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Article

Generalizing Coherent States with the Fox *H* Function

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Abstract

In the present scenario, coherent states of a quantum harmonic oscillator are generalized with positive Fox *H* auxiliary functions. The novel generalized coherent states provide canonical coherent states and Mittag-Leffler or Wright generalized coherent states, as particular cases, and resolve the identity operator, over the Fock space, with a weight function that is the product of a Fox *H* function and a Wright generalized hypergeometric function. The novel generalized coherent states, or the corresponding truncated generalized coherent states, are characterized by anomalous statistics for large values of the number of excitations: the corresponding decay laws exhibit, for determined values of the involved parameters, various behaviors that depart from exponential and inverse-power-law decays, or their product. The analysis of the Mandel *Q* factor shows that, for small values of the label, the statistics of the number of excitations becomes super-Poissonian, or sub-Poissonian, by simply choosing sufficiently large values of one of the involved parameters. The time evolution of a generalized coherent state interacting with a thermal reservoir and the purity are analyzed.

Keywords: quantum harmonic oscillator; generalized coherent states; Fox *H*-functions; Wright generalized hypergeometric functions; purity; thermal reservoir

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1. Introduction

A quantum harmonic oscillator is fundamental for the description of the most varied systems in the framework of quantum theory. In this regard, a basal example is represented by the quantization of a single electromagnetic field mode. Coherent states (CSs) are special states of a quantum harmonic oscillator that are characterized by minimum uncertainty and exhibit quasi-classical properties in the time evolution. These states find the most varied applications in mathematical physics, quantum optics, and quantum information. See Refs. [1–12].

During the last decades, CSs have been generalized in various ways. See Refs. [4,6,8,13–17]. Klauder has generalized CSs by requiring the constraints of normalizability, continuity in the label which characterizes these states, and resolution of the identity operator over the canonical Fock space with a positive weight function [4–6,8]. Mittag-Leffler or Wright generalized coherent states (GCSs) are some examples of Klauder's generalization [13–16]. These states are named after the normalization factors that are represented by the Mittag-Leffler function or Wright functions of the square modulus of the label, respectively [18–21]. Mittag-Leffler or Wright GCSs exhibit anomalous statistics for large values of the number of excitations.

Truncated coherent states (TCSs) are obtained from canonical CSs by considering the Fock space to be finite-dimensional (truncated). Refer to [22–32] for details. If compared to canonical CSs, TCSs exhibit further properties [27,32]. Truncated generalized coherent states (TGCSs) are defined by requesting that the three above-reported conditions introduced by Klauder hold over the truncated Fock space. Thus, TGCSs resolve the identity operator over the truncated Fock space with a positive weight function [22–30,32]. Mittag-Leffler and Wright TGCSs are special examples of TGCSs [16].

The experimental realization of CSs in mechanical resonators [33], or with lasers [34], might perturb these quantum states and change their properties. For example, the photon number statistics of real lasers deviate from the canonical Poisson distribution that is predicted theoretically [34,35]. Furthermore, consider a single-mode quantized field of light that interacts with a low-dissipative Kerr-like nonlinear medium. Perturbations of the initial CS are found to be constructive or destructive against the dissipative processes, according to their nature [36]. Perturbations of canonical Schrödinger cat states, given by Wright generalized Schrödinger cat states, maintain, over determined time intervals, the regularities that the canonical Schrödinger cat states exhibit if they are exposed to amplitude damping noise [37]. This property does not hold for perturbations of canonical Schrödinger cat states that are given by Mittag-Leffler generalized Schrödinger cat states.

As a continuation of the above-reported line of research, in the present scenario, we intend to determine GCSs that result in Mittag-Leffler or Wright GCSs and canonical CSs, as particular cases. In this way, the novel GCSs provide perturbations of canonical CSs that include the perturbations of Mittag-Leffler or Wright type, as particular cases. By definition, the novel GCSs are required to resolve the identity operator with a positive weight function. The novel GCSs allow for an examination of whether and how the physical properties of canonical CSs are affected by perturbations that might arise from the preparation procedure.

Mittag-Leffler and Wright functions are particular cases of the Fox H function [38–46]. The weight function characterizing Mittag-Leffler GCSs is a product of a power law, a stretched exponential, and a Mittag-Leffler function [13,14]. Instead, the weight function characterizing Wright GCSs is a product of a Wright function and a Fox H function [15–17]. The Fox H function is a special function that is defined via the Mellin-Barnes integrals. This function finds applications in the most varied areas of mathematics, statistics, and physics. In this regard, refer to [43,45–50]. In the present scenario, we intend to define novel GCSs by choosing positive Fox H functions as auxiliary function [51]. The novel GCSs are required to describe the above-mentioned perturbations of canonical CSs, as particular cases. Possibly, this approach provides novel forms of weight functions that resolve the identity operator over the Fock space with the novel class of GCSs [4–6,8,13–17]. We intend to study the distribution of the number of excitations that characterizes the novel GCSs, the Mandel Q factor [52], and the evolution of the novel GCSs induced by the interaction with a zero-temperature reservoir or a thermal reservoir [37,53–56].

The paper is organized as follows. GCSs and TGCSs are introduced in Section 2 for the sake of clarity and completeness. Section 3 is devoted to the generalization of CSs and TCSs by choosing positive Fox H-functions as auxiliary functions. In Section 4, we analyze the statistics of the number of excitations of novel GCSs via the Mandel Q factor. The dissipative processes induced by the interaction with a zero-temperature reservoir are analyzed in Section 5. In Section 6, the time evolution and the purity are studied in the case where the GCS interacts with a thermal reservoir. A summary of the results and conclusions is reported in Section 7. Details of the calculations are provided in Appendix A.

2. GCSs and TGCSs

For the sake of clarity and completeness, in the present Section, we report the definition and the main properties of GCSs. The Fock basis $\mathcal{F}$ of a quantum harmonic oscillator is composed of the eigenstates of the quantum number operator, $\mathcal{F} \equiv \{|0\rangle, |1\rangle, \ldots\}$. The eigenstates are mutually orthogonal and normalized to unity, $\langle j||k\rangle = \delta_{j,k}$, for every $j, k \in \mathbb{N}_0$, where $\mathbb{N} \equiv \{1, 2, \ldots\}$, $\mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$, and $\delta_{j,k}$ is the Kronecker symbol.

By definition, the class $\{|z; g\rangle, \ \forall z \in \mathbb{C}\}$ of GCSs is required to fulfill the conditions of normalizability, continuity in the label, and the resolution of the identity with a positive weight function. This class is generated by the arithmetic function $g(n)$ of the natural variable n, and it is defined over the Fock basis $\mathcal{F}$ as below,

$$|z; g\rangle = \left[N_{\rm g}\left(|z|^2\right)\right]^{-1/2} \sum_{n=0}^{+\infty} \frac{z^n}{\sqrt{g(n)}} |n\rangle, \tag{1}$$

for every $z \in \mathbb{C} \setminus \{0\}$. Thus, the arithmetic function $g(n)$ is required to be positive, $g(n) > 0$ for every $n \in \mathbb{N}_0$. The GCS $|0; g\rangle$ is the ground state of the quantum harmonic oscillator or, equivalently, the state with no excitation (vacuum state), $|0; g\rangle = |0\rangle$. By definition, the normalization factor $N_{\rm g}(|z|^2)$, given by the expression below,

$$N_{\rm g}(u) = \sum_{n=0}^{+\infty} \frac{u^n}{g(n)}, \tag{2}$$

is required to be positive and finite,

$$0 < N_{\rm g}(u) < +\infty, \tag{3}$$

for every $u > 0$. Condition (3) of normalizability is realized for every $z \in \mathbb{C} \setminus \{0\}$ if only

$$\lim_{n \to +\infty} \sup_{n' \geq n} \left[g(n')\right]^{-1/n'} = 0. \tag{4}$$

Due to Condition (4), the following power series: $\sum_{n=0}^{\infty} \zeta^n / g(n)$ exists and is a continuous function of the complex variable ζ for every $\zeta \in \mathbb{C}$. Thus, the following limit: $\mathrm{Re}\langle z||z_0\rangle \to 1$, holds as $z \to z_0$, and the required continuity in the label, $|z; g\rangle \to |z_0; g\rangle$, is realized for every $z, z_0 \in \mathbb{C}$.

By definition, the class $\{|z; g\rangle, \ \forall z \in \mathbb{C}\}$ of GCSs resolves the identity operator I over the Fock basis $\mathcal{F}$ if a weight function $U_{\rm g}(u)$, positive on the set of the positive real numbers $\mathbb{R}^+$, exists such that the following relation holds [4,5,8,13,14]:

$$\int_{\mathbb{R}^2} U_{\rm g}\left(|z|^2\right) |z; g\rangle\langle z; g| d^2 z = I, \tag{5}$$

where $d^2 z = d\,\mathrm{Re}(z) d\,\mathrm{Im}(z)$. The resolution of the identity operator is determined by the properties of the arithmetic function $g(n)$ [4,5,8,13,14,57]. In fact, let the auxiliary function $f(u)$ be defined on $\mathbb{R}^+$ via the weight function $U_{\rm g}(u)$ and the normalization factor $N_{\rm g}(u)$ as below [4,5,8,13,14,57]:

$$f(u) \equiv \pi \frac{U_{\rm g}(u)}{N_{\rm g}(u)}, \tag{6}$$

for every $u > 0$. The resolution of the identity operator, Equation (5), holds over the Fock space $\mathcal{F}$ if

$$\hat{f}(n+1) = g(n), \tag{7}$$

for every $n \in \mathbb{N}_0$. The function $\hat{f}(s)$ is the Mellin transform of the auxiliary function $f(u)$,

$$\hat{f}(s) = \int_0^\infty f(u) u^{s-1} du, \tag{8}$$

for every value of the complex variable s such that the involved integral exists [58–62].

The probability $P_g\left(n, |z|^2\right)$ that the GCS $|z; g\rangle$ is characterized by n excitations, i.e., the state $|n\rangle$, is

$$P_g\left(n, |z|^2\right) = \frac{|z|^{2n}}{N_g(|z|^2) g(n)}, \tag{9}$$

for every $n \in \mathbb{N}_0$, and $z \in \mathbb{C} \setminus \{0\}$, while $P_g(n, 0) = \delta_{n,0}$, for every $n \in \mathbb{N}_0$.

The truncated Fock basis $\mathcal{F}_{\mathfrak{d}}$ of a quantum harmonic oscillator is defined as follows: $\mathcal{F}_{\mathfrak{d}} \equiv \{|0\rangle, \ldots, |\mathfrak{d}\rangle\}$, for every $\mathfrak{d} \in \mathbb{N}$. The corresponding truncated Fock space is $(\mathfrak{d}+1)$-dimensional. The class $\{|z; g; \mathfrak{d}\rangle, \ \forall z \in \mathbb{C}\}$ of TGCSs is defined over the truncated Fock basis $\mathcal{F}_{\mathfrak{d}}$ as below:

$$|z; \mathfrak{d}; g\rangle = \left[N_{\mathfrak{d},g}\left(|z|^2\right)\right]^{-1/2} \sum_{n=0}^{\mathfrak{d}} \frac{z^n}{\sqrt{g(n)}} |n\rangle, \tag{10}$$

for every $z \in \mathbb{C} \setminus \{0\}$ and $\mathfrak{d} \in \mathbb{N}$, while $|0; \mathfrak{d}, g\rangle = |0\rangle$ for every $\mathfrak{d} \in \mathbb{N}$. The normalization factor $N_{\mathfrak{d},g}(|z|^2)$ is given by the form below,

$$N_{\mathfrak{d},g}(u) = \sum_{n=0}^{\mathfrak{d}} \frac{u^n}{g(n)}, \tag{11}$$

for every $u > 0$ and $\mathfrak{d} \in \mathbb{N}$. TGCSs are required to resolve the identity operator I over the truncated Fock basis $\mathcal{F}_{\mathfrak{d}}$,

$$\int_{\mathbb{R}^2} U_{\mathfrak{d},g}\left(|z|^2\right) |z; \mathfrak{d}; g\rangle \langle z; \mathfrak{d}; g| d^2 z = I. \tag{12}$$

Relations (6) and (7) hold for the weight function $U_{\mathfrak{d},g}(u)$ and the normalization factor $N_{\mathfrak{d},g}(u)$, for every $\mathfrak{d} \in \mathbb{N}$, if the Mellin transform $\hat{f}(s)$ of the positive auxiliary function $f(u)$ exists for $1 \leq \operatorname{Re} s \leq \mathfrak{d} + 1$.

The probability $P_g\left(n, \mathfrak{d}, |z|^2\right)$ that the GCS $|z; \mathfrak{d}; g\rangle$ is characterized by n excitations, i.e., the state $|n\rangle$, is

$$P_g\left(n, \mathfrak{d}, |z|^2\right) = \frac{|z|^{2n}}{N_{\mathfrak{d},g}(|z|^2) g(n)}, \tag{13}$$

for every $n = 0, \ldots, \mathfrak{d}$, while $P_g(n, \mathfrak{d}, 0) = \delta_{n,0}$, for every $n = 0, \ldots, \mathfrak{d}$, and $\mathfrak{d} \in \mathbb{N}$.

3. GCSs Characterized by Positive Fox *H* Auxiliary Functions

In the present Section, we intend to investigate whether canonical CSs and Mittag-Leffler or Wright GCSs can be generalized further via Fox H auxiliary functions. For the sake of clarity and completeness, we report below the definition of the Fox H function and conditions under which Fox H functions are positive.

Briefly, the Fox H function is defined as below [38–43]:

$$H^{m,n}_{p,q}\left[z \left| \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{matrix} \right. \right] = \frac{1}{2\pi\imath} \int_{\mathcal{C}} \Xi^{m,n}_{p,q}\left[s \left| \begin{matrix} (\alpha_j, A_j)_1^p \\ (\beta_j, B_j)_1^q \end{matrix} \right. \right] z^{-s} ds, \tag{14}$$

where

$$\Xi_{p,q}^{m,n}\left[s\left|\begin{array}{c}(\alpha_j, A_j)_1^p\\(\beta_j, B_j)_1^q\end{array}\right.\right] = \frac{\prod_{j=1}^{m}\Gamma(\beta_j + B_j s)\prod_{j=1}^{n}\Gamma(1-\alpha_j - A_j s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j - B_j s)\prod_{j=n+1}^{p}\Gamma(\alpha_j + A_j s)}. \tag{15}$$

Poles of the Gamma functions $\Gamma(\beta_1 + B_1 s), \ldots, \Gamma(\beta_m + B_m s)$ are required to differ from the poles of the Gamma functions $\Gamma(1-\alpha_1 - A_1 s), \ldots, \Gamma(1-\alpha_n - A_n s)$. This property is provided by the following inequality:

$$A_j\left(l + \beta_{j'}\right) \neq B_{j'}(\alpha_j - l' - 1), \tag{16}$$

that is requested to hold for every $j = 1, \ldots, n$, $j' = 1, \ldots, m$, and $l, l' \in \mathbb{N}_0$. The empty products coincide with unity. The allowed values of the indexes n and m are $0 \leq n \leq p$, $0 \leq m \leq q$, and $A_i, B_j \in \mathbb{R}^+$, $\alpha_i, \beta_j \in \mathbb{C}$ for every $i = 1, \cdots, p$ and $j = 1, \cdots, q$, where $\mathbb{C}$ is the set of the complex numbers. The following notation is adopted for the sake of shortness: $(x_j)_1^l \equiv x_1, \ldots, x_l$ for every $l \in \mathbb{N}$, while $(x_j)_1^l \equiv 1$ for $l = 0$; $(x_j, X_j)_1^l \equiv (x_1, X_1), \ldots, (x_l, X_l)$ for every $l \in \mathbb{N}$, while $(x_j, X_j)_1^l \equiv 1$ for $l = 0$; and $((x_j, X_j), (y_j, Y_j))_1^l \equiv (x_1, X_1), (y_1, Y_1), \ldots, (x_l, X_l), (y_l, Y_l)$ for every $l \in \mathbb{N}$. Refer to [42,43] for the existence condition, the domain of analiticity and the contour path $\mathcal{C}$ that is adopted in the definition (14) of the Fox H function.

The Wright generalized hypergeometric function is a special case of the Fox H function and is defined by the below-reported power series [42,43],

$$\begin{aligned}{}_p\Psi_q\left[z\left|\begin{array}{c}(\alpha_j, A_j)_1^p\\(\beta_k, B_k)_1^q\end{array}\right.\right] &= \sum_{n=0}^{+\infty}\frac{\prod_{j=1}^{p}\Gamma(\alpha_j + A_j n)}{\prod_{k=1}^{q}\Gamma(\beta_k + B_k n)}\frac{z^n}{n!}\\ &= H_{p,q+1}^{1,p}\left[-z\left|\begin{array}{c}(1-\alpha_j, A_j)_1^p\\(0,1), (1-\beta_k, B_k)_1^q\end{array}\right.\right],\end{aligned} \tag{17}$$

for every $z \in \mathbb{C}\setminus\{0\}$, $\alpha_j, \beta_k \in \mathbb{C}$, $A_j, B_k \in \mathbb{R}_+$. The Wright generalized hypergeometric function is an entire function of the complex variable z for every $z \in \mathbb{C}\setminus\{0\}$ if $\mu > -1$, where

$$\mu = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j. \tag{18}$$

The Mellin transform of the Fox H function [42,43]:

$$\int_0^{+\infty} u^{s-1} H_{p,q}^{m,n}\left[u\left|\begin{array}{c}(\alpha_j, A_j)_1^p\\(\beta_j, B_j)_1^q\end{array}\right.\right] du = \Xi_{p,q}^{m,n}\left[s\left|\begin{array}{c}(\alpha_j, A_j)_1^p\\(\beta_j, B_j)_1^q\end{array}\right.\right], \tag{19}$$

exists for

$$-\min_{j=1,\ldots,m}\left\{\frac{\Re(\beta_j)}{B_j}\right\} < \Re(s) < \min_{j=1,\ldots,n}\frac{1-\Re(\alpha_j)}{A_j}, \tag{20}$$

if $\nu > 0$, where

$$\nu = \sum_{j=1}^{n} A_j - \sum_{j=n+1}^{p} A_j + \sum_{j=1}^{m} B_j - \sum_{j=m+1}^{q} B_j. \tag{21}$$

A class of Fox H functions, positive on $\mathbb{R}^+$, is given by the form below,

$$H^{m',n'}_{p',q'}\left[u \middle| \begin{array}{c} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_{j'}, B'_j\right)_1^{q'} \end{array}\right] > 0, \tag{22}$$

for every $u > 0$. The indexes m', n', p', q' are

$$m' = n_1 + n_2 + n_3, \tag{23}$$
$$n' = n_3 + n_4, \tag{24}$$
$$p' = n_2 + n_3 + n_4, \tag{25}$$
$$q' = n_1 + n_2 + n_3 + n_4, \tag{26}$$

where n_1, n_2, n_3, n_4 are natural numbers such that

$$n_1 \geq 1, \text{ or } n_3 \geq 1. \tag{27}$$

The involved parameters are

$$\left(\alpha'_j, A'_j\right)_1^{p'} = \left(1 - r_j, a''_j\right)_1^{n_3}, \left(1 - v_j, a'''_j\right)_1^{n_4}, \left(d_j, a'_j\right)_1^{n_2}, \tag{28}$$
$$\left(\beta'_j, B'_{j'}\right)_1^{q'} = \left(b_j, a_j\right)_1^{n_1}, \left(c_j, a'_j\right)_1^{n_2}, \left(o_j, a''_j\right)_1^{n_3}, \left(1 - w_j, a'''_j\right)_1^{n_4}. \tag{29}$$

By definition, the constraint below,

$$A'_j\left(\beta'_{j'} + l\right) \neq B'_{j'}\left(\alpha'_j - l' - 1\right), \tag{30}$$

is required to hold for every $j = 1, \ldots, n'$, $j' = 1, \ldots, m'$, and $l, l' \in \mathbb{N}_0$. Additionally, the parameters $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, $a'_1, \ldots, a'_n$, $c_1, \ldots, c_n$, $d_1, \ldots, d_n$, $a''_1, \ldots, a''_n$, $o_1, \ldots, o_n$, $r_1, \ldots, r_n$, $a'''_1, \ldots, a'''_n$, $v_1, \ldots, v_n$, $w_1, \ldots,$ and w_n are required to fulfill the following relations:

$$a_j > 0,\ b_j \geq 0, \quad j = 1, \ldots, n_1, \tag{31}$$
$$a'_j > 0,\ c_j \geq 0,\ d_j \geq c_j + 1, \quad j = 1, \ldots, n_2, \tag{32}$$
$$a''_j, r_j > 0,\ o_j \geq 0, \quad j = 1, \ldots, n_3, \tag{33}$$
$$a'''_j, v_j > 0,\ w_j \geq v_j + 1, \quad j = 1, \ldots, n_4. \tag{34}$$

for every $(n_1, n_2, n_3, n_4) \in \mathbb{S}$, where $\mathbb{S} \equiv \mathbb{N}_0^4 \setminus \{(0,0,0,0)\}$. Let the parameter ν' be the value of the parameter ν, defined by Equation (21), that characterizes the positive Fox H function involved in relation (22). The parameter ν' is positive,

$$\nu' = \sum_{j=1}^{n_1} a_j + 2\sum_{j=1}^{n_3} a''_j > 0, \tag{35}$$

if condition (27) holds. Instead, the parameter ν' vanishes, $\nu' = 0$, for $n_1 = n_3 = 0$.

At this stage, we are equipped to process special forms of the auxiliary function $f(u)$ that are represented by Fox H functions. In fact, consider the following expression of the auxiliary function:

$$f_{\mathrm{H}}(u) = H^{q',0}_{p',q'}\left[u \middle| \begin{array}{c} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_{j'}, B'_j\right)_1^{q'} \end{array}\right], \tag{36}$$

for every $u > 0$. The involved indices and parameters are defined by relations (23)–(35), with $n_3 = n_4 = 0$. Thus, the function $f_{\mathrm{H}}(u)$ is a Fox H function that is positive on $\mathbb{R}^+$, i.e., $f_{\mathrm{H}}(u) > 0$, for every $u > 0$. The parameter ν' of the Fox H function $f_{\mathrm{H}}(u)$ is positive, $\nu' > 0$. Thus, the Mellin transform $\hat{f}_{\mathrm{H}}(n+1)$ of the function $f_{\mathrm{H}}(u)$, given by Equation (36), exists for every $n \in \mathbb{N}_0$. The arithmetic function $g_{\mathrm{H}}(n)$, corresponding to the auxiliary function $f_{\mathrm{H}}(u)$, is determined via Equation (7),

$$g_{\mathrm{H}}(n) = \left(\prod_{j=1}^{n_1} \Gamma\left(\bar{b}_j + a_j n\right)\right)\left(\prod_{k=1}^{n_2} \frac{\Gamma\left(\bar{c}_k + a'_k n\right)}{\Gamma\left(\bar{d}_k + a'_k n\right)}\right), \tag{37}$$

for every $n \in \mathbb{N}_0$, where $\bar{b}_j = b_j + a_j$, for every $j = 1, \dots, n_1$, and $\bar{c}_k = c_k + a'_k$, $\bar{d}_k = d_k + a'_k$, for every $k = 1, \dots, n_2$. Note that Condition (4) holds due to the asymptotic behavior of the Gamma function [42]. According to the above-reported properties, the Fox H function $f_H(u)$, given by Equation (36), represents an auxiliary function that is legitimate for the definition of GCSs.

The normalization factor $N_{\mathrm{H}}\left(|z|^2\right)$ is given by a Wright generalized hypergeometric function,

$$N_{\mathrm{H}}(u) = {}_{p'+1}\Psi_{q'}\left[u \,\middle|\, \begin{matrix} (1,1), \left(\bar{d}_j, a'_j\right)_1^{n_2} \\ \left(\bar{b}_j, a_j\right)_1^{n_1}, \left(\bar{c}_j, a'_j\right)_1^{n_2} \end{matrix}\right], \tag{38}$$

for every $u > 0$. The weight function $U_{\mathrm{H}}\left(|z|^2\right)$, corresponding to the auxiliary function $f_{\mathrm{H}}(u)$ given by Equation (36), results in being a product of a Wright generalized hypergeometric function and a Fox H function,

$$\begin{aligned} U_{\mathrm{H}}(u) = \pi^{-1}{}_{p'+1}\Psi_{q'}&\left[u \,\middle|\, \begin{matrix} (1,1), \left(\bar{d}_j, a'_j\right)_1^{n_1} \\ \left(\bar{b}_j, a_j\right)_1^{n_1}, \left(\bar{c}_j, a'_j\right)_1^{n_2} \end{matrix}\right] \\ &\times H^{q',0}_{p',q'}\left[u \,\middle|\, \begin{matrix} \left(d_j, a'_j\right)_1^{n_2} \\ \left(b_j, a_j\right)_1^{n_1}, \left(c_j, a'_j\right)_1^{n_2} \end{matrix}\right], \end{aligned} \tag{39}$$

for every $u > 0$.

We are finally able to state that the set $\{|z; g_{\mathrm{H}}\rangle, \ \forall z \in \mathbb{C}\}$ is a legitimate class of GCSs, obtained with positive Fox H auxiliary functions. This class of GCSs resolves the identity operator over the Fock space $\mathcal{F}$. The positive weight function is the product of a Wright hypergeometric function and a Fox H function.

3.1. Special Cases

The above-reported analysis shows that the set $\{|z, g_{\mathrm{H}}\rangle, \ \forall z \in \mathbb{C}\}$ is a legitimate class of GCSs. This class is generated by the arithmetic function $g_{\mathrm{H}}(n)$ that is defined by choosing positive Fox H-functions as the auxiliary function $f(u)$. The present class of GCSs provides canonical CSs and Mittag-Leffler or Wright GCSs as special cases. In fact, canonical CSs are obtained for $n_1 = 1$, $n_2 = 0$, $b_1 = 0$, and $a_1 = 1$, i.e., $g_{\mathrm{H}}(n) = \Gamma(n+1)$ for every $n \in \mathbb{N}_0$. Mittag-Leffler GCSs are obtained for $n_1 = 1$ and $n_2 = 0$, i.e., $g_{\mathrm{H}}(n) = \Gamma\left(a_1 n + \bar{b}_1\right)$ for every $n \in \mathbb{N}_0$. Wright GCSs are obtained for $n_1 = 2$, $n_2 = 0$, $b_1 = 0$, and $a_1 = 1$, i.e., $g_{\mathrm{H}}(n) = n!\Gamma\left(a_2 n + \bar{b}_2\right)$ for every $n \in \mathbb{N}_0$. Mittag-Leffler GCSs provide perturbations of the canonical CSs as the parameters a_1 and b_1 continuously depart from unity. Similarly, Wright GCSs provide perturbations of the canonical CSs for small, nonvanishing values of the parameter a_2 [36]. The CSs generalized with positive Fox H auxiliary functions are equipped to describe perturbations of CSs that include perturbations of Mittag-Leffler and Wright type, as particular cases. As a matter of fact, a theoretical description of the perturbation of canonical CSs allows for an investigation into how the physical properties of canonical CSs are affected by their perturbations.

For the sake of clarity, we report below an example of the above-introduced GCSs and use of the resolution of the identity operator to express the states of the Fock basis $\mathcal{F}$. Consider the (not-normalized) quantum state defined by the expression below in the Fock basis $\mathcal{F}$,

$$\sum_{n=0}^{\infty} \frac{\Gamma(\bar{d}+a'n)z^n}{\Gamma(\bar{b}+an)\Gamma(\bar{c}+a'n)}|n\rangle, \tag{40}$$

for every $z \in \mathbb{C} \setminus \{0\}$, with $a, a' > 0$, $b, c \geq 0$, $d \geq c+1$, $\bar{b} = b+a$, $\bar{c} = c+a'$, $\bar{d} = d+a'$. By adopting the above-used notation, the quantum state (40) is a GCS obtained from the auxiliary function $f_1(u)$, given by the following positive Fox H function:

$$f_1(u) = H^{2,0}_{1,2}\left[u \left| \begin{array}{c} (d_1, a'_1) \\ (b_1, a_1), (c_1, a'_1) \end{array} \right.\right] > 0, \tag{41}$$

for every $u > 0$, with $a_1 = a$, $b_1 = b$, $a'_1 = a'$, $c_1 = c$, and $d_1 = d$. The corresponding arithmetic function $g_1(n)$ is

$$g_1(n) = \Gamma(\bar{b}_1 + a_1 n)\frac{\Gamma(\bar{c}_1 + a'_1 n)}{\Gamma(\bar{d}_1 + a'_1 n)}, \tag{42}$$

for every $n \in \mathbb{N}_0$. The normalization factor $N_1(|z|^2)$ is given by the following Wright generalized hypergeometric function:

$$N_1(u) = {}_2\Psi_2\left[u \left| \begin{array}{c} (1,1), (\bar{d}, a') \\ (\bar{b}, a), (\bar{c}, a') \end{array} \right.\right], \tag{43}$$

for every $u > 0$. The weight function $U_1(|z|^2)$, corresponding to the auxiliary function $f_1(u)$, results in being a product of a Wright generalized hypergeometric function and a Fox H function,

$$U_1(u) = \pi^{-1}{}_2\Psi_2\left[u \left| \begin{array}{c} (1,1), (\bar{d}, a') \\ (\bar{b}, a), (\bar{c}, a') \end{array} \right.\right] H^{2,0}_{1,2}\left[u \left| \begin{array}{c} (d, a') \\ (b, a), (c, a') \end{array} \right.\right], \tag{44}$$

for every $u > 0$. The set $\{|z; g_1\rangle, \ \forall z \in \mathbb{C}\}$ of GCSs resolves the identity operator according to Equation (5) in case the arithmetic function $g(n)$ coincides with the arithmetic function $g_1(n)$. For example, every element of the Fock basis $\mathcal{F}$ is expressed as follows:

$$|n\rangle = \pi^{-1}\sqrt{\frac{\Gamma(\bar{d}+a'n)}{\Gamma(\bar{b}+an)\Gamma(\bar{c}+a'n)}} \times \int_{\mathbb{R}^2} z^{*n} \sqrt{{}_2\Psi_2\left[|z|^2 \left| \begin{array}{c} (1,1), (\bar{d}, a') \\ (\bar{b}, a), (\bar{c}, a') \end{array} \right.\right] H^{2,0}_{1,2}\left[|z|^2 \left| \begin{array}{c} (d, a') \\ (b, a), (c, a') \end{array} \right.\right]} |z, g_1\rangle d^2 z, \tag{45}$$

for every $n \in \mathbb{N}_0$.

Figures 1 and 2 display the Fox H function, appearing in relation (41), for various values of the independent variable and the involved parameters. The positivity condition of the auxiliary function is crucial for the definition of the GCSs and the resolution of the identity operator [4,5,8,13,14,57]. The values of the parameters that realize the positivity condition according to Figures 1 and 2 agree with the values provided by relations (22)–(34).

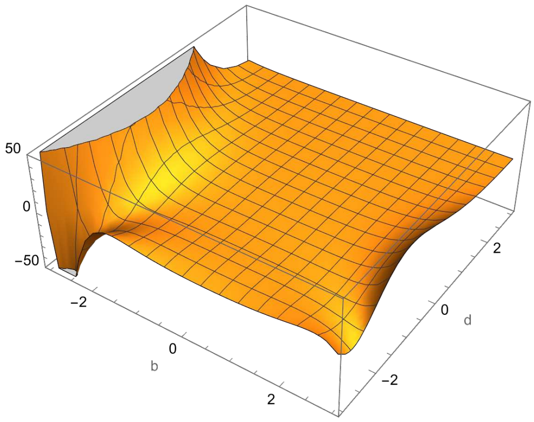

Figure 1. The Fox H-function involved in relation (41) for $|z|^2 = a = a' = 1$, $c = 1 - b$, and $-3 \leq b, d \leq 3$. This function is positive or negative according to the values of the involved parameters.

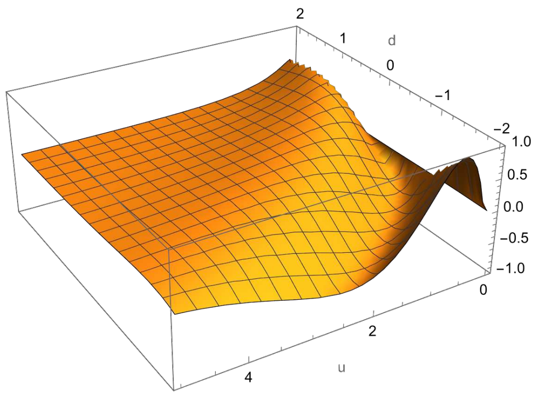

Figure 2. The Fox H-function involved in relation (41) for $a = a' = 1$, $b = c = 1/2$, $-2 \leq d \leq 2$, and $0 \leq |z|^2 \leq 5$. This function is positive or negative according to the values of the variable u and the involved parameters.

3.2. Statistics of the Number of Excitations

The probability $P_{\mathrm{H}}\left(n, |z|^2\right)$ that the generalized coherent state $|z; g_{\mathrm{H}}\rangle$ coincides with n excitations, i.e., the state $|n\rangle$ of the Fock basis, is

$$P_{\mathrm{H}}\left(n, |z|^2\right) = \frac{|z|^{2n}}{N_{\mathrm{H}}(|z|^2) g_{\mathrm{H}}(n)}, \tag{46}$$

for every $n \in \mathbb{N}_0$, and $z \in \mathbb{C} \setminus \{0\}$, while $P_{\mathrm{H}}(n, 0) = \delta_{n,0}$, for every $n \in \mathbb{N}_0$.

For large values of the number of excitations, $n \gg 1$, the probability $P_{\mathrm{H}}\left(n, |z|^2\right)$ is described by the following asymptotic form:

$$P_{\mathrm{H}}\left(n, |z|^2\right) \sim \frac{\gamma' |z|^{2n}}{N_{\mathrm{H}}(|z|^2)} \left(\frac{e}{n}\right)^{\mu' n} (\kappa')^{-n} n^{-\sigma}, \tag{47}$$

where

$$\gamma' = (2\pi)^{-\rho'} \exp\left(\sum_{j=1}^{q'} \bar{\beta}_j - \sum_{j=1}^{p'} \bar{\alpha}_j\right) \frac{\prod_{j=1}^{p'} \left(A'_j\right)^{\bar{\alpha}_j - (1/2)}}{\prod_{j=1}^{q'} \left(B'_j\right)^{\bar{\beta}_j - (1/2)}}, \tag{48}$$

$$\rho' = \frac{q' - p'}{2}, \tag{49}$$

$$\kappa' = \left(\prod_{j=1}^{p'} \left(A'_j\right)^{-A'_j}\right)\left(\prod_{j=1}^{q'} \left(B'_j\right)^{B'_j}\right), \tag{50}$$

$$\sigma = \sum_{j=1}^{q'} \bar{\beta}_j - \sum_{j=1}^{p'} \bar{\alpha}_j + \frac{p' - q'}{2}, \tag{51}$$

for every allowed value of the involved parameters.

3.3. Truncated Coherent States Generalized with Positive Fox H Functions

At this stage, we consider the case where the Fock basis $\mathcal{F}_{\mathfrak{d}}$ of a quantum harmonic oscillator is truncated: $\mathcal{F}_{\mathfrak{d}} \equiv \{|0\rangle, \ldots, |\mathfrak{d}\rangle\}$, for every $\mathfrak{d} \in \mathbb{N}$. Let the auxiliary function $f(u)$ be represented by a positive Fox H-function defined by relations (22)–(35) [51],

$$f_{\mathrm{H}}(u) = H^{m',n'}_{p',q'}\left[u \left| \begin{matrix} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_j, B'_j\right)_1^{q'} \end{matrix} \right.\right], \tag{52}$$

for every $u > 0$. Let the following conditions hold: $\nu > 0$, and

$$\mathfrak{d} + 1 < \min_{j=1,\ldots,n'} \left\{\frac{1 - \alpha'_j}{A'_j}\right\}, \tag{53}$$

with $\mathfrak{d} \in \mathbb{N}$. Then, the corresponding values $g_H(0), \ldots, g_{\mathrm{H}}(\mathfrak{d})$ are obtained from Equation (7),

$$g_{\mathrm{H}}(n) = \frac{\prod_{j=1}^{m'} \Gamma\left(\bar{\beta}_j + B'_j n\right) \prod_{j=1}^{n'} \Gamma\left(1 - \bar{\alpha}_j - A'_j n\right)}{\prod_{j=m'+1}^{q'} \Gamma\left(1 - \bar{\beta}_j - B'_j n\right) \prod_{j=n+1}^{p'} \Gamma\left(\bar{\alpha}_j + A'_j n\right)}, \tag{54}$$

for every $n = 0, \ldots, \mathfrak{d}$. The corresponding normalization factor, $N_{\mathfrak{d},\mathrm{H}}\left(|z|^2\right)$, is given by Equation (11) in case the terms $g(0), \ldots, g(\mathfrak{d})$ are obtained from Equation (54) for every $n = 0, \ldots, \mathfrak{d}$. The corresponding weight function $U_{\mathfrak{d},\mathrm{H}}\left(|z|^2\right)$ is given by the form below,

$$U_{\mathfrak{d},H}(u) = \pi^{-1} N_{\mathfrak{d},H}(u) H^{m',n'}_{p',q'}\left[u \left| \begin{matrix} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_{j'}, B'_j\right)_1^{q'} \end{matrix} \right.\right], \tag{55}$$

for every $u > 0$.

The probability $P_{\mathrm{H}}\left(n, \mathfrak{d}, |z|^2\right)$ that the GCS $|z; \mathfrak{d}; g_{\mathrm{H}}\rangle$ is characterized by n excitations, i.e., the state $|n\rangle$, is

$$P_{\mathrm{H}}\left(n, \mathfrak{d}, |z|^2\right) = \frac{|z|^{2n}}{N_{\mathfrak{d},H}(|z|^2) g_H(n)}, \tag{56}$$

for every $n = 0, \ldots, \mathfrak{d}$, and $z \in \mathbb{C} \setminus \{0\}$, while $P_{\mathrm{H}}(n, \mathfrak{d}, 0) = \delta_{n,0}$, for every $n = 0, \ldots, \mathfrak{d}$, and $\mathfrak{d} \in \mathbb{N}$.

For large values of the number of excitations, $\mathfrak{d} \geq n \gg 1$, the probability $P_{\mathrm{H}}\left(n, \mathfrak{d}, |z|^2\right)$ is properly approximated by the following asymptotic form:

$$P_{\mathrm{H}}\left(n, \mathfrak{d}, |z|^2\right) \simeq \frac{\gamma'' |z|^{2n}}{N_{\mathfrak{d},H}(|z|^2)} \left(\frac{e}{n}\right)^{\mu' n} (\kappa')^{-n} n^{-\sigma}, \tag{57}$$

where

$$\gamma'' = \left(\frac{2\pi}{e}\right)^{q' - m' - n'} \gamma',$$

for any allowed values of the involved parameters.

4. Sub- and Super-Poissonian Statistics of the Number of Excitations

The distribution of the number of excitations for a canonical CS is given by purely Poisson statistics. The deviation from this canonical condition is estimated by the Mandel Q parameter [52]. The Mandel Q parameter is defined in terms of the expectation values of $\hat{N}^2$, the square of the number operator, and $\hat{N}$, the number operator,

$$Q = \frac{\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2}{\langle \hat{N} \rangle} - 1. \tag{58}$$

The distribution of the number of excitations is super-Poissonian if the Mandel parameter is positive, $Q > 0$. In this case, the variance is larger than the mean value of the number of excitations, $\left[\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2\right] > \langle \hat{N} \rangle$. The distribution of the number of excitations is purely Poisson if the Mandel parameter vanishes, $Q = 0$. In this case, the variance coincides with the mean value of the number of excitations, $\left[\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2\right] = \langle \hat{N} \rangle$. The distribution of the number of excitations is sub-Poissonian if the Mandel parameter is negative, $Q < 0$. In this case, the variance is smaller than the mean value of the number of excitations, $\left[\langle \hat{N}^2 \rangle - \langle \hat{N} \rangle^2\right] < \langle \hat{N} \rangle$. Negative values of the Mandel parameter are related to the non-classical nature of the system.

For the GCSs generated by the arithmetic function $g(n)$, the Mandel parameter $Q\left(|z|^2, g\right)$ is given by the following form [16]:

$$Q\left(|z|^2, g\right) = |z|^2 \left[\frac{\sum_{n=0}^{\infty} (n+1)(n+2) |z|^{2n} / g(n+2)}{\sum_{n=0}^{\infty} (n+1) |z|^{2n} / g(n+1)} - \frac{\sum_{n=0}^{\infty} (n+1) |z|^{2n} / g(n+1)}{\sum_{n=0}^{\infty} |z|^{2n} / g(n)} \right], \tag{59}$$

for every $z \in \mathbb{C} \setminus \{0\}$. For the GCSs generated by the arithmetic function $g_{\mathrm{H}}(n)$, the Mandel parameter $Q\left(|z|^2, g_{\mathrm{H}}\right)$ is expressed in terms of the Wright generalized hypergeometric function,

$$Q\left(|z|^2, g_{\mathrm{H}}\right) = |z|^2 \left\{ \frac{{}_{p'+1}\Psi_{q'}\left[u \middle| \begin{array}{c} (3,1), \left(\alpha_j + 3A'_j, A'_j\right)_1^{p'} \\ \left(\beta + 3B'_j, B'_j\right)_1^{q'} \end{array}\right]}{{}_{p'+1}\Psi_{q'}\left[u \middle| \begin{array}{c} (2,1), \left(\alpha_j + 2A'_j, A'_j\right)_1^{p'} \\ \left(\beta + 2B'_j, B'_j\right)_1^{q'} \end{array}\right]} - \frac{{}_{p'+1}\Psi_{q'}\left[u \middle| \begin{array}{c} (2,1), \left(\alpha_j + 2A'_j, A'_j\right)_1^{p'} \\ \left(\beta + 2B'_j, B'_j\right)_1^{q'} \end{array}\right]}{{}_{p'+1}\Psi_{q'}\left[u \middle| \begin{array}{c} (1,1), \left(\alpha_j + A'_j, A'_j\right)_1^{p'} \\ \left(\beta + B'_j, B'_j\right)_1^{q'} \end{array}\right]} \right\}, \tag{60}$$

for every $z \in \mathbb{C} \setminus \{0\}$. The Mandel parameter of GCSs is studied for large and small values of the label in ref. [16]. The Mandel parameter tends to the opposite of unity for large values of the label. This behavior is confirmed by the statistics of the GCSs under study,

$$Q\left(|z|^2, g_{\mathrm{H}}\right) \to -1, \tag{61}$$

for large values of the label, $|z| \to +\infty$. In accordance with the general case, for the GCSs under study, the distributions of the number of excitations is sub-Poissonian at large values of the label.

For small, nonvanishing values of the label, the Mandel parameter of the GCSs under study is positive (negative),

$$Q\left(|z|^2, g_{\mathrm{H}}\right) >_{(<)} 0, \tag{62}$$

as $|z| \to 0^+$, with $z \neq 0$, if the following constraint holds [16]:

$$\frac{g_{\mathrm{H}}(0) g_{\mathrm{H}}(2)}{g_{\mathrm{H}}^2(1)} <_{(>)} 2. \tag{63}$$

Condition (63) is fulfilled by the following values of the involved parameters:

$$a_j = a'_k = 1, \quad b_j = \frac{2 - 2^{\varepsilon_j}}{2^{\varepsilon_j} - 1}, \quad c_k = \frac{2 - 2^{\varsigma_k}}{2^{\varsigma_k} - 1}, \quad d_k = \frac{2 - 2^{\vartheta_k}}{2^{\vartheta_k} - 1}, \tag{64}$$

in case

$$\sum_{j=1}^{n_1} \varepsilon_j + \sum_{k=1}^{n_2} (\varsigma_k - \vartheta_k) <_{(>)} 1, \tag{65}$$

with $0 < \varepsilon_j, \varsigma_k \leq 1$ and $0 < \vartheta_k \leq \log_2(2 - 2^{-\varsigma_k})$ for every $j = 1, \ldots, n_1$, with $n_1 \geq 1$ and $k = 1, \ldots, n_2$.

According to relation (65), the GCSs under study exhibit super-Poissonian statistics of the number of excitations, $Q\left(|z|^2, g_{\mathrm{H}}\right) > 0$, for small, nonvanishing values of the label, $|z| \ll 1$, with $z \neq 0$, in cases where the involved parameters are given by Condition (64) with $\varepsilon_j \simeq \varsigma_k \simeq 0$, for every $j = 1, \ldots, n_1$, with $n_1 \geq 1$, and $k = 1, \ldots, n_2$. This condition is provided by the following values of the involved parameters: $b_j \gg 1$, $d_k \geq 1 + c_k \gg 2$, for every $j = 1, \ldots, n_1$, with $n_1 \geq 1$, and $k = 1, \ldots, n_2$. Instead, the statistics of the number of excitations are sub-Poissonian, $Q\left(|z|^2, g_{\mathrm{H}}\right) < 0$, for small, nonvanishing values of the label, $|z| \ll 1$, with $z \neq 0$, in cases where the involved parameters are given by condition (64) and the following conditions hold: a value j' of the index j and a value k' of the index k exist, at least, such that $\varepsilon_{j'} \simeq \varsigma_{k'} \simeq 1$, and $\vartheta_k \simeq 0$, for every $k = 1, \ldots, n_2$. These relations are equivalent to the following values of the involved parameters: $b_{j'} \simeq c_{k'} \simeq 0$, and $d_k \gg 1$, for every $k = 1, \ldots, n_2$. The statistics of the number of excitations are sub-Poissonian for small, nonvanishing values of the label also in cases where the involved parameters are given by Condition (64) with $n_1 \geq 2$, and two values j' and j'' of the index j exist, at least, such that $\varepsilon_{j'} \simeq \varepsilon_{j''} \simeq 1$, and $\vartheta_k \simeq 0$, for every $k = 1, \ldots, n_2$. These conditions are equivalent to the following values of the involved parameters: $b_{j'} \simeq b_{j''} \simeq 0$, and $d_k \gg 1$, for every $k = 1, \ldots, n_2$. More generally, the statistics of the number of excitations are sub-Poissonian for small, nonvanishing values of the label in any case where

$$\sum_{j=1}^{n_1} \varepsilon_j + \sum_{k=1}^{n_2} \varsigma_k > 1,$$

with $n_1 \geq 1$, and $\vartheta_k \simeq 0$, for every $k = 1, \ldots, n_2$.

Consider GCSs defined via Equations (23)–(26), (28), (29), (31)–(34), and (36), with $n_1 \geq 1$, and $n_3 = n_4 = 0$. Super-Poissonian statistics of the number of excitations are realized for small, nonvanishing values of the label if one index j' exists, at least, such that the value of the parameter $A'_{j'}$ is sufficiently large, $A'_{j'} \gg 1$. Similarly, sub-Poissonian statistics of the number of excitations are realized for small, nonvanishing values of the label if one index j'' exists, at least, such that the value of the parameter $B'_{j''}$ is sufficiently large, $B'_{j''} \gg 1$. Notice that, with respect to Condition (64), the above-reported conditions include values of the parameters a_j and a'_k that differ from unity.

In summary, the GCSs under study exhibit sub-Poissonian statistics of the number of excitations for large values of the label. In fact, the corresponding Mandel parameter is expressed in terms of Wright generalized hypergeometric functions and is negative for large values of the label. Instead, the statistics are sub- or super-Poissonian for small, nonvanishing values of the label according to the values of the involved parameters. Each statistic can be obtained by choosing sufficiently large values of one of the involved parameters.

Figures 3 and 4 display the ratio $g_{\mathrm{H}}(0)g_{\mathrm{H}}(2)/g_{\mathrm{H}}^2(1)$ for particular values of the involved parameters. Super-Poissonian (sub-Poissonian) statistics of the number of excitations is realized for small, nonvanishing values of the label in case the displayed ratio is smaller (larger) than the value 2, i.e., Condition (63). The values of the parameters realizing the super-Poissonian or sub-Poissonian statistics according to Figures 3 and 4 agree with the theoretical values produced by relations (64) and (65).

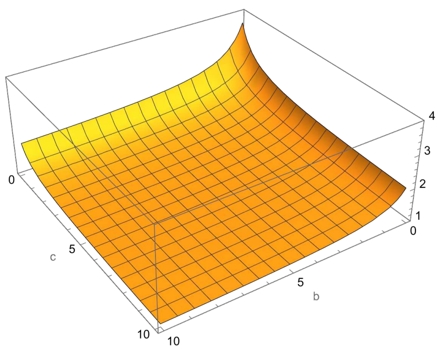

Figure 3. The quantity $g_{\mathrm{H}}(0)g_{\mathrm{H}}(2)/g_{\mathrm{H}}^2(1)$, appearing in relation (63), in case $n_1 = n_2 = 1$, $a_1 = a'_1 = 1$, $b_1 = b$, $c_1 = c$, and $d_1 = 10$ for $0 \leq b \leq 10$ and $0 \leq c \leq 10$. According to relations (62) and (63), super-Poissonian statistics of the number of excitations are obtained for small, nonvanishing values of the label in case $b, c \gg 1$, and sub-Poissonian statistics are found in case $b, c \simeq 0$.

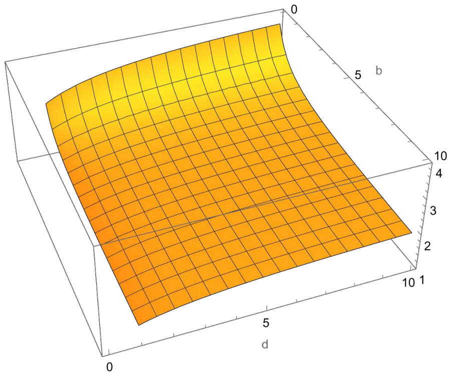

Figure 4. The quantity $g_{\rm H}(0)g_{\rm H}(2)/g^2_{\rm H}(1)$, appearing in relation (63), in case $n_1 = n_2 = 1$, $a_1 = a'_1 = 1$, $b_1 = b$, and $c_1 = 0$ for $0 \leq b \leq 10$ and $1 \leq d \leq 10$. According to relations (62) and (63), super-Poissonian statistics of the number of excitations are obtained for small, nonvanishing values of the label in case $b \gg 1$ and $d \simeq 1$, and sub-Poissonian statistics are found in case $b \simeq 0$ and $d \gg 1$.

5. Dissipative Effects

At this stage, we evaluate the dissipative effects that a memoryless environment produces over the CSs generalized with a Fox H function [37,53–56]. For the sake of clarity and consistency, we report below the dissipative evolution of a GCS interacting with such an environment by following Refs. [37,55,56].

Briefly, the dissipative evolution of an initial state $\rho(0)$, weakly interacting with a zero-temperature reservoir, results in the mixed state $\rho(t)$, given by the form below, in the interaction picture by adopting the Born and Markov approximation,

$$\rho(t) = \sum_{l=0}^{\infty} \Lambda_l(t)\rho(0)\Lambda_l^{\dagger}(t), \tag{66}$$

for every $t \geq 0$. The effect operators $\Lambda_0(t), \Lambda_1(t), \ldots$ mimic the loss of zero, one, or more excitations, or, equivalently, decay events, and are given by the following form in the canonical Fock basis $\mathcal{F}$:

$$\Lambda_l(t) = \sum_{l'=l}^{\infty} \sqrt{\frac{l'![p(t)]^{l'-l}[1-p(t)]^l}{l!(l'-l)!}}\, |l'-l\rangle\langle l'|, \tag{67}$$

for every $t \geq 0$ and $l \in \mathbb{N}_0$. The function $p(t)$ is the exponentially damped survival probability of the initial state,

$$p(t) = \exp(-vt), \tag{68}$$

for every $t \geq 0$. The positive parameter v, describing the exponential decay, is determined by the coupling between the system and environment and the correlation time of the environment.

The loss of zero, one, or more excitations by a GCS $|z;g\rangle$ produces states that, despite the excitation loss, remain within the family of GCSs [37],

$$\Lambda_l(t)|z;g\rangle = z^l \mathfrak{K}_l\left[p(t), |z|^2, g_l\right] |z\sqrt{p(t)}; g_l\rangle, \tag{69}$$

for every $t \geq 0$, $z \in \mathbb{C} \setminus \{0\}$ and $l \in \mathbb{N}_0$. The function $\mathfrak{K}_l\left[p(t), |z|^2, g_l\right]$ is given by the following expression:

$$\mathfrak{K}_l\left[p(t), |z|^2, g_l\right] = \sqrt{\frac{[1 - p(t)]^l N_{g,l}\left[|z|^2 p(t)\right]}{l!\, N_{g,0}\left(|z|^2\right)}}. \quad (70)$$

The transformed GCS $|z\sqrt{p(t)}; g_l\rangle$ is generated by the arithmetic function $g_l(n)$, given by

$$g_l(n) = \frac{g_0(n+l)}{(n+1)\cdots(n+l)}, \quad (71)$$

for every $l, n \in \mathbb{N}_0$. Note that $g_0(n) = g(n)$ for every $n \in \mathbb{N}_0$. The normalization factor $N_{g,l}(u)$ is given by Equation (2) for every $u > 0$ in case the arithmetic function $g(n)$ is substituted with the arithmetic function $g_l(n)$ for every $l \in \mathbb{N}_0$.

The class $\{|z; g_l\rangle, \ \forall\, z \in \mathbb{C}\}$ of GCSs produced by l decay events resolves the identity operator with the weight function $U_{g,l}(u)$, given by the expression below,

$$U_{g,l}(u) = N_{g,l}(u) \int_u^\infty \cdots \int_{u_{l-1}}^\infty \frac{U_{g,0}(u_l)}{N_{g,0}(u_l)}\, du_l \ldots du_1, \quad (72)$$

for every $u > 0$, and $l \in \mathbb{N}$, with $u_0 = u$.

According to Equation (66), the time evolution of the initial GCS $|z; g\rangle$, i.e., $\rho(0) = |z; g\rangle\langle z; g|$, is

$$\rho(t) = \sum_{l=0}^{\infty} \Xi_l\left[p(t), |z|^2, g_l\right] |z\sqrt{p(t)}; g_l\rangle\langle z\sqrt{p(t)}; g_l|, \quad (73)$$

for every $t \geq 0$. The statistical mixture $\rho(t)$ is composed by the transformed GCSs $|z\sqrt{p(t)}; g_0\rangle$, $|z\sqrt{p(t)}; g_1\rangle, \ldots$, with weights $\Xi_0\left(p(t), |z|^2, g_0\right)$, $\Xi_1\left(p(t), |z|^2, g_1\right), \ldots$, given by

$$\Xi_l\left[p(t), |z|^2, g_l\right] = |z|^{2l} \mathfrak{K}_l^2\left[p(t), |z|^2, g_l l\right] = \frac{|z|^{2l}[1 - p(t)]^l N_{g,l}\left[|z|^2 p(t)\right]}{l!\, N_{g,0}\left(|z|^2\right)}, \quad (74)$$

for every $t \geq 0$, $l \in \mathbb{N}_0$, and $z \in \mathbb{C} \setminus \{0\}$.

We are finally equipped to describe the dissipative processes that affect CSs generalized with the Fox H function that interacts with a zero-temperature reservoir. The GCS $|z; g_{\mathrm{H}}\rangle$ is transformed by l decay events into the GCS $|z; g_{\mathrm{H},l}\rangle$ for every $z \in \mathbb{C} \setminus \{0\}$, described by Equations (69) and (70), for every $l \in \mathbb{N}_0$. The involved arithmetic function $g_{\mathrm{H},l}(n)$ is obtained from Equations (37) and (71),

$$g_{\mathrm{H},l}(n) = \frac{\Gamma(1+n)}{\Gamma(1+l+n)} \left(\prod_{j=1}^{n_1} \Gamma\left(b_{j,l} + a_j n\right)\right) \left(\prod_{k=1}^{n_2} \frac{\Gamma\left(c_{k,l} + a'_k n\right)}{\Gamma\left(d_{k,l} + a'_k n\right)}\right), \quad (75)$$

for every $l, n \in \mathbb{N}_0$. The involved parameters are defined as follows:

$$b_{j,l} = b_j + (l+1)a_j, \quad c_{k,l} = c_k + (l+1)a'_k, \quad d_{k,l} = d_k + (l+1)a'_k, \quad (76)$$

for every $j = 1, \ldots, n_1$, with $n_1 \in \mathbb{N}$, and $k = 1, \ldots, n_2$, with $n_2, l \in \mathbb{N}_0$. Note that $g_{\mathrm{H},0}(n) = g_{\mathrm{H}}(n)$ for every $n \in \mathbb{N}_0$.

The (positive) auxiliary function $f_{\mathrm{H},l}(u)$, corresponding to the arithmetic function $g_{\mathrm{H},l}(n)$, is given by the following Fox H function:

$$f_{\mathrm{H},l}(u) = H^{q'+1,0}_{p'+1,q'+1}\left[u\left|\begin{array}{c}(l+1,1),\left(\alpha'_{j,l},A'_j\right)_1^{p'}\\(1,1),\left(\beta'_{j,l},B'_j\right)_1^{q'}\end{array}\right.\right], \tag{77}$$

for every $u > 0$ and $l \in \mathbb{N}_0$. The involved parameters are

$$\left(\alpha'_{j,l},A'_j\right)_1^{p'} = \left(d_{k,l},a'_k\right)_1^{n_2}, \tag{78}$$

$$\left(\beta'_{j,l},B'_{j'}\right)_1^{q'} = \left(b_{j,l},a_j\right)_1^{n_1},\left(c_{k,l},a'_k\right)_1^{n_2}, \tag{79}$$

for every $l,n,n_2 \in \mathbb{N}_0$ and $n_1 \in \mathbb{N}$.

The normalization factor $N_{\mathrm{H},l}(u)$ of the transformed GCS $|z;g_{\mathrm{H},l}\rangle$ is given by a Wright generalized hypergeometric function,

$$N_{\mathrm{H},l}(u) = {}_{p'+1}\Psi_{q'}\left[u\left|\begin{array}{c}(l+1,1),\left(\alpha'_{j,l},A'_j\right)_1^{p'}\\\left(\beta'_{j,l},B'_j\right)_1^{q'}\end{array}\right.\right], \tag{80}$$

for every $u > 0$. The weight function $U_{\mathrm{H},l}(u)$, corresponding to the auxiliary function $f_{\mathrm{H},l}(u)$, results in being a product of a Wright generalized hypergeometric function and a Fox H function,

$$\begin{aligned}U_{\mathrm{H},l}(u) = \pi^{-1}{}_{p'+1}\Psi_{q'}&\left[u\left|\begin{array}{c}(l+1,1),\left(\alpha'_{j,l},A'_j\right)_1^{p'}\\\left(\beta'_{j,l},B'_j\right)_1^{q'}\end{array}\right.\right]\\&\times H^{q'+1,0}_{p'+1,q'+1}\left[u\left|\begin{array}{c}(l+1,1),\left(\alpha'_{j,l},A'_j\right)_1^{p'}\\(1,1),\left(\beta'_{j,l},B'_j\right)_1^{q'}\end{array}\right.\right],\end{aligned} \tag{81}$$

for every $u > 0$.

The exponentially damped survival probability of the initial state, $p(t)$, vanishes over long times, $t \gg 1/v$. Thus, every time-dependent GCS generated by zero, one, or more decay events, $|z\sqrt{p(t)};g_{\mathrm{H},l}\rangle$, for every l with a nonvanishing label, $z \in \mathbb{C}\setminus\{0\}$ for every $l,n \in \mathbb{N}_0$ tends to the vacuum state, $|z\sqrt{p(t)};g\rangle \to |0\rangle$, for $t \gg 1/v$. Consequently, the amplitude damping noise ultimately reduces the initial GCS $|z;g_{\mathrm{H}}\rangle$ to the vacuum state, $\rho(t) \to |0\rangle\langle 0|$, over long times, $t \gg 1/v$ [37].

6. Thermal Reservoir

The time evolution of a quantum state, weakly interacting with a thermal reservoir at temperature T, is described in the Born and Markov approximation and in the interaction picture by the following form:

$$\rho(t) = \exp(\mathfrak{L}t)\rho(0), \tag{82}$$

for every $t \geq 0$, where $\rho(0)$ is the initial state and $\rho(t)$ is the evolved state at the time t. The superoperator $\mathfrak{L}$ is a function of the creation operator $\mathfrak{a}^\dagger$ and annihilation operator $\mathfrak{a}$

of the quantum harmonic oscillator and the tempearture T, and it acts over the general density operator ϱ as follows [63]:

$$\mathfrak{L}\varrho = v'(1+\mathfrak{n}_T)\left(2\mathfrak{a}\varrho\mathfrak{a}^\dagger - \mathfrak{a}^\dagger\mathfrak{a}\varrho - \varrho\mathfrak{a}^\dagger\mathfrak{a}\right) + v'\mathfrak{n}_T\left(2\mathfrak{a}^\dagger\varrho\mathfrak{a} - \mathfrak{a}\mathfrak{a}^\dagger\varrho - \varrho\mathfrak{a}\mathfrak{a}^\dagger\right). \quad (83)$$

The parameter $\mathfrak{n}_T$ represents the average number of excitations at the effective temperature T,

$$\mathfrak{n}_T = \left(\exp\left(\frac{\hbar\omega}{K_B T}\right) - 1\right)^{-1}, \quad (84)$$

for every $T > 0$, where ω is the frequency of the quantum harmonic oscillator, $\hbar$ is the Planck constant, and K_B is the Boltzmann constant. The positive constant v' represents the decay rate.

For the sake of clarity and convenience, we report below the time evolution of a canonical CS, weakly interacting with the thermal reservoir, following Refs. [64,65],

$$\exp(\mathfrak{L}t)|z\rangle\langle z| = \sum_{j=0}^{+\infty} \frac{(\mathfrak{n}(t,T))^j}{(1+\mathfrak{n}(t,T))^{j+1}} \mathfrak{D}\left(z\exp\left(-v't\right)\right)|j\rangle\langle j|\mathfrak{D}^\dagger\left(z\exp\left(-v't\right)\right), \quad (85)$$

for every $t \geq 0$ and $z \in \mathbb{C}$. The time-dependent Glauber's displacement operator $\mathfrak{D}(z\exp(-v't))$ is defined in terms of the creation and annihilation operators via the following form:

$$\mathfrak{D}(z) = \exp\left(z\mathfrak{a}^\dagger - z^*\mathfrak{a}\right), \quad (86)$$

holding for every $z \in \mathbb{C}$. The time-dependent function $\mathfrak{n}(t,T)$ is given by

$$\mathfrak{n}(t,T) = \mathfrak{n}_T(1 - \exp(-2v't)), \quad (87)$$

for every $t \geq 0$ and $T > 0$. Note that $\mathfrak{n}(+\infty, T) = \mathfrak{n}_T$ for every $T > 0$. According to Equations (85) and (86), the interaction with the thermal reservoir reduces over long times, $t \gg 1/v'$, the initial CS $|z\rangle$ to the thermal state ϱ_T, given by the following form in the Fock basis $\mathcal{F}$:

$$\varrho_T = \sum_{j=0}^{+\infty} \frac{\mathfrak{n}_T^j}{(1+\mathfrak{n}_T)^{j+1}} |j\rangle\langle j|, \quad (88)$$

for every $T > 0$. The thermal state ϱ_T is a statistical mixture composed of the states of the Fock basis $\mathcal{F}$, and the corresponding weights are determined by the average number $\mathfrak{n}_T$ of excitations at the effective temperature T.

At this stage, we analyze the evolution of a GCS that weakly interacts with the thermal reservoir at temperature T. The time evolution of the GCSs characterized by positive Fox H auxiliary functions is obtained from the time evolution of a GCS by substituting the arithmetic function $g(n)$ with the arithmetic function $g_{\mathrm{H}}(n)$. Two formal descriptions of the time evolution of the GCS are presented below by relying on the completeness property of canonical CSs, the Glauber-Sudarshan P representation, and Glauber' s displacement operators [2,3,64–68].

Let the GCS $|z,g\rangle$ be the initial state of the quantum harmonic oscillator interacting with the thermal reservoir, i.e., $\rho(0) = |z,g\rangle\langle z,g|$. The interaction transforms the initial GCS $|z,g\rangle$ in the mixed state $\rho(t)$ that is described in the Fock basis $\mathcal{F}$ by the following form:

$$\rho(t) = \exp(\mathfrak{L}t)|z,g\rangle\langle z,g| = \sum_{l_1=0}^{+\infty}\sum_{l_2=0}^{+\infty} \Xi_{l_1,l_2}(z,z^*,t,T)|l_1\rangle\langle l_2|, \tag{89}$$

for every $t \geq 0$, where

$$\Xi_{l_1,l_2}(z,z^*,t,T) = \sum_{j_1=0}^{+\infty}\sum_{j_2=0}^{+\infty} \frac{z^{j_1} z^{*j_2}}{\sqrt{j_1! j_2! g(j_1) g(j_2)} N_g\left(|z|^2\right)} \Lambda_{l_1,l_2,j_1,j_2}(t,T), \tag{90}$$

for every $l_1, l_2 \in \mathbb{N}_0$, $z \in \mathbb{C}$, and $t \geq 0$. The function $\Lambda_{l_1,l_2,j_1,j_2}(t,T)$ mimics the effect of the thermal reservoir on the GCS,

$$\Lambda_{l_1,l_2,j_1,j_2}(t,T) = \sum_{k=0}^{+\infty} \frac{\mathfrak{n}(t,T)^k}{(1+\mathfrak{n}(t,T))^{k+1}} \Upsilon_{l_1,l_2,j_1,j_2,k}(t), \tag{91}$$

for every $l_1, l_2, j_1, j_2 \in \mathbb{N}_0$, $t \geq 0$, and $T > 0$. The time-dependent function $\Upsilon_{l_1,l_2,j_1,j_2,k}(t)$ is derived from the expression below by considering ζ and ζ^* as independent variables [2,3,65–68],

$$\Upsilon_{l_1,l_2,j_1,j_2,k}(t) = \frac{\partial^{j_1+j_2}}{\partial\zeta^{j_1}\partial\zeta^{*j_2}} \left(\exp(\zeta\zeta^*) U_{l_1,l_2,k}(\zeta,\zeta^*,t)\right)\Big|_{\zeta=0,\,\zeta^*=0} \tag{92}$$

for every $l_1, l_2, j_1, j_2 \in \mathbb{N}_0$, and $t \geq 0$. The time-dependent function $U_{l_1,l_2,k}(\zeta,\zeta^*,t)$ is defined via Glauber' s displacement operator [2,3,65–68],

$$U_{l_1,l_2,k}(\zeta,\zeta^*,t) = \langle l_1|\mathfrak{D}\left(\zeta\exp\left(-v't\right)\right)|k\rangle\langle k|\mathfrak{D}^\dagger\left(\zeta\exp\left(-v't\right)\right)|l_2\rangle, \tag{93}$$

and results in the following explicit expression:

$$\begin{aligned} U_{l_1,l_2,k}(\zeta,\zeta^*,t) = {} & \exp\left(-\zeta\zeta^*\exp\left(-2v't\right)\right) \\ & \left(\Theta_1(l_1-k)\sqrt{\frac{k!}{l_1!}}\left(\zeta\exp\left(-v't\right)\right)^{l_1-k} L_k^{(l_1-k)}\left(\zeta\zeta^*\exp\left(-2v't\right)\right)\right. \\ & \left. +\Theta_2(k-l_1)\sqrt{\frac{l_1!}{k!}}\left(-\zeta^*\exp\left(-v't\right)\right)^{k-l_1} L_{l_1}^{(k-l_1)}\left(\zeta\zeta^*\exp\left(-2v't\right)\right)\right) \\ & \left(\Theta_1(k-l_2)\sqrt{\frac{l_2!}{k!}}\left(-\zeta\exp\left(-v't\right)\right)^{k-l_2} L_{l_2}^{(k-l_2)}\left(\zeta\zeta^*\exp\left(-2v't\right)\right)\right. \\ & \left. +\Theta_2(l_2-k)\sqrt{\frac{k!}{l_2!}}\left(\zeta^*\exp\left(-v't\right)\right)^{l_2-k} L_k^{(l_2-k)}\left(\zeta\zeta^*\exp\left(-2v't\right)\right)\right), \end{aligned} \tag{94}$$

for every $l_1, l_2, k \in \mathbb{N}_0$, and $t \geq 0$, where $\Theta_1(\tau) = 1$ for every $\tau \geq 0$, $\Theta_1(\tau) = 0$ for every $\tau < 0$, and $\Theta_2(\tau) = 1$ for every $\tau > 0$, $\Theta_2(\tau) = 0$ for every $\tau \leq 0$. Form $L_k^{(l)}(\zeta)$ represents the Laguerre polynomials [69] for every $l, k \in \mathbb{N}_0$ and $\zeta \in \mathbb{C}$.

The resolution of the identity operator with the canonical CSs provides the following form for the time evolution of the GCS $|z,g\rangle$ interacting with the thermal reservoir:

$$\rho(t) = \pi^{-2}\int_{\mathbb{R}^2} d^2\zeta \int_{\mathbb{R}^2} d^2\chi \langle\zeta||z,g\rangle\langle z,g||\chi\rangle \exp(\mathfrak{L}t)|\zeta\rangle\langle\chi|, \tag{95}$$

for every $t \geq 0$ and $z \in \mathbb{C}$. The latter term, $\exp(\mathfrak{L}t)|\zeta\rangle\langle\chi|$, is studied in ref. [64],

$$\exp(\mathfrak{L}t)|\zeta\rangle\langle\chi| = \langle\chi||\zeta\rangle A(\zeta,\chi,t,T) \sum_{j=0}^{+\infty} \frac{(\mathfrak{n}(t,T))^j}{(1+\mathfrak{n}(t,T))^{j+1}} \mathfrak{D}\left(\bar{\zeta}(t,T)\right)|j\rangle\langle j|\mathfrak{D}^{\dagger}(\bar{\chi}(t,T)), \tag{96}$$

for every $t \geq 0$, $\zeta, \chi \in \mathbb{C}$, and $T > 0$. The above-reported time-dependent functions are defined as below [64],

$$\bar{\zeta}(t,T) = \frac{\exp(-\upsilon' t)}{2\mathfrak{n}(t,T)+1}((\mathfrak{n}(t,T)+1)\zeta + \mathfrak{n}(t,T)\chi), \tag{97}$$

$$\bar{\chi}(t,T) = \frac{\exp(-\upsilon' t)}{2\mathfrak{n}(t,T)+1}(\mathfrak{n}(t,T)\zeta + (\mathfrak{n}(t,T)+1)\chi), \tag{98}$$

$$A(\zeta,\chi,t,T) = \exp\left(\frac{1}{2}\left(\left|\bar{\zeta}(t,T)\right|^2 + |\bar{\chi}(t,T)|^2\right) - \bar{\zeta}(t,T)\bar{\chi}^*(t,T) + \mathfrak{n}(t,T)\left|\bar{\zeta}(t,T) - \bar{\chi}(t,T)\right|^2\right), \tag{99}$$

for every $t \geq 0$, $T > 0$, and $\zeta, \chi \in \mathbb{C}$. The terms involving the displacement operators that appear in Equation (96) are given by the forms below,

$$\langle j'|\mathfrak{D}\left(\bar{\zeta}(t,T)\right)|j\rangle = \Theta_1(j'-j)\sqrt{\frac{j!}{j'!}}\left(\bar{\zeta}(t,T)\right)^{j'-j} \exp\left(-\frac{\left|\bar{\zeta}(t,T)\right|^2}{2}\right) L_j^{(j'-j)}\left(\left|\bar{\zeta}(t,T)\right|^2\right) + \Theta_2(j-j')\sqrt{\frac{j'!}{j!}}\left(-\bar{\zeta}^*(t,T)\right)^{j-j'} \exp\left(-\frac{\left|\bar{\zeta}(t,T)\right|^2}{2}\right) L_{j'}^{(j-j')}\left(\left|\bar{\zeta}(t,T)\right|^2\right), \tag{100}$$

$$\langle j|\mathfrak{D}^{\dagger}(\bar{\chi}(t,T))|j''\rangle = \Theta_1(j-j'')\sqrt{\frac{j''!}{j!}}(-\bar{\chi}(t,T))^{j-j''} \exp\left(-\frac{|\bar{\chi}(t,T)|^2}{2}\right) L_{j''}^{(j-j'')}\left(|\bar{\chi}(t,T)|^2\right) + \Theta_2(j''-j)\sqrt{\frac{j!}{j''!}}(\bar{\chi}^*(t,T))^{j''-j} \exp\left(-\frac{|\bar{\chi}(t,T)|^2}{2}\right) L_j^{(j''-j)}\left(|\bar{\chi}(t,T)|^2\right), \tag{101}$$

for every $j', j, j'' \in \mathbb{N}_0$, $t \geq 0$, $T > 0$, and $\zeta, \chi \in \mathbb{C}$.

According to the above-performed analysis, the time evolution of a canonical CS of a quantum harmonic oscillator, interacting with a thermal reservoir, is generally different from the time evolution of a GCS or a CS generalized with the Fox H function. Qualitatively, this difference is due to the fact that canonical CSs are eigenstates of the annihilation operator of the quantum harmonic oscillator, while GCSs are not [36,37]. However, the effect of the thermal reservoir dominates over the initial condition over long times, $t \gg 1/\upsilon'$, and the long-time evolution of canonical CSs and CSs generalized with the Fox H function, and GCSs tend to the thermal state ϱ_T.

Purity

The purity $\Pi(\varrho)$ of a general quantum state ϱ is defined via the trace operation of the operator ϱ^2 [70],

$$\Pi(\varrho) = \mathrm{tr}\left(\varrho^2\right).$$

The purity measures how much the quantum state ϱ is mixed. The purity of pure states is equal to unity and represents the upper bound of this measure. Instead, the lower bound of the purity is the reciprocal of the dimension of the corresponding Hilbert space.

The mixed state $\rho(t)$, time evolution of the initial GCS $|z,g\rangle$ interacting with the thermal reservoir, is characterized by the following expression of the purity:

$$\Pi(\rho(t)) = \pi^{-4} \int_{\mathbb{R}^2} d^2\zeta_1 \int_{\mathbb{R}^2} d^2\chi_1 \int_{\mathbb{R}^2} d^2\zeta_2 \int_{\mathbb{R}^2} d^2\chi_2 \langle\zeta_1||z,g\rangle\langle z,g||\chi_1\rangle\langle\zeta_2||z,g\rangle\langle z,g||\chi_2\rangle$$
$$\times \langle\chi_1||\zeta_1\rangle\langle\chi_2||\zeta_2\rangle \operatorname{tr}\big((\exp(\mathfrak{L}t)|\zeta_1\rangle\langle\chi_1|)(\exp(\mathfrak{L}t)|\zeta_2\rangle\langle\chi_2|)\big), \tag{102}$$

for every $z \in \mathbb{C}$ and $t \geq 0$, where

$$\operatorname{tr}\big((\exp(\mathfrak{L}t)|\zeta_1\rangle\langle\chi_1|)(\exp(\mathfrak{L}t)|\zeta_2\rangle\langle\chi_2|)\big) = \langle\chi_1||\zeta_1\rangle\langle\chi_2||\zeta_2\rangle A(\zeta_1,\chi_1,t,T)A(\zeta_2,\chi_2,t,T)$$
$$\sum_{j=0}^{+\infty}\sum_{j_1=0}^{+\infty}\sum_{j_2=0}^{+\infty}\sum_{l=0}^{+\infty} \frac{(\mathfrak{n}(t,T))^{j_1+j_2}}{(1+\mathfrak{n}(t,T))^{j_1+j_2+2}} \langle j|\mathfrak{D}\big(\bar{\zeta}_1(t,T)\big)|j_1\rangle\langle j_1|\mathfrak{D}^\dagger(\bar{\chi}_1(t,T))|l\rangle$$
$$\times \langle l|\mathfrak{D}\big(\bar{\zeta}_2(t,T)\big)|j_2\rangle\langle j_2|\mathfrak{D}^\dagger(\bar{\chi}_2(t,T))|j\rangle, \tag{103}$$

for every $t \geq 0$, $\zeta_1, \chi_1, \zeta_2, \chi_2 \in \mathbb{C}$, and $T > 0$. The terms involving Glauber's displacement operators are given by Equations (100) and (101).

Over long times, $t \gg 1/v'$, the purity is determined uniquely by the average number of excitations $\mathfrak{n}_T$,

$$\Pi(\rho(+\infty)) = \frac{1}{1+2\mathfrak{n}_T}, \tag{104}$$

for every $T > 0$. Particularly, the above-reported asymptotic value tends to unity, $\Pi(\rho(+\infty)) \to 1^-$, at a low, nonvanishing temperature, $T \ll \hbar\omega/K_B$. Instead, the asymptotic value vanishes, $\Pi(\rho(+\infty)) \to 0^+$, at a high temperature, $T \gg \hbar\omega/K_B$.

Let the initial state of the quantum harmonic oscillator be the GCS $|z, g_{\mathrm{H}}\rangle$, i.e., $\rho(0) = |z, g_{\mathrm{H}}\rangle\langle z, g_{\mathrm{H}}|$. The purity of the corresponding evolved state, $\Pi(\rho(t))$, is given by Equation (102) for every $z \in \mathbb{C}$ and $t \geq 0$ by substituting the arithmetic function $g(n)$ with the arithmetic function $g_{\mathrm{H}}(n)$. Instead, let the initial state be the canonical CS $|z\rangle$. The purity of the evolved state, $\Pi(\exp(\mathfrak{L}t)|z\rangle\langle z|)$, is described by a simplified form,

$$\Pi(\exp(\mathfrak{L}t)|z\rangle\langle z|) = \sum_{j=0}^{+\infty}\sum_{j_1=0}^{+\infty}\sum_{j_2=0}^{+\infty}\sum_{l=0}^{+\infty} \frac{(\mathfrak{n}(t,T))^{j_1+j_2}}{(1+\mathfrak{n}(t,T))^{j_1+j_2+2}} \langle j|\mathfrak{D}\big(z\exp(-v't)\big)|j_1\rangle$$
$$\times\langle j_1|\mathfrak{D}^\dagger\big(z\exp(-v't)\big)|l\rangle\,\langle l|\mathfrak{D}\big(z\exp(-v't)\big)|j_2\rangle\langle j_2|\mathfrak{D}^\dagger\big(z\exp(-v't)\big)|j\rangle, \tag{105}$$

for every $t \geq 0$ and $z \in \mathbb{C}$.

In summary, the purity of the evolved GCS differs, in general, from the purity of the evolved canonical CS. Again, this is due to the fact that, in general, GCSs are not eigenstates of the annihilation operator of the quantum harmonic oscillator. However, the purity of the evolved canonical CS tends, over long times, $t \gg 1/v'$, to the common asymptotic value, $\Pi(\rho(+\infty))$, given by Equation (104) for every $T > 0$ for any GCS. The common asymptotic value of the purity is determined uniquely by the average number of excitations at temperature T and tends to the maximum value, unity, at vanishing temperature, but vanishes at high temperatures.

7. Summary and Conclusions

CSs of a quantum harmonic oscillator are fundamental states of minimum uncertainty that exhibit a Poisson distribution of the number of excitations. CSs find applications in the most varied scenarios, from quantum optics to mechanical devices.

Theoretical generalizations of CSs are performed in various ways. Klauder's generalization of CSs is performed by requiring the conditions of normalizability, continuity in the label, and the resolution of the identity operator with a (positive) weight function [4,5]. This approach has led to various generalizations of CSs that are performed with special functions, including Mittag-Leffler and Wright functions [4,5,13–17,36]. These GCSs exhibit various distributions of the number of excitations, ranging from super-Poissonian to the non-classical sub-Poissonian statistics. Additionally, Wright generalized Schrödinger cat states evolve under amplitude damping noise similarly to the canonical Scrödinger cat states. This property opens to possible applications in quantum information processing [37,55,56].

The Fox H function is a special function that produces Mittag-Leffler and Wright functions as particular cases [38–49]. Therefore, in the present scenario, we have adopted this special function to perform a further generalization of CSs. We have found that the resulting GCSs provide canonical CSs and Mittag-Leffler or Wright GCSs as particular cases. Additionally, these GCSs are characterized by anomalous distributions of the number of excitations that result in products of exponential and power laws and powers of the term n^{-n} for large numbers. Thus, the novel GCSs are equipped to describe a large variety of anomalous statistics in the framework of purely quantum theory. The corresponding Mandel Q factor consists of ratios of Wright generalized hypergeometric functions that are negative for large values of the label and, therefore, witness (non-classical) sub-Poissonian statistics. Instead, for small, nonvanishing values of the label, the Mandel Q factor is positive or negative, according to the values of the involved parameters. In these cases, the statistics are super- or sub-Poissonian, respectively, according to the values of the involved parameters. Additionally, the super-Poissonian regime is obtained for small, nonvanishing values of the label by simply choosing sufficiently large values of just one parameter. The same property holds for the realization of the sub-Poissonian statistics. The GCSs under study resolve the identity operator with a weight function that is the product of a Wright generalized hypergeometric function and a Fox H function.

The interaction with a zero-temperature reservoir turns the initial GCS into a mixture of GCSs of different natures. The mixed state tends, ultimately, to the vacuum state over sufficiently long times. The evolution of a GCS, induced by the interaction with a thermal reservoir, and the purity are described via Glauber's displacement operators. Over long times, every GCS tends to the thermal state, and the final value of the purity is less than unity. However, this asymptotic value tends to unity at low temperature but vanishes at high temperature. The above-described behaviors also characterize the CSs generalized with the Fox H function.

In conclusion, the CSs generalized with the Fox H auxiliary functions describe various perturbations of canonical CSs, resolve the identity operator with novel forms of the weight function, and exhibit classical or non-classical properties and anomalous statistics for large numbers of excitations. These peculiarities might help to describe anomalous phenomena in the framework of purely quantum theory. The present theoretical construct might help to investigate if and how the realization of CSs in the most varied scenarios changes their properties by analyzing this kind of perturbation.

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Appendix A. Details

The normalization factor $N_{\mathrm{H}}(u)$, given by Equation (38), is obtained from the general form (2) by considering the arithmetic function $g_{\mathrm{H}}(n)$, given by Equation (37). The corresponding power series provides the Wright generalized hypergeometric function via

Equation (17). The weight function $U_{\mathrm{H}}(u)$, given by Equation (39), is derived from Equations (6), (36), and (38).

The reciprocal of the arithmetic function $g_{\mathrm{H}}(n)$ results in the following ratio of products of Gamma functions:

$$\frac{1}{g_{\mathrm{H}}(n)}=\frac{\prod_{j=1}^{p'}\Gamma\left(\bar{\alpha}_j+A'_jn\right)}{\prod_{k=1}^{q'}\Gamma\left(\bar{\beta}_k+B_kn\right)} \tag{A1}$$

for every $n\in\mathbb{N}_0$. The asymptotic behavior of the term $1/g_{\mathrm{H}}(n)$, as $n\to+\infty$, is obtained by evaluating the asymptotic behavior of each Gamma function appearing on the right side of Equation (A1). The following asymptotic form [42]:

$$\Gamma(az+b)\sim(2\pi)^{1/2}\left(\frac{az}{e}\right)^{az}(az)^{b-(1/2)}, \tag{A2}$$

holds for every $a>0$, $b,z\in\mathbb{C}$, with $|\arg z|<\pi$, as $z\to\infty$. Relation (A2) provides the asymptotic behavior of each Gamma function appearing in Equation (A1) in case $z=n$ and $n\to+\infty$. The asymptotic behavior of the probability $P_{\mathrm{H}}\left(n,|z|^2\right)$, given by Equation (46), is obtained from the asymptotic behavior of the term $1/g_{\mathrm{H}}(n)$, as $n\to+\infty$. In this way, the asymptotic form (47) of the probability $P_{\mathrm{H}}\left(n,|z|^2\right)$, as $n\to+\infty$, and the expressions (48)–(51) of the involved parameters are evaluated in a straightforward manner.

Form (57) is obtained from the above-reported approach by observing that the term $g_{\mathrm{H}}(n)$, given by Equation (54), is positive for every $n=0,\ldots,\mathfrak{d}$. Thus, the term $1/g_{\mathrm{H}}(n)$ coincides with the ratio of the products of the absolute values of the involved Gamma functions, for every $n=0,\ldots,\mathfrak{d}$. Hence, for large values of the number of excitations, $\mathfrak{d}\geq n\gg1$, the term $1/g_{\mathrm{H}}(n)$ is properly approximated by the asymptotic expression of the term $1/|g_{\mathrm{H}}(n)|$, as $n\to+\infty$. This asymptotic expression is evaluated via the following asymptotic form [42]:

$$|\Gamma(x)|\sim(2\pi)^{1/2}\left(\frac{|x|}{e}\right)^{x}|x|^{-1/2}, \tag{A3}$$

holding for any $x\in\mathbb{R}$, as $|x|\to\infty$. In this way, form (57) is found.

The present generalization of CSs relies on the positivity of the Fox H function on $\mathbb{R}^+$, i.e., relation (22). This property is realized via the values of the involved indices and parameters that are described by relations (23)–(34). For the sake of clarity, we report below the main steps that provide these relations, following Ref. [51]. Consider the elementary functions $\varphi_{a,b}(u),\phi_{a,b,c}(u),\psi_{a,b,c}(u),\eta_{a,b,c}(u)$, defined on $\mathbb{R}^+$, as below,

$$\varphi_{a,b}(u)=\frac{u^{b/a}}{a}\exp\left(-u^{1/a}\right),\quad u>0, \tag{A4}$$

for every $a>0$ and $b\geq0$;

$$\begin{aligned}\phi_{a,b,c}(u)&=\frac{u^{b/a}\left(1-u^{1/a}\right)^{c-b-1}}{a\Gamma(c-b)},\quad 0<u<1,\\ \phi_{a,b,c}(u)&=0,\quad u\geq1,\end{aligned} \tag{A5}$$

for every $a>0$, $b\geq0$, and $c\geq b+1$;

$$\psi_{a,b,c}(u)=\frac{\Gamma(b+c)}{a}u^{b/a}\left(1+u^{1/a}\right)^{-b-c},\quad u>0, \tag{A6}$$

for every $a, c > 0$ and $b \geq 0$;

$$\eta_{a,b,c}(u) = \frac{u^{(1-c)/a}}{a\Gamma(c-b)} \left(u^{1/a} - 1\right)^{c-b-1}, \quad u > 1,$$
$$\eta_{a,b,c}(u) = 0, \quad 0 < u \leq 1, \tag{A7}$$

for every $a, b > 0$ and $c \geq b + 1$. The values of the involved parameters are chosen in such a way that the functions $\varphi_{a,b}(u), \phi_{a,b,c}(u), \psi_{a,b,c}(u), \eta_{a,b,c}(u)$ behave properly as $u \to 0, 1, +\infty$.

The Mellin convolution product $(\theta_1 \vee \theta_2)(u)$ of the functions $\theta_1(u)$ and $\theta_2(u)$ is defined on $\mathbb{R}^+$ by the following form:

$$(\theta_1 \vee \theta_2)(u) = \int_0^{+\infty} \frac{1}{u'} \theta_1\left(\frac{u}{u'}\right) \theta_2(u') du', \tag{A8}$$

for every $u > 0$ in case the involved integral exists. Let the function $f(u)$ be defined on $\mathbb{R}^+$ via the following Mellin convolution products:

$$f(u) \equiv \Big(\varphi_{a_1,b_1} \vee \cdots \vee \varphi_{a_{n_1},b_{n_1}} \vee \phi_{a'_1,c_1,d_1} \vee \cdots \vee \phi_{a'_{n_2},c_{n_2},d_{n_2}}$$
$$\vee \psi_{a''_1,o_1,r_1} \vee \cdots \vee \psi_{a''_{n_3},o_{n_3},r_{n_3}} \vee \eta_{a'''_1,v_1,w_1} \vee \cdots \vee \eta_{a'''_{n_4},v_{n_4},w_{n_4}}\Big)(u), \tag{A9}$$

and let Conditions (31)–(34) hold for the involved parameters. By definition, the function $f(u)$ is positive, $f(u) > 0$ for every $u > 0$ and every $(n_1, n_2, n_3, n_4) \in \mathbb{N}_0^4$, except for the following cases: $n_1, n_3, n_4 = 0$, and $n_2 \geq 1$, or $n_1, n_2, n_3 = 0$, and $n_4 \geq 1$. The Mellin convolution product introduced in Equation (A9) is continuous, as the involved functions are uniformly continuous and bounded on $\mathbb{R}^+$. Thus, the following equality holds:

$$f(u) = H^{m',n'}_{p',q'} \left[u \middle| \begin{array}{c} \left(\alpha'_j, A'_j\right)_1^{p'} \\ \left(\beta'_j, B'_j\right)_1^{q'} \end{array} \right], \tag{A10}$$

for every u, u', u'' such that $u'' \geq u \geq u' > 0$, as the Mellin transforms of the functions involved in Equation (A10) coincide over a common non-empty strip. Hence, relation (22) holds for every $u > 0$ if Conditions (31)–(34) are fulfilled. Refer to [51] for details.

Expression (60) of the Mandel parameter is obtained from Equation (59) by using form (37) of the arithmetic function $g(n)$. The asymptotic form (61) is evaluated in a straightforward manner from Equation (60) by considering the Wright generalized hypergeometric function as a particular case of the Fox H function [42], Equation (17).

Relation (63) is equivalent to the following inequality involving ratios of positive Gamma functions:

$$\left(\prod_{j=1}^{n_1} \frac{\Gamma(\bar{b}_j + 2a_j)}{\Gamma(\bar{b}_j + a_j)}\right) \left(\prod_{k=1}^{n_2} \frac{\Gamma(\bar{c}_k + 2a'_k)}{\Gamma(\bar{c}_k + a'_k)} \frac{\Gamma(\bar{d}_k + a'_k)}{\Gamma(\bar{d}_k + 2a'_k)}\right)$$
$$<_{(>)} 2 \left(\prod_{j=1}^{n_1} \frac{\Gamma\left(\bar{b}_j + a'_j\right)}{\Gamma(\bar{b}_j)}\right) \left(\prod_{k=1}^{n_2} \frac{\Gamma(\bar{c}_k + a'_k)}{\Gamma(\bar{c}_k)} \prod_{k=1}^{n_2} \frac{\Gamma(\bar{d}_k)}{\Gamma(\bar{d}_k + a'_k)}\right). \tag{A11}$$

The recurrence relation for the Gamma function [71], $\Gamma(z+1) = z\Gamma(z)$, holding for $\mathrm{Re}\, z > 0$, provides the following form:

$$\left(\prod_{j=1}^{n_1} \frac{b_j+2}{b_j+1}\right)\left(\prod_{k=1}^{n_2} \frac{c_k+2}{c_k+1}\right) <_{(>)} 2\prod_{k=1}^{n_2} \frac{d_k+2}{d_k+1}, \tag{A12}$$

in case $a_j = a'_k = 1$, for every $j = 1, \ldots, n_1$, with $n_1 \geq 1$ and $k = 1, \ldots, n_2$. The parametrization (64) provides relation (65) from Condition (A12). Thus, relation (65) is equivalent to relation (63) in any case where $a_j = a'_k = 1$, for every $j = 1, \ldots, n_1$, with $n_1 \geq 1$, and $k = 1, \ldots, n_2$.

Relation (63) is realized for sufficiently large values of one involved parameter in the general case where GCSs are defined via Equations (23)–(26), (28), (29), (31)–(34), and (36), with $n_1 \geq 1$ and $n_3 = n_4 = 0$. In fact, super-Poissonian statistics are derived for small, nonvanishing values of the label via the following vanishing asymptotic behavior [71]:

$$\lim_{A'_{j'} \to +\infty} \frac{\Gamma\left(\bar{\alpha}_{j'} + A'_{j'}\right)}{\Gamma\left(\bar{\alpha}_{j'}\right)} \Bigg/ \frac{\Gamma\left(\bar{\alpha}_{j'} + 2A'_{j'}\right)}{\Gamma\left(\bar{\alpha}_{j'} + A'_{j'}\right)} = 0^+, \tag{A13}$$

holding for $A'_{j'} \gg 1$. Instead, sub-Poissonian statistics are derived for small, nonvanishing values of the label via the following vanishing asymptotic behavior [71]:

$$\lim_{B'_{j''} \to +\infty} \frac{\Gamma\left(\bar{\beta}_{j''} + 2B'_{j''}\right)}{\Gamma\left(\bar{\beta}_{j''} + B'_{j''}\right)} \Bigg/ \frac{\Gamma\left(\bar{\beta}_{j''} + B'_{j''}\right)}{\Gamma\left(\bar{\beta}_{j''}\right)} = +\infty, \tag{A14}$$

holding for $B'_{j''} \gg 1$. Relations (75)–(81) are obtained in a straightforward manner from Equations (6), (7), and (14)–(39).

The operator $|j_1\rangle\langle j_2|$ is expressed in terms of canonical CSs via the following integral form [2,3,65,66,68]:

$$|j_1\rangle\langle j_2| = \int_{\mathbb{R}^2} d^2\zeta \frac{\exp(\zeta\zeta^*)}{\sqrt{j_1! j_2!}} \frac{\partial^{j_1+j_2}}{\partial\zeta^{j_1}\partial\zeta^{*j_2}} \delta^2(\zeta)|\zeta\rangle\langle\zeta|, \tag{A15}$$

for every $j_1, j_2 \in \mathbb{N}_0$, where $\delta^2(\zeta) = \delta(Re\,\zeta)\delta(Im\,\zeta)$, for every $\zeta \in \mathbb{C}$. The function $\delta(\zeta)$ represents the Dirac delta function. Relation (A15) provides expressions (89)–(93) by considering the time evolution of canonical CSs interacting with a thermal reservoir [64],

$$\exp(\mathfrak{L}t)|\zeta\rangle\langle\zeta| = \sum_{k=0}^{+\infty} \frac{\mathfrak{n}(t,T)^k}{(1+\mathfrak{n}(t,T))^{k+1}} \mathfrak{D}(\zeta\exp(-v't))|k\rangle\langle k|\mathfrak{D}^\dagger(\zeta\exp(-v't)), \tag{A16}$$

for every $t \geq 0$, $\zeta \in \mathbb{C}$, and $T > 0$. See Refs. [2,3] for the Glauber's displacement operators.

Expression (95) is obtained from the following representation of the initial condition:

$$|z,g\rangle\langle z,g| = \pi^{-2} \int_{\mathbb{R}^2} d^2\zeta \int_{\mathbb{R}^2} d^2\chi |\zeta\rangle\langle\zeta||z,g\rangle\langle z,g||\chi\rangle\langle\chi|, \tag{A17}$$

for every $\zeta \in \mathbb{C}$. Relations (96)–(99) are evaluated in Ref. [64]. Relations (100) and (101) are obtained in a straightforward manner from the expression of the term $\langle l|\mathfrak{D}(\zeta)|k\rangle$ holding for every $\zeta \in \mathbb{C}$ and every $l, k \in \mathbb{N}_0$ such that $l \geq k$.

Expression (102) of the purity is obtained from the trace operation of the term $\rho^2(t)$, given the form below,

$$\rho^2(t) = \pi^{-4} \int_{\mathbb{R}^2} d^2\zeta_1 \int_{\mathbb{R}^2} d^2\chi_1 \int_{\mathbb{R}^2} d^2\zeta_2 \int_{\mathbb{R}^2} d^2\chi_2 \langle\zeta_1||z,g\rangle\langle z,g||\chi_1\rangle\langle\zeta_2||z,g\rangle\langle z,g||\chi_2\rangle$$
$$(\exp(\mathfrak{L}t)|\zeta_1\rangle\langle\chi_1|)(\exp(\mathfrak{L}t)|\zeta_2\rangle\langle\chi_2|), \quad \text{(A18)}$$

for every $z \in \mathbb{C}$, and $t \geq 0$. The trace operation given by Equation (103) is obtained in a straightforward manner from the term $(\exp(\mathfrak{L}t)|\zeta_1\rangle\langle\chi_1|)(\exp(\mathfrak{L}t)|\zeta_2\rangle\langle\chi_2|)$, given by the expression below,

$$(\exp(\mathfrak{L}t)|\zeta_1\rangle\langle\chi_1|)(\exp(\mathfrak{L}t)|\zeta_2\rangle\langle\chi_2|) = \langle\chi_1||\zeta_1\rangle\langle\chi_2||\zeta_2\rangle A(\zeta_1,\chi_1,t,T)A(\zeta_2,\chi_2,t,T)$$
$$\times \sum_{j_1=0}^{+\infty}\sum_{j_2=0}^{+\infty} \frac{(\mathfrak{n}(t,T))^{j_1+j_2}}{(1+\mathfrak{n}(t,T))^{j_1+j_2+2}} \mathfrak{D}\big(\bar{\zeta}_1(t,T)\big)|j_1\rangle\langle j_1|\mathfrak{D}^\dagger(\bar{\chi}_1(t,T))\mathfrak{D}\big(\bar{\zeta}_2(t,T)\big)|j_2\rangle\langle j_2|\mathfrak{D}^\dagger(\bar{\chi}_2(t,T)), \quad \text{(A19)}$$

for every $t \geq 0$, $\zeta_1, \chi_1, \zeta_2, \chi_2 \in \mathbb{C}$, and $T > 0$.

The long-time vanishing behaviour, $t \gg 1/v'$, of the functions $\bar{\zeta}(t,T)$ and $\bar{\chi}(t,T)$ provides Equation (104). This concludes the demonstration of the present results.

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