

# On the tau invariants in instanton and monopole Floer theories

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## Abstract

We unify two existing approaches to the *tau* invariants in instanton and monopole Floer theories, by identifying  $\tau_G$ , defined by the second author via the *minus* flavors  $\underline{\text{KHI}}^-$  and  $\underline{\text{KHM}}^-$  of the knot homologies, with  $\tau_G^\sharp$ , defined by Baldwin and Sivek via cobordism maps of the 3-manifold homologies induced by knot surgeries. We exhibit several consequences, including a relationship with Heegaard Floer theory, and use our result to compute  $\underline{\text{KHI}}^-$  and  $\underline{\text{KHM}}^-$  for twist knots.

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## 1 | INTRODUCTION

Among the Floer invariants of 3-manifolds, it is now known that various flavors of Heegaard Floer homology, monopole Floer homology, and embedded contact homology are isomorphic, while their relationship with instanton Floer homology remains a major open question.

The relationships between Floer invariants of knots in 3-manifolds are even less understood: Knot instanton Floer homology is not known to be isomorphic to the other knot homologies, and while it is known that the usual knot monopole Floer homology is isomorphic to the *hat* flavor of knot Heegaard Floer homology (tensored with the mod-2 Novikov field  $\mathcal{R}$ ) as graded modules over  $\mathcal{R}$  [11, 17, 22, 37]:

$$\underline{\text{KHM}}(Y, K; \mathcal{R}) \cong \widehat{\text{HFK}}(Y, K; \mathbb{F}_2) \otimes \mathcal{R}, \quad (1.1)$$

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no analogous statement is known for the more powerful *minus* flavor of knot Heegaard Floer homology  $\text{HFK}^-(Y, K; \mathbb{F}_2)$ , which is a graded module over  $\mathbb{F}_2[U]$  rather than  $\mathbb{F}_2$ . In fact, the *minus* flavors  $\text{KHM}^-$  and  $\text{KHI}^-$  of knot monopole and instanton Floer homologies have been defined only recently by the second author [26] using contact handle attachment maps of sutured manifolds, based on work of Baldwin and Sivek [3] and inspired by work of Etnyre, Vela-Vick, and Zarev [13]. As such, many basic structural properties of  $\text{KHM}^-$  and  $\text{KHI}^-$  are yet unknown.

For example, a key property of  $\text{HFK}^-$  for knots  $K \subset S^3$  is its unique  $\mathbb{F}_2[U]$ -summand, the negative of whose maximal Alexander  $\mathbb{Z}$ -grading is a concordance invariant  $\tau_{\text{H}}(K)$ .<sup>†</sup> In fact,  $\tau_{\text{H}}$  defines a homomorphism  $\tau_{\text{H}} : \mathcal{C} \rightarrow \mathbb{Z}$  from the smooth concordance group  $\mathcal{C}$ . Moreover,  $|\tau_{\text{H}}(K)|$  also gives a lower bound on the smooth 4-genus  $g_4(K)$ . More generally,  $\tau_{\text{H}}$  can be defined for null-homologous knots  $K$  in a connected, oriented, closed 3-manifold  $Y$ , with a choice of a Seifert surface  $S$ . Inspired by this, the second author [26] similarly defines  $\tau_{\text{M}}(Y, K, S)$  and  $\tau_{\text{I}}(Y, K, S)$  to be the negative<sup>‡</sup> of the maximal Alexander  $\mathbb{Z}$ -grading of the non- $U$ -torsion elements of  $\text{KHM}^-$  and  $\text{KHI}^-$ .<sup>§</sup> However, for knots  $K \subset S^3$ , these have not been shown to be concordance invariants or to give 4-genus bounds.

In a different approach to the *tau* invariants, Baldwin and Sivek [6] define a concordance invariant  $\nu_{\text{I}}^{\sharp}$  using cobordism maps between the framed instanton Floer homology  $\text{I}^{\sharp}$  of  $S^3$  and of the integer surgeries  $S_n^3(K)$  along  $K$ , and homogenize  $\nu_{\text{I}}^{\sharp}$  to obtain a concordance invariant  $\tau_{\text{I}}^{\sharp}$ .<sup>¶</sup> They show that  $|\tau_{\text{I}}^{\sharp}(K)| \leq g_4(K)$ , and that  $2\tau_{\text{I}}^{\sharp}$  gives a homomorphism  $2\tau_{\text{I}}^{\sharp} : \mathcal{C} \rightarrow \mathbb{R}$  that is, in fact, a *slice-torus invariant*, as defined by Lewark [23] following Livingston [27]; however, defined via a homogenization process,  $\tau_{\text{I}}^{\sharp}$  is not known to be an integer (or even a rational number). Nonetheless, these properties of  $\tau_{\text{I}}^{\sharp}$  are sufficient for Baldwin and Sivek to use to determine  $\text{I}^{\sharp}$  of all nonzero rational surgeries on 20 of the 35 nontrivial prime knots in  $S^3$  through eight crossings, and establish several other results. While it is not explicitly stated, a concordance invariant  $\tau_{\text{M}}^{\sharp}$  can be similarly defined in the monopole Floer theory, via the *tilde* flavor  $\widetilde{\text{HM}}(S_n^3(K); \mathcal{R})$ . By construction,  $\tau_{\text{I}}^{\sharp}$  and  $\tau_{\text{M}}^{\sharp}$  are defined only for knots  $K \subset S^3$ .

This article represents the natural first step in understanding the structures of  $\text{KHM}^-$  and  $\text{KHI}^-$  and their comparisons with  $\text{HFK}^-$ . In the following, we shall replace the subscripts M and I (for “monopole” and “instanton”) in  $\tau_{\text{M}}$  and  $\tau_{\text{I}}$  by the subscript G (for “gauge-theoretic”) in  $\tau_{\text{G}}$ , when the statement applies to both theories. To begin, our main theorem identifies the *tau* invariants, answering the question posed in (a previous version of) [6].

**Theorem 1.2.** *For all knots  $K \subset S^3$ , we have  $\tau_{\text{G}}(K) = \tau_{\text{G}}^{\sharp}(K)$ .*

We immediately have the following corollaries in the instanton setting.

**Corollary 1.3.** *For all knots  $K \subset S^3$ , the invariant  $\tau_{\text{I}}^{\sharp}(K)$  is an integer. In other words,  $\tau_{\text{I}}^{\sharp}$  defines a homomorphism  $\tau_{\text{I}}^{\sharp} : \mathcal{C} \rightarrow \mathbb{Z}$ .*

<sup>†</sup> Technically,  $\tau_{\text{H}}$  — usually simply denoted  $\tau$  — may depend on the coefficient ring. In this article, we always take  $\tau_{\text{H}}(K)$  to mean  $\tau_{\text{H}}(K; \mathbb{F}_2)$ .

<sup>‡</sup> There is no negative sign in Definition 2.5 because we have reversed the orientation of the ambient 3-manifold there.

<sup>§</sup> When  $Y = S^3$ , we abbreviate these by  $\tau_{\text{M}}(K)$  and  $\tau_{\text{I}}(K)$ .

<sup>¶</sup> In [6],  $\tau_{\text{I}}^{\sharp}$  is simply denoted as  $\tau^{\sharp}$ ; we add the subscript I to separate it from the monopole version  $\tau_{\text{M}}^{\sharp}$ .

**Corollary 1.4** (cf. [6, Proposition 5.4]). *For all knots  $K \subset S^3$ , we have  $|\tau_I(K)| \leq g_4(K)$ .*

As mentioned above, Baldwin and Sivek [6, Theorem 1.6] show that  $2\tau_I^\sharp$  is a slice-torus invariant, and use this to show that  $\tau_I^\sharp(K)$  agrees with  $g_4(K)$  when  $K$  is quasi-positive. Moreover, as Lewark [23] proves that slice-torus invariants agree with the negative of the signature for alternating knots, they obtain  $\tau_I^\sharp(K) = -\sigma(K)/2$  for such knots.<sup>†</sup> Lewark also proves that the values of all slice-torus invariants agree on homogeneous knots, which gives  $\tau_I^\sharp(K) = \tau_H(K)$  for such knots.

In the monopole setting, the statements in the preceding paragraph can be readily proved for  $\tau_M^\sharp$  also. Thus, Theorem 1.2 immediately implies the following for knots in  $S^3$ .

**Corollary 1.5** (cf. [6, Theorem 1.6]). *The invariant  $2\tau_G$  is a slice-torus invariant. If  $K$  is a quasi-positive knot, then  $\tau_G(K) = g_4(K)$ . If  $K$  is an alternating knot, then  $\tau_G(K) = -\sigma(K)/2$ . If  $K$  is a homogeneous knot, then  $\tau_G(K) = \tau_H(K)$ .*<sup>‡</sup>

In fact, in the monopole setting, we can strengthen this last statement to hold for all knots.

**Theorem 1.6.** *For all knots  $K \subset S^3$ , we have  $\tau_M(K) = \tau_H(K)$ .*

*Proof.* Baldwin and Sivek [6, Section 10] detail how the Heegaard Floer  $\tau_H$  invariant can also be expressed as the homogenization of a concordance invariant coming from surgeries, as explained to them by Jennifer Hom. (One may reasonably denote such an invariant by  $\tau_H^\sharp$ .) They then use this to show that if

$$\dim_{\mathbb{C}} I^\sharp(Y; \mathbb{C}) = \dim_{\mathbb{F}_2} \widehat{\text{HF}}(Y; \mathbb{F}_2)$$

holds for all  $Y$  obtained via integer surgery along a knot in  $S^3$ , then  $\tau_I^\sharp(K) = \tau_H^\sharp(K) = \tau_H(K)$  for all  $K \subset S^3$  [6, Proposition 1.24]. The exact same proof can be adapted to show that if

$$\text{rk}_{\mathcal{R}} \widetilde{\text{HM}}(Y; \mathcal{R}) = \dim_{\mathbb{F}_2} \widehat{\text{HF}}(Y; \mathbb{F}_2) \tag{1.7}$$

holds for all  $Y$  obtained via integer surgery, then  $\tau_M^\sharp(K) = \tau_H^\sharp(K) = \tau_H(K)$ . But (1.7) is simply the isomorphism between monopole and Heegaard Floer homologies for 3-manifolds [11, 17, 37]. Thus, our claim follows from Theorem 1.2.  $\square$

The significance of Theorem 1.6 is that it represents the first step toward proving the generalization of the isomorphism between  $\underline{\text{KHM}}$  and  $\widehat{\text{HFK}}$  in (1.1) to the *minus* flavor.

**Conjecture 1.8.** *Let  $Y$  be a connected, oriented, closed 3-manifold, and let  $K \subset Y$  be an oriented, nullhomologous knot. Then there is an isomorphism of graded modules over  $\mathcal{R}[U]$ :*

$$\underline{\text{KHM}}^-(Y, K; \mathcal{R}) \cong \text{HFK}^-(Y, K; \mathbb{F}_2) \otimes \mathcal{R}[U].$$

<sup>†</sup> We follow the convention where the right-handed trefoil has signature  $-2$ .

<sup>‡</sup> These facts combined show that  $\tau_I = \tau_H$  for all prime knots through nine crossings, except possibly  $9_{42}$ ,  $9_{44}$ , and  $9_{48}$ .

*Corollary 1.5* has another implication, as pointed out to the authors by Steven Sivek.

**Corollary 1.9.** *Suppose that  $\Lambda \subset (S^3, \xi_{\text{std}})$  is a Legendrian knot of smooth knot type  $K$ , then*

$$tb(\Lambda) + |r(\Lambda)| \leq 2\tau_G(K) - 1.$$

*Proof.* Consider the positive and negative transverse push-offs  $\Theta_{\pm}(\Lambda)$ , which have self-linking numbers  $sl(\Theta_{\pm}(\Lambda)) = tb(\Lambda) \mp r(\Lambda)$ , respectively. By [6, Theorem 6.1],  $sl(\Theta) \leq 2\tau_G^{\sharp}(K) - 1$  for all transverse representatives  $\Theta$  of  $K$ . ([6, Theorem 6.1] is a statement for  $\tau_1^{\sharp}$ , but the same argument works for  $\tau_M^{\sharp}$ .) Thus, the result follows from Theorem 1.2. (Note that [6, Theorem 6.1] is, in fact, the key ingredient in proving that  $\tau_1^{\sharp}(K) = g_4(K)$  for quasi-positive knots  $K$ .)  $\square$

*Remark 1.10.* The analogous statement that

$$tb(\Lambda) + |r(\Lambda)| \leq 2\tau_H(K) - 1, \quad (1.11)$$

first proved by Plamenevskaya [35], implies Corollary 1.9 for  $\tau_M$  via Theorem 1.6.

Below, we describe the strategy to prove Theorem 1.2. To simplify our notation, we first set up some conventions for the rest of the article.

## Conventions

The coefficient ring for monopole Floer homologies is always taken to be the mod-2 Novikov field, and that for instanton Floer homologies is always taken to be the field  $\mathbb{C}$  of complex numbers. In both cases, we shall denote the coefficient ring by  $\mathcal{R}$ . Similar to  $\tau_G$ , we shall denote both  $\underline{\text{SHM}}$  and  $\underline{\text{SHI}}$  by  $\underline{\text{SHG}}$  when a statement applies to both sutured monopole and sutured instanton Floer homologies, and likewise denote by  $\underline{\text{KHG}}$  (resp.  $\underline{\text{KHG}}^-$ ) the knot monopole and instanton Floer homologies  $\underline{\text{KHM}}$  (resp.  $\underline{\text{KHM}}^-$ ) and  $\underline{\text{KHI}}$  (resp.  $\underline{\text{KHI}}^-$ ).

### 1.1 | Strategy

The astute reader may have noticed that we did not state the concordance invariance of  $\tau_G$ , or its additivity under connected sum, as a corollary of Theorem 1.2. The reason is that, in order to prove Theorem 1.2, we shall, in fact, *first* prove the concordance invariance of  $\tau_G$ .

**Proposition 1.12.** *For all knots  $K \subset S^3$ , the integer  $\tau_G(K)$  is a concordance invariant.*

To establish Proposition 1.12, we shall also prove the key property that  $\underline{\text{KHG}}^-$  has a unique  $\mathcal{R}[U]$ -summand (also known as an infinite  $U$ -tower) for knots  $K \subset S^3$ , analogous to  $\text{HF}K^-$ :

**Proposition 1.13.** *For all knots  $K \subset S^3$ ,  $\underline{\text{KHG}}^-(S^3, K)$  has a unique  $\mathcal{R}[U]$ -summand.*

After establishing Proposition 1.12, we shall turn to the additivity of  $\tau_G$  under connected sum.

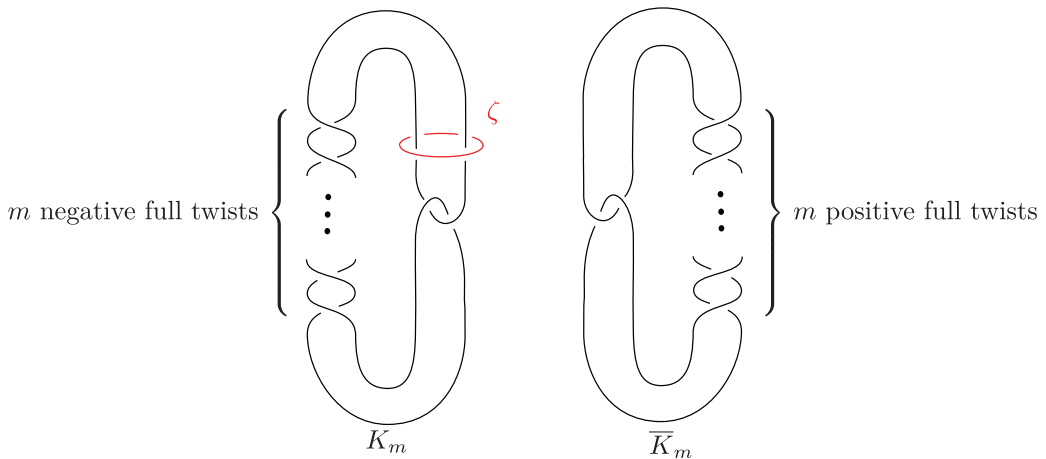


FIGURE 1 The twist knots  $K_m$  and  $\overline{K}_m$ . In Section 6, we shall perform surgery along the curve  $\zeta$  in the proof of Theorem 1.16.

**Proposition 1.14.** *For all pairs of knots  $K_1, K_2 \subset S^3$ , we have  $\tau_G(K_1 \# K_2) = \tau_G(K_1) + \tau_G(K_2)$ .*

The rest of the proof of Theorem 1.2 can be described roughly as follows. Recall that  $\text{KHG}^-$  is defined in terms of a directed system of  $\text{SHG}$  of the knot complement  $S^3(K)$  with sutures  $\Gamma_n$ , over different values of  $n$ , where  $\Gamma_n$  denotes a pair of parallel sutures on the boundary torus with  $n$  full twists. First, using bypass and surgery exact triangles involving  $-S^3(K)$ , we reformulate  $\tau_G$  in terms of the twisting coefficient  $n_0$  for which  $\text{SHG}$  of  $-S^3(K)$  with  $-\Gamma_{n_0}$  sutures uniquely attains minimum rank. (This is conceptually similar to Baldwin and Sivek’s notion of *V-shaped* knots.) Next, noting that whether the inequality  $\text{rk}_{\mathcal{R}} \text{SHG}(-S^3(K), -\Gamma_{n+1}) > \text{rk}_{\mathcal{R}} \text{SHG}(-S^3(K), -\Gamma_n)$  holds is equivalent to the (non-)vanishing of certain surgery cobordism maps involving  $-S^3(K)$ , further analysis using surgery exact triangles allows us to relate  $\tau_G$  to the (non-)vanishing of surgery cobordism maps involving  $I^\#$  or  $\overline{\text{HM}}$  of  $-S^3_{-n}(K)$ , and thence to  $\nu_G^\#$ , giving the inequality

$$2\tau_G(K) - 1 \leq \nu_G^\#(K) \leq 2\tau_G(K) + 1$$

whenever  $\nu_G^\#(K) \neq 0$ . A homogenization argument, using the fact that  $\tau_G$  is a concordance homomorphism, completes the proof.

### 1.2 | Examples

Let  $K_m \subset S^3$  be the twist knot with a positive clasp and  $m$  negative full twists (or  $-m$  positive full twists if  $m < 0$ ), and let  $\overline{K}_m$  denote its mirror image; see Figure 1. (In the notation of Baldwin and Sivek [6], their  $K_n$  corresponds to our  $K_{-n/2}$  when  $n$  is even, and to our  $\overline{K}_{(n+1)/2}$  when  $n$  is odd.) Baldwin and Sivek [6] compute<sup>†</sup>

$$\nu_1^\#(K_m) = \begin{cases} 0 & \text{for } m \leq 0, \\ 1 & \text{for } m > 0, \end{cases} \quad \nu_1^\#(\overline{K}_m) = \begin{cases} 0 & \text{for } m \leq 0, \\ -1 & \text{for } m > 0, \end{cases} \tag{1.15}$$

<sup>†</sup> They only compute half of these, but the antisymmetry of  $\nu_1^\#$  under mirroring gives the other half.

and use it to fully determine  $\dim_{\mathbb{C}} \mathbb{I}^{\sharp}(S^3_{p/q}(K_m))$ . One can also compute  $\tau_1^{\sharp}(K_m)$  (and hence  $\tau_1^{\sharp}(\overline{K}_m)$ ) as follows: Since twist knots are alternating, [6, Corollary 1.10] says that  $\tau_1^{\sharp}(K_m) = -\sigma(K_m)/2$ , and the signature  $\sigma(K_m)$  can be directly computed from the  $2 \times 2$  Seifert matrix. This gives

$$\tau_1^{\sharp}(K_m) = \begin{cases} 0 & \text{for } m \leq 0, \\ 1 & \text{for } m > 0, \end{cases} \quad \tau_1^{\sharp}(\overline{K}_m) = \begin{cases} 0 & \text{for } m \leq 0, \\ -1 & \text{for } m > 0. \end{cases}$$

(One can also use (1.15) to compute  $\tau_1^{\sharp}(K_m)$  without computing  $\sigma(K_m)$ , using [6, Theorem 3.7, Proposition 5.4, and Corollary 1.10].) With this in hand, to illustrate Theorem 1.2, we provide a direct and complete computation of  $\underline{\text{KHG}}^{-}(-S^3, K_m)$  and  $\tau_G$  for this infinite family.

**Theorem 1.16.** *We abbreviate by  $\mathcal{R}$  the  $\mathcal{R}[U]$ -module  $\mathcal{R}[U]/U$ , and denote Alexander gradings by subscripts, and direct sums by superscripts.*

(1) *For  $m \leq 0$ , we have*

$$\underline{\text{KHG}}^{-}(-S^3, K_m) \cong \underline{\text{KHG}}^{-}(-S^3, \overline{K}_m) \cong \mathcal{R}[U]_0 \oplus \mathcal{R}_1^{-m} \oplus \mathcal{R}_0^{-m},$$

$$\tau_G(K_m) = \tau_G(\overline{K}_m) = 0.$$

(2) *For  $m > 0$ , we have*

$$\underline{\text{KHG}}^{-}(-S^3, K_m) \cong \mathcal{R}[U]_1 \oplus \mathcal{R}_1^{m-1} \oplus \mathcal{R}_0^m, \quad \underline{\text{KHG}}^{-}(-S^3, \overline{K}_m) \cong \mathcal{R}[U]_{-1} \oplus \mathcal{R}_1^m \oplus \mathcal{R}_0^{m-1},$$

$$\tau_G(K_m) = 1, \quad \tau_G(\overline{K}_m) = -1.$$

### 1.3 | Future work

As this article represents the first step in our major goal to understand the structures of  $\underline{\text{KHG}}^{-}$ , we present here some open questions that arise naturally from our discussion.

First, while Corollary 1.5 gives  $\tau_G(K)$  for all alternating knots  $K \subset S^3$ , one may reasonably hope to fully determine  $\underline{\text{KHG}}^{-}(S^3, K)$  for such  $K$ . Indeed, we do so in Theorem 1.16 for twist knots, which are alternating. In knot Heegaard Floer homology, if  $K$  is alternating with signature  $\sigma = \sigma(K)$ , and symmetrized Alexander polynomial  $\Delta_K(t) = \sum_i a_i \cdot t^i$ , then

$$\text{HFK}^{-}(S^3, K; \mathbb{F}_2) \cong \mathbb{F}_2[U]_{\frac{\sigma}{2}} \oplus \left( \bigoplus_{i \leq \frac{\sigma}{2}} (\mathbb{F}_2[U]/U)_i^{|b_i|-1} \right) \oplus \left( \bigoplus_{i < \frac{\sigma}{2}} (\mathbb{F}_2[U]/U)_i^{|b_i|} \right),$$

where  $b_i = \sum_{j \geq 0} a_{i+j}$ ; see [29], and also [33, Corollary 10.3.2].

**Question 1.17.** Is there an analogous formula for  $\underline{\text{KHG}}^{-}(S^3, K)$  for alternating knots  $K$ , or at least two-bridge knots  $K$ ?

Another question is the mirroring of knots. It follows from our work that if  $\bar{K}$  is the mirror of  $K$ , then  $\tau_G(\bar{K}) = -\tau_G(K)$ . In knot Heegaard Floer homology, one has the following more precise formula: If

$$\text{HFK}^-(S^3, K; \mathbb{F}_2) \cong \mathbb{F}_2[U]_{-\tau} \oplus \left( \bigoplus_{i=1}^k (\mathbb{F}_2[U]/U)_{S_i}^{n_i} \right),$$

then

$$\text{HFK}^-(S^3, \bar{K}; \mathbb{F}_2) \cong \mathbb{F}_2[U]_{\tau} \oplus \left( \bigoplus_{i=1}^k (\mathbb{F}_2[U]/U)_{n_i - S_i}^{n_i} \right),$$

where  $\tau = \tau_H(K)$ ; see [31, Section 3.5], and also [33, Proposition 7.4.3].

**Question 1.18.** Is there an analogous formula for  $\underline{\text{KHG}}^-(S^3, \bar{K})$  in terms of  $\underline{\text{KHG}}^-(S^3, K)$ ?

Aside from symmetry, there are questions concerning the behavior of  $\underline{\text{KHG}}^-$  under crossing changes and with respect to skein relations. In particular, if  $K_-$  is the result of changing a positive crossing in  $K_+$  to a negative crossing, then there exists graded  $\mathbb{F}_2[U]$ -module maps

$$C_- : \text{HFK}^-(S^3, K_+) \rightarrow \text{HFK}^-(S^3, K_-), \quad C_+ : \text{HFK}^-(S^3, K_-) \rightarrow \text{HFK}^-(S^3, K_+),$$

such that  $C_- \circ C_+$  and  $C_+ \circ C_-$  is each equal to multiplication by  $U$ . Exploiting this, Alishahi and Eftekhary [1] define a  $U$ -torsion order invariant  $l(K)$  that gives a lower bound on the unknotting number  $u(K)$ . A generalized version is used by Juhász, Miller, and Zemke [16] to obtain an obstruction to connected knot cobordisms with a given number of local maxima.

**Question 1.19.** Are there analogous maps for  $\underline{\text{KHG}}^-(S^3, K_+)$  and  $\underline{\text{KHG}}^-(S^3, K_-)$ , and consequently a  $U$ -torsion order invariant in knot instanton and monopole Floer theory?

Note that a positive answer to Question 1.19 would imply that  $0 \leq \tau_G(K_+) - \tau_G(K_-) \leq 1$ , a fact that can be deduced from Corollary 1.5; see [27, Corollary 3].

In regard to the oriented skein relation, Kronheimer and Mrowka [19, Theorem 3.1] prove that  $\underline{\text{KHI}}$  satisfies an exact triangle relating  $K_+$ ,  $K_-$ , and their oriented resolution  $K_0$ . The analogous relation in knot Heegaard Floer homology is satisfied by both  $\widehat{\text{HFK}}$  and  $\text{HFK}^-$  [32], and like its instanton counterpart, has proved to be a very useful tool. One may thus ask:

**Question 1.20.** Does  $\underline{\text{KHI}}^-$  satisfy an oriented skein relation?

In order to answer Question 1.20, one must necessarily generalize the definition of  $\underline{\text{KHI}}^-$  to links with multiple components. This has been carried out by the first and second authors [14, Section 6.2], who define  $\underline{\text{KHG}}^-(Y, L)$  for nullhomologous links  $L \subset Y$ .

**Question 1.21.** Can  $\tau_G$  be generalized to a multiset of values for links  $L \subset Y$ , and if so, what properties does it satisfy?

The analogous notion in knot Heegaard Floer theory is that of the  $\tau_H$ -set of a link [10, 15, 33]. Notably, Hedden and Raoux [15, Theorem 2] prove that the  $\tau_H$ -set of a link  $L \subset Y$  satisfy many interesting properties previously known for  $\tau_H(S^3, K)$ , including concordance invariance in  $Y$ , crossing-change inequalities in  $Y$ , 4-genus bounds in  $Y$ , and, in the case  $L \subset S^3 = \partial W$  where  $W$  is a definite 4-manifold, an inequality for surfaces in  $W$  bound by  $L$ .

Finally, we turn to Legendrian knot invariants. For a Legendrian knot  $\Lambda \subset (Y, \xi)$  of smooth knot type  $K$ , Baldwin and Sivek [5, 7] define a class  $\widehat{\mathcal{L}}_M(\Lambda) \in \underline{\text{KHM}}(-Y, K)$ , and show it to be equivalent to the LOSS invariant  $\widehat{\mathcal{L}}_H(\Lambda) \in \widehat{\text{HFK}}(-Y, K)$ .<sup>†</sup> Notably, their work implies that  $\widehat{\mathcal{L}}_H$  gives an obstruction to the existence of exact Lagrangian cobordisms between Legendrian knots — without adjectives such as *decomposable* or *regular* — for which there is currently no proof purely in Heegaard Floer theory. On the other hand,  $\widehat{\mathcal{L}}_H$  has a generalization  $\mathcal{L}_H(\Lambda) \in \text{HFK}^-(Y, K)$ , which is a non- $U$ -torsion class that is mapped to  $\widehat{\mathcal{L}}_H(\Lambda)$  under the natural map  $\text{HFK}^-(Y, K) \rightarrow \widehat{\text{HFK}}(-Y, K)$ . Etnyre, Vela-Vick, and Zarev [13] place  $\mathcal{L}_H$  in the context of  $\text{HFK}^-$  as the limit of a directed system of SFH; following this strategy, we may also define a Legendrian invariant  $\mathcal{L}_M(\Lambda) \in \underline{\text{KHM}}(-Y, K)$ , which is mapped to  $\widehat{\mathcal{L}}_M(\Lambda)$  under the natural map  $\underline{\text{KHM}}(-Y, K) \rightarrow \widehat{\text{KHM}}(-Y, K)$ . By the naturality in monopole Floer theory [2],  $\mathcal{L}_M$  is a well-defined class — and not only a class defined up to isomorphism — in  $\underline{\text{KHM}}(-Y, K)$ .

**Question 1.22.** Is the Legendrian invariant  $\mathcal{L}_M(\Lambda) \in \underline{\text{KHM}}(-Y, K)$  effective in distinguishing Legendrian knots, or in obstructing exact Lagrangian cobordisms?

## 1.4 | Organization

We review the definitions of  $\underline{\text{KHG}}^-$ ,  $\tau_G$ , and  $\tau_G^\#$  in Section 2. In Section 3, we prove Proposition 1.13 and Proposition 1.12, establishing that  $\tau_G$  is a concordance invariant; in Section 4, we prove Proposition 1.14, the additivity of  $\tau_G$ . We then carry out the argument described in Section 1.1 to prove Theorem 1.2, identifying the *tau* invariants in Section 5. Finally, we compute  $\underline{\text{KHG}}^-$  and  $\tau_G$  for twist knots in Section 6, proving Theorem 1.16.

## 2 | PRELIMINARIES

### 2.1 | KHG and naturality

In this article, we shall focus on oriented, based knots  $(K, p) \subset S^3$  and  $(K, p) \subset -S^3$ . As in [2, Section 8], by the knot complement  $S^3(K)$  and the meridional sutures  $\Gamma_\mu$ , we mean the following: Let  $D^2$  be the unit disk in the complex plane with boundary  $S^1 = \partial D^2$ , and let  $\varphi : S^1 \times D^2 \rightarrow S^3$  be an embedding such that  $\varphi(S^1 \times \{0\}) = K$  and  $\varphi(\{1\} \times \{0\}) = p$ ; then

$$(S^3(K), \Gamma_\mu) = (S^3 \setminus \text{Int}(\text{Im}(\varphi)), \mu_\varphi^+ \cup -\mu_\varphi^-),$$

where  $\mu_\varphi^\pm$  is the oriented meridian  $\varphi(\{\pm 1\} \times \partial D^2)$  on  $\partial S^3(K)$ . Of course, this definition does not quite make sense yet, as it depends on the choice of  $\varphi$ . In work of Kronheimer and Mrowka [20],

<sup>†</sup> For  $\Lambda \subset (S^3, \xi_{\text{std}})$ , the Alexander grading of  $\widehat{\mathcal{L}}_H(\pm\Lambda)$  is  $(tb(\Lambda) \mp r(\Lambda) + 1)/2$ , which offers another proof of (1.11).



the balanced sutured manifold  $(S^3(K), \Gamma_\mu)$  is used to construct the knot instanton and monopole Floer homologies:<sup>†</sup>

$$\underline{\text{KHG}}(S^3, K, p) = \underline{\text{SHG}}(S^3(K), \Gamma_\mu).$$

The sutured instanton and monopole Floer homologies  $\underline{\text{SHG}}(M, \gamma)$ , defined in general for balanced sutured manifolds  $(M, \gamma)$ , themselves depend on the choice of a *closure*, a closed 3-manifold  $Y$  obtained by gluing an auxiliary piece to  $(M, \gamma)$  and then identifying the remaining boundary components, together with a distinguished surface  $R \subset Y$ . Kronheimer and Mrowka assign modules to each such closure, and show that these modules are all isomorphic. By refining the notion of closures, Baldwin and Sivek [2] prove that there are, in fact, *canonical isomorphisms* relating these modules — well defined up to multiplication by a unit in  $\mathcal{R}$  — and use them to build a *projectively transitive system*  $\underline{\text{SHG}}$  for balanced sutured manifolds. By abuse of notation, whenever we write  $\underline{\text{SHG}}$  in the sequel, we shall mean the canonical module associated to  $\underline{\text{SHG}}$ .<sup>‡</sup>

Coming back to  $\underline{\text{KHG}}(S^3, K, p)$ , while the definition of  $(S^3(K), \Gamma_\mu)$  above depends on  $\varphi$ , Baldwin and Sivek [2, Proposition 8.2] further prove that there are canonical isomorphisms relating  $\underline{\text{SHG}}$  of the sutured manifolds  $(S^3(K), \Gamma_\mu)$  constructed using different embeddings  $\varphi$  and  $\varphi'$ . This proof hinges on the fact that the basepoint  $p$  is fixed, and explains the notation  $\underline{\text{KHG}}(S^3, K, p)$ . Once again, this leads to a projectively transitive system  $\underline{\text{KHG}}(S^3, K, p)$ , and we shall take  $\underline{\text{KHG}}(S^3, K, p)$  to mean the associated canonical module.

## 2.2 | $\underline{\text{KHG}}^-$ and $\tau_G$

In this subsection, we recall the construction of  $\underline{\text{KHG}}^-$  and  $\tau_G$  by the second author [26].

Let  $S \subset S^3$  be an oriented, minimal-genus Seifert surface of  $K$ . The surface  $S$  induces a framing on the boundary of the knot complement  $S^3(K)$  and hence longitude  $\lambda$  (whose orientation agrees with that of  $K$ ). Let  $\varphi : S^1 \times D^2 \rightarrow S^3$  be as before, with  $\varphi(S^1 \times \{0\}) = K$  and  $\varphi(\{1\} \times \{0\}) = p$ ; then define the balanced sutured manifold

$$(S^3(K), \Gamma_n) = (S^3 \setminus \text{Int}(\text{Im}(\varphi)), \lambda_{\varphi,n}^+ \cup -\lambda_{\varphi,n}^-),$$

where  $\lambda_{\varphi,n}^+$  is the oriented longitude  $\varphi(\{e^{it} \times e^{i(-nt)}\}_{t \in [0, 2\pi)})$ , and  $\lambda_{\varphi,n}^-$  is the oriented longitude  $\varphi(\{e^{it} \times e^{i(-nt+\pi)}\}_{t \in [0, 2\pi)})$ , on  $\partial S^3(K)$ . Note that  $\Gamma_n \subset \partial S^3(K)$  is the union of two disjoint, parallel, oppositely oriented simple closed curves of slope  $-n$  (or, equivalently, of class  $\pm([\lambda] - n[\mu]) \in H_1(\partial S^3(K))$ ). Like  $(S^3(K), \Gamma_\mu)$ , the sutured manifold  $(S^3(K), \Gamma_n)$  depends on the choice of  $\varphi$ . By an argument similar to that of [2, Proposition 8.2], there are canonical isomorphisms relating  $\underline{\text{SHG}}$  of  $(S^3(K), \Gamma_n)$   $\varphi'$  constructed using different embeddings  $\varphi$  and  $\varphi'$ .

Note that, while our exposition so far focuses on  $(S^3(K), \Gamma_\mu)$  and  $(S^3(K), \Gamma_n)$ , a similar construction gives the balanced sutured manifolds  $(-S^3(K), -\Gamma_\mu)$  and  $(-S^3(K), -\Gamma_n)$ , which we shall use extensively.

<sup>†</sup> The basepoint  $p$  is omitted in the Kronheimer–Mrowka definition.

<sup>‡</sup> Technically, the canonical module is only defined for an honestly (i.e., not projectively) transitive system; for projectively transitive systems, one would only obtain a module modulo multiplication by a unit in  $\mathcal{R}$ , which is only a set. Instead, we choose to interpret  $\underline{\text{SHG}}$  as an actual  $\mathcal{R}$ -module, whose *elements* are well defined only up to multiplication by a unit in  $\mathcal{R}$ .

To define  $\underline{\text{KHG}}^-$ , maps  $\psi_{\pm, n+1}^n$  are defined in [26],<sup>†</sup> which fit into a commutative diagram

$$\begin{array}{ccccccc}
 \cdots & \longrightarrow & \underline{\text{SHG}}(-S^3(K), -\Gamma_n) & \xrightarrow{\psi_{-, n+1}^n} & \underline{\text{SHG}}(-S^3(K), -\Gamma_{n+1}) & \longrightarrow & \cdots \\
 & & \downarrow \psi_{+, n+1}^n & & \downarrow \psi_{+, n+2}^{n+1} & & \\
 \cdots & \longrightarrow & \underline{\text{SHG}}(-S^3(K), -\Gamma_{n+1}) & \xrightarrow{\psi_{-, n+2}^{n+1}} & \underline{\text{SHG}}(-S^3(K), -\Gamma_{n+2}) & \longrightarrow & \cdots
 \end{array} \tag{2.1}$$

Each horizontal row forms a *directed system* of  $\mathcal{R}$ -modules.<sup>‡</sup> (Note that we choose to work primarily with  $(-S^3(K), -\Gamma_n)$  instead of  $(S^3(K), \Gamma_n)$ , because the definition of  $\psi_{\pm, n+1}^n$  makes use of a *contact element*  $\phi_\xi \in \underline{\text{SHG}}(-M, -\gamma)$  [3, 4], defined for a contact structure  $\xi$  on  $(M, \gamma)$ .)

**Definition 2.2** [26, Definition 5.4]. The *minus knot monopole or instanton Floer homology*  $\underline{\text{KHG}}^-(-S^3, K, p)$  is the direct limit of the directed system in (2.1), which is an  $\mathcal{R}$ -module whose elements are well defined up to multiplication by a unit in  $\mathcal{R}$ . The collection of maps  $\{\psi_{\pm, n+1}^n\}_{n \in \mathbb{Z}_+}$  defines a map on the direct limit

$$U : \underline{\text{KHG}}^-(-S^3, K, p) \rightarrow \underline{\text{KHG}}^-(-S^3, K, p),$$

which gives  $\underline{\text{KHG}}^-(-S^3, K, p)$  an  $\mathcal{R}[U]$ -module structure.

In the following paragraphs, we describe the grading on  $\underline{\text{KHG}}^-(-S^3, K, p)$ . Our description shall be brief; for more details, see [26, Section 3 and 4].

Fix the balanced sutured manifold  $(-S^3(K), -\Gamma)$ , where  $\Gamma$  is either  $\Gamma_\mu$  or  $\Gamma_n$  for some  $n$ . Now realize  $S$  as a properly embedded surface  $(\tilde{S}, \partial\tilde{S}) \subset (-S^3(K), -\partial(S^3(K)))$ ; then  $\partial\tilde{S} \cap \Gamma$  must consist of exactly  $2k$  points for some  $k$ . The realization of  $S$  as  $\tilde{S}$ , in fact, involves a choice, corresponding to the value of  $k$ . By isotoping  $\tilde{S}$  near its boundary, one could create a new pair of intersection points with  $\Gamma$ ; this is called the *positive* or *negative stabilization* of  $\tilde{S}$  depending on the isotopy. We denote by  $\tilde{S}^q$  (resp.  $\tilde{S}^{-q}$ ) the result of performing  $q$  positive (resp. negative) stabilizations on  $\tilde{S}$ . (When  $q = 1$ , we also denote these by  $\tilde{S}^\pm$ .) It is proved [26, Theorem 3.4] that  $(\tilde{S}, \partial\tilde{S}) \subset (-S^3(K), -\partial S^3(K))$  induces a  $\mathbb{Z}$ -grading on  $\underline{\text{SHG}}(-S^3(K), -\Gamma)$  whenever  $\partial\tilde{S}$  intersects  $\Gamma$  at  $2k$  points, where  $k$  is odd, and a formula [26, Proposition 4.9] is given that relates the  $\mathbb{Z}$ -gradings associated to Seifert surfaces related by stabilizations: For all  $r \in \mathbb{Z}$ , we have

$$\underline{\text{SHG}}(-S^3(K), -\Gamma, \tilde{S}^{q+2r}, i) \cong \underline{\text{SHG}}(-S^3(K), -\Gamma, \tilde{S}^q, i + r), \tag{2.3}$$

where  $\underline{\text{SHG}}(-S^3(K), -\Gamma, \tilde{S}, i)$  denotes the summand in grading  $i \in \mathbb{Z}$ .

Now fix  $(-S^3(K), -\Gamma_n)$  for some  $n$ . Since the longitude  $\lambda$  is the boundary of the Seifert surface  $S$ , and  $\Gamma_n$  is of class  $\pm([\lambda] - n[\mu])$ , it follows that  $S$  has a realization  $(S_n, \partial S_n) \subset (-S^3(K), -\partial S^3(K))$  such that  $\partial S_n \cap \Gamma_n$  consists of exactly  $2n$  points. Then, for  $n$  odd (resp. even), we obtain  $\mathbb{Z}$ -gradings

<sup>†</sup> Technically, these maps are well defined only up to multiplication by a unit in  $\mathcal{R}$ , for the same reason as before. Here and in the rest of the article, we say that  $f = g$  if the maps  $f$  and  $g$  on  $\underline{\text{SHG}}$  (or consequently  $\underline{\text{KHG}}^-$ ) agree up to multiplication by a unit in  $\mathcal{R}$ . In particular, (2.1) commutes up to multiplication by a unit in  $\mathcal{R}$ .

<sup>‡</sup> As before, this is really a directed system of “ $\mathcal{R}$ -modules whose elements are well defined only up to multiplication by a unit in  $\mathcal{R}$ .” One could take the alternative viewpoint of choosing an honest  $\mathcal{R}$ -module representative for each  $\underline{\text{SHG}}(-S^3(K), -\Gamma_n)$  by specifying an embedding  $\varphi$  and a closure  $(Y, R)$ ; however, more work would be necessary to take care of the fact that  $\psi_{\pm, n+1}^n$  is only well defined (up to multiplication by a unit) for “compatible” closures of  $(-S^3(K), \Gamma_n)$  and  $(-S^3(K), \Gamma_{n+1})$ , which necessarily have auxiliary pieces of the same genus.

induced by the surfaces  $S_n$  (resp.  $S_n^-$ ); for brevity, we write  $S_n^{\tau(n)}$  for  $S_n$  when  $n$  is odd, and  $S_n^-$  when  $n$  is even; that is,  $\tau(n) = 0$  or  $-1$ . For  $(-S^3(K), -\Gamma_\mu)$ , we obtain a  $\mathbb{Z}$ -grading induced by the surface  $S_\mu$  that intersects  $\Gamma_\mu$  at exactly 2 points.

It is then proved [26, Propositions 5.5 and 5.6] that, after an appropriate grading shift<sup>†</sup>

$$\sigma(n) = \frac{n - 1 - \tau(n)}{2}, \tag{2.4}$$

the maps  $\psi_{-,n+1}^n$  in the directed system in (2.1) become grading-preserving maps, that is, they each decompose into maps

$$\psi_{-,n+1}^n : \underline{\text{SHG}}(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, i) [\sigma(n)] \rightarrow \underline{\text{SHG}}(-S^3(K), -\Gamma_{n+1}, S_{n+1}^{\tau(n+1)}, i) [\sigma(n+1)]$$

for  $i \in \mathbb{Z}$ . Thus, the Seifert surface  $S$  induces a  $\mathbb{Z}$ -grading on  $\underline{\text{KHG}}^-(-S^3, K, p)$ , known as the *Alexander grading*. The maps  $\psi_{+,n+1}^n$ , and hence the action of  $U$  on  $\underline{\text{KHG}}^-$ , is then of degree  $-1$ . For knots inside  $S^3$  or  $-S^3$ , the Alexander grading is independent of the choice of the Seifert surface  $S$ ; we shall therefore suppress  $S$  from the notation. Thus, we obtain a decomposition

$$\underline{\text{KHG}}^-(-S^3, K, p) = \bigoplus_{i \in \mathbb{Z}} \underline{\text{KHG}}^-(-S^3, K, p, i),$$

where  $\underline{\text{KHG}}^-(-S^3, K, p, i)$  denotes the summand in Alexander grading  $i \in \mathbb{Z}$ .

Inspired by the *tau* invariant in knot Heegaard Floer homology defined by Ozsváth and Szabó [30], we have the following definition.

**Definition 2.5** [26, Definition 5.7]. For a knot  $K \subset S^3$ , the *instanton or monopole tau invariant* is defined as

$$\tau_G(K) = \max \{i \in \mathbb{Z} \mid \text{there is a homogeneous, non-}U\text{-torsion element } x \in \underline{\text{KHG}}^-(-S^3, K, p, i)\}.$$

(Here, a non- $U$ -torsion element  $x$  is one such that  $U^j x \neq 0$  for all  $j \geq 0$ .)

In the sequel, we shall often compute the rank of  $\underline{\text{KHG}}^-(-S^3, K, p, i)$  in a specific Alexander grading  $i \in \mathbb{Z}$  as an  $\mathcal{R}$ -module. We claim that this completely determines the  $\mathcal{R}$ -module isomorphism type of  $\underline{\text{KHG}}^-(-S^3, K, p, i)$ : For  $\underline{\text{KHI}}^-$ , this is clear, since the module is a vector space over  $\mathbb{C}$ . For  $\underline{\text{KHM}}^-$ , our claim is a consequence of the following lemma.

**Proposition 2.6.** *For any based knot  $(K, p)$  in  $S^3$  and any  $i \in \mathbb{Z}$ , the  $\mathcal{R}$ -module  $\underline{\text{KHM}}^-(-S^3, K, p, i)$  is free and of finite rank.*

*Proof.* From [26, Proposition 5.10], we know that there exists a sufficiently large  $n \in \mathbb{Z}$ , such that

$$\underline{\text{KHM}}^-(-S^3, K, p, i) \cong \underline{\text{SHM}}(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, j)$$

for some  $j \in \mathbb{Z}$ , which shows that the  $\mathcal{R}$ -module is of finite rank. (Here,  $i$  might not be equal to  $j$  because of the grading shift in the definition of  $\underline{\text{KHM}}^-$  that we mentioned earlier.)

<sup>†</sup> The difference between this and the formula for  $\sigma(n)$  on [26, p. 1401] is due to a typographical error in the cited article.

To prove that it is free, recall that by work of Kronheimer and Mrowka [20, Lemma 4.9], for a balanced sutured manifold  $(M, \gamma)$  and a coefficient ring  $\mathcal{R}$  of characteristic 0, we have an isomorphism of  $\mathcal{R}$ -modules

$$\underline{\text{SHM}}(M, \gamma; \mathcal{R}) \cong \underline{\text{SHM}}(M, \gamma; \Gamma_\eta) \cong \underline{\text{SHM}}(M, \gamma; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{R}, \tag{2.7}$$

which respects the grading. (Here,  $\Gamma_\eta$  denotes a local system whose fiber at every point is  $\mathcal{R}$ , and is unrelated to the sutures  $\Gamma_\mu$  and  $\Gamma_n$ .) Sivek [36, Section 2.2] extends  $\underline{\text{SHM}}$  to mod-2 coefficients, which gives the isomorphism analogous to (2.7) for the Novikov ring  $\mathcal{R}$  of characteristic 2.  $\square$

To simplify notation, we shall omit the basepoint  $p$  from the notation involving  $\underline{\text{KHG}}^-$  in the sequel; however, we emphasize again that the basepoint is a necessary input for naturality results that allow  $\underline{\text{KHG}}^-$  to be well defined.

### 2.3 | $\tau_G^\sharp$

We now recall the definition of  $\tau_G^\sharp$  by Baldwin and Sivek [6]. For simplicity of notation, we first focus on  $\tau_1^\sharp$ . First, for a knot  $K \subset S^3$ , let  $N(K)$  be the smallest integer  $n \geq 0$  for which the cobordism map

$$I^\sharp(X_n, \nu_n) : I^\sharp(S^3) \rightarrow I^\sharp(S_n^3(K))$$

vanishes, where  $X_n$  is the trace of  $n$ -surgery along  $K$ , and  $\nu_n$  (unrelated to  $\nu_1^\sharp$ ) is some properly embedded surface in  $X_n$ .

**Definition 2.8** [6, Definition 3.5]. For a knot  $K \subset S^3$ , define  $\nu_1^\sharp(K) \in \mathbb{Z}$  by the equation

$$\nu_1^\sharp(K) = N(K) - N(\overline{K}).$$

It is proved [6, Theorem 3.7] that  $\nu_1^\sharp(K)$  depends only on the smooth concordance class of  $K$ , and satisfies the smooth 4-genus bound

$$|\nu_1^\sharp(K)| \leq \max(2g_4(K) - 1, 0).$$

It is then shown [6, Theorem 5.1] that  $\nu_1^\sharp$  defines a quasi-morphism from the smooth concordance group  $\mathcal{C}$  to  $\mathbb{Z}$ , that is, that  $\nu_1^\sharp$  satisfies

$$|\nu_1^\sharp(K_1 \sharp K_2) - \nu_1^\sharp(K_1) - \nu_1^\sharp(K_2)| \leq 1$$

for all knots  $K_1, K_2 \subset S^3$ , and subsequently make the following definition:

**Definition 2.9.** For a knot  $K \subset S^3$ , define  $\tau_1^\sharp(K) \in \mathbb{R}$  as the homogenization

$$\tau_1^\sharp(K) = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\nu_1^\sharp(\sharp nK)}{n}.$$

One then has [6, Proposition 5.4] that this concordance invariant defines a group homomorphism  $\tau^\sharp : C \rightarrow \mathbb{R}$  and satisfies the smooth 4-genus bound

$$|\tau_1^\sharp(K)| \leq g_4(K).$$

Finally,  $\tau_M^\sharp(K)$  can be defined completely analogously, where  $N(K)$  would instead be the smallest integer  $n \geq 0$  such that the cobordism map

$$\widetilde{\text{HM}}(X_n) : \widetilde{\text{HM}}(S^3) \rightarrow \widetilde{\text{HM}}(S_n^3(K))$$

vanishes.

### 3 | CONCORDANCE INVARIANCE OF $\tau_G$

In this section, we prove the concordance invariance of  $\tau_G$ , establishing Proposition 1.12. Throughout the section, we have a knot  $K \subset S^3$  and the sutures  $\Gamma_n$  and  $\Gamma_\mu$  on  $\partial S^3(K)$ , as described in the previous section.

Fix  $n \in \mathbb{Z}_+$ ; on  $\partial S^3(K)$ , we pick a meridional curve  $\alpha$  such that  $\alpha$  intersects the sutures  $\Gamma_n$  twice. Let  $[-1, 0] \times \partial S^3(K) \subset S^3(K)$  be a collar of  $\partial S^3(K)$  inside the knot complement  $S^3(K)$ , and endow a  $[-1, 0]$ -invariant tight contact structure on  $[-1, 0] \times \partial S^3(K)$ , so that each slice  $\{t\} \times \partial S^3(K)$  for  $t \in [-1, 0]$  is convex and the dividing set is (isotopic to)  $\Gamma_n$ . By the Legendrian Realization Principle, we can push  $\alpha$  into the interior of the collar  $[-1, 0] \times \partial S^3(K)$  and get a Legendrian curve  $\beta$ . With respect to the surface framing, the curve  $\beta$  has  $tb = -1$ . (When talking about framings of  $\beta$ , we will always refer to the surface framing with respect to  $\partial S^3(K)$ .)

Following Baldwin and Sivek [3], since  $\alpha$  intersects the sutures  $\Gamma_n$  twice, after making  $\alpha$  Legendrian, we can glue a contact 2-handle to  $(S^3(K), \Gamma_n)$  along  $\alpha$ , and get a new balanced sutured manifold  $(M, \gamma)$ . Suppose that  $(Y, R)$  is a closure of  $(S^3(K), \Gamma_n)$  in the sense of Kronheimer and Mrowka [20] such that  $g(R)$  is sufficiently large; then, by work of Baldwin and Sivek [5], we know that a closure  $(Y_0, R)$  of  $(M, \gamma)$  can be obtained from  $(Y, R)$  by performing 0-surgery along the curve  $\beta$ . Note that, inside  $Y$ ,  $\beta$  is disjoint from  $R$ , and so, the surgery can be made disjoint from  $R$ ; this means that the surface  $R$  survives in  $Y_0$ . Now let  $(M_{-1}, \Gamma_n)$  be the balanced sutured manifold obtained from  $(S^3(K), \Gamma_n)$  by performing a  $(-1)$ -surgery along  $\beta$ . Note that  $\beta$  is contained in the interior of  $S^3(K)$ , and so, the surgery does not affect the boundary or the sutures.

Clearly, if we perform  $(-1)$ -surgery along  $\beta$  on  $Y$ , we will get a closure  $(Y_{-1}, R)$  of the balanced sutured manifold  $(M_{-1}, \Gamma_n)$ . The surgery exact triangle proved by Kronheimer, Mrowka, Ozsváth, and Szabó [21, Theorem 2.4], generalized to the sutured setting, gives the exact triangle

$$\begin{array}{ccc} \underline{\text{SHG}}(-M_{-1}, -\Gamma_n) & \xrightarrow{\quad} & \underline{\text{SHG}}(-S^3(K), -\Gamma_n) \\ & \swarrow & \searrow \\ & \underline{\text{SHG}}(-M, -\gamma) & \end{array} \quad C_{h,n}$$

*Remark 3.1.* Compared to the one in [21], the surgery exact triangle here seems to go in the reverse direction; this is because the orientations on the sutured manifolds have been reversed.

We now determine that  $(M, \gamma)$  and  $(M_{-1}, \Gamma_n)$  are familiar balanced sutured manifolds. First,  $(M, \gamma)$  is obtained from  $(S^3(K), \Gamma_n)$  by attaching a contact 2-handle along a meridional curve  $\alpha$ , and so, it is nothing but  $(S^3(1), \delta)$ , where  $S^3(1)$  is obtained from  $S^3$  by removing a 3-ball, and  $\delta$  is a connected simple closed curve on the spherical boundary of  $S^3(1)$ . For  $(M_{-1}, \Gamma_n)$ , note that  $\beta$  and  $K$  are inside the 3-sphere  $S^3$ , and  $\beta$  is a meridian around  $K$ . Thus,  $(-1)$ -surgery along  $\beta$  on  $S^3(K)$  will result in the same 3-manifold  $S^3(K)$ , while the framing on its boundary will increase by 1. In other words, we have  $(M_{-1}, \Gamma_n) \cong (S^3(K), \Gamma_{n-1})$ . (Recall that the slope of  $\Gamma_n$  is  $-n$ ). Thus, the above exact triangle becomes

$$\begin{array}{ccc}
 \underline{\text{SHG}}(-S^3(K), -\Gamma_{n-1}) & \xrightarrow{\quad\quad\quad} & \underline{\text{SHG}}(-S^3(K), -\Gamma_n) \\
 & \swarrow & \searrow^{C_{h,n}} \\
 & \underline{\text{SHG}}(-S^3(1), -\delta) &
 \end{array} \tag{3.2}$$

**Lemma 3.3.** Denote by  $\overline{tb}(K)$  the maximal Thurston–Bennequin number among all Legendrian representatives  $\Lambda \subset (S^3, \xi_{\text{std}})$  of the smooth knot type  $K$ . If  $n \geq -\overline{tb}(K)$ , then the map  $C_{h,n}$  is surjective, and hence,

$$\text{rk}_{\mathcal{R}} \underline{\text{SHG}}(-S^3(K), -\Gamma_n) = \text{rk}_{\mathcal{R}} \underline{\text{SHG}}(-S^3(K), -\Gamma_{n-1}) + 1. \tag{3.4}$$

*Proof.* Since  $n \geq -\overline{tb}(K)$ , we can isotope  $K$  to a Legendrian  $\Lambda \subset (S^3, \xi_{\text{std}})$  with  $tb(\Lambda) = -n$ . We can remove a standard Legendrian neighborhood of  $\Lambda$ ; then the dividing set on the boundary of the complement is the sutures  $\Gamma_n$ . Hence, when we glue back a contact 2-handle, we get  $(S^3(1), \delta)$  with the standard tight contact structure. By work of Baldwin and Sivek [3, 4], we know that the corresponding contact element is a generator of

$$\underline{\text{SHG}}(-S^3(1), -\delta) \cong \mathcal{R}.$$

Since the contact 2-handle attaching map  $C_{h,n}$  preserves the contact element, we see that  $C_{h,n}$  is surjective. □

We now digress to prove that there is a unique  $\mathcal{R}[U]$ -summand in  $\underline{\text{KHG}}^-(S^3, K)$ .

*Proof of Proposition 1.13.* Suppose that  $S$  is a minimal-genus Seifert surface of  $K$ , and let  $g = g(S)$ . The main portion of this proof will be to show that a pattern emerges for  $\underline{\text{SHG}}(-S^3(K), -\Gamma_n)$  for sufficiently large  $n$ , with gradings taken into account. More precisely, we shall use bypass exact triangles to show that the rank of  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k})$  increases by a fixed positive integer  $r$  whenever the nonnegative integer  $k$  increases by 1, as expressed in the following: For  $k$  odd, we have

$$\begin{aligned}
 & \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}, i) \\
 \cong & \begin{cases} 0 & \text{for } i > 2g + (k - 1)/2, \\ \underline{\text{SHG}}(-S^3, -\Gamma_{2g}, S_{2g}^-, i - (k - 1)/2) & \text{for } (k + 1)/2 \leq i \leq 2g + (k - 1)/2, \\ \mathcal{R}^r & \text{for } -(k - 1)/2 \leq i \leq (k - 1)/2, \\ \underline{\text{SHG}}(-S^3, -\Gamma_{2g}, S_{2g}^-, i + (k + 1)/2) & \text{for } -2g - (k - 1)/2 \leq i \leq -(k + 1)/2, \\ 0 & \text{for } i < -2g - (k - 1)/2; \end{cases} \tag{3.5}
 \end{aligned}$$

while for  $k$  even, we have

$$\begin{aligned} & \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^-, i) \\ \cong & \begin{cases} 0 & \text{for } i > 2g + k/2, \\ \underline{\text{SHG}}(-S^3, -\Gamma_{2g}, S_{2g}^-, i - k/2) & \text{for } k/2 + 1 \leq i \leq 2g + k/2, \\ \mathcal{R}^r & \text{for } -k/2 + 1 \leq i \leq k/2, \\ \underline{\text{SHG}}(-S^3, -\Gamma_{2g}, S_{2g}^-, i + k/2) & \text{for } -2g - k/2 + 1 \leq i \leq -k/2, \\ 0 & \text{for } i < -2g - k/2 + 1. \end{cases} \end{aligned} \tag{3.6}$$

To begin, as described in [26, p. 1360], the maps  $\psi_{\pm, n+1}^n$  fit into bypass exact triangles proved by Baldwin and Sivek [8, Theorem 1.21]:

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K), -\Gamma_{n-1}) & \xrightarrow{\psi_{\pm, n}^{n-1}} & \underline{\text{SHG}}(-S^3(K), -\Gamma_n) \\ & \swarrow \psi_{\pm, n-1}^\mu \quad \searrow \psi_{\pm, \mu}^n & \\ & \underline{\text{SHG}}(-S^3(K), -\Gamma_\mu). & \end{array} \tag{3.7}$$

(Note that (3.7) is, in fact, two different bypass exact triangles, one for positive bypasses and one for negative bypasses, written together. The same is true for (3.8) and (3.9) below.) Examining the proof of [26, Proposition 5.5], one obtains the graded versions of the exact triangles above: Let  $S_n$  and  $S_\mu$ , as well as their positive and negative stabilizations, be as in Section 2.2; then, for  $n$  odd, we have

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K), -\Gamma_{n-1}, S_{n-1}^\pm, i) & \xrightarrow{\psi_{\pm, n}^{n-1}} & \underline{\text{SHG}}(-S^3(K), -\Gamma_n, S_n, i) \\ & \swarrow \psi_{\pm, n-1}^\mu \quad \searrow \psi_{\pm, \mu}^n & \\ & \underline{\text{SHG}}(-S^3(K), -\Gamma_\mu, S_\mu^{\mp n \pm 1}, i); & \end{array} \tag{3.8}$$

while for  $n$  even, we have

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K), -\Gamma_{n-1}, S_{n-1}^{\pm 2}, i) & \xrightarrow{\psi_{\pm, n}^{n-1}} & \underline{\text{SHG}}(-S^3(K), -\Gamma_n, S_n^\pm, i) \\ & \swarrow \psi_{\pm, n-1}^\mu \quad \searrow \psi_{\pm, \mu}^n & \\ & \underline{\text{SHG}}(-S^3(K), -\Gamma_\mu, S_\mu^{\mp n \pm 2}, i). & \end{array} \tag{3.9}$$

We shall in general be applying (3.8) and (3.9) with  $n = 2g + k$ , where  $k > 0$ . The key observation is that the homology group in the bottom rows of (3.8) and (3.9) is zero for many gradings  $i$ , which give us an isomorphism in the top row. Precisely, it is well known (e.g., see [20]) that for  $|i| > g$ ,

$$\underline{\text{SHG}}(-S^3(K), -\Gamma_\mu, S_\mu, i) = 0,$$

and so by the grading shift in (2.3), for  $k$  odd, we have

$$\begin{aligned} \underline{\text{SHG}}(-S^3(K), -\Gamma_\mu, S_\mu^{-2g-k+1}, i) &= 0 && \text{for } i < -g + (g + (k - 1)/2) = (k - 1)/2, \\ \underline{\text{SHG}}(-S^3(K), -\Gamma_\mu, S_\mu^{2g+k-1}, i) &= 0 && \text{for } i > g + (-g - (k - 1)/2) = -(k - 1)/2; \end{aligned}$$

thus, the positive (resp. negative) bypass exact triangle in (3.8) splits for  $i < (k - 1)/2$  (resp.  $i > -(k - 1)/2$ ), and we obtain, for  $i < (k - 1)/2$ ,

$$\begin{aligned} \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}, i) &\cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1}, S_{2g+k-1}^+, i) \\ &\cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1}, S_{2g+k-1}^-, i + 1), \end{aligned} \tag{3.10}$$

where the last isomorphism follows also from (2.3), and for  $i > -(k - 1)/2$ ,

$$\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}, i) \cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1}, S_{2g+k-1}^-, i). \tag{3.11}$$

Similarly, for  $k$  even, the positive and negative bypass exact triangles in (3.9), respectively, give, for  $i - 1 < (k - 2)/2$  (i.e., for  $i < k/2$ ),

$$\begin{aligned} \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^-, i) &\cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^+, i - 1) \\ &\cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1}, S_{2g+k-1}^2, i - 1) \\ &\cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1}, S_{2g+k-1}, i), \end{aligned} \tag{3.12}$$

and for  $i > -(k - 2)/2 = -k/2 + 1$ ,

$$\begin{aligned} \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^-, i) &\cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1}, S_{2g+k-1}^{-2}, i) \\ &\cong \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1}, S_{2g+k-1}, i - 1). \end{aligned} \tag{3.13}$$

Now for  $k$  odd, by setting  $n = 2g + k$  in [28, Theorem 2.21 (1)], we see that the  $\mathcal{R}$ -module  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}, i)$  is supported only in gradings  $-2g - (k - 1)/2 \leq i \leq 2g + (k - 1)/2$ . This, together with (3.12) and (3.13), implies that for  $k$  even,  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^-, i)$  is supported only in gradings  $-2g - k/2 + 1 \leq i \leq 2g + k/2$ . We call these the *possible gradings*.

Therefore, in essence, what (3.10) and (3.12) say is that the summands of  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k})$  in the bottom  $2g + k - 1$  possible gradings are respectively isomorphic to the summands of  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1})$  in the bottom  $2g + k - 1$  possible gradings (possibly with a grading shift), and (3.11) and (3.13) give the analogous statement for the top  $2g + k - 1$  possible gradings. (Since  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1})$  has  $4g + k - 1$  possible gradings, this means that summands in  $2(2g + k - 1) - (4g + k - 1) = k - 1$  “middle” gradings are “sampled” twice.) As  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k})$  has  $4g + k$  possible gradings, these isomorphisms completely determine  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k})$  in terms of  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k-1})$ , except in the case  $k = 1$ , wherein the middle grading  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+1}, S_{2g+1}, 0)$  is not determined. Simply letting

$$r = \text{rk}_{\mathcal{R}} \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+1}, S_{2g+1}, 0),$$

we establish (3.5) and (3.6) by inducting on  $k$ .



Now, taking  $k$  to be sufficiently large, (3.4) implies that the integer  $r$  must, in fact, be 1. Thus, for a fixed grading  $i = -g - m \leq -g$ , we have that

$$\begin{aligned} & \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^{\tau(2g+k)}, i)[\sigma] \\ & \cong \begin{cases} \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^{\tau(2g+k)}, -m + (k - 1)/2) & \text{if } k \text{ is odd,} \\ \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}^{\tau(2g+k)}, -m + k/2) & \text{if } k \text{ is even,} \end{cases} \\ & \cong \mathcal{R} \end{aligned}$$

whenever  $k \geq m + 1$ , where the last isomorphism follows from (3.5) and (3.5). By definition, this means that

$$\underline{\text{KHG}}^-(-S^3, K, i) \cong \mathcal{R}$$

for all  $i \leq -g$ . Finally, this together with [26, Corollary 5.11] implies that there is a submodule (and at most one such submodule) in  $\underline{\text{KHG}}^-(-S^3, K)$  isomorphic to  $\mathcal{R}[U]$ . Since  $\mathcal{R}$  is a field in our context, we conclude that it is, in fact, a unique  $\mathcal{R}[U]$ -summand.  $\square$

In the following, we will continue to denote by  $g$  the genus  $g(K)$  of a knot  $K$ .

Strictly speaking, we did not have to prove Proposition 1.13 for the arguments of this section. Its significance, however, is that it explains the definition of  $\tau_G$  as a natural, unique definition for knots in  $S^3$ .

Having proved that  $\underline{\text{KHG}}^-(-S^3, K)$  has a unique  $\mathcal{R}[U]$ -summand, we now return to the main setup of the section to prove the concordance invariance of  $\tau_G$ . Our next major goal is to recharacterize  $\tau_G$  in terms of the (non-)vanishing of a map on  $\underline{\text{KHG}}^-(-S^3, K)$  induced by the maps  $C_{h,n}$ . We begin with the following lemma.

**Lemma 3.14.** *The maps*

$$C_{h,n} : \underline{\text{SHG}}(-S^3(K), -\Gamma_n) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta),$$

which appear in the exact triangle (3.2), induce a surjective map

$$C_h : \underline{\text{KHG}}^-(-S^3, K) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta).$$

Furthermore,  $C_h$  commutes with the action of  $U$  on  $\underline{\text{KHG}}^-(-S^3, K)$ .

*Proof.* The lemma follows from Lemma 3.3 and the following two commutative diagrams, one for positive bypasses and one for negative bypasses:

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3, -\Gamma_n) & \xrightarrow{\psi_{\pm, n+1}^n} & \underline{\text{SHG}}(-S^3, -\Gamma_{n+1}) \\ & \searrow C_{h,n} & \swarrow C_{h, n+1} \\ & \underline{\text{SHG}}(-S^3(1), -\delta) & \end{array}$$

To prove these commutative diagrams, recall that the maps  $\psi_{\pm, n+1}^n$  are constructed via bypass attachments, which can be interpreted as contact handle attachments (see [34, Section 3] and

[3, Section 5]), and so, is  $C_{h,n}$ . The commutativity for the diagrams follows from the observation that the contact handle attaching regions for  $\psi_{\pm,n+1}^n$  and  $C_{n,h}$  are disjoint from each other.  $\square$

As a quick aside, we exhibit an immediate consequence of Lemma 3.14 as follows. Let

$$\Psi : \underline{\text{KHG}}^-(-S^3, K) \rightarrow \underline{\text{KHG}}^-(-S^3, K)$$

be the map induced by the collection of maps  $\{\psi_n\}$  under the directed system in (2.1), where

$$\psi_n : \underline{\text{SHG}}(-S^3(K), -\Gamma_{n-1}) \rightarrow \underline{\text{SHG}}(-S^3(K), -\Gamma_n)$$

is the map in the exact triangle (3.2). Then we have the following.

**Corollary 3.15.** *There is an exact triangle*

$$\begin{array}{ccc} \underline{\text{KHG}}^-(-S^3, K) & \xrightarrow{\Psi} & \underline{\text{KHG}}^-(-S^3, K) \\ & \searrow & \swarrow C_h \\ & \underline{\text{SHG}}(-S^3(1), -\delta) & \end{array}$$

*Proof.* The maps  $C_{h,n}$  in the exact triangle (3.2) commute with the maps  $\psi_{-n+1}^n$  in the directed system, and so, we can pass to the direct limit and still have an exact triangle.  $\square$

The significance of Corollary 3.15 is the following. There is an exact triangle in Heegaard Floer theory that involves the modules that are analogous to those appearing in Corollary 3.15:

$$\begin{array}{ccc} \text{HFK}^-(-S^3, K) & \xrightarrow{U-1} & \text{HFK}^-(-S^3, K) \\ & \searrow & \swarrow \\ & \widehat{\text{HF}}(-S^3) & \end{array}$$

One key difference is that, in this context, the map in the top row is defined algebraically. Thus, we are led to ask the following natural question.

**Question 3.16.** Does  $\Psi$  admit an interpretation as  $U - 1$ , where  $U$  denotes the action of  $U$ ?

We believe that establishing a positive answer to this question would have topological applications.

In any case, we are now ready to recharacterize  $\tau_G$ .

**Proposition 3.17.** *The invariant  $\tau_G(K)$  admits an alternative definition:*

$$\tau_G(K) = \max \left\{ i \in \mathbb{Z} \mid \text{the restriction of } C_h \text{ to } \underline{\text{KHG}}^-(-S^3, K, i) \text{ is nontrivial} \right\}.$$

*Proof.* We claim that an element  $[x] \in \underline{\text{KHG}}^-(-S^3, K)$  is not  $U$ -torsion if and only if  $C_h([x]) \neq 0$ .

First, note that the  $U$  map commutes with the map  $C_h$  by Lemma 3.14:  $C_h \circ U = C_h$ . It follows immediately that if  $C_h([x]) \neq 0$ , then  $[x]$  is not  $U$ -torsion.

Conversely, let  $[x] \in \underline{\text{KHG}}^-(-S^3, K)$  be a non- $U$ -torsion element; then it is represented by an element  $x \in \underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k}, S_{2g+k}, i + g + (k - 1)/2)$  such that

$$\psi_{+,2g+k+l}^{2g+k+l-1} \circ \dots \circ \psi_{+,2g+k+1}^{2g+k}(x) \neq 0$$

for all even  $l \in \mathbb{Z}_+$ . (Note that this implies a statement for odd  $l$  as well.) Taking into account the gradings as in the first rows of (3.8) and (3.9), for a given, even  $l \in \mathbb{Z}_+$ , this is an element of

$$\underline{\text{SHG}}\left(-S_3(K), -\Gamma_{2g+k+l}, S_{2g+k+l}, i + g + \frac{k - 1}{2} - l\right).$$

Now the idea is that, for large  $n$ , the map  $C_{h,n}$  is an isomorphism when restricted to the “middle” possible gradings; and we can ensure that our element lies in those “middle” gradings by taking  $l$  to be sufficiently large. Precisely, from the proof of [28, Proposition 4.26],  $C_{h,2g+k+l}$  is an isomorphism when restricted to  $\underline{\text{SHG}}(-S^3(K), -\Gamma_{2g+k+l}, S_{2g+k+l}, j)$  for

$$-\frac{k - 1}{2} - l \leq j \leq \frac{k - 1}{2} + l.$$

Since we chose  $k \geq -g - i + 1$ , we have that

$$-\frac{k - 1}{2} - l \leq i + g + \frac{k - 1}{2} - l$$

for all  $l$ ; and if we take  $l \geq 2g$ , then we will have

$$i + g + \frac{k - 1}{2} - l \leq 2g + \frac{k - 1}{2} \leq \frac{k - 1}{2} + l.$$

Then, for these choices, we see that

$$C_{h,2g+k}(x) = C_{h,2g+k+l} \circ \psi_{+,2g+k+l}^{2g+k+l-1} \circ \dots \circ \psi_{+,2g+k+1}^{2g+k}(x) \neq 0,$$

which implies that  $C_h([x]) \neq 0$ . The proposition follows immediately. □

*Remark 3.18.* By the same argument as in the proof of Proposition 2.6, we can show that in Proposition 3.17, the map  $C_h$  being nontrivial is equivalent to it being surjective.

With the alternative definition of  $\tau_G$ , we can now prove that it is a concordance invariant.

*Proof of Proposition 1.12.* Suppose that  $K_0$  and  $K_1$  are concordant; then there exists a properly embedded annulus  $A \subset [0, 1] \times S^3$  such that

$$(\{0\} \times S^3, A \cap \{0\} \times S^3) \cong (S^3, K_0), (\{1\} \times S^3, A \cap \{1\} \times S^3) \cong (S^3, K_1).$$

The idea of the proof is that  $A$  induces a grading-preserving cobordism map  $F_A : \underline{\text{KHG}}^-(-S^3, K_0) \rightarrow \underline{\text{KHG}}^-(-S^3, K_1)$  that commutes with  $C_h$ , which will imply the result for  $\tau_G$  via Proposition 3.17.

The first step is to analyze the cobordism map induced by  $A$  on  $\underline{\text{SHG}}(-S^3(K_0), \Gamma_n)$ . For each  $n$ , the pair  $([0, 1] \times S^3, A)$  induces a cobordism  $W_n$  from  $Y_{0,n}$  to  $Y_{1,n}$ , where  $Y_{i,n}$  is a closure of  $(-S^3(K_i), -\Gamma_n)$ , and  $W_n$  induces a map

$$F_{A,n} : \underline{\text{SHG}}(-S^3(K_0), -\Gamma_n) \rightarrow \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n)$$

as follows. There are two ways to describe  $W_n$ , which are both useful; below, we briefly recall both of these descriptions from [25].

First, take a parametrization of  $A \cong [0, 1] \times S^1$ . Then, a tubular neighborhood of  $A \subset [0, 1] \times S^3$  can be identified with  $A \times D^2 \cong [0, 1] \times S^1 \times D^2$ , with

$$(A \times D^2) \cap (\{0, 1\} \times S^3) \cong \{0, 1\} \times S^1 \times D^2.$$

Thus, we know that

$$\partial([0, 1] \times S^3 \setminus (A \times D^2)) \cong -S^3(K_0) \cup ([0, 1] \times S^1 \times \partial D^2) \cup S^3(K_1). \tag{3.19}$$

Choosing a closure  $Y_{0,n}$  of  $(-S^3(K_0), -\Gamma_n)$ , we can write

$$W_n \cong -([0, 1] \times S^3 \setminus (A \times D^2)) \cup ([0, 1] \times (Y_{0,n} \setminus S^3(K_0))),$$

via a natural identification

$$[0, 1] \times S^1 \times \partial D^2 \cong [0, 1] \times \partial S^3(K_0).$$

A second description of  $W_n$  is as follows. As

$$\partial S^3(K_0) \cong \partial S^3(K_1) \cong S^1 \times D^2,$$

from (3.19),  $([0, 1] \times S^3) \setminus (A \times D^2)$  can be obtained from  $([0, 1] \times S^3(K_1))$  by attaching a set of four-dimensional handles  $\mathcal{H}$  to the interior of  $\{1\} \times S^3(K_0)$ , as in [25, Lemma 3.9]. Thus, as above, choosing a closure  $Y_{0,n}$  of  $(-S^3(K_0), -\Gamma_n)$ , we can attach the same set of handles  $\mathcal{H}$  to  $\{1\} \times Y_{0,n} \subset [0, 1] \times Y_{0,n}$ , and the result is again  $W_n$ .

We break down the rest of the proof into four claims, as detailed below.

*Claim 1.* The maps  $F_{A,n}$  give rise to a map<sup>†</sup>

$$F_A : \underline{\text{KHG}}^-(-S^3, K_0) \rightarrow \underline{\text{KHG}}^-(-S^3, K_1).$$

To prove the claim, it suffices to show that we have a commutative diagram

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K_0), -\Gamma_n) & \xrightarrow{F_{A,n}} & \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n) \\ \downarrow \psi_{-,n+1}^n & & \downarrow \psi_{-,n+1}^n \\ \underline{\text{SHG}}(-S^3(K_0), -\Gamma_{n+1}) & \xrightarrow{F_{A,n+1}} & \underline{\text{SHG}}(-S^3(K_1), -\Gamma_{n+2}). \end{array}$$

<sup>†</sup> The basepoints  $p_i$  for  $K_i$  are specified by  $p_i = \{i\} \times p$  in the parametrization  $A \cong [0, 1] \times S^1$ .

The commutativity of this diagram follows from the fact that the attaching regions for the handles associated to  $F_{A,n}$  and to  $\psi_{-n+1}^n$  are disjoint: When constructing  $F_{A,n}$ , we attached handles to  $[0, 1] \times Y_{0,n}$  along the region  $\{1\} \times \text{Int } S^3(K_0)$ , while when constructing the map  $\psi_{-n+1}^n$ , we attached handles to  $[0, 1] \times Y_{i,n}$  along the region  $\{1\} \times [0, 1] \times \partial S^3(K_i)$ ; see [25, Section 3]).

*Claim 2.*  $F_A$  commutes with the  $U$  map on  $\underline{\text{KHG}}^-$ . The proof of this claim is completely analogous to one for Claim 1, with  $\psi_+$  instead of  $\psi_-$ .

The two claims above show that  $F_A$  is a homomorphism of  $\mathcal{R}[U]$ -modules.

*Claim 3.* There is a commutative diagram

$$\begin{array}{ccc} \underline{\text{KHG}}^-(-S^3, K_0) & \xrightarrow{F_A} & \underline{\text{KHG}}^-(-S^3, K_1) \\ & \searrow C_h & \swarrow C_h \\ & \underline{\text{SHG}}(-S^3(1), -\delta) & \end{array}$$

where  $C_h$  is defined as in Lemma 3.14.

To prove the claim, it suffices to prove that the following diagram commutes for all  $n$ :

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K_0), -\Gamma_n) & \xrightarrow{F_{A,n}} & \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n) \\ \downarrow C_{h,n} & & \downarrow C_{h,n} \\ \underline{\text{SHG}}(-S^3(1), -\delta) & \xrightarrow{\text{Id}} & \underline{\text{SHG}}(-S^3(1), -\delta). \end{array} \tag{3.20}$$

As above, suppose that we have a closure  $Y_{0,n}$  for  $(-S^3(K_0), -\Gamma_n)$ . Let  $Y_{1,n}$  be the corresponding closure for  $(-S^3(K_1), -\Gamma_n)$  as in the construction of  $W_n$  above. Recall from the construction of  $C_{h,n}$  that it is the map associated to a 2-handle attached along a meridian curve  $\alpha \subset \partial S^3(K_0)$ ; we can push  $\alpha$  slightly into the interior and get a curve  $\beta$ . Then we get a closure  $Y'_0$  for  $(-S^3(1), -\delta)$  by performing 0-surgery on  $Y_{0,n}$  along  $\beta$ . Note that the difference between  $S^3(K_0)$  and  $S^3(K_1)$  is contained in the interior, and so, we also have the curve  $\beta \subset S^3(K_1) \subset Y_{1,n}$ . Thus, we can obtain another closure  $Y'_1$  for  $(-S^3(1), -\delta)$ . We can form a cobordism  $W'_n$  from  $Y'_0$  to  $Y'_1$  by attaching the set of four-dimensional handles  $\mathcal{H}$  as in the proof of Claim 1 to  $Y'_0 \times \{1\} \subset Y'_0 \times [0, 1]$ , and the attaching region is contained in  $\text{Int}(S^3(K_0)) \subset Y'_0$ . Hence, there is a commutative diagram just as in the proof of Claim 1:

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K_0), -\Gamma_n) & \xrightarrow{F_{A,n}} & \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n) \\ \downarrow C_{h,n} & & \downarrow C_{h,n} \\ \underline{\text{SHG}}(-S^3(1), -\delta) & \xrightarrow{F'_A} & \underline{\text{SHG}}(-S^3(1), -\delta), \end{array}$$

where  $F'_A$  is the map induced by the cobordism  $W'_n$ .

So, to prove (3.20), it suffices to show that  $W'_n$  is actually a product  $[0, 1] \times Y'_0$ , which will imply that  $F'_A = \text{Id}$ . To do this, recall that  $W'_n$  is obtained from  $[0, 1] \times Y'_0$  by attaching a set of handles  $\mathcal{H}$ , while the attachment regions are contained in  $\text{Int } S^3(K_0) \subset \text{Int } S^3(1) \subset \{1\} \times Y'_0$ . This means

that we can split  $W'_n$  into two parts

$$W'_n \cong W''_n \cup ([0, 1] \times (Y'_0 \setminus S^3(1))),$$

where  $W''_n$  is obtained from  $[0, 1] \times S^3(1)$  by attaching the set of handles  $\mathcal{H}$ . Recall that  $(S^3(1), \delta)$  is obtained from  $(S^3(K_0), \Gamma_n)$  by attaching the contact 2-handle  $h$ , and so, topologically,

$$S^3(1) \cong S^3(K_0) \cup B^3.$$

Note the 3-ball  $B^3$  is attached to  $S^3(K_0)$  along part of the boundary, and the set of handles  $\mathcal{H}$  is attached to  $[0, 1] \times S^3(1)$  within the region  $\text{Int}(S^3(K_0)) \subset \{1\} \times S^3(1)$ , and so, the two attaching regions are disjoint. Thus, we have

$$\begin{aligned} W''_n &\cong [0, 1] \times S^3(1) \cup \mathcal{H} \\ &\cong ([0, 1] \times (S^3(K_0) \cup B^3)) \cup \mathcal{H} \\ &\cong ([0, 1] \times S^3(K_0)) \cup \mathcal{H} \cup ([0, 1] \times B^3) \\ &\cong (([0, 1] \times S^3) \setminus (A \times D^2)) \cup ([0, 1] \times B^3). \end{aligned}$$

Here,  $[0, 1] \times B^3$  is glued to  $([0, 1] \times S^3) \setminus (A \times D^2)$  along a thickened annulus. From here, it is straightforward to check that the resulting manifold  $W''_n$  is diffeomorphic to  $[0, 1] \times S^3(1)$ .

*Claim 4.* The map

$$F_A : \underline{\text{KHG}}^-(-S^3, K_0) \rightarrow \underline{\text{KHG}}^-(-S^3, K_1)$$

preserves the grading.

By definition, we know that for any fixed  $j \in \mathbb{Z}$ , we can pick a large enough odd  $n$  so that, for  $i = 0, 1$ ,

$$\underline{\text{KHG}}^-(-S^3, K_i, j) \cong \underline{\text{SHG}}\left(-S^3(K_i), -\Gamma_{i,n}, S_{i,n}, j + \frac{n-1}{2}\right).$$

(Here,  $\Gamma_{i,n}$  is a set of sutures on  $-S^3(K_i)$  of slope  $-n$ , and  $S_{i,n}$  is a minimal-genus Seifert surface of  $K_i$  that intersects  $\Gamma_{i,n}$  at exactly  $2n$  points.) Hence, to show that  $F_A$  preserves the grading, we need only to show that  $F_{A,n}$  preserves the grading. Note that we can identify the boundaries:

$$\partial S^3(K_0) \cong \partial S^3(K_1)$$

via the parametrization  $A \cong [0, 1] \times S^1$ , and we can assume that under the above identification,

$$S_{0,n} \cap \partial S^3(K_0) \cong S_{1,n} \cap \partial S^3(K_1).$$

Now let  $Y_{0,n}$  be a closure of  $(-S^3(K_0), -\Gamma_n)$ , and let  $\bar{S}_{0,n}$  be the closure of  $S_{0,n}$  in  $Y_{0,n}$ , as in the construction of gradings; see [26, Section 3]. Then we have a corresponding closure  $Y_{1,n}$  for

$(-S^3(K_1), -\Gamma_n)$ , inside which there is the closure  $\bar{S}_{1,n}$  of  $S_{1,n}$ . To describe this surface, recall that

$$Y_{1,n} \cong -S^3(K_1) \cup_{\partial S^3(K_0) \cong \partial S^3(K_1)} (Y_{0,n} \setminus S^3(K_0))$$

as in the construction of  $W_n$  at the beginning of the proof; then concretely,  $\bar{S}_{1,n}$  is defined to be

$$\bar{S}_{1,n} = S_{1,n} \cup (\bar{S}_{0,n} \setminus S^3(K_0)).$$

Using the Mayer–Vietoris sequence, we see that

$$H_2([0, 1] \times S^3 \setminus (A \times D^2)) = 0.$$

Therefore, the closed surface  $-S_{0,n} \cup A \cup S_{1,n} \subset ([0, 1] \times S^3) \setminus (A \times D^2)$  bounds a 3-chain  $c \subset ([0, 1] \times S^3) \setminus (A \times D^2)$ . Now inside  $W_n$ , let

$$d = c \cup \left( [0, 1] \times \left( \bar{S}_{0,n} \setminus S^3(K_0) \right) \right),$$

where the two pieces are glued along

$$A \cong [0, 1] \times S^1 \cong [0, 1] \times \partial(\bar{S}_{0,n} \setminus S^3(K_0)).$$

It is straightforward to check that

$$\partial d \cong -\bar{S}_{0,n} \cup \bar{S}_{1,n}.$$

Hence, we conclude that

$$[\bar{S}_{0,n}] = [\bar{S}_{1,n}] \in H_2(W_n),$$

whence it follows that  $F_{A,n}$  preserves the grading.

The four claims above together prove the existence of a grading-preserving homomorphism  $F_A : \underline{\text{KHG}}^-(-S^3, K_0) \rightarrow \underline{\text{KHG}}^-(-S^3, K_1)$  of  $\mathcal{R}[U]$ -modules that commutes with the map  $C_h$ . By Proposition 3.17,  $\tau_G$  is the maximum grading for which  $C_h$  is nontrivial, and thus, our proof is complete.  $\square$

Having achieved our main goal of the section, we end it with an application to *ribbon concordance*, which is a knot concordance that admits a handle decomposition with only 0-, 1-, but not 2-handles. In recent work of Daemi, Lidman, Vela-Vick, and the third author [12], it is proved that the map on  $\underline{\text{KHI}}$  associated to a ribbon concordance is injective. We may quickly extend this result to  $\underline{\text{KHI}}^-$ .

**Corollary 3.21.** *Suppose that  $A$  is a ribbon concordance from  $K_1$  to  $K_2$  in  $[0, 1] \times S^3$ ; then the map*

$$F_A : \underline{\text{KHG}}^-(-S^3, K_0) \rightarrow \underline{\text{KHG}}^-(-S^3, K_1)$$

*defined in the proof of Proposition 1.12 is injective.*

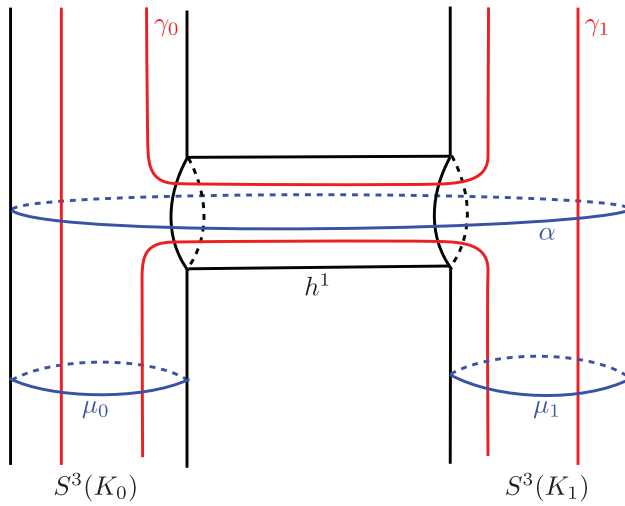


FIGURE 2 The contact handles:  $h_0^2$  is attached along  $\mu_0$ ,  $h_1^2$  is attached along  $\mu_1$ , and  $h_2^2$  is attached along  $\alpha$ .

*Proof.* By [12, Theorem 4.4], the map

$$F_{A,n} : \underline{\text{SHG}}(-S^3(K_0), -\Gamma_n) \rightarrow \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n)$$

is injective for all  $n \in \mathbb{Z}$ . Passing to the direct limit, we see that  $F_A$  is also injective. □

These results may be compared to that of Zemke [38], who first proves the analogous statement for both  $\widehat{\text{HFK}}$  and  $\text{HFK}^-$ .

#### 4 | ADDITIVITY OF $\tau$ UNDER CONNECTED SUM

In this subsection, we prove the additivity of the  $\tau_G$  under connected sum, establishing Proposition 1.14. To begin, we establish the superadditivity of  $\tau_G$ .

**Proposition 4.1.** *Suppose that  $K_0$  and  $K_1$  are two knots in  $S^3$ ; then*

$$\tau_G(K_0 \# K_1) \geq \tau_G(K_0) + \tau_G(K_1).$$

*Proof.* Suppose that  $K_0$  and  $K_1$  are two knots in  $S^3$ , and suppose that  $m$  and  $n$  are two sufficiently large, odd integers. Suppose further that  $S_0$  and  $S_1$  are minimal-genus Seifert surfaces of  $K_0$  and  $K_1$ , respectively. We can attach a 1-handle  $h^1$  to connect the two balanced sutured manifolds  $(S^3(K_0), \Gamma_m)$  and  $(S^3(K_1), \Gamma_n)$ . Let  $(M_0, \gamma_0)$  be the resulting balanced sutured manifold; then we have

$$C_{h^1} : \underline{\text{SHG}}(-S^3(K_0), -\Gamma_m) \otimes \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n) \xrightarrow{\cong} \underline{\text{SHG}}(-M_0, -\gamma_0). \tag{4.2}$$

On  $(M_0, \gamma_0)$ , we can attach a contact 2-handle  $h_2^2$  along the curve  $\alpha$ , as depicted in Figure 2, and the resulting balanced sutured manifold is  $(S^3(K_0 \# K_1), \Gamma_{m+n})$ . (This  $\alpha$  is not the same as the one



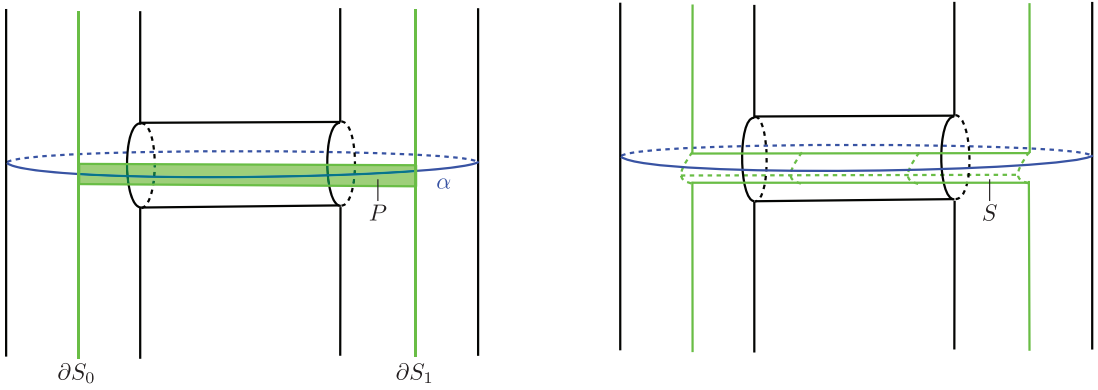


FIGURE 3 Left: The strip  $P$  as the shaded region. Right: Pushing off the interior of  $P$  into the interior of  $M_0$  in the construction of  $S$ .

in Figure 1.) Thus, there is a map

$$C_{h_2^2} : \underline{\text{SHG}}(-M_0, -\gamma_0) \rightarrow \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n}).$$

Inside  $(M_0, \gamma_0)$ , there is a surface  $S_0 \sqcup S_1$ , whose associated grading is the one we are interested in. However, the surface  $S_0 \sqcup S_1$  intersects the curve  $\alpha$ , along which we attach the 2-handle  $h_2^2$ , and so, it does not survive in  $(S^3(K_0 \# K_1), \Gamma_{m+n})$  as a properly embedded surface. To circumvent this problem, we add to it a strip  $P$ , as described in the next paragraph.

See Figure 3. Pick a strip  $P \subset \partial M_0$ , which serves as a two-dimensional 1-handle attached to the surfaces  $S_0$  and  $S_1$ . Let  $S$  be the union  $S_0 \cup S_1 \cup P$ , with the interior of  $P$  being pushed off into the interior of  $M_0$ ; then  $S$  is a properly embedded surface inside  $(M_0, \gamma_0)$  and is disjoint from  $\alpha$ . Thus, after attaching the contact 2-handle along  $\alpha$ ,  $S$  survives in  $S^3(K_0 \# K_1, \Gamma_{m+n})$ , and it is obvious that  $S$  is a Seifert surface of  $K_0 \# K_1$ . Since  $\alpha \cap S = \emptyset$ , the map  $C_{h_2^2}$  preserves the gradings induced by  $S$  and its stabilizations. To compare the gradings induced by  $S_0 \sqcup S_1$  and  $S$ , note that their difference, the two-dimensional 1-handle  $P$ , is chosen to be on  $\partial M_1$ . Hence, we know that

$$[S_0, \partial S_0] + [S_1, \partial S_1] = [S, \partial S] \in H_2(M_0, \partial M_0).$$

In [14, Section 4], the first and second authors prove that the gradings induced by  $S_0 \sqcup S_1$  and  $S$  differ by an overall grading shift. To pin down the exact grading shift, observe that the decomposition of  $(M, \gamma)$  along  $S_0 \sqcup S_1$  and  $S$  are both taut; this fact allows us to identify the maximal nonvanishing gradings. Thus, combining with the fact that  $C_{h_2^2}$  preserves the grading, we have the following lemma.

**Lemma 4.3.** *Suppose that  $m$  and  $n$  are sufficiently large, odd integers. Then, for all  $i, j \in \mathbb{Z}$ , the map  $C_{h_2^2} \circ C_{h_1}$  shifts the grading as follows:*

$$\begin{aligned} C_{h_2^2} \circ C_{h_1} : \underline{\text{SHG}}(-S^3(K_0), -\Gamma_m, S_0, i) \otimes \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n, S_1, j) \\ \rightarrow \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n}, S^-, i + j + 1). \end{aligned}$$

□

Let  $\mu_0 \subset \partial S^3(K_0)$  and  $\mu_1 \subset \partial S^3(K_1)$  be meridians of  $K_0$  and  $K_1$ , respectively. See Figure 2. We can attach contact 2-handles  $h_0^2$  and  $h_1^2$  along  $\mu_0$  and  $\mu_1$ , respectively; the resulting balanced sutured manifolds are both  $(S^3(1), \delta)$ . Thus, we have maps

$$C_{h_0^2} : \underline{\text{SHG}}(-S^3(K_0), -\Gamma_m) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta),$$

$$C_{h_1^2} : \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta).$$

The curves  $\mu_0$  and  $\mu_1$  are disjoint from the contact handles  $h^1$  and  $h_2^2$ , and so, they survive in  $(S^3(K_0 \# K_1), \Gamma_{m+n})$ . Both  $\mu_0$  and  $\mu_1$  become meridians of  $K_0 \# K_1$ , and so, the contact 2-handle attaching maps associated to them (viewed as attachment maps from  $(-S^3(K_0 \# K_1), -\Gamma_{m+n})$ ) are the same:

$$C^\# = C_{h_0^2} = C_{h_1^2} : \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n}) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta).$$

The commutativity of contact handle attachments then gives us the following commutative diagram:

$$\begin{array}{ccccc}
 (K_0, \Gamma_m) \otimes (S^3(1), \delta) & \xleftarrow{\text{Id} \otimes C_{h_1^2}} & (K_0, \Gamma_m) \otimes (K_1, \Gamma_n) & \xrightarrow{C_{h_0^2} \otimes \text{Id}} & (K_1, \Gamma_n) \otimes (S^3(1), \delta) \\
 \downarrow \cong C_{h^1} & & \downarrow \cong C_{h^1} & & \downarrow \cong C_{h^1} \\
 & & (M_0, \gamma_0) & & \\
 & \swarrow C_{h_1^2} & \downarrow C_{h_2^2} & \searrow C_{h_0^2} & \\
 (K_0, \Gamma_m) & & (K_0 \# K_1, \Gamma_{m+n}) & & (K_1, \Gamma_n) \\
 & \searrow C_{h_0^2} & \downarrow C^\# & \swarrow C_{h_1^2} & \\
 & & (S^3(1), \delta) & & 
 \end{array} \tag{4.4}$$

Here and below, for the sake of space, we often denote  $\underline{\text{SHG}}(-S^3(1), -\delta)$  by  $(S^3(1), \delta)$ , denote  $\underline{\text{SHG}}(-S^3(K), -\Gamma)$  by  $(K, \Gamma)$ , and denote  $\underline{\text{SHG}}(-M_0, -\gamma_0)$  by  $(M_0, \gamma_0)$ :

Since  $m$  and  $n$  are chosen to be odd and sufficiently large, by [26, Proposition 5.10], elements in  $\underline{\text{KHG}}^-(-S^3, K_0)$  and  $\underline{\text{KHG}}^-(-S^3, K_1)$  of sufficiently large gradings can be found in  $\underline{\text{SHG}}(-S^3(K_0), -\Gamma_m)$  and  $\underline{\text{SHG}}(-S^3(K_1), -\Gamma_n)$  respectively, as in the previous section. In particular, let  $x_0 \in \underline{\text{SHG}}(-S^3(K_0), -\Gamma_m)$  be an element representing a non- $U$ -torsion element in  $\underline{\text{KHG}}^-(-S^3, K_0)$  of maximal grading; then, by Proposition 3.17,

$$\text{gr}_{S_0}(x_0) = \tau_G(K_0) + \frac{m-1}{2}, \quad C_{h_0^2}(x_0) \neq 0,$$

where  $\text{gr}_{S_0}$  means the grading with respect to  $S_0$ , and the term  $(m-1)/2$  represents the grading shift in the definition of  $\underline{\text{KHG}}^-$ . Similarly, we can pick  $y_0 \in \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n)$  to represent a non- $U$ -torsion element in  $\underline{\text{KHG}}^-(-S^3, K_1)$  of maximal grading; then

$$\text{gr}_{S_1}(y_0) = \tau_G(K_1) + \frac{n-1}{2}, \quad C_{h_1^2}(y_0) \neq 0.$$

Let

$$z_0 = C_{h_2} \circ C_{h_1}(x_0 \otimes y_0) \in \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n});$$

then we know from Lemma 4.3 that

$$\text{gr}_{S^-}(z_0) = \tau_G(K_0) + \tau_G(K_1) + \frac{m+n}{2}.$$

From the commutative diagram (4.4), we know that

$$C^\#(z_0) = C^\# \circ C_{h_2} \circ C_{h_1}(x_0 \otimes y_0) = C_{h_0} \circ C_{h_1} \circ (\text{Id} \otimes C_{h_2})(x_0 \otimes y_0) = C_{h_0}(x_0) \neq 0, \tag{4.5}$$

where the third equality uses the fact that  $C_{h_1^2}(y_0) \neq 0$ . Hence, by Proposition 3.17, we have

$$\tau_G(K_0 \# K_1) + \frac{m+n}{2} \geq \text{gr}_{S^-}(z_0) = \tau_G(K_0) + \tau_G(K_1) + \frac{m+n}{2},$$

from which the proposition follows.

We now upgrade the inequality in Proposition 4.1 to an equality.

*Proof of Proposition 1.14.* We keep all notation from the proof of Proposition 4.1. In particular, we have an element

$$z_0 = C_{h_2} \circ C_{h_1}(x_0 \otimes y_0) \in \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n}),$$

where  $x_0 \in \underline{\text{SHG}}(-S^3(K_0), -\Gamma_m)$  and  $y_0 \in \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n)$  represent non- $U$ -torsion elements in  $\underline{\text{KHG}}^-(-S^3, K_0)$  and  $\underline{\text{KHG}}^-(-S^3, K_1)$  of maximal gradings, respectively.

By (4.5), we see that  $z_0$ , in fact, corresponds to a non- $U$ -torsion element

$$z_0^- \in \underline{\text{KHG}}^-(-S^3, K_0 \# K_1).$$

If we assume the contrary of the proposition, that is,

$$\tau_G(K_0 \# K_1) > \tau_G(K_0) + \tau_G(K_1),$$

then we are assuming that  $z_0^-$  is not the starting point of the unique infinite  $U$ -tower; in other words, it has a preimage under  $U$ . Translating back to  $\underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n})$ , this means that there is an element

$$z_1 \in \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n-1})$$

such that

$$\psi_{+,m+n}^{m+n-1}(z_1) = z_0.$$

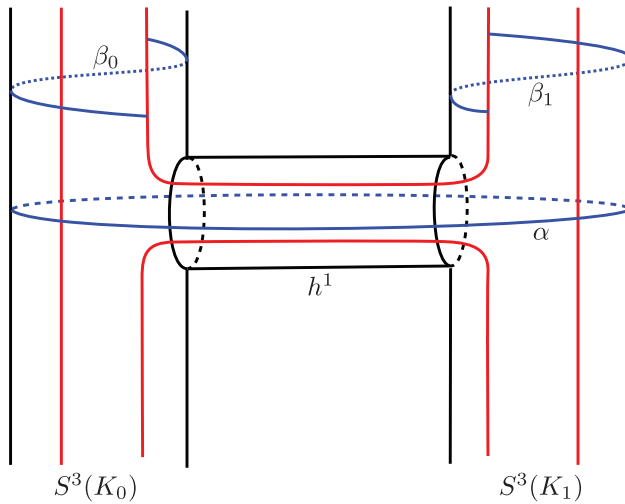


FIGURE 4 The arcs  $\beta_0$  and  $\beta_1$ , along which bypasses are attached, viewed in  $(S^3(K_0 \# K_1), \Gamma_{m+n})$ .

By the positive bypass exact triangle in (3.7), we see that

$$\psi_{+, \infty}^{m+n}(z_0) = 0.$$

We claim that this will lead to a contradiction.

Indeed, consider the maps

$$\psi_{+, \mu}^m : \underline{\text{SHG}}(-S^3(K_0), -\Gamma_m) \rightarrow \underline{\text{SHG}}(-S^3(K_0), -\Gamma_\mu),$$

$$\psi_{+, \mu}^n : \underline{\text{SHG}}(-S^3(K_1), -\Gamma_n) \rightarrow \underline{\text{SHG}}(-S^3(K_1), -\Gamma_\mu),$$

which each fit into the positive bypass exact triangle in (3.7). Let  $\beta_0 \in \partial S^3(K_0)$  and  $\beta_1 \in \partial S^3(K_1)$  be the arcs along which bypasses corresponding to these maps are attached; we may view  $\beta_0$  and  $\beta_1$  in  $(S^3(K_0 \# K_1), \Gamma_{m+n})$ , as in Figure 4. (Since  $\beta_0$  and  $\beta_1$  are both disjoint from the 1-handle  $h^1$  and the 2-handle  $h_2^2$ , they survive in  $(S^3(K_0 \# K_1), \Gamma_{m+n})$ .)

Inside  $(S^3(K_0 \# K_1), \Gamma_{m+n})$ , the arcs  $\beta_1$  and  $\beta_2$  are isotopic; thus, they both correspond to the bypass map

$$\psi_{+, \mu}^{m+n} : \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_{m+n}) \rightarrow \underline{\text{SHG}}(-S^3(K_0 \# K_1), -\Gamma_\mu).$$

For concreteness, suppose that this bypass map is constructed via a bypass attached along  $\beta_2$ . Since  $\beta_2$  is disjoint from  $h^1$  and  $h_2^2$ , there is a commutative diagram as follows:

$$\begin{CD} (K_0, \Gamma_m) \otimes (K_1, \Gamma_n) @>{\text{Id} \otimes \psi_{+, \mu}^n}>> (K_0, \Gamma_m) \otimes (K_1, \Gamma_\mu) \\ @V{C_{h_2^2} \circ C_{h^1}}VV @VV{C_{h_2^2} \circ C_{h^1}}V \\ (K_0 \# K_1, \Gamma_{m+n}) @>{\psi_{+, \mu}^{m+n}}>> (K_0 \# K_1, \Gamma_\mu) \end{CD}$$

(Here, we are using the simplified notation as in (4.4).)

From the commutativity, we know that

$$\begin{aligned} \psi_{+,\mu}^{m+n}(z_0) &= \psi_{+,\mu}^{m+n} \circ C_{h_2^2} \circ C_{h^1}(x_0 \otimes y_0) \\ &= C_{h_2^2} \circ C_{h^1} \circ (\text{Id} \otimes \psi_{+,\mu}^n)(x_0 \otimes y_0) \\ &= C_{h_2^2} \circ C_{h^1}(x_0 \otimes y_\mu), \end{aligned}$$

where  $y_\mu = \psi_{+,\mu}^n(y_0)$ . Since  $y_0$  corresponds to a non- $U$ -torsion element in  $\text{KHG}^-(-S^3, K_1)$  of maximal grading, we know that  $y_0 \notin \text{Im} \psi_{+,\mu}^{n-1}$ , and so, by the exactness of (3.7), we know that  $y_\mu \neq 0$ .

Now we claim that the following diagram commutes:

$$\begin{array}{ccc} (K_0, \Gamma_m) \otimes (K_1, \Gamma_\mu) & \xrightarrow{\psi_{+,\mu}^m \otimes \text{Id}} & (K_0, \Gamma_\mu) \otimes (K_1, \Gamma_\mu) \\ \downarrow C_{h_2^2} \circ C_{h^1} & & \downarrow \cong \\ (K_0 \# K_1, \Gamma_\mu) & \xrightarrow{=} & (K_0 \# K_1, \Gamma_\mu). \end{array} \tag{4.6}$$

(The isomorphism in the right column arises from the fact that the two sutured manifolds have the same closure; the same is true for  $C_{h^1}$  on the left column, but we display it explicitly so that  $C_{h_2^2}$  makes sense.) Since  $x_\mu = \psi_{+,\mu}^m(x_0) \neq 0$  (as  $x_0$  represents a non- $U$ -torsion element in  $\text{KHG}^-(-S^3, K_0)$  of maximal grading), this will show that

$$\psi_{+,\mu}^{m+n}(z_0) = C_{h_2^2} \circ C_{h^1}(x_0 \otimes y_\mu) = \psi_{+,\mu}^m(x_0) \otimes y_\mu = x_\mu \otimes y_\mu \neq 0,$$

giving us the desired contradiction.

The rest of the proof is devoted to proving the commutativity of (4.6). Let  $(M_1, \gamma_1)$  be the result of attaching the handle  $h^1$  to  $(S^3(K_0), \Gamma_m) \sqcup (S^3(K_1), \Gamma_\mu)$ . (This gives us a map

$$C_{h^1} : \text{SHG}(-S^3(K_0), -\Gamma_m) \otimes \text{SHG}(-S^3(K_1), -\Gamma_\mu) \rightarrow \text{SHG}(-M_1, -\gamma_1),$$

similar to (4.2), but with sutures  $-\Gamma_\mu$  instead of  $-\Gamma_n$  on  $-S^3(K_1)$ .) Our strategy is to analyze the contact 2-handle attachment along  $\alpha$ , corresponding to  $C_{h_2^2}$  and viewed in  $(-M_1, -\gamma_1)$ , and compare it to the bypass attachment along  $\beta_0$ , corresponding to  $\psi_{+,\mu}^m$ . A bypass attachment is, in fact, the composition of a contact 1-handle and a contact 2-handle (see, for example, [3, Section 5]); in our context, we shall work with the preclosures of the sutured manifolds (see [3, Section 4.2] for details of the relevant constructions), where the contact 1-handle associated to  $\psi_{+,\mu}^m$  will be identified with a part of the auxiliary surface associated to  $(M_1, \gamma_1)$ , and the attaching curve of the contact 2-handle associated to  $\psi_{+,\mu}^m$  will be identified with an isotopic copy of  $\alpha$ .

See Figure 5. Because we have the sutures  $\Gamma_\mu$  on  $\partial S^3(K_1)$ , we see that after the 1-handle  $h^1$  is added, one component of  $\gamma_1$  is simply a meridian on the  $\partial S^3(K_1)$  part of the boundary (of  $S^3(K_0) \# K_1$ ), while the other component, which intersects the  $\partial S^3(K_0)$  part of the boundary, also wraps around the  $\partial S^3(K_1)$  part of the boundary like a meridian. We may thus view a part of this latter component, an arc  $\hat{\gamma}_1$ , as isotopic to a part of  $\alpha$ , which we call  $\hat{\alpha}$ , relative to their endpoints. More precisely, while the arcs  $\hat{\gamma}_1$  and  $\hat{\alpha}$  do not have the same endpoints; however, from Figure 5, one can pair up the endpoints obviously by short arcs  $\zeta$ .

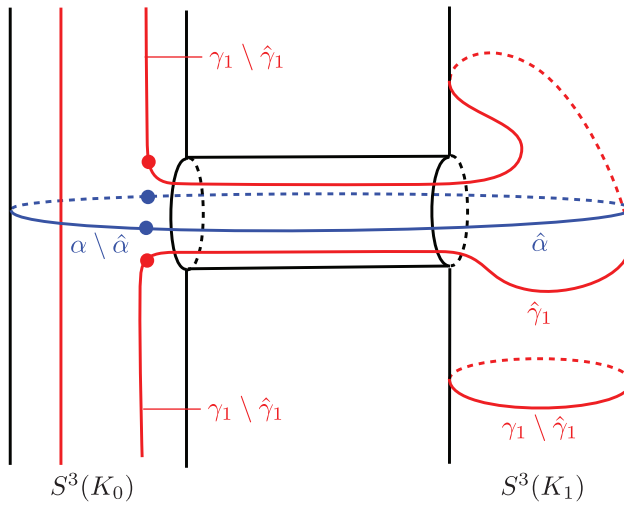


FIGURE 5 The arcs  $\hat{\gamma}_1$  and  $\hat{\alpha}$ , which we think of as isotopic relative to their endpoints. Their endpoints are denoted by the red and blue dots, respectively. The short arcs  $\zeta$  are omitted.

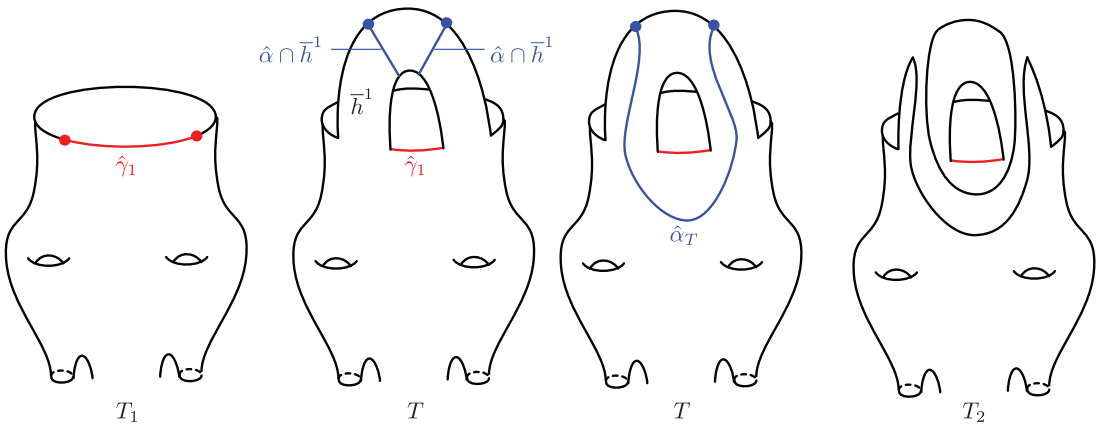


FIGURE 6 Constructing an auxiliary surface  $T_2$  for  $(M_2, \gamma_2)$ , from an auxiliary surface  $T_1$  for  $(M_1, \gamma_1)$ . In the second diagram, only a part of  $\hat{\alpha}$  is on  $T$ ; we isotope all of  $\hat{\alpha}$  onto  $T$  in the third diagram. As shown, the auxiliary surfaces may have nonzero genus; their irrelevant boundary components are omitted.

Suppose that  $T_1$  is a connected auxiliary surface of  $(M_1, \gamma_1)$ ; then we can form the preclosure

$$\tilde{M} = M_1 \cup [-1, 1] \times T_1.$$

From [3, Section 4.2.2], there is an auxiliary surface  $T$  for  $(S^3(K_0), \Gamma_m) \sqcup (S^3(K_1), \Gamma_\mu)$ , obtained from  $T_1$  by attaching a two-dimensional 1-handle  $\bar{h}^{-1}$ , which corresponds to the three-dimensional 1-handle  $h^1$ , as in Figure 6, so that we also have

$$\tilde{M} = (S^3(K_0) \sqcup S^3(K_1)) \cup ([-1, 1] \times T).$$

In this description, we can think of  $h^1$  as a thickening of  $\bar{h}^{-1} \subset T$ .

From [3, Section 4.2.3], attaching the contact 2-handle  $h_2^2$  along  $\alpha$  corresponds to performing a 0-surgery along a push-off of  $\alpha$  on the level of preclosures. Since  $\hat{\alpha}$  is isotopic to  $\hat{\gamma}_1$  relative to their endpoints, we can isotope  $\hat{\alpha}$  onto  $\hat{\gamma}_1$  (using  $\zeta$ ) and hence onto  $T$ , to give a properly embedded arc  $\hat{\alpha}_T \subset T$ , as depicted in Figure 6. The product neighborhood of  $\hat{\alpha}_T$  corresponds to a contact 1-handle  $h_0^1$  attached to  $(S^3(K_0), \Gamma_m) \sqcup (S^3(K_1), \Gamma_\mu)$ ; this is the contact 1-handle associated to  $\psi_{+,\mu}^m$ . Let  $(M_2, \gamma_2)$  be the balanced sutured manifold obtained by attaching  $h_0^1$ , and let  $T_2 = T \setminus \hat{\alpha}$ ; then  $T_2$  is an auxiliary surface for  $(M_2, \gamma_2)$ , and thus,

$$\tilde{M} = (M_2, \gamma_2) \cup [-1, 1] \times T_2.$$

Because  $\hat{\alpha}_T$  is isotopic to  $\hat{\alpha}$ , we can think of  $h_0^1$  as attached to  $(S^3(K_0), \Gamma_m) \sqcup (S^3(K_1), \Gamma_\mu)$  along the two end points of  $\alpha \setminus \hat{\alpha}$  on  $\Gamma_m \subset \partial S^3(K_0)$ . We further isotope  $\hat{\alpha}_T$  to an arc  $\hat{\alpha}_{h_0^1}$ , on the boundary of  $h_0^1$ , that intersects  $\gamma_2$  exactly once, and let

$$\alpha' = (\alpha \setminus \hat{\alpha}) \cup \hat{\alpha}_{h_0^1} \subset \partial M_2.$$

Then, the 0-surgery (with respect to the surface framing) along a push-off of  $\alpha$  corresponds to a 0-surgery along a push-off of  $\alpha'$ , and hence to a contact 2-handle attachment along  $\alpha'$ . The 1-handle  $h_0^1$  and the 2-handle attached along  $\alpha'$  together correspond to a bypass attached along  $\alpha \setminus \hat{\alpha}$ .

Now under the same identification of the endpoints as before — by the short arcs  $\zeta$  — we see that  $\alpha \setminus \hat{\alpha}$  is isotopic to the arc  $\beta_0$  relative to their endpoints (if we allow the endpoints to move along  $\Gamma_m$ ), viewed on  $(S^3(K_0), \Gamma_m)$ ; compare Figure 4 and Figure 5. (They are not isotopic when viewed on  $(S^3(K_0 \# K_1), \Gamma_{m+n})$ .) Thus, we see that the map  $C_{h_2^2}$  corresponds to the map associated to a bypass attached along  $\beta_0$ , which is  $\psi_{+,\mu}^m$ , and the proposition follows.  $\square$

Having achieved our goal of the section, we end it by spelling out an immediate corollary.

**Corollary 4.7.** *For all knots  $K \subset S^3$ , we have  $\tau_G(\bar{K}) = -\tau_G(K)$ , where  $\bar{K}$  is the mirror image of  $K$ .*

*Proof.* This is a direct consequence of Proposition 1.12 and Proposition 1.14.  $\square$

## 5 | IDENTIFYING THE TAU INVARIANTS

In this section, we identify the invariants  $\tau_G$  and  $\tau_G^\sharp$ , proving Theorem 1.2. While the instanton and monopole Floer theories are formally similar, there are some differences in their definitions. For example, the definition of  $\text{SHM}(M, \gamma)$  involves a decomposition into  $\text{Spin}^c$  structures of  $Y$  (where  $(Y, R)$  is a closure of  $(M, \gamma)$ ), which are in bijection with  $H^2(Y)$  (see [18]); the definition of  $\text{SHI}(M, \gamma)$  involves a generalized eigenspace decomposition by actions of surfaces, corresponding to  $H^2(Y)/\text{Tors}$  (see, e.g., [20, Corollary 7.6]). To identify  $\tau_G$  with  $\tau_G^\sharp$ , we have to work directly with these objects above. As  $\tau_G^\sharp$  is defined only in the instanton setting in [6], we focus on  $\tau_1$  and  $\tau_1^\sharp$  throughout the section, and discuss the changes necessary for the monopole setting.

## 5.1 | A conjugation symmetry for $\underline{\text{SHI}}(-S^3(K), -\Gamma_n)$

One key ingredient we shall need is a symmetry on  $\underline{\text{SHI}}(-S^3(K), -\Gamma_n)$  that is analogous to the  $\text{Spin}^c$  conjugation symmetry in monopole and Heegaard Floer theories. In particular, this will give us an isomorphism between the homology in grading  $i$  with the homology in grading  $-i$ .

**Proposition 5.1.** *Suppose that  $n$  is odd, which implies that  $\tau(n) = 0$ . For any  $i \in \mathbb{Z}$ , we have an isomorphism*

$$\underline{\text{SHI}}\left(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, i\right) \cong \underline{\text{SHI}}\left(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, -i\right).$$

*Proof.* If  $(Y, R)$  is a closure of  $(-S^3(K), -\Gamma_n)$  such that  $S_n^{\tau(n)}$  extends to a closed surface  $\bar{S}_n$ , then  $(Y, -R)$  is a closure of  $(-S^3(K), \Gamma_n)$ . Denote by  $\text{Eig}(\mu(R), i)$  the generalized  $i$ -eigenspace of  $\mu(R)$ . Then,

$$\underline{\text{SHI}}\left(-S^3(K), -\Gamma_n\right) = \text{Eig}(\mu(R), 2g(R) - 2);$$

taking gradings into consideration,

$$\underline{\text{SHI}}\left(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, i\right) = \text{Eig}(\mu(R), 2g(R) - 2) \cap \text{Eig}(\mu(\bar{S}_n), 2i).$$

Similarly,

$$\underline{\text{SHI}}\left(-S^3(K), \Gamma_n, S_n^{\tau(n)}, i\right) = \text{Eig}(\mu(-R), 2g(R) - 2) \cap \text{Eig}(\mu(\bar{S}_n), 2i).$$

Since

$$\text{Eig}(\mu(R), 2g(R) - 2) = \text{Eig}(\mu(-R), 2 - 2g(R))$$

holds in general, we have

$$\begin{aligned} \underline{\text{SHI}}\left(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, i\right) &= \text{Eig}(\mu(R), 2g(R) - 2) \cap \text{Eig}(\mu(\bar{S}_n), 2i) \\ &= \text{Eig}(\mu(-R), 2 - 2g(R)) \cap \text{Eig}(\mu(\bar{S}_n), 2i) \\ &\cong \text{Eig}(\mu(-R), 2g(R) - 2) \cap \text{Eig}(\mu(\bar{S}_n), -2i) \\ &= \underline{\text{SHI}}\left(-S^3(K), \Gamma_n, S_n^{\tau(n)}, -i\right). \end{aligned}$$

where the isomorphism in the third line follows from [9, Lemma 2.3]. The isomorphisms above commute with cobordism maps.

Now since  $\partial S^3(K)$  is a torus, we can isotope  $\Gamma_n$  to  $-\Gamma_n$ . Hence, there is a diffeomorphism

$$f : (S^3(K), \Gamma_n) \rightarrow (S^3(K), -\Gamma_n),$$



which restricts to the identity outside a collar of the boundary. Hence, under this diffeomorphism, the surface  $S_n^{\tau(n)} = S_n$  is preserved:  $f(S_n) = S_n$ . Thus, this diffeomorphism induces an isomorphism

$$\underline{\text{SHI}}(-S^3(K), \Gamma_n, S_n, -i) \cong \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, -i).$$

Combining this with the paragraph above, we have

$$\begin{aligned} \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, i) &\cong \underline{\text{SHI}}(-S^3(K), \Gamma_n, S_n^{\tau(n)}, -i) \\ &\cong \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, -i), \end{aligned}$$

which is what we wanted to prove. □

We have the following corollary, which is analogous to the fact that  $\text{HFK}^-(-K)$  is isomorphic to  $\text{HFK}^-(K)$ , where  $-K$  denotes the reverse of  $K$ . We shall not need this corollary in the sequel.

**Corollary 5.2.** *We have  $\underline{\text{KHI}}^-(-S^3, K) \cong \underline{\text{KHI}}^-(-S^3, -K)$ . In particular,  $\tau_1(K) = \tau_1(-K)$ .*

*Proof.* The longitude and meridian for  $-K$  are the same as those for  $K$  with their orientations reversed; this means that given a sutured manifold  $(-S^3(K), -\Gamma_n)$  in the directed system associated to  $K$ , the corresponding sutured manifold for  $-K$  is  $(-S^3(K), \Gamma_n)$ . Also, if  $S_n^{\tau(n)}$  is a Seifert surface for  $K$ , then  $-S_n^{\tau(n)}$  is a Seifert surface for  $-K$ . As in the proof of Proposition 5.1, by [9, Lemma 2.3], for odd  $n$ , we have that

$$\underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, i) \cong \underline{\text{SHI}}(-S^3(K), \Gamma_n, -S_n^{\tau(n)}, i).$$

A similar argument can be made for even  $n$ , with the modification that we need to switch between negative and positive stabilizations under the symmetry. Fitting these into the directed systems, one also needs to switch between positive and negative bypass maps under the symmetry. In any case, the isomorphisms above commute with the bypass cobordism maps, and so, we have an isomorphism of the directed systems, meaning that  $\underline{\text{KHI}}^-(-S^3, K) \cong \underline{\text{KHI}}^-(-S^3, -K)$ . □

## 5.2 | $\tau_1$ revisited

Recall that the *tau* invariant was defined in Definition 2.5 and subsequently reformulated in Proposition 3.17. Below, we give yet another reformulation; roughly speaking, we translate the characterization of  $\tau_1$  in Proposition 3.17 from the  $\underline{\text{KHI}}^-$  context to the  $\underline{\text{SHI}}$  context, and then use the symmetry in Proposition 5.1 to switch to viewing  $\tau_1$  as a minimal rather than maximal grading.

Recall from Section 3 that  $\alpha$  is a meridian of  $K$  on  $\partial S^3(K)$  that intersects the sutures  $\Gamma_n$  twice. Let  $\beta \subset \text{int}(S^3(K))$  be a push-off of  $\alpha$  into the interior of  $(S^3(K), \Gamma_n)$ , and let  $(N, \Gamma_n)$  be obtained from  $(S^3(K), \Gamma_n)$  by performing a 0-surgery along  $\beta$ .

Our first key observation is that, by [3, Section 4.2.3], the map

$$C_{h,n} : \underline{\text{SHI}}(-S^3(K), -\Gamma_n) \rightarrow \underline{\text{SHI}}(-M, -\gamma) = \underline{\text{SHI}}(-S^3(1), -\delta)$$

in (3.2) associated to a 2-handle attachment along  $\alpha$  in Section 3 can be identified with the map

$$F_{\beta,n} : \underline{\text{SHI}}(-S^3(K), -\Gamma_n) \rightarrow \underline{\text{SHI}}(-N, -\Gamma_n)$$

associated to 0-surgery along  $\beta$ . Here,  $(-S^3(1), -\delta)$  is not diffeomorphic to  $(-N, -\Gamma_n)$ , but they differ by a 1-handle attachment along two points on  $\delta$ , and hence have the same closures; thus, their sutured instanton Floer homologies are canonically identified. In particular, we know that

$$\underline{\text{SHI}}(-N, -\Gamma_n) \cong \mathbb{C}.$$

From now on, we shall often refer to (3.2) but have in mind  $F_{\beta,n}$  in place of  $C_{h,n}$ . We may now state the reformulation of  $\tau_1$ .

**Proposition 5.3.** *For  $n$  odd and sufficiently large,*

$$\tau_1(K) = \max \left\{ i \in \mathbb{Z} \mid \text{the restriction of } F_{\beta,n} \text{ to } \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, i) \text{ is nontrivial} \right\} - \frac{n-1}{2},$$

$$\tau_1(K) = -\min \left\{ i \in \mathbb{Z} \mid \text{the restriction of } F_{\beta,n} \text{ to } \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, i) \text{ is nontrivial} \right\} - \frac{n-1}{2}.$$

*Proof.* The first statement follows directly from Proposition 3.17, with the grading shift of  $(n-1)/2$  coming from (2.4) in the definition of  $\underline{\text{KHI}}^-$ . Since the diffeomorphism  $f$  in the proof of Proposition 5.1 restricts to the identity outside a collar of  $\partial S^3(K)$ , we can take  $\beta$  to be inside the region where  $f$  is the identity. Hence, the isomorphism

$$\underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, i) \cong \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, -i)$$

in Proposition 5.1 intertwines the maps  $F_{\beta,n}$ . This implies that the maximum grading for which the restriction of  $F_{\beta,n}$  is nontrivial is minus the minimum grading for which the restriction of  $F_{\beta,n}$  is nontrivial.  $\square$

### 5.3 | The sutured manifold $(S^3(K), \Gamma_n)$

Let us now discuss the strategy of identifying  $\tau_1$  with  $\tau_1^\sharp$ . Recall that  $\tau_1^\sharp$  is  $1/2$  times the homogenization of  $\nu_1^\sharp$ . Baldwin and Sivek explain in [6, p. 16] that the sequence of integers  $(\dim_{\mathbb{C}} I^\sharp(S_n^3(K)))_{n \in \mathbb{Z}}$  satisfies the following:

- consecutive values always differ by  $\pm 1$ , that is,  $|\dim_{\mathbb{C}} I^\sharp(S_n^3(K)) - \dim_{\mathbb{C}} I^\sharp(S_{n-1}^3(K))| = 1$ ; and
- either the sequence is unimodal, with a unique minimum at  $n = \nu_1^\sharp(K)$ , which they call *V-shaped*, or  $\nu_1^\sharp(K) = 0$  and there are two minima at  $n = \pm 1$ , which they call *W-shaped*.

Our goal is to study, analogously, the sequence  $(\dim_{\mathbb{C}} \underline{\text{SHI}}(-S^3(K), -\Gamma_n))_{n \in \mathbb{Z}}$ , which turns out to be similar to the sequence above and but is always *V-shaped*. In this subsection, we prove this assertion, and relate  $\tau_1$  with the slope  $n_0$  at which the unique minimum occurs. In the next subsection, we shall use  $n_0$  to relate  $\tau_1$  with  $\nu_1^\sharp$  and pass to the homogenization to get  $\tau_1^\sharp$ .

For brevity, for a given  $K$ , let us denote

$$d_n = \dim_{\mathbb{C}} \underline{\text{SHI}}(-S^3(K), -\Gamma_n).$$

First, we prove that  $d_n$  differs from  $d_{n-1}$  by  $\pm 1$ .

**Lemma 5.4.** *For all  $n \in \mathbb{Z}$ ,  $|d_n - d_{n-1}| = 1$ .*

*Proof.* This follows directly from (3.2) and the fact that  $\underline{\text{SHI}}(-S^3(1), -\delta) \cong \mathbb{C}$ . □

Next, we prove that  $(d_n)_{n \in \mathbb{Z}}$  is  $V$ -shaped.

**Lemma 5.5.** *For all  $n \in \mathbb{Z}$ , if  $d_n > d_{n-1}$ , then  $d_{n+1} > d_n$ ; if  $d_n < d_{n-1}$ , then  $d_{n-1} < d_{n-2}$ . Consequently, the sequence  $(d_n)_{n \in \mathbb{Z}}$  has a unique minimum.*

*Proof.* First, consider the surgery exact triangle (3.2), with  $F_{\beta,n}$  in place of  $C_{h,n}$ :

$$\begin{array}{ccc}
 \underline{\text{SHI}}(-S^3(K), -\Gamma_{n-1}) & \xrightarrow{\quad\quad\quad} & \underline{\text{SHI}}(-S^3(K), -\Gamma_n) \\
 & \swarrow G_{\beta,n-1} & \searrow F_{\beta,n} \\
 & \underline{\text{SHI}}(-N, -\Gamma_n) & 
 \end{array} \tag{5.6}$$

Here, as in Section 5.2,  $(N, \Gamma_n)$  is obtained from  $(S^3(K), \Gamma_n)$  by 0-surgery along  $\beta$ . Since  $\underline{\text{SHI}}(-N, -\Gamma_n) \cong \underline{\text{SHI}}(-S^3(1), -\delta) \cong \mathbb{C}$ , we see that the condition  $d_n > d_{n-1}$  is equivalent to  $F_{\beta,n} \neq 0$ , and also to  $G_{\beta,n-1} \equiv 0$ ; similarly, the condition  $d_n < d_{n-1}$  is equivalent to  $F_{\beta,n} \equiv 0$ , and also to  $G_{\beta,n-1} \neq 0$ .

The idea now is to combine this surgery exact triangle with either one of the two bypass exact triangles associated to  $\psi_{+,n+1}^n$  and  $\psi_{-,n+1}^n$ :

$$\begin{array}{ccc}
 \underline{\text{SHI}}(-S^3(K), -\Gamma_n) & \xrightarrow{\psi_{\pm,n+1}^n} & \underline{\text{SHI}}(-S^3(K), -\Gamma_{n+1}) \\
 & \swarrow & \searrow \\
 & \underline{\text{SHI}}(-S^3(K), -\Gamma_{\mu}) & 
 \end{array}$$

Since  $\beta$  lies in the interior of  $S^3(K)$ , the triangle above is intertwined by the maps  $F_{\beta,n}$ :

$$\begin{array}{ccccc}
 \underline{\text{SHI}}(-S^3(K), -\Gamma_n) & \xrightarrow{\psi_{\pm,n+1}^n} & & \underline{\text{SHI}}(-S^3(K), -\Gamma_{n+1}) & \\
 \downarrow F_{\beta,n} & \swarrow & \searrow & \downarrow F_{\beta,n+1} & \\
 & \underline{\text{SHI}}(-S^3(K), -\Gamma_{\mu}) & & & \\
 \downarrow & \downarrow & \downarrow & \downarrow & \\
 \underline{\text{SHI}}(-N, -\Gamma_n) & \xrightarrow{\quad\quad\quad} & & \underline{\text{SHI}}(-N, -\Gamma_{n+1}) & \\
 & \swarrow & \searrow & \swarrow & \searrow \\
 & \underline{\text{SHI}}(-N, -\Gamma_{\mu}) & & & 
 \end{array}$$

Note that  $N \cong S^1 \times S^2(K)$  is a solid torus with meridional disk  $D$ , where  $\partial D = \alpha \subset \partial(S^1 \times S^2(K)) = \partial S^3(K)$ . In other words,  $D$  is a boundary-compressing disk, and so  $(-N, -\Gamma_\mu)$  is not taut. Therefore,

$$\underline{\text{SHI}}(-N, -\Gamma_\mu) = 0,$$

and we have the following commutative diagram:

$$\begin{array}{ccc} \underline{\text{SHI}}(-S^3(K), -\Gamma_n) & \xrightarrow{\psi_{\pm, n+1}^n} & \underline{\text{SHI}}(-S^3(K), -\Gamma_{n+1}) \\ \downarrow F_{\beta, n} & & \downarrow F_{\beta, n+1} \\ \underline{\text{SHI}}(-N, -\Gamma_n) & \xrightarrow{\cong} & \underline{\text{SHI}}(-N, -\Gamma_{n+1}) \end{array}$$

Thus, if  $F_{\beta, n} \neq 0$ , then  $F_{\beta, n+1} \neq 0$ . This means that if  $d_n > d_{n-1}$ , then  $d_{n+1} > d_n$ .

Similarly, the bypass exact triangles are intertwined by the maps

$$G_{\beta, n} : \underline{\text{SHI}}(-N, -\Gamma_{n+1}) \rightarrow \underline{\text{SHI}}(-S^3(K), -\Gamma_n),$$

which appear in (5.6), from which we get the following commutative diagram:

$$\begin{array}{ccc} \underline{\text{SHI}}(-S^3(K), -\Gamma_{n-2}) & \xrightarrow{\psi_{\pm, n-1}^n} & \underline{\text{SHI}}(-S^3(K), -\Gamma_{n-1}) \\ \uparrow G_{\beta, n-2} & & \uparrow G_{\beta, n-1} \\ \underline{\text{SHI}}(-N, -\Gamma_{n-1}) & \xrightarrow[I_{n-1}]{\cong} & \underline{\text{SHI}}(-N, -\Gamma_n) \end{array} \tag{5.7}$$

From this, we conclude that if  $G_{\beta, n-1} \neq 0$ , then  $G_{\beta, n-2} \neq 0$ . This means that if  $d_n < d_{n-1}$ , then  $d_{n-1} < d_{n-2}$ .

Finally, the inequalities imply that  $(d_n)_{n \in \mathbb{Z}}$  has at most one minimum. Since  $d_n$  is a dimension, we have  $d_n \geq 0$  for all  $n$ , which means that  $(d_n)_{n \in \mathbb{Z}}$  does indeed have a unique minimum.  $\square$

Let  $n_0$  be the index at which the sequence  $(d_n)_{n \in \mathbb{Z}}$  attains its unique minimum. We now turn to relating  $n_0$  with  $\tau_\Gamma$ .

Consider the map  $G_{\beta, n} : \underline{\text{SHI}}(-N, -\Gamma_{n+1}) \rightarrow \underline{\text{SHI}}(-S^3(K), -\Gamma_n)$  in (5.6). Using again the fact that  $\underline{\text{SHI}}(-N, -\Gamma_{n+1}) \cong \mathbb{C}$ , we define the element

$$x_n = G_{\beta, n}(\mathbf{1}_{n+1}) \in \underline{\text{SHI}}(-S^3(K), -\Gamma_n),$$

where  $\mathbf{1}_{n+1}$  is a generator of  $\underline{\text{SHI}}(-N, -\Gamma_{n+1})$ . Recall that  $\underline{\text{SHI}}(-N, -\Gamma_{n+1})$  is well defined only up to a unit; if we like, we could choose concrete representatives of each term, and choose  $\mathbf{1}_n$  so that  $I_n(\mathbf{1}_n) = \mathbf{1}_{n+1}$ , where  $I_n : \underline{\text{SHI}}(-N, -\Gamma_n) \rightarrow \underline{\text{SHI}}(-N, \Gamma_{n+1})$  is the map in (5.7). That commutative

diagram ensures that

$$x_{n+1} = \psi_{\pm, n+1}^n(x_n). \tag{5.8}$$

Note that there are two assertions here, one for  $\psi_{+, n+1}^n$  and one for  $\psi_{-, n+1}^n$ , and each one holds up to multiplication by a (possibly different) unit in  $\mathbb{C}$ . We will need both assertions later.

We are interested in the “width” of  $x_n$  in terms of the grading. Precisely, writing

$$x_n = \sum_{i \in \mathbb{Z}} x_{n,i},$$

where  $x_{n,i} \in \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n^{\tau(n)}, i)$ , we define the “width” to be the maximum supported grading of  $x_n$  less the minimum supported grading:

$$l_n = \begin{cases} \max\{i \in \mathbb{Z} \mid x_{n,i} \neq 0\} - \min\{i \in \mathbb{Z} \mid x_{n,i} \neq 0\} + 1 & \text{if } x_n \neq 0; \\ 0 & \text{if } x_n = 0. \end{cases}$$

The reason we are interested in  $l_n$  is the following. On the one hand, for large, positive, odd  $n$ , the value of  $l_{-n}$  is related to  $\tau_1(\bar{K})$  via Proposition 5.3. (Here,  $\bar{K}$  denotes the mirror of  $K$ ; the appearances of the mirror and the negative sign before  $n$  are related to the fact that there is a duality between the maps  $F_{\beta, n}$  and  $G_{\beta, -n}$ .) On the other hand, as  $-n$  increases,  $l_{-n}$  strictly decreases until it reaches zero, and the value of  $-n$  when  $l_{-n}$  reaches zero determines  $n_0$ . We first prove the first assertion.

**Lemma 5.9.** *For  $n$  odd and sufficiently large,*

$$l_{-n} = 2\tau_1(\bar{K}) + n.$$

Here,  $\bar{K}$  is the mirror of  $K$ .

*Proof.* First, the unimodality of  $(d_n)_{n \in \mathbb{Z}}$  from Lemma 5.5 implies that, for  $n$  sufficiently large,  $d_{-n+1} < d_{-n}$  must hold, or equivalently,  $G_{\beta, -n} \neq 0$ ; this implies that  $x_{-n} \neq 0$ , and so,  $l_{-n} = \max\{i \in \mathbb{Z} \mid x_{-n,i} \neq 0\} - \min\{i \in \mathbb{Z} \mid x_{-n,i} \neq 0\} + 1$ .

There is an orientation-preserving diffeomorphism

$$(-S^3(K), -\Gamma_{-n}, S_n) \cong (S^3(\bar{K}), \Gamma_n, S_n).$$

Hence, we have the following commutative diagram:

$$\begin{array}{ccc} \underline{\text{SHI}}(-S^3(K), -\Gamma_{-n}, S_n, i) & \xrightarrow{\cong} & \underline{\text{SHI}}(S^3(\bar{K}), \Gamma_n, S_n, i) \\ \uparrow G_{\beta, -n, i} & & \uparrow \bar{G}'_{\beta, n, i} \\ \underline{\text{SHI}}(-N, -\Gamma_{-n}) & \xrightarrow{\cong} & \underline{\text{SHI}}(N, \Gamma_n) \end{array}$$

Here, the bar on  $\overline{G}'_{\beta,n,i}$  reminds us that it is a map associated to  $\overline{K}$ , and  $G_{\beta,-n,i}$  is the component of  $G_{\beta,-n}$  that lands in the grading- $i$  summand (and similarly for  $\overline{G}'$ ). From [25, Theorem 1.7], we have natural isomorphisms

$$\underline{\text{SHI}}(S^3(\overline{K}), \Gamma_n, S, i) \cong \underline{\text{SHI}}(-S^3(\overline{K}), \Gamma_n, S, i)^\vee, \quad \underline{\text{SHI}}(N, \Gamma_n) \cong \underline{\text{SHI}}(-N, \Gamma_n)^\vee,$$

where  $V^\vee$  denotes the vector space dual to  $V$ . These isomorphisms fit into a commutative diagram:

$$\begin{array}{ccc} \underline{\text{SHI}}(S^3(\overline{K}), \Gamma_n, S, i) & \xrightarrow{\cong} & \underline{\text{SHI}}(-S^3(\overline{K}), \Gamma_n, S, i)^\vee \\ \uparrow \overline{G}'_{\beta,n,i} & & \uparrow \overline{F}^\vee_{\beta,n,i} \\ \underline{\text{SHI}}(N, \Gamma_n) & \xrightarrow{\cong} & \underline{\text{SHI}}(-N, \Gamma_n)^\vee \end{array}$$

Here, the map  $\overline{F}^\vee_{\beta,n,i}$  is the dual of the map

$$\overline{F}_{\beta,n,i} : \underline{\text{SHI}}(-S^3(\overline{K}), \Gamma_n, S, i) \rightarrow \underline{\text{SHI}}(-N, \Gamma_n).$$

The reason that the diagram above commutes is that  $\overline{G}'_{\beta,n}$  and  $\overline{F}_{\beta,n}$  are induced by the same cobordism with opposite orientations. Thus, we have

$$\begin{aligned} \max \left\{ i \in \mathbb{Z} \mid x_{-n,i} \neq 0 \right\} &= \max \left\{ i \in \mathbb{Z} \mid G_{\beta,-n,i} \neq 0 \right\} = \max \left\{ i \in \mathbb{Z} \mid \overline{G}'_{\beta,n,i} \neq 0 \right\} \\ &= \max \left\{ i \in \mathbb{Z} \mid \overline{F}^\vee_{\beta,n,i} \neq 0 \right\} = \max \left\{ i \in \mathbb{Z} \mid \overline{F}_{\beta,n,i} \neq 0 \right\} \\ &= \tau_1(\overline{K}) + \frac{n-1}{2}, \end{aligned}$$

where the last equality follows from Proposition 5.3. Similarly,

$$-\min \left\{ i \in \mathbb{Z} \mid x_{-n,i} \neq 0 \right\} = \tau_1(\overline{K}) + \frac{n-1}{2}.$$

Summing these and adding 1, we get

$$l_{-n} = \max \left\{ i \in \mathbb{Z} \mid x_{-n,i} \neq 0 \right\} - \min \left\{ i \in \mathbb{Z} \mid x_{-n,i} \neq 0 \right\} + 1 = 2\tau_1(\overline{K}) + n,$$

as claimed. □

Next, we prove the following lemma.

**Lemma 5.10.** *For  $n \in \mathbb{Z}$ , if  $l_n > 0$ , then  $l_{n+1} \leq l_n - 1$ ; if  $l_n = 0$ , then  $l_{n+1} = 0$ .*

*Proof.* The second claim follows directly from the definition, since if  $x_n = 0$ , then  $x_{n+1} = \psi_{\pm, n+1}^n(x_n) = 0$ . We focus on the first claim, where we assume  $x_n \neq 0$ . First, if  $x_{n+1} = 0$ , then  $l_{n+1} = 0$  and the inequality holds; thus, we may also assume that  $x_{n+1} \neq 0$ .

Recall, from the discussion following (2.4), that with the  $\mathbb{Z}$ -gradings on  $\text{SHI}(-S^3(K), -\Gamma_n)$  induced by  $S_n^{\tau(n)}$ , the map  $\psi_{-, n+1}^n$  preserves grading while  $\psi_{+, n+1}^n$  decreases grading by 1. Thus, for  $j > \max\{i \in \mathbb{Z} \mid x_{n,i} \neq 0\} - 1$ , the graded version of (5.8) becomes

$$x_{n+1,j} = \psi_{+, n+1}^n(x_{n,j+1}) = \psi_{+, n+1}^n(0) = 0,$$

while for  $j < \min\{i \in \mathbb{Z} \mid x_{n,i} \neq 0\}$ ,

$$x_{n+1,j} = \psi_{-, n+1}^n(x_{n,j}) = \psi_{-, n+1}^n(0) = 0.$$

This means that

$$l_{n+1} \leq \left( \max \left\{ i \in \mathbb{Z} \mid x_{n,i} \neq 0 \right\} - 1 \right) - \min \left\{ i \in \mathbb{Z} \mid x_{n,i} \neq 0 \right\} + 1 = l_n - 1,$$

as claimed. □

We now combine the two claims to relate  $n_0$  with  $\tau_1(\bar{K})$ .

**Corollary 5.11.** *We have the inequality*

$$n_0 \leq 2\tau_1(\bar{K}).$$

*Proof.* From its definition, it is clear that  $l_n > 0$  whenever  $x_n \neq 0$ . Fix some  $n$  that is odd and sufficiently large, so that Lemma 5.9 holds and  $l_{-n} = 2\tau_1(\bar{K}) + n$ . Letting  $m = 2\tau_1(\bar{K}) + n$ , we may inductively apply Lemma 5.10  $m$  times to conclude that

$$l_{2\tau_1(\bar{K})} \leq \max \{l_{-n} - m, 0\} = \max \left\{ 2\tau_1(\bar{K}) + n - (2\tau_1(\bar{K}) + n), 0 \right\} = 0.$$

Thus, we see that  $x_{2\tau_1(\bar{K})} = 0$ , or equivalently,  $G_{\beta, 2\tau_1(\bar{K})} \equiv 0$ . As explained in the text following (5.6), this is equivalent to the condition that  $d_{2\tau_1(\bar{K})+1} > d_{2\tau_1(\bar{K})}$ . This can occur only when  $2\tau_1(\bar{K}) \geq n_0$ . □

## 5.4 | Identifying $\tau_I$ with $\tau_I^\sharp$

**Lemma 5.12.** *If  $v^\sharp(K) \neq 0$ , then*

$$v^\sharp(K) > -2\tau_1(\bar{K}) - 2.$$

*Proof.* To simplify the notation, we set  $n = 2\tau_1(\bar{K}) + 2$  throughout this proof. By Corollary 5.11, we have  $d_{n-1} > d_{n-2}$ , or equivalently,  $F_{\beta, n-1} \neq 0$ , as explained in the text after (5.6).

Our goal is to relate  $F_{\beta, n-1}$  to a cobordism map in the definition of  $\nu^\sharp(K)$ . To do so, take a curve of class  $n\mu - \lambda$  on  $\partial S^3(K)$ , and push it off into the interior of  $S^3(K)$  to obtain a curve  $\eta$ , such that  $\eta$  has linking number 1 with  $\beta$  (meaning that it is closer to  $\partial S^3(K)$  than  $\beta$ ). Consider the surgery exact triangle associated to  $\eta$ :

$$\begin{array}{ccc} \underline{\text{SHI}}(-S^3(K), -\Gamma_{n-1}) & \xrightarrow{F_{\eta, n-1}} & \underline{\text{SHI}}(-Y_0, -\Gamma_{n-1}) \\ & \searrow & \swarrow \\ & \underline{\text{SHI}}(-Y_{-1}, -\Gamma_{n-1}) & \end{array}$$

Here,  $(Y_0, \Gamma_{n-1})$  (resp.  $(Y_{-1}, \Gamma_{n-1})$ ) is obtained from  $(S^3(K), \Gamma_{n-1})$  by 0-surgery (resp.  $-1$ -surgery) along  $\eta$ , where the surgery coefficient is taken with respect to the surface framing induced by  $\partial S^3(K)$ .

We can, in fact, determine these Floer homology groups: First, as explained in Section 5.2, we can identify  $F_{\eta, n-1}$  with a map  $C_{h, n-1} : \underline{\text{SHI}}(-S^3(K), \Gamma_{n-1}) \rightarrow \underline{\text{SHI}}(-P(1), -\delta)$  associated to a 2-handle attachment along  $n\mu - \lambda$  (of which  $\eta$  is a push-off). Again,  $(-P(1), -\delta)$  is not diffeomorphic to  $(-Y_0, -\Gamma_{n-1})$ , but they have the same closures and hence the same Floer homologies. Here,  $P$  is obtained from  $S^3(K)$  by performing a Dehn filling along  $n\mu - \lambda$ , and so, it is nothing but  $S^3_{-n}(K)$ . After attaching the 2-handle, the boundary becomes a sphere, which is why we have  $P(1)$ ; the sutures become connected, which is why we have  $\delta$ . Thus,

$$\underline{\text{SHI}}(-Y_0, -\Gamma_{n-1}) \cong \underline{\text{SHI}}(-P(1), -\delta) \cong \mathbb{I}^\sharp(-S^3_{-n}(K)).$$

Second, since  $\eta$  is boundary parallel, performing a  $(-1)$ -surgery along  $\eta$  is equivalent to performing a Dehn twist along  $n\mu - \lambda$  on  $\partial S^3(K)$ . This means that  $(Y_{-1}, \Gamma_{n-1})$  is diffeomorphic to  $(S^3(K), \Gamma_\mu)$ . Combining, we have the following exact triangle:

$$\begin{array}{ccc} \underline{\text{SHI}}(-S^3(K), -\Gamma_{n-1}) & \xrightarrow{C_{h, n-1}} & \mathbb{I}^\sharp(-S^3_{-n}(K)) \\ & \searrow & \swarrow \\ & \underline{\text{SHI}}(-S^3(K), -\Gamma_\mu) & \end{array}$$

We can also perform a 0-surgery along  $\beta$  and obtain a commutative diagram:

$$\begin{array}{ccccc} \underline{\text{SHI}}(-S^3(K), -\Gamma_{n-1}) & \xrightarrow{C_{h, n-1}} & \mathbb{I}^\sharp(-S^3_{-n}(K)) & & \\ \downarrow F_{\beta, n-1} & \searrow & \swarrow & \downarrow V_{-n} & \\ & \underline{\text{SHI}}(-S^3(K), -\Gamma_\mu) & & & \\ & \downarrow & & & \\ \underline{\text{SHI}}(-N, -\Gamma_{n-1}) & \xrightarrow{\quad} & \mathbb{I}^\sharp(-S^3) & & \\ & \swarrow & \downarrow & \swarrow & \\ & \underline{\text{SHI}}(-N, -\Gamma_\mu) & & & \end{array}$$



Note that the map  $V_{-n}$  on  $I^\sharp$  really does coincide with the map on SHI of the 0-surgeries, since the sutured manifolds involved have the same closures. Here,  $N$  is the manifold we encountered in Section 5.2. We have that  $\text{SHI}(-N, -\Gamma_\mu) = 0$ , because  $N$  is irreducible and  $(-N, -\Gamma_\mu)$  is not taut. Thus, we obtain the commutative diagram:

$$\begin{array}{ccc}
 \text{SHI}(-S^3(K), -\Gamma_{n-1}) & \xrightarrow{C_{n,n-1}} & I^\sharp(-S^3_{-n}(K)) \\
 \downarrow F_{\beta,n-1} & & \downarrow V_{-n} \\
 \text{SHI}(-M, -\Gamma_{n-1}) & \xrightarrow{\cong} & I^\sharp(-S^3)
 \end{array}$$

This implies that, since  $F_{\beta,n-1} \neq 0$ , we have that  $V_{-n} \neq 0$ . (Recall that  $n = 2\tau_1(\bar{K}) + 2$ .)

Finally, we note that  $V_{-n}$ , a cobordism map associated to the 0-surgery along  $\beta$ , is dual to the map

$$W_{-n} : I^\sharp(S^3) \rightarrow I^\sharp(S^3_{-n}(K))$$

associated to the same cobordism upside down. Hence,  $W_{-n} \neq 0$ . Recall from Definition 2.8 that  $\nu_1^\sharp(K) = N(K) - N(\bar{K})$ , where  $N(K)$  is the smallest nonnegative integer for which the cobordism  $W_{N(K)} : I^\sharp(S^3) \rightarrow I^\sharp(S^3_{N(K)}(K))$  vanishes. By [6, Proposition 3.3], if  $\nu_1^\sharp(K) \neq 0$ , then precisely, one of  $N(K)$  and  $N(\bar{K})$  is nonzero, and

$$\left( \dim_{\mathbb{C}} I^\sharp(S^3_m(K)) \right)_{m \in \mathbb{Z}}$$

is unimodal with minimum precisely at  $m = \nu_1^\sharp(K)$ . By [6, (3.1) and the proof of Proposition 3.2], this means that  $W_m : I^\sharp(S^3) \rightarrow I^\sharp(S^3_m(K))$  vanishes exactly when  $m \geq \nu_1^\sharp(K)$ . Thus, we obtain that

$$-2\tau_1(\bar{K}) - 2 = -n < \nu_1^\sharp(K),$$

as claimed. □

**Corollary 5.13.** *If  $\nu_1^\sharp(K) \neq 0$ , then*

$$2\tau_1(K) - 2 < \nu_1^\sharp(K) < 2\tau_1(K) + 2.$$

*Proof.* Since  $\tau_1$  is a concordance homomorphism, we know that  $\tau_1(K) = -\tau_1(\bar{K})$ . Hence, the first inequality follows directly from Lemma 5.12. For the second inequality, we use the fact from [6, Section 3] that  $-\nu_1^\sharp(K) = \nu_1^\sharp(\bar{K})$  and use Lemma 5.12 on  $\bar{K}$  to get

$$\nu_1^\sharp(\bar{K}) > -2\tau_1(K) - 2,$$

from which our inequality immediately follows. □

We are now ready to identify  $\tau_1$  with  $\tau_1^\sharp$ .

*Proof of Theorem 1.2 in the instanton setting.* If  $\tau_1^\#(K) \geq 1$ , then for all positive  $n$ , we have  $\tau_1^\#(\#nK) \geq n$ , and from [6, Proposition 5.4], we have  $|2\tau_1^\#(\#nK) - \nu_1^\#(\#nK)| \leq 1$ , and, in particular,  $\nu_1^\#(\#nK) > 0$ . So, we can apply Corollary 5.13 to  $\#nK$  to conclude that

$$2n\tau_1(K) - 2 = 2\tau_1(\#nK) - 2 < \nu_1^\#(\#nK) < 2\tau_1(\#nK) + 2 = 2n\tau_1(K) + 2.$$

From the definition of  $\tau_1^\#(K)$  (Definition 2.9), we have

$$\lim_{n \rightarrow \infty} \left( \tau_1(K) - \frac{1}{n} \right) \leq \tau_1^\#(K) = \frac{1}{2} \lim_{n \rightarrow \infty} \frac{\nu_1^\#(\#nK)}{n} \leq \lim_{n \rightarrow \infty} \left( \tau_1(K) + \frac{1}{n} \right),$$

and so,

$$\tau_1^\#(K) = \tau_1(K).$$

If  $\tau_1^\#(K) \leq 1$ , then we can pick a knot  $K_0$  with  $\tau_1^\#(K_0) = 1$  (e.g., the right-handed trefoil). Since  $\tau_1^\#$  is a concordance homomorphism, we have

$$\tau_1^\#(K) = \tau_1^\#(K \# nK_0) - \tau_1^\#(\#nK_0) = \tau_1^\#(K \# nK_0) - n.$$

Thus, for  $n$  large enough, we have  $\tau_1^\#(K \# nK_0) \geq 1$ , and so, the paragraph above shows that both

$$\tau_1^\#(K \# nK_0) = \tau_1(K \# nK_0)$$

and

$$\tau_1^\#(\#nK_0) = \tau_1(\#nK_0) = n.$$

Since  $\tau_1$  is also a concordance homomorphism, we conclude that

$$\tau_1^\#(K) = \tau_1(K),$$

as desired.  $\square$

Finally, we state the necessary changes for the monopole setting.

*Proof of Theorem 1.2 in the monopole setting.* The proof is similar to that in the instanton setting, with a different proof for the following statement: In Proposition 5.1, the symmetry isomorphism for odd  $n$ ,

$$\underline{\text{SHI}}(-S^3(K), \Gamma_n, S_n, i) \cong \underline{\text{SHI}}(-S^3(K), -\Gamma_n, S_n, -i)$$

follows from a symmetry in the generalized eigenspaces associated to  $\mu(R)$ . The analogous statement,

$$\underline{\text{SHM}}(-S^3(K), \Gamma_n, S_n, i) \cong \underline{\text{SHM}}(-S^3(K), -\Gamma_n, S_n, -i),$$

follows from the conjugation symmetry in the Spin<sup>c</sup> decomposition in monopole Floer theory.  $\square$

## 6 | COMPUTATION FOR TWIST KNOTS

In this section, we compute  $\underline{\text{KHG}}^-$  for the family of knots  $K_m$  as in Figure 1, proving Theorem 1.16. We divide Theorem 1.16 into four propositions: Proposition 6.6, Proposition 6.9, Proposition 6.10, and Proposition 6.12

Note that, in particular,  $K_1$  is the right-handed trefoil,  $K_0$  is the unknot, and  $K_{-1}$  is the figure-eight knot.

From the Seifert algorithm, we can easily construct a genus-1 Seifert surface for  $K_m$ , which we denote by  $S_m$ . Hence,  $g(K_m) = 1$ . Also, it is straightforward to compute the (symmetrized) Alexander polynomial of  $K_m$  to be

$$\Delta_{K_m}(t) = mt + (1 - 2m) + mt^{-1}. \tag{6.1}$$

First, we will compute  $\underline{\text{KHG}}(-S^3, K_m)$ . Suppose that  $(S^3(K_m), \Gamma_\mu)$  is the balanced sutured manifold obtained by taking meridional sutures on knot complements. There is a curve  $\zeta \subset \text{Int } S^3(K_m)$  as in Figure 1 so that we have a surgery exact triangle:

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K_m), -\Gamma_\mu) & \longrightarrow & \underline{\text{SHG}}(-S^3(K_{m+1}), -\Gamma_\mu) \\ & \searrow & \swarrow \\ & \underline{\text{SHG}}(-Q, -\Gamma_\mu) & \end{array}$$

Here,  $K_m$  is described as above, and  $Q$  is obtained from  $S^3(K_m)$  by performing a 0-Dehn surgery along  $\zeta$ . We can use the surface  $S_{m,\mu}$  that intersects the suture  $\Gamma_\mu$  twice to construct a grading on the sutured monopole and instanton Floer homologies. Let  $S_{m,n}$  be an isotopy of  $S_{m,\mu}$  so that  $S_{m,n}$  intersects the suture  $\Gamma_n$  exactly  $2n$  times. Since  $\zeta$  is disjoint from  $S_{m,\mu}$ , all the Seifert surfaces  $S_{m,\mu}$  and  $S_{m,n}$  survive in  $Q$ , which we call  $S_\mu$  and  $S_n$ , respectively. Also, there is a graded version of the exact triangle (note that we omit the surfaces from the following exact triangle):

$$\begin{array}{ccc} \underline{\text{SHG}}(-S^3(K_m), -\Gamma_\mu, i) & \longrightarrow & \underline{\text{SHG}}(-S^3(K_{m+1}), -\Gamma_\mu, i) \\ & \searrow & \swarrow \\ & \underline{\text{SHG}}(-Q, -\Gamma_\mu, i) & \end{array} \tag{6.2}$$

Since  $S_{m,\mu}$  has genus one and intersects the suture twice, all the graded sutured monopole and instanton Floer homologies in (6.2) could only possibly be nontrivial for  $-1 \leq i \leq 1$ . To understand what is  $\underline{\text{SHG}}(-Q, -\Gamma_\mu)$ , from [19] and [24], the surgery exact triangle (3.7) is just the same as the oriented skein exact triangle and  $\underline{\text{SHG}}(-Q, -\Gamma)$  is isomorphic to the knot monopole or instanton Floer homology of the oriented smoothing of  $K_m$ , which is a Hopf link. Applying oriented Skein relation again on Hopf links, we can conclude that

$$\text{rk}_{\mathcal{R}}(\underline{\text{SHG}}(-Q, -\Gamma_\mu)) \leq 4. \tag{6.3}$$

For the monopole and instanton knot Floer homologies of  $K_1$  (trefoil), we could look at the surgery exact triangle along the curve  $\zeta$  in Figure 7 and argue in the same way as in [19] to conclude

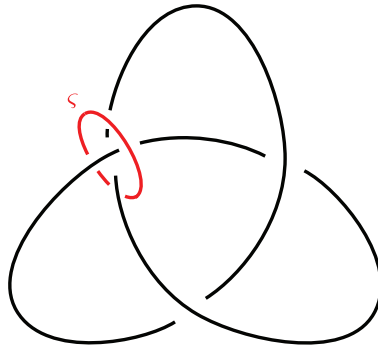


FIGURE 7 The trefoil and the circle  $\zeta$ .

$$\text{rk}_{\mathcal{R}}(\underline{\text{SHG}}(-S^3(K_1), -\Gamma_{\mu}) \leq 3.$$

Using the Alexander polynomial in (6.1) and [19, 20], we know that

$$\underline{\text{SHG}}(-S^3(K_1), -\Gamma_{\mu}, S_{1,\mu}, i) \cong \mathcal{R} \quad (6.4)$$

for  $i = -1, 0, 1$  and it vanishes in all other gradings.

Now let  $m = 1$  in (6.2). We know from (6.1) that

$$\text{rk}_{\mathcal{R}}(\underline{\text{SHG}}(-S^3(K_2), -\Gamma_{\mu}) \geq 7.$$

Then, from the exactness and inequalities (6.3) and (6.4), we know that

$$\text{rk}_{\mathcal{R}}(\underline{\text{SHG}}(-Q, -\Gamma_{\mu})) = 4.$$

After further examining each gradings, we know that

$$\underline{\text{SHG}}(-Q, -\Gamma_{\mu}, S_{\mu}, i) = \begin{cases} \mathcal{R} & \text{for } i = 1, -1, \\ \mathcal{R}^2 & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by using the same argument and the induction, we can compute, for  $m > 0$ , that

$$\underline{\text{SHG}}(-S^3(K_m), -\Gamma_{\mu}, S_{m,\mu}, i) = \begin{cases} \mathcal{R}^m & \text{for } i = 1, -1, \\ \mathcal{R}^{2m-1} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6.5)$$

Since  $K_0$  is the unknot, we can use the same technique to compute for,  $m \leq 0$ , that

$$\underline{\text{SHG}}(-S^3(K_m), -\Gamma_{\mu}, S_{m,\mu}, i) = \begin{cases} \mathcal{R}^{-m} & \text{for } i = 1, -1, \\ \mathcal{R}^{1-2m} & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

gr		$\Gamma_\mu$	$\xrightarrow{\psi_{+,1}^\mu}$	$\Gamma_1$	$\xrightarrow{\psi_{+,2}^1}$	$\Gamma_2$
1		$\mathcal{R}^{-m}$				$\mathcal{R}^{-m}$
0		$\mathcal{R}^{1-2m}$		$\mathcal{R}^{-m}$		$\mathcal{R}^b$
-1		$\mathcal{R}^{-m}$		$\mathcal{R}^a$		$\mathcal{R}^c$
-2				$\mathcal{R}^{-m}$		$\mathcal{R}^{-m}$

FIGURE 8 The map  $\psi_{+,2}^1$  for  $K_m$ . Each row is the positive bypass exact triangle in a particular grading. The leftmost column indicates the gradings. We use letters like  $a, b$ , and  $c$  to indicate that, *a priori*, we do not know what the rank is.

gr		$\Gamma_\mu$	$\xrightarrow{\psi_{+,2}^\mu}$	$\Gamma_2$	$\xrightarrow{\psi_{+,3}^2}$	$\Gamma_3$
2		$\mathcal{R}^{-m}$				$\mathcal{R}^{-m}$
1		$\mathcal{R}^{1-2m}$		$\mathcal{R}^{-m}$	$\xrightarrow{(\psi_{+,3}^2)_1}$	$\mathcal{R}^{1-m}$
0		$\mathcal{R}^{-m}$		$\mathcal{R}^{1-m}$	$\xrightarrow{(\psi_{+,3}^2)_0}$	$\mathcal{R}$
-1				$\mathcal{R}^{a+m}$		$\mathcal{R}^{a+m}$
2				$\mathcal{R}^{-m}$		$\mathcal{R}^{-m}$

FIGURE 9 The map  $\psi_{+,3}^2$  for  $K_m$ . We denote by  $(\psi_{+,3}^2)_i$  the restriction of the map  $\psi_{+,3}^2$  to the grading  $i$ .

Now we are ready to compute the minus version. Recall that the Seifert surface induces a framing on the boundary of the knot complements as well as  $Q$ . Write  $\Gamma_n$  the suture consists of two curves of slope  $-n$ . We have a graded version of by-pass exact triangles (3.9) for even  $n$  as well as (3.8) for odd  $n$ .

A simple case to analyze is when  $m < 0$ . For the knot  $K_m$  with  $m < 0$ , take  $n = 2$  in (3.8); we have Figure 8.

Here, as in [26, Section 4], the top and bottom nonvanishing grading of  $\text{SHG}(-S^3(K_m), -\Gamma_n)$  can be computed via sutured manifold decomposition and coincide with the top and bottom nonvanishing grading of  $\text{SHG}(-S^3(K_m), -\Gamma_\mu)$ .

From the graded exact triangles on the rows of the table and an extra exact triangle (3.2), we know that

$$b \geq 1 - m, \quad c \geq a + m, \quad b + c \leq a + 1.$$

Hence, the only possibility is  $b = 1 - m, c = a + m$ . Now take  $n = 3$  in (3.9); we have Figure 9.

Here,  $\underline{\text{SHG}}(-S^3(K_m), -\Gamma_3)$  can be computed by taking  $k = 1$  (note  $g = g(K_m) = 1$ ) in (3.5). We know from [26, Section 5.2] that

$$\underline{\text{KHG}}^-(-S^3, K_m, i) \cong \underline{\text{SHG}}(-S^3(K_m), -\Gamma_3, S_{m,3}, i + 1)$$

for  $i = 1, 0$ , and  $-1$ , and the  $U$  maps on  $\underline{\text{KHG}}^-(-S^3, K_m, i)$  for  $i = 1$  and  $2$  coincide with the maps  $\psi_{+,2}^i$  as in Figure 9. From the exactness, we know that  $U$  map is actually zero at grading 1 and has a kernel of rank  $-m$  at grading 0. Hence, we conclude the following.

**Proposition 6.6.** *Suppose  $m \leq 0$  and the knot  $K_m$  is described as above. Then,*

$$\underline{\text{KHG}}^-(-S^3, K_m) \cong \mathcal{R}[U]_0 \oplus (\mathcal{R}_1)^{-m} \oplus (\mathcal{R}_0)^{-m},$$

and hence,  $\tau_G(K_m) = 0$ .

To compute  $\underline{\text{KHG}}^-$  of  $K_m$  for  $m > 0$ , we first deal with the case  $m = 1$ . Now  $K_1$  is a right-handed trefoil, which has  $tb(K_1) = 1$ , and hence, from Lemma 3.3, we know that

$$\text{rk}_{\mathcal{R}} \underline{\text{SHG}}(-S^3(K_1), -\Gamma_1) = \text{rk}_{\mathcal{R}} \underline{\text{SHG}}(-S^3(K_1), -\Gamma_0) + 1.$$

Now let us compute  $\underline{\text{SHG}}(-S^3(K_1), -\Gamma_0)$ . Pick  $S_0$  to be a genus 1 Seifert surface of  $K$  so that  $S_0$  is disjoint from  $\Gamma_0$ . We can use the surface  $S_0^-$ , a negative stabilization of  $S_0$  as in [26, Definition 3.1] to construct a grading on  $\underline{\text{SHG}}(-S^3(K_1), -\Gamma_0)$ . From the construction of grading and the adjunction inequality, there could only be three nonvanishing grading  $-1, 0$ , and  $1$ . For the grading 1 part, we can apply [26, Lemma 3.2 and Lemma 4.2] and get

$$\underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, S_0^-, 1) \cong \underline{\text{SHG}}(M', \gamma'),$$

where the balanced sutured manifold  $(M', \gamma')$  is obtained from  $(-S^3(K), -\Gamma_0)$  by a (sutured manifold) decomposition along the surface  $S_0$ . Since  $K$  is a fibred knot, the underlining manifold  $M'$  is just a product  $[-1, 1] \times S_0$ . The suture  $\gamma'$  is not just  $\{0\} \times \partial S$  but is actually three parallel copies of  $\{0\} \times \partial S$  on  $[-1, 1] \times \partial S$ . We can find an annulus  $A \subset [-1, 1] \times \partial S$  that contains the suture  $\gamma'$ . Then, we can push the interior of  $A$  into the interior of  $S \times [-1, 1]$  and get a properly embedded surface. If we further decompose  $(M', \gamma')$  along (the pushed off of)  $A$ , then we get a disjoint union of a product balanced sutured manifold  $(S \times [-1, 1], \partial S \times \{0\})$  with a solid torus with four longitudes as the suture. The sutured monopole and instanton Floer homologies of the first are both of rank 1 and the second of rank 2, as in [19] and [24]. Hence, we conclude

$$\underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, S_0^-, 1) \cong \mathcal{R}^2.$$

For the other two gradings, note that from the grading shifting property in [26, Proposition 4.9], we have

$$\begin{aligned} \underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, S_0^-, i) &= \underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, S_0^+, i - 1) \\ &= \underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, (-S_0)^-, 1 - i). \end{aligned}$$

gr	$\Gamma_\mu \xrightarrow{\psi_{+,1}^\mu} \Gamma_1 \xrightarrow{\psi_{+,2}^1} \Gamma_2$	$\Gamma_\mu \xrightarrow{\psi_{-,1}^\mu} \Gamma_1 \xrightarrow{\psi_{-,2}^1} \Gamma_2$
2		$\mathcal{R} \quad \mathcal{R}$
1	$\mathcal{R} \quad \quad \quad 1$	$\mathcal{R} \quad \mathcal{R}^3 \quad \mathcal{R}^b$
0	$\mathcal{R} \quad \mathcal{R} \quad \mathcal{R}^b$	$\mathcal{R} \quad \mathcal{R} \quad \mathcal{R}^c$
-1	$\mathcal{R} \quad \mathcal{R}^3 \quad \mathcal{R}^c$	$\mathcal{R} \quad \quad \quad \mathcal{R}$
-2	$\quad \quad \quad \mathcal{R} \quad \mathcal{R}$	

FIGURE 10 The map  $\psi_{+,2}^1$  (on the left) and  $\psi_{-,2}^1$  (on the right) for  $K_1$ .

The second equality follows from the basic observation that if we reverse the orientation of the surface  $S_0^+$ , then we get  $(-S_0)^-$ . Hence,

$$\underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, S_0^-, -1) = \underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, (-S_0)^-, 2) = 0$$

by the adjunction inequality and

$$\underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, S_0^-, 0) = \underline{\text{SHG}}(-S^3(K_1), -\Gamma_0, (-S_0)^-, 1) \cong \mathcal{R}^2.$$

by the same argument as above. Thus, as a conclusion,

$$\underline{\text{SHG}}(-S^3(K_1), -\Gamma_1) \cong \mathcal{R}^5.$$

Similarly, there are only three possible nonvanishing gradings  $-1, 0, 1$ . We have already known that the homology at top and bottom gradings are of rank 1 each, so the middle grading has rank 3. Let  $n = 2$  in (3.8); we have Figure 10.

From the exactness, we know that  $b = c = 2$ . The rest of the computation is straightforward and we conclude that

$$\underline{\text{KHG}}^-(-S^3, K_1) \cong \mathcal{R}[U]_1 \oplus \mathcal{R}_0.$$

Now we have the map

$$C_{1,h,1} : \underline{\text{SHG}}(-S^3(K_1), -\Gamma_1) \rightarrow \underline{\text{SHG}}(-S^3(1), \delta)$$

and by the description of  $\underline{\text{KHG}}^-(-S^3, K_1)$  above, Proposition 3.17, and the fact that  $C_{1,h,n}$  commutes with  $\psi_{-,n}$  (Claim 1 in the proof of Proposition 1.12), we know that

$$C_{1,h,1} : \underline{\text{SHG}}(-S^3(K_1), -\Gamma_1, 1) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta)$$

is surjective, and, since  $\underline{\text{SHG}}(-S^3(K_1), -\Gamma_1, 1)$  has rank 1, it is actually an isomorphism (for the monopole case, the argument is essentially the same as in the proof of Proposition 2.6). Now we go back to the surgery exact triangle in (6.2), which corresponds to surgeries on the curve  $\zeta \subset$

Int  $S^3(K_m)$ . Since  $\zeta$  is disjoint from the boundary, and as above, disjoint from all Seifert surfaces  $S_{m,n}$ , we have the following exact triangle for any  $m$  and  $n$  (where we again omit the surfaces):

$$\begin{array}{ccc}
 \underline{\text{SHG}}(-S^3(K_m), -\Gamma_n, i) & \xrightarrow{\quad\quad\quad} & \underline{\text{SHG}}(-S^3(K_{m+1}), -\Gamma_n, i) \\
 & \searrow \quad \swarrow & \\
 & \underline{\text{SHG}}(-Q, -\Gamma_n, i) &
 \end{array} \tag{6.7}$$

There are contact 2-handle attaching maps

$$C_{m,h,n} : \underline{\text{SHG}}(-S^3(K_m), -\Gamma_n) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta),$$

where the contact 2-handle is attached along a meridional curve on the knot complements. We can attach a contact 2-handle along the same curve on the boundary of  $Q$ , and the handle attaching maps commute with the maps in the exact triangle (6.7). Thus, we have a diagram:

$$\begin{array}{ccccc}
 \underline{\text{SHG}}(-S^3(K_m), -\Gamma_n, i) & \xrightarrow{\quad\quad\quad} & \underline{\text{SHG}}(-S^3(K_{m+1}), -\Gamma_n, i) & & \\
 \downarrow C_{m,h,n} & \swarrow & \tau_{m,n,i} \swarrow & \searrow C_{m+1,h,n} & \\
 & \underline{\text{SHG}}(-Q, -\Gamma_n, i) & & & \\
 \downarrow C_{m,h,n} & \xrightarrow{\phi_\infty} & \underline{\text{SHG}}(-S^3(1), -\delta) & \xrightarrow{\quad\quad\quad} & \underline{\text{SHG}}(-S^3(1), -\delta) \\
 & \swarrow \phi_0 & \downarrow C_{Q,h,n} & \swarrow \phi_1 & \\
 & \underline{\text{SHG}}(-S^2 \times S^1(1), -\delta) & & &
 \end{array} \tag{6.8}$$

Here,  $S^2 \times S^1$  is obtained from  $S^3$  by performing a 0-surgery along the unknot  $\zeta$ . The balanced sutured manifold  $(S^2 \times S^1(1), \delta)$  is obtained from  $S^2 \times S^1$  by removing a 3-ball and assigning a connected simple closed curve on the spherical boundary as the suture. Its sutured monopole and instanton Floer homologies are computed in [4] and [24] and are both of rank 2. Thus, the exactness tells us that  $\phi_\infty = 0$ ,  $\phi_1$  is injective, and  $\phi_0$  is surjective.

Now take  $m = 0, n = 1$ , and  $i = 1$ , we know that

$$\underline{\text{SHG}}(-Q, -\Gamma_1, S_1, 1) \cong \underline{\text{SHG}}(-S^3(K_1), -\Gamma_1, S_{1,1}, 1) \cong \mathcal{R},$$

and  $C_{Q,h,n}$  is injective. Then, take  $m$  to be an arbitrary nonnegative integer and  $n = 1, i = 1$  in (6.8). From (6.5), we know that

$$\underline{\text{SHG}}(-S^3(K_m), -\Gamma_\mu, S_{m,\mu}, 1) \cong \mathcal{R}^m.$$

By performing sutured manifold decompositions along  $S_{m,n}$  and applying [26, Lemma 4.2], we know that

$$\underline{\text{SHG}}(-S^3(K_m), -\Gamma_1, S_{m,1}, 1) \cong \underline{\text{SHG}}(-S^3(K_m), -\Gamma_\mu, S_{m,\mu}, 1) \cong \mathcal{R}^m.$$



$$\begin{array}{c|ccc}
 g\Gamma & \Gamma_\mu & \xrightarrow{\psi_{+,1}^\mu} & \Gamma_1 & \xrightarrow{\psi_{+,2}^1} & \Gamma_2 \\
 1 & \mathcal{R}^m & & & & \mathcal{R}^m \\
 0 & \mathcal{R}^{2m-1} & & \mathcal{R}^m & \xrightarrow{(\psi_{+,2}^1)_0} & \mathcal{R}^b \\
 -1 & \mathcal{R}^m & & \mathcal{R}^a & & \mathcal{R}^c \\
 -2 & & & \mathcal{R}^m & & \mathcal{R}^m
 \end{array}$$

FIGURE 11 The map  $\psi_{+,2}^1$  for  $K_m$ .

Recall from above discussions, we have

$$\underline{\text{SHG}}(-Q, -\Gamma_1, S_1, 1) \cong \mathcal{R},$$

so in the exact triangle (6.8), we know that  $\tau_{m,1,1}$  is surjective. Then, we can use the commutativity part of (6.8) and conclude that

$$C_{m+1,h,n} : \underline{\text{SHG}}(-S^3(K_{m+1}), -\Gamma_1, S_{m,1}, 1) \rightarrow \underline{\text{SHG}}(-S^3(1), -\delta)$$

is surjective. From the fact that  $\psi_{\pm,n+1}^n$  commutes with  $C_{h,n}$  as in Claims 1 and 2 in the proof of Proposition 1.12, we know that this surjectivity means that the unique  $U$  tower in  $\underline{\text{KHG}}^-(-S^3, K_m, p_m)$  starts at grading 1:

$$\tau_G(K_m) = 1$$

for  $m > 0$ .

Take  $n = 2$  in (3.8); then we have Figure 11.

The fact that  $\tau_G(K_m) = 1$  means that  $(\psi_{+,2}^1)_0 \neq 0$ , as  $(\psi_{+,2}^1)_0$  corresponds to the  $U$  map at grading 1 part of  $\underline{\text{KHG}}^-(-S^3, K_m, p_m)$ . Thus, from the exactness, we know that

$$b \geq m + 1, c \geq a - m.$$

From the exact triangle (3.2), we know that

$$b + c \leq a + 1$$

and hence  $b = m + 1, c = a - m$ . Thus, we conclude the following.

**Proposition 6.9.** *Suppose  $m > 0$  and  $K_m$  is as above. Then*

$$\underline{\text{KHG}}^-(-S^3, K_m) \cong \mathcal{R}[U]_1 \oplus (\mathcal{R}_1)^{m-1} \oplus (\mathcal{R}_0)^m,$$

and hence,  $\tau_G(K_m) = 1$ .

$$\begin{array}{c|ccc}
 \text{gr} & & \Gamma_\mu \xrightarrow{\psi_{+,1}^\mu} & \Gamma_1 \xrightarrow{\psi_{+,2}^1} & \Gamma_2 \\
 1 & & \mathcal{R} & & \mathcal{R}^m \\
 0 & & \mathcal{R} & \mathcal{R}^m \xrightarrow{(\psi_{+,2}^1)_0} & \mathcal{R}^b \\
 -1 & & \mathcal{R}^m & \mathcal{R}^a & \mathcal{R}^c \\
 -2 & & & \mathcal{R} & \mathcal{R}
 \end{array}$$

FIGURE 12 The map  $\psi_{+,2}^1$  for  $\bar{K}_1$ .

We could also compute the  $\text{KHG}^-$  of the knots  $\bar{K}_m$ , the mirror image of  $K_m$ . For  $m \leq 0$ , the computation is exactly the same as before, and we conclude:

**Proposition 6.10.** *Suppose  $m \leq 0$  and the knot  $\bar{K}_m$  is as above. Then*

$$\text{KHG}^-(-S^3, \bar{K}_m) \cong \mathcal{R}[U]_0 \oplus (\mathcal{R}_1)^{-m} \oplus (\mathcal{R}_0)^{-m},$$

and hence,  $\tau_G(\bar{K}_m) = 0$ .

For  $m > 0$ , we have a diagram similar to (6.8), as follows.

$$\begin{array}{ccc}
 \text{SHG}(-S^3(\bar{K}_{m+1}), -\Gamma_n, i) & \xrightarrow{\hspace{10em}} & \text{SHG}(-S^3(\bar{K}_m), -\Gamma_n, i) \\
 & \swarrow & \searrow^{\tau_{m,n,i}} \\
 & \text{SHG}(-Q, -\Gamma_n, i) &
 \end{array} \tag{6.11}$$

Let us first compute the case  $m = 1$ , when  $\bar{K}_m$  is the left-handed trefoil. In this case, take  $n = 2$  in (3.8); then we get Figure 12.

The left-handed trefoil is not right veering in the sense of [8], so from their discussion, we conclude that  $(\psi_{+,2}^1)_0 = 0$ . (This is how they prove that the second top grading of the instanton knot Floer homology of a non-right-veering knot is nontrivial. Though they only work in the instanton case, the monopole case is exactly the same.) Thus, we conclude that  $b = 0$ .

In (6.11), let  $m = 0, n = 2, i = 0$ . Note the grading is induced by  $S_{m,2}^+$ , that is, a Seifert surface of the knot  $\bar{K}_m$  that intersects the suture  $\Gamma_2$  transversely at four points and with a positive stabilization. With the gradings as in the first row of (3.9), we have

$$\text{SHG}(-S^3(\bar{K}_1), -\Gamma_2, S_{1,2}^{\tau(2)}, 0) = \mathcal{R}^b = 0, \quad \text{SHG}(-S^3(\bar{K}_0), -\Gamma_2, S_{0,2}^{\tau(2)}, 0) \cong \mathcal{R}.$$

Here,  $\bar{K}_0$  is the unknot and we have computed the  $\text{SHG}$  of a solid torus with any possible sutures in [26, Section 4.4]. Thus, we conclude that

$$\text{SHG}(-Q, -\Gamma_2, S_2^{\tau(2)}, 0) \cong \mathcal{R}.$$

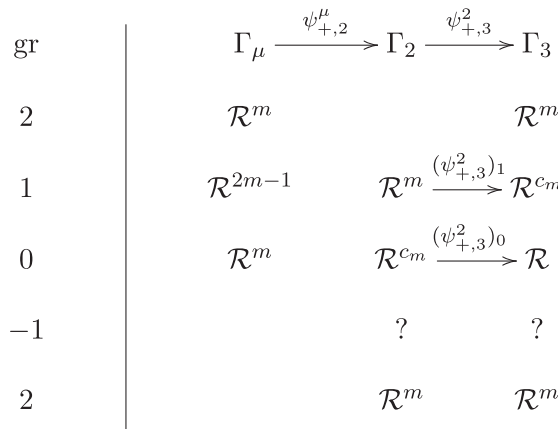


FIGURE 13 The map  $\psi_{+,3}^2$  for  $K_m$ .

Use the exactness and the induction, then we have

$$\underline{\text{SHG}}(-S^3(\bar{K}_m), \Gamma_2, 0) \cong \mathcal{R}^{c_m}, \quad c_m \leq m - 1.$$

For the knot  $\bar{K}_m$ , take  $n = 3$  in (3.9); then we have Figure 13.

Thus, we conclude from the exactness that  $c_m = m - 1$ ,  $(\psi_{+,3}^2)_1 = 0$ , and  $(\psi_{+,3}^2)_0 = 0$ . As above, the two maps  $(\psi_{+,3}^2)_1$  and  $(\psi_{+,3}^2)_0$  correspond to the  $U$  maps of  $\underline{\text{KHG}}^-(-S^3, \bar{K}_m)$  at grading 1 and 0, respectively. Hence, we conclude:

**Proposition 6.12.** *Suppose  $m > 0$  and the knot  $\bar{K}_m$  is as above. Then,*

$$\underline{\text{KHG}}^-(-S^3, \bar{K}_m) \cong \mathcal{R}[U]_{-1} \oplus (\mathcal{R}_1)^m \oplus (\mathcal{R}_0)^{m-1},$$

and hence,  $\tau_G(\bar{K}_m) = 0$ .

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## REFERENCES

1. A. Alishahi and E. Eftekhary, *Knot Floer homology and the unknotting number*, *Geom. Topol.* **24** (2020), no. 5, 2435–2469. MR 4194296
2. J. A. Baldwin and S. Sivek, *Naturality in sutured monopole and instanton homology*, *J. Differential Geom.* **100** (2015), no. 3, 395–480. MR 3352794
3. J. A. Baldwin and S. Sivek, *A contact invariant in sutured monopole homology*, *Forum Math. Sigma* **4** (2016), e12, 82. MR 3510331
4. J. A. Baldwin and S. Sivek, *Instanton Floer homology and contact structures*, *Selecta Math. (N.S.)* **22** (2016), no. 2, 939–978. MR 3477339
5. J. A. Baldwin and S. Sivek, *Invariants of Legendrian and transverse knots in monopole knot homology*, *J. Symplectic Geom.* **16** (2018), no. 4, 959–1000. MR 3917725
6. J. A. Baldwin and S. Sivek, *Framed instanton homology and concordance*, *J. Topol.* **14** (2021), no. 4, 1113–1175.
7. J. A. Baldwin and S. Sivek, *On the equivalence of contact invariants in sutured Floer homology theories*, *Geom. Topol.* **25** (2021), no. 3, 1087–1164. MR 4268162
8. J. A. Baldwin and S. Sivek, *Khovanov homology detects the trefoils*, *Duke Math. J.* **171** (2022), no. 4, 885–956. MR 4393789
9. J. A. Baldwin and S. Sivek, *Instantons and L-space surgeries*, *J. Eur. Math. Soc. (JEMS)* **25** (2023), no. 10, 4033–4122. MR 4634689
10. A. Cavallo, *The concordance invariant tau in link grid homology*, *Algebr. Geom. Topol.* **18** (2018), no. 4, 1917–1951. MR 3797061
11. V. Colin, P. Ghiggini, and K. Honda, *The equivalence of Heegaard Floer homology and embedded contact homology III: from hat to plus*, Preprint, version 2, arXiv:1208.1526, 2017.
12. A. Daemi, T. Lidman, D. S. Vela-Vick, and C.-M. Michael Wong, *Ribbon homology cobordisms*, *Adv. Math.* **408** (2022), Paper No. 108580, 68 pages. MR 4467148
13. J. B. Etnyre, D. S. Vela-Vick, and R. Zarev, *Sutured Floer homology and invariants of Legendrian and transverse knots*, *Geom. Topol.* **21** (2017), no. 3, 1469–1582. MR 3650078
14. S. Ghosh and Z. Li, *Decomposing sutured monopole and instanton Floer homologies*, *Selecta Math. (N.S.)* **29** (2023), no. 3, Paper No. 40, 60. MR 4587642
15. M. Hedden and K. Raoux, *Knot Floer homology and relative adjunction inequalities*, *Selecta Math. (N.S.)* **29** (2023), no. 1, Paper No. 7, 48. MR 4507976
16. A. Juhász, M. Miller, and I. Zemke, *Knot cobordisms, bridge index, and torsion in Floer homology*, *J. Topol.* **13** (2020), no. 4, 1701–1724. MR 4186142
17. Ç. Kutluhan, Y.-J. Lee, and C. H. Taubes, *HF=HM, V: Seiberg-Witten Floer homology and handle additions*, *Geom. Topol.* **24** (2020), no. 7, 3471–3748. MR 4194309
18. P. Kronheimer and T. Mrowka, *Monopoles and three-manifolds*, *New Mathematical Monographs*, vol. 10, Cambridge University Press, Cambridge, 2007. MR 2388043
19. P. Kronheimer and T. Mrowka, *Instanton Floer homology and the Alexander polynomial*, *Algebr. Geom. Topol.* **10** (2010), no. 3, 1715–1738. MR 2683750
20. P. Kronheimer and T. Mrowka, *Knots, sutures, and excision*, *J. Differential Geom.* **84** (2010), no. 2, 301–364. MR 2652464
21. P. Kronheimer, T. Mrowka, P. Ozsváth, and Z. Szabó, *Monopoles and lens space surgeries*, *Ann. of Math. (2)* **165** (2007), no. 2, 457–546. MR 2299739
22. Y. Lekili, *Heegaard-Floer homology of broken fibrations over the circle*, *Adv. Math.* **244** (2013), 268–302. MR 3077873
23. L. Lewark, *Rasmussen’s spectral sequences and the  $\mathfrak{sl}_N$ -concordance invariants*, *Adv. Math.* **260** (2014), 59–83. MR 3209349

24. Z. Li, *Contact structures, excisions and sutured monopole Floer homology*, *Algebr. Geom. Topol.* **20** (2020), no. 5, 2553–2588. MR 4171573
25. Z. Li, *Gluing maps and cobordism maps in sutured monopole and instanton Floer theories*, *Algebr. Geom. Topol.* **21** (2021), no. 6, 3019–3071. MR 4344877
26. Z. Li, *Knot homologies in monopole and instanton theories via sutures*, *J. Symplectic Geom.* **19** (2021), no. 6, 1339–1420. MR 4450625
27. C. Livingston, *Computations of the Ozsváth-Szabó knot concordance invariant*, *Geom. Topol.* **8** (2004), 735–742. MR 2057779
28. Z. Li and F. Ye, *Instanton Floer homology, sutures, and Heegaard diagrams*, *J. Topol.* **15** (2022), no. 1, 39–107. MR 4407491
29. P. Ozsváth and Z. Szabó, *Heegaard Floer homology and alternating knots*, *Geom. Topol.* **7** (2003), 225–254. MR 1988285
30. P. Ozsváth and Z. Szabó, *Knot Floer homology and the four-ball genus*, *Geom. Topol.* **7** (2003), 615–639. MR 2026543
31. P. Ozsváth and Z. Szabó, *Holomorphic disks and knot invariants*, *Adv. Math.* **186** (2004), no. 1, 58–116. MR 2065507
32. P. Ozsváth and Z. Szabó, *On the skein exact sequence for knot Floer homology*, Preprint, version 1, arXiv:0707.1165, 2007.
33. P. S. Ozsváth, A. I. Stipsicz, and Z. Szabó, *Grid homology for knots and links*, *Mathematical Surveys and Monographs*, vol. 208, American Mathematical Society, Providence, RI, 2015. MR 3381987
34. B. Ozbagci, *Contact handle decompositions*, *Topology Appl.* **158** (2011), no. 5, 718–727. MR 2774055
35. O. Plamenevskaya, *Bounds for the Thurston-Bennequin number from Floer homology*, *Algebr. Geom. Topol.* **4** (2004), 399–406. MR 2077671
36. S. Sivek, *Monopole Floer homology and Legendrian knots*, *Geom. Topol.* **16** (2012), no. 2, 751–779. MR 2928982
37. C. H. Taubes, *Embedded contact homology and Seiberg-Witten Floer cohomology V*, *Geom. Topol.* **14** (2010), no. 5, 2961–3000. MR 2746727
38. I. Zemke, *Knot Floer homology obstructs ribbon concordance*, *Ann. of Math. (2)* **190** (2019), no. 3, 931–947. MR 4024565