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Stability walls in heterotic theories

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Abstract: We study the sub-structure of the heterotic Kähler moduli space due to the presence of non-Abelian internal gauge fields from the perspective of the four-dimensional effective theory. Internal gauge fields can be supersymmetric in some regions of the Kähler moduli space but break supersymmetry in others. In the context of the four-dimensional theory, we investigate what happens when the Kähler moduli are changed from the supersymmetric to the non-supersymmetric region. Our results provide a low-energy description of supersymmetry breaking by internal gauge fields as well as a physical picture for the mathematical notion of bundle stability. Specifically, we find that at the transition between the two regions an additional anomalous U(1) symmetry appears under which some of the states in the low-energy theory acquire charges. We compute the associated D-term contribution to the four-dimensional potential which contains a Kähler-moduli dependent Fayet-Iliopoulos term and contributions from the charged states. We show that this D-term correctly reproduces the expected physics. Several mathematical conclusions concerning vector bundle stability are drawn from our arguments. We also discuss possible physical applications of our results to heterotic model building and moduli stabilisation.

Keywords: Superstrings and Heterotic Strings, Differential and Algebraic Geometry, Superstring Vacua, M-Theory

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1 Introduction

Heterotic compactifications on Calabi-Yau manifolds necessarily have gauge field expectation values (vevs) in the internal dimensions. This feature, which is a consequence of demanding that the total charge for the Neveu-Schwarz form should vanish on the internal compact space, gives rise to much of the complexity and structure of these theories [1, 2]. One of the interesting properties of these gauge field vevs concerns their supersymmetry preserving properties. The internal field strength, F, is usually chosen so as to preserve $\mathcal{N} = 1$ supersymmetry in the four-dimensional effective theory. This can be imposed by demanding that the ten-dimensional gaugino supersymmetry variations vanish. This leads to the conditions

$$g^{a\bar{b}} F_{\bar{b}a} = 0 \, , \quad F_{ab} = F_{\bar{a}\bar{b}} = 0 \tag{1.1}$$

which are known as the Hermitian Yang-Mills equations (here, a and $\bar{b}$ are holomorphic and anti-holomorphic indices on the Calabi-Yau, respectively). However, if one chooses gauge fields satisfying the Hermitian Yang-Mills equations and, hence, preserving supersymmetry at one point in moduli space, and then changes the values of the moduli, one can find that eqs. (1.1) fail to have a solution and supersymmetry becomes broken [3, 4]. More specifically, it is possible to demarcate regions in Kähler moduli space where the gauge field vevs can preserve supersymmetry and regions where they necessarily break it [5, 6]. Given the explicit dependence of the eqs. (1.1) on the metric, and hence the Kähler moduli, such a behaviour is perhaps not surprising.

What happens in the effective field theory when the moduli evolve such that the gauge fields break supersymmetry? One can see from the dimensional reduction of the ten-dimensional effective action of the $E_8 \times E_8$ heterotic theory that there will be a positive definite potential in the non-supersymmetric region of Kähler moduli space. The argument goes as follows. Consider the following three terms in the ten-dimensional effective action.

$$S_{\text{partial}} = -\frac{1}{2\kappa_{10}^2}\frac{\alpha'}{4}\int_{\mathcal{M}_{10}} \sqrt{-g}\left\{\text{tr}\left(F^{(1)}\right)^2 + \text{tr}\left(F^{(2)}\right)^2 - \text{tr}R^2\right\} . \tag{1.2}$$

The notation here is standard [2] with the field strengths $F^{(1)}$ and $F^{(2)}$ being associated to the two E_8 factors in the gauge group. One consequence of the ten-dimensional Bianchi Identity,

$$dH = -\frac{3\alpha'}{\sqrt{2}}\left(\text{tr}F^{(1)} \wedge F^{(1)} + \text{tr}F^{(2)} \wedge F^{(2)} - \text{tr}R \wedge R\right) , \tag{1.3}$$

is its integrability condition,

$$\int_{M_6} J \wedge \left(\text{tr } F^{(1)} \wedge F^{(1)} + \text{tr } F^{(2)} \wedge F^{(2)} - \text{tr } R \wedge R\right) = 0 , \tag{1.4}$$

where J is the Kähler form. Now, suppose that we begin with a supersymmetric field configuration, and then vary the Kähler moduli while keeping the other moduli fixed. The background gauge field strengths are then $(1,1)$ forms to lowest order (as is the curvature two-form). Using this observation, and the fact that we are working, again to lowest order, with a Ricci flat metric on a manifold of SU(3) holonomy, equation (1.4) can be rewritten as follows:

$$\int_{M_{10}} \sqrt{-g}\left(\text{tr}\left(F^{(1)}\right)^2 + \text{tr}\left(F^{(2)}\right)^2 - \text{tr}R^2 - \text{tr}\left(F^{(1)}_{a\bar{b}}g^{a\bar{b}}\right)^2 - \text{tr}\left(F^{(2)}_{a\bar{b}}g^{a\bar{b}}\right)^2\right) = 0 . \tag{1.5}$$

Using this relation in (1.2), we arrive at the following result

$$S_{\text{partial}} = -\frac{1}{2\kappa_{10}^2}\frac{\alpha'}{4}\int_{\mathcal{M}_{10}} \sqrt{-g}\left\{\text{tr}\left(F^{(1)}_{a\bar{b}}g^{a\bar{b}}\right)^2 + \text{tr}\left(F^{(2)}_{a\bar{b}}g^{a\bar{b}}\right)^2\right\} . \tag{1.6}$$

The terms in eq. (1.6) form a part of the ten-dimensional theory which does not contain any four-dimensional derivatives. It therefore contributes, upon dimensional reduction, to the potential of the four dimensional theory. In the case of a supersymmetric field configuration,

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the terms in the integrand of (1.6) vanish, these being precisely the squares of the first equation in (1.1). Thus, in this case, no potential is generated. However, if the Kähler moduli are varied such that the gauge field vevs are no longer supersymmetric, (1.6) no longer vanishes and we obtain a positive definite contribution to the potential energy seen in four dimensions. Thus, we are led to a picture of a perturbative potential which, while positive definite in the non-supersymmetric regions of moduli space, vanishes precisely where the gauge field vevs preserve supersymmetry.

Beyond what is described above, it might seem difficult to write down the exact expression for this potential in terms of the moduli fields. Naively it seems like we need to know the metric and gauge connection on the Calabi-Yau 3-fold. These quantities are of course unknown, except possibly numerically [7–10]. In fact, however, one can analytically derive the exact form of this potential as an explicit function of the moduli fields. This will be the main focus of the present paper.

Before we can discuss the explicit form of this potential, it is useful to briefly review the mathematical language normally used to describe supersymmetry within a heterotic compactification. The question of whether a supersymmetric vacuum exists can be answered by a mathematical analysis of the associated holomorphic vector bundle, V, based on the Donaldson-Uhlenbeck-Yau theorem [3, 4] and the notion of slope-stability. We will explicitly carry this out later in the paper. For the purpose of the present general discussion it suffices to know that the supersymmetry properties of V are governed by a (maximally) destabilizing sub-bundle $\mathcal{F} \subset V$ and a number associated it, called the slope $\mu(\mathcal{F})$, which is a function of the Kähler moduli, t^i, of the Calabi-Yau manifold. The vector bundle V is slope-stable and, hence, the associated gauge field is supersymmetric, in the part of Kähler moduli space where $\mu(\mathcal{F}) < \mu(V)$, and it is unstable and supersymmetry is broken where $\mu(\mathcal{F}) > \mu(V)$. The boundary between those regions, defined by $\mu(\mathcal{F}) = \mu(V)$, divides the Kähler cone into regions of preserved and broken supersymmetry. Such a co-dimension one "boundary" will be referred to as a stability "wall" in the Kähler cone. In the following sections, we will demonstrate that, in fact, the potential given in (1.6), reproduces this structure.

We are now in a position to summarize our main results. For concreteness, we will illustrate the structure of our effective theory for a bundle V with an internal gauge group $G = \mathrm{SU}(3)$, but analogous statements hold for other $\mathrm{SU}(n)$ groups. In this case, $\mu(V) = 0$. For a general point in the supersymmetric region of the Kähler moduli space, that is for $\mu(\mathcal{F}) < 0$, the low-energy gauge group is E_6 (times a possible hidden gauge group which is not relevant to our discussion), the commutant of $\mathrm{SU}(3)$ within E_8. The matter field content consists of a certain number of families and anti-families in $\mathbf{27}$ and $\overline{\mathbf{27}}$ representations, respectively, plus a number of singlet fields which can be interpreted as the bundle moduli of V. For specific examples, the number of these multiplets can be computed from the bundle cohomology of V and we will do this later in the paper. So far, this is simply the field content of a standard heterotic Calabi-Yau compactification.

Next, we consider the theory at the stability wall, that is, the boundary between supersymmetric and non-supersymmetric regions in the Kähler cone where $\mu(\mathcal{F}) = 0$. Here, we find that the structure group of the bundle "degenerates" to $S(\mathrm{U}(2) \times \mathrm{U}(1))$ and, hence, the low-energy gauge group enhances from E_6 to $E_6\times\mathrm{U}(1)$. This theory has the same chiral asymmetry between **27** and $\overline{\mathbf{27}}$ multiplets as the theory at a generic supersymmetric point in moduli space (although their individual numbers may change), bundle moduli for the $S(\mathrm{U}(2)\times\mathrm{U}(1))$ bundle and additional singlet fields C^L. The families/anti-families and the C^L fields carry a charge under the additional U(1) symmetry. It is well-known, in the context of heterotic compactifications [11, 12, 14], that a low-energy U(1) symmetry which arises from of a U(1) factor in the internal gauge group is anomalous in the Green-Schwarz sense. The U(1) vector field is massive as a consequence of the Higgs mechanism. In addition, associated to this U(1) is a D-term which contains a Fayet-Illiopolous (FI) contribution.[1] In our case, we find that the U(1) D-term takes the following form at lowest order in the expansions of heterotic M-theory, and close to the boundary between the supersymmetric and non-supersymmetric regions:

$$D^{\mathrm{U}(1)} = f(t^i) - \sum_{M,\bar{N}} Q^M G_{M\bar{N}} C^M \bar{C}^{\bar{N}} \ . \tag{1.7}$$

Here $G_{L\bar{M}}$ is a positive definite metric and Q^L are the U(1) charges of the fields C^L. The FI term, $f(t^i)$, takes the form (up to a positive constant of proportionality)

$$f(t^i) \sim \frac{\mu(\mathcal{F})}{\mathcal{V}} \tag{1.8}$$

with $\mathcal{V}$ the Calabi-Yau volume and $\mu(\mathcal{F})$ is the slope parameter (described above) of a sub-bundle. The associated D-term potential is the explicit form of the potential described in equation (1.6).

Let us discuss this D-term (1.7) in the various regions of the Kähler cone. At the stability wall, $\mu(\mathcal{F})=0$, the FI term vanishes and, hence, the fields C^M have a vanishing vacuum expectation value. The combination of Kähler moduli perpendicular to the stability wall receives a mass from the FI term and represents the Higgs particle. Its axionic superpartner is absorbed by the U(1) vector field. All of the C^L fields are massless at the stability wall. Now we move into the region $\mu(\mathcal{F})<0$ where supersymmetry should be preserved. In this region, the FI term is negative and the fields C^M develop a compensating vev to set $D^{\mathrm{U}(1)}=0$. Of course, this only works if there is at least one negative U(1) charge Q^L and we will verify that this is indeed the case. In this way, we find that supersymmetry is preserved in the region $\mu(\mathcal{F})<0$, as expected. One might also ask about matching the number of states we observe in this theory to the results obtained from a standard analysis of the supersymmetric region. We find that when the fields C^L develop a vev, the U(1) gauge boson receives an additional contribution to its mass and eventually becomes so massive that it should be dropped from the low-energy spectrum. In this way, we recover the E_6 symmetry at a generic superymmetric point. Further, due to the non-vanishing C^L vevs, one combination of fields, predominately made up from C^L fields, now

[1] We note that for internal gauge fields with structure group $G=\mathrm{U}(1)$, it is known [11, 12, 14, 15] that eq. (1.6) leads to a D-term potential associated with a Green-Schwarz anomalous U(1) symmetry [12, 14]. In fact, it is not difficult to derive this D-term potential from eq. (1.6). In the present paper, however, we are interested in the case of non-Abelian internal gauge groups, specifically $G=\mathrm{SU}(n)$.

becomes the Higgs multiplet and should be removed from the spectrum. For a matching of states between the theory at the stability wall and at a generic supersymmetric point in moduli space we need, therefore, that the number of $S(\mathrm{U}(2) \times \mathrm{U}(1))$ bundle moduli plus the number of C^L fields equals the number of SU(3) bundle moduli plus one. Again, we will explicitly verify that this is true in general. What happens if we move into the region $\mu(\mathcal{F}) > 0$ where we expect supersymmetry to be broken? The above D-term will only lead to broken supersymmetry in this region if the C^L fields cannot compensate for the, now positive, FI term. In other words, *all* of the charges, Q^L, need to be negative if our D-term is to reproduce the supersymmetry properties of the gauge bundle as derived in higher dimensions. We will show that this is indeed always the case. In summary, the above D-term reproduces all of the expected features of supersymmetry breaking induced by internal gauge fields, a subject usually studied in the context of algebraic geometry. As such, it provides a physical picture for the mathematical notion of slope stability for vector bundles and it opens up a range of physical applications, for example in relation to heterotic model building and moduli stabilisation.

In the remainder of this paper, we derive the potential described above, in detail, from first principles. In the next section, we discuss the ten-dimensional picture, by introducing the mathematical description of supersymmetric and non-supersymmetric gauge field vevs in terms of vector bundles via the theorem of Donaldson, Uhlenbeck, and Yau [3, 4]. We describe how one may study any given model to see if it preserves or breaks supersymmetry at a given point in moduli space. Section 3 uses this technology to show, from a ten-dimensional perspective, how supersymmetric and non-supersymmetric regions, with stability walls between them, arise in the Kähler cone. In section 4, we describe the four-dimensional effective description of this phenomenon and derive the D-term (1.7). In section 5, we confirm the picture described in this introduction by studying the vacuum space of the four-dimensional effective theory. Higher order corrections are explored in section 6. In section 7, we conclude and discuss further work. Certain mathematical details and a conjecture are provided in appendix A. In appendix B, we provide another detailed example of a bundle exhibiting a stability wall in the Kähler cone and the explicit field theory describing it.

2 Vector bundle stability in heterotic compactifications

A supersymmetric heterotic string compactification requires the geometric input of a complex three-dimensional Calabi-Yau manifold, X, and a holomorphic vector bundle, V, defined over X. The gauge connection, A, on V with associated field strength, F, must satisfy the Hermitian Yang-Mills equations (1.1). On a holomorphic vector bundle, V, one can always choose a connection with a purely $(1,1)$ field strength, F, so that the last two conditions in (1.1), $F_{ab} = F_{\overline{a}\overline{b}} = 0$, are satisfied. To solve the first equation (1.1), $g^{a\overline{b}}F_{\overline{b}a} = 0$, is more difficult, at least for the case of non-Abelian bundle structure groups. However, for Calabi-Yau manifolds, there exists a powerful way of transforming this equation into a problem in algebraic geometry. For Kähler manifolds, the Donaldson-Uhlenbeck-Yau theorem [3, 4] states that on each *poly-stable* holomorphic vector bundle V, there exists a unique connection satisfying the Hermitian Yang-Mills equation (1.1). Thus, to verify that our vector bundle is consistent with supersymmetry we need to verify that it possesses the property of poly-stability.

The concept of stability of a bundle (or coherent sheaf), $\mathcal{F}$, over a Kähler three-fold, X, is defined by means of a quantity called the *slope*:

$$\mu(\mathcal{F}) \equiv \frac{1}{\mathrm{rk}(\mathcal{F})} \int_X c_1(\mathcal{F}) \wedge J \wedge J \; . \tag{2.1}$$

Here, J is the Kähler form on X, and $\mathrm{rk}(\mathcal{F})$ and $c_1(\mathcal{F})$ are the rank and the first Chern class of $\mathcal{F}$, respectively. A bundle V is now called stable (resp. semi-stable) if for all sub-sheaves $\mathcal{F} \subset V$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(V)$ the slope satisfies

$$\mu(\mathcal{F}) < \mu(V) \qquad (\text{resp. } \mu(\mathcal{F}) \leq \mu(V)) \; . \tag{2.2}$$

A bundle is poly-stable if it can be decomposed into a direct sum of stable bundles ($V = \bigoplus_n V_n$), which all have the same slope ($\mu(V_i) = \mu(V)$). It follows that every stable bundle is poly-stable and, in turn, every poly-stable bundle is semi-stable.[2] Thus, as a series of implications: stable $\Rightarrow$ poly-stable $\Rightarrow$ semi-stable.

In this work, we will consider holomorphic vector bundles with structure group $\mathrm{SU}(n)$ with $n = 3, 4, 5$. Since the slope of these bundles vanishes($c_1(V) = 0$ for $\mathrm{SU}(n)$ bundles), in order for V to be stable we must have that all proper sub-sheaves, $\mathcal{F}$, of V have strictly negative slope. Thus if $\mathcal{F} \subset V$ we require,

$$\mu(\mathcal{F}) < 0 \; . \tag{2.3}$$

But what qualifies a sheaf $\mathcal{F}$ to be a sub-sheaf of V? This is simply the condition that it has smaller rank and that there exists an embedding $\mathcal{F} \hookrightarrow V$. The space of homomorphisms between $\mathcal{F}$ and V, denoted $\mathrm{Hom}_X(\mathcal{F}, V)$, is isomorphic to the space of global holomorphic sections $H^0(X, \mathcal{F}^* \otimes V)$. Hence, we have that

$$V \text{ stable } \iff \mu(\mathcal{F}) < 0 \quad \forall\, \mathcal{F} \text{ s.t. } 0 < \mathrm{rk}(\mathcal{F}) < n \text{ and } 0 \subset \mathcal{F} \subset V \; . \tag{2.4}$$

To begin our study of stability, we will first re-write the slope condition (2.1) into a form better suited to our purposes. Given a basis of harmonic $(1,1)$ forms J_i on X, where $i, j = 1, \ldots, h^{1,1}(X)$, we expand the Kähler form as $J = t^i J_i$ with the t^i being the Kähler moduli. Inserting this into eq. (2.1), the slope of a sheaf $\mathcal{F}$ can then be written as

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} d_{ijk} c_1^i(\mathcal{F}) t^j t^k \, , \tag{2.5}$$

where the $d_{ijk} = \int_X J_i \wedge J_j \wedge J_k$ are the triple intersection numbers of X, and $c_1(\mathcal{F}) = c_1^i(\mathcal{F}) J_i$. It is useful to define the "dual Kähler moduli", s_i, by

$$s_i \equiv d_{ijk} t^j t^k \; . \tag{2.6}$$

[2]Note that the converse to these statements do not hold. That is, not every semi-stable bundle is poly-stable, etc.

The slope then turns into

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} s_i c_1^i(\mathcal{F}) \; , \tag{2.7}$$

and is, hence, given by a simple dot product between the first Chern class of $\mathcal{F}$ and the dual Kähler moduli s_i. As stated above, we are interested in bundles V with structure group $G = \mathrm{SU}(n)$ so that $c_1(V) = 0$ and $\mu(V) = 0$. Using eq. (2.7), stability for such a bundle V then amounts to the condition

$$\mu(\mathcal{F}) = \frac{1}{rk(\mathcal{F})} c_1^i(\mathcal{F}) s_i < 0 \; , \tag{2.8}$$

for all $\mathcal{F} \subset V$. Hence, for a given de-stabilizing sub-sheaf $\mathcal{F} \subset V$, eq. (2.8) divides s_i space into two regions (the condition $\mu(\mathcal{F}) = 0$ defines a co-dimension 1 hyperplane in s_i space). To understand stability of a bundle, we need to analyse, for all relevant sub-sheaves $\mathcal{F} \subset V$, how these regions relate to the Kähler cone (the allowed set of Kähler parameters s_i). Concretely, this amounts to finding a region in the Kähler cone which is not de-stabilised by any sub-sheaf $\mathcal{F} \subset V$.

A choice of a vector s_i is referred to as a 'polarization'. That is, the bundle which is stable with respect to all polarizations is stable everywhere in the Kähler cone. However, viewed from the perspective of physics, this is actually a stronger condition than we require. In a heterotic compactification, we shall define our low-energy effective theory perturbatively around a particular vacuum corresponding to some point in moduli space. So, it is sufficient to show that the bundle is stable *somewhere* in the Kähler cone (with the hope that we may eventually stabilize the moduli within this region). In previous work [16, 17], several of the authors made use of this viewpoint to formulate stability criteria for bundles defined over Calabi-Yau manifolds with $h^{1,1}(X) > 1$. The resulting algorithm is a generalization of the stability condition given by Hoppe [18] which, in its original form, applies to Calabi-Yau manifolds with $h^{1,1}(X) = 1$. In ref. [17], we describe this algorithmic method for determining the stable regions of a bundle. We will not repeat the details of this analysis here, but rather highlight some of its important features in the following.

2.1 Algorithmic testing of slope stability

Vector bundle stability is a notoriously difficult property to prove. The main obstacle arises in classifying all possible sub-sheaves, $\mathcal{F}$, of the bundle V. There are no general techniques known for identifying such sub-sheaves or for computing their topological properties such as Chern classes (and, hence, their slopes from eq. (2.1)). However, as described in ref. [17], despite these obstacles, progress can be made by systematically constraining the possible sub-sheaves, $\mathcal{F} \subset V$.

2.1.1 Sub-line bundles and stability

In this section we will demonstrate that, in order to prove stability at any given point in Kähler moduli space, it is sufficient to test the slope criteria (2.2) for all *sub-line bundles* $\mathcal{L}$ of certain anti-symmetric powers, $\wedge^k V$, of the bundle V.

To begin, consider a rank-n vector bundle V over a projective variety X. If $\mathcal{F}$ is a sub-sheaf of V then it injects into V via the resolution

$$0 \to \mathcal{F} \to V \to \mathcal{K} \to 0\,, \tag{2.9}$$

with $\mathrm{rk}(\mathcal{F}) < \mathrm{rk}(V)$ and $\mathcal{K} = V/\mathcal{F}$. We shall consider such sub-sheaves one rank at a time. First, we observe that since V is a vector bundle, it is torsion-free and, thus, has no rank-zero sub-sheaves. So, we begin with the case of a rank one sub-sheaf. Since $\mathcal{F}$ is torsion free, there is an injection

$$\mathcal{F} \xrightarrow{i} \mathcal{F}^{**} \tag{2.10}$$

where $\mathcal{F}^{**}$ is the double-dual of $\mathcal{F}$. A locally free coherent sheaf, $\mathcal{L}$, is isomorphic to its double-dual, that is $\mathcal{L}^{**} \approx \mathcal{L}$. Since $\mathcal{F}$ is rank one and torsion free, it can be shown that $\mathcal{F}^{**}$ is locally free and, hence, a line bundle [19]. Dualizing the sequence (2.9) we obtain

$$0 \to \mathcal{K}^* \to V^* \to F^* \to \mathcal{E}xt^1(\mathcal{K}, O_X) \to \mathcal{E}xt^1(V, O_X) \tag{2.11}$$

where $\mathcal{E}xt^1$ is the sheaf Ext on X. We now observe that $\mathcal{E}xt^1(V, O_X) = 0$ since V is locally free. Moreover, there exists an open subset $U \subset X$ so that $\mathcal{K}|_U$ is locally free, hence $\mathcal{E}xt^1(\mathcal{K}, O_X)$ is a torsion sheaf on X. Therefore, $\mathcal{E}xt^1(\mathcal{K}, O_X)^* = 0$. Thus dualizing (2.11), and using (2.10) we have

$$\mathcal{F} \subset \mathcal{F}^{**} \subset V^{**} \approx V\ . \tag{2.12}$$

It is straightforward to show that $\mu(\mathcal{F}) = \mu(\mathcal{F}^{**})$. Thus, instead of checking the slope condition (2.2) for all rank-one torsion-free sub-sheaves of V, it suffices to check it for all sub-line bundles. But what about sub-sheaves of higher rank?

Let $\mathcal{F}$ be a torsion free sub-sheaf of rank k (with $1 < k < n$). Once again, we have an inclusion $0 \to \mathcal{F} \to V$ which in turn induces a mapping

$$\wedge^k \mathcal{F} \to \wedge^k V \tag{2.13}$$

which can also be shown to be an injection [20]. By definition of the anti-symmetric tensor power $\wedge^k$, $\wedge^k\mathcal{F}$ is a rank one sheaf. Since $\mathcal{F}$ is torsion free, so is $\wedge^k\mathcal{F}$ [21]. Next, by an argument similar to the one given above (in and around (2.10)), we can argue that there is a line bundle $\mathcal{L}$ associated to $\wedge^k\mathcal{F}$, namely $\mathcal{L} = (\wedge^k\mathcal{F})^{**}$. Note that in general for a rank n bundle V,

$$c_1(\wedge^k V) = \binom{n-1}{k-1} c_1(V)\ . \tag{2.14}$$

Thus we observe that for SU(n) bundles, which have $c_1(V) = 0$, it follows that $c_1(\wedge^k V) = 0$ as well. Likewise, we see that applied to a rank k, sub-line bundle, $\mathcal{F}$, (2.14) gives us $\mu(\wedge^k\mathcal{F}) = k\mu(\mathcal{F})$. Therefore, for each rank k de-stabilizing sub-sheaf of V we have a corresponding de-stabilizing sub-line bundle $\mathcal{L} \subset \wedge^k V$. Thus in proving stability of an SU(n) vector bundle V, we need only show that if $\mathcal{L} \subset \wedge^k V$, then

$$\mu(\mathcal{L}) < \mu(\wedge^k V) = 0 \tag{2.15}$$

for all k with $0 < k < n$. Since line bundles are classified by their first Chern class on a Riemannian manifold, this is a dramatic simplification of the problem of stability. Rather than the untenable problem of considering all sub-sheaves, we have only to analyze and constrain the well-defined set of line bundle sub-sheaves of $\wedge^k V$.

2.1.2 Constraints on line bundle sub-sheaves

What constraints can we place on the line bundles which must be considered in examining the stability of an SU(n) bundle V? Using the results of the previous subsection, we begin by considering a line bundle sub-sheaf $\mathcal{L}$ of $\wedge^k V$. We present several simple characteristics that distinguish line bundle sub-sheaves of stable SU(n) bundles.

First, as discussed in (2.4), by definition, if $\mathcal{L} \subset \wedge^k V$ then

$$\mathrm{Hom}_X(\mathcal{L}, \wedge^k V) \neq 0 \ . \tag{2.16}$$

Therefore, we have a non-trivial cohomology condition to check for any candidate line bundle sub-sheaf[3] of V. Note that in this section, we will consider the mapping of $\mathcal{L} \hookrightarrow \wedge^k V$ for *generic* values of the bundle moduli of V.

The second observation is that for SU(n) bundles, if V is stable then $H^0(X, V) = H^0(X, V^*) = 0$. Indeed, if $H^0(X, V)$ were non-vanishing, then it is clear that $\mathrm{Hom}_X(\mathcal{O}, V) \cong H^0(X, \mathcal{O}^* \otimes V) = H^0(X, V) \neq 0$ and, hence, that the trivial sheaf $\mathcal{O}$ would de-stabilize V for any choice of Kähler moduli. A similar argument holds for V^* which is stable exactly if V is. For this reason, checking that $H^0(X, V) = H^0(X, V^*) = 0$ for an SU(n) bundle V is a useful first test for stability which we can carry out before proceeding further. Assuming this has been verified, it is clear that all possible de-stabilizing line bundle sub-sheaves, $\mathcal{L} \subset V$ (or $\mathcal{L} \subset V^*$), must satisfy $H^0(X, \mathcal{L}) = 0$. Furthermore, if an SU(n) bundle is stable then its anti-symmetric tensor powers, $\wedge^k V$, are at least semi-stable [19, 21]. As a result, by scanning for possible line bundle sub-sheaves of $\wedge^k V$ for all values of k, we can definitively determine the region of stability of the SU(n) bundle V. If we discover that for a fixed polarization, $\wedge^k V$ is destabilized by a line bundle $\mathcal{L}$, then by the observations above, we know that V itself is unstable for this choice of Kähler form.

To summarize, the method of analyzing the stability of an SU(n) bundle at any given point in Kähler moduli space proceeds as follows.

- **Check that $H^0(X, V) = H^0(X, V^*) = 0$.**

  Should this not be the case, the bundle is unstable everywhere in Kähler cone and we can stop.

- **Consider all possible line bundles $\mathcal{L}$, as classified by their first Chern class.**

  The results of the previous subsection assure us that we need only consider line bundles rather than all sheaves of rank $k < n$.

[3] Note that $\mathrm{Hom}_X(\mathcal{L}, \wedge^k(V)) \neq 0$ implies that $\mathcal{L}$ is a line bundle sub-sheaf rather than a sub-line bundle of V. This follows from the fact that while injective maps exist, the image of $\mathcal{L}$ in V may not be a bundle. Equivalently, it is possible that $V/\mathcal{L}$ is not always a bundle [22].

- **Discard all line bundles, $\mathcal{L}$, for which $\mathbf{Hom}(\mathcal{L}, \wedge^k \mathbf{V}) = 0$ for all $\mathbf{k} < \mathbf{n}$.**

  If $Hom(\mathcal{L}, \wedge^k V) = 0$, such a line bundle is not a sub sheaf of $\wedge^k V$ for any $k < n$ and thus need not be considered.[4] As a simplification, for $k = 1, n-1$, we can discard all line bundles with $H^0(X, \mathcal{L}) \neq 0$. Indeed, since we have already verified that $H^0(X, V) = H^0(X, V^*) = 0$, such line bundles cannot inject into V and $\wedge^{n-1} V \simeq V^*$.

- **Check the slope of the remaining line bundles.**

  We must check the slope $\mu(\mathcal{L})$ of the remaining line bundles at the point in Kähler moduli space we are considering. If there exist no line bundles such that $\mu(\mathcal{L}) \geq \mu(V) = 0$, then V is slope-stable at this point in Kähler moduli space.[5]

3 Stability walls in the Kähler cone

The stability condition (2.8) clearly depends on the choice of Kähler parameters and thus a bundle need not be stable throughout its entire Kähler cone. Furthermore, the choice of bundle moduli can affect which potentially de-stabilizing sub-sheaves inject into V. In principle then, "walls" between regions of stability/instability such as those depicted in the (dual) Kähler cone in figure 1 can occur. In the neighborhood of such stability walls, the supersymmetric structure of the low energy effective theory must be studied in more detail than in the stable region. We begin by exploring the structure of stability walls in Kähler moduli space.

While this discussion can be applied to a Kähler cone of any size, to illustrate this concept, we will consider a two-dimensional Kähler cone (that is, $h^{1,1}(X) = 2$) given by the positive quadrant in the (s_1, s_2) plane of dual Kähler moduli. Suppose that V is an $SU(n)$ bundle V and that a stability wall[6] of the form shown in figure 1 is generated by a de-stabilizing sub-sheaf $\mathcal{F} \subset V$ with $c_1(\mathcal{F}) = -kJ_1 + mJ_2$, where $k > 0$ and $m > 0$.[7] From eq. (2.8), the slope of such a sub-sheaf is given by

$$\mu(\mathcal{F}) = \frac{1}{\text{rk}(\mathcal{F})} c_1^i(\mathcal{F}) s_i = \frac{1}{\text{rk}(\mathcal{F})}(-ks_1 + ms_2)\ . \tag{3.1}$$

This means that, for all Kähler parameters (s_1, s_2) where $\mu(\mathcal{F}) > 0$, that is, for $s_2/s_1 > k/m$, the bundle is unstable while for $\mu(\mathcal{F}) < 0$, or $s_2/s_1 < k/m$, it is potentially stable, subject, of course, to other possible destabilizing sub-sheaves. For example, in addition, there may exist a sub-sheaf with first Chern class given by $c_1(\mathcal{F}) = pJ_1 - qJ_2$, where $p > 0$

[4]As an additional simplifying technique, we note that while scanning for possible line bundle sub-sheaves of $\wedge^k V$, if it is true that $H^0(X, \wedge^k V) = 0\ \forall\ k$, we can eliminate any line bundles for which $H^0(X, \mathcal{L}) \neq 0$.

[5]For a fixed polarization, if $\mu(\mathcal{L}) > 0$, for some line bundle $\mathcal{L} \subset \wedge^k V$, then V is unstable. However, if $\mu(\mathcal{L}) = 0$, one cannot conclude that V itself is necessarily stable/unstable. This last case must be analyzed on an individual basis.

[6]In general, for an $h^{1,1}(X)$-dimensional Kähler moduli space, the stability wall will be a $(h^{1,1}(X) - 1)$-dimensional hyperplane.

[7]Note that for an $h^{1,1}(X)$-dimensional, positive Kähler cone, if $\mathcal{F}$ is to define a stability wall, $c_1^i(\mathcal{F})$ must contain at least one negative and one positive component. For this reason, a bundle defined on a manifold with $h^{1,1}(X) = 1$ is stable everywhere or nowhere.

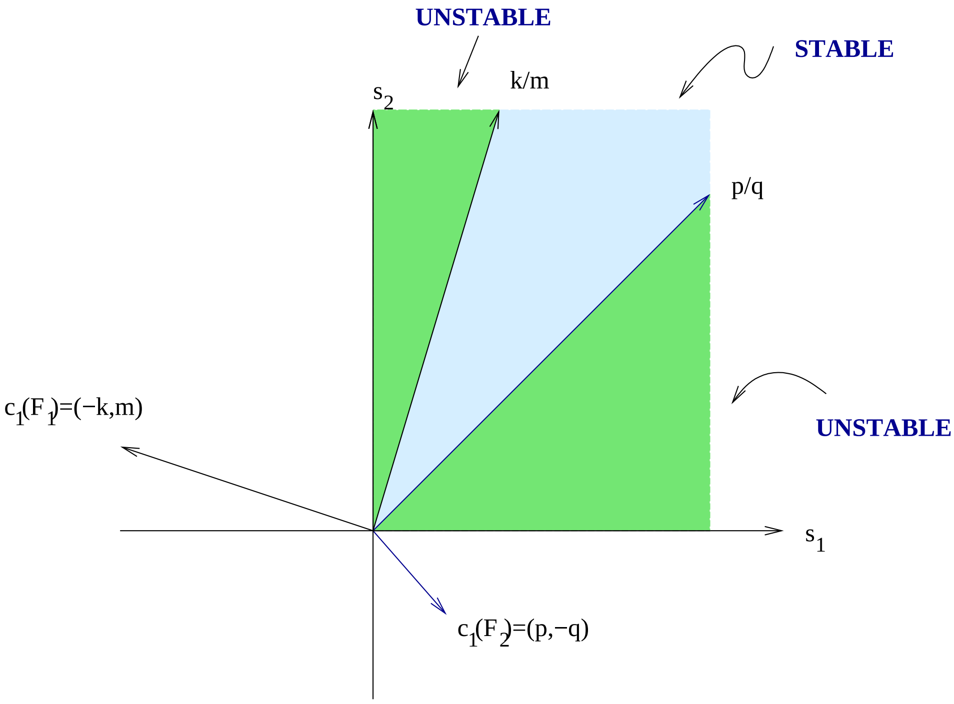

Figure 1. A two-dimensional dual Kähler cone, defined by $s_1 \geq 0$ and $s_2 \geq 0$, where $s_i = d_{ijk}t^j t^k$. Shown are two de-stabilizing sub-sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ with first Chern classes given by $c_1(\mathcal{F}_1) = (-k, m)$ and $c_1(\mathcal{F}_2) = (p, -q)$ for some integers k, m, p, q. The bundle V is stable between the lines with slopes k/m and p/q.

and $q > 0$, which would yield a lower boundary line with slope p/q. If these two sub-sheaves are the "maximally destabilizing" ones on either side of the Kähler cone, then the bundle is supersymmetric for all values $p/q < s_2/s_1 < k/m$. For a two-dimensional Kähler cone, the supersymmetric region of a general bundle will be defined by these upper and lower boundaries as illustrated in figure 1. For the present discussion, we will focus our attention on the theory near one of these boundary lines.

What happens on the line with slope k/m itself? There, the bundle is manifestly semi-stable since $\mu(\mathcal{F}) = \mu(V) = 0$. However, to decide whether the low energy theory is supersymmetric or not, we must consider not only our position in Kähler moduli space, but in bundle moduli space as well. If we examine this line in Kähler moduli space while remaining at an arbitrary point in bundle moduli space for which V is an *indecomposable* rank n bundle, then supersymmetry will be broken. This must be the case since supersymmetric vacua exist if and only if the bundle is poly-stable. A semi-stable bundle can only be poly-stable if it is a direct sum of stable bundles. Therefore, the stability wall in Kähler moduli space will only correspond to a supersymmetric solution if the bundle decomposes into a direct sum $V \to \mathcal{F} \oplus \mathcal{K}$ where $\mathrm{rk}(\mathcal{F}) + \mathrm{rk}(\mathcal{K}) = \mathrm{rk}(V)$ and $c_1(\mathcal{F}) = -c_1(\mathcal{K})$. Such bundle decompositions near a wall of semi-stability were discussed for $K3$ manifolds in ref. [5].

At this special "decomposable" locus in bundle moduli space, the bundle is split and poly-stable. While the topological quantities of V remain the same at this locus, other important features of the bundle and the corresponding low energy theory can change. For instance, at this decomposable locus, the structure group of an $\mathrm{SU}(n)$ bundle will become $S(U(n_1) \times U(n_2))$ with $n_1 = \mathrm{rk}(\mathcal{F})$ and $n_2 = \mathrm{rk}(\mathcal{K})$. As we shall discuss in detail in the next section, this change in the structure group of V will also alter the visible gauge symmetry of the four-dimensional theory. For instance, if $\mathrm{rk}(V) = 3$, then the commutant of $S(\mathrm{U}(2) \times \mathrm{U}(1))$ in E_8 is no longer E_6, but is enhanced to $E_6 \times \mathrm{U}(1)$.

Before one can study such supersymmetric theories further, it is prudent to ask whether such a decomposable point exists? Fortunately, it can be shown that if there exists a sub-sheaf $\mathcal{F}$ of V which injects into V, then there will always exist a natural decomposition of V into direct sum $\mathcal{F} \oplus V/\mathcal{F}$. If we define the relationship between $\mathcal{F}$ and V via an 'extension' short exact sequence

$$0 \to \mathcal{F} \to V \to V/\mathcal{F} \to 0\,, \tag{3.2}$$

then it is well-known that the space of possible extensions is given by $\mathcal{E}xt^1(V/\mathcal{F}, \mathcal{F})$ [23]. Furthermore, the zero-element of the Ext group corresponds to the decomposable locus $V = \mathcal{F} \oplus \mathcal{K}$, where $\mathcal{K} = V/\mathcal{F}$. Finally, note that if a single indecomposable sub-sheaf $\mathcal{F}$ defines a stability wall in moduli space, then, by definition, it (and $(V/\mathcal{F})^*$) must be stable.[8]

The decomposition, $V \to \mathcal{F} \oplus V/\mathcal{F}$, can be best described by considering a Jordan-Hölder filtration of V [24]. Points on a moduli space of strictly semi-stable bundles do not correspond to unique objects, rather they represent an "S-Equivalence class" [19, 24, 25]. Two bundles are S-equivalent if their Jordan-Hölder graded sums $\mathrm{Gr}(V) = \mathcal{F}_1 \oplus V/\mathcal{F}_1 \oplus \ldots$ are isomorphic. For any S-equivalence class, there is a unique poly-stable representative up to isomorphism. That is, there is a unique graded sum in which the summands are stable bundles and, as a result, we can consider the decomposition of V into this sum on the stability wall. We will now present a simple example of a Calabi-Yau manifold X and a bundle V which exhibits a stability wall.

3.1 A stability wall example

Up to this point, our entire discussion has been completely general. Let us now exemplify our previous comments by considering a bundle defined on the complete intersection Calabi-Yau manifold [26],

$$X = \left[\begin{array}{c|c} \mathbb{P}^1 & 2 \\ \mathbb{P}^3 & 4 \end{array}\right]\,, \tag{3.3}$$

defined by a polynomial of bi-degree $(2, 4)$ in the ambient space $\mathbb{P}^1 \times \mathbb{P}^3$. This manifold has two Kähler moduli, so $h^{1,1}(X) = 2$. A basis of harmonic $(1,1)$ forms is given by the Kähler forms J_1 and J_2 of the ambient projective spaces $\mathbb{P}^1$ and $\mathbb{P}^3$ (pulled back to X). We denote the corresponding Kähler moduli by t^1 and t^2. The Kähler cone is the positive quadrant $t^1 \geq 0$ and $t^2 \geq 0$ and the non-zero triple intersection numbers are given by $d_{122} = 4$ and

[8] It is possible that $\mathcal{F}$ could be a direct sum of stable objects with the same slope. In this case, we would simply obtain a further decomposition for V, that is, $V \sim \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \ldots$.

$d_{222} = 2$. From eq. (2.6), we can calculate the dual Kähler moduli s_1 and s_2 and we find

$$s_1 = 4(t^2)^2 \,, \quad s_2 = 8t^1t^2 + 2(t^2)^2 \,. \tag{3.4}$$

Hence, expressed in terms of these dual Kähler moduli, the Kähler cone is the positive quadrant above the line $s_2/s_1 = 1/2$. Line bundles on X are characterised by two integers, k and l, and are denoted by $\mathcal{O}_X(k,l)$. Their first Chern class is given by $c_1(\mathcal{O}_X(k,l)) = kJ_1 + lJ_2$.

We will define a rank 3 monad bundle [16, 27, 28] on this space by the short exact sequence

$$0 \to V \to \mathcal{O}_X(1,0) \oplus \mathcal{O}_X(1,-1) \oplus \mathcal{O}_X(0,1)^{\oplus 2} \xrightarrow{f} \mathcal{O}_X(2,1) \to 0 \,. \tag{3.5}$$

The bundle V is defined as the kernel of the map f. This map is derived from polynomials of bi-degree $((1,1),(1,2),(2,0),(2,0))$ (mapping sections of $\mathcal{O}_X(1,0)\oplus\mathcal{O}_X(1,-1)\oplus\mathcal{O}_X(0,1)^{\oplus 2}$ to sections of $\mathcal{O}_X(2,1)$). The rank of V is three and $c_1(V) = 0$ so that the structure group is generically SU(3). At a generic point in moduli space, the only non-vanishing cohomology of this SU(3) bundle is $h^1(X,V) = 2$. This means there are two families in **27** multiplets and no anti-families in $\overline{\mathbf{27}}$. The moduli space of V has dimension

$$h^1(X, V \otimes V^*) = 22, \tag{3.6}$$

so that we have 22 E_6 singlet fields which should be interpreted as bundle moduli.[9] We select the bundle (3.5) solely because it provides a straightforward example of a bundle with both supersymmetric and non-supersymmetric regions in its moduli space, and make no attempt here to consider models with fully realistic particle spectra.

To analyze the stability of this rank three bundle we must consider the potentially de-stabilizing rank one and two sub-sheaves. As discussed in the previous subsections, this may be done more simply by considering potentially de-stabilizing line bundle sub-sheaves of V and $\wedge^2 V \cong V^*$.

Beginning with rank one sub-sheaves, we consider all sub-line bundles of V. One can verify that $H^0(X,V) = 0$ for the bundle (3.5) and that all line bundles $\mathcal{O}_X(k,l)$, where $k,l \geq 0$ have sections. Hence, such semi-positive line bundles need not be considered. Further, semi-negative line bundles $\mathcal{O}_X(k,l)$, where $k,l \leq 0$ always have a negative slope in the interior of the Kähler cone and are irrelevant. It is, therefore, clear that the only line bundles we need to consider are those with 'mixed' positive/negative entries in their first Chern classes. That is, $\mathcal{L}$ is given by $\mathcal{O}_X(-k,m)$ or $\mathcal{O}_X(p,-q)$ for $k,m,p,q > 0$. We seek such line bundles for which $\mathrm{Hom}_X(\mathcal{L},V) \neq 0$. A straightforward but lengthy analysis (see [16, 17] for details) yields that if $\mathcal{L} = \mathcal{O}_X(-k,m)$ then $\mathrm{Hom}_X(\mathcal{L},V) \neq 0$ for $k \geq 3$ and $m = 1$. Further, $\mathcal{O}_X(p,-q)$ does not inject for any values of p,q. Hence, the "maximally destabilizing" rank one sub-sheaf corresponds to the line bundle $\mathcal{L}_1 = \mathcal{O}_X(-3,1)$ and we have the short exact sequence

$$0 \to \mathcal{L}_1 \to V \to V/\mathcal{L}_1 \to 0 \,. \tag{3.7}$$

[9] See [28, 29] for general formulae for the spectra and moduli of monad bundles.

This implies that above a line with slope $s_2/s_1 = 3$ in the Kähler cone, the bundle is definitely unstable while it may be stable below this line.

However, we still need to consider rank two destabilizing sub-sheaves or, equivalently, rank one line bundle sub-sheaves of $\wedge^2 V$. As before, we find that no lower boundary exists, that is, $\mathrm{Hom}_X(O(p,-q), \wedge^2 V) = 0$ for all values of $p, q > 0$. For the upper boundary, we consider sub-bundles of the form $\mathcal{L} = \mathcal{O}_X(-k, m)$ in $\wedge^2 V$. Since V is an $\mathrm{SU}(n)$ bundle we have $\wedge^2 V \simeq V^*$. This means we can extract information about $\wedge^2 V$ from the dual

$$0 \to \mathcal{O}_X(-2,-1) \to \mathcal{O}_X(-1,0) \oplus \mathcal{O}_X(-1,1) \oplus \mathcal{O}_X(0,-1)^{\oplus 2} \to V^* \to 0 \tag{3.8}$$

of the monad sequence (3.5). Twisting this sequence by $\mathcal{L}^* = \mathcal{O}_X(k, -m)$ we get

$$\begin{aligned} 0 &\to \mathcal{O}_X(k-2, -1-m) \\ &\to \mathcal{O}_X(k-1,-m) \oplus \mathcal{O}_X(k-1, 1-m) \oplus \mathcal{O}_X(k, -1-m)^{\oplus 2} \to \mathcal{L}^* \otimes V^* \to 0 \; . \end{aligned} \tag{3.9}$$

One can verify from this sequence that $\mathrm{Hom}_X(\mathcal{L}, \wedge^2 V) \cong H^0(X, \mathcal{L}^* \otimes V^*) \neq 0$ only for $\mathcal{L} = \mathcal{O}(-k, 1)$ and $k \geq 1$. Hence, the maximally destabilizing line bundle is $\mathcal{L}_2 = \mathcal{O}_X(-1,1)$ and we have

$$0 \to \mathcal{L}_2 \to V^* \to V^*/\mathcal{L}_2 \to 0 \; . \tag{3.10}$$

Thus, V^* is stable only below the line with slope $s_2/s_1 = 1$. Equivalently, this implies that there is a rank two sub-sheaf, $\mathcal{F}$ of V with $c_1(\mathcal{F}) = -J_1 + J_2$ and

$$0 \to \mathcal{F} \to V \to V/\mathcal{F} \to 0 \; . \tag{3.11}$$

Since the rank two sub-sheaf $\mathcal{F}$ de-stabilizes a larger region of the dual Kähler cone then the rank one sub-sheaf $\mathcal{L}_1$, the existence of $\mathcal{L}_1 = \mathcal{O}_X(-3,1)$ is irrelevant here. While there are an infinite number of sub-sheaves that de-stabilize some portion of the Kähler cone, for this bundle there is only one relevant stability wall which is determined by the rank two sub-sheaf $\mathcal{F} \subset V$. From eq. (3.1), the slope of this sub-sheaf is given by

$$\mu(\mathcal{F}) = \frac{1}{2}(-s_1 + s_2) \; , \tag{3.12}$$

and it follows that V is stable below the line with slope $s_2/s_1 = 1$. The dual Kähler cone, together with the region of stability, are plotted in figure 2.

The discussion of the last two sections is rather mathematical in nature. It would be desirable to have more physical insight into what is going on, and to be able to describe stability walls in the Kähler cone in terms of the four-dimensional effective action. To this end, in the next section, we will study the effective four-dimensional theory describing fluctuations about the stability wall, that is, the locus in moduli space where the bundle structure group decomposes. We will then use these results, and the ones of the present section, to discuss what happens physically as one crosses a wall of stability.

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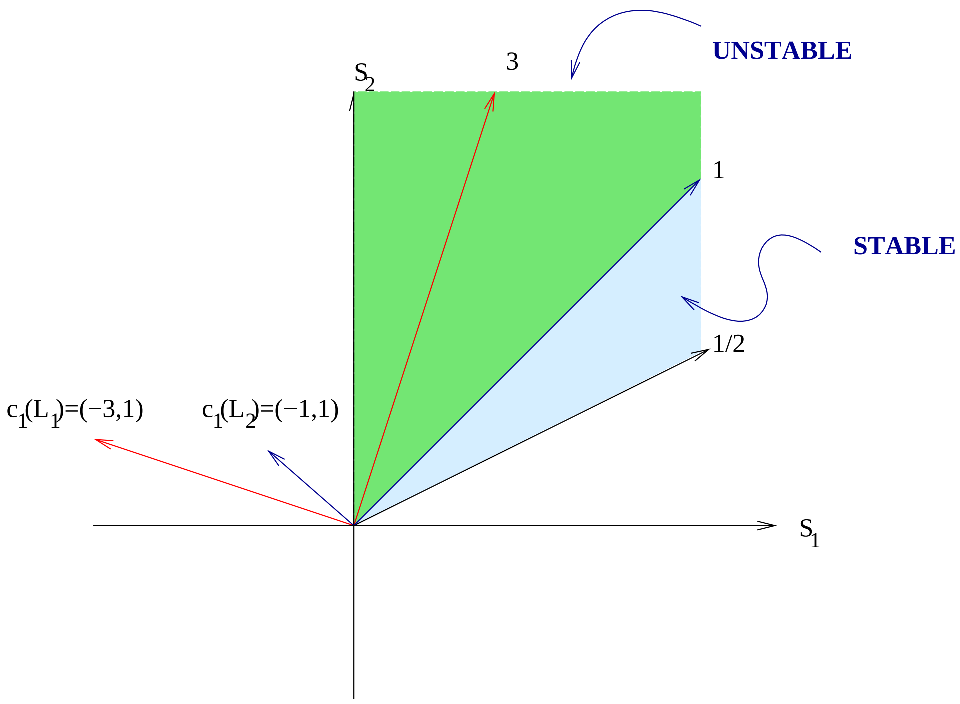

Figure 2. The dual Kähler cone and the regions of stability/instability for the monad bundle described in section 3.1. Here L_1 and L_2 are line bundle sub-sheaves of V and $\wedge^2(V)$ respectively. The boundaries of the dual Kähler cone are denoted by the s_2-axis and the line with slope $1/2$.

4 Effective field theory at the decomposable locus

In this section, we will compute the potential in the four-dimensional effective theory near the locus in bundle moduli space where the sequence $0 \to \mathcal{F} \to V \to \mathcal{K} \to 0$ becomes the trivial extension, that is, where the bundle decomposes as $V = \mathcal{F} \oplus \mathcal{K}$. To perform such a computation, the first thing we need to know is the low energy spectrum. We will describe this in two stages; first presenting those fields which descend from ten-dimensional gauge fields before continuing to describe those which arise from other sources.

4.1 Four-dimensional spectrum from the gauge sector

As stated in the previous section, it is clear from the sequence $0 \to \mathcal{F} \to V \to \mathcal{K} \to 0$, and the fact that $c_1(V)$ vanishes, that $\mathcal{F}$ and $\mathcal{K}$ have equal and opposite first Chern class. Thus, at the decomposable point in bundle moduli space the structure group of V is $S(U(n_1) \times U(n_2))$, where $n_1 + n_2 = n$. It will turn out that this is not the most convenient way in which to express this group for what follows, in particular for the calculation of the spectrum. As such, we will now carry out a little bit of group theory in order to obtain a more suitable form.

Locally, at the level of Lie algebras, $S(U(n_1) \times U(n_2))$ is equivalent to $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1)$. Elements of the former group are defined by a pair (A, B), where A and B are $n_1 \times n_1$ and $n_2 \times n_2$ unitary matrices respectively, satisfying the condition $\det A \det B = 1$. Elements of $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1)$ are defined by a triplet $(\mathcal{A}, \mathcal{B}, \mathcal{E})$, where $\mathcal{A}$ and $\mathcal{B}$ are $n_1 \times n_1$ and $n_2 \times n_2$ special unitary matrices respectively and $\mathcal{E}$ is the U(1) phase. We may define a map $\sigma : \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1) \to S(U(n_1) \times U(n_2))$ by $(\mathcal{A}, \mathcal{B}, \mathcal{E}) \to (A, B) = (\mathcal{E}^{n_2}\mathcal{A}, (\mathcal{E}^*)^{n_1}\mathcal{B})$ and it is easy to verify that this map is onto and that $\mathrm{Ker}(\sigma) \cong \mathbb{Z}_{n_1 n_2}$. Hence, globally $S(U(n_1) \times U(n_2)) \cong (\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1))/\mathbb{Z}_{n_1 n_2}$. To understand the matter content of the low energy heterotic theory we must consider the branching of the adjoint of E_8 under the bundle structure group and its commutant. In the standard texts [30], these branchings are given in terms of $\mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \times \mathrm{U}(1)$ rather than $S(U(n_1) \times U(n_2))$ which is why we have discussed the relation between those two groups.

For the sake of brevity we will only consider one possible structure group in the main text of this paper. We shall detail in full the case $\mathrm{SU}(3) \to S(\mathrm{U}(2) \times \mathrm{U}(1))$ and note that all other $\mathrm{SU}(n)$ decompositions follow in an entirely analogous manner.[10] We consider, then, the case where we have an SU(3) structure group at a generic point in moduli space, degenerating to $\mathrm{SU}(2) \times \mathrm{U}(1)$ at the stability wall about which we construct our low energy theory. This gives us a low energy gauge group $E_6 \times \mathrm{U}(1)$ at this locus. Under the decomposition $E_8 \supset E_6 \times \mathrm{SU}(2) \times \mathrm{U}(1)$ the adjoint of E_8 decomposes as follows.

$$\begin{aligned}\mathbf{248} = &(\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{2})_{-3} + (\mathbf{1},\mathbf{2})_3 + (\mathbf{1},\mathbf{3})_0 + (\mathbf{78},\mathbf{1})_0 \\ &+(\mathbf{27},\mathbf{1})_2 + (\mathbf{27},\mathbf{2})_{-1} + (\overline{\mathbf{27}},\mathbf{1})_{-2} + (\overline{\mathbf{27}},\mathbf{2})_1\end{aligned} \tag{4.1}$$

In the above decomposition, the first number in the bracket is the E_6 representation, the second number is that of a SU(2) representation and the subscript is the U(1) charge. We note that our sign conventions differ somewhat from those of [30].

The field content of the low energy theory is determined by the first and zeroth cohomologies of various combinations of $\mathcal{F}$ and $\mathcal{K}$ as determined by the decomposition (4.1). The first cohomologies tell us about scalars and the zeroth about gauge bosons in the four-dimensional effective theory. We must remember that the groups SU(2) and U(1) in the above branching are not directly the structure groups of $\mathcal{F}$ and $\mathcal{K}$ in the decomposition $V = \mathcal{F} \oplus \mathcal{K}$. Rather, since $\mathcal{F}$ is a rank two bundle with non-vanishing first Chern class its structure group is U(2). Further, the structure group of $\mathcal{K}$ is U(1) with the additional constraint that $c_1(\mathcal{F}) + c_1(\mathcal{K}) = 0$,[11] so that the overall structure group of $\mathcal{F} \oplus \mathcal{K}$ is $S(\mathrm{U}(2) \times \mathrm{U}(1))$. The proceeding group theory discussion tells us that the elements of this structure group are given by $(\mathcal{E}\mathcal{A}, (\mathcal{E}^*)^2)$ where $(\mathcal{A}, \mathcal{E}) \in \mathrm{SU}(2) \times \mathrm{U}(1)$. We have summarised the information about the various representations and cohomologies, associated to low-energy chiral multiplets, in table 1. Note that the charges given as a subscript in the first column refer to the $\mathrm{U}(1) \subset \mathrm{SU}(2) \times \mathrm{U}(1)$ while the charges in the last column refer to

[10] For example, there are two possible decompositions for an SU(3) bundle. First, we have $\mathrm{SU}(3) \to S(\mathrm{U}(2) \times \mathrm{U}(1)) \approx \mathrm{SU}(2) \times \mathrm{U}(1)$, corresponding to $V \to \mathcal{F} \oplus \mathcal{K}$, a sum of a rank two and a rank one bundle. There is a second possibility, namely $\mathrm{SU}(3) \to S(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)) \approx \mathrm{U}(1) \times \mathrm{U}(1)$, corresponding to a decomposition into three line bundles: $V \to \mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3$. In this latter case, one would find two additional low energy U(1) symmetries. In the interests of brevity, we will only detail the case of a single U(1) here.

[11] Note that we have assumed here that it is the rank 2 sub-sheaf which injects everywhere and destabilizes V. The same analysis can be repeated assuming that it is the rank 1 sub-sheaf which is destabilizing without changing the result. This is because the only information which will enter the considerations of this subsection is the nature of the bundle *at the decomposable locus in bundle moduli space.*

Representation	Cohomology	Physical U(1) charge
$(\mathbf{1},\mathbf{2})_{-3}$	$H^1(X,\mathcal{F}\otimes\mathcal{K}^*)$	$-3/2$
$(\mathbf{1},\mathbf{2})_{3}$	$H^1(X,\mathcal{F}^*\otimes\mathcal{K})$	$3/2$
$(\mathbf{1},\mathbf{3})_{0}$	$H^1(X,\mathcal{F}\otimes\mathcal{F}^*)$	0
$(\mathbf{27},\mathbf{1})_{2}$	$H^1(X,\mathcal{K})$	1
$(\mathbf{27},\mathbf{2})_{-1}$	$H^1(X,\mathcal{F})$	$-1/2$
$(\overline{\mathbf{27}},\mathbf{1})_{-2}$	$H^1(X,\mathcal{K}^*)$	-1
$(\overline{\mathbf{27}},\mathbf{2})_{1}$	$H^1(X,\mathcal{F}^*)$	$1/2$

Table 1. Representations, cohomologies and U(1) charges associated to the zero modes which arise at the stability wall. The first column gives the representation under $E_6\times \mathrm{SU}(2)\times \mathrm{U}(1)$, the second column is the relevant cohomology involving $\mathcal{F}$ and $\mathcal{K}$, and the last column is the charge of the states under the U(1) which is in the commutant of $S(\mathrm{U}(2)\times\mathrm{U}(1))$ in E_8.

the U(1) in the commutant of $S(\mathrm{U}(2)\times\mathrm{U}(1))$ in E_8. Let us interpret the fields that appear here carefully. The $(\mathbf{27},\mathbf{1})_2$, $(\overline{\mathbf{27}},\mathbf{1})_{-2}$, $(\mathbf{27},\mathbf{2})_{-1}$ and $(\overline{\mathbf{27}},\mathbf{2})_1$ multiplets unambiguously represent matter fields while the $(\mathbf{1},\mathbf{3})_0$ multiplet clearly corresponds to moduli of the $S(\mathrm{U}(2)\times\mathrm{U}(1))$ bundle. The remaining two cohomologies, however, are a little bit more subtle to interpret. From the point of view of the theory at stability wall - the point of view we are considering here - these fields are charged under a visible sector gauge group (the enhanced U(1)) and, hence, they are matter fields. However, it would also not be unreasonable to regard them as bundle moduli. In general, we think of the cohomology $H^1(X,V\otimes V^*)$ as representing bundle moduli.[12] At the stability wall, where the bundle V decomposes as $V=\mathcal{F}\oplus\mathcal{K}$, this bundle cohomology splits into various parts as

$$H^1(X,V\otimes V^*) = H^1(X,\mathcal{F}\otimes\mathcal{F}^*)\oplus H^1(X,\mathcal{F}^*\otimes\mathcal{K})\oplus H^1(X,\mathcal{F}\otimes\mathcal{K}^*)\;. \tag{4.2}$$

Here, we have used that $\mathcal{K}$ is a line bundle in the case we are considering and that $H^1(X,\mathcal{O}) = 0$ on a Calabi-Yau manifold. Thus, it is not unreasonable to interpret $H^1(X,\mathcal{F}^*\otimes\mathcal{K})$ and $H^1(X,\mathcal{F}\otimes\mathcal{K}^*)$ as giving rise to bundle moduli. Thinking about the perturbations such degrees of freedom would contribute to the higher dimensional gauge field, we see that they describe the deformations of the split bundle where $\mathcal{F}$ and $\mathcal{K}$ are mixed into one another; that is, they parametrize movement in moduli space away from the decomposable locus. However, we stress that, in this work, the effective field theory that we will derive will describe perturbations around the decomposable locus, and hence, we will think of the charged fields in $H^1(X,\mathcal{F}^*\otimes\mathcal{K})$ and $H^1(X,\mathcal{F}\otimes\mathcal{K}^*)$ as matter. In the following, these fields will be denoted by C^L.

4.2 Four dimensional spectrum from the gravitational sector

In addition to the fields of the previous subsection, we have the usual low energy moduli from the gravitational sector of the eleven-dimensional theory. It is important to note

[12] More precisely, as a vector space, $H^1(X,V\otimes V^*)$ can be viewed as the tangent space to bundle moduli space.

that some of these moduli are also charged under the U(1) symmetry in the low energy gauge group even though they do not descend from higher-dimensional gauge fields. The moduli fields which are not associated to the $E_8 \times E_8$ gauge fields include the dilaton, the (complexified) Kähler moduli, the complex structure moduli and possible five-brane moduli. It turns out that, of these fields, only the complex structure moduli are not charged under the U(1) symmetry. For now, we will focus on tree level results where only the Kähler moduli are of importance. The other fields will come into play in section 6 where we calculate what, in the weakly coupled language, correspond to one-loop corrections. For reasons which will become clear, the following arguments will be carried out using the language of the strongly-coupled $E_8 \times E_8$ heterotic string [31]–[35], that is, M-theory on the orbifold $S^1/\mathbb{Z}_2$. However, analogous arguments leading to the same results can be presented starting with the weakly-coupled ten-dimensional theory [2].

In terms of higher-dimensional fields, the Kähler moduli T^i, where $i, j, k = 1, \ldots, h^{1,1}(X)$ can be written as follows.

$$T^i = t^i + 2i\chi^i \tag{4.3}$$

Here t^i are the Kähler parameters of the Calabi-Yau manifold, which we have already encountered in our bundle stability analysis, and χ^i are the associated T-axions which descend from the M-theory three-form as

$$C_{11a\bar{b}} = \chi^i J_{ia\bar{b}} \; . \tag{4.4}$$

We recall that $\{J_i\}$ is a basis of harmonic $(1,1)$ forms on the Calabi-Yau manifold, chosen to be dual to a basis $\{\mathcal{C}^i\}$ of the second Calabi-Yau homology such that

$$\frac{1}{v^{1/3}} \int_{\mathcal{C}^i} J_j = \delta^i_j \; , \tag{4.5}$$

where v is an arbitrary coordinate volume of the Calabi-Yau space. The index 11 refers to the coordinate of the $S^1/\mathbb{Z}_2$ orbifold, and $a, b, \ldots$ and $\bar{a}, \bar{b}, \ldots$ denote holomorphic and anti-holomorphic Calabi-Yau indices.

It is a well-known fact that anti-symmetric tensor fields in heterotic theories transform under $E_8 \times E_8$ gauge transformations [2]. Consider a local infinitesimal gauge transformation,

$$\delta A_A = -D_A \epsilon \; , \tag{4.6}$$

where the derivative is covariant and ϵ is the gauge transformation parameter. Under such a change of gauge the two-form C_{11AB} transforms as

$$\delta C_{11AB} = -\left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \frac{1}{4\pi} \delta(x^{11}) \mathrm{tr}(\epsilon F_{AB}) \; , \tag{4.7}$$

where $A, B = 0, \ldots, 9$ label the coordinates transverse to the $S^1/\mathbb{Z}_2$ orbifold. Let us concentrate first on the internal components of equation (4.7) by writing $\delta C_{11a\bar{b}} = \delta\chi^i J_{ia\bar{b}}$. Integration over $\mathcal{C}^i \times S^1/Z_2$ then leads to the following gauge transformation

$$\delta\chi^i = -\frac{\epsilon_S \epsilon_R^2}{16\pi} \int_{\mathcal{C}^i} \mathrm{tr}(\epsilon F) \tag{4.8}$$

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for the T-axions, where we have introduced the dimensionless $\mathcal{O}(\kappa_{11}^{2/3})$ combination of constants

$$\epsilon_S \epsilon_R^2 = \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \frac{8}{4\pi\rho v^{1/3}} \tag{4.9}$$

and $\pi\rho$ is the coordinate volume of the $S^1/\mathbb{Z}_2$ interval. The constants ϵ_S and ϵ_R are the usual expansion parameters defining four-dimensional heterotic M-theory [36]. In weakly coupled language, $\epsilon_s \epsilon_R^2 = 8\pi\alpha'/(4v_{10})^{1/3}$, where v_{10} is the Calabi-Yau coordinate volume in the 10-dimensional theory. Hence, the effects considered here are order α' but at tree level. Corrections which are one-loop from a weakly coupled perspective will be discussed in section 6. Normally, the transformation (4.8) does not lead to a non-trivial gauge transformation of the T-axions under the visible sector gauge group. This is because if F is nonzero, in order for χ to have a non-trivial transformation according to (4.8), then F breaks the associated gauge symmetry at the compactification scale and so it does not appear as a factor in the visible sector gauge group. However, in our case we have a U(1) factor in the structure group of our bundle which, due to its self commutation, is both visible and hidden at the same time. In particular, F can have a non-trivial vev in the U(1) direction without breaking the associated visible sector gauge symmetry.

Thus, for our case, we have a non-trivial U(1) transformation for the moduli T^k. If we consider a gauge transformation associated with the additional U(1) seen in the visible sector, with a gauge parameter denoted by $\tilde{\epsilon}$, we may rewrite (4.8) in terms of the first Chern $c_1(\mathcal{F})$ of the de-stabilizing sub-sheaf $\mathcal{F}$ as follows.

$$\delta\chi^i = -\frac{3}{16}\epsilon_S \epsilon_R^2 \tilde{\epsilon}\, c_1^i(\mathcal{F}) \tag{4.10}$$

In addition, the singlet matter fields C^L carry a U(1) charge Q^L and transform linearly as

$$\delta C^L = -i\tilde{\epsilon} Q^L C^L \,. \tag{4.11}$$

It is known [11–13] that a low-energy U(1) symmetry in heterotic compactifications which arises as the commutant of a U(1) factor in the internal bundle structure group is generally anomalous in the Green-Schwarz sense. In this case, the triangle anomalies in the four-dimensional theory are cancelled by an anomalous variation of the gauge kinetic function, as usual. Perhaps not so well-known, but explained in detail in ref. [14], is that this includes an anomalous variation of the T-modulus dependent threshold corrections of the gauge kinetic function which transforms under (4.10). We stress that this is different from the perhaps better known "universal" anomalous U(1) where the triangle anomaly is cancelled by a variation of the dilaton only [11, 13]. This universal anomaly arises when the anomalous U(1) symmetry has no internal counterpart in the bundle structure group. Such a situation can arise in the SO(32) heterotic string but not in smooth compactifications of the $E_8 \times E_8$ theory. Hence, in the present context we are always dealing with a "non-universal" anomalous U(1) symmetry which transforms the T-moduli as in eq. (4.10). In general, amomalous U(1) symmetries are associated to FI terms in the four-dimensional theory. While the universal heterotic U(1) implies the well-known dilaton-dependent FI term [11, 13], the present non-universal case leads to a T-dependent FI term [14] at leading order. We will now derive this FI term explicitly.

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4.3 The four dimensional potential

The potential of an $\mathcal{N} = 1$ supersymmetric theory contains two types of contribution: those from D and F-terms. As we will see later, F-terms are less relevant in our context, so we focus on D-terms and their associated potential. To do this, we need to know the Kähler potential of the fields involved. Having deferred loop corrections to section 6, we concentrate here on the leading order which only involves the T-moduli $T^i = t^i + 2i\chi^i$ and the singlet matter fields C^L. We will work in the usual approximation keeping only leading terms in C^L and in inverse powers of the T-moduli. The usual Kähler potential for the T-moduli is given by

$$\kappa_4^2 K_T = -\ln \mathcal{V}\,, \quad \mathcal{V} = \frac{1}{6}\mathcal{K}\,, \tag{4.12}$$

where $\mathcal{V}$ is the Calabi-Yau volume and $\mathcal{K}$ is the cubic polynomial

$$\mathcal{K} = d_{ijk}t^i t^j t^k = \frac{1}{8} d_{ijk}(T^i + \bar{T}^i)(T^j + \bar{T}^j)(T^k + \bar{T}^k)\,. \tag{4.13}$$

The C^L part of the Kähler potential has the form

$$K_{\text{matter}} = G_{LM} C^L \bar{C}^M\,. \tag{4.14}$$

Here G_{LM} is the matter field space metric, which depends on the various moduli in the theory. The precise form of this metric will not be needed but it will be important that it is positive definite.

We now have all of the information we require to compute the D-term contribution to the four dimensional theory's potential. Given that we have identified the transformation properties of our low-energy fields, in particular under the U(1) symmetry, this derivation is standard and can, in a somewhat different context, be found in the literature (see for example, [14, 15]). Nevertheless, we will carry this out explicitly, to present a complete and coherent argument. According to the usual structure of four-dimensional $\mathcal{N} = 1$ supergravity, the D-terms are determined by the following equations [37].

$$g_{I\bar{J}} \bar{X}^{\bar{J}\eta} = i\frac{\partial}{\partial M^I} D^\eta \tag{4.15}$$

$$g_{I\bar{J}} X^{I\eta} = -i\frac{\partial}{\partial \bar{M}^{\bar{J}}} D^\eta \tag{4.16}$$

Here, the M^I represent all of the fields in the theory, $g_{I\bar{J}}$ the complete field space metric, and η is an index labeling the adjoint of the gauge group. The quantities X are the holomorphic Killing vectors which generate those analytic isometries of the Kähler field space which can be gauged. Under such a gauge transformation, the fields M^I then transform as

$$\delta M^I = -\epsilon^\eta X^{I\eta}\,, \tag{4.17}$$

where ϵ^η are the gauge parameters. We can now determine the Killing vector for the U(1) symmetry by comparing this expression with the field transformations (4.10) and (4.11) which we have derived from the higher-dimensional theory. This leads to

$$X^i = i\frac{3}{8}\epsilon_S \epsilon_R^2 c_1^i(\mathcal{F}) \tag{4.18}$$

$$X^L = iQ^L C^L\,. \tag{4.19}$$

Inserting this Killing vector into (4.15) and solving for the associated U(1) D-term we obtain

$$D^{\mathrm{U}(1)} = \frac{3}{16}\frac{\epsilon_S \epsilon_R^2}{\kappa_4^2}\frac{\mu(\mathcal{F})}{\mathcal{V}} - \sum_{L,\bar{M}} Q^L G_{L\bar{M}} C^L \bar{C}^{\bar{M}} \,, \tag{4.20}$$

where $\kappa_4^2 = \kappa_{11}^2/(v2\pi\rho)$ is the four-dimensional Planck constant. Here, we have neglected contributions to this D-term from **27** and $\overline{\mathbf{27}}$ multiplets charged under the U(1) symmetry. As long as E_6 remains unbroken these further contributions vanish and, for our explicit example, this will indeed be enforced by the E_6 D-terms. We note that the above D-term consists of a FI piece which is proportional to the slope

$$\mu(\mathcal{F}) = \frac{1}{2}c_1^i(\mathcal{F})s_i = \frac{1}{2}d_{ijk}c_1^i(\mathcal{F})t^j t^k = \frac{1}{8}d_{ijk}c_1^i(\mathcal{F})(T^j + \bar{T}^j)(T^k + \bar{T}^k) \,, \tag{4.21}$$

of the destabilizing sub-sheaf $\mathcal{F}$ and a standard matter field piece. We also recall that $\mathcal{V}$ is the Calabi-Yau volume given in eq. (4.12).

5 Stability walls in the effective theory

In this section, we will study the vacuum structure of the effective theory derived in the previous section. Our aim is to show how this four-dimensional, field theory based analysis reproduces features seen in the mathematical, ten-dimensional analysis of section 3. In other words, we would like to show how the abstract mathematical concept of bundle stability and its implications for supersymmetry can be understood in a physical way, from our four-dimensional effective theory. It is clear from the expression (4.20) for the D-term that the nature of the four-dimensional vacuum space crucially depends on the charges Q^L of the matter field singlets C^L. We begin with a general discussion and then illustrate the main points with the example discussed in section 3.1.

We need to understand how the interplay between the FI and matter field terms in eq. (4.20) can reproduce the expected pattern of broken or unbroken supersymmetry. A crucial observation is that the FI term is proportional, with a positive constant of proportionality, to the slope, $\mu(\mathcal{F})$, of the destabilizing sub-sheaf $\mathcal{F}$. We recall from our previous discussion that this slope is negative in the part of the Kähler moduli space where the bundle is stable and hence supersymmetric, and that it is positive where the bundle breaks supersymmetry. The stability wall which separates these two regions in Kähler moduli space is defined by $\mu(\mathcal{F}) = 0$. Given these features of the FI term, one can ask how the D-term (4.20) for $\mu(\mathcal{F}) < 0$ can vanish and hence preserve supersymmetry as we would expect. To achieve this, the FI term obviously has to be cancelled by the matter field contribution in (4.20) through a suitable adjustment of the matter field vevs. This will work precisely if there is *at least one negatively charged matter field* C^L, with $Q^L < 0$ present. On the other hand, if the D-term (4.20) is to be non-zero and thus break supersymmetry for $\mu(\mathcal{F}) > 0$, as we expect it should, *all matter fields need to be negatively charged*; that is, there should be no matter fields with $Q^L > 0$. The D-term (4.20) then becomes a sum of two positive definite terms in the non-supersymmetric region and there is no way in which they can cancel each other.

Hence, for the D-term to correctly describe the expected pattern of supersymmetry breaking the zero modes at the stability wall are constrained in a specific way. Let us focus on our main class of examples, namely bundles V with SU(3) structure group which decompose as $V = \mathcal{F} \oplus \mathcal{K}$, where $\mathcal{F}$ is the rank two de-stabilizing sub-sheaf (and $\mathcal{K}$ is locally free). Then we can indeed show that the required constraints on the particle spectrum are satisfied. We recall from table 1 that the singlet matter fields C^L correspond to the cohomology groups $H^1(X, \mathcal{F} \otimes \mathcal{K}^*)$ and $H^1(X, \mathcal{F}^* \otimes \mathcal{K})$, where the former leads to negative and the latter to positive charge. Then one can show the following

Lemma 1. *Let V be a holomorphic vector bundle with structure group* SU(3) *defined over X, a Calabi-Yau* 3*-fold. If $\mathcal{F}$ is a rank* 2*, stable sub-sheaf of V, defining the "wall" in the dual Kähler cone given by $\mu(\mathcal{F}) = 0$, such that V is stable for $\mu(\mathcal{F}) < 0$ and unstable for $\mu(\mathcal{F}) > 0$, then $H^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*) \neq 0$ and $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) = 0$ (for any effective field theory describing only V).*[13]

The proof of this lemma (generalized to SU(n) bundles) is provided in appendix A. It states that all singlet matter fields C^L result from the cohomology group $H^1(X, \mathcal{F} \otimes \mathcal{K}^*)$ and therefore, from table 1, are negatively charged, as required. The fact that all of the fields C^L carry U(1) charges of the same sign means, of course, that the U(1) symmetry is anomalous. This is in line with expectations and we know that this triangle anomaly is cancelled by the four-dimensional version of the Green-Schwarz mechanism. Since we are dealing with a non-universal anomalous U(1), as discussed, this involves an anomalous variation of the threshold correction to the gauge kinetic function induced by the transformation (4.10) of the T-axions. Details of this can be found in ref. [14].

We would now like to discuss the D-term (4.20) and its associated vacuum space and particle masses in more detail. This will provide us with a general picture of how the theory at the stability wall relates to the standard heterotic low-energy theory at a generic point in the supersymmetric part of Kähler moduli space. As mentioned before, we will focus on the part of the moduli space where E_6 is unbroken, so that we do not need to consider vevs of $\mathbf{27}$ and $\overline{\mathbf{27}}$ multiplets. Hence, the fields of central interest are the T-moduli $T^i = t^i + 2i\chi^i$ and the singlet matter fields C^L. It is clear that the D-term (4.20) gives mass to precisely one real combination of the Kähler moduli t^i and the matter fields C^L, the Higgs field. Expanding (4.20) around a vacuum (that is, a vanishing D-term, $D^{\mathrm{U}(1)} = 0$) by writing $t^i = \langle t^i \rangle + \delta t^i$ and $C^L = \langle C^L \rangle + \delta C^L$, we find that this massive linear combination is given by

$$D^{\mathrm{U}(1)} = -\frac{3}{16}\frac{\epsilon_S \epsilon_R^2}{\kappa_4^2} G_{jk} c_1^j(\mathcal{F}) \delta t^k - \sum_{L,\bar{M}} Q^L G_{L\bar{M}} \left(\left\langle C^L \right\rangle \delta \bar{C}^{\bar{M}} + \delta C^L \left\langle \bar{C}^{\bar{M}} \right\rangle \right) , \tag{5.1}$$

where

$$G_{ij} = -\frac{\partial^2 \ln \mathcal{V}}{\partial t^i \partial t^j} \tag{5.2}$$

[13] If $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) \neq 0$ then the bundle defined by the extension $\mathcal{E}xt^1(\mathcal{F}, V/\mathcal{F}) = H^1(X, \mathcal{F}^* \otimes V/\mathcal{F})$ is *not* isomorphic to V. This case corresponds to a branch structure in the effective field theory which provides a transition to a new vector bundle and will be explored in more detail in [38].

is the Kähler moduli space metric, expressed in terms of the Calabi-Yau volume $\mathcal{V}$ as defined in eq. (4.12). In this discussion, we are ignoring terms which are higher order in $\langle C^L \rangle$ and inverse powers of t^i. The Goldstone mode, the corresponding linear combination of T-axions χ^i and C^L phases, is absorbed by the U(1) vector boson in the super-Higgs effect. Since supersymmetry is unbroken, the mass of the linear combination (5.1) and the U(1) vector boson must be equal and they can be computed from eq. (5.1) or from the χ^i and C^L kinetic terms. Either way one finds the mass is given by

$$m^2_{\mathrm{U}(1)} = \frac{1}{s}\left(\frac{(3\epsilon_S\epsilon_R^2)^2}{256\kappa_4^2} c_1^i(\mathcal{F}) c_1^j(\mathcal{F}) G_{ij} + \sum_{L,\bar{M}} Q^L Q^{\bar{M}} G_{L\bar{M}} \langle C\rangle^L \langle \bar{C}\rangle^{\bar{M}}\right), \qquad (5.3)$$

where $s = \mathrm{Re}(S)$ is the real part of the dilaton. To obtain this result from eq. (5.1) it is necessary to canonically normalise the kinetic terms $\frac{1}{4\kappa_4^2} G_{ij}\partial\delta t^i \partial\delta t^j$ and $G_{L\bar{M}}\partial\delta C^L \partial\delta\bar{C}^{\bar{M}}$.

Let us discuss this result, beginning at a point on the stability wall. At the stability wall, $\mu(\mathcal{F}) = 0$ and it follows from eq. (4.20) that $\langle C^L\rangle = 0$ in order to have a vanishing D-term. Hence, at the stability wall the Higgs field is a linear combination of Kähler moduli δt^i only, while the Goldstone mode consists of T-axions χ^i. The U(1) and Higgs mass are then given by the first term in eq. (5.3) which scales like $1/(st^2)$ for a typical Kähler modulus t. This is to be compared with the mass of a typical gauge sector massive mode which scales as $1/(st)$. We see that the U(1) and Higgs masses are suppressed by a factor $1/t$ and, hence, that in the large radius limit and close to the stability wall it is consistent to keep these fields in the low energy theory.

What happens as we move away from the stability wall into the supersymmetric region? From eq. (4.20) the matter field vevs $\langle C^L\rangle$ are now non-vanishing and their fluctuations δC^L contribute to the Higgs fields and the Goldstone mode. Once we move sufficiently away from the stability wall, so that $\mu(\mathcal{F}) = \mathcal{O}(t^2)$, eq. (4.20) implies $G_{L\bar{M}} C^L \bar{C}^{\bar{M}} \sim 1/t$. Hence, far away from the stability wall, the U(1) mass (5.3) scales as $1/(st)$ and becomes comparable to a typical heavy gauge sector mass. In this limit, we should, therefore, remove the U(1) vector multiplet and the Higgs multiplet from the low-energy theory. In this way, we recover the standard E_6 gauge group at a generic supersymmetric point in the Kähler moduli space.

What about the matching of chiral multiplets to the usual analysis? First, we note that far away from the stability wall the Higgs multiplet becomes predominantly a linear combination of the matter fields C^L. This means that there are massless "T-moduli" in this region, which are slightly corrected versions of the naively defined fields, consistent with the expectation from standard heterotic compactifications. As for the E_6 singlet matter fields, at the stability wall we have $h^1(X, \mathcal{F}\otimes(V/\mathcal{F})^*)$ fields C^L and $h^1(X, \mathcal{F}\otimes\mathcal{F}^*)$ bundle moduli. Away from the stability wall, one combination of C^L fields is removed from the low-energy theory so that we remain with $h^1(X, \mathcal{F}\otimes(V/\mathcal{F})^*) + h^1(X, \mathcal{F}\otimes\mathcal{F}^*) - 1$ singlet fields. To match standard heterotic compactifications, this must equal $h^1(X, V\otimes V^*)$, the number of bundle moduli at a generic supersymmetric point in Kähler moduli space. That is is indeed always the case is stated in the following lemma.

Lemma 2. *Let V be a holomorphic vector bundle with structure group* SU(3) *defined over X, a Calabi-Yau 3-fold. If $\mathcal{F}$ is a rank 2, stable sub-sheaf of V, defining the "wall" in the dual Kähler cone given by $c_1^i(\mathcal{F})s_i = 0$, such that V is stable for $\mu(\mathcal{F}) < 0$ and unstable for $\mu(\mathcal{F}) > 0$, and further, $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) = 0$, then*

$$h^1(X, V \otimes V^*) = h^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*) + h^1(X, \mathcal{F} \otimes \mathcal{F}^*) - 1\,, \tag{5.4}$$

where $h^1(X, V \otimes V^*)$ is the generic dimension of bundle moduli space when V is a stable bundle.

The proof of this lemma can be found in appendix A.

In summary, we see that the D-term (4.20) correctly reproduces all of the expected physical features of gauge bundle supersymmetry. Specifically, the D-term vanishes and, hence, preserves supersymmetry precisely in the region where the gauge bundle is stable while it is non-vanishing in the region where the bundle is unstable. We have seen that in the large radius limit and at the stability wall it is consistent to keep the massive U(1) vector multiplet and the Higgs multiplet in the low energy theory. Away from the stability wall, however, these fields develop heavy masses and have to be dropped. In this way, we recover the usual heterotic effective theory at a generic, supersymmetric point in moduli space.

It is clear that the physics of the non-supersymmetric region of the Kähler cone is dominated by the potential wall due to the non-vanishing D-term (4.20). Since there is no perturbative vacuum in this region, we shall refrain from discussing the mass spectrum in this part of field space. However, to finish this section we shall make a few comments about the regime of the non-supersymmetric region where our effective field theory analysis is valid. In addition to the usual expansions of heterotic M-theory, which will be discussed in more detail in section 6, validity of our approach requires that the potentials present should be below the compactification scale. Furthermore, the C^L field vevs should not be too large, since we have assumed they were small in deriving the effective potential in this section. Obviously, both of these conditions are satisfied close to the transition between the supersymmetric and non-supersymmetric regions of moduli space and, as such, the above discussion can be trusted. Far into the non-supersymmetric region one may not expect a four dimensional description to exist at all. The potential grows in size as we penetrate inside this zone until eventually it becomes of the same mass scale as heavy states which have been truncated in our analysis. To give some idea of scale, let us examine the size of the potential in the non-supersymmetric region when all C^L vevs vanish. At a typcial non-supersymmetric point in field space, the ratio of this potential to the fourth power of a typical mass of a heavy gauge sector state is of order s, the dilaton, when working in string units. As such, in a valid regime of the effective theory where s is large, one should typically not include regions with such a potential in the four-dimensional theory. Close to the boundary with the supersymmetric region (where the D-term potential vanishes exactly), however, the potential is surpressed from its usual scale by the smallness of $\mu(\mathcal{F})^2/\mathcal{V}^{4/3}$, which smoothly increases from zero as we enter the non-supersymmetric part of the Kähler cone. Thus, we can trust our analysis and investigate this potential in the four-dimensional theory in the region near to the boundary where $(\mu(\mathcal{F})^2/\mathcal{V}^{4/3})s \ll 1$.

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Representation	Cohomology	Physical U(1)charge	Dimension of Cohomology
$(\mathbf{1},\mathbf{2})_{-3}$	$H^1(X,\mathcal{F}\otimes\mathcal{K}^*)$	$-3/2$	16
$(\mathbf{1},\mathbf{2})_{3}$	$H^1(X,\mathcal{F}^*\otimes\mathcal{K})$	$3/2$	0
$(\mathbf{1},\mathbf{3})_{0}$	$H^1(X,\mathcal{F}\otimes\mathcal{F}^*)$	0	7
$(\mathbf{27},\mathbf{1})_{2}$	$H^1(X,\mathcal{K})$	1	0
$(\mathbf{27},\mathbf{2})_{-1}$	$H^1(X,\mathcal{F})$	$-1/2$	2
$(\overline{\mathbf{27}},\mathbf{1})_{-2}$	$H^1(X,\mathcal{K}^*)$	-1	0
$(\overline{\mathbf{27}},\mathbf{2})_{1}$	$H^1(X,\mathcal{F}^*)$	$1/2$	0

Table 2. Particle content of the model defined by the bundle (3.5) at the decomposable locus where $V = \mathcal{F}\oplus\mathcal{K}$, with $\mathcal{F}$ defined by (5.5) and $\mathcal{K} = \mathcal{O}_X(1,-1)$.

5.1 An example

To illustrate the above general discussion, let us return to the example of section 3.1. Recall, that we have defined the monad bundle, (3.5) on the complete intersection Calabi-Yau manifold (3.3). As mentioned in section 3.1, we find that the SU(3) bundle, V, decomposes as $V \to \mathcal{F}\oplus\mathcal{K}$ where $\mathcal{K} = \mathcal{O}_X(1,-1)$ is a line bundle. The de-stabilizing sub-sheaf $\mathcal{F}\subset V$ has rank two[14] and is described by the monad

$$0 \to \mathcal{F} \to \mathcal{O}_X(1,0)\oplus\mathcal{O}_X(0,1)^{\oplus 2} \to \mathcal{O}_X(2,1) \to 0\,. \tag{5.5}$$

The locus in the moduli space of V where it decomposes as $V = \mathcal{F}\oplus\mathcal{K}$ corresponds to setting to setting to zero the bi-degree $(1,2)$ polynomials in the monad map, f, given in (3.5). Using the results of refs. [17, 28], we can calculate the dimensions of the cohomology groups of $\mathcal{F}$ and $\mathcal{K}$ listed in table 1. The results are summarised in table 2. The *only* matter fields present which are charged under the additional U(1) symmetry appear in the first and fifth row in the table. They both have negative charge under the four-dimensional U(1), listed in the third column. In particular, this means that the singlet matter fields C^L, which correspond to the first row in the table, are all negatively charged, in accordance with Lemma 1. Further, in this particular model it turns out that the **27** matter multiplets are also negatively charged. This means, by gauge invariance, that the F-term part of the potential vanishes. Having only **27** but no $\overline{\mathbf{27}}$ multiplets means the **27** vevs will be forced to zero by the E_6 D-terms. Hence, they do not contribute to the U(1) D-term (4.20). For the present example and all models with similar particle content, the U(1) D-term (4.20) therefore describes the full vacuum space. In general, models with positively charged E_6 multiplets or anti-families in $\overline{\mathbf{27}}$ exist. For such models one would expect superpotential terms or D-flat directions with non-vanishing **27** and $\overline{\mathbf{27}}$ vevs, leading to a more complicated structure of the vacuum space. As explained before, for such models the U(1) D-term (4.20) describes the part of the vacuum space where E_6 is unbroken.

[14] In this example $\mathcal{F}$ is a bundle that injects into V everywhere in moduli space, while $\mathcal{K}$ is a line-bundle and *only* injects at the decomposable point. That $\mathcal{F}$ is indeed a bundle and not simply a sheaf has been checked explicitly using the computer algebra packages [39].

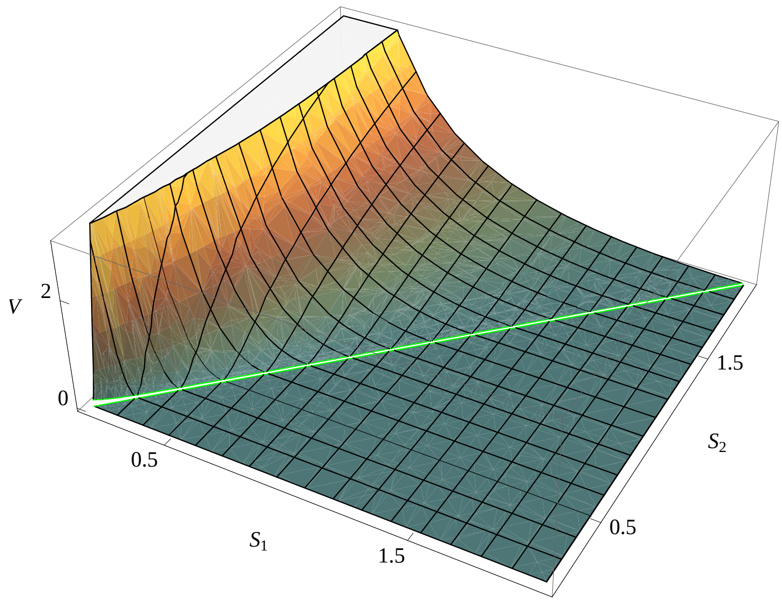

Figure 3. The potential in the dual Kähler cone as a functions of the two dual Kähler variables. The potential has been minimized with respect to the C^L fields (which are not plotted here). The flat region of the potential is where the bundle is stable. The positive definite potential wall which one encounters upon entering the region where the bundle is unstable can clearly be seen, arising at the line with slope = 1.

A generalised expression, including the family and anti-family degrees of freedom in the D-term, can trivially be derived.

From eqs. (4.20) the D-term for this example reads

$$D^{\mathrm{U}(1)} = \frac{3}{16}\frac{\epsilon_S \epsilon_R^2}{\kappa_4^2}\frac{\mu(\mathcal{F})}{\mathcal{V}} + \frac{3}{2}\sum_{L,\bar{M}=1}^{16} G_{L\bar{M}} C^L \bar{C}^{\bar{M}} \,, \tag{5.6}$$

where, from eqs. (3.12), (3.4), the slope is given by

$$\mu(\mathcal{F}) = \frac{1}{2}(-s_1 + s_2)\,, \quad s_1 = 4(t^2)^2\,, \quad s_2 = 8t^1 t^2 + 2(t^2)^2\,. \tag{5.7}$$

For the volume we have

$$\mathcal{V} = 2t^1(t^2)^2 + \frac{1}{3}(t^2)^3\,. \tag{5.8}$$

In figure 3 we plot the D-term potential (5.6) as a function of the dual Kähler cone variables s_1 and s_2, defined in eq. (3.4). The C^L vevs, which are not plotted due to lack of dimensions, have been chosen to be at their minimum. The potential rising from zero in the unstable region is clearly visible, as is the stability wall determined by the line with slope 1 (in agreement with the bundle stability regions shown in figure 2). Figure 4 shows the D-term potential from eq. (5.6) as a function of the coordinate s_1, with s_2 chosen such that the plot traces a line perpendicular to the stability wall in the Kähler cone, and the radius $|C|$ of a representative singlet matter field. This figure makes it clear that there is no "boundary"

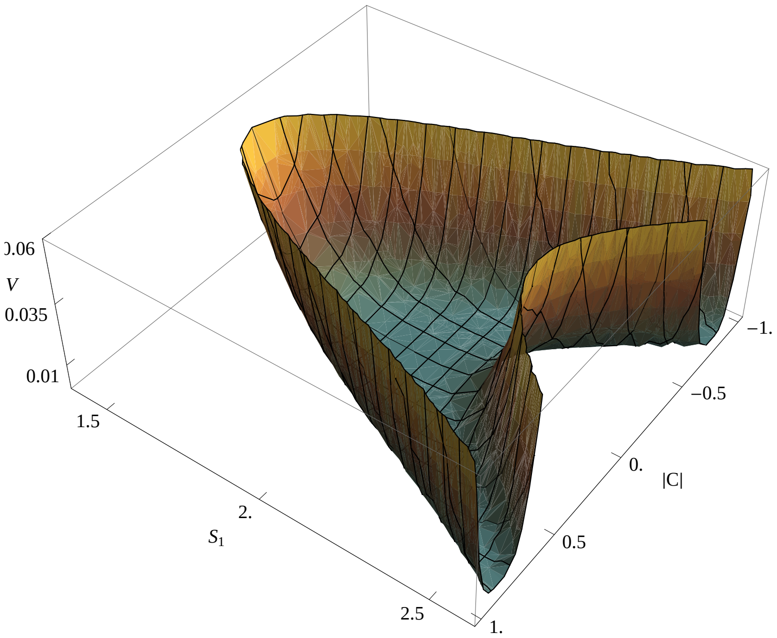

Figure 4. The D-term potential from eq. (5.6), as a function of s_1, a dual Kähler modulus, and the absolute value, $|C|$, of a representative singlet matter field C. In this plot we have chosen $s_2 = 4 - s_1$ so that we are examining a line in Kähler moduli space perpendicular to the boundary between the supersymmetric and non-supersymmetric regions. The boundary itself is found at $s_1 = 2$ in this diagram. Since the exact form of the Kähler potential for the matter fields is not known, a simple, canonical form has been chosen for illustrative purposes.

to the vacuum space at the stability wall in Kähler moduli space if one considers the full field space of the theory.

At the stability wall, where $t^2 = 4t^1$ and $\langle C^L \rangle = 0$, the variation of the D-term (5.6) becomes

$$D^{\mathrm{U}(1)} = \frac{9\epsilon_S \epsilon_R^2}{640\kappa_4^2} \frac{1}{(t^1)^2} (4\delta t^1 - \delta t^2) \ . \tag{5.9}$$

This shows that it is the combination $4\delta t^1 - \delta t^2$ of Kähler moduli perpendicular to the stability wall which becomes massive at this point, as is also evident from figure 4. The mass of this linear combination is given by

$$m^2_{\mathrm{U}(1)} = \frac{3(\epsilon_S \epsilon_R^2)^2}{256\kappa_4^2} \frac{1}{s(t^1)^2} \ . \tag{5.10}$$

This expression shows explicitly the aforementioned $1/t^2$ scaling of the U(1) vector and Higgs masses which justifies keeping these states in the low-energy theory close to the stability wall. As discussed earlier, far away from the stability wall the Higgs multiplet becomes pre-dominantly a linear combination of the C^L multiplets and there are two massless Kähler moduli as one would expect at a generic point in the supersymmetric region. From table 2, we have 16 singlet matter fields C^L and 7 bundle moduli at the stability wall. With one of the C^L becoming massive one would expect $16 + 7 - 1 = 22$ bundle moduli

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at a generic supersymmetric point in moduli space and this is indeed the number we have computed for this example, see eq. (3.6). This illustrates the general statement in Lemma 2. An additional example, with an unrelated manifold and method of bundle construction is provided in appendix B.

5.2 Results in bundle stability from the effective theory

So far, we have used mathematical information on vector bundle stability to construct a low-energy description of bundle supersymmetry. Now that we have established such a picture, let us reverse our approach and see if can recover some of the mathematical results in bundle stability from the effective field theory. A key fact to remember is the interpretation of the charged matter fields, C^L, as bundle moduli of the SU(3) bundle - as described in section 4. When these fields vanish the bundle decomposes as $V = \mathcal{F} \oplus \mathcal{K}$, and has structure group $S(\mathrm{U}(2) \times \mathrm{U}(1))$. For non-vanishing C^L vevs, at a generic point in moduli space, the bundle no longer splits up into a direct sum of sub-bundles, and the structure group reverts to SU(3).

First, in what we would expect to be the supersymmetric region of Kähler moduli space, the fields C^L *must* acquire a vev if the D-term is to vanish. From this observation, we reproduce the fact that the bundle will only produce a supersymmetric vacuum in the so-called "stable" region of Kähler moduli space, if it is at a *generic* (that is, non-split) point in its moduli space, that is, if the structure group is SU(3). If, in the normally "stable" region of Kähler moduli space, the bundle moduli move to the decomposable locus where the structure group is $S(\mathrm{U}(2) \times \mathrm{U}(1))$, the D-term is non-vanishing and supersymmetry is broken. This is all in perfect agreement with the algebro-geometric analysis presented in section 2.

As we learned in section 2, at the stability wall in Kähler moduli space, in order to have a supersymmetric theory, the bundle must be split and semi-stable; that is, it must decompose into a direct sum of stable bundles of the same slope. From (4.20) we again see this behaviour reproduced. The FI term vanishes on this line in Kähler moduli space. Hence, the vanishing of the D-term required by supersymmetry forces the C^L field vevs to vanish - taking us precisely to the split point in bundle moduli space.

As before, our discussion here allows us to go further than has been previously possible and discuss what happens in the region where supersymmetry is spontaneously broken as well. Although the D-term (4.20) cannot vanish in this part of moduli space, for fixed Kähler moduli, the D-term potential can be minimized by vanishing fields C^L. Thus, the bundle will relax to the decomposable locus in bundle moduli space throughout this region, as well as at the stability wall, in the absence of non-perturbative effects.

As a final comment, it is interesting to note that the D-term (4.20) does not depend on the complex structure fields. Thus, it should also be true that the stability regions derived in section 3 are not dependent on the choice of complex structure, for those bundles which can give rise to supersymmetric theories in four dimensions. This result, which is somewhat surprising from a mathematical perspective, will be discussed further in the appendix.

6 Higher order corrections

In the analysis of proceeding sections we have worked to first order in the strong coupling expansion parameter, ϵ_S, and the square of the matter fields. The strong coupling expansion parameter ϵ_S itself, as opposed to the combination $\epsilon_S \epsilon_R^2$ which is what was defined in (4.9) and has appeared heretofore, is given by the following [31, 36],

$$\epsilon_S = \left(\frac{\kappa_{11}}{4\pi}\right)^{2/3} \frac{2\pi\rho}{v^{2/3}} \,. \tag{6.1}$$

In the weakly coupled langauge, we have been working, up to this point, at string theory tree-level. One can do better than this and work out those $\mathcal{O}(\epsilon_S^2)$ corrections that correspond to string one-loop corrections.[15] In particular, the D-term given in (4.20), receives ϵ_S^2 corrections which can be calculated. Corrections to the matter field part of the D-term (4.20) are uninteresting. The only fact that we have used about this term is the positive definite nature of the matter field metric $G_{L\bar{M}}$, and this will not be changed by such corrections. However, the $\mathcal{O}(\epsilon_S^2)$ corrections to the FI term are of some interest and we now proceed to derive these. At lowest order, the T-moduli had a non-trivial U(1) transformation while all other moduli fields were invariant. As we will see, at higher order, the dilaton S and the five-brane position moduli Z^α, where $\alpha = 1, \ldots, N$ numbers the different five-branes, also transform non-trivially. We start by defining these four-dimensional superfields in terms of the underlying geometric fields. The definition of the T-moduli, $T^i = t^i + 2i\chi^i$, is as previously given (see eq. (4.3)). For the dilaton and the five-brane moduli we have [42, 43]

$$S = V_0 + \pi\epsilon_S \sum_{\alpha=1}^{N} \beta_i^\alpha t^i z_\alpha^2 + i\left(\sigma + 2\pi\epsilon_S \sum_{\alpha}^{N} \beta_i^\alpha \chi^i z_\alpha^2\right) \tag{6.2}$$

$$Z^\alpha = \beta_i^\alpha \left(t^i z_\alpha + 2i(-n_\alpha^i \nu_\alpha + \chi^i z_\alpha)\right) \,. \tag{6.3}$$

Here, V_0 is the Calabi-Yau volume averaged over the orbifold and σ is the dilatonic axion, the dual of the four-dimensional two-form $B_{\mu\nu} = C_{11\mu\nu}$. Further, z_α is the distance from the left orbifold fixed plane to the α-th five-brane, the β_i^α are the charges associated to the α-th five-brane and $n_\alpha^i = \beta_i^\alpha / (\sum_i (\beta_i^\alpha)^2)$. The fields ν_α are axions located on the five-brane world-volumes. In order to compute the corrections to the D-term, we need to consider the U(1) transformations of the fields at order ϵ_S^2. For the T-moduli and the matter fields, these transformations are given in eqs. (4.10) and (4.11) with no further corrections at $\mathcal{O}(\epsilon_S^2)$. The transformation of the dilaton and five-brane position superfields are slightly more subtle in their origin. To discuss the dilaton, we consider the relevant terms in the four-dimensional effective action which involve the two-form $B_{\mu\nu} = C_{11\mu\nu}$. These terms are [14]

$$S_{4d,B} = -\frac{1}{2\kappa_4^2} \int_{\mathcal{M}_4} \left[V_0^2 H \wedge *H + \frac{3}{4}\pi\epsilon_S^2\epsilon_R^2 c_1^i(\mathcal{F})\beta_i B \wedge F \right] \,, \tag{6.4}$$

[15] Corrections corresponding to higher orders in α' would require knowledge of the Kähler potential for bundle moduli, which is only known for special cases [40]. For a discussion of higher order corrections in α' to the supersymmetry/slope stability condition in the Type II context, see e.g. [41].

where $H = dB + \dots$ and the dots indicate a Chern-Simons three-form which is irrelevant for the present discussion. Further, $F = dA$ is the field strength of the U(1) gauge field A and the integer charges β_i of the E_8 sector under consideration are defined as

$$\beta_i = \frac{1}{16\pi^2}\int_X \left(\mathrm{tr}F\wedge F - \frac{1}{2}\mathrm{tr}R\wedge R\right)\wedge J_i \; . \tag{6.5}$$

In order to dualise the two-form B to the dilatonic axion σ, we set $H_0 = dB$ and add to the above action the term

$$\frac{1}{\kappa_4^2}\int_{\mathcal{M}_4} H_0 \wedge d\sigma \; . \tag{6.6}$$

By integrating out H_0, we find the kinetic term

$$S_{4d,\mathrm{dual}} = -\frac{1}{\kappa_4^2}\int_{\mathcal{M}_4}\left(\frac{1}{V_0^2}\Sigma\wedge *\Sigma\right) \tag{6.7}$$

for the dilatonic axion σ, where the "field strength" Σ is defined as

$$\Sigma = d\sigma - \frac{3}{8}\pi\epsilon_S^2\epsilon_R^2 c_1^i(\mathcal{F})\beta_i A \; . \tag{6.8}$$

This field strength needs to be invariant under U(1) gauge transformations with $\delta A = -D\tilde{\epsilon}$, which implies the following transfomation law for the dilatonic axion.

$$\delta\sigma = -\frac{3}{8}\pi\epsilon_S^2\epsilon_R^2 c_1^i(\mathcal{F})\beta_i\,\tilde{\epsilon} \; . \tag{6.9}$$

The five-brane axions ν_α do not transform under U(1) transformations, so the χ^i transformation (4.10) and the above σ transformation (6.9) are all we have to take into account at the component field level. Note that, from eqs. (6.2), (6.3) and (4.3), this implies non-trivial transformations for all superfields S, Z^α and T^i. In particular, the five-brane moduli superfields Z^α in eq. (6.3) pick up a non-trivial transformation through their dependence on the T-axions χ^i.

Taking these new field transformations into account, we may now calculate the correction to our D-term, (4.20), at order ϵ_S^2. For this we need the relevant corrections to the Kähler potential. In eq. (4.12) we have already given the Kähler potential for the T-moduli which remains unchanged at the orders we require. The Kähler potential for the dilaton and the five-brane moduli is given by

$$K_S = -\ln\left[S + \bar{S} - \pi\epsilon_S\sum_{\alpha=1}^{N}\frac{(Z^\alpha+\bar{Z}^\alpha)^2}{\beta_i^\alpha(T^i+\bar{T}^i)}\right] \tag{6.10}$$

Given these expressions, we may follow exactly the same procedure as in section 4 to obtain the corrected D-term

$$D^{\mathrm{U}(1)} = f - \sum_{L\bar{M}} Q^L G_{L\bar{M}} C^L \bar{C}^{\bar{M}} \; . \tag{6.11}$$

Here, the FI term f is given by

$$f = f^{(0)} + f^{(1)} \tag{6.12}$$

$$f^{(0)} = \frac{3}{16}\frac{\epsilon_S \epsilon_R^2}{\kappa_4^2}\frac{\mu(\mathcal{F})}{\mathcal{V}} \tag{6.13}$$

$$f^{(1)} = \frac{3\pi\epsilon_S^2\epsilon_R^2}{8\kappa_4^2}\frac{1}{S+\bar{S}}\left[\beta_i c_1^i(\mathcal{F}) + \pi\sum_{\alpha=1}^{N}\frac{(Z^\alpha+\bar{Z}^\alpha)^2}{(\beta_i^\alpha(T^i+\bar{T}^i))^2}\beta_i^\alpha c_1^i(\mathcal{F})\right] \tag{6.14}$$

We see that the leading contribution, $f^{(0)}$, to the FI term precisely reproduces our previous result (4.20) while the correction term $f^{(1)}$ is surpressed by an extra power of ϵ_S, as expected. As mentioned earlier, the second term in (6.11) will also receive corrections. However, since these small corrections cannot change the sign of this term they are of no immediate interest to us here.

The $\mathcal{O}(\epsilon_S^2)$ correction $f^{(1)}$ to the FI term depends on fields other than the Kähler moduli. This means that the position of the stability wall in the Kähler cone will change slightly as we change, for example, the value of the dilaton or the five-brane moduli Z^α. Naively, this suggests that we have lost the link, as espoused in the rest of the paper, between the mathematical stability analysis and the four-dimensional effective field theory. However, this is not the case.

The crucial point is that the four-dimensional fields which appear in the above expression are not quite those which are "experienced by the gauge fields". In heterotic M-theory, the vacuum solution in eleven dimensions includes a warping in the eleventh direction which introduces dependence of the Kähler moduli on the orbifold coordinate. In other words, the six dimensional manifold changes shape slightly as we traverse the $S^1/\mathbb{Z}_2$ orbifold direction. The four dimensional Kähler moduli t^i which appear in the above expressions (for example, in eq. (6.11)) are the orbifold *average* of these varying Kähler parameters. The gauge fields of our bundle, however, reside on one of the orbifold fixed planes at either end of the interval. Thus, in performing the stability analysis of sections 2 and 3, it is not the averaged quantities which are relevant, but the Kähler moduli of the Calabi-Yau 3-fold at the relevant orbifold fixed plane. It is precisely the difference between those Kähler parameters at the orbifold fixed plane and the averaged ones which accounts for the correction given in equation (6.14). This may be checked explicitly using the expressions for the warping of heterotic M-theory given in refs. [32, 33, 35, 42]. Note that, in the case of Abelian bundles, such corrections have been discovered elsewhere in the literature [15, 44].

To make this precise, let us drop the requirement that we write the FI term in terms of four-dimensional superfields. Instead, we introduce the Kähler moduli $\tilde{s}_i$ of the Calabi-Yau manifold on the relevant orbifold fixed plane (as opposed to the averaged Kähler moduli s_i) and denote by $\tilde{\mu}(\mathcal{F}) = c_1^i(\mathcal{F})\tilde{s}_i/2$ and $\tilde{\mathcal{V}}$ the corresponding slope and volume. Then one can show that the corrected D-term (6.11) can be written as

$$D^{\mathrm{U}(1)} = \frac{3}{16}\frac{\epsilon_S\epsilon_R^2}{\kappa_4^2}\frac{\tilde{\mu}(\mathcal{F})}{\tilde{\mathcal{V}}} - \sum_{L,\bar{M}} Q^L G_{L\bar{M}} C^L \bar{C}^{\bar{M}} \tag{6.15}$$

All correction terms have disappeared and the FI term is proportional to the slope computed for Kähler parameters on the orbifold plane, where the bundle is actually defined. This is precisely the slope one would define in a mathematical context. Hence, our interpretation of the U(1) D-term in terms of gauge bundle stability is completely unchanged by higher order corrections.

7 Conclusions and further work

In this paper, we have explored in detail the structure of heterotic theories near a stability wall, separating regions in Kähler moduli space where a non-Abelian internal gauge bundle preserves or breaks supersymmetry. We have found four-dimensional effective theories valid near such boundaries which provide us with an explicit low-energy description of bundle supersymmetry breaking and with a physical picture for the mathematical notion of slope stability. A key observation in our analysis is that at a stability wall the structure group of the internal gauge bundle decomposes and acquires a U(1) factor. This leads to an additional U(1) symmetry in the four-dimensional effective theory which is Green-Schwarz anomalous. The associated U(1) D-term consists of a FI term and a matter field term and it controls the supersymmetry properties of the bundle from a four-dimensional point of view. Specifically, the FI term is proportional to the slope $\mu(\mathcal{F})$ of the destabilizing subsheaf $\mathcal{F} \subset V$ of the internal vector bundle V. For negative slope the bundle V is stable. In the four-dimensional theory this is reproduced, since non-trivial vacuum expectation values of U(1) charged matter fields compensate the FI term so that the U(1) D-term vanishes and supersymmetry is preserved. For positive slope, that is an unstable bundle V, the FI term changes sign. As all U(1) charges have the same sign, the FI term cannot be cancelled by matter field vevs in this case and supersymmetry is broken. In four dimensions, the relation between the theory at the stability wall and at a generic supersymmetric point is governed by the super-Higgs effect. As one moves away from the stability wall the U(1) vector field mass increases and has to be removed from the low-energy theory, together with the associated Higgs multiplet. The implied matching of degrees of freedom can be precisely reproduced by a cohomology calculation. We have also shown that our results are robust under corrections suppressed from the leading effects by a power of ϵ_S (the strong coupling expansion parameter), corresponding to string one-loop corrections. While the FI term does receive corrections at this order, they have a simple interpretation in terms of 11-dimensional geometry. While the standard four-dimensional Kähler moduli $t^i = \mathrm{Re}(T^i)$ measure the *average* Calabi-Yau size across the orbifold, the gauge bundle and its stability properties are sensitive to the Calabi-Yau moduli, $\tilde{t}^i$, on the relevant orbifold fixed plane. The order ϵ_S^2 corrections to the FI term simply accounts for the difference between those two types of moduli when the D-term is expressed in terms of the standard four-dimensional fields t^i. In other words, the order ϵ_S^2 terms disappear when the D-term is written in terms of $\tilde{t}^i$. Hence, these one-loop corrections do *not* suggest a modification of the mathematical notion of bundle stability but simply reflect the fact that the gauge fields are localised in the orbifold direction. We stress that the basic picture we provide here, while illustrated for the sake of clarity with vector bundles with SU(3) structure group decomposing into

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$S(\mathrm{U}(2) \times \mathrm{U}(1))$, is very general. We expect its main features to holds for any Calabi-Yau three-fold and for any construction of vector bundles. Indeed, the validity of our approach has been checked in a large number of disparate examples.

Our results suggest many further directions for research, some mathematical in nature and some physical. It would be of great interest to study various generalisations and extensions of the mechanism described in this paper. In the present paper, we have focused, when describing examples, on simple cases with two Kähler moduli, so that the stability walls in Kähler moduli space are lines. We stress, however, that the phenomenon we have described is much more general and appears in Kähler cones of any dimensionality greater than one. In general, the stable region is a sub-cone of the Kähler cone with each co-dimension one face giving rise to a D-term of the type we have described. At each generic point on the stability wall only one of these D-term will be relevant. However, for more than two Kähler moduli co-dimension one faces can intersect so that there are special loci on the stability wall where two or more D-terms need to be considered at a time. Further study of more complicated examples would be an interesting future line of research. Further generalisation could involve considering more complicated splitting types at the stability wall, such as $\mathrm{SU}(3) \to S(\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1))$, and $\mathrm{SU}(n)$ bundle structure groups with $n > 3$. Indeed, the authors have already studied such cases in detail and hope to present examples of this type in future work. An interesting observation is that the four-dimensional effective field theory only depends on the structure of the gauge bundle at the split locus in moduli space. This suggests that phenomena similar to the ones described here can link nominally different bundles together via smooth transitions in physical moduli space. The authors are currenty actively investigating this effect.

From a more phenomenological perspective, the potential we provide may be of some interest in moduli stabilization [45]. Its perturbative nature means that this potential is relatively steep. Thus, if one were to balance it against a non-perturbative potential, such as that due to membrane instantons, one might be able to obtain a naturally small scale of supersymmetry breaking. An investigation of whether such an idea is phenomenologically viable is underway. Global remnants of the anomalous U(1) symmetry at the stability wall may have implications for the structure of the theory even at a generic supersymmetric point in moduli space. For example, one might be able to conclude that certain superpotential terms are forbidden. Such considerations may be used to constrain the type of vector bundles which can lead to realistic low-energy models.

Finally one can imagine attempting to use the analysis described in this work to investigate what may be said about bundle stability purely from the point of view of the four-dimensional effective theory. One goal of such work would be to give a simple set of rules, for example based on four-dimensional anomaly cancellation, which would guarantee that a given vector bundle on a Calabi-Yau manifold is stable in a certain region of moduli space.

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A Two lemmas and a conjecture

In this section, we will state the two lemmas used in section 5 (regarding the dimensions of certain cohomology groups) somewhat more formally and provide proofs. These results will be an example of the types of cohomology conditions one can derive in the context of slope stability. Similar conditions can be derived when different SU(n) bundle decompositions are considered or when additional enhanced U(1) symmetries are present. Furthermore, we will make a conjecture regarding the complex structure dependence of a stability wall.

Let X be a Calabi-Yau three-fold with Kähler form J and V a holomorphic vector bundle defined over X with structure group SU(n), where $n = 3, 4, 5$. We will consider a case in which a single sub-sheaf $\mathcal{F} \subset V$ of rank $n-1$ de-stabilizes V in some part of the Kähler moduli space of X. We define the slope, $\mu(\mathcal{F})$, of $\mathcal{F}$ for a given polarization $J = t^k J_k$ by

$$\mu(\mathcal{F}) = \frac{1}{\mathrm{rk}(\mathcal{F})} \int_X c_1(\mathcal{F}) \wedge J \wedge J \, . \tag{A.1}$$

Let us further suppose that $\mathcal{F}$ itself is slope-stable and has a slope such that it destabilizes only part of the Kähler cone (as in figure 1 in section 3). Thus, V is stable for polarizations J with $\mu(\mathcal{F}) < 0$ and unstable for polarizations J with $\mu(\mathcal{F}) > 0$. The two regions are separated by a stability wall in Kähler moduli space where $\mu(\mathcal{F}) = 0$ and V is semi-stable. Using the short exact sequence

$$0 \to \mathcal{F} \to V \to V/\mathcal{F} \to 0 \, , \tag{A.2}$$

we note, as in sections 3 and 4, that we can write $V = \mathcal{F} \oplus V/\mathcal{F}$ as an element in its S-equivalence class.

Our physical four-dimensional picture of bundle stability suggests certain conditions on bundle cohomology which we now discuss. Due to the Fayet-Iliopoulos (FI) D-term (4.20) derived in this paper, the preservation of supersymmetry in the effective theory depends upon the existence (or absence) of certain charged matter fields (the fields C^L in (4.20)) described by $H^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*)$ and $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F})$. Specifically, in order to preserve supersymmetry in the region of moduli space with $\mu(\mathcal{F}) < 0$, the fields C^L described by $H^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*)$ must acquire a vacuum expectation value and cancel the FI term in (4.20), hence setting the potential to zero in this region of Kähler moduli space. In particular, this means that such fields must exist and hence $H^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*) \neq 0$. On the other hand, if the region of moduli space for which $\mu(\mathcal{F}) > 0$ is to have broken supersymmetry, there must be *no* fields C^L described by $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) = 0$. Stating this more formally, we must have the following Lemma:

Lemma I. *Let V be a holomorphic vector bundle with structure group* SU(n) *($n = 3, 4, 5$) defined over X, a Calabi-Yau* 3*-fold with Kähler form J. If $\mathcal{F}$ is a rank $(n-1)$, stable sub-sheaf of V, defining a "stability wall" in in the Kähler cone given by $\mu(\mathcal{F}) = 0$, such that V is stable for $\mu(\mathcal{F}) < 0$ and unstable for $\mu(\mathcal{F}) > 0$ (and $V/\mathcal{F}$ is locally free), then $H^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*) \neq 0$ and $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) = 0$ (for any effective field theory describing V).*

Proof. We begin with the first condition $H^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*) \neq 0$. Consider twisting the sequence (A.2) by the line bundle $\mathcal{K}^*$, where $\mathcal{K} = (V/\mathcal{F})^{**} \approx V/\mathcal{F}$. This leads to the short exact sequence

$$0 \to \mathcal{F} \otimes \mathcal{K}^* \to V \otimes \mathcal{K}^* \to \mathcal{K} \otimes \mathcal{K}^* \to 0 \, . \tag{A.3}$$

Then the associated long exact sequence in cohomology contains the terms

$$0 \to H^0(X, \mathcal{F} \otimes \mathcal{K}^*) \to H^0(X, V \otimes \mathcal{K}^*) \to H^0(X, \mathcal{K} \otimes \mathcal{K}^*) \to H^1(X, \mathcal{F} \otimes \mathcal{K}^*) \to \dots \tag{A.4}$$

Because V is stable for $\mu(\mathcal{F}) < 0$, it must follow that at a generic point in the bundle moduli space of V, $H^0(X, V \otimes \mathcal{K}^*) = 0$ (otherwise $\mathcal{K}$ would be a sub-sheaf of V and would destabilize V). Furthermore, since $\mathcal{K}$ is a line-bundle on a Calabi-Yau manifold, $H^0(X, \mathcal{K} \otimes \mathcal{K}^*) = 1$. As a result, we have

$$0 \to H^0(X, \mathcal{K} \otimes \mathcal{K}^*) \to H^1(X, \mathcal{F} \otimes \mathcal{K}^*) \to \dots \tag{A.5}$$

and it is clear that we *must* have $H^1(X, \mathcal{F} \otimes \mathcal{K}^*) \neq 0$ in order to avoid a contradiction. However, since the value of this cohomology is unaffected as we move to the decomposable locus in the moduli space of V (as described in section 3), we see that $H^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*) \neq 0$ is satisfied, as expected.

We turn now to the second cohomology condition that we must investigate. In order for the theory to break supersymmetry above the line with $\mu(\mathcal{F}) = 0$, it must be the case that $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) = 0$. This too follows immediately from the definition of a stability boundary. Suppose that $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) \neq 0$, then there exists a non-trivial extension:

$$0 \to V/\mathcal{F} \to \tilde{V} \to \mathcal{F} \to 0 \, . \tag{A.6}$$

But, by definition, this implies that there exists an injective map from $V/\mathcal{F}$ to $\tilde{V}$ at generic points in moduli space (away from the decomposable locus). Thus, we must ask, can $\tilde{V}$ be isomorphic to V? If this is the case, then $V/\mathcal{F}$ is a sub-sheaf of V that destabilizes V in the region $\mu(\mathcal{F}) < 0$. But by construction, we know that $\mathcal{F}$ destabilizes V in the region with $\mu(\mathcal{F}) > 0$, hence the bundle is stable nowhere in Kähler moduli space. This is a contradiction, since we are considering the situation in which the line $\mu(\mathcal{F}) = 0$ defines the boundary of a stable/unstable transition in the moduli space. Thus, if V is stable for $\mu(\mathcal{F}) < 0$, then the extension $\mathcal{E}xt^1(V/\mathcal{F}, \mathcal{F}) = H^1(X, \mathcal{F}^* \otimes V/\mathcal{F})$ defining $\tilde{V}$ is *not* isomorphic to V. Thus, if $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) \neq 0$ we are considering an effective theory in which branch structure is present, connecting more than one vector bundle. However, for the statement of this lemma, we shall consider only the effective theory describing V. □

Note that similar vanishing conjectures could be formulated for other possible bundle decompositions of V. However, in the presence of additional U(1) gauge fields and different decompositions of E_8 under these symmetries, each case must be investigated on an individual basis.

Next, we turn to the proof of the second lemma in section 5. It states the following:

Lemma II. *Let V be a holomorphic vector bundle with structure group* SU(n) *defined over X, a Calabi-Yau* 3*-fold with Kähler form J. If $\mathcal{F}$ is a rank $n-1$, stable sub-sheaf of V, defining a stability wall in in the Kähler cone given by $\mu(\mathcal{F}) = 0$, such that V is stable for $\mu(\mathcal{F}) < 0$ and unstable for $\mu(\mathcal{F}) > 0$ (and $V/\mathcal{F}$ is locally free), and $H^1(X, \mathcal{F}^* \otimes V/\mathcal{F}) = 0$, then*

$$h^1(X, V \otimes V^*) = h^1(X, \mathcal{F} \otimes (V/\mathcal{F})^*) + h^1(X, \mathcal{F} \otimes \mathcal{F}^*) - 1\,, \tag{A.7}$$

where $h^1(X, V \otimes V^*)$ is the generic dimension of bundle moduli space when V is a stable bundle.

Proof. Consider once again the short exact sequence (A.2) which defines the sub-sheaf $\mathcal{F}$. In order to relate the generic (stable) bundle moduli of V to the possible deformations of $\mathcal{F} \oplus \mathcal{K}$, we will compute $h^1(X, V \otimes V^*)$ using (A.2). To begin, we consider the following three short exact sequences that follow directly from (A.2).

$$0 \to \mathcal{F} \otimes V^* \to V \otimes V^* \to \mathcal{K} \otimes V^* \to 0 \tag{A.8}$$
$$0 \to \mathcal{F} \otimes \mathcal{K}^* \to \mathcal{F} \otimes V^* \to \mathcal{F} \otimes \mathcal{F}^* \to 0 \tag{A.9}$$
$$0 \to \mathcal{K} \otimes \mathcal{K}^* \to \mathcal{K} \otimes V^* \to \mathcal{K} \otimes \mathcal{F}^* \to 0 \tag{A.10}$$

From these sequences we can consider long exact sequences in cohomology. We begin with (A.9). Using the results of Lemma I, and the fact that for this class of examples V is stable for $\mu(\mathcal{F}) < 0$, we have $H^0(X, \mathcal{F} \otimes V^*) = 0$ and $H^2(X, \mathcal{F} \otimes \mathcal{K}^*) = 0$. Thus,

$$0 \to H^0(X, \mathcal{F} \otimes \mathcal{F}^*) \to H^1(X, \mathcal{F} \otimes \mathcal{K}^*) \to H^1(X, \mathcal{F} \otimes V^*) \to H^0(X, \mathcal{F} \otimes \mathcal{F}^*) \to 0\,. \tag{A.11}$$

Next, from (A.10), we note that since $\mathcal{K}$ is a line bundle, $\mathcal{K} \otimes \mathcal{K}^* \approx \mathcal{O}$ and hence, $H^1(X, \mathcal{K} \otimes \mathcal{K}^*) = 0$ and by Lemma I, we have that $H^1(X, \mathcal{K} \otimes \mathcal{F}^*) = 0$. Further, we have $H^0(X, \mathcal{K} \otimes \mathcal{F}^*) = 0$ since $\mathcal{F}$ is stable. Hence, it follows that

$$h^0(X, \mathcal{K} \otimes V^*) = 1 \quad \text{and} \quad h^1(X, \mathcal{K} \otimes V^*) = 0\,. \tag{A.12}$$

Substituting this information into the cohomology sequence for (A.8), we find

$$0 \to H^0(X, V \otimes V^*) \to H^0(X, \mathcal{K} \otimes V^*) \to H^1(X, \mathcal{F} \otimes V^*) \to H^1(X, V \otimes V^*) \to 0 \tag{A.13}$$

Then, in terms of dimensions:

$$h^1(X, V \otimes V^*) = h^0(X, V \otimes V^*) - h^0(X, \mathcal{K} \otimes V^*) + h^1(X, \mathcal{F} \otimes V^*) \tag{A.14}$$

and upon substitution

$$h^0(X, V \otimes V^*) - 1 + h^0(X, \mathcal{F} \otimes \mathcal{F}) + h^1(\mathcal{F} \otimes \mathcal{F}) + h^1(X, \mathcal{F} \otimes \mathcal{K}^*)\,. \tag{A.15}$$

Finally, since V and $\mathcal{F}$ are stable $h^0(X, V \otimes V^*) = 1 = h^1(X, \mathcal{F} \otimes \mathcal{F}^*)$ and we arrive at the result,

$$h^1(X, V \otimes V^*) = h^1(X, \mathcal{F} \otimes \mathcal{K}^*) + h^1(\mathcal{F} \otimes \mathcal{F}^*) - 1 \tag{A.16}$$

as required. □

We end this section by a statement of a conjecture. This is less easy to verify than the cohomology conditions described above, though we have found it to be true in all the cases that we have investigated. The central result of this paper is the form of the FI D-term given in (4.20) which reproduces the notion of vector bundle stability for a supersymmetric, anomaly free[16] bundle. As discussed in section 5, the form of this potential clearly does not depend on the complex structure moduli of the Calabi-Yau manifold X. In addition, using the techniques of section 3, we have searched through numerous examples, and have yet to find a complex structure dependent boundary wall for an anomaly free bundle. As a result, we posit the conjecture:

Conjecture. *Let V be an anomaly-free holomorphic vector bundle with structure group* SU(n) *($n = 3, 4, 5$) defined over X, a Calabi-Yau 3-fold. If there exists a wall of semi-stability of V in Kähler moduli space (defining the boundary between stable and unstable regions), then the position of this wall is independent of the complex structure moduli of X.*

This conjecture is a consequence of our field-theoretical approach to slope-stability but it is not obvious to the authors how to prove it from an algebraic geometry viewpoint.

B Another example

To highlight the versatility of the formalism developed in this paper, in this section we will sketch another example bundle, its regions of stability in the Kähler cone and the effective field theory modeling this behavior.

We shall once again consider a bundle defined on a complete intersection Calabi-Yau manifold, X. The so-called 'bi-cubic' 3-fold:

$$X = \left[\begin{array}{c|c} \mathbb{P}^2 & 3 \\ \mathbb{P}^2 & 3 \end{array} \right] , \tag{B.1}$$

defined by a polynomial of bi-degree $(3, 3)$ in the ambient space $\mathbb{P}^2 \times \mathbb{P}^2$. As in our previous example, $h^{1,1}(X) = 2$ and the Kähler cone is the positive quadrant $t^1 \geq 0$ and $t^2 \geq 0$. The non-zero triple intersection numbers are given by $d_{122} = 3$ and $d_{112} = 3$. It follows that the dual Kähler moduli s_1 and s_2 are

$$s_1 = 3t^2(2t^1 + t^2) , \quad s_2 = 3t^1(2t^2 + t^1) . \tag{B.2}$$

[16] Recall that a vector bundle V in the $E_8 \times E_8$ heterotic theory defines an anomaly free superymmetric theory if $ch_2(TX) - ch_2(V) = W$ where W is an effective class of X [2, 46, 47]. This condition is necessary here, as without it anti five-branes or a non-supersymmetric hidden bundle would be required to make the reduction from eleven to four dimensions consistent. This would result in a theory which was not supersymmetric in four dimensions [48, 49], and as such the analysis of this paper would not apply.

Hence, the dual Kähler cone in this case is the entire positive quadrant. We shall define line bundles $\mathcal{O}_X(m,n)$ on this space using the same notation as in section 3.1.

On the given manifold, we define a bundle, V, by extension [50],

$$0 \to W \to V \to \mathcal{L} \to 0\,, \tag{B.3}$$

where $\mathcal{L}$ is a line bundle and W is a rank 2, U(2) monad bundle defined as follows

$$0 \to W \to \mathcal{O}_X(2,0)^{\oplus 3} \to \mathcal{O}_X(2,2) \to 0 \tag{B.4}$$
$$\mathcal{L} = \mathcal{O}_X(-4,2).$$

Since the first chern classes of $\mathcal{L}$ and W satisfy

$$c_1(W) = -c_1(\mathcal{L}) \tag{B.5}$$

the extension bundle V defined by (B.3) has $c_1(V) = 0$ and hence defines an SU(3) bundle. Furthermore, V is a non-trivial extension of $\mathcal{L}$ by W (i.e. V is not simply the sum $W \oplus \mathcal{L}$ since $\mathcal{E}xt^1(\mathcal{L}, W) \neq 0$). The spectrum of the four dimensional E_6 theory associated to V consists of 18 **27** matter fields and 18 $\overline{\mathbf{27}}$'s for a net chiral asymmetry of zero. In addition, there are generically $h^1(X, V \otimes V^*) = 530$ bundle moduli.

We can now ask, what are the regions of stability of V in the Kähler cone? A simple analysis using the techniques of section 2.1.2 verifies first that W is an everywhere stable U(2) bundle, and furthermore, that W is generically the *only* de-stabilizing sub-sheaf of V. Thus, since $c_1(W) = 4J_1 - 2J_2$, V itself is stable above the line with slope $s_2/s_1 = 2$ and unstable beneath it. We will now reproduce this geometric result from the point of view of the effective field theory developed in this work.

As was argued in section 3, at the line of semi-stablility in the dual Kähler cone defined by $s_2 = 2s_1$, V will be forced away from an SU(3) configuration towards the structure group $S(\mathrm{U}(2) \times \mathrm{U}(1))$ (and the four dimensional symmetry will be enhanced to $E_6 \times \mathrm{U}(1)$). As in the example given in 5.1, V decomposes as $V = \mathcal{F} \oplus \mathcal{K}$ where in this case $\mathcal{F} = W$ and $\mathcal{K} = \mathcal{L}$ (as defined above in (B.4)). Note that the split locus is simply the zero of the group $\mathcal{E}xt^1(\mathcal{L}, W)$ which describes the space of possible extensions. Using the results of [16, 27, 28] to compute the cohomology of W and $\mathcal{L}$ on the bi-cubic, and the representation decomposition given in table 1, we find that non-vanishing massless spectrum of V at the decomposable locus is given by

$$h^1(X, W)_{-1/2} = 18 \qquad h^1(X, \mathcal{L}^*)_{-1} = 18\,, \tag{B.6}$$
$$h^1(X, W \otimes W)_0 = 9 \quad h^1(X, W \otimes \mathcal{L}^*)_{-3/2} = 522\,. \tag{B.7}$$

The subscript on the cohomology denotes the U(1) charge of the fields. We may now write down the U(1) D-term (4.20) contribution to the potential

$$D^{\mathrm{U}(1)} = \frac{3}{16}\frac{\epsilon_S \epsilon_R^2}{\kappa_4^2}\frac{\mu(W)}{\mathcal{V}} + \frac{3}{2}\sum_{L,\bar{M}=1}^{16} G_{L\bar{M}} C^L \bar{C}^{\bar{M}}\,, \tag{B.8}$$

where here, from (B.4), the slope is given by

$$\mu(W) = \frac{1}{2}(4s_1 - 2s_2) \tag{B.9}$$

while for the volume we have

$$\mathcal{V} = \frac{3}{2}(t^1 t^2)(t^1 + t^2)\ . \tag{B.10}$$

The relevant charged matter fields C^L in this case are the fields in $H^1(X, W \otimes \mathcal{L}^*)$, since the fields associated to $H^1(X, W)$ and $H^1(X, \mathcal{L}^*)$ will have vevs forced to zero by the requirement that E_6 remains unbroken. As we would predict based from the algebro-geometric results of the stability analysis, negatively charged matter is present so that the vevs of the charged fields can adjust to cancel the FI term when $\mu(\mathcal{F}) < 0$, setting the D-term to zero. Thus, supersymmetry is preserved for the region of dual Kähler moduli space defined by $s_2 > 2s_1$. However, since there is no positively charged matter available, for the region of moduli space where $\mu(\mathcal{F}) > 0$, the FI term cannot be cancelled and supersymmetry is broken. This is in agreement with what we would expect from the general results of Lemma 1.

Finally, for this example, we can verify the general predictions of Lemma 2 by considering the number of bundle moduli associated to V at a generic point in its moduli space as well as at the decomposable locus. According to Lemma 2, we would expect there to be one extra light modulus at the stability wall. For the bundle V defined by (B.3), at a generic point in its moduli space, $h^1(X, V \otimes V^*) = 530$. Moreover, using the results of (B.6) we observe that at the decomposable locus, the number of bundle moduli is given by $h^1(X, W \otimes W^*) + h^1(X, W \otimes \mathcal{L}^*) = 531$. Thus, as described in section 5, as we move in Kähler moduli space away from the stability wall, one degree of freedom is made massive by the Higgs mechanism, (5.3), as expected.

References

[1] P. Candelas, G.T. Horowitz, A. Strominger and E. Witten, *Vacuum configurations for superstrings*, *Nucl. Phys.* **B 258** (1985) 46 [SPIRES].

[2] M.B. Green, J.H. Schwarz and E. Witten, *Superstring theory. Vol. 2: Loop amplitudes, anomalies and phenomenology*, Cambridge Monographs On Mathematical Physics, Cambridge University Press, Cambridge U.K. (1987), pg. 596 [SPIRES].

[3] K. Uhlenbeck and S.-T. Yau, *On the existence of Hermitian Yang-Mills connections in stable bundles*, *Comm. Pure App. Math.* **39** (1986) 257.

[4] S. Donaldson, *Anti self-dual Yang-Mills connections over complex algebraic surfaces and stable vector bundles*, *Proc. London Math. Soc.* **50** (1985) 1 [SPIRES].

[5] E.R. Sharpe, *Kähler cone substructure*, *Adv. Theor. Math. Phys.* **2** (1999) 1441 [hep-th/9810064] [SPIRES].

[6] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, *The edge of supersymmetry: stability walls in heterotic theory*, *Phys. Lett.* **B 677** (2009) 190 [arXiv:0903.5088] [SPIRES].

[7] S.K. Donaldson, *Some numerical results in complex differential geometry*, math.DG/0512625.

JHEP09(2009)026

[8] M.R. Douglas, R.L. Karp, S. Lukic and R. Reinbacher, *Numerical solution to the hermitian Yang-Mills equation on the Fermat quintic*, *JHEP* **12** (2007) 083 [hep-th/0606261] [SPIRES].

[9] M.R. Douglas, R.L. Karp, S. Lukic and R. Reinbacher, *Numerical Calabi-Yau metrics*, *J. Math. Phys.* **49** (2008) 032302 [hep-th/0612075] [SPIRES].

[10] V. Braun, T. Brelidze, M.R. Douglas and B.A. Ovrut, *Calabi-Yau metrics for quotients and complete intersections*, *JHEP* **05** (2008) 080 [arXiv:0712.3563] [SPIRES].

[11] M. Dine, N. Seiberg, X.G. Wen and E. Witten, *Nonperturbative effects on the string world sheet*, *Nucl. Phys.* **B 278** (1986) 769 [SPIRES].

[12] M. Dine, N. Seiberg and E. Witten, *Fayet-Iliopoulos terms in string theory*, *Nucl. Phys.* **B 289** (1987) 589 [SPIRES].

[13] J. Distler and B.R. Greene, *Aspects of (2,0) string compactifications*, *Nucl. Phys.* **B 304** (1988) 1 [SPIRES].

[14] A. Lukas and K.S. Stelle, *Heterotic anomaly cancellation in five dimensions*, *JHEP* **01** (2000) 010 [hep-th/9911156] [SPIRES].

[15] R. Blumenhagen, G. Honecker and T. Weigand, *Loop-corrected compactifications of the heterotic string with line bundles*, *JHEP* **06** (2005) 020 [hep-th/0504232] [SPIRES].

[16] L.B. Anderson, *Heterotic and M-theory compactifications for string phenomenology*, arXiv:0808.3621 [SPIRES].

[17] L.B. Anderson, Y.H. He and A. Lukas, *Vector bundle stability in heterotic monad models*, to appear.

[18] H. Hoppe, *Generischer spaltungstypumun zweite Chernklassesstabiler Vektorraumbundel vom rang 4 auf P4*, *Math. Z.* **187** (1884) 345.

[19] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*, Springer Verlag, New York U.S.A. (1991).

[20] R. Donagi, B.A. Ovrut, T. Pantev and R. Reinbacher, SU(4) *instantons on Calabi-Yau Threefolds with $Z_2 \times Z_2$ fundamental group*, *JHEP* **01** (2004) 022 [hep-th/0307273] [SPIRES].

[21] S. Kobayashi, *Differential geometry of complex vector bundles*, Princeton University Press, Princeton U.S.A. (1987).

[22] D. Grayson, private communication.

[23] R. Hartshorne, *Algebraic geometry, Springer*, GTM 52, Springer-Verlag (1977); P. Griffith and J. Harris, *Principles of algebraic geometry*, (1978).

[24] D. Huybrechts and M. Lehn, *The geometry of the moduli space of stable of sheaves*, Aspects of Mathematics, E 31 (1997).

[25] S. Bradlow, *Hermitian-Einstein inequalities and Harder-Narasimhan filtrations*, alg-geom/9506008.

[26] T. Hubsch, *Calabi-Yau manifolds — A bestiary for physicists*, World Scientific, Singapore (1994).

[27] L.B. Anderson, Y.-H. He and A. Lukas, *Heterotic compactification, an algorithmic approach*, *JHEP* **07** (2007) 049 [hep-th/0702210] [SPIRES].

JHEP09(2009)026

[28] L.B. Anderson, Y.-H. He and A. Lukas, *Monad bundles in heterotic string compactifications*, *JHEP* **07** (2008) 104 [arXiv:0805.2875] [SPIRES].

[29] L.B. Anderson, J. Gray, D. Grayson, Y.-H. He and A. Lukas, *Yukawa couplings in heterotic compactification*, arXiv:0904.2186 [SPIRES].

[30] R. Slansky, *Group theory for unified model building*, *Phys. Rept.* **79** (1981) 1 [SPIRES].

[31] E. Witten, *Strong coupling expansion of Calabi-Yau compactification*, *Nucl. Phys.* **B 471** (1996) 135 [hep-th/9602070] [SPIRES].

[32] A. Lukas, B.A. Ovrut and D. Waldram, *On the four-dimensional effective action of strongly coupled heterotic string theory*, *Nucl. Phys.* **B 532** (1998) 43 [hep-th/9710208] [SPIRES].

[33] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *The universe as a domain wall*, *Phys. Rev.* **D 59** (1999) 086001 [hep-th/9803235] [SPIRES].

[34] R. Donagi, A. Lukas, B.A. Ovrut and D. Waldram, *Holomorphic vector bundles and non-perturbative vacua in M-theory*, *JHEP* **06** (1999) 034 [hep-th/9901009] [SPIRES].

[35] A. Lukas, B.A. Ovrut, K.S. Stelle and D. Waldram, *Heterotic M-theory in five dimensions*, *Nucl. Phys.* **B 552** (1999) 246 [hep-th/9806051] [SPIRES].

[36] A. Lukas, B.A. Ovrut and D. Waldram, *Non-standard embedding and five-branes in heterotic M-theory*, *Phys. Rev.* **D 59** (1999) 106005 [hep-th/9808101] [SPIRES].

[37] J. Wess and J. Bagger, *Supersymmetry and supergravity*, Princeton University Press, Princeton U.S.A. (1992), pg. 259 [SPIRES].

[38] L.B. Anderson, J. Gray, A. Lukas and B. Ovrut, to appear.

[39] G.-M. Greuel, G. Pfister and H. Schönemann, *Singular: a computer algebra system for polynomial computations*, Centre for Computer Algebra, University of Kaiserslautern, Germany (2001), available at http://www.singular.uni-kl.de/;
D. Grayson and M. Stillman, *Macaulay 2, a software system for research in algebraic geometry*, available at http://www.math.uiuc.edu/Macaulay2/;
J. Gray, Y.-H. He, A. Ilderton and A. Lukas, *STRINGVACUA: a mathematica package for studying vacuum configurations in string phenomenology*, *Comput. Phys. Commun.* **180** (2009) 107 [arXiv:0801.1508] [SPIRES]; *A new method for finding vacua in string phenomenology*, *JHEP* **07** (2007) 023 [hep-th/0703249] [SPIRES];
J. Gray, Y.H. He and A. Lukas, *Algorithmic algebraic geometry and flux vacua*, *JHEP* **09** (2006) 031 [hep-th/0606122] [SPIRES];
J. Gray, *A simple introduction to Grobner basis methods in string phenomenology*, arXiv:0901.1662 [SPIRES].

[40] J. Gray and A. Lukas, *Gauge five brane moduli in four-dimensional heterotic models*, *Phys. Rev.* **D 70** (2004) 086003 [hep-th/0309096] [SPIRES];
J. Gray, *An explicit example of a moduli driven phase transition in heterotic models*, *Phys. Rev.* **D 72** (2005) 066004 [hep-th/0406241] [SPIRES].

[41] M.R. Douglas, B. Fiol and C. Romelsberger, *Stability and BPS branes*, *JHEP* **09** (2005) 006 [hep-th/0002037] [SPIRES].

[42] M. Brändle, *Aspects of branes in (heterotic) M-theory*, Dissertation, Humboldt-Universität zu Berlin, Germany.

[43] M. Brandle and A. Lukas, *Five-branes in heterotic brane-world theories*, *Phys. Rev.* **D 65** (2002) 064024 [hep-th/0109173] [SPIRES].

[44] R. Blumenhagen, S. Moster and T. Weigand, *Heterotic GUT and standard model vacua from simply connected Calabi-Yau manifolds*, *Nucl. Phys.* **B 751** (2006) 186 [hep-th/0603015] [SPIRES].

[45] V. Braun and B.A. Ovrut, *Stabilizing moduli with a positive cosmological constant in heterotic M-theory*, *JHEP* **07** (2006) 035 [hep-th/0603088] [SPIRES].

[46] M.R. Douglas, R. Reinbacher and S.-T. Yau, *Branes, bundles and attractors: bogomolov and beyond*, math/0604597 [SPIRES].

[47] F. Bogomolov, *Holomorphic tensors and vector bundles on projective varieties*, *Math. USSR Izv.* **13** (1979) 499.

[48] J. Gray, A. Lukas and B. Ovrut, *Flux, gaugino condensation and anti-branes in heterotic M-theory*, *Phys. Rev.* **D 76** (2007) 126012 [arXiv:0709.2914] [SPIRES].

[49] J. Gray, A. Lukas and B. Ovrut, *Perturbative anti-brane potentials in heterotic M-theory*, *Phys. Rev.* **D 76** (2007) 066007 [hep-th/0701025] [SPIRES].

[50] R. Donagi, Y.-H. He, B.A. Ovrut and R. Reinbacher, *Moduli dependent spectra of heterotic compactifications*, *Phys. Lett.* **B 598** (2004) 279 [hep-th/0403291] [SPIRES]; *The spectra of heterotic standard model vacua*, *JHEP* **06** (2005) 070 [hep-th/0411156] [SPIRES];
V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *A heterotic standard model*, *Phys. Lett.* **B 618** (2005) 252 [hep-th/0501070] [SPIRES]; *Vector bundle extensions, sheaf cohomology and the heterotic standard model*, *Adv. Theor. Math. Phys.* **10** (2006) 4 [hep-th/0505041] [SPIRES];
V. Braun, Y.-H. He and B.A. Ovrut, *Stability of the minimal heterotic standard model bundle*, *JHEP* **06** (2006) 032 [hep-th/0602073] [SPIRES];
V. Braun, Y.-H. He, B.A. Ovrut and T. Pantev, *The exact MSSM spectrum from string theory*, *JHEP* **05** (2006) 043 [hep-th/0512177] [SPIRES].