CLIFFORD ALGEBRA AND THE CONSTRUCTION OF A THEORY OF ELEMENTARY PARTICLE FIELDS

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(Received: April, 1994)

Abstract. We present a review of the development of a theory of elementary particle fields. Instead of a mathematical model based in a mathematical group, we show that we can actually develop a theory which, as a consequence, points to a mathematical structure. Clifford algebra is used as the basic tool.

We show that an extended representation of the Multivector Clifford algebra allows, first, a series of factorizations of the Laplacian operator, and, second, generates 3 families of elementary particles with the experimentally observed lepton and quark content for each family and the experimentally observed electroweak color interactions and other related properties. The factorizations $\nabla^2 = (\Gamma^{\mu}_{(f)}\partial^{(d)}_{\mu})^*(\Gamma^{\nu}_{(f)}\partial^{(d)}_{\mu})$ and the related Dirac-like equations

$$\Gamma^{\mu}_{(f)}\partial^{(d)}_{\mu}\psi_{(d,f)} = 0$$

are studied, its symmetries given. The $\Gamma^{\mu}_{(f)}$ generate the 3 families, the $\partial^{(d)}_{\mu}$ generate the observed lepton and quark content of the families.

In contrast to the usual approach to the standard model the properties for the different fields of the model are consequences of the relative properties of the equations, among themselves and in relation to spacetime, and therefore, they do not need to be postulates of the theory.

1. Introduction

In the years 1980-1983 it became apparent that besides the accepted SU(3)⊗SU(2)⊗U(1) structures of the elementary fields corresponding to a family of elementary particles, there were 3 possible families, and perhaps more, each one repeating the group structure of the fundamental family. All the experimental analysis, in the decade elapsed since that time, confirms that scheme. The construction of the basic field as composite of other, more fundamental fields, pointed to the need of combining the gauge, or interaction, fields with the study of the basic fields and moreover to the need of incorporating basic concepts like "confinement" and "colorless composites" together with the consequences of the basic group. Of paramount importance is violation of parity in weak interactions, the massless character of neutrinos and their associated left (right) handedness.

For us all the phenomenological concepts and models fit together in such a way that, if an appropriate mathematical framework is used, we could develop a theory of elementary particles and their interaction fields which should then be the foundation of the set of phenomenological laws.

Here we show that this task is now possible and that a useful mathematical tool which reduces the need for additional, or ad hoc definitions, is the Clifford algebra approach to the mathematical formulation of spacetime and the basic fields.

In fact a usual approach in mathematical physics is to use the concept of spacetime as a frame of reference for the description of the matter and their interaction fields. Spacetime, having a multivector structure and containing a spinor (and dual spinor) space, not only describes our perception of the physical nature but is also a powerful mathematical tool. Adopting spacetime as a basic frame of reference for physical phenomena should imply that its structure and symmetries corresponds to the observed characteristics of the matter and interaction fields. If a contradiction or insuficience were found a wider reference frame should then be constructed and used, but this does not seem to be the actual case.

We have several motivations for the analysis presented here which follow from studies we have performed in the last 14 years [Keller 1991]:

1. Given spacetime and its multivector Clifford algebra, $C\ell_{1,3}$ or its complexification $C\ell_{0,5}$, we can ask: which fields may exist in it obeying the Klein-Gordon wave equation $\Delta\psi = -a^2\psi$, with: $(a^2 \geq 0)$? Introducing the fields from first principles and guiding our analysis of thore fields (to make connection with experiment) from the accepted form of the rtandard phenomenology.
2. In the standard model, if we consider the fields that may exist in spacetime according to 1): do we need to add isospace to spacetime? After all the natural tangent space T_M to spacetime $R^{1(3}$ contains 16 elements and the T_M to the complex spacetime $C\ell_{0,5}$ contains 32 elements.

The elements γ^A of $R_{1,3}$ are the dimentionless Grassmann numbers

$$\begin{gathered}
\mathbf{1},\ \gamma_\mu,\ \gamma_\mu\gamma_\nu = g_{\mu\nu} + \gamma_{\mu\nu},\ g_{\mu\nu} = diag(\mathbf{1},-\mathbf{1},-\mathbf{1},-\mathbf{1}),\\
\gamma_{\mu\nu} = -\gamma_{\nu\mu}, \gamma_\rho\gamma_{\mu\nu} = g_{\rho\mu}\gamma_\nu - g_{\rho\nu}\gamma_\mu + \gamma_{\rho\mu\nu}\\
\text{and}\quad \gamma_\lambda\gamma_{\mu\nu\rho} = g_{\lambda\mu}\gamma_{\nu\rho} - g_{\lambda\nu}\gamma_{\mu\rho} + g_{\lambda\rho}\gamma_{\mu\nu} + \gamma_5 \quad \text{or} \quad \gamma_5 = \gamma_{0123},\\
\text{all } \{\mu,\nu,\lambda,\rho\} = 0,1,2,3.
\end{gathered}$$

The complexification of $C\ell_{1,3} \to C\ell_{0,5}$ can be denoted by $\{\gamma^A + i\gamma^A;\quad \gamma^A \subset C\ell_{1,3}\}$. All multivectors act as operators among themselves and on the ψ's describing the matter and interaction fields, the definitions are such that $\gamma^0, i\gamma^{12}$ and $i\gamma^5$ are hermitian. Multivectors are defined from the vectors γ^μ through their Grassmann outer product $\gamma^{\mu\nu\cdots} = \gamma^\mu \wedge \gamma^\nu \wedge \ldots$ (See Chisholm and Common (1986) or Micali, Boudet and Helmstetter (1992)).

We have discussed elsewhere the use of multivectors as generators of Lie groups, see for example [Keller and Rodríguez-Romo 1990, 1991a] where we analyse the construction within $C\ell_{0,5}$ of frequently used groups as for example S0(2,3), SU(3) or SU(2). Also the integration of spinors and multivectors into a geometric superalgebra [Keller and Rodríguez 1992].

Here we show that the basic phenomenology, and the essential lefthandedness of the neutrino, can all be combined in a generalization of the Dirac equation and the postulate that all physical possibilites implied should be included.

2. Chiral symmetry in spacetime

We assume that a local observer describes spacetime by an orthonormal tetrad a) $(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = 1$. In this frame b) $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is both the duality transform operator and the pseudoscalar $(\gamma^5)^2 = -1$. It is important that if another observer uses a different coordinate system, related by a Lorentz transformation L, the fundamental properties $(i\gamma^5)^2 = 1$ and $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ are also preserved, together with a).

The handedness operator $H = i\gamma^5$ can be used to construct the chirality projectors P_R and P_L:

$$P_R + P_L = 1,\ P_R P_R = P_R;\ P_L P_L = P_L,\ P_R P_L = P_L P_R = 0,$$

where $P_R = \frac{1}{2}(1 + i\gamma^5)$, $P_L = \frac{1}{2}(1 - i\gamma^5)$ or, as discussed here below,

$$P_{R,L} = \frac{1}{2}(1 \pm H).$$

If a coordinate transformation $\gamma^5 \to (\gamma^5)'$ is allowed where a), and consequently b), is not preserved (that is if the determinant ς of the transformation is not $\varsigma = +1$) then $H \neq i(\gamma^5)'$ showing that a chirality operator $H = i(\gamma^5)'/\varsigma$,with $H^2 = 1$ in all frames has to be used.

H is in fact an invariant dimentionless quantity, it obeys $H^2 = 1$ in all frames of reference. Even if the handedness of the frame F' is changed relative to frame F' because $(\gamma^5)' = \varsigma\gamma^5$. Given that $g' = \varsigma^2 g$ and then the effect of the sign of ς is lost, we cannot define H in terms of $\sqrt{|g|}$, we have to define it in relation to the "handedness (F)" of a given frame F and then the use of ς ensures that in a change to F' we obtain handedness(F') = sign(ς)• handedness(F). In terms of this, relative, handedness definition we could write

$$H = \text{handedness}\,(F) i\gamma^5/\sqrt{|g|} \tag{A1}$$

which is equivalent to our definition $H = i\gamma^5$ when and only when the conditions mentioned in the text are satisfied. γ^5 and $\sqrt{|g|}$ could both, simultaneously, be considered to have (length)^4 dimention and H still would be a dimentionless quantity.

Here we will assume $H = i\gamma^5$, because of the restriction a) and the assumption that we have selected a "right" handeded frame of reference. The P_R and P_L can better be considered numbers of a new mathematical field, with basis **1** and **H**, in an hypercomplexification of the Clifford algebra. **H**($= H$) is coordinate invariant.

3. Chiral symmetry theory of elementary particles

Using spinors, vectors and multivectors [Fock and Ivanenko (1929), see also Keller 1991, Keller and Rodríguez-Romo 1991b, Hestenes 1966, Casanova 1976, Keller and Viniegra 1992, Keller and Rodríguez 1992] we will now construct a theory for lepton and quark fields using the possible multivector generalization of the Dirac factorization of the Laplacian (d'Alembert operator $\nabla^2 = \partial^\mu\partial_\mu$). We start, as a guiding concept, by considering the Klein-Gordon equation operator and its factorization

$$(\partial^\mu\partial_\mu + m^2) \quad = (D^\dagger + mi)(D - mi) \tag{1}$$

which requires that

$$-D^\dagger m + mD = 0 \qquad \text{and} \qquad D^\dagger D = \partial^\mu\partial_\mu = \nabla^2 \tag{2}$$

we can have then a set of choices, either

1. any value of m and $D^\dagger = D$ (the standard Dirac operator D_0), or
2. for the case where m = 0 the possibility $D^\dagger \neq D$ also becomes acceptable. Here we will use the field generated by **1** and **H**.

In multivector algebra the Dirac operator is the standard vector operator (using the vectors γ^μ) $D \to D_0 = \gamma^\mu\partial_\mu$. (Sometimes $D \to \gamma^0 D_0 = \gamma^{0\mu}\partial_\mu$ is used).

The basic requirement $D^\dagger D = DD^\dagger = \partial^\mu\partial_\mu$ limits the choices of D, it can be taken to be written in the Lorentz invariant form

$$D_{(d,f)} = \Gamma^\mu_{(f)}\partial^{(d)}_\mu, \quad \text{also} \quad D_{(d,f)}\psi_{(d,f)} = 0, \tag{3}$$

in order to show the relation to the Dirac's original factorization in the simplest possible form. Here the $\Gamma^{\mu}_{(f)}$ are operators on the ψ which can be represented by generalized Dirac γ^{μ} matrices, see below. The limitation is so strong that the only possible choice is where the multivector $i\gamma^5$ (or the invariant $\mathbf{H}$), which has the same action on all γ^{μ}, that is $i\gamma^5\gamma^{\mu} = -\gamma^{\mu}i\gamma^5$, is used see [Keller 1982, 1984, 1986 and 1991, pag. 158 and following], this is particularly interesting because chirality comes naturally into the theory.

We construct the following (Lorentz invariant but coordinate system dependent) operator

$$\partial_{\mu}^{(d)} = \left\{ \mathbf{1} \cos(n + t_{\mu}^{d})\frac{\pi}{2} + \mathbf{H} \sin(n + t_{\mu}^{d})\frac{\pi}{2} \right\} \partial_{\mu} \qquad (4)$$

condition (2) requires n and t_{μ}^{d} integer and it results in the simplest multivectors. Here, to take the electron as reference we use $n = -1$.

With this choice of presentation we can have the "diagonal" structure:

$$\partial_{\mu}^{(d)} = \begin{cases} \partial_{\mu} & if \;\; n + t_{\mu}^{d} \text{ are even} \\ i\gamma^5\partial_{\mu} & if \;\; n + t_{\mu}^{d} \text{ are odd} \end{cases} \qquad (5)$$

The standard $\gamma^{\mu} = \Gamma^{\mu}_{(1)}$ matrices which correspond to an irreducible representation of $C\ell_{1,3}$ are found to be useful to write the wave equations of the first or fundamental family $(e_R^-, e_L^-, \nu_L, \{u_L, d_L; color\})$ of elementary particles. The electron requires a combination of two fields $e^- = (e_R^-, e_L^-)$ for the standard phenomenology of electroweak-color interactions.

The study of families other than the electron family suggested that, a more general, non reducible representations of $C\ell_{1,3}$, could in fact be needed. They are collectively denoted by $\Gamma^{\mu}_{(f)}$. In Clifford algebra their Lorentz transformations $\Gamma^{\mu}_{f} \rightarrow (\Gamma^{\mu}_{f})'$ do not change the $\partial^{(d)}$. From our basic postulates the Γ^{μ} can all be written as exterior products of the $\gamma^{\mu}, \gamma^5, i\gamma^5$ and $\mathbf{1}$, . A fundamental representation would be for example [see Królikowski 1990]

$$\Gamma^{\mu}_{f} = \gamma^{\mu} \otimes (\mathbf{1} \otimes \mathbf{1} \otimes \ldots)_{2(f-1)products} \qquad (6)$$

other equivalent, but different, representations, being also possible. We call these representations of the Clifford algebra "capital representation" [see Keller 1993]. The corresponding spinors would then be the, totally antisymmetric, exterior products

$$\psi_{(f)} = \psi(x) \wedge (\psi_1 \wedge \psi_2 \wedge \ldots)_{2(f-1)products}.$$

For a local theory (assumed here) the first factor $\psi(x)$ is the only one that carries spacetime position dependence.

Then the ψ_i are $2(f-1)$ non null, normalized constant Dirac spinors which correspond to extra, mathematical, internal degrees of freedom of the diracon fields. For the fundamental $s = 1/2$ fields their spin should add to zero (f odd integer). The total antisymmetry of $\psi_{(f)}$ limits the value of f to $f = 1, 2, 3$ otherwise the exterior product is null.

The degeneracy n_f of the representations of the $\Gamma^{\mu}_{(f)}$ gives statistical weight to each family: $n_1 = 1$, $n_2 = 4$ and $n_3 = 24$. This will result in factors for the terms of the mass matrix.

The elementary fields thus described are mathematically composite, but still elementary in the sense that they cannot be decomposed experimentally into their components. No size of the particle is required by the theory, they are representations of the basic elementary fermion equations, no spacetime structure is involved, there is only the mathematical complexity of the wave function. Each family has an internal relationship identical to the fundamental family $f = 1$ and the same $SU(3)_{color} \otimes SU(2) \otimes U(1)$ symmetry. No additional gauge interaction field is needed to relate the different families. They are algebraic families of otherwise structureless leptons and quarks. The algebra of the $\Gamma^{\mu}_{(f)}$ has been developed and studied by Królikowski (1990), as well as the consequences for the phenomenology of the elementary particle families.

4. Chiral geometry theory of charge isospin and color

For the quarklike diracons, an introductory analysis to study the consequences of (3), we use a reference frame F in such a way that a local reference direction is defined to be $\gamma_\ell = (\gamma_1 + \gamma_2 + \gamma_3)\sqrt{3}$ and the notation $\gamma^D_\mu \equiv i\gamma_5\gamma_\mu$ is used. Such that we can explicitly exhibit the vector-(imaginary) axial vector momentum admixture and show that it is a constant (independent of the "color" of the diracon field).

Let us write in detail the "energy momentum multivector" $\mathbf{p}$ of every diracon field d, including the different "colors" red (r), blue (b), or green (g) of the quarks, according to Table I, ($\mathbf{p}_{dir} = p^0\gamma_0 + p^\ell\gamma^{dir}_\ell$):

$$
\begin{array}{lll}
\text{electron } e: & \mathbf{p}_e & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2 + \gamma_3)/\sqrt{3} \\
 & \mathbf{p}_{\bar{u}}^r & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2 + \gamma_3)/\sqrt{3} \\
\text{quark } \bar{u}: & \mathbf{p}_{\bar{u}}^b & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2^D + \gamma_3)/\sqrt{3} \\
 & \mathbf{p}_{\bar{u}}^g & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2 + \gamma_3^D)/\sqrt{3} \\
 & \mathbf{p}_d^r & = p^0\gamma_0 + p^\ell(\gamma_1 + \gamma_2^D + \gamma_3^D)/\sqrt{3} \\
\text{quark } d: & \mathbf{p}_d^b & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2 + \gamma_3^D)/\sqrt{3} \\
 & \mathbf{p}_d^g & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2^D + \gamma_3)/\sqrt{3} \\
\text{neutrino } \nu: & \mathbf{p}_\nu & = p^0\gamma_0 + p^\ell(\gamma_1^D + \gamma_2^D + \gamma_3^D)/\sqrt{3}
\end{array} \tag{7}
$$

Here p^ℓ is the three-momentum and p^0 is the energy. We can see that the energy-momentum vectors are all in different phases of the $p_\mu \to p_\mu^D$ rotations, with none, one, two or three vector rotations.

Let us now consider a gauge energy-momentum vector field $A^\mu\gamma_\mu$, in the Coulomb gauge $A^0 = 0$, added to the diracon fields with coupling constant proportional to Q_e, modifying the *vector* part of the momentum, with the energy-momentum components given in the same proportion to the time part and to the spatial parts (calling $\gamma_\perp$ a vector perpendicular to the direction of motion γ_ℓ). For the electron

$$\mathbf{p}' = p^0\gamma_0 + (p^\ell + Q_e A^\ell)\gamma_\ell + Q_e A^\perp\gamma_\perp \tag{8}$$

has components

$$\text{timelike } \gamma_0 \cdot p = p^0, \text{ spacelike parallel } \gamma_\ell \cdot \mathbf{p}' = p^\ell + Q_e A^\ell,$$

$$\text{spacelike perpendicular } \gamma_\perp \cdot \mathbf{p}' = Q_e A^\perp. \tag{9}$$

All of them are scalar quantities.

However, for a $\bar{u}$ quark (taking, for example, a red quark, the result being invariant with respect to color),

$$\gamma_\ell \cdot \gamma_\ell^{\bar{u}} = \frac{1}{\sqrt{3}}(\gamma_1 + \gamma_2 + \gamma_3) \cdot \frac{1}{\sqrt{3}}(\gamma_1^D + \gamma_2 + \gamma_3) = \frac{2}{3} + \frac{1}{3}i\gamma_5 \tag{10}$$

the scalar components will be affected by a factor of $\frac{2}{3}$, and following the same procedure for a down quark, the scalar components will be affected by a factor of $\frac{1}{3}$, and for a neutrino the scalar components will be affected by a factor 0.

Then if we make the obvious definition that the scalar part of the gauge field, treated on an equal basis for the electrons and for the quarks of the neutrino, is to be considered as gauged by the electromagnetic field A, the "electric charges" have to be Q_e, $\frac{2}{3}Q_e$, $\frac{1}{3}Q_e$, and 0, respectively. The pseudoscalar (proportional to $i\gamma_5$) parts are to be treated on a different basis, and will be shown to correspond to the weak and color interactions.

In the full Lagrangian, introduced and discussed in [Keller 1991], a first term equivalent to the standard Dirac matter-field Lagrangian

$$\mathcal{L}_m = i\bar{\psi}\gamma^\mu D_\mu\psi \quad (\text{here } \partial_\mu \to D_\mu \text{after gauging}) \tag{11}$$

is to be replaced by the corresponding expression for diracons:

$$\mathcal{L}_d = i\bar{\psi}\gamma^\mu_d D_\mu\psi \tag{12}$$

It is in this term of the Lagrangian where we have to introduce an electromagnetic gauging with a coefficient e for the electron field, $2e/3$ for the (anti) up-quark field, $e/3$ for the down-quark field, and 0 for the neutrino field. Then in the gauge theory we are constructing, the charges for the $U(1)$ part of the gauge fields are the (postulated usually) integer, fractional, or zero values of the standard theory. In general our method will allow us to *develop* a gauge theory instead of postulating it as in the standard approaches. In this form we are showing the physical origin of the various couplings of the gauge fields, and the role played by $i\gamma_5$ in it, as a part of the symmetry-constrained Dirac particle theory.

For this purpose the A field discussed above will have to be enlarged and split into contributions, usually called B and W^3 in the literature, and new "charges" T^3 and Y are introduced with the standard notation

$$Q = T^3 + Y/2 \tag{13}$$

but the assignment of T^3 and Y to give our values of Q will be straightforward and its physical origin clear.

It is convenient to start with a rearrangement of the set of diracon fields in groups which will show an explicit $SU(2) \times SU(3) \subset \text{spin}(8)$ symmetry as shown in Table I on page 387.

To start, we explore the $SU(2)$ relations; for each given family we can see that the addition of a set of symmetry coefficients $\{W^-\} = (0, -1, -1, -1)$, modulus -2, to the first row produces the last row and their addition to any one of the first group of three up-quark fields produces one of the group of three down-quark fields. That is: the same chiral phase change that takes the neutrino field into a left electron field will change an up quark into a down quark. The reverse process proceeds in the corresponding way. The "neutral" interaction will arise from a change in the phase of one of the partner fields canceling that of the change of the other.

In the language of bilinear spinor operators, creation-anihilation, we could write all these processes in terms of spinors: if $\{\chi_\nu, \chi_u, \chi_d, \chi_e\} = \chi_a$ represent the neutrino, up-quark, down-quark, and electron fields, respectively, and their respective dual fields are $\{\chi^\dagger_\nu, \chi^\dagger_u, \chi^\dagger_d, \chi^\dagger_e\} = \chi^\dagger_a$, with the orthogonality condition $\chi^\dagger_a\psi_b = \delta_{ab}$, then the processes above can be described by

$$\hat{W}^- = w^-(\chi_e\chi_\nu^\dagger + \chi_d\chi_u^\dagger) \tag{14}$$

$$\hat{W}^+ = w^+(\chi_\nu\chi_e^\dagger + \chi_u\chi_d^\dagger) \tag{15}$$

and the neutral interaction (to be combined with the electromagnetic) is

$$\hat{W}^3 = w^3\tfrac{1}{2}(\hat{W}^+\hat{W}^- - \hat{W}^-\hat{W}^+) \tag{16}$$

provided that, in order to account for the spin $h/2\pi$ of the gauge fields, in all cases the spins of each spinor operator of the product are opposite, i.e., that the spinor of the electron field created is opposite to that of the neutrino field annihilated, etc. Then these processes correspond to vector interactions with total spin one, equal to the change in spin of the field during the interaction.

What we will show below is the correspondence between the interaction fields and each product of an interaction operator, written here in a formal way. We should add at this stage that, besides spin, energy-momentum is being exchanged during the interaction; for example, a photon interacting with an electron, with energy-momentum exchange q, could be written

$$\hat{A} = \sum_p \hat{\chi}_{e(p+q,\mp s\pm 1)}\hat{\chi}_e^\dagger(p,\mp s) \tag{17}$$

stating that the electromagnetic interaction annihilates an electron of momentum p and spin component s and creates an electron of momentum $p+q$ and of opposite spin.

The *color* interaction will change one of the spacelike t_i^d indexes of the quarks from the value 1 to 0 and produce a value 1 for one of the other indexes (which was zero previously), or change the axial vector momentum of two of those indexes simultaneously to a total of the eight operations $\{1 \to 2, 1 \to 3, 2 \to 3, 2 \to 1, 3 \to 1, 3 \to 2, 11 \to 22, 22 \to 33\}$, corresponding to the $SU(3)$ color symmetry; we can also write these results in a formal operator way if we add a color subindex to the quark fields; then

$$\hat{G}_{ab} = \hat{\chi}_{q,a}\hat{\chi}_{q,b}^\dagger \tag{18}$$

will correspond to a gluonic interaction changing color b into color a.

All these interactions in our diracon fields and in our chiral phase language correspond to a change in the free particle wave function

$$\psi_d = u\exp(p^d \cdot x + \phi_d^0) = u\exp(\phi_d) \tag{19}$$

with u a spinor and the de Broglie phases ϕ_d being the sum of the scalar and the pseudoscalar parts of the products of the vector x with the momenta given by equations (12). The de Broglie phases are gauged by the ϕ_d^0 which also contain scalar and pseudoscalar parts. For the leptons the de Broglie phases are

$$\phi_{\text{electron}} = p^\mu x_\mu + \phi_e^0 , \tag{20}$$

$$\phi_{\text{neutrino}} = p^0 x_0 + i\gamma_5 p^k x_k + \phi_\nu^0, \qquad k = 1,2,3 \tag{21}$$

The spinor u for the electron can be left- or right-handed, whereas for the neutrino, in order to satisfy equations (2-3), only the left-handed field is possible.

In order to preserve rotational symmetry, for each one of the quarks we need to show explicitly the gauge phase $\phi_{q,a}^0$ ensuring that the overall de Broglie phase is space-symmetric. This requires a complicated vector notation. If a space index is k (with values 1,2,3), a references space index is $r = 1,2,3$, and a color index is a or b (with values r, b, g), we have a set of three multivectors [vector $+i$ axial vector, $i = (-1)^{1/2}$],

$$e_k^a = c_k^{ar}\gamma_r; \qquad c_k^{ar} = \cos\omega_{rk}[\cos(\pi t_r^a/2) + i\gamma_5 \sin(\pi t_r^a/2)] \tag{22}$$

for each color a of a given quark, direction k in space, and quantum number t^a in Table I, for reference space direction r, this reference space direction at an angle ω_{rk} with the observer's space coordinates k. This is a more general notation than that of equation (7), where, for simplicity, the particle was taken to move in a direction with all $\cos\omega_{rv} = 1/\sqrt{3}$. The c_k^{ar} are then the sum of a scalar and (i times) a pseudoscalar.

For the purpose of our formalism we need a duality-symmetric set of coefficients b_k^{ar} such that $c_k^{ar} + b_k^{ar} = \cos\omega_{rk}$, the ordinary cosine directors (no axial vector mixing).

In terms of the multivectors (22) the de Broglie phases for the quarks are

$$\text{up quark} \qquad \phi_{u,a} = p^0 x_0 + c_k^{ar} p^k x_r + b_k^{ar} \phi^k x_r + \phi_{u,a}^0 \tag{23}$$

$$\text{down quark} \qquad \phi_{d,b} = p^0 x_0 + c_k^{br} p^k x_r + b_k^{br} \phi^k x_r + \phi_{d,b}^0 \tag{24}$$

The constants c_k^{ar} are different for up quarks and for down quarks, corresponding to the t_d^a quantum numbers.

Now, the phase angles ϕ_d^0 can either change the scalar-pseudoscalar structure of the de Broglie phases or leave them with the same structure. In the first case we have a change of the particle's nature (the resulting wave function will obey a different wave equation), and in the second case we have a type-conserving interaction. For this purpose we construct a Lagrangian which is invariant to the changes of the phase structure of the different

TABLE I. Allowed Sets of Symmetry-Constrained Quantum Numbers $\{t'^d_\mu \equiv t^d_\mu + n\}$ for Chiral Fields Corresponding to the Electron Family. Satisfying the Generalized Dirac Equation $D_{(f,d)}\psi_{(f,d)} = 0$. The quantum numbers n, t^d_μ, and operator $D_{(f,d)}$ are defined in equations (3)-(4) in the text. They correspond to the choice of e^- as reference. The charges are given by the average value $(t'_1 + t'_2 + t'_3)/3t'_0$ as described by the explanation of (72) in [Keller 1991]. The isospin pairs are connected by a change in the t'_μ such that $|t^{d'}_\mu - t^d_\mu| = (2,1,1,1) \bmod 2$, and the color triplets by a change in the t'_μ such that $t^d_\mu - t^{d'}_\mu = t^{d'}_\nu - t^d_\nu$.

t'_0	t'_1	t'_2	t'_3	Q	2I	Color	Name
-1	-1	-1	-1	-1	-1	-	electron
1	0	1	1	$+\frac{2}{3}$	1	r	
1	1	0	1	$+\frac{2}{3}$	1	b	up quark
1	1	1	0	$+\frac{2}{3}$	1	g	
-1	-1	0	0	$-\frac{1}{3}$	0	r	
-1	0	-1	0	$-\frac{1}{3}$	0	b	down quark
-1	0	0	-1	$-\frac{1}{3}$	0	g	
1	0	0	0	0	0	-	neutrino

$\phi_d = \rho^u_d \chi_\mu + \phi^0_d$ shown above. We have done this in [Keller 1991] using matrix notation for isospin to conform to the usual expression of the standard theory.

Here we should remember that the idempotents $\frac{1}{2}(1 \pm i\gamma^5)$ correspond to the operators selecting handedness (or chirality) in spacetime algebra. The set of t^d_μ are then restricted forms of handling the chiral symmetry of the different fields. The relative chiral symmetries of the fields are the relevant quantities. The properties are relative properties, only the relations are meaningful not the actual components which are frame of reference dependent (or even coordinate dependent if general transformations are allowed). The group of these relations (see Table I) is the mathematical structure of physical interest. It is a $SU(2) \otimes SU(3)_c$ structure for each f. The $U(1)$ additional symmetry is related to the standard gauge freedom of the wave function.

In Table I Q = charge and I = isospin. Color and name refer to standard nomenclature. See [Close 1979 , Field 1979, Okun 1982, or Halzen and Martin 1984].

The basic equations for the set of spinor fields being (with $D_{(d,f)}$ explicitly defined above)

$$D_{(d,f)}\psi_{(d,f)} = 0 \ , \qquad \psi_{(d,f)} = D^{\dagger}_{(d,f)}\Phi \quad \text{and} \quad \partial^{\mu}\partial_{\mu}\Phi = 0 \tag{25}$$

where the subindex d stands for symmetry constrained Dirac fields (Diracons), it is given the values $(\text{electron})_{left}$, electron_{right}, $u_r, u_b, u_g, d_r, d_b, d_g$ and ν, for the first family, to conform to standard phenomenology and the subindex f refers to the family number.

We have shown [Keller 1991] that they constitute a set with all the known properties of each elementary particle's family, the fields they represent can be:

— massless or massive in the particular case of $e_L + e_R$

— charged (integer or fractional).

and it is discussed in [Keller 1991, pages 158 and following], that the collection of the fields constructed with (5(and (6) have weak charge and color, and in general the characteristics usually postulated on phenomenological basis, like composites being colorless, confinement, etc. these being immediate consequences of the defining equations.

Because of the appearance, or not, of the $i\gamma^5$ factors in (5), the fields have definite chiral properties. Only one type (for each family) of field in the theory may have simultaneously both chiralities and therefore can be, as a free field, massive, charged (reference charge ± 1) and weak charged: this is, for the first family, identified as the electron field.

We should stress, again, that in Table I properties are not assigned they are relative and are properties of the gauged Lagrangian. See [Keller 1991 pages 161 and following] for a full discussion of this point.

— The resulting theory is a **chiral geometry theory of charge, isospin and color.**

The theory has a Lagrangian formulation that reproduces all aspects of the standard theory. Higgs particles have not, in its first approximation (see below) the same motivation as in the standard theory. Confinement results, within the theory, from the requirement that the Lorentz symmetry should not be broken even at local level. The same requirement gives rise to the colorless condition for hadrons, the new feature is that hadrons should be both globally and locally colorless. Fractional charges are also a natural consequence of the gauging properties of the Lagrangian.

Mass results from vector and axial vector gauging, this procedure conserves the succesfull role of the Higgs field in the standard theory, weak bosons acquiring the same mass.

The theory shows the reason for chirality being a basic property of nature as shown by the set of elementary particles. This can be clearly seen with the gauging of the diracons equations

$$D_{(d,f)} = \Gamma^{\mu}_{(f)}[\partial^{(d)}_{\mu} - \mathbf{I}\frac{e^{\cdot}}{\hbar}A^{\mu}_{(d,f)}(x)] \tag{26}$$

the gauging fields having the multivector composition

$$\begin{aligned} A^{\mu}_{d}(x) = A^{\mu}_{d,\,\text{scalar (electromagnetic)}} + i\gamma^5\, A^{\mu}_{d,\text{pseudoscalar (weak, color)}} \\ + A^{\mu}_{\alpha\beta,\text{tensor(gravity)}}\gamma^{\alpha\beta} + \gamma^{\alpha}\, A^{\mu}_{\alpha,\,poincar\acute{e}} + \gamma^5\gamma^{\beta}\, A^{\mu}_{\beta,\,poincar\acute{e}} \end{aligned} \tag{27}$$

That is, the gauging has electromagnetic, weak, color and gravity parts. Then the wave function becomes upon gauging (φ a reference spinor).

$$\psi_d(x) = B\exp\{\mathbf{I}(p^{\mu}_{d}x_{\mu} + \phi_d(x))\}\varphi \tag{28}$$

with the phase factor being a multivector

$$\phi_d(x) = \phi_{d,\text{scalar}}(x)\mathbf{1} + \phi_{d,\text{pseudoscalar}}(x)i\gamma^5 + \phi_{d,\alpha\beta}(x)\gamma^{\alpha\beta} + \phi_{d,\,po\acute{i}ncar\acute{e}} \tag{29}$$

the particular, relative, combinations for the phase factor of the $i\gamma^5$ terms generate isospin and color and the $\gamma^{\alpha\beta}$ generate the local Lorentz transformations which are a consequence of gravity. To get a more common formulation of the theory we take first $\mathbf{I} = \gamma^5$ and second replace it by its eigenvalues $\pm$ i. The usefullness of γ^5 stems from the fact that it commutes with $\mathbf{1}$, $\gamma^{\alpha\beta}$ and $i\gamma^5$, (or H = handededness $(F)i\gamma^5/\sqrt{|g|}$). The symmetries of $\phi_{d,\text{scalar}}(x) + \phi_{d,\text{pseudoscalar}}(x)\, i\gamma^5$ generate the well known $SU(3)_c \otimes [SU(2) \otimes U(1)]_{ew}$ standard theory. The mass matrix for the $f > 1$ families of elementary particles has a very interesting form in its first approximation:

$$\hat{m}_{(f,d)} = N_f m_d(\,5.75 + \text{effect of nondiagonal terms}) \tag{30}$$

with $N_f = n_f c_f^2$ and $m_d = m_o(n_c)_d Q_d^4$, where n_f is the degeneracy of the family's wave function, $c_f = 2f - 1$ the number of spinors in the outer product of ψ, m_o the electron mass, n_c the number of color degrees of freedom: 1 (for ν and e^-) and 3 (for the quarks) and Q_d the charge of the lepton or quark field. Then the masses are all, in a first approximation, proportional to the electron mass. The factor Q_d^4 suggests that the mass matrix is directly related to the electromagnetic, gauge, field as of a self interaction origin. The creation of a pair of elementary particles at a given point, and its subsequent separation, involves the creation of their gauge fields, Q_d^2 is the factor for the energy required to create the particle's electromagnetic field, an inseparable field from the concept of the existence of the particle, whereas Q_d^4 should correspond to self interaction.

The phase factor (29) may contain additional terms in the vector + axial vector part of the Clifford algebra. In particular the possibility of a vector contribution $(m/4)\gamma^\mu\chi_\mu$ will result in the term called the "frame field" by Chisholm and Farwell (1992) generating the mass of the matter field.

5. The basic set of equations

It is interesting that the fundamentals of the theory can be summarized in the set of equations (25 and 26) labeled by (f, d) which should be treated together and with the corresponding equations for the gauge fields.

The comparison of the matter fields to see their relative properties is mathematically a spin (8) $\oplus$ spin (1) model for each family of elementary particles. This substantiates the work of Chisholm and Farwell as a further evidence that we have presented here a **theory of elementary particles**.

6. Conclusions

In the theory we have presented here the physical properties are now a constitutive part of the wave equations. The relative properties are clearly shown [Keller 1991] when supermatrices describe a collection of fields. Off diagonal terms couple them among themselves.

We have seen that spacetime and its T_M (complex) allows enough degrees of freedom to construct a theory of elementary particles and their interactions. Specially important is that all known interactions are properly described. No additional isospin space is therefore needed.

Nucleons like proton or neutron and mesons are, within this theory, composite fields but elementary particles. In fact these composite "elementary" particles cannot, even if enough energy is available, be split into smaller components; the requirement of rotational invariance forces the "colorless" combination of quarks, even to the smallest possible experimental probe size.

Acknowledgements

The author has the support of the Sistema Nacional de Investigadores (México). The technical assistance of Mrs. A. Irma Vigil de Aragón is also gratefully acknowledged.

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