BOUNDARY VALUE PROBLEMS

IN

QUANTUM COSMOLOGY

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This essay contains a part of the work done during my first four terms of research for the Ph. D. degree. The dissertation is original when no reference is made to the work of other authors, and it is not the same as any work that has been submitted for consideration for any degree, diploma or other qualification at any other University. The research was carried out in the Department of Applied Mathematics and Theoretical Physics of the University of Cambridge (U. K.), financially supported by St. John's College.

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CONTENTS

Chapter One

INTRODUCTION TO QUANTUM GRAVITY AND QUANTUM COSMOLOGY

1.1 Quantum Gravity : Various Approaches, Achievements and Unsolved Problems

This dissertation deals with quantum cosmology, where one applies some mathematical techniques of interest to quantum gravity so as to get a better understanding of the early universe. It is therefore very important to emphasize some basic points about the problem of quantum gravity at the very beginning of our work. In our opinion, the two main motivations for studying the quantum gravity problem are the following :

(a) the singularity theorems of Penrose, Hawking and Geroch show that Einstein's general relativity leads to the occurrence of singularities in cosmology in a rather generic way (Hawking 1966a, b, 1967, Geroch 1967, Hawking and Penrose 1970, Hawking and Ellis 1973). One might therefore define the quantum gravity era as the one when "physics" is confined to a region whose linear size is of the order of 10^{-33} cm. In other words we are asking the questions : is there a theory which describes gravitational interactions on these length scales ? Does this theory lead to the avoidance of singularities in a generic way ?

(b) the electroweak and strong interactions are described by renormalizable quantum field theories (Warr 1988). However, Einstein's general relativity cannot be renormalized

(Duff 1982). Some authors (De Witt 1964) tried to rearrange and sum infinite subclasses of Feynman graphs, but in so doing the effective propagators may be shown to pick up new poles which destroy unitarity (Warner 1982).

In order to study these problems, many efforts have been produced so far. The main approaches seem to be : (1) covariant (De Witt 1964, 1967b, c); (2) canonical (De Witt 1967a, Ashtekar 1988); (3) path integral (Hawking 1979a, b); (4) asymptotic quantization (Ashtekar 1987); (5) quantization of supergravity (van Niewvenhuizen 1981, Warner 1982, D'Eath 1984); (6) higher derivative theories (Stelle 1977, Barth and Christensen 1983, Boulware 1984); (7) lattice theories (Menotti and Pelissetto 1987); (8) application of Regge calculus (Rocek and Williams 1982, Warner 1982); (9) string theories (Green, Schwarz and Witten 1987); (10) twistor theory (Penrose 1975, 1987). A full account of all these approaches would require by itself a book, but a few important comments on some ideas can be made here.

(I) Using the Arnowitt-Deser-Misner formalism for general relativity (Misner, Thorne and Wheeler 1973, Mac Callum 1975, Regge et al. 1976), one makes a $3+1$ split of the spacetime metric, which may be locally cast in the form :

$$ds^2 = -(N^2 - N_i N^i)dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j \tag{1.1.1}$$

Using the Gauss-Codazzi equations (Lightman et al. 1974, Regge et al. 1976), one finds that the action for general relativity (York 1972, Gibbons and Hawking 1977, York 1986)

$$16\pi G I_g = \int_M R\sqrt{-g}\,d^4x + 2\int_{\partial M} \chi\sqrt{h}\,d^3x \tag{1.1.2}$$

gives rise to the Lagrangian :

$$L = \int_{\partial M} N\sqrt{h}\left(\chi_{ij}\chi^{ij} - \chi^2 + {}^{(3)}R\right) d^3x \tag{1.1.3}$$

Thus one finds the primary constraints (De Witt 1967a) : $\pi = \frac{\delta L}{\delta N} = 0$, $\pi^i = \frac{\delta L}{\delta N_i} = 0$. Requiring the preservation in time of these constraints (Dirac 1964), one finds the secondary constraints :

$$H = \sqrt{h}\left(\chi_{ij}\chi^{ij} - \chi^2 - {}^{(3)}R\right) = 0 \tag{1.1.4}$$

$$H^i = -2\pi^{ij}{}_{,j} - h^{il}\left(2h_{jl,k} - h_{jk,l}\right)\pi^{jk} = 0 \tag{1.1.5}$$

where $\pi^{ij} = \frac{\delta L}{\delta h_{ij}} = -\sqrt{h}\left(\chi^{ij} - h^{ij}\chi\right)$. The constraint (1.1.4) is called Hamiltonian constraint. On quantization, Poisson brackets become commutators, and the constraint equations become conditions on the state vector ψ (De Witt 1967a) :

$$\pi\psi = 0 \quad \pi^i\psi = 0 \tag{1.1.6}$$

$$H\psi = 0 \quad H^i\psi = 0 \tag{1.1.7}$$

Now, it is indeed true that the $3+1$ split of the metric may seem contrary to the whole spirit of relativity (Hawking 1979b, Ashtekar 1987). It is also true that the underlying manifold structure has been assumed to be R x Σ (where Σ is a three-manifold) and usually kept fixed (see, however, Ashtekar 1987), whereas one would expect quantum gravity to allow also for those topologies which are not a product (Hawking 1979b). But the main problem is due to the difficulty in solving the quantum constraints. In fact, the equation $H\psi = 0$ (the Wheeler-De Witt equation) is an equation on a space, called superspace (Fisher 1970, Francaviglia 1975), whose points are equivalence classes of metrics related each other by the action of the diffeomorphism group of a compact spacelike three-surface. More precisely, the superspace $S(M)$ is defined as : $S(M) = Riem(M)/Diff(M)$. In this notation, M is a compact, connected, orientable, Hausdorff C^{∞} three-manifold without boundary. $Riem(M)$ is the space of C^{∞} Riemannian metrics on M, and $Diff(M)$ is the group of C^{∞} orientation preserving diffeomorphisms of M. Thus one has to deal with an infinite number of degrees of freedom, and in addition operator ordering problems are found to arise, because the Hamiltonian H is a quadratic function of the momenta π^{ij}. Later on, we shall see how supergravity can be cast in Hamiltonian form, and how this formalism can be used so as to study quantum cosmological problems. However, it is to be recalled that progress has been done in the last few years in our understanding of the canonical structure of general relativity owing to Ashtekar (Ashtekar 1988). In his new formalism, the space-time metric is a secondary object, while the new configuration variable is a $SU(2)$ connection on a three-manifold. The momentum conjugate to this variable is a $SU(2)$ soldering form which turns internal $SU(2)$ indices into $SU(2)$ spinor indices. Therefore we are now able to use in quantum gravity some techniques which were already very useful for other gauge theories, and there is now hope to solve the quantum constraints in a non perturbative way. It is especially remarkable that, in terms of the Ashtekar's variables, the constraints are polynomial.

(II) The basic postulate of the path integral approach to quantum gravity (Hawking 1979a, b) is that the probability amplitude of going from a 3-metric h_{ij} and a matter field configuration ϕ on a spacelike surface Σ to a 3-metric h'_{ij} and a field configuration ϕ' on a spacelike surface Σ' is given formally by :

$$< h'_{ij}, \phi' \mid h_{ij}, \phi >= \int_{C'} D\left[g_{\mu\nu}\right] D\left[\Phi\right] e^{iI[g_{\mu\nu},\Phi]} \tag{1.1.8}$$

where C' is the class of all 4-metrics inducing h_{ij} on Σ and h'_{ij} on Σ', and of all field configurations matching ϕ on Σ and ϕ' on Σ'. In computing the amplitude (1.1.8), we have to fix the gauge, and it is worth discussing this problem. In so doing we shall closely follow Teitelboim 1983. The Hamiltonian of the action for the gravitational field in a closed universe (here studied for simplicity) is :

$$I = \int_{t_1}^{t_2} \int d^3x \left(\pi^{ij} \dot{h}_{ij} - N^{\perp} H_{\perp} - N^i H_i \right) \tag{1.1.9}$$

where $N^{\perp}$ is equal to $g^{-\frac{1}{2}}$ times the usual lapse N. Denoting by $\{\ ,\ \}$ the Poisson brackets, we have that the action (1.1.9) is invariant under the gauge transformation :

$$\delta h_{ij} = \{h_{ij}, H[\epsilon^{\mu}]\} \tag{1.1.10}$$

$$\delta \pi^{ij} = \{\pi^{ij}, H[\epsilon^{\mu}]\} \tag{1.1.11}$$

plus a more complicated relation involving $\delta N^{\perp}$ and δN^{i}, whose form is not strictly needed here. In (1.1.10-11), one has :

$$H[\epsilon^{\mu}] = \int d^3x \left(\epsilon^{\perp} H_{\perp} + \epsilon^{i} H_{i}\right) \tag{1.1.12}$$

with the boundary condition :

$$\epsilon^{\perp}(x,t_2) = \epsilon^{\perp}(x,t_1) = 0 \tag{1.1.13}$$

Moreover, in view of the fact that we are fixing the three-geometries at the end points, we also have to require that :

$$\epsilon^{i}(x,t_2) = \epsilon^{i}(x,t_1) = 0 \tag{1.1.14}$$

Let us now consider the intervals $I_1 =]t_1, t^{'}[$ and $I_2 =]t^{'}, t_2[$, and let us require that :

$$\dot{N}^{\perp} = 0 \quad N^{i} = 0 \quad \forall\, t \in I_1 \tag{1.1.15}$$

$$\dot{N}^{i} = 0 \quad N^{\perp} = 0 \quad \forall\, t \in I_2 \tag{1.1.16}$$

These conditions imply that $N^{\perp} = 0 \ \forall t > t^{'}$. However, in so doing we allow for an adjustable change of spatial coordinates during I_2. Namely, the dependence of N^{i} on x is not fixed during I_2, which in turn allows to set any given coordinate system on the final surface $\Sigma^{'}$. Another way of fixing the gauge, together with a detailed study of the ghost action and of the path integral can be found in Teitelboim 1983 as well. The application of the ghost formalism to quantum cosmology can be found in Laflamme 1988 and Halliwell 1988. Following Hawking, we now assume that an analytical continuation to the Euclidean regime (where $\tau = it$) is possible, so as to deal with probability amplitudes of the form :

$$< h'_{ij}, \phi' \mid h_{ij}, \phi >= \int_C D\left[g_{\mu\nu}\right] D\left[\Phi\right] e^{-I_E\left[g_{\mu\nu},\Phi\right]} \tag{1.1.17}$$

where I_E is the action integral for gravitation and matter fields in the Euclidean regime, and where the metrics belonging to the class C are now required to be Euclidean, namely positive-definite. Indeed, there is not general agreement on this approach because it has several drawbacks. In fact :

(i) the measure in (1.1.8) has not yet been given a rigorous mathematical meaning (Allen 1983, Glimm and Jaffe 1987). In particular, translational invariance in Feynman-type integrals may be violated (see Tarski 1980 and literature given therein);

(ii) there are topological obstructions to a Wick rotation if the Euler number is not zero (Warner 1982). Therefore, it is very difficult to perform a meaningful analytical continuation from the Lorentzian to the Euclidean regime, or viceversa (Hawking 1984a);

(iii) in defining formally (1.1.17), we are also summing over all possible topologies. This idea may seem very appealing, but there is no proof that a spacetime foam (Hawking 1979a, b) exists. Indeed, some authors (De Witt and Anderson 1986) disagree with Hawking about the idea that the spacetime topology might change;

(iv) the Euclidean action of the gravitational field is not positive-definite (Hawking 1979b). Thus, the integrand on the right-hand side of (1.1.17) is not exponentially damped, but it may blow up exponentially (see, however : Gibbons, Hawking and Perry 1978, Schleich 1987).

Despite all these problems, at least two very important results have been obtained using path integrals :

(i) the thermodynamic properties of black holes have been derived in a very powerful way (Gibbons and Hawking 1977). It is to be emphasized that black holes provide us with the first example in physics where entropy is related to a purely geometric quantity such as the area of the event horizon (Birrell and Davies 1982);

(ii) gravitational instantons are complete non singular positive-definite solutions of the Euclidean Einstein field equations $R_{\mu\nu} = \Lambda g_{\mu\nu}$. They play an important role in quantum gravity because thay are assumed to give the dominant contribution to the calculation of the right-hand side of (1.1.17). Compact, asymptotically flat and asymptotically Euclidean instantons were studied in detail (Hawking 1979b, Pope 1980, 1981). The fixed points of the Euclidean action are of two types : isolated points called nuts and 2-surfaces called bolts. Gibbons and Hawking were also able to relate numbers and types of nuts and bolts to the topological invariants of the instantons. Moreover, they worked out a formula for the gravitational action of the instantons in terms of the areas of the bolts and certain nut charges and potentials (Gibbons and Hawking 1979).

The path integral approach is also the most used in studying cosmological problems. This will be discussed in the next section. Here we just wish to remark that maybe the most "revolutionary" approach to quantum gravity is still the one based on twistors. When we talk about quantum gravity, we might think space-time points remain well-defined whereas the spacelike, null or timelike nature of a vector is subject to quantum uncertainties in view of a "quantization" of the metric tensor. But in twistor theory the situation is reversed. What is well defined is the concept of null direction, whereas the concept of space-time point becomes fuzzy (Penrose 1975). Twistors in flat space are defined in some detail in section 3.6. A lot of progress has been done in twistor theory over the past fifteen years (Huggett and Tod 1985, Penrose 1986), but unfortunately it is not yet clear how to apply twistor theory in getting a consistent theory of quantum gravity.

1.2 Quantum Cosmology : Motivations and Some Recent Developments

The aim of quantum cosmology is to study the early universe and its evolution so as to shed new light on the following problems : (1) boundary conditions in cosmology (Hawking 1982); (2) singularity problem (Hawking 1984a); (3) other problems of the hot big bang model (Hawking 1984a, Linde 1984, Brandenberger 1985); (4) origin of structure in the universe (Halliwell and Hawking 1985); (5) large scale properties of the observed universe (Hawking and Luttrell 1984a, b, Amsterdamski 1985); (6) arrow of time in cosmology (Hawking 1985, Page 1985, Laflamme 1988).

A review of all problems studied so far cannot be attempted here, but the basic ideas and techniques must be briefly recalled. Let us assume for simplicity that only scalar fields are present (see Concluding Remarks about this point). The basic object in quantum cosmology is a wave functional ψ. In a closed universe, this functional can only depend on the three-geometry and on the matter field configuration on a compact spacelike three-surface (De Witt 1967a, Hartle and Hawking 1983). It is given by the Euclidean path integral :

$$\psi\left[h_{ij},\phi_0\right]=\int_C D\left[g_{\mu\nu}\right] D\left[\Phi\right] e^{-I_E\left[g_{\mu\nu},\Phi\right]} \tag{1.2.1}$$

One can show (Hawking 1984a) that the right-hand side of (1.2.1) satisfies the Wheeler-De Witt equation :

$$\left[-G_{ijkl}\frac{\delta^2}{\delta h_{ij}\delta h_{kl}}+\sqrt{h}\left(-{}^{(3)}R+16\pi T_{00}\left(\frac{\delta}{\delta\phi},\phi\right)\right)\right]\psi\left[h_{ij},\phi\right]=0 \tag{1.2.2}$$

and the momentum constraints :

$$\left(\frac{\delta\psi}{\delta h_{ij}}\right)_{|i}=8\pi T^{0j}\psi \tag{1.2.3}$$

More precisely, also this one is a delicate point, and a partial proof of this result has been given only recently in Halliwell 1988. In (1.2.2), $G_{ijkl}=\frac{1}{2\sqrt{h}}\left(h_{ik}h_{jl}+h_{il}h_{jk}-h_{ij}h_{kl}\right)$, and (1.2.3) just expresses the fact that ψ does not depend on the particular three-metric (Hawking 1984a). We may think that (1.2.1) is obtained from (1.1.17) by letting the initial three-surface shrink to zero (Horowitz 1985). Thus ψ would be the amplitude for the universe to appear from "nothing" (Hartle and Hawking 1983). However, the interpretation of ψ is an open question, and is still receiving careful consideration (Halliwell 1987). Nobody knows how to solve exactly the Wheeler-De Witt equation on superspace, but in the last few years progress has been made in choosing the boundary conditions and computing ψ in approximate models.

It is indeed clear that, in computing (1.2.1), we must specify the class C of four-metrics and matter fields involved in the path integral. To this purpose, Hartle and Hawking (1983) made the following proposal. The path integral (1.2.1) should be taken only over all compact Euclidean four-metrics which induce h_{ij} on Σ, and all matter field configurations which are regular on the compact four-manifolds having Σ as their only boundary. It is to be emphasized that also alternative proposals have been made (see for example Vilenkin 1986, 1988). Moreover, an usual criticism of quantum field theorists is that regular matter field configurations should not dominate in the path integral, because they have zero measure. However, in this essay we shall try to work out some consequences of the Hartle-Hawking proposal, and thus it is worth summing up the original argument of these authors (Esposito 1988). One is assuming the dominant contribution to (1.2.1) is given by solutions of the Euclidean Einstein field equations : $R_{\mu\nu} = \Lambda g_{\mu\nu}$. When Λ is > 0, the four-metric is to be compact. In fact, a theorem due to Myers (Milnor 1962) states that in an n-dimensional Riemannian manifold M whose Ricci curvature is $\geq \frac{(n-1)}{r}$ (r being a positive constant) everywhere, every geodesic whose length is $> \pi\sqrt{r}$ contains conjugate points and hence is not minimal. Moreover, if M is assumed to be geodesically complete, the Hopf-Rinow theorem states that any two points of M can be joined by a minimal geodesic. We also know that in the geodesically complete manifolds any closed bounded set is compact. Thus, in view of these theorems, a Riemannian geodesically complete four-dimensional manifold M whose Ricci curvature is $\geq \frac{3}{r}$ everywhere, is compact with a diameter $\leq \pi\sqrt{r}$.

If Λ is ≤ 0, there are non compact solutions of $R_{\mu\nu} = \Lambda g_{\mu\nu}$; the most symmetric are flat Euclidean space and Euclidean antiDeSitter space. The latter is the maximally symmetric space $SO(4,1)/SO(4)$, namely a four-hyperboloid of constant negative curvature, with R^4 topology. Thus its curvature tensor is : $R_{abcd} = \lambda\left(g_{ac}g_{bd} - g_{ad}g_{bc}\right)$, and moreover : $R_{ab} = 3\lambda g_{ab}$, $R = 12\lambda$, where λ is < 0. In this space, if we identify τ and $\tau+\beta$, the quantum fields can be held at any temperature $\frac{1}{\beta}$ (Allen 1983). One can show (Hawking 1984b) that connected asymptotically Euclidean (namely flat Euclidean space out of a compact region) or antiDeSitter metrics which have an inner boundary at Σ do not give the dominant contribution to (1.2.1), because there is a scale transformation which violates a certain inequality which ought to hold. Therefore one is left with disconnected metrics given by a compact part with boundary at Σ and an asymptotically Euclidean or antiDeSitter metric without inner boundary. But Hawking (1984a, b) argues that these metrics are suitable for scattering problems, which is not what happens in cosmology. Some authors have also used the choice of compact metrics so as to set boundary conditions for the numerical solutions of the Wheeler-De Witt equation in models with only a finite number of degrees of freedom, called minisuperspace models (see for example Laflamme and Shellard 1987). However, there is not yet general agreement about these numerical techniques.

Another important point is how the Hartle-Hawking (hereafter referred to as HH) Euclidean path integral enables one to describe classical Lorentzian spacetimes. The basic idea is that a semiclassical approximation of ψ is possible, so that :

$$\psi \sim \sum_k A_k e^{-(I_E)_k} \tag{1.2.4}$$

where $(I_E)_k$ is the Euclidean action computed along a compact solution of the Euclidean field equations inducing h_{ij} on Σ. If $(I_E)_k$ is complex and $Re\ (I_E)_k$ is slowly varying compared with $Im\ (I_E)_k$ (Louko 1988a), each component of the sum (1.2.4) behaves as Ce^{iS}, where, setting : $S = S(q_1, ..., q_N)$, one has : $\frac{\partial S}{\partial q_i} \gg \frac{\partial (lnC)}{\partial q_i}\ \forall i = 1, ..., N$. For example, in a minisuperspace model with a massive or massless scalar field ϕ (Hawking 1984a, b, Laflamme and Shellard 1987, Esposito and Platania 1988) and a single scale factor a, a wavefunction of the form Ce^{iS} is peaked about the first integral : $p_a = \frac{\partial S}{\partial a}$, $p_\phi = \frac{\partial S}{\partial \phi}$, where S is the analytical continuation of the action for compact four-metrics and regular matter fields. This first integral consists of two first-order ordinary differential equations and so the solution involves just two arbitrary constants. Therefore the HH wavefunction is peaked about a set of solutions which are a two-parameter subset of the general solution of the field equations, which involves three arbitrary parameters. It has been shown that this is a subset of inflationary solutions (see again : Hawking 1984a, b, Laflamme and Shellard 1987, Esposito and Platania 1988). Nevertheless, Vilenkin disagrees about this point.

A brief mention of the singularity problem in the HH approach can be found in the Concluding Remarks of this essay. Let us now recall the basic results about the canonical formalism for supergravity.

1.3 Canonical Formalism for Supergravity

In chapter three we will need the Hamiltonian formulation of simple supergravity in four dimensions. This is why we are here aiming to sum up the main ideas of this theory. In so doing, we will closely follow D'Eath 1984.

Simple supergravity is a supersymmetric theory of gravitation. Thus, the gravitational field represented by a spin 2 particle, the graviton, will acquire a supersymmetric partner represented by a spin $\frac{3}{2}$ particle, called gravitino (both these particles are as yet unobserved). From now on, we shall omit the word "simple" for convenience of notation. The basic quantities in the action integral of supergravity are the tetrad, given by the Hermitian spinor-valued one-form $e^{AA'}_{\ \ \ \mu}$, the spin $\frac{3}{2}$ field, given by the spinor-valued one-form $\psi^{A}_{\ \mu}$, and its Hermitian conjugate $\overline{\psi}^{A'}_{\ \ \mu}$. The $e^{AA'}_{\ \ \ \mu}$ commute with all other variables, whereas $\psi^{A}_{\ \mu}$ and $\overline{\psi}^{A'}_{\ \ \mu}$ anticommute among themselves. In terms of the connection forms : $\omega^{AA'BB'}_{\ \ \ \ \ \ \ \mu} = \omega^{AB}_{\ \ \ \mu}\epsilon^{A'B'} + \overline{\omega}^{A'B'}_{\ \ \ \ \mu}\epsilon^{AB}$, the derivative operator D_{μ} acting on spinor-valued forms is defined by : $D_{\mu}e^{AA'}_{\ \ \ \nu} = \partial_{\mu}e^{AA'}_{\ \ \ \nu} + \omega^{A}_{\ B\mu}e^{BA'}_{\ \ \ \nu} + \overline{\omega}^{A'}_{\ \ B'\mu}e^{AB'}_{\ \ \ \nu}$, $D_{\mu}\psi^{A}_{\ \nu} = \partial_{\mu}\psi^{A}_{\ \nu} + \omega^{A}_{\ B\mu}\psi^{B}_{\ \nu}$. The torsion is thus given by the spinor-valued two-form :

$$S^{AA'}_{\ \ \ \mu\nu} = D_{[\mu}e^{AA'}_{\ \ \ \nu]} = -4\pi i\overline{\psi}^{A'}_{\ \ [\mu}\psi^{A}_{\ \nu]} \qquad (1.3.1)$$

Moreover, $R^{AB}_{\ \ \ \mu\nu} = 2\left(\partial_{[\mu}\omega^{AB}_{\ \ \ \nu]} + \omega^{A}_{\ C[\mu}\omega^{CB}_{\ \ \ \nu]}\right)$ is the definition of the spinor-valued curvature two-forms, so that the curvature scalar is : $R = e_{AA'}^{\ \ \ \ \mu}e_{B}^{\ A'\nu}R^{AB}_{\ \ \ \mu\nu} + H.C.$ The action integral of supergravity is :

$$I = \int d^4x \left[\frac{(det\ e)R}{16\pi} + \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \left(\overline{\psi}^{A'}{}_{\mu} e_{AA'\nu} D_\rho \psi^A{}_\sigma + H.C. \right) \right] \tag{1.3.2}$$

The action (1.3.2) is to be supplemented by boundary terms at spatial infinity and on all boundary surfaces which might be present. The auxiliary fields do not appear in (1.3.2) because they vanish classically and are set to zero in the Hamiltonian formalism. If boundary surfaces are absent, (1.3.2) is invariant under the following transformations:

(a) local Lorentz transformations :

$$\delta e^{AA'}{}_{\mu} = N^A{}_B e^{BA'}{}_{\mu} + \overline{N}^{A'}{}_{B'} e^{AB'}{}_{\mu} \tag{1.3.3}$$

$$\delta \psi^A{}_{\mu} = N^A{}_B \psi^B{}_{\mu} \quad \delta \overline{\psi}^{A'}{}_{\mu} = \overline{N}^{A'}{}_{B'} \overline{\psi}^{B'}{}_{\mu} \tag{1.3.4}$$

where $N^{AB} = N^{(AB)}$.

(b) coordinate transformations :

$$\delta e^{AA'}{}_{\mu} = \xi^\nu \partial_\nu e^{AA'}{}_{\mu} + e^{AA'}{}_{\nu} \partial_\mu \xi^\nu \tag{1.3.5}$$

$$\delta \psi^A{}_{\mu} = \xi^\nu \partial_\nu \psi^A{}_{\mu} + \psi^A{}_{\nu} \partial_\mu \xi^\nu \tag{1.3.6}$$

(c) local supersymmetry transformations :

$$\delta e^{AA'}{}_{\mu} = -4\pi i \left(\epsilon^A \overline{\psi}^{A'}{}_{\mu} + \overline{\epsilon}^{A'} \psi^A{}_{\mu} \right) \tag{1.3.7}$$

$$\delta\psi^{A}{}_{\mu} = \frac{1}{2\pi}D_{\mu}\epsilon^{A} \quad \delta\overline{\psi}^{A'}{}_{\mu} = \frac{1}{2\pi}D_{\mu}\overline{\epsilon}^{A'} \tag{1.3.8}$$

In (1.3.7-8), ϵ^{A} and $\overline{\epsilon}^{A'}$ are anticommuting fields which depend on the spacetime position. Defining $p_{AA'}{}^{i} = \frac{\delta I}{\delta \dot{e}^{AA'}{}_{i}}$, the basic dynamical variables of the theory are : $e^{AA'}{}_{i}$, $p_{AA'}{}^{i}$, $\psi^{A}{}_{i}$ and $\overline{\psi}^{A'}{}_{i}$. The remaining variables N, N^{i} (see (A3-4) and (A6)), $\psi^{A}{}_{0}$ and $\overline{\psi}_{0}^{A'}$ can be freely specified. Defining :

$$\pi_{A}{}^{i} = -\frac{1}{2}\epsilon^{ijk}\overline{\psi}^{A'}{}_{j}e_{AA'k} \tag{1.3.9}$$

$$J_{AB} = e_{(A}{}^{A'i}p_{B)A'i} + \psi_{(A}{}^{i}\pi_{B)i} \tag{1.3.10}$$

one finds the primary constraints :

$$J_{AB} = 0 \quad \overline{J}_{A'B'} = 0 \tag{1.3.11}$$

in view of the invariance of (1.3.2) under (1.3.3-4). The Hamiltonian H is found to be : $H = \int d^{3}x \, \left[N\left({}_{1}H_{\perp}\right) + N^{i}\left({}_{1}H_{i}\right) + \psi^{A}{}_{0}\left({}_{1}S_{A}\right) + \left({}_{1}\overline{S}_{A'}\right)\overline{\psi}^{A'}{}_{0} + M_{AB}J^{AB} + \overline{M}_{A'B'}\overline{J}^{A'B'}\right]$.

In this relation, $\psi^{A}{}_{0}$, $\overline{\psi}^{A'}{}_{0}$, M_{AB} and $\overline{M}_{A'B'}$ act as Lagrange multipliers. The quantities $({}_{1}H_{\perp})$, $({}_{1}H_{i})$ and $({}_{1}S_{A})$ depend only on the basic variables $e^{AA'}{}_{i}$, $p_{AA'}{}^{i}$, $\psi^{A}{}_{i}$ and $\overline{\psi}^{A'}{}_{i}$. Their complicated expression may be found in D'Eath 1984. There also primary constraints in view of the vanishing of the momenta conjugate to N, N^{i}, $\psi^{A}{}_{0}$ and $\overline{\psi}^{A'}{}_{0}$. Requiring the preservation in time of these primary constraints (Dirac 1964), one finds the secondary constraints :

$$\left({}_1H_\perp\right) = 0 \quad \left({}_1H_i\right) = 0 \quad \left({}_1S_A\right) = 0 \quad \left({}_1\overline{S}_{A'}\right) = 0 \tag{1.3.12}$$

The last two constraints in (1.3.12) are called supersymmetry constraints. Much more details about the Hamiltonian formulation of supergravity can be found in D'Eath 1984 and in Ashtekar 1988. A detailed analysis of the algebra of supersymmetry constraints in supersymmetric minisuperspace models can be found in Hughes 1989.

1.4 An Outline of This Work

At the very beginning of each subsequent chapter, we have summarized in detail all results obtained. For the sake of clarity, we are aiming to sum up again these results here, but emphasizing much more some foundational issues.

In computing the semiclassical approximation of ψ (see (1.2.4)) we are writing the four-metric as (Louko 1988a) : $g_{\mu\nu} = g^c_{\mu\nu} + \gamma_{\mu\nu}$, where $g^c_{\mu\nu}$ is a solution of the classical field equations obeying the HH boundary conditions, and $\gamma_{\mu\nu}$ is a perturbation about $g^c_{\mu\nu}$. The action is then expanded as : $I = I^c + I^{(2)} + \ldots$, where $I^c = I\left(g^c_{\mu\nu}\right)$ and $I^{(2)}$ is quadratic in $\gamma_{\mu\nu}$. Neglecting all higher order terms, we find :

$$\psi_{HH} = Pe^{-I^c} \quad P = \int d\left[\gamma_{\mu\nu}\right] e^{-\left(I^{(2)}\right)} \tag{1.4.1}$$

Thus the prefactor P is given by the Gaussian integral (1.4.1), which can be regularized using the zeta-function method (see appendix B). At first, one has to make sure that (1.4.1) is taken only over the physical degrees of freedom of the problem. This can be done using the Hamiltonian formulation of the theory and choosing a gauge condition (in the next chapters we will use this method). A hint to the alternative method using Faddeev-Popov determinants can be found in Schleich 1985. Writing : $I^{(2)} = \int \gamma^{\mu\nu} A \gamma_{\mu\nu} \sqrt{g} d^4x$ (where A is a positive-definite second order differential operator) and : $d\left[\gamma_{\mu\nu}\right] = \pi_n \left(\mu dc_n\right)$, $\gamma_{\mu\nu} = \sum_n c_n \gamma_{\mu\nu}^{(n)}$, one finds that (Schleich 1985, Louko 1988a) :

$$P = e^{\left[\frac{1}{2}\varsigma(0) ln\left(\pi\mu^2\right) + \frac{1}{2}\varsigma'(0)\right]} = Da^{\varsigma(0)} \tag{1.4.2}$$

in perturbing about flat Euclidean space with a three-sphere boundary of radius a. In (1.4.2), D is a constant. In this one-loop calculation, the perturbative modes are required to be regular in the origin, and are set equal to zero on S^3 (Dirichlet conditions). As $a \to 0$, the prefactor P had been found to diverge in the pure gravity case (Schleich 1985). Thus, at first we have studied the following problem : is the amplitude at one-loop finite in a quantum cosmology based on a supersymmetric theory of gravitation ? Namely, as $a \to 0$, is P constant in the case of $N = 1$ supergravity ? Indeed, $N = 1$ supergravity is known to be 1-loop and 2-loop finite (Warner 1982). Therefore, one would expect this property still holds true in the presence of a compact boundary such as a three-sphere (namely we study a spatially homogeneous and isotropic closed minisuperspace model without coupling to super-matter). Our main result is that there is no analytical evidence in favour of this conjecture, and the prefactor P for this theory seems to be still diverging as $a \to 0$.

In deriving the contribution to P coming from the spin $\frac{3}{2}$ field which represents the gravitino, it turns out that it is enlightening to perform at first a similar but simpler calculation for the spin $\frac{1}{2}$ field. This is why in chapter two we work out P for the Einstein-Dirac theory. In so doing we pay special attention to the way of dealing with first-order differential operators and to the asymptotic expansion of the heat kernel using relations worked out by Olver (1954) and applied in a simpler case by Stewartson and Waechter (1971). In the Einstein-Dirac theory, the boundary value problem of interest to quantum cosmology is to find the solution of the Euclidean Dirac equation such that $\phi_A^{(+)}$, $\chi_A^{(+)}$, $\widetilde{\phi}_{A'}^{(+)}$, $\widetilde{\chi}_{A'}^{(+)}$ match prescribed values on the final surface S_F, and ϕ_A, χ_A, $\widetilde{\phi}_{A'}$, $\widetilde{\chi}_{A'}$ are regular on the interior of S_F. In this notation, the $(+)$ parts correspond to the modes with unbarred coefficients in the expansion in spinor harmonics on S^3, multiplying unbarred harmonics having positive eigenvalues for the three-dimensional Dirac operator on the bounding S^3. D'Eath and Halliwell (1987) did show that one has to fix the unbarred variables on S_F so as to have a regular solution in the massless limit. Requiring Dirichlet boundary conditions on all modes, one finds that : $P = Da^{-\frac{11}{360}}$. We also introduce Hawking's local boundary conditions on S^3 : $\sqrt{2}n^{AA'}\psi_A = \overline{\psi}^{A'}$, and we show these conditions lead to a good classical boundary value problem.

In chapter three, at first we extend the procedure of chapter two to the $N = 1$ supergravity model. The physical degrees of freedom in (1.4.1) are picked out imposing the supersymmetry constraints (see (1.3.12)) and choosing the gauge condition : $e_{AA'}{}^{j}\psi^{A}{}_{j} = 0$. Setting the coefficients of these degrees of freedom equal to zero on S^3, we find that the contribution to the prefactor due to the spin $\frac{3}{2}$ field is $Da^{\frac{289}{360}}$, which does not cancel the one due to the gravitational field. We then turn our attention to Hawking's local boundary conditions on S^3 involving field strengths and normals : $2^s n^{AA'} ... n^{LL'} \phi_{A...L} = \pm \overline{\phi}^{A'...L'}$ for spin $s > 0$, and : $\phi = \pm \overline{\phi}$, $n^{AA'} \nabla_{AA'} \phi = \mp n^{BB'} \nabla_{BB'} \overline{\phi}$ for a complex scalar field. The consideration of these boundary conditions is suggested by the existence of a one to one map between solutions of the massless free-field equations for adjacent spins s and $s + \frac{1}{2}$. This map is obtained using spin lowering and spin raising operators (see section (3.6)), and the conditions under which these operators preserve Hawking's conditions are derived. These conditions are shown to imply that : $\varsigma(0) = -\frac{1}{180}$ or $-\frac{1}{360}$ for spin 0, and $\varsigma(0) = -\frac{77}{180}$ or $\frac{29}{180}$ for spin 1. In fact, for a complex scalar field one is led to study a Dirichlet problem for $Re\ \phi$ and a Neumann problem for $Im\ \phi$ or viceversa, and for the spin 1 field strength one is led to set to zero on S^3 either the components of the magnetic field or the components of the electric field. For spin $\frac{1}{2}$, $\varsigma(0)$ is harder to compute, because the corresponding eigenvalue equation is found to be very complicated : $J_{n+1}(Ea) = \pm J_{n+2}(Ea)$. One can easily work out the solutions to this equation when $n \rightarrow \infty$ and Ea is fixed, or when n is fixed and $| Ea | \rightarrow \infty$. But this information is not sufficient in working out numerically $\varsigma(0)$. One of the main problems is that the recurrence relations for generating Bessel functions of higher order become a source of large errors when the argument is comparable with the order. The only possibility of solving completely this problem via analytical techniques seems to lie in generalizing a method used by Stewartson, Waechter and Kennedy for a scalar field with an arbitrarily shaped boundary in R^4. The problem is however complicated here by the coupled nature of the spinorial field equations and boundary conditions. Eventually, a positive result could be applied also in studying the boundary conditions involving the spin

$\frac{3}{2}$ and the spin 2 field strengths. However, in that case one more problem arises. In fact, the action integral of the gravitational field is not expressed in terms of the field strength, and we have not yet been able to define in a suitable way a momentum conjugate to the Weyl spinor. Nevertheless, at this stage we can already remark that, had supersymmetry been "working", the value of $\zeta(0)$ should be the same for all values of spin. Thus in particular $\zeta(0)$ for spin 0 should be equal to $\zeta(0)$ for spin 1, whereas it turns out this is not true. This in turn seems to imply there is not one-loop finiteness for the amplitudes of our problem in the presence of a S^3 boundary, as anticipated before. As a partial justification of this result, we have finally proved why, in the comparison spin 0 vs spin $\frac{1}{2}$, the spin lowering operator does not lead to the same eigenvalues.