary part of Ω is positive. The stationary distribution given by Eq. (18) is a solution of Eq. (17), but not a solution of Eq. (16). However, we assume that the solution Eq. (18) satisfies Eq. (16) on the average and separate the stationary part from Eq. (15). Then the perturbed function ψ_1 satisfies Eq (15) where ψ is replaced by ψ_1. The self-force term $eV(\theta, t)$ is expressed in the usual way by use of the longitudinal coupling impedance $Z(\omega)$ in the form[13,14]

$$eV(\theta, t) = -e\omega_0 \sum_{p=-\infty}^{\infty} Z(p\omega_0 + \Omega)\,\tilde{\rho}\left(p + \frac{\Omega}{\omega_0}\right) \times \exp\left[+ i\left(p + \frac{\Omega}{\omega_0}\right) - i\Omega t\right] \tag{24}$$

where

$$\tilde{\rho}(p') = \frac{1}{2\pi}\frac{\omega_s}{k_0}\int\!\!\int \psi_1(r, \phi)\exp[-ip'r\cos\phi]\,r\,dr\,d\phi. \tag{25}$$

Now we put Eq. (23) into Eq. (15), multiply both-hand sides by $1/(2\pi)\exp(-im\phi)$ and integrate over ϕ from 0 to 2π. Then we obtain a coupled integro-differential equation for $R_m(r)$. It is, however, more convenient to introduce an action variable x defined by

$$x = r^2/2\sigma_\theta^2. \tag{26}$$

Then we obtain the integro-differential equation

$$\begin{aligned}
&-i(\Omega - m\omega_s)R_m + \frac{e\omega_0}{T\sigma_\theta^2} m \frac{d\psi_0}{dx} \sum_{p=-\infty}^{\infty} \frac{Z(p\omega_0 + \Omega)}{p + \Omega/\omega_0} \\
&\times i^m J_m\left((p + \Omega/\omega_0)\sqrt{2}\sigma\sqrt{x}\right) \sum_{n=-\infty}^{\infty} i^{-n} \int R_n(x') J_n\left((p + \Omega/\omega_0)\sqrt{2}\sigma\sqrt{x'}\right) dx' \\
&= \frac{2}{\tau}\left\{R_m + x\left[\frac{dR_m}{dx} - \frac{1}{2}\frac{dR_{m-2}}{dx} - \frac{1}{2}\frac{dR_{m+2}}{dx}\right] + \frac{1}{4}[(m-2)R_{m-2} - (m+2)R_{m+2}]\right\} \\
&+ \frac{D}{\sigma_\theta^2}\left(\frac{k_0}{\omega_s}\right)^2 \left\{x\frac{d^2R_m}{dx^2} + \frac{1}{2}\frac{dR_m}{dx} - \frac{1}{2}x\frac{d^2R_{m-2}}{dx^2} - \frac{1}{4}\frac{dR_{m-2}}{dx}\right. \\
&- \frac{1}{2}x\frac{d^2R_{m+2}}{dx^2} - \frac{1}{4}\frac{dR_{m+2}}{dx} - \frac{1}{4x}[(m-2)R_{m-2} - (m+2)R_{m+2}] \\
&+ \frac{1}{2}\left[(m-2)\frac{dR_{m-2}}{dx} - (m+2)\frac{dR_{m+2}}{dx}\right] + \frac{1}{2}\left[\frac{dR_m}{dx} + \frac{1}{2}\frac{dR_{m-2}}{dx} + \frac{1}{2}\frac{dR_{m+2}}{dx}\right] \\
&\left.- \frac{1}{4x}\left[m^2R_m + \frac{(m-2)^2}{2}R_{m-2} + \frac{(m+2)^2}{2}R_{m+2}\right]\right\}.
\end{aligned} \tag{27}$$

This is the basic equation which we must solve. It is a generalization of Sacherer's integral equation to the case of the Fokker-Planck equation. When $\tau \to \infty$ and $D \to 0$, Eq. (27) reduces to Sacherer's integral equation. We note from Eq. (19) that

$$\frac{D}{\sigma_\theta^2}\left(\frac{k_0}{\omega_s}\right)^2 = \frac{2}{\tau} \tag{28}$$

and that the damping terms and the diffusion terms can be combined. The terms involving R_{m+2} and R_{m-2} in Eq. (27) appear because of the terms involving $\sin^2\phi$, $\cos^2\phi$ and $\sin 2\phi$ in the Fokker-Planck Eq. (15).

3. SOLUTION OF THE FOKKER-PLANCK EQUATION

We solve the basic Eq. (27) by expanding the radial function $R_m(r)$ in terms of orthogonal functions. This method was used by Besnier[11] to solve Sacherer's integral equation without mode coupling and the method was extended to solve the case of mode coupling by Suzuki, *et al.*[6] In accordance with their method, we expand $R_m(x)$ as

$$R_m(x) = \exp(-x) \sum_{k=0}^{\infty} a_k^{(m)} f_k^{(|m|)}(x), \tag{29}$$

where

$$f_k^{(|m|)}(x) = \sqrt{\frac{k!}{(|m| + k)!}}\, x^{|m|/2} L_k^{(|m|)}(x). \tag{30}$$

Here the $L_k^{(|m|)}(x)$ are generalized Laguerre polynomials.[15] The functions $f_k^{(|m|)}(x)$'s satisfy an orthogonality relation.

$$\int_0^\infty \exp(-x) f_k^{(|m|)}(x) f_l^{(|m|)}(x)\, dx = \delta_{kl}. \tag{31}$$

The convergence of the series in Eq. (29) is checked by computer calculation.

The expansion in Eq. (29) assumes that $R_m(x)$ behaves as $x^{|m|/2}$ for small x. This assumption is valid for the Vlasov equation [$\tau \to \infty$ and $D \to 0$ in Eq. (27)] because $J_m((p + \Omega/\omega_0)\sqrt{2}\sigma\sqrt{x}) \sim x^{|m|/2}$ for small x, but it must be justified for the Fokker-Planck Eq. (27). If we put the form in Eq. (29) into Eq. (27), we easily find that terms of order lower than $x^{|m|/2}$ cancel each other and the lowest-order term in x that appears in Eq. (27) is $x^{|m|/2}$. Thus the expansion in Eq. (29) is justified.

We insert Eq. (29) into Eq. (27), and use the relations[15]

$$x\frac{d^2 g_k^{(|m|)}}{dx^2} + (x + 1)\frac{d g_k^{(|m|)}}{dx} + \left(k + \frac{|m|}{2} + 1 - \frac{m^2}{4x}\right) g_k^{(|m|)} = 0 \tag{32}$$

and

$$\frac{d g_k^{(|m|)}}{dx} = -g_k^{(|m|)} + \frac{|m|}{2x} g_k^{(|m|)} + \frac{k}{x} g_k^{(|m|)} - \frac{\sqrt{k(|m| + k)}}{x} g_{k-1}^{(|m|)}, \tag{33}$$

where

$$g_k^{(|m|)}(x) = \exp(-x) f_k^{(|m|)}(x),$$

and obtain

$$\begin{aligned}
&-i(\Omega - m\omega_s)\exp(-x) \sum a_k^{(m)} f_k^{(|m|)}(x) - \frac{Ne^2\alpha\omega_0}{2\pi\nu_s E\sigma_\theta^2 T} m \exp(-x) \\
&\times \sum_{p=-\infty}^{\infty} \frac{Z(p\omega_0 + \Omega)}{p + \Omega/\omega_0} i^m \times J_m((p + \Omega/\omega_0)\sqrt{2}\sigma\sqrt{x}) \sum_{n=-\infty}^{\infty} \sum_{h=0}^{\infty} i^{-n} a_h^{(n)} \\
&\times \int_0^\infty \exp(-x') f_h^{(n)}(x') J_n((p + \Omega/\omega_0)\sqrt{2}\sigma\sqrt{x'})\, dx' \\
&= \frac{2}{\tau}\exp(-x)\left[-\sum_{k=0}^{\infty} \left(\frac{|m|}{2} + k\right) a_k^{(m)} f_k^{(|m|)}(x) + \frac{1}{2}\sum_{k=0}^{\infty} \left(k + \frac{|m-2|}{2} - \frac{m-2}{2}\right)\right. \\
&\times a_k^{(m-2)} f_k^{(|m-2|)}(x) - \frac{m-1}{4x} \sum_{k=0}^{\infty} \{m - 2 - |m-2| - 2k\} a_k^{(m-2)} f_k^{(|m-2|)}(x) \\
&- \frac{m-1}{2x} \sum_{k=0}^{\infty} \sqrt{k(|m-2| + k)}\, a_k^{(m-2)} f_{k-1}^{(|m-2|)}(x) + \frac{1}{2}\sum_{k=0}^{\infty} \left\{k + \frac{|m+2|}{2} + \frac{m+2}{2}\right\} \\
&\times a_k^{(m+2)} f_k^{(|m+2|)}(x) - \frac{m+1}{4x} \sum_{k=0}^{\infty} \{m + 2 + |m+2| + 2k\}\, a_k^{(m+2)} f_k^{(|m+2|)}(x) \\
&\left. + \frac{m+1}{2x} \sum_{k=0}^{\infty} \sqrt{k(|m+2| + k)}\, a_k^{(m+2)} f_{k-1}^{(|m+2|)}(x) \right].
\end{aligned} \tag{34}$$

We multiply both sides of Eq. (34) by $f_l^{(|m|)}(x)$, integrate over x and obtain

$$
\begin{aligned}
&-i(\Omega - m\omega_s)a_l^{(m)} - \frac{Ne^2\alpha}{2\pi v_s E\sigma_\theta{}^2 T} m \\
&\quad\times \sum_{p=-\infty}^{\infty} \frac{Z(p\omega_0 + \Omega)}{p + \Omega/\omega_0} i^m I_{ml}(p + \Omega/\omega_0) \sum_{n=\infty}^{\infty} \sum_{h=0}^{\infty} i^{-n} a_h^{(n)} I_{nh}(p + \Omega/\omega_0) \\
&= -\frac{|m| + 2l}{\tau} a_l^{(m)} + \frac{2}{\tau}\left[\frac{1}{2}\sum_{k=0}^{\infty}\left(k + \frac{|m-2|}{2} - \frac{m-2}{2}\right) a_k^{(m-2)} K_{k,l}^{|m-2|,|m|,0}\right. \\
&\quad - \frac{m-1}{4}\sum_{k=0}^{\infty}(m - 2 - |m-2| - 2k) a_k^{(m-2)} K_{k,l}^{|m-2|,|m|,-1} \\
&\quad - \frac{m-1}{2}\sum_{k=0}^{\infty}\sqrt{k(|m-2| + k)}\, a_k^{(m-2)} K_{k-1,l}^{|m-2|,|m|,-1} \\
&\quad + \frac{1}{2}\sum_{k=0}^{\infty}\left(k + \frac{|m+2|}{2} + \frac{m+2}{2}\right) a_k^{(m+2)} K_{k,l}^{|m+2|,|m|,0} \\
&\quad - \frac{m+1}{4}\sum_{k=0}^{\infty}(m + 2 + |m+2| + 2k) a_k^{(m+2)} K_{k,l}^{|m+2|,|m|,-1} \\
&\quad \left. + \frac{m+1}{2}\sum_{k=0}^{\infty}\sqrt{k(|m+2| + k)}\, a_k^{(m+2)} K_{k-1,l}^{|m+2|,|m|,-1}\right],
\end{aligned} \tag{35}
$$

where

$$I_{ml}(p') = \int_0^\infty \exp(-x) f_l^{(|m|)}(x) J_m(p'\sqrt{2}\sigma_\theta\sqrt{x})\, dx \tag{36}$$

and

$$K_{k,l}^{\alpha,\beta,N} = \int_0^\infty \exp(-x) x^N f_k^{(\alpha)}(x) f_l^{(\beta)}(x) dx. \tag{37}$$

The integral in Eq. (36) is evaluated as[6]

$$I_{nl}(p') = \frac{1}{\sqrt{(n+l)!\,l!}}\left(\frac{\sigma_\theta p'}{\sqrt{2}}\right)^{n+2} \exp\left(-\frac{(\sigma_\theta p')^2}{2}\right), (n > 0) \tag{38}$$

$$I_{-nl}(p') = (-1)^n I_{nl(p')}, \tag{39}$$

by using the series expansion of the Bessel function and the Rodrigues formula for the generalized Laguerre polynomials.[15] The integrals in Eq. (37) are evaluated in an Appendix.

Now, we introduce a matrix element M_{nh}^{ml} defined by

$$M_{nh}^{ml} = \sum_{p=-\infty}^{\infty} \frac{Z(p\omega_0 + \Omega)}{p + \Omega/\omega_0} i^{m-n} I_{ml}(p + \Omega/\omega_0) I_{nh}(p + \Omega/\omega_0) \tag{40}$$

and use the scaling parameter ξ of Chao and Gareyte[5] defined by

$$\xi = \frac{\alpha I}{\nu_s^2 E}, \tag{41}$$

where $I = Ne/T$ is the current per bunch. Then if we define

$$\lambda = \Omega/\omega_s, \tag{42}$$

Eq. (35) is transformed into the matrix equation

$$\begin{aligned}
\left(\lambda - m + i\frac{|m| + 2l}{\tau\omega_s}\right) a_l^{(m)} = \sum_{n=-\infty}^{\infty} \sum_{k=0}^{\infty} \Bigg\{ & i\frac{m\xi e}{2\pi\sigma_\theta^2} M_{nk}^{ml} + \frac{2i}{\tau\omega_s}\delta_{m-2,n} \\
& \times \left[\frac{1}{2}\left(k + \frac{|m-2|}{2} - \frac{m-2}{2}\right) K_{k,l}^{|m-2|,|m|,0} \right. \\
& - \frac{m-1}{4}(m - 2 - |m-2| - 2k) K_{k,l}^{|m-2|,|m|,-1} \\
& \left. - \frac{m-1}{2}\sqrt{k(|m-2| + k)} K_{k-1,l}^{|m-2|,|m|,-1} \right] \\
& + \frac{2i}{\tau\omega_s}\delta_{m+2,n} \left[\frac{1}{2}\left(k + \frac{|m+2|}{2} + \frac{m+2}{2}\right) K_{k,l}^{|m+2|,|m|,0} \right. \\
& - \frac{m+1}{4}(m + 2 + |m+2| + 2k) K_{k,l}^{|m+2|,|m|,-1} \\
& \left. + \frac{m+1}{2}\sqrt{k(|m+2| + k)} K_{k-1,l}^{|m+2|,|m|,-1} \right] \Bigg\} a_k^{(n)}.
\end{aligned} \tag{43}$$

Equation (43) is a matrix equation which we solve by computation. We note from Eq. (43) that when the self-force is neglected, *i.e.*, $M_{nk}^{ml} = 0$, and when we neglect mode coupling, i.e., we neglecct terms involving $\delta_{m-2,n}$ and $\delta_{m+2,n}$, the eigenvalue is

$$\lambda = m - i\frac{|m| + 2l}{\tau\omega_s}. \tag{44}$$

It is thus suggested that the stability criterion when we use the Vlasov equation is

$$\text{(growth rate)} < \frac{|m|}{\tau}. \tag{45}$$

This point will be studied in more detail in Section 4.

4. ROBINSON INSTABILITY DUE TO RF CAVITIES

In this section, we calculate the instability phenomena when mode coupling is neglected. Namely, we take only one azimuthal mode m in Eq. (43) and neglect the coupling between different azimuthal modes. Equation (43) is then simplified as

$$\left(\lambda - m + i\frac{|m| + 2l}{\tau\omega_s}\right) a_l^{(m)} = i\frac{m\xi e}{2\pi\sigma_\theta^2}\sum_{k=0}^{\infty} M_{mk}^{ml} a_k^{(m)}. \tag{46}$$

We assume a resonator impedance $Z(\omega)$ given by

$$Z(\omega) = \frac{R}{1 - iQ\left(\dfrac{\omega}{\omega_r} - \dfrac{\omega_r}{\omega}\right)}, \tag{47}$$

where R is the shunt impedance, Q is the quality factor and $\omega_r/2\pi$ is the resonant frequency of a cavity. The matrix element M_{mk}^{ml} given by Eq. (40) is expressed as

$$M_{mk}^{ml} = \frac{1}{\sqrt{(m+k)!k!(m+l)!l!}} \sum_{p=-\infty}^{\infty} \frac{Z(p + m\nu_s)}{p + m\nu_s}\left(\frac{\sigma_\theta(p + m\nu_s)}{\sqrt{2}}\right)^{2m+2k+2l} \times \exp[-\sigma_\theta^2(p + m\nu_s)^2], \tag{48}$$

where the coherent frequency Ω is replaced by $m\omega_s$. The quantity $Z(p + m\nu_s)/(p + m\nu_s)$ is written as

$$\frac{Z(p + m\nu_s)}{p + m\nu_s} = \frac{iR}{2Q\sqrt{1 - 1/4Q^2}}\left[\frac{1}{p - p_1} - \frac{1}{p - p_2}\right], \tag{49}$$

where

$$p_1 = p_r\sqrt{1 - 1/4Q^2} - m\nu_s - i\frac{p_r}{2Q} \tag{50}$$

$$p_2 = -p_r\sqrt{1 - 1/4Q^2} - m\nu_s - i\frac{p_r}{2Q} \tag{51}$$

$$p_r = \frac{\omega_r}{\omega_0}. \tag{52}$$

In these expressions, a single-bunch case is assumed.

Now we evaluate the infinite sum $S_{2m}(p_1)$ defined by

$$S_{2m}(p_1) = \sum_{p=-\infty}^{\infty} \frac{z^{2m}\exp(-z^2)}{p - p_1}, \tag{53}$$

where

$$z = \sigma_\theta(p + m\nu_s). \tag{54}$$

The sum in Eq. (53) was evaluated by Zotter[16] to good approximation as

$$S_{2m}(p_1) = z_1^{2m} S_0(p_1) + \sum_{l=0}^{m-1} \Gamma(l + \tfrac{1}{2}) z_1^{2m-2l-1}, \tag{55}$$

where

$$S_0(p_1) = -\pi \exp(-z_1{}^2) \cot \pi p_1 - i\pi[w(z_1) - \exp(-z_1{}^2)] \tag{56}$$

$$z_1 = \sigma_\theta(p_1 + m\nu_s). \tag{57}$$

$w(z)$ is the complex error function[15] and Γ is the gamma function.[15] The matrix element in Eq. (48) is then written as

$$M^{ml}_{mk} = \frac{1}{\sqrt{(m+k)!k!(m+l)!l!}} \frac{iR}{2Q\sqrt{1 - 1/4Q^2}\,2^{(m+k+l)}} \times [S_{2(m+k+l)}(p_1) - S_{2(m+k+l)}(p_2)]. \tag{58}$$

We apply this formalism to a model higher-order mode impedance of a TRISTAN rf cavity. The parameters used are $E = 8$ GeV, $p_r = 7000.1$ (695.3 MHz), $R = 800\ M\Omega$ and $Q = 6 \times 10^4$. In this case, strong Robinson instability occurs. The results are shown in Figures 1a to 1c for $m = 1$ (dipole), $m = 2$ (quadrupole) and $m = 3$ (sextupole) modes, respectively. The synchrotron tune ν_s is arbitrarily chosen to be 0.02 for the cases of $m = 1$, $m = 2$ and 0.01 for the case of $m = 3$. The corresponding natural bunch lengths are 1.57 cm and 3.13 cm, respectively. In the figures, the solid curves show the solution of the Fokker-Planck equation, while the dashed curves show the solution of the Vlasov equation. Five radial modes ($k, l = 0 \sim 4$) are taken into account. Calculations using ten radial modes have also been done, but the results do not change. This shows the good convergence of the expansion in terms of generalized Laguerre polynomials.

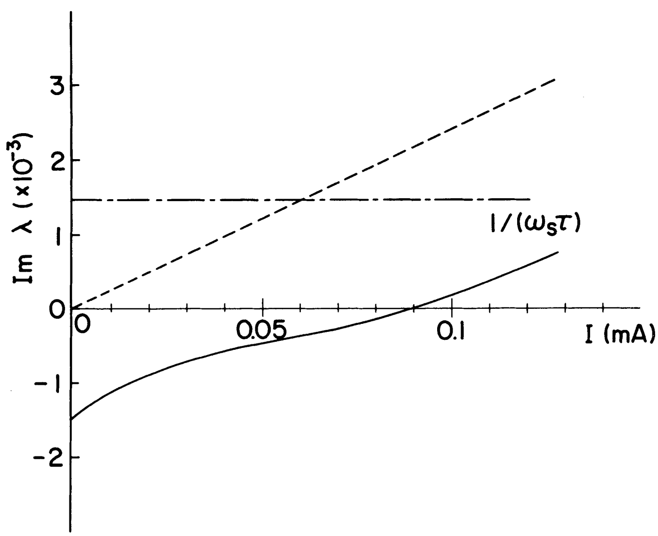

FIGURE 1a Imaginary part of coherent oscillation frequency λ in unit of synchrotron frequency ω_s for the dipole mode $m = 1$. $E = 8$ GeV, $\nu_s = 0.02$ and the bunch length is 1.57 cm. The solid curve shows the solution of the Fokker-Planck equation and the dashed curve shows the solution of the Vlasov equation.

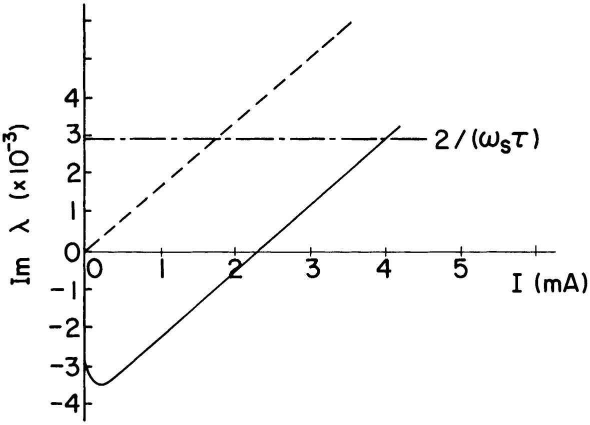

FIGURE 1b Same as (a) for the quadrupole mode $m = 2$.

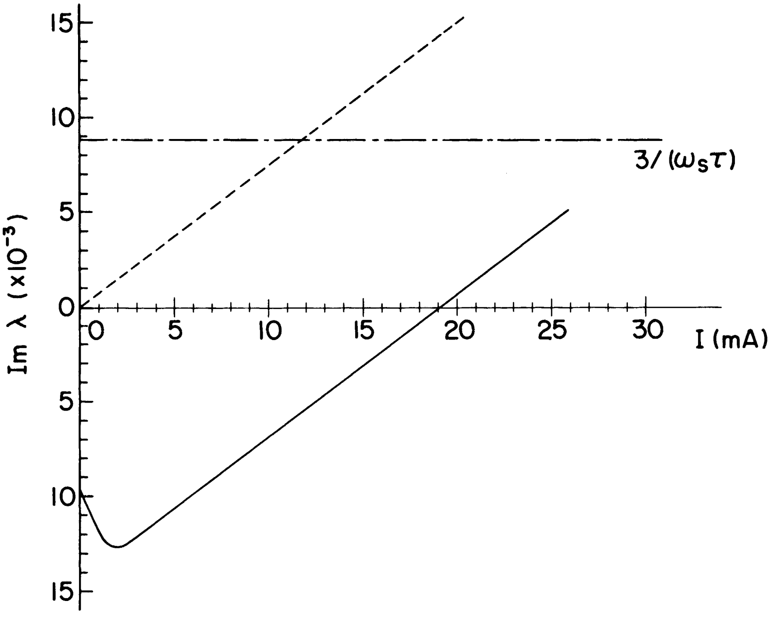

FIGURE 1c Same as (a) for the sextupole mode $m = 3$. ν_s is chosen to be 0.01 and the bunch length is 3.13 cm.

We see from Figures 1a to 1c that the solution of the Vlasov equation with the stability criterion of Eq. (45)

$$(\text{growth rate}) < |m|/\tau$$

suggested in Section 3 is an approximate measure of the solution of the Fokker-Planck equation although the Fokker-Planck equation gives a higher threshold current. Thus this simplified approach can be employed for practical purposes, although the solution of the Fokker-Planck equation is not so difficult compared with the solution of the Vlasov equation. The criterion of Eq. (45) is also suggested by the work of Renieri.[8] In general, the Fokker-Planck equation is expected to give a higher threshold current than the Vlasov-equation approach because different radial modes mix in the matrix-eigenvalue problem and higher radial modes ($l \neq 0$) give a stronger damping rate than that given by Eq. (45), as is evident from Eq. (44). Finally we should remark that the higher-order mode impedance in the above calculation is only a model impedance. In reality, there are many impedance peaks and their eigenfrequencies have a statistical distribution arising from the construction errors of cavities. Thus the growth rate obtained in the above calculation has nothing to do with reality. The statistical calculations for the TRISTAN rf cavities are done in Refs. 17 and 18 using the Vlasov equation.

5. MODE COUPLING AND ANOMALOUS BUNCH LENGTHENING

For a broadband impedance, the summation over harmonics p in the matrix element M_{nh}^{ml} given by Eq. (40) can be replaced by an integral. When only one azimuthal mode m is taken into account, the contribution of the real part of the coupling impedance to M_{mh}^{ml} vanishes identically and only the imaginary part contributes, because the product $I_{ml}(p)I_{mk}(p)$ is an even function of p and

$$\text{Re}\, Z(-\omega) = \text{Re}\, Z(\omega) \tag{59}$$

$$\text{Im}\, Z(-\omega) = -\text{Im}\, Z(\omega). \tag{60}$$

The matrix element iM_{mk}^{ml} is real and symmetric so that no instability occurs. When we take different azimuthal modes together, the imaginary part of the coupling impedance contributes to terms with $m - n =$ even, and the real part contributes to terms with $m - n =$ odd. When the real parts of the coherent frequencies of the two different azimuthal modes become equal, strong instability occurs, as explained by Sacherer.[2] These statements are for the Vlasov-equation approach, but the same thing can be said for the Fokker-Planck equation approach if we make the additional remark that additional damping and mode coupling terms appear due to synchrotron radiation. We study in this section the contribution of these additional terms.

The mode coupling leads to anomalous bunch lengthening in electron storage rings. The effect was studied in detail by using the Vlasov equation in a previous report[6] and comparison was made between theory and the experimental data in SPEAR II.[19] We study the same phenomena using the Fokker-Planck equation formalism developed in Section 3. The coupling impedance of SPEAR II proposed by Wilson, *et al.*,[19,20] is used. We calculate the threshold current of anomalous bunch lengthening in SPEAR II by using both the Vlasov and the Fokker-Planck equations. The result is shown in Figure 2 for the case $\nu_s = 0.044$. We take into account eight azimuthal modes ($m = -4$ to 4) and five radial modes. Calculation using ten radial modes have also been done, but the result does not change. This again shows the good convergence of the expansion in terms of generalized Laguerre polynomials. The calculation using the Fokker-Planck equation and that using the Vlasov equation give almost identical results and no essential difference is seen between the two theories. This is explained by the fact that the instability due to mode coupling is quite a fast process and the slow processes of

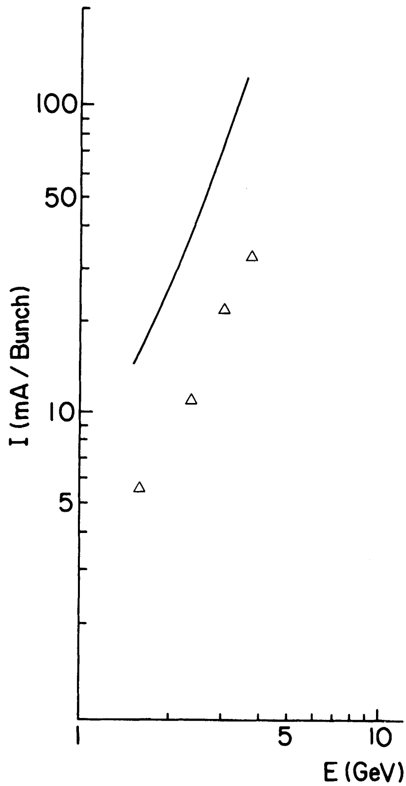

FIGURE 2 Threshold current of anomalous bunch lengthening for SPEAR II. ν_s is taken to be 0.044. The triangles show the experimental data taken from Ref. 19. The Fokker-Planck and Vlasov equations give almost identical results.

radiation damping and excitation play no essential part. The quantitative disagreement between the theory and the experiment in SPEAR II by a factor of about three to four, which was found in the previous report,[6] remains unexplained even when radiation effects are taken into account.

In order to study the above conclusion in the energy range of TRISTAN, we have calculated the threshold current of anomalous bunch lengthening in TRISTAN. The coupling impedance is assumed to come only from vacuum chambers and the effect of rf cavities is neglected for simplicity. Thus the result is not a good estimate of the bunch-lengthening effect, but the difference between the Vlasov and the Fokker-Planck equation treatments will be made manifest. We assume the same coupling impedance as that of SPEAR II, except that an improvement factor of 0.2 per unit length is taken into account.[21]

The result is shown in Figure 3, where ν_s is assumed to be 0.1. The Vlasov approach gives almost the same results as the Fokker-Planck approach and the conclusion obtained before that the two approaches give almost identical results is confirmed also in the energy range of TRISTAN. With $\nu_s = 0.1$, the natural bunch length at 8 GeV is 0.31 cm. This is a very small value and some anomalous bunch lengthening is expected to occur with this short bunch length.

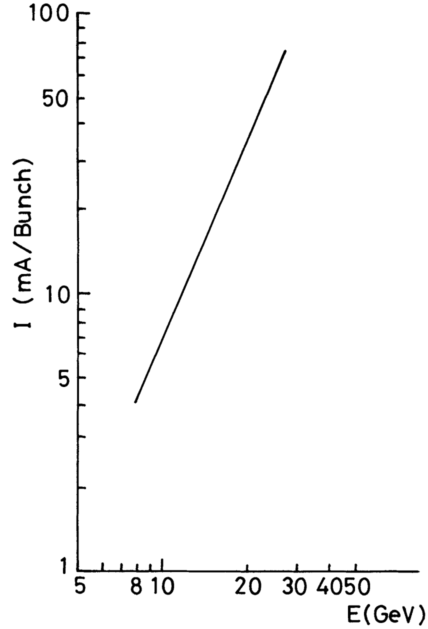

FIGURE 3 Threshold current of anomalous bunch lengthening in TRISTAN. Only the vacuum chamber impedance is taken into account. ν_s is taken to be 0.1. The Fokker-Planck aand Vlasov equations give almost identical results.

6. CONCLUSIONS

The Fokker-Planck equation for the synchrotron motion is expressed using action-angle variables. The equation is solved by expansion in terms of generalized Laguerre polynomials. The convergence of the expansion is confirmed by computer calculation to be very rapid. The method of solution parallels that of Sacherer's integral equations, which are based on the Vlasov equation, and a direct comparison can be made between the two approaches.

For a sharply peaked impedance, the Robinson instability occurs. The approximate stability criterion of Eq. 45, that the growth rate calculated by the Vlasov equation should be smaller than $|m|/\tau$, where m is the azimuthal mode number and τ is the radiation damping time, is shown to hold by the calculation using the Fokker-Planck equation. So a simplified approach of Sacherer's integral equations can be used for electron beam instabilities when combined with the above criterion. We note, however, that a previously used criterion that the growth rate should be smaller than the radiation damping rate is not always correct and that higher azimuthal and radial modes are more strongly damped.

For a broadband impedance, no instability occurs even with the Fokker-Planck equation when mode coupling is neglected. The situation is the same as with the Vlasov-equation treatment. When two different azimuthal-mode frequencies become equal, a very fast instability occurs. The threshold of this instability is almost identical to the one given by the Vlasov-equation approach. This result is explained by the fact that the instability caused by mode coupling is a fast process compared with the radiation damping and excitation processes and that the radiation effect plays little role in the fast instability. The discrepancy found before[6] between the theory and the experimental data in Spear II thus remains unexplained even when the radiation effect is taken into account.

ACKNOWLEDGMENT

The author wishes to thank Mr. K. Yokoya for helpful discussions. He also wishes to thank Prof. T. Nishikawa, Prof. T. Kamei and Prof. Y. Kimura for their interest in this work and continual encouragement.

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APPENDIX

We evaluate the integrals defined by Eq. (37), i.e.,

$$K_{k,l}^{\alpha,\beta,N} = \int_0^\infty \exp(-x) x^N f_k^{(\alpha)}(x) f_l^{(\beta)}(x)\, dx, \tag{61}$$

where $f_k^{(\alpha)}(x)$'s are the orthogonal functions defined by Eq. (30), i.e.,

$$f_k^{(\alpha)}(x) = \sqrt{\frac{k!}{(\alpha + k)!}}\, x^{\alpha/2} L_k^{(\alpha)}(x). \tag{62}$$

We have to evaluate the integral in Eq. (61) only for cases where $\beta = \alpha, \beta = \alpha + 2$ and $N = 0$ or $N = -1$ because only the suffices k and l interchange when $\beta = \alpha - 2$.

We first consider the case where $\beta = \alpha$. We calculate the case where $k \leqq l$. In this case, we use the series expansion of the Laguerre polynomial[15]

$$L_k^\alpha(x) = \sum_{r=0}^{k} (-1)^r \binom{\alpha + k}{k - r} \frac{x^r}{r!} \tag{63}$$

for $L_k^\alpha(x)$ and the Rodrigues formula[15]

$$L_l^\alpha(x) = \frac{\exp(x)x^{-\alpha}}{l!} \frac{d^l}{dx^l} \{\exp(-x)x^{\alpha+l}\} \tag{64}$$

for $L_l^\alpha(x)$. Then the integral in Eq. (61) is expressed as

$$K_{k,l}^{\alpha,\alpha,N} = \sum_{r=0}^{k} (-1)^r \binom{\alpha + k}{k - r} \frac{1}{r!} \frac{1}{l!} \times \sqrt{\frac{k!}{(\alpha + k)!}} \sqrt{\frac{l!}{(\alpha + l)!}}$$

$$\int_0^\infty x^{r+N} \frac{d^l}{dx^l} \{e^{-x} x^{\alpha+l}\} dx. \tag{65}$$

Integrating by parts l times, we obtain

$$K_{k,l}^{\alpha,\alpha,N} = \sqrt{\frac{k!}{(\alpha + k)!}} \sqrt{\frac{l!}{(\alpha + l)!}} \sum_{r=0}^{k} (-1)^r \binom{\alpha + k}{k - r} \frac{1}{r!} \frac{1}{l!}$$

$$\times (-1)^l (r + N)(r + N - 1) \cdot \cdots \cdot (r + N - l + 1) \times (r + N + \alpha)! \tag{66}$$

From Eq. 66, we easily obtain

$$K_{k,l}^{\alpha,\alpha,0} = \delta_{kl}, \tag{67}$$

which shows the normalization of the functions $f_k^{(\alpha)}(x)$'s and

$$K_{k,l}^{\alpha,\alpha,-1} = \sqrt{\frac{(\alpha + k)!\,l!}{(\alpha + l)!\,k!}} \frac{1}{\alpha} \qquad (k \leqq l). \tag{68}$$

For the case where $k > l$, we easily obtain

$$K_{k,l}^{\alpha,\alpha,-1} = \sqrt{\frac{(\alpha + l)!\,k!}{(\alpha + k)!\,l!}} \frac{1}{\alpha} \qquad (k > l). \tag{69}$$

Now, we consider the case where $\beta = \alpha + 2$. Also in this case, we combine the series expansion and the Rodrigues formula for the generalize Laguerre polynomials and integrate by parts. Then we obtain the following results in the same way as for the case where $\beta = \alpha$.

$$K_{k,l}^{\alpha,\alpha+2,0} = \begin{cases} \sqrt{\dfrac{(\alpha+k)!\,l!}{(\alpha+2+l)!\,k!}}\,(\alpha+1) & (k \leqq l) \\ -\delta_{k,l+1}\sqrt{\dfrac{(l+1)}{(\alpha+l+2)}} & (k > l) \end{cases} \tag{70}$$

$$K_{k,l}^{\alpha,\alpha+2,-1} = \begin{cases} \sqrt{\dfrac{(\alpha+k)!\,l!}{(\alpha+l+2)!\,k!}}\,(l+1-k) & (k \leqq l) \\ 0 & (k > l) \end{cases} \tag{71}$$