SPECTRALLY ORDERED LIE ALGEBRAS

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Abstract

We study ordered vector spaces which are at the same time Lie algebras and satisfy certain compatibility conditions between the algebraic and the order structure. The main point is that the ordering of the Lie algebra is completely determined by the induced ordering on a Cartan algebra.

Introduction

A physical quantity in quantum mechanics is usually described by a self-adjoint operator in Hilbert space. All the information on the physical properties is provided by the spectral resolution of the corresponding operator. The quantities of a whole physical system are then described by the space A of self adjoint elements in a C^*-algebra. Alfsen and Shultz noted in [AS76] that the spectral theory of A can be described fully in terms of A and its positive cone A^+. Petrov suggests in [Pe85] to describe a quantum mechanical system by an ordered vector space L, which in addition carries the stucture of a Lie algebra, related to the dynamics and the symmetries of the system. He asks for the invariance of the positive cone and an order unit under the group G of inner automorphisms of L. In addition the notions of compatibility arising from the order and the algebraic structure have to aggree in his approach.

In this note we establish some basic results concerning the structure of finite dimensional Lie algebras of the aforementioned type. For the simplicity of the exposition we do not use Alfsen and Shultz's version of a spectral theory for ordered vector spaces, but the more algebraic version of Riedel (see [Ri83]). At the present stage of Petrov's approach there doesn't seem to be a physical reason to prefer any of the (inequivalent) versions.

Definitions

Let L be an ordered vector space and L^+ the positive cone with respect to the order. For any element $x \in L$ we define the *order ideal* I_x generated by $I_x = (L^+ - \mathbb{R}\cdot x) \cap (\mathbb{R}\cdot x - L^+)$. An element $e \in L$ is called an *order unit* if $I_e = L$. We assume that L has order units and fix one. A *fundamental unit* is an element $p \in [0, e] = \{x \in L: 0 \leq x \leq e\}$ for which $I_p \cap I_{e-p} = \{0\}$ holds. We denote the set of all fundamental units in L by $\mathcal{F}_L$. If $p \in \mathcal{F}_L$ then an element $x \in L$ is called *compatible* with p, written $p \parallel x$, if $x \in I_p \oplus I_{e-p}$. The *$\mathcal{F}$-commutant* $\mathcal{K}_{\mathcal{F}}(x)$ of an element $x \in L$ is the set $\mathcal{K}_{\mathcal{F}}(x) = \{p \in \mathcal{F}_L: p \parallel x\}$. Finally we call $\mathcal{B}_{\mathcal{F}}(x) = \{p \in \mathcal{K}_{\mathcal{F}}(x): p \parallel q \text{ for all } q \in \mathcal{F}_L\}$ the *$\mathcal{F}$-bicommutant* of $x \in L$. The basic condition which allows one to do a spectral theory on ordered vector spaces is the *spectral condition*:

SC1 Any bounded increasing sequence in L has a supremum in L.

SC2 For all $x \in L$ there is a $p \in \mathcal{B}_{\mathcal{F}}(x)$ such that $\pi_p x \geq 0$ and $\pi_{e-p} x \leq 0$ where $\pi_p: I_p \oplus I_{e-p} \to I_p$ and $\pi_{e-p}: I_p \oplus I_{e-p} \to I_p$ are the canonical projections.

A set $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ of fundamental units is called a *spectral set* if the following conditions hold:

SS1 $\lambda \leq \mu \Rightarrow e_\lambda \leq e_\mu$ for all $\lambda, \mu \in \mathbb{R}$.

SS2 $e_\lambda = \inf_{\lambda < \mu} e_\mu$ in L for all $\lambda \in \mathbb{R}$.

SS3 $e = \sup_{\lambda \in \mathbb{R}} e_\lambda$ in L and $0 = \inf_{\lambda \in \mathbb{R}} e_\lambda$ in L.

A spectral set $\{e_\lambda\}_{\lambda \in \mathbb{R}}$ is called a *spectral resolution* of an element $x \in L$ if $e_\lambda \in \mathcal{K}_{\mathcal{F}}(x)$ for all $\lambda \in \mathbb{R}$ and $\pi_{e_\lambda} x \leq \lambda e_\lambda$ as well as $\pi_{e-e_\lambda} x \geq \lambda(e - e_\lambda)$. The spectral theory for ordered vector spaces as laid down in [Ri83] implies the existence and uniqueness of spectral resolutions. This allows to extend the notion of compatibility to arbitrary elements of L by saying x and y are compatible, written again $x \parallel y$, if x is compatible with all the elements in the spectral resolution of y. Thus it makes sense to define the *commutant* $\mathcal{K}(x) = \{y \in L: y \parallel x\}$ of x and the *bicommutant* $\mathcal{B}(x) = \{y \in L: y \parallel z \text{ for all } z \in \mathcal{K}(x)\}$ of x. The results of [Ri83] then show that $\mathcal{K}_{\mathcal{F}}(x) = \mathcal{K}(x) \cap \mathcal{F}_L$ and $\mathcal{B}_{\mathcal{F}}(x) = \mathcal{B}(x) \cap \mathcal{F}_L$.

In the sequel we will assume that L is a finite dimensional real Lie algebra. It will be called *spectrally ordered* if the following conditions hold.

SO1 (L, L^+, e) is an ordered vector space with order unit satisfying the spectral condition.

SO2 L^+ is invariant under the inner automorphisms of L.

SO3 e is fixed under the inner automorphisms of L.

SO4 For all $x, y \in L$ we have $[x, y] = 0$ if and only if $x \parallel y$.

Cartan algebras

If L is spectrally ordered then the order unit is central in L and contained in the interior of the positive cone L^+. In fact, if we differentiate the equation $e^{\mathrm{ad}\, x} e = e$ we find $[x, e] = 0$ for all $x \in \mathbb{R}$. Moreover, since e is an order unit, we know that $L = I_e \subseteq (L^+ - \mathbb{R}{\cdot}e)$ so that $e \in \operatorname{int} L^+$. If we now use the results of [HH86] we find that any spectrally ordered Lie algebra must be *compact*, that is, its group of inner automorphisms is compact. There is a complete classification of *invariant* (under inner automorphisms) convex cones in Lie algebras in terms of their intersection with an arbitrary *compactly embedded* Cartan subalgebra (see [HHL87]). In compact Lie algebras all Cartan algebras are compactly embedded. In this case the classification result is the following.

1. Theorem. *Let L be a compact Lie algebra and H a Cartan subalgebra of L. Moreover let $G = \langle e^{\mathrm{ad}\, x} : x \in L \rangle$ be the group of inner automorphisms of L and $Z(H, G)$ and $N(H, G)$ the centralizer and the normalizer of H in G respectively. Consider the finite group $\mathcal{W}(H, L) = N(H, G)/Z(H, G)$ which operates on H. Then the map $W \mapsto W \cap H$ is a bijection between the set of G-invariant convex cones W in L with $W - W = L$ and $W \cap -W = \{0\}$ and the set of $\mathcal{W}(H, L)$-invariant convex cones C in H with $C - C = H$ and $C \cap -C = \{0\}$. The inverse of the map is $C \mapsto \operatorname{conv}\{G{\cdot}C\}$.* ∎

This suggests that we try to describe (L, L^+, e) via (H, H^+, e), where $H^+ = L^+ \cap H$. In particular we want to see how the spectral and compatibility conditions of (L, L^+, e) are reflected in (H, H^+, e).

Until further notice we will assume in the following that (L, L^+, e) is a fixed spectrally ordered Lie algebra and H is a fixed Cartan algebra in L.

2. Lemma. *Let $h \in H$, then we have*

(i) $H \subseteq \mathcal{K}(h)$.

(ii) $\mathcal{K}(H) = H$.

(iii) $\mathcal{B}(h) \subseteq H$.

(iv) $\mathcal{B}(h) = H$ *if and only if h is regular, which is the case if and only if* $\mathcal{K}(h) = H$.

Proof. $y \in \mathcal{K}(H)$ if and only if $y \parallel x$ for all $x \in H$, which by (SO4) is equivalent to $y \in Z(H,L) = H$. This implies (ii), whereas (i) is clear in view of this argument. Now we note that if $y \in \mathcal{B}(h)$ then $y \parallel x$ for all $x \in \mathcal{K}(h)$ so that $y \parallel x$ for all $x \in H$ by (i), which then shows $y \in \mathcal{K}(H) = H$ and thus proves (iii). Finally we remark that h is regular if and only if $Z(h,L) = H$ hence if and only if $H = \{y \in L : y \parallel h\} = \mathcal{K}(h)$. But this in turn is equivalent to $\mathcal{B}(h) = \mathcal{K}(H) = H$ and the lemma is proved. ∎

Lemma 2 allows us to conclude that the spectral resolution $\{e_\lambda\}$ of an element $h \in H$ consists entirely of elements of H. In fact [Ri83,Prop.3.9] says that all the e_λ are contained in $\mathcal{B}(h) \subseteq H$.

Our next goal is to show that the fundamental units in the spectral resolution of $h \in H$ are not only elements of H, but also fundamental units of the ordered vector space (H, H^+) with order unit $e \in Z(L) \cap \operatorname{int} L^+ \subseteq \operatorname{int} H^+$. To that end recall that in a compact Lie algebra we can always find an inner product which is invariant under inner automorphisms. We choose one on L and consider the orthogonal projection $P_H : L \to H$ onto H. This projection can be defined in terms of an integral over $Z(H,G)$ and therefore leaves the cone L^+ invariant. We claim that

$$(1) \qquad P_H([0,x]) = [0,x] \cap H = [0,x]_H = \{y \in H : 0 \le y \le x\}.$$

In fact if $y \in [0,x] = L^+ \cap (x - L^+)$ then $P_H(y) \subseteq (L^+ \cap H) \cap \big(x - (L^+ \cap H)\big)$ because of $P_H(L^*) \subseteq L^+ \cap H$ and $P_H(x) = x$. But then $P_H([0,x]) \subseteq [0,x] \cap H \subseteq [0,x]_H = P_H([0,x]_H) \subseteq P_H([0,x])$.

Now we can show that

$$(2) \qquad H \cap \mathcal{F}_L \subseteq \mathcal{F}_H.$$

By (1) it suffices to show that $J_p \cap J_{e-p} = \{0\}$ for $p \in \mathcal{F}_L \cap H$ where $J_p = (H^+ - \mathbb{R}\cdot p) \cap (\mathbb{R}\cdot p - H^+)$ and $J_{e-p} = \big(H^+ - \mathbb{R}\cdot(e-p)\big) \cap \big(\mathbb{R}\cdot(e-p) - H^+\big)$ are the respective order ideals in H. But it is clear that $J_p \subseteq I_p$ and $J_{e-p} \subseteq I_{e-p}$ so that (2) follows. Note that it now also follows from (1) that the spectral resolution of an element $h \in H$ is not only a spectral set in L consisting of elements of H, but is again a spectral set if viewed as a set of fundamental units in the ordered space (H, H^+, e). Even more is true:

3. Lemma. *Let $\{e_\lambda\}$ be a spectral resolution of $h \in H$ in L. Then $\{e_\lambda\}$ is a spectral resolution of h in H.*

Proof. We just remarked that $\{e_\lambda\}$ is a spectral set in (H, H^+, e). Moreover we have $e_\lambda \in \mathcal{K}(h)$ for all $h \in H$ by Lemma 2. Thus it only remains to show that the canonical projections $\psi_{e_\lambda} : J_{e_\lambda} \oplus J_{e-e_\lambda} \to J_{e_\lambda}$ and $\psi_{e-e_\lambda} : J_{e_\lambda} \oplus J_{e-e_\lambda} \to J_{e-e_\lambda}$ satisfy $\psi_{e_\lambda}(h) \le \lambda e_\lambda$ and $\psi_{e-e_\lambda}(h) \ge \lambda(e - e_\lambda)$. We fix a λ and let $h = x' + x'' \in J_{e_\lambda} \oplus J_{e-e_\lambda}$. Recall that $J_{e_\lambda} \subseteq I_{e_\lambda}$ and $J_{e-e_\lambda} \subseteq I_{e-e_\lambda}$. Moreover the maps ψ_{e_λ} and ψ_{e-e_λ} are the restrictions of π_{e_λ} and π_{e-e_λ} so that $\psi_{e_\lambda}(h) = x' = \pi_{e_\lambda}(h) \le \lambda e_\lambda$ and $\psi_{e-e_\lambda}(h) = x'' = \pi_{e-e_\lambda}(h) \ge \lambda(e - e_\lambda)$ in L hence , by (1), in H. ∎

Now we can prove the main result of this section.

4. Theorem. *Let (L, L^+, e) be a spectrally ordered Lie algebra and H a Cartan subalgebra of L, then (H, H^+, e) with $H^+ = L^+ \cap H$ is a spectrally ordered abelian Lie algebra.*

Proof. First we show that (H, H^+, e) satisfies the spectral condition. To do this, consider $h \in H$ with a spectral resolution $\{e_\lambda\}$ in L, hence in H. Choosing $\lambda = 0$ we find $\psi_{e_0}(h) \leq 0$ and $\psi_{e-e_0}(h) \geq 0$. Therefore $p = e - e_0 \in \mathcal{K}(h)$ satisfies $\psi_p(h) \geq 0$ and $\psi_{e-p}(h) \leq 0$, which show that (H, H^+, e) satisfies condition (SB1) from [Ri83]. Thus we can apply [Ri83, Cor.3.21] to conclude that (H, H^+, e) satisfies the spectral condition. It remains to show that (H, H^+, e) also satisfies (SO1) through (SO4). But we have just shown that (S01) holds and conditions (SO2) and (SO3) are trivially satisfied. For (SO4) it suffices to show that two elements h and h' in H, which are compatible in L, are also compatible in H. Considering spectral resolutions in L, we may assume that h and h' are fundamental units in L. Since h and h' are compatible in L we know that $h' \in I_h \oplus I_{e-h}$. This means that we can write h' as $a + b$ with $a = x_a - r_a h = s_a h - y_a \in I_h$ and $b = x_b - r_b(e-h) = s_b(e-h) - y_b \in I_h$, where $r_a, r_b, s_a, s_b \in \mathbb{R}$ and $x_a.x_b, y_a, y_b \in L^+$. If we now apply P_H to h', a and b, and recall that $P_H(L^+) \subseteq H^+$, we find that $h' = P_H(a) + P_H(b)$ and $P_H(a) \in J_h$ as well as $P_H(b) \in J_{e-h}$ which is exactly what we had to show. ■

Note that Theorem 4 implies also the converse of inclusion of (2):

$$H \cap \mathcal{F}_L = \mathcal{F}_H. \tag{2'}$$

In fact, since (H, H^+, e) satisfies the spectral condition, spectral resolutions in H are unique, hence must agree with the spectral resolution coming from L. Since the spectral resolution of a fundamental unit h in H is singleton, this single element must also form the spectral resolution of h in L, which in turn means that h is also a fundamental unit in L. But Theorem 4 yields even more information.

5. Corollary. *Let (L, L^+, e) be a spectrally ordered Lie algebra and H a Cartan algebra in L, then H^+ is a simplicial cone, that is, a polyhedral cone spanned by* $\dim H$ *extremal rays.*

Proof. Since (H, H^+, e) is spectrally ordered abelian it follows from [AS76] that it is, as an ordered vector space, isomorphic to some $\big(C(X), C(X)^+, f\big)$ for a compact space X. Thus the claim follows since H is finite dimensional. ■

Let us summarize:

6. Theorem. *Let (L, L^+, e) be a spectrally ordered Lie algebra and H a Cartan algebra in L. Then $H^+ = L^+ \cap H$ is a $\mathcal{W}(H, L)$-invariant simplicial cone in H which contains e in its interior. The triple (H, H^+, e) is a spectrally ordered* *abelian Lie algebra whose fundamental units are precisely those fundamental units of* (L, L^+, e) *which are contained in* H. *The positive cone* L^+ *of* L *can be retrieved from* H^+ *via* $L^+ = \text{conv}\{G \cdot H\}$, *where* G *is the (compact) group of inner automorphisms of* L. ■

Conclusion

Theorem 6 yields a series of rather stringent necessary conditions on (H, H^+, e) for the ordered Lie algebra with order unit (L, L^+, e) to be spectrally ordered. What remains to be done, is to find a good way to check when a fundamental unit in H is a fundamental unit in L. This, together with the fact that any element of L is contained in some conjugate of H would pave the way to a converse of Theorem 6 (see [Ri83, Prop.3.20 and Cor.3.21]), and hence to a classification of finite dimensional spectrally ordered Lie algebras.

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