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Revisiting the Poincaré and Little Groups with Physical Applications

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Abstract

This paper elucidates the Lorentz group, a fundamental subgroup of the Poincaré group. The orbits and little groups associated with the Lorentz group are described in detail, along with their corresponding properties. The Poincaré group is presented. Another fundamental aspect of the Poincaré group is Wigner's little groups obtained from this group. An in-depth discussion on the cases of both massive and massless relativistic particles within the context of little groups is given. Our examination extends to the properties of various special groups associated with the Poincaré group. Applications of these groups are elaborated by physical examples taken from high-energy physics and optics from both classical and quantum domains. Specifically, covariant harmonic oscillators including entangled states, proton form factors, and the parton picture as proposed by Feynman are discussed. In this context, laser cavities and shear states are also addressed. We lay out the underlying mathematics that connects these apparently disparate realms of physics.

Keywords: Lorentz group; Poincaré group; little groups; covariant harmonic oscillators; proton form factors; Feynman's parton picture; laser cavities; quantum shear states



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1. Introduction

It had long been thought that one promising approach to unifying special relativity and quantum mechanics was to develop a covariant formulation of quantum field theory. This, however, has not been the only approach. Indeed, P.A.M. Dirac, in his 1949 paper [1], suggested that the construction of a representation of the Poincaré group (alternatively called the inhomogeneous Lorentz group) might be the right place to begin. For studying the Poincaré group, Wigner's 1931 book on atomic spectra, containing many physical ideas, and the paper he wrote in 1939 on the inhomogeneous Lorentz group [2,3] provide a good starting point.

The limitations of formulating a covariant quantum field theory became apparent when the extended charge distribution of the proton [4] was discovered and the quark theory was subsequently formulated [5–7]. Considering that quarks are permanently confined inside hadrons, any theory unifying special relativity and quantum theory has to account for the bound state of quarks inside the hadron [8]. We contend that the same space-time symmetry should be inherent in any bound-state model. One of the purposes of this paper is to show that the principles contained in Wigner's 1939 [2] paper and in Dirac's 1949 [1] paper provide us the means to build a relativistic bound-state model that can explain the features of hadrons currently observed in high-energy laboratories.

It was Wigner who, in 1927 [9], introduced group theory into physics. In the present time, there is more and more interest in constructing explicit representations of groups that are now used in physics. Most of the groups used in physics are Lie groups. They provide a better understanding of physical phenomena through symmetries in almost all branches of physics. It is well known that the angular momentum and the spin operators are described by the rotation group and $SU(2)$ in quantum mechanics [10]. Almost all gauge theories rely on Lie groups and the associated Lie algebras [11], including the Lorentz and Poincaré groups [12] addressed in this paper. Likewise, particle physics utilizes these formulations for the space-time symmetries of relativistic particles [13,14]. Classical and quantum optics are no exceptions to this enterprise [12]. It is also worthwhile to mention a completely different field from those we shall discuss here, namely, statistical mechanics and thermodynamics [15,16].

A Lie group [17] is a group that is also a smooth manifold, where the group operations, specifically, the multiplication and the inversion, are both smooth maps. A closed set of generators, as we shall see below, determines the properties of Lie groups. This closed set is entitled the Lie algebra of the group.

One can obtain the Lie group from the Lie algebra by exponentiation. Consider any complex valued square matrix such as X ; then,

$$\exp X = \sum_{n=0}^{\infty} \frac{1}{n!} X^n. \quad (1)$$

From this the group element, the correspondence to any parameter σ can be derived from

$$G(\sigma) = \exp[-i\sigma X]. \quad (2)$$

The algebra of a particular Lie group can be obtained by differentiation:

$$i \frac{d}{d\sigma} G(\sigma) \Big|_{\sigma=0} = X_{\sigma}. \quad (3)$$

In Section 2, we review Lorentz transformations, which have become the bedrock formulation for special theory of relativity. We present the Lie algebra and give a representation in terms of the four-by-four matrices that result from it. We further present the little groups of the Lorentz group and the orbits that are also associated with this group. In Section 3, we show that the group of Lorentz transformations has a covering group, known as $SL(2, c)$.

Section 4 consists of a discussion of the subgroups contained in the Lorentz group, which itself is a subgroup of the Poincaré group. Specifically, we examine the subgroups $SO(3)$ and the covering group $SU(2)$, along with $Sp(2)$, $SU(1, 1)$, and $SO(2, 1)$. These groups are special groups, all having $\det(+1)$. On the other hand, we note that $O(3)$ has two connected components, one with $\det(+1)$ and the other with $\det(-1)$. Reflections belong to $O(3)$, but not to $SO(3)$. The Lie algebra obtained from the connected component to the identity is the same as the Lie algebra obtained from $SO(3)$; however, this algebra does not encompass the other connected component of $O(3)$ having $\det(-1)$.

Section 5 presents the Poincaré group, and representations of the Poincaré group are constructed. We then discuss Wigner's little groups that are the subgroups of the Poincaré group. In addition to the little groups discussed in Section 2, specifically, the $SO(3)$ group and the $SO(2, 1)$ group for massive and imaginary mass particles, here we note that there exists an $E(2)$ -like subgroup of the Poincaré group for massless particles. We also give the two-by-two representations of these little groups to the end that $SO(3)$ and $SO(2, 1)$ become the $SO(3)$ -like and the $SO(2, 1)$ -like groups of Wigner [2]. Although it is well known that the rotational symmetry governed by the $E(2)$ -like group corresponds to the helicity of

the photon, translational degrees of freedom of this group needed to be elucidated further. This section also clarifies the physical meaning of these translational degrees of freedom.

In Section 6, we give physical examples where the mathematics described in the preceding sections is used. In 1949 [1], Dirac suggested that finding a representation of the Poincaré group is to be undertaken by Lorentz-covariant quantum mechanics. In that regard, a formalism for covariant harmonic oscillators that is compatible with Dirac's general strategy to develop a relativistic quantum mechanics is rendered. Thus, we show how the Lorentz group is manifested through coupled covariant harmonic oscillators. In that context, we discuss the proton form factor, which is an ongoing research topic in high-energy physics, both theoretically and experimentally. For that matter, we discuss quarks and Feynman's partons as being two different manifestations of the same covariant entity. We also give two more applications from classical and quantum optics, namely, resonators in laser cavities and quantum shear states, respectively.

Section 7 is devoted to conclusions. In Appendix A, we provide the two-by-two transformation matrices of $SL(2, c)$ and their corresponding four-by-four transformation matrices in $SO(3, 1)$. These, given in Table A1, are derived by exponentiation using the generators given within the text.

2. The Lorentz Group

We start by considering the four dimensional spacetime manifold with coordinates

$$x_\mu = (t, -z, -x, -y). \quad (4)$$

Conventionally, Lorentz transformations are defined to be the group that preserves the inner product $x^\mu x_\mu = (t^2 - z^2 - x^2 - y^2)$, with $x_\mu = \eta_{\mu\nu} x^\nu$, where the Minkowski metric is taken to be $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$. Specifically, we have

$$\eta_{\alpha\beta} a^\alpha{}_\mu a^\beta{}_\nu = \eta_{\mu\nu}, \quad (5)$$

where $a^\alpha{}_\mu$ stands for the components of the transformation matrix. This group is called $SO(3, 1)$ when Lorentz transformations are restricted to being proper, i.e., to the condition that $\det a^\alpha{}_\mu = 1$. Here, they will also be constrained by $a^0{}_0 \geq 0$. This is known as the proper orthochronous Lorentz group.

When transformations are applied to the column vector $(t, z, x, y)^T$, the matrix form of the generators is

$$\begin{aligned} J_1 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}, & K_1 &= \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \\ J_2 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & K_2 &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\ J_3 &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix}, & K_3 &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (6)$$

They form six independent generators for the Lie algebra of the Lorentz group [18].

The transformation matrices of the Lorentz group can be written in exponential form:

$$A = \exp \left[-i \sum_{i=1}^3 (\theta_i J_i + \eta_i K_i) \right]. \quad (7)$$

Here, J_i are the generators of rotations and K_i are proper Lorentz boosts. In Appendix A, we tabulate the explicit matrix forms for each transformation matrix obtained through exponentiation: $\exp[-i\theta_i J_i]$ and $\exp[-i\eta_i K_i]$.

Generally, when considering the generators above that form the Lie algebra of the Lorentz group, one can apply them to an arbitrary coordinate variable function:

$$J_i = -i\varepsilon_{ijk} x^j \left(\frac{\partial}{\partial x^k} \right) \quad \text{and} \quad K_i = i \left[t \left(\frac{\partial}{\partial x^i} \right) - x_i \left(\frac{\partial}{\partial t} \right) \right], \quad (8)$$

with $x^i = (z, x, y)$ and $i = 3, 1, 2$. These generators satisfy these commutation relations:

$$[J_i, J_j] = i\varepsilon_{ijk} J_k, \quad [J_i, K_j] = i\varepsilon_{ijk} K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk} J_k. \quad (9)$$

The first commutator shows that the J_i are indeed the generators of the rotation subgroup of $SO(3, 1)$, the proper Lorentz group. The second and third commutators indicate that the boosts by themselves do not form a group. Multiplying two pure boosts will produce both a boost and a rotation.

We can combine the six generators shown in Equation (8) into the covariant form

$$L_{\mu\nu} = i \{ x_\mu \left(\frac{\partial}{\partial x^\nu} \right) - x_\nu \left(\frac{\partial}{\partial x^\mu} \right) \}, \quad (10)$$

where

$$J_i = \frac{1}{2} \varepsilon_{ijk} L_{jk} \quad \text{and} \quad K_i = L_{0i}. \quad (11)$$

Considering transformations on x_μ and $\partial/\partial x^\nu$, $L_{\mu\nu}$ behaves like a second-rank tensor:

$$L'_{\mu\nu} = a^\rho{}_\mu a^\sigma{}_\nu L_{\rho\sigma} \quad (12)$$

when a Lorentz transformation is applied. Then, for $L_{\mu\nu}$, the commutation relations are as follows [19]:

$$[L_{\mu\nu}, L_{\rho\sigma}] = -i(\eta_{\mu\rho} L_{\nu\sigma} + \eta_{\nu\sigma} L_{\mu\rho} - \eta_{\mu\sigma} L_{\nu\rho} - \eta_{\nu\rho} L_{\mu\sigma}). \quad (13)$$

Orbits of the Lorentz Group

Let X be a vector space and p a point in that vector space. Now, if $G(p)$ is a group, then the *little group* of G at p is the maximal subgroup leaving p invariant. The *orbit* of G at p is the set of points V that can be reached applying G to the single point p in X .

The little groups of the proper Lorentz group are a subset of transformations $a^\mu{}_\nu$, which leave the four-vector p^μ invariant

$$p^\mu = a^\mu{}_\nu p^\nu. \quad (14)$$

The orbits of proper Lorentz group the p^μ are such that

$$p^\mu p_\mu = \pm m^2, \quad (15)$$

generating surfaces in this four dimensional space. Here, p^μ is identified as the four-momentum and written as a column vector $(p^0, p^3, p^1, p^2)^T$.

The four-momentum is time-like $p^\mu p_\mu > 0$, light-like $p^\mu p_\mu = 0$, or space-like $p^\mu p_\mu < 0$. By analyzing the little group for a single typical four-vector on each orbit, these orbits allow us to investigate every potential little group.

When considering the proper Lorentz group, six orbits can be distinguished. Two time-like surfaces that have positive m^2 compose the first two orbits. These have, respectively, positive and negative values of p_0 . The forward and backward light cones, respectively, constitute the third and fourth orbits. The fifth orbit has negative values of m^2 and, therefore, is a space-like surface. The sixth orbit, where $p_0 = p_1 = p_2 = p_3 = 0$, is the origin in this space.

The linear equations in Equation (14) have to be solved so as to find representations of the little groups such that for each representative p^μ , the transformation matrix $a^\mu{}_\nu$ is to satisfy Equation (5). The number of independent parameters is reduced by this latter condition to three.

- (a) For time-like $p^\mu_{\pm m} = (\pm m, 0, 0, 0)$, we have $a^0{}_i = a^i{}_0 = 0$. The little group is, thus, the rotation group.
- (b) For light-like $p^\mu_\omega = (\omega, \omega, 0, 0)$, this four-vector is invariant under rotations around the z -axis.
- (c) For, again, light-like $p^\mu_{-\omega} = (-\omega, \omega, 0, 0)$, the rotation matrix is the same.
- (d) For space-like $p^\mu_q = (0, q, 0, 0)$, this four-vector remains invariant under rotations around the z -axis and boosts along either the x - or y -axis. Together, these form the three-dimensional Lorentz group $SO(2, 1)$, satisfying the required condition.
- (e) For $p^\mu_0 = (0, 0, 0, 0)$, the entire Lorentz group leaves this zero momentum invariant, where the origin is the orbit.

These orbits are shown in Figure 1.

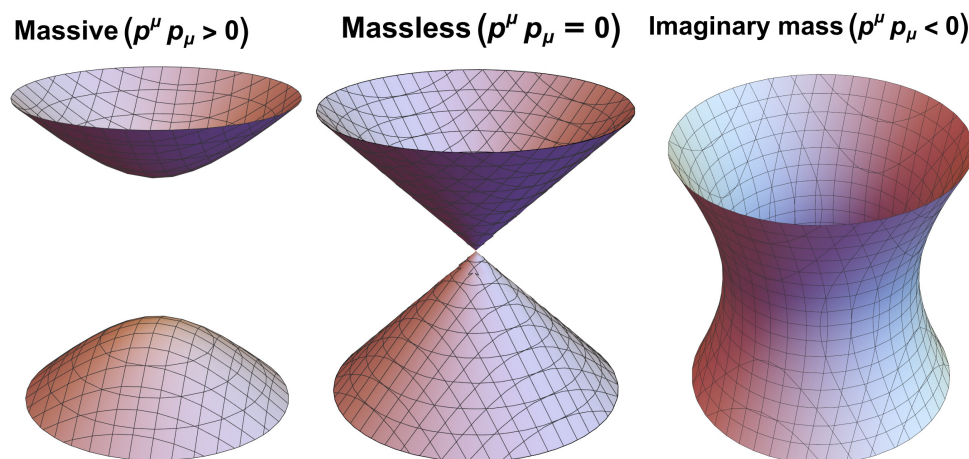


Figure 1. There are six orbits of the Lorentz group that encompass all possible four-momenta. Each of those four-momenta take one of the following forms: time-like ($p^\mu p_\mu > 0$), light-like ($p^\mu p_\mu = 0$), space-like ($p^\mu p_\mu < 0$), or zero. Zero momentum is represented by the double cone’s vertex [12].

To be used later in Section 6, we give a particular little group transformation for a relativistic massive particle moving in the z -direction whose initial four-momentum is $P_i = (E, P_z, 0, 0)^T$. When boosted back to its rest frame with $B_z(\eta)^{-1}$, it takes the form $(E/\cosh \eta, 0, 0, 0)^T$, with $\tanh \eta = P_z/E$. Then, in the rest frame, this can be rotated around the y -axis, where we make use of the respective boost and rotation matrices in Appendix A. Finally, it is boosted to its initial momentum with $B(\eta)$. The chain of process yields

$$B_z(\eta)R_y(\theta)B_z(\eta)^{-1}P_i = P_i, \tag{16}$$

which is a little group transformation.

We shall discuss, in Section 5, the little groups of the Poincaré group. There, we shall see that not all the little groups of the Poincaré group coincide with the little groups of the Lorentz group, due to the fact that the Lorentz group is a subgroup of the Poincaré group.

3. The Covering Group of the Lorentz Group: $SL(2, c)$

Let us consider the group generated by the following six two-by-two matrices [20,21]:

$$J_i = (1/2)\sigma_i \quad \text{and} \quad K_i = (i/2)\sigma_i, \quad (17)$$

where σ_i are the Pauli spin matrices defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (18)$$

The explicit forms of the operators in Equation (17) are

$$\begin{aligned} J_1 &= \begin{pmatrix} 0 & 1/2 \\ 1/2 & 0 \end{pmatrix}, & K_1 &= \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}, \\ J_2 &= \begin{pmatrix} 0 & -i/2 \\ i/2 & 0 \end{pmatrix}, & K_2 &= \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \\ J_3 &= \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, & K_3 &= \begin{pmatrix} i/2 & 0 \\ 0 & -i/2 \end{pmatrix}. \end{aligned} \quad (19)$$

These matrices satisfy the commutation relations

$$[J_i, J_j] = i\varepsilon_{ijk}J_k, \quad [J_i, K_j] = i\varepsilon_{ijk}K_k, \quad [K_i, K_j] = -i\varepsilon_{ijk}J_k. \quad (20)$$

These relations are identical to those of the Lorentz group. $SL(2, c)$ is homomorphic (the mapping is onto but not one-to-one) to $SO(3, 1)$, namely, it is the double cover of $SO(3, 1)$.

The group of two-by-two matrices whose generators are defined in Equation (17) is of the form

$$W = \exp \left[-i \sum_{i=1}^3 (\theta_i J_i + \eta_i K_i) \right] \quad (21)$$

and has the same algebraic properties as that of the Lorentz group. Specifically, matrices in Equation (17) are the generators of the $SL(2, c)$ group. These generators are traceless, so the determinant of W is one. Therefore, $SL(2, c)$ is an unimodular ($\det = 1$) group. On the other hand, $SL(2, c)$ is not a unitary group, given that the K_i matrices are not Hermitian. We point out that if the sign of K_i is changed in Equation (20), the Lie algebra remains invariant. In the case of $SO(3, 1)$, the sign of the boost generators K_i is unambiguously defined in terms of the space and time variables. However, when it comes to $SL(2, c)$, one has to consider both signs. Since the sign change can be easily implemented, the positive sign as given in Equation (19) will be used, unless otherwise required.

Now, we examine the expression of Equation (21) and note that K_i is just i times J_i . Thus, W of Equation (21) can be rewritten as

$$W = \exp \left[-\frac{i}{2} (\zeta_1 \sigma_1 + \zeta_2 \sigma_2 + \zeta_3 \sigma_3) \right], \quad (22)$$

where $\zeta_i = \theta_i + i\eta_i$. This implies that three generators and three complex parameters are sufficient to work with the $SL(2, c)$ group.

In order to see that matrices belonging to $SL(2, c)$ indeed perform Lorentz transformations, we consider the two-by-two coordinate matrix

$$X = \sigma_0 x_0 + \sigma_i x_i, \quad (23)$$

where σ_0 is the two-by-two unit matrix. Then,

$$\det(X) = t^2 - z^2 - x^2 - y^2 \quad (24)$$

is the quadratic form which should remain invariant. The group elements act on X through the following transformation:

$$X' = WXW^\dagger. \quad (25)$$

Since W and W^\dagger are unimodular, $\det(X)$ is preserved. Therefore, transformations of the form above are Lorentz transformations.

4. Subgroups of the Lorentz Group

In this section, we give the subgroups of the Lorentz group. For this purpose, it is sufficient to look at the generators of Lorentz group in Equation (6) and investigate which of those can be grouped together to form a closed set.

First, we note that the rotation generators J_i form a closed set, and, therefore, they constitute a subgroup, namely, the $SO(3)$ or the proper rotation group. These generators are Hermitian, while the boost generators are not.

Now, we consider the two-by-two representation of the Lorentz group, specifically, the $SL(2, c)$ group. The subgroup $SU(2)$ is a three-parameter group and is generated by the Pauli spin matrices. The generators J_i in Equation (17) satisfy the well-known commutation relations:

$$[J_i, J_j] = i\varepsilon_{ijk}J_k. \quad (26)$$

Just like the group $SL(2, c)$ is to $SO(3, 1)$, the group $SU(2)$ is the double cover of $SO(3)$. If W in Equation (25) is restricted to the $SU(2)$ group, the t variable is not effected, so the transformation acts only on $\sigma_i x_i$ and is equivalent to a rotation in the three-dimensional (z, x, y) space. It is known how the $SU(2)$ group acts on spinors through unitary transformations that correspond to rotations in the spin space.

Additionally, Lorentz transformations along the x - and y -directions and of the rotation on the xy -plane around the axis perpendicular to this plane form the group $SO(2, 1)$. The generators of this group are K_1, K_2 , and J_3 , as given in Equation (6). They satisfy the commutation relations:

$$[K_1, K_2] = -iJ_3, \quad [K_2, J_3] = iK_1, \quad [K_1, J_3] = -iK_2. \quad (27)$$

It can also be observed from Equation (19) that K_1, K_3 , and J_2 satisfy the commutation relations:

$$[K_1, K_3] = iJ_2, \quad [K_3, J_2] = -iK_1, \quad [K_1, J_2] = iK_3. \quad (28)$$

Since these generators are pure imaginary, the transformation matrices have real elements. This is the two-dimensional symplectic group $Sp(2)$. Alternatively stated, it is the $SL(2, r)$ group. The groups $Sp(2)$ and $SO(2, 1)$ are locally isomorphic.

From Equation (19), we choose

$$J_3 = \frac{1}{2}\sigma_3, \quad K_1 = \frac{i}{2}\sigma_1, \quad K_2 = \frac{1}{2}\sigma_2. \quad (29)$$

These are the generators of the group $SU(1,1)$. They satisfy the same commutation relations as those of Equation (27). So, the groups $SU(1,1)$ and $SO(2,1)$ are also locally isomorphic.

As for the final set of generators, we choose K_2, K_3 , and J_1 . Their commutation relations satisfy

$$[K_2, K_3] = -iJ_1, \quad [K_2, J_1] = -iK_3, \quad [K_3, J_1] = iK_2, \quad (30)$$

where the group they generate is recognized as an $SU(1,1)$ -like group.

The Squeeze-Rotation and the Shear-Squeeze Representations of the $Sp(2)$ Group

At this point, it is worthwhile to give the *squeeze-rotation* representation of the $Sp(2)$ group. The generators are given as

$$J_2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad K_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad K_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (31)$$

Now, we consider

$$X_1 = K_1 - J_2, \quad X_2 = K_1 + J_2, \quad X_3 = 2K_3 \quad (32)$$

then we have

$$X_1 = \begin{pmatrix} 0 & i \\ 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 \\ i & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \quad (33)$$

They satisfy the commutation relations

$$[X_1, X_2] = iX_3, \quad [X_1, X_3] = -2iX_1, \quad [X_2, X_3] = 2iX_2. \quad (34)$$

Shear transformation matrices can be decomposed as

$$\exp(-i\alpha X_1) = [\exp(-i\theta_1 J_2)]^{-1} \exp(-i\lambda K_3) \exp(-i\theta_2 J_2) \quad (35)$$

$$\exp(-i\beta X_2) = [\exp(-i\theta_3 J_2)]^{-1} \exp(-i\rho K_3) \exp(-i\theta_4 J_2) \quad (36)$$

with

$$\theta_1 = \pi - \cot^{-1}(\alpha/2), \quad \theta_2 = \cot^{-1}(\alpha/2), \quad \lambda = \cosh^{-1}(1 + \alpha^2/2), \quad (37)$$

$$\theta_3 = \cot^{-1}(\beta/2), \quad \theta_4 = \pi - \cot^{-1}(\beta/2), \quad \rho = \cosh^{-1}(1 + \beta^2/2). \quad (38)$$

This is another representation of the $Sp(2)$ group. This particular representation is called the *shear-squeeze* representation. This representation builds the necessary algebraic structure of $T(d)$ and $M(R)$ matrices of classical optics that we shall give in Section 6.3, as well as the quantum shear states provided in Section 6.4.

5. Poincaré Group and Wigner's Little Groups

Here we define the Poincaré or inhomogeneous Lorentz group. This group consists of both translations and the Lorentz group. It was Eugene Wigner in his landmark 1939 paper [2] who first discussed the *little groups* that are subgroups of the Poincaré group. The little groups are defined as those groups whose transformations leave the four-momentum of the relativistic particle unchanged.

5.1. Poincaré Group

The Poincaré group [22] is composed of the group of inhomogeneous Lorentz transformations that operate on the four-dimensional Minkowski space in the following form:

$$x'^{\mu} = a^{\mu}_{\nu} x^{\nu} + b^{\mu}. \quad (39)$$

Here, a^{μ}_{ν} is the Lorentz transformation matrix defined in Section 2, while b^{μ} consists of the translation parameters. A typical five-by-five transformation matrix applicable to the column vector $(x^{\nu}, 1)^T$ is of the form

$$(M^{\mu}_{\nu}) = \begin{pmatrix} a^{\mu}_{\nu} & b^{\mu} \\ 0 & 1 \end{pmatrix}. \quad (40)$$

The commutation relations for generators of the Poincaré group are given as follows [19]:

$$[P_{\mu}, P_{\nu}] = 0 \quad (41)$$

$$[L_{\mu\nu}, P_{\rho}] = -i(\eta_{\mu\rho} P_{\nu} - \eta_{\nu\rho} P_{\mu}) \quad (42)$$

$$[L_{\mu\nu}, L_{\rho\sigma}] = -i(\eta_{\mu\rho} L_{\nu\sigma} - \eta_{\nu\rho} L_{\mu\sigma} + \eta_{\mu\sigma} L_{\rho\nu} - \eta_{\nu\sigma} L_{\rho\mu}). \quad (43)$$

Here, $L_{\mu\nu}$ are the generators of the Lorentz group, while P_{μ} are translation generators. Since P_{μ} does not commute with $L_{\mu\nu}$, the Poincaré group cannot be expressed as a direct product of the Lorentz and translation groups. However, the Lorentz group is a subgroup of the Poincaré group.

The generators of the Poincaré group are composed of the generators of the Lorentz group given in Equation (10), together with the translations generated by

$$P_{\mu} = i \left(\frac{\partial}{\partial x^{\mu}} \right) = i \left(\frac{\partial}{\partial x^{0'}} - \frac{\partial}{\partial x^{i'}} \right). \quad (44)$$

5.2. Wigner's Little Groups

Wigner's little groups are the subgroups of the Poincaré group whose transformations leave the four-momentum of a relativistic particle invariant. In addition to the Lorentz group, the Poincaré group involves translations. As a preparation to Wigner's little groups, we first consider the Euclidean or $E(2)$ group consisting of translations and a rotation in a two-dimensional Euclidean plane. The transformation equations are defined as follows [12]:

$$x' = x \cos \alpha - y \sin \alpha + u, \quad y' = x \sin \alpha + y \cos \alpha + v. \quad (45)$$

Translations in the xy -plane are represented by the parameters u and v .

The generators of this group satisfy

$$[P_1, P_2] = 0, \quad [J_3, P_1] = -iP_2, \quad [J_3, P_2] = iP_1, \quad (46)$$

where P_1 and P_2 generate translations along the x and y -directions, respectively.

We observe that not all little groups of the Poincaré group coincide with the little groups of the Lorentz group. In addition to the little groups we have presented in Section 2, which are explicitly given in items (a)–(e), namely, the rotation group and the $SO(2, 1)$ group for massive and imaginary mass particles, for massless particles with $p_{\omega}^{\mu} = (\omega, \omega, 0, 0)$,

there exists a translation-like subgroup of an $E(2)$ -like group, whose matrix is in the following form [2]:

$$T(u, v) = \begin{pmatrix} 1 + (u^2 + v^2)/2 & -(u^2 + v^2)/2 & u & v \\ (u^2 + v^2)/2 & 1 - (u^2 + v^2)/2 & u & v \\ u & -u & 1 & 0 \\ v & -v & 0 & 1 \end{pmatrix}. \quad (47)$$

Furthermore, there is the Hermitian conjugate of this matrix to be employed when $p_{-\omega}^\mu = (-\omega, \omega, 0, 0)$.

In view of Equation (3), the expressions for the generators of translations, N_1 and N_2 , can be obtained by taking the derivative with respect to u and v of Equation (47) separately and evaluating at zero. They can be formed through a particular combination of the generators of the Lorentz group as given in Equation (6) in the following way:

$$J_3, \quad N_1 = K_1 - J_2, \quad N_2 = K_2 + J_1. \quad (48)$$

The little group obtained from the above generators is a subgroup of the Poincaré group. The commutation relations of N_1 , N_2 , and J_3 are identical to those of the $E(2)$ group given in Equation (46). Specifically, they constitute the Lie algebra of an $E(2)$ -like group.

Einstein's special theory of relativity was introduced for point-like particles without any internal structures. On the other hand, in addition to governing the symmetries of the four-momentum, Wigner's little groups also dictate the internal space-time symmetries of relativistic particles. For that matter, for instance, for spin 1/2 massless particles, it is the little group obtained from generators of the $SL(2, c)$ following the formulation of Equation (48). However, since all types of neutrinos have been observed to have non-zero mass [23], we are still on watch for the existence of massless spin 1/2 particles.

On the other side of the coin, photons, being spin 1 particles, can safely be regarded to be the most fundamental particle in almost all branches of physics. When it comes to massless particles, one is bound to deal with the fact that there are no Lorentz frames in which they are at rest. Therefore, to be able to tackle this issue, we are compelled to go beyond the Lorentz group and consider $E(2)$ -like little groups of the Poincaré group.

For this purpose we consider the transformation having the form

$$x' = x \cos \alpha - y \sin \alpha, \quad y' = x \sin \alpha + y \cos \alpha, \quad z' = z + ux + vy. \quad (49)$$

We see here that $(x^2 + y^2)$ is invariant under this transformation, but that z can vary from $-\infty$ to $+\infty$. Therefore, this defines a *cylindrical transformation*. It requires little effort to explain the rotational degree of freedom as the helicity of the photon. The interpretation of the translational degrees of freedom has been given in [14,24] as the gauge degrees of freedom for the photon. The physical aspect of this transformation is illustrated in Figure 2.

Again, we resort to [2] to elucidate on the gauge transformation and consider the matrix

$$D(\gamma, \phi) = \exp[-i\gamma(N_1 \sin \phi + N_2 \cos \phi)] \quad (50)$$

where N_1 and N_2 are as given in Equation (48). This acts on the four-potential of a photon $A^\mu = (A_0, A_z, A_x, A_y)$ moving along the z direction. The Lorentz gauge condition $p^\mu A_\mu = 0$ yields $A_0 = A_z$; then, the four-potential becomes

$$A_{LC}^\mu = (A_0, A_0, A_x, A_y). \quad (51)$$

Subsequently, $D(\gamma, \phi)A_{LC}^\mu$ leads to the invariance of A_x and A_y , while the A_0 component becomes

$$A'_0 = A_0 + \gamma(A_x \cos(\phi) + A_y \sin(\phi)). \tag{52}$$

This expression clearly lays out how N_1 and N_2 are involved in gauge transformations.

In terms of the differential operators acting on functions, in [14], the translation-like generators are introduced as

$$Q_1 = -ix \frac{\partial}{\partial z} \quad \text{and} \quad Q_2 = -iy \frac{\partial}{\partial z}, \tag{53}$$

including the rotation generator J_3 ; these satisfy

$$[Q_1, Q_2] = 0, \quad [J_3, Q_1] = iQ_2, \quad [J_3, Q_2] = -iQ_1, \tag{54}$$

which are like those of Equation (46). This is the Lie algebra for the *cylindrical group*.

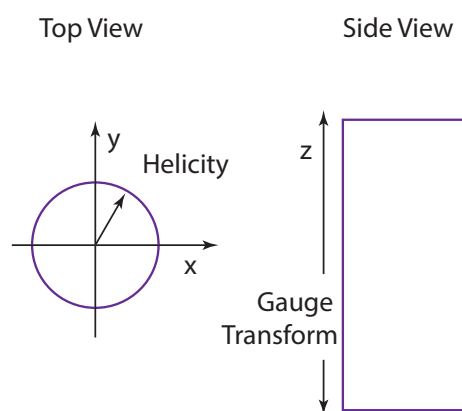


Figure 2. The three parameters of the cylindrical group have physical interpretations. First, the top view accounts for the rotational degree of freedom and corresponds to the helicity of the photon. Then, the side view depicts up and down translations on the z-axis where they represent the gauge degrees of freedom [14].

5.3. Wigner Four-Momentum-Matrices

In addition to massless particles, there are also massive and imaginary mass particles. When a massive particle is at rest, Wigner observed that the particle still has three rotational degrees of freedom. Therefore, the rotation group defined in Section 2 becomes Wigner’s rotation-like subgroup, $SO(3)$ -like. For imaginary mass particles, which we cannot physically detect, Wigner defined an $SO(2, 1)$ -like subgroup. In this regard, the group $SO(2, 1)$ was also discussed in Section 2. Now that we have three types of little groups, this section will explore how to express them comprehensively.

A Minkowski space-time four-vector can be expressed by a two-by-two matrix as

$$X = \begin{pmatrix} t + z & x - iy \\ x + iy & t - z \end{pmatrix}, \tag{55}$$

and a $SL(2, c)$ matrix of the form of Equation (23) performs Lorentz transformations on this vector. Similarly, the momentum four-vector p_μ can be expressed by a two-by-two matrix as

$$P = \begin{pmatrix} E + p_z & p_x - ip_y \\ p_x + ip_y & E - p_z \end{pmatrix}, \tag{56}$$

whose determinant is

$$E^2 - p_z^2 - p_x^2 - p_y^2 \tag{57}$$

and is equal to m^2 , 0, or $-m^2$, depending on the nature of the mass of the particle. If a massive particle moves only along the z-direction, this matrix becomes

$$\begin{pmatrix} E + p & 0 \\ 0 & E - p \end{pmatrix}. \tag{58}$$

This matrix takes the form

$$P_m = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{59}$$

when the particle is at rest. On the other hand, $E = p$ for massless particles, while for tachyon particles [25–27], the mass is imaginary. Thus, for all these cases, there are three distinct two-by-two matrix representations proportional to

$$P_m = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad P_{im} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{60}$$

where m is conveniently factored out. These matrices are collectively called Wigner four-momentum-matrices and denoted by $P_w = \{P_m, P_0, P_{im}\}$. For all P_w matrices,

$$Z(\phi) = \exp \left[-\frac{i}{2} \phi \sigma_3 \right] = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \tag{61}$$

satisfies

$$P_w = Z(\phi) P_w Z(\phi)^\dagger. \tag{62}$$

More generally, consider the matrices that leave the four-momentum-matrices invariant

$$P = W P W^\dagger. \tag{63}$$

Internal space-time symmetries of relativistic particles are dictated by these W matrices. They constitute Wigner’s little group. In this particular case, they transform four-momentum-matrices P_w of Equation (60) by

$$P_w = W_i P_w W_i^\dagger, \tag{64}$$

rendering P_w to remain invariant. Corresponding Wigner matrices W_i are tabulated in Table 1. These matrices are unimodular with real elements. Therefore, they are classified within the group $Sp(2)$.

Table 1. Four-momenta are represented by two-by-two matrices, labeled as P_w , whose determinants are positive, zero, and negative for massive, massless, and imaginary-mass particles, respectively. P_w remain invariant by their respective W_i transformations. Therefore, matrices W_i belong to Wigner’s little group [28].

Particle Mass	Wigner Four-Vector P_w	Wigner Transformation Matrix W_i
Massive	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$
Massless	$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & -\gamma \\ 0 & 1 \end{pmatrix}$
Imaginary-mass	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$

6. Examples

The Poincaré group and the subgroups thereof are extensively used in physical sciences [29]. Here, we have chosen four illustrative examples from different disciplines of physics. The underlying mathematics that connects these different physics realms is discussed. We choose to work with natural units, where $\hbar = c = 1$.

6.1. Applications to Quantum Mechanics: Lorentz-Covariant Harmonic Oscillators and Entangled Excited States

The harmonic oscillator has long been used in physics. It has been shown, in particular, that the harmonic oscillator wave function without time-like excitations had a probability interpretation [30] and these form the vector spaces for the Poincaré group [12,13]. In this section, the Lorentz transformation properties of harmonic oscillators are examined. The properties of the Lorentz-invariant differential equation are also investigated, as the entangled excited states are generated by solutions of this differential equation. The role these play in illuminating the phenomena observed in high-energy physics experiments will be shown in Section 6.2.

6.1.1. Lorentz-Covariant Harmonic Oscillators

We consider the Hamiltonian of the form where x_1 represents t , x_2 represents z , p_1 represents p_0 , and p_2 represents p_z :

$$H_{\text{inv}} = \frac{1}{2}(p_1^2 + x_1^2) - \frac{1}{2}(p_2^2 + x_2^2) = \frac{1}{2}(x_1^2 - x_2^2) + \frac{1}{2}(p_1^2 - p_2^2). \quad (65)$$

This Lorentz invariant Hamiltonian was first introduced by Yukawa [31] to describe non-local fields in a relativistic manner, which is in the most elementary form, yet it is able to describe the intrinsic properties of elementary particles. Later, it was used by Feynman et al. [32]. Currently, we observe a growing interest in the relationships among covariant harmonic oscillators, relativistic quantum mechanics, and quantum field theory in the literature [33–35].

The Hamiltonian in Equation (65) is invariant under the Lorentz transformation (boost along the z -axis), which is a squeeze transformation:

$$\begin{aligned} x_1 &\rightarrow x_1 \cosh \eta + x_2 \sinh \eta, & x_2 &\rightarrow x_2 \cosh \eta + x_1 \sinh \eta, \\ p_1 &\rightarrow p_1 \cosh \eta + p_2 \sinh \eta, & p_2 &\rightarrow p_2 \cosh \eta + p_1 \sinh \eta. \end{aligned} \quad (66)$$

Rewriting in Dirac's [1] light-cone coordinates as

$$x_{\pm} = (x_1 \pm x_2)/\sqrt{2} \quad \text{and} \quad p_{\pm} = (p_1 \pm p_2)/\sqrt{2} \quad (67)$$

and using the transformation

$$x_{\pm} \rightarrow e^{\pm\eta} x_{\pm}, \quad p_{\pm} \rightarrow e^{\pm\eta} p_{\pm}, \quad (68)$$

the harmonic oscillator wave function becomes coupled. Since the space and momentum variables are seen to be transformed in the same way, this transformation is like Einstein's Lorentz boost. Hence, this transformation is called the Lorentz transformation. Indeed, this is how the covariant harmonic oscillator functions are Lorentz-boosted.

The Lorentz-covariant wave function for the space-time coordinates has the form

$$f_{\eta}(x_1, x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} \left[(x_1 + x_2)^2 e^{-2\eta} + (x_1 - x_2)^2 e^{2\eta} \right] \right\}. \quad (69)$$

The momentum-energy wave function for this covariant harmonic oscillator is

$$g_\eta(p_1, p_2) = \frac{1}{2\pi} \int \exp[i(p_1x_1 - p_2x_2)]f_\eta(x_1, x_2)dx_1 dx_2, \tag{70}$$

which then becomes

$$g_\eta(p_1, p_2) = \frac{1}{\sqrt{\pi}} \exp\left\{-\frac{1}{4}\left[(p_1 + p_2)^2e^{-2\eta} + (p_1 - p_2)^2e^{2\eta}\right]\right\}. \tag{71}$$

It should be noted that the wave function for the space-time and momentum-energy coordinates are transformed in the same way. This is not true for two coupled oscillators. This is illustrated in Figure 3.

These wave functions enable us to construct a representation of Wigner’s little group for the localized bound-state wave function. It enables us also to explain Gell-Mann’s quark model and Feynman’s parton model as two special cases of one Lorentz covariant entity, as shown in Section 6.2.2.

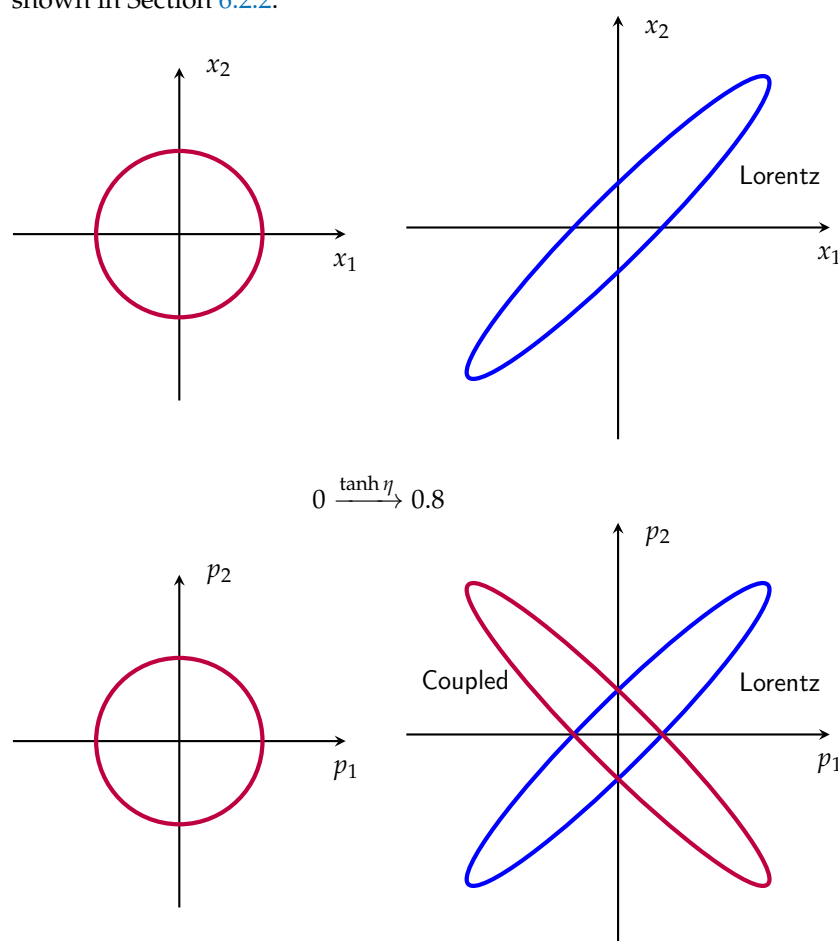


Figure 3. Shown in the figure are coupled and covariant oscillators. This illustrates that although coupled and covariant oscillators are transformed in the same way in the space-time coordinates, they are transformed differently in the momentum-energy coordinates [12].

6.1.2. Entangled Excited States

We can write the Lorentz invariant harmonic oscillator equation as

$$\frac{1}{2} \left\{ \left(x_1^2 - \frac{\partial^2}{\partial x_1^2} \right) - \left(x_2^2 - \frac{\partial^2}{\partial x_2^2} \right) \right\} \chi_n(x_1)\chi_m(x_2) = (n - m)\chi_n(x_1)\chi_m(x_2) \tag{72}$$

where this is a quadratic equation in two independent variables x_1 and x_2 , which behave as time-like and space-like variables, respectively. This equation is Lorentz invariant; however, the solution which is $\chi_n(x_1)\chi_m(x_2)$ is subject to the conditions of Lorentz covariance. If we consider a single excitation state, which has no time-like excitations allowed, the wave function can be written as

$$\Psi_0^n(x_1, x_2) = \chi_0(x_1)\chi_n(x_2) = \left[\frac{1}{\pi 2^n n!} \right]^{\frac{1}{2}} H_n(x_2) \exp \left\{ -\frac{1}{2} (x_1^2 + x_2^2) \right\}. \quad (73)$$

This function, in the Lorentzian regime, is squeezed when the x_1 and x_2 variables are replaced in the wave function by

$$x_1 \rightarrow x_1 \cosh \eta - x_2 \sinh \eta, \quad x_2 \rightarrow x_2 \cosh \eta - x_1 \sinh \eta. \quad (74)$$

We can then expand this squeezed function in form of a series

$$\chi_0(x'_1)\chi_n(x'_2) = \sum_{k'k} A_{k'k}(n) \chi_k(x_1)\chi_{k'}(x_2) \quad (75)$$

with $k' - k = n$.

In view of the details given in [12,36], where the generators of the Hermite polynomials have been used, the coefficient A_{nk} can be calculated, which results in the wave function

$$\chi_0(x'_1)\chi_n(x'_2) = \left[\frac{1}{\cosh \eta} \right]^{n+1} \sum_k \left[\frac{(n+k)!}{n!k!} \right]^{\frac{1}{2}} (\tanh \eta)^k \chi_k(x_1)\chi_{k+n}(x_2). \quad (76)$$

Using the ground state, $n = 0$, this expression, which should be equal to Equation (69), leads to

$$\frac{1}{\cosh \eta} \sum_k (\tanh \eta)^k \chi_k(x_1)\chi_k(x_2) = \frac{1}{\sqrt{\pi}} \exp \left\{ -\frac{1}{4} [(x_1 + x_2)^2 e^{-2\eta} + (x_1 - x_2)^2 e^{2\eta}] \right\}. \quad (77)$$

This gives us the series expansion of a squeezed Gaussian form not separable in the x_1 and x_2 variables. Therefore, this wave function is *entangled* in terms of continuous variables. This formulation is important in quantum information, where it is explored within the framework of quantum optics. There, it is responsible for the entanglement that arises from the infinite means of superpositions where photons in equal numbers can be distributed in each mode [37]. That this Gaussian entanglement is also applicable to various dynamical systems is well known [38–40].

6.2. Applications to High Energy Physics: Proton Form Factor and Feynman's Parton Model

The discovery of Hofstadter and McAllister [4] in 1955 showed that the proton had an internal charge distribution that then required the use of the *proton form factor*, which describes how electron–proton scattering deviates from the Rutherford formula. This quantity depends on the momentum transferred during the scattering. Additionally, the quark model proposed by Gell-Mann [5–7] viewed the proton and any hadron as having the charge distribution composed of two (mesons) or three (baryons) quarks when at rest. What happens when the hadron travels at relativistic speeds? The answer proposed by Feynman [41,42] in 1969 was that a fast-moving hadron could be viewed as a collection of many *partons* that had properties different from the properties of quarks. For example, although there appear to be two or three quarks inside a static hadron, the number of partons in a hadron moving rapidly appears to be infinite.

6.2.1. The Proton Form Factor

After the experimental evidence [4] confirmed the extended charge distribution of the proton, but before the quark model, Markov [43] attempted to calculate the proton form factor using Yukawa's [31] harmonic oscillator formalism. After the emergence of the quark model [5–7], Ginzburg and Man'ko considered obtaining bound states for quarks using the relativistic harmonic oscillator [44]. The form factor of the proton was successfully calculated in 1970 by Fujimura et al. [45], without mentioning the relativistic harmonic oscillator of Yukawa [31], even though they used it. By using the relativistic harmonic oscillator wave functions presented here, they were able to obtain the *dipole cut-off* of the form factor universally observed in high-energy experiments. Licht and Pagnamenta [46], also in 1970, derived the proton form factor using Lorentz-contracted oscillator wave functions. By using the Breit coordinate system, they were able to bypass the time-separation variable that is present in covariant formalism [46]. Feynman et al. [32] confirmed, in 1971, the earlier suggestion that Yukawa made by observing that the experimental mass spectrum of the hadron could be explained using the degeneracies of the three-dimensional harmonic oscillator. They quoted Fujimura et al. [45] but did not mention the 1953 article of Yukawa [31].

Using the Lorentz Breit frame, the proton enters along the positive z -axis and, after interaction with the electron, exits along the negative z -axis [12,47]. This is shown in Figure 4. As the momentum transfer increases substantially, the cross section now deviates from Rutherford scattering. This is what was first observed in 1955 [4]. As a consequence, the observed cross section decreases as

$$\frac{1}{(\text{momentum transfer})^8}. \quad (78)$$

The reason for this deviation is that the proton is not a point particle and inside the proton, the electric charge is distributed with a finite radius. The proton form factor describes the portion of the scattering amplitude that is due to the electric charge distribution. Therefore this proton form factor should decrease as

$$\frac{1}{(\text{momentum transfer})^4}. \quad (79)$$

This decrease in behavior is known in the literature as the dipole cut-off, which now constitutes one of the major branches of high-energy physics, prompting, in the past, some far-reaching theoretical models, including the quark model [7]. Here, we approach the problem using the harmonic oscillator formalism described above, by showing that a consequence of the coherence between the incoming wave function and the contraction of the proton wave function is the dipole cut-off.

The transfer of the four-momentum is

$$(E, p) - (E, -p) = (0, 2p). \quad (80)$$

Here, p is the momentum of the proton and the energy is $\sqrt{p^2 + m^2}$. Then, we can write the proton form factor, where we note that the momentum is in equal but opposite directions, as

$$F(p) = \int e^{2ipz} \psi_{\eta}^*(t, z) \psi_{-\eta}(t, z) dt dz. \quad (81)$$

Using the ground state harmonic oscillator wave function [12,47] the integral above becomes

$$F(p) = \frac{1}{\pi} \int e^{2ipz} \exp \left\{ -\cosh(2\eta) (t^2 + z^2) \right\} dt dz. \quad (82)$$

After integration over t , we have

$$F(p) = \frac{1}{\sqrt{\pi \cosh(2\eta)}} \int e^{2ipz} \exp \left\{ -z^2 \cosh(2\eta) \right\} dz. \quad (83)$$

In Equation (83), the Gaussian factor can be seen to shrink by $1/\sqrt{\cosh(2\eta)}$, which is a result of the Lorentz squeeze. Now, completing the integral in Equation (83), we obtain

$$F(p) = \frac{1}{\cosh(2\eta)} \exp \left\{ \frac{-p^2}{\cosh(2\eta)} \right\}. \quad (84)$$

The Gaussian factor, as p^2 becomes large, can be seen to be a constant. The form factor decrease of $1/p^2$ follows from the factor $1/\cosh(2\eta)$. This decrease is much slower than if the exponential cut-off was considered without the squeeze effect.

Now, we have to consider that there is still a difference between Equation (84) and the experimental data observed. This is because there are three quarks in the proton; therefore, there are two harmonic oscillator modes. Feynman et al. [32], considering this fact, worked out this problem and showed that it leads to the $(1/p^2)^2$ cut-off, consistent with that observed in high-energy experiments.

Although there are some reports of deviations from the exact dipole cut-off, it is worth noting that the Lorentz squeeze resulted in a polynomial decrease in momentum transfer. Here, we started from the fundamental principles of special relativity and quantum mechanics.

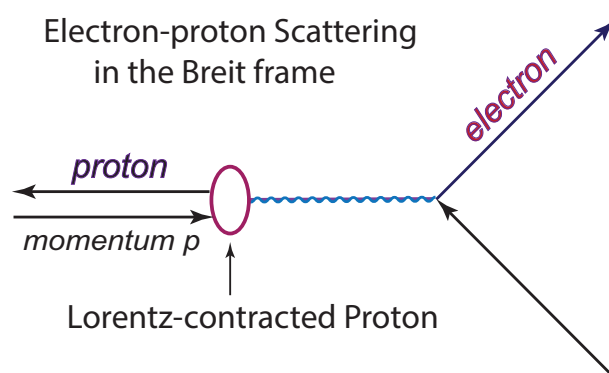


Figure 4. Breit frame for scattering of an electron by a nucleon. The incoming nucleon, after scattering the electron with no change in nucleon momentum, goes out along along the incoming axis but in the opposite direction [28].

6.2.2. The Parton Picture

It is widely accepted that hadrons are bound states of quarks. There is convincing evidence of this if one considers the hadron mass spectra [32]. But this bound state picture is only applicable to those observers who are in the ground state. The following question arises: What does an observer in another Lorentz frame perceive? This question led to Feynman making the following systematic observations:

- (a) The picture is only valid if the hadrons are moving at close to light speed.
- (b) The partons behave as independent free particles, and the interaction time of the quarks becomes dilated between the quarks.
- (c) The hadron appears to have a widespread momentum distribution of partons.
- (d) The parton number appears to be much greater than that of quarks or even infinite.

Each of the above phenomena, especially (b) and (c) together, appears paradoxical in view of the fact that the hadron is thought to be a bound state of quarks. The solution to this issue lies in the Lorentz squeeze property of hadrons. To resolve this paradox, we can use the momentum-energy wave function given in Equation (71). Since we are using

the harmonic oscillator, as the hadron gains speed, both the momentum-energy and the space-time wave functions have, as we have seen in Section 6.1.1, the same form. These wave-functions, thus, have the same Lorentz squeeze properties, as shown in Figure 5. When we calculate the hadron structure function from this formalism, it is in reasonable agreement with experimental data [48].

The squeezed-wave functions, as the spring constant appears weaker, have a wide-spread distribution. Thus, the quarks inside the hadron appear free. The squeezed-wave functions describing the quarks, if confined to a narrow region, become compressed along the light cone axis and appear as massless particles. This effect is a consequence of Lorentz covariance.

Another puzzling aspect is that the quarks in the rest state are coherent, while the partons appear to be incoherent. The squeeze along the light cone axis means that the interaction time among the quarks becomes dilated. As the wave function spreads, the distance between the ends of the harmonic oscillator well spread apart. This effect was first noticed by Feynman [41,42] and is again observed in all high-energy hadron experiments. The oscillation period increases as e^η . However, as the signal is traveling in a direction opposite of the hadron, the external signal is traveling along the negative light-cone axis. This is also shown in Figure 5.

Since the hadron is contracted along the negative light-cone axis, the interaction time is decreased by $e^{-\eta}$. Then, we can take the ratio of the interaction time to the period of oscillation, which becomes $e^{-2\eta}$. Each proton coming out of the large hadron collider has an energy of 13 TeV. Therefore, we have a ratio of $1.25 \cdot 10^{-9}$, which is a very small number. This results in the external signal being unable to sense the interaction of the quarks with each other. We can see that the covariant harmonic oscillator, by providing one Lorentz covariant entity, produces the quark and parton models as two limiting cases.

This distribution resembles black-body radiation, where there is an infinite number of photon distributions. Then, with the covariant harmonic formalism, one entity is provided, which makes it possible to conclude that the quark and parton models are two distinct manifestations of the same Lorentz-covariant quantity [12,47,49].

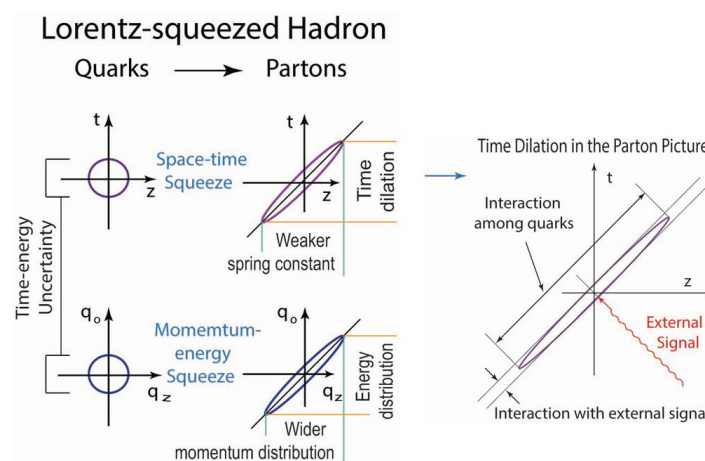


Figure 5. Since Lorentz-squeezed momentum-energy and space-time wave functions have the same deformation properties, as the speed of the hadron approaches that of light, both of these wave functions, as can be seen, become concentrated along positive light-cone axes, respectively. The concentrations along the light cone are what lead to Feynman’s parton picture [50]. As the external signal is moving in the opposite direction to that of the hadron, it travels along the negative light-cone axis. Thus, this signal has an interaction time with the bound state that is much shorter than the oscillation period of the quarks inside the hadron. Hence, this effect is often known as Feynman’s time dilation [28,41,42,49,51].

6.3. Application to Classical Optics: Laser Cavity

We consider a laser cavity consisting of two identical concave mirrors facing each other with a radius of curvature R and separated by a distant d , as shown in Figure 6. A round trip of the beam inside the cavity can be expressed by the $ABCD$ matrix as

$$[ABCD] = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix} \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}, \tag{85}$$

where the basic matrices are

$$T(d) = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad M(R) = \begin{pmatrix} 1 & 0 \\ -2/R & 1 \end{pmatrix}, \tag{86}$$

namely, the translation and mirror matrices, respectively [52–56].

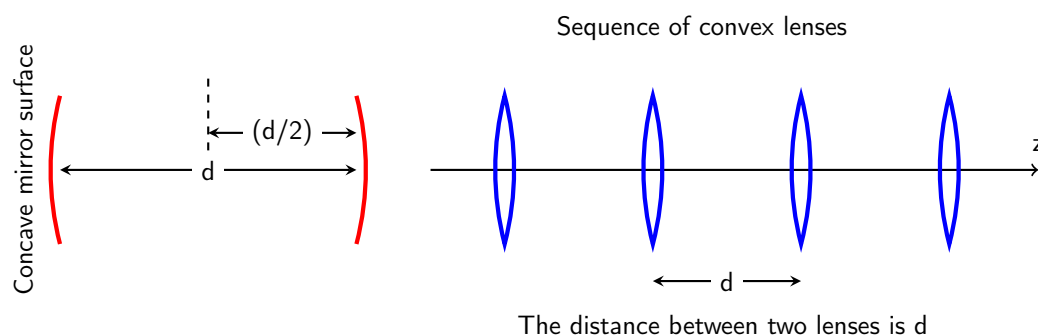


Figure 6. The picture on left illustrates a resonator in a laser cavity. The light beam within the cavity is reflected back and forth for a very large number of times. The mathematical formulation is equivalent when the beam passes through a very large number of lenses. The figure on the right represents this analogous situation [57].

In a laser cavity, the amplification of light is achieved by multiple reflections, the process which is formulated by taking the N^{th} power of the $[ABCD]$ matrix, where N is very large. However, evaluating the N^{th} power is more tractable, if the matrix is in an equidiagonal form. To this end, the $[ABCD]$ matrix is rewritten as

$$[ABCD] = T(-d/2)T(d/2)M(R)[T(d/2)]^2M(R)T(d/2)T(d/2), \tag{87}$$

which can be expressed in the form of a similarity transformation:

$$[ABCD] = \begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 - d/R & d - d^2/2R \\ -2/R & 1 - d/R \end{pmatrix} \right]^2 \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix}. \tag{88}$$

Extracting the core of the above sequence, we have

$$\begin{pmatrix} 1 - d/R & d - d^2/2R \\ -2/R & 1 - d/R \end{pmatrix} = \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix} \begin{pmatrix} 1 - d/R & 1 - d/2R \\ -2d/R & 1 - d/R \end{pmatrix} \begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix}. \tag{89}$$

Then,

$$[ABCD] = \begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix} \left[\begin{pmatrix} 1 - d/R & 1 - d/2R \\ -2d/R & 1 - d/R \end{pmatrix} \right]^2 \begin{pmatrix} 1/\sqrt{d} & 0 \\ 0 & \sqrt{d} \end{pmatrix} \begin{pmatrix} 1 & d/2 \\ 0 & 1 \end{pmatrix} \tag{90}$$

or, in a compact form,

$$[ABCD] = E C^2 E^{-1}, \tag{91}$$

where

$$C = \begin{pmatrix} 1 - d/R & 1 - d/2R \\ -2d/R & 1 - d/R \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} 1 & -d/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{d} & 0 \\ 0 & 1/\sqrt{d} \end{pmatrix}. \quad (92)$$

Since the components of the C matrix are dimensionless numbers, it can be rewritten as

$$C(\gamma, \eta) = \begin{pmatrix} \cos(\gamma/2) & e^\eta \sin(\gamma/2) \\ -e^{-\eta} \sin(\gamma/2) & \cos(\gamma/2) \end{pmatrix}, \quad (93)$$

with

$$\cos(\gamma/2) = 1 - \frac{d}{R} \quad \text{and} \quad e^\eta = \sqrt{\frac{2R-d}{4d}}. \quad (94)$$

Here, physical parameters d and R are constrained by $d \leq R$, where both are positive. This restriction is commonly referred to as the *stability criteria* in the literature [53,55].

Now, C^2 can be decomposed as a similarity transformation:

$$\begin{aligned} C^2(\gamma, \eta) &= \begin{pmatrix} \cos \gamma & e^\eta \sin \gamma \\ -e^{-\eta} \sin \gamma & \cos \gamma \end{pmatrix} \\ &= \begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix} \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} e^{-\eta/2} & 0 \\ 0 & e^{\eta/2} \end{pmatrix} \end{aligned} \quad (95)$$

which is $B(\eta) R(2\gamma) B(\eta)^{-1}$, in a compact form. This expression, as we have described through Equation (16), can be recognized as the two-by-two representation of the little group for a massive particle moving in the z -direction, and the decomposition is sometimes referred as the *Wigner decomposition* in the literature [58].

Hence, the $ABCD$ matrix becomes

$$[ABCD] = [E B(\eta)] R(2\gamma) [E B(\eta)]^{-1}. \quad (96)$$

Now, it is possible to express the repeated application of this process as

$$[ABCD]^N = [E B(\eta)] R(2N\gamma) [E B(\eta)]^{-1}, \quad (97)$$

which is the formulation describing multiple reflections of the light beam in a laser cavity.

6.4. Applications to Quantum Optics: Shear States

Two of the most important concepts in the quantum theory of light are the coherent and squeezed states [59]. They both sustain minimum uncertainty product $(\Delta x)(\Delta p)$. On the other hand, for shear states, as lesser-known states, they are as canonical as the other two. The uncertainty product is meaningfully preserved, when t has some finite value. In the phase space, the time evolution is described as [36]

$$\begin{pmatrix} q' \\ p' \end{pmatrix} = \begin{pmatrix} 1 & t/m \\ 0 & 1 \end{pmatrix} \begin{pmatrix} q \\ p \end{pmatrix}, \quad (98)$$

where it is apparent that the time evolution matrix belongs to $E(2)$ -like group.

Quantum image processing, a growing subfield of quantum computing and quantum information, focuses on processing visual data. By and large, it relies on knowledge of quantum mechanics. It is well known that a series of three alternating shear transformations produces a rotation of an image [28,60]. It is possible to employ quantum circuits based on shear operators to process quantum states representing images [61].

The relation between sheared states and a squeezed states can best be perceived by considering the Wigner functions in the phase space [62]. We start with the vacuum state Wigner function:

$$W_0(q, p) = \frac{1}{\pi} \exp\left\{-(q^2 + p^2)\right\}. \tag{99}$$

When (p, q) are translated by the real and imaginary part, respectively, of a complex number, we obtain a coherent state. Nevertheless, we proceed with the vacuum state, and employ the shear transformation of Equation (98). Then this function becomes

$$W_{sh}(q, p) = \frac{1}{\pi} \exp\left\{-[(q - 2ap)^2 + p^2]\right\} \tag{100}$$

with $t/m = 2a$. Let us next consider the rotated and squeezed form of Equation (99):

$$W_{sq}(q, p) = \frac{1}{\pi} \exp\left\{-[e^{-2\eta}(q \cos \theta + p \sin \theta)^2 + e^{2\eta}(-q \sin \theta + p \cos \theta)^2]\right\}. \tag{101}$$

Now, it can be seen that a sheared Wigner function can be obtained from above by rearranging the parameters as

$$\tan(2\theta) = \frac{1}{a}, \quad e^{2\eta} = 1 + 2a^2 + 2a\sqrt{a^2 + 1}, \quad e^{-2\eta} = 1 + 2a^2 - 2a\sqrt{a^2 + 1}. \tag{102}$$

Therefore, to put it briefly, a sheared vacuum is a very special case of a squeezed vacuum. A visual representation of these transformations is given in Figure 7.

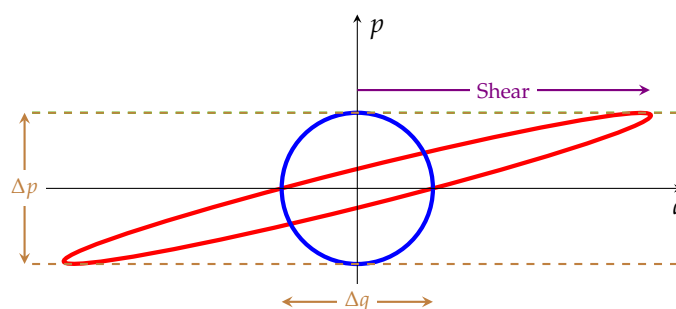


Figure 7. The circle in the figure corresponds to the vacuum state. The rotated ellipse is for the squeezed vacuum. They both have the same area. The ellipse is rotated with a specific angle as in Equation (102), in order to coincide with a sheared state. From the axis viewpoint, the p -axis remains invariant, while q -axis is expanded; thus, a circle subjected to a shear transformation becomes an ellipse [36].

Now, sheared states will be constructed in the Fock space. In this space, creation and annihilation operators are defined as

$$a^\dagger|n\rangle = \sqrt{n + 1}|n + 1\rangle \quad \text{and} \quad a|n\rangle = \sqrt{n}|n - 1\rangle. \tag{103}$$

In particular, an annihilation or step-down operator decreases by one the number of particles in a given state, while its adjoint, the creation or step-up operator, increases by one the number of particles in a given state. These operators satisfy the following commutation relations:

$$[a, a^\dagger] = 1, \quad [a, a] = 0, \quad [a^\dagger, a^\dagger] = 0. \tag{104}$$

They appear in a variety of physical disciplines such as in quantum mechanics [62], quantum optics [63], and quantum field theory [64] and are regarded as an essential part of their program.

In order to construct shear states, we shall make use of the *squeeze-rotation* generators of $Sp(2)$. First, we note that there are four operators, $aa, a^\dagger a^\dagger, aa^\dagger$ and $a^\dagger a$, subject to the restrictions of Equation (104), reducing them to three, which can be combined in a Hermitian matrix [65]:

$$\begin{pmatrix} \frac{aa^\dagger + a^\dagger a}{2} & aa \\ a^\dagger a^\dagger & \frac{aa^\dagger + a^\dagger a}{2} \end{pmatrix}. \quad (105)$$

This leads to three independent operators,

$$\hat{f}_2 = \frac{1}{4}(a^\dagger a + aa^\dagger), \quad \hat{K}_1 = -\frac{1}{4}(aa + a^\dagger a^\dagger), \quad \hat{K}_3 = -\frac{i}{4}(aa - a^\dagger a^\dagger), \quad (106)$$

satisfying the same commutation relations as those of Equation (28):

$$[\hat{K}_1, \hat{K}_3] = i\hat{f}_2, \quad [\hat{K}_1, \hat{f}_2] = i\hat{K}_3, \quad [\hat{K}_3, \hat{f}_2] = -i\hat{K}_1. \quad (107)$$

In view of Equation (32), the *shear-squeeze* representation is written as

$$\hat{X}_1 = -\frac{1}{4}(a - a^\dagger)^2, \quad \hat{X}_2 = -\frac{1}{4}(a + a^\dagger)^2, \quad \hat{X}_3 = -\frac{i}{2}(aa - a^\dagger a^\dagger). \quad (108)$$

In the Fock space, shear states can be constructed by applying shear transformations to the vacuum state:

$$\exp(-i\alpha\hat{X}_1)|0\rangle = \exp\left\{i\frac{\alpha}{4}(a - a^\dagger)^2\right\}|0\rangle \quad (109)$$

$$\exp(-i\beta\hat{X}_2)|0\rangle = \exp\left\{i\frac{\beta}{4}(a + a^\dagger)^2\right\}|0\rangle \quad (110)$$

where the power series expansion reads as

$$\sum \frac{1}{n!} \left(\frac{i\alpha}{4}\right)^n (a - a^\dagger)^{2n} |0\rangle, \quad \sum \frac{1}{n!} \left(\frac{i\beta}{4}\right)^n (a + a^\dagger)^{2n} |0\rangle, \quad (111)$$

respectively. The series is constructed; however, the computation of $(a \pm a^\dagger)^{2n}|0\rangle$ can be rather involved. This difficulty can be circumvented by decomposing the shear operator into squeeze and rotation operators. To achieve this, we refer to Equations (35) and (36). When these relations are taken into account, shear operators can now be written in a more comprehensible form:

$$\exp(-i\alpha\hat{X}_1) = \exp\left\{i\alpha(a - a^\dagger)^2/4\right\} = \exp(i\theta_1\hat{f}_2) \exp(-i\lambda\hat{K}_3) \exp(-i\theta_2\hat{f}_2), \quad (112)$$

$$\exp(-i\beta\hat{X}_2) = \exp\left\{i\beta(a + a^\dagger)^2/4\right\} = \exp(i\theta_3\hat{f}_2) \exp(-i\rho\hat{K}_3) \exp(-i\theta_4\hat{f}_2). \quad (113)$$

From Equations (37) and (38), we already have the relations between the shear parameters α, β and the rotation parameters θ_i , along with the squeeze parameters λ, ρ .

These operators are applied to the vacuum state. Then, $\exp(-i\theta_2\hat{f}_2)|0\rangle$ and $\exp(-i\theta_4\hat{f}_2)|0\rangle$ become $\exp(i\theta_2/4)|0\rangle$ and $\exp(i\theta_4/4)|0\rangle$, respectively. Therefore, sheared vacuum states become

$$\exp(-i\alpha\hat{X}_1)|0\rangle = \exp(i\theta_2/4) \exp(i\theta_1\hat{f}_2) \exp(-i\lambda\hat{K}_3)|0\rangle, \quad (114)$$

$$\exp(-i\beta\hat{X}_2)|0\rangle = \exp(i\theta_4/4) \exp(i\theta_3\hat{f}_2) \exp(-i\rho\hat{K}_3)|0\rangle. \quad (115)$$

These equations indicate that the sheared vacuum is a squeezed vacuum followed by a rotation.

7. Conclusions

In this article, we studied the Lorentz and Poincaré group, the subgroups, and their properties. Among these subgroups, little groups are examined in detail for both massive and massless particles.

We gave various examples with a view to contend that having similar transformation properties establishes connections between seemingly unrelated disciplines of physics. For instance, let us examine Equation (77), and rename the variables x_1 and x_2 as t and z ; then, we observe that the right-hand side of the equation falls in the realm of special theory of relativity, while the left-hand side comes from quantum optics. We also note that the squeezed Gaussian function of this equation is crucial for constructing the wave functions of hadrons moving at relativistic speeds.

Additionally, we illustrate how these properties facilitate calculations and allow us to extract further information from the system under consideration. We see that Wigner decomposition of the little group transformation rendered the calculations tractable in periodic systems, where we had to evaluate a large number of cycles. For instance, we examined a laser cavity whose primary matrices belong to $Sp(2)$. Further, we used the squeeze-rotation generators of $Sp(2)$ to construct quantum shear states of light, revealing that the sheared vacuum is a squeezed vacuum followed by a rotation.

The Poincaré group and the subgroups are rooted in the backbone formulations of quantum mechanics along with relativistic quantum mechanics of extended objects, as well as contemporary optics. We analyzed systems sharing the same underlying symmetries across disciplines, giving us the means to transfer knowledge between them. We hope that this approach revives interest in subjects that may otherwise appear to be dormant.

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Appendix A

For ease of use, we tabulate the transformation matrices that are obtained from the generators of $SL(2, c)$ and $SO(3, 1)$.

Table A1. The two-by-two transformation matrices of $SL(2, c)$ and their corresponding four-by-four transformation matrices in $SO(3, 1)$. The four-by-four matrices are applicable to the Minkowski space with the coordinate system $x^\mu = (t, z, x, y)$, and the Minkowski metric is chosen as $\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1)$ [28].

Exponentiation	Two-by-Two	Four-by-Four
$R_x(\theta) = \exp[-i\theta J_1]$	$\begin{pmatrix} \cos(\theta/2) & i \sin(\theta/2) \\ i \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta & 0 & \cos \theta \end{pmatrix}$
$B_x(\lambda) = \exp[-i\lambda K_1]$	$\begin{pmatrix} \cosh(\lambda/2) & \sinh(\lambda/2) \\ \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	$\begin{pmatrix} \cosh \lambda & 0 & \sinh \lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh \lambda & 0 & \cosh \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$R_y(\theta) = \exp[-i\theta J_2]$	$\begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$
$B_y(\lambda) = \exp[-i\lambda K_2]$	$\begin{pmatrix} \cosh(\lambda/2) & -i \sinh(\lambda/2) \\ i \sinh(\lambda/2) & \cosh(\lambda/2) \end{pmatrix}$	$\begin{pmatrix} \cosh \lambda & 0 & 0 & \sinh \lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \lambda & 0 & 0 & \cosh \lambda \end{pmatrix}$
$R_z(\phi) = \exp[-i\phi J_3]$	$\begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \phi & -\sin \phi \\ 0 & 0 & \sin \phi & \cos \phi \end{pmatrix}$
$B_z(\eta) = \exp[-i\eta K_3]$	$\begin{pmatrix} e^{\eta/2} & 0 \\ 0 & e^{-\eta/2} \end{pmatrix}$	$\begin{pmatrix} \cosh \eta & \sinh \eta & 0 & 0 \\ \sinh \eta & \cosh \eta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

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