Worldgraph Approach to Amplitudes

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Abstract of the Dissertation

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String theory uses the first-quantized method (quantum mechanics) to derive scattering amplitudes. External states are considered as vertex operators inserted on the worldsheet generated by an internal string and the amplitudes are calculated perturbatively by calculating vacuum expectation values (vevs) of these vertex operators on worldsheets with different topologies. This approach is different from the common approach of particle theories in which the second-quantized method (quantum field theory) is adopted to calculate amplitudes. A natural question to ask is whether there exists a first-quantization formalism for particles that gives particle amplitudes to all orders. This thesis presents the recent research in answering this question.

In the first-quantized approach for particles, amplitudes are considered as the vevs of vertex operators inserted on different graphs generated by an internal particle. I refer to these graphs as worldgraphs and first-quantized approach for particles as worldgraph ap-

proach. To evaluate these vevs, vertex operators for several external states and Green functions on different worldgraphs are needed. In this thesis, various vertex operators are considered and a general method to obtain scalar Green functions on different worldgraphs is obtained. Some examples of the worldgraph approach to amplitudes in scalar theory and Yang-Mills theory are presented.

To Xinhui and Niuniu

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Chapter 1

Worldgraph method

1.1 Introduction

The so-called worldgraph approach to amplitudes we will develop in this thesis represents the unrenormalized scattering amplitudes at a certain loop level by a first-quantized path integral, namely a quantum mechanical transition probability. The idea, to our knowledge, was first suggested by Feynman [1], Nambu [2] and Schwinger [3] in as early as the 1950's. However, it was not developed much over the following 20 years. There are mainly two reasons for this. (1) It is limited to dealing with tree and 1-loop amplitudes. A tree amplitude can be considered as a particle moving through a series of potentials and a 1-loop amplitude can be considered as a particle moving through potentials and going back to its starting point. On the other hand, higher loop amplitudes are obviously more difficult to fit in this framework. (2) The quantum theory of fields (second-quantized method) was invented and gave an general way of constructing a theory and calculating its scattering amplitudes to all orders.

String theory has revived the development of the first-quantized method, as it uses first-quantization as a key method to calculate the string scattering amplitudes [4] and this calculation can be carried out, in principle, at all loop levels (for a review see [5]). Here a string scattering amplitude is represented by a path integral of fields living on the worldsheet with vertex operators (representing the external lines) inserted. The method is not limited to calculating only tree and 1-loop amplitudes, because string amplitudes at different loop

levels are characterized by integration over worldsheet with different topologies in the action. The first-quantized method in string theory is much more tractable than the method given by a (second-quantized) string field theory, where infinite number of graphs need to be calculated to recover one single amplitude given by the first-quantized method. In fact the second-quantized method hasn't been fully developed for string theory yet.

One will naturally conjecture that the first-quantized method will also be useful in calculating the particle amplitudes. Indeed, from a certain limit of string theory, Bern and Kosower obtained a set of first-quantized rules for calculating the scattering amplitudes in Yang-Mills theory at the 1-loop level [6]. This new method was shown to be equivalent to the ordinary second-quantized method and much more efficient when calculating 1-loop gluon-gluon scattering [7]. Later Strassler gave an alternative derivation of the same rules directly from the first-quantized formalism of the particle theory [8].

1.2 General formalism

We start with the theory of scalar fields with cubic interaction (ϕ^3 theory). An unrenormalized N-point amplitude in ϕ^3 theory at a certain loop level can be represented by the following path integral,

$$\mathscr{A}\left(\mathrm{M}, N\right) = \int \frac{\mathscr{D}e\mathscr{D}X}{V_{\mathrm{rep}}} \exp\left\{-S_{\mathrm{cl}}^{\mathrm{M}}\left[e, X\right]\right\} \prod_{i=1}^{N} \mathcal{O}_i\left[e, X\right] \tag{1.1}$$

where M means the particular topology of the worldgraph being considered, N is the number of external lines, V_{rep} is the volume of the reparametrization group, and $S_{\mathrm{cl}}^{\mathrm{M}}\left[e, X\right]$ is the classical action of a free particle on the space M,

$$S_{\mathrm{cl}}^{\mathrm{M}}\left[e, X\right] = \int_{\mathrm{M}} d\tau \left(\frac{1}{2} e^{-1} \dot{X}^{\mu} \dot{X}_{\mu} + \frac{1}{2} e m^2\right) \tag{1.2}$$

where X is the coordinate scalar and e is the worldline vielbein. V_i's are the vertex operators that represents external lines,

$$\mathcal{O}_i\left[e, X\right] = \int_{\mathrm{M}} d\tau e W_i\left(\tau\right) = \int_{\mathrm{M}} d\tau e \exp\left(i k_i \cdot X\left(\tau\right)\right) \tag{1.3}$$

To fix the reparametrization symmetry, we can set the vielbein e to 1 and introduce the coordinate ghosts b and c. However, by doing this, we have left some part of the symmetry unfixed (the "Killing group", like the conformal Killing group in string theory), as well as over-fixed some non-symmetric transformation (the "moduli space", also like in string theory). To repair these mismatches by hand, we add integrals over the moduli space and take off some of the integrals over the topology. The general form of the amplitude (1.1) after fixing the reparametrization symmetry is (up to a constant factor from possible fixing of the discrete symmetry which arises due to the indistinguishable internal lines)

$$\begin{aligned}\mathscr{A}(\mathrm{M},N) &= \prod_{a=1}^{\mu}\int_{\mathrm{F}_a} dT_a \int \mathscr{D}X\mathscr{D}b\mathscr{D}c \\ &\qquad \times \exp\left\{-S_{\mathrm{gf}}^{\mathrm{M}}\left[X,b,c\right]\right\}\prod_{i\in C} cW_i\left(\tau_i\right)\prod_{i\notin C}\mathcal{O}_i\left[1,X\right]\end{aligned}$$

where C denotes the external lines whose position has to be fixed to break the residual symmetry, and in the particle case the moduli T_a are just the "proper times" represented by the lengths of the edges in the worldgraph of topology M. $S_{\mathrm{gf}}^{\mathrm{M}}\left[X,b,c\right]$ is the gauge-fixed action

$$S_{\mathrm{gf}}^{\mathrm{M}}\left[X,b,c\right] = \int_{\mathrm{M}} d\tau \left(\frac{1}{2}\dot{X}^{\mu}\dot{X}_{\mu} + \frac{1}{2}m^2 + b\dot{c}\right) \tag{1.4}$$

And the measure of the path integral implicitly contains appropriate ghost insertions to make the path integral over the ghost fields well-defined and normalized to 1.

Further evaluation by the usual method of 1D field theory gives the following expression (with the delta function produced by the zero mode integral suppressed since it trivially enforces the total momentum conservation in the

calculation of the scattering amplitude),

$$\mathscr{A}(\mathrm{M},N) = \prod_{a=1}^{\mu}\int_{\mathrm{F}_a} dT_a\,\mathscr{V}_{\mathrm{M}}(T_a)\int_{\mathrm{M}}\left(\prod_{k\notin\mathrm{C}} d\tau_k\right) \times\exp\left[-\frac{1}{2}\sum_{i,j} k_i\cdot k_j G_{\mathrm{M}}(\tau_i,\tau_j)\right] \tag{1.5}$$

where $\mathscr{V}_{\mathrm{M}}(T_a)$ is the amplitude of the vacuum bubble diagram,

$$\mathscr{V}_{\mathrm{M}}(T_a) = \int \mathscr{D}\mathrm{X}\exp\left[-\frac{1}{2}\int_{\mathrm{M}} d\tau\left(\dot{X}^\mu\dot{X}_\mu + m^2\right)\right] \tag{1.6}$$

and G_{M} is the Green function which satisfies the following differential equation on the worldline of topology M:

$$\ddot{G}_{\mathrm{M}}(\tau,\tau') = -\delta(\tau-\tau') + \rho \tag{1.7}$$

where ρ is a constant, of which the integral over the whole 1D space gives 1, i.e., the inverse of the total volume (length) of the 1D space

$$\rho = \frac{1}{\int_{\mathrm{M}} d\tau}$$

This constant is required by the compactness of the worldline space.

The scattering amplitude (1.5) is a general expression for a worldgraph of any topology with an arbitrary number of external lines. The form of the vacuum bubble amplitude and Green function depends on the topology. For example, the amplitude of the one-loop 1PI graph with N external lines (fig. 1.1) is

$$\mathscr{A}(\bigcirc, N) = \int_0^\infty dT\,\mathscr{V}_{\bigcirc}\int_{\bigcirc}\left(\prod_{c=1}^{N-1} d\tau_c\right)\exp\left[-\frac{1}{2}\sum_{i,j} k_i\cdot k_j G_{\bigcirc}(\tau_i,\tau_j)\right]$$

where

$$\mathscr{V}_{\bigcirc}(T) = \exp\left(-\frac{1}{2}Tm^2\right)T^{-D/2}$$

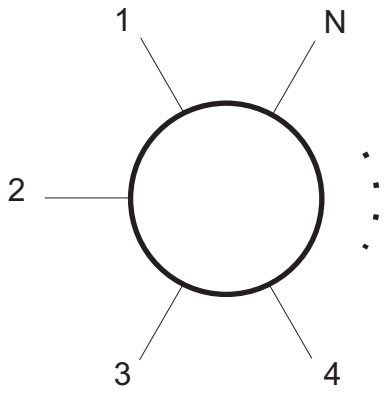

Figure 1.1: One-loop diagram with N external lines. The topology of a circle has one modulus, the circumference T of the loop. Also there is one residual symmetry, which has to be fixed by taking off one of the proper-time integrals.

and

$$G_{\bigcirc}(\tau,\tau') = -\frac{1}{2}\left[|\tau-\tau'| - \frac{(\tau-\tau')^2}{T}\right]$$

Another example is the amplitude of the two-loop 1PI graph with N external lines (fig. 1.2),

$$\begin{aligned}\mathscr{A}(\ominus,N) &= \prod_{a=1}^{3}\int_0^\infty dT_a\,\mathscr{V}_\ominus(T_1,T_2,T_3)\\ &\quad\times\int_\ominus\left(\prod_{c=1}^{N}d\tau_c\right)\exp\left[-\frac{1}{2}\sum_{i,j}k_i\cdot k_j G_\ominus(\tau_i,\tau_j)\right]\end{aligned}$$

where $\mathscr{V}_\ominus$ and $G_\ominus$ will be determined in the following sections.

The expression for the amplitudes in Yang-Mills theory is similar, but with some new features.

Firstly, although it is possible to use the first-quantized method to directly calculate the full Yang-Mills amplitudes, we choose to, for simplicity, perform the usual "color-ordering" technique (for a review, see [9]) first to separate out the group theory factors and use the first-quantized method to deal with the color-ordered (partial) amplitudes. The patial amplitudes are planar and this means that the integrated vertex operators are now not always integrated over the whole worldgraph space, but only the part that keeps the resulting graph planar.

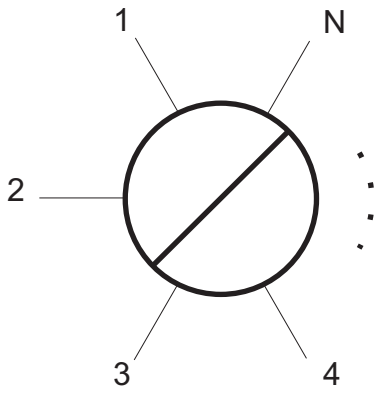

Figure 1.2: Two-loop diagrams with N external lines. The topology of this kind has three moduli, the lengths T_1, T_2 and T_3 of the three edges. And there is no unfixed residual symmetry.

Secondly, the fixed vertex operators may not be simply the integrand of the integrated vertex operators. The form of the vertex operators will depend on the vacuum one chooses. Further at 2-loop or higher-loop, additional operators (the picture changing operators) will be needed.

Thirdly, in addition to the bosonic Green function and the bosonic vacuum bubbles, one will need the fermionic Green functions and the fermionic vacuum bubbles. Unlike the bosonic case, the general formulas of the fermionic Green functions and vacuum bubbles on arbitrary worldgraph are not known yet. We will work out several special cases in this thesis.

Chapter 2

Scalar theory

2.1 Introduction

After early results of first-quantized rules at 1-loop level, the generalization to the rules for scalar theory at multi-loop level was then studied by Schmidt and Schubert [10] and later by Roland and Sato [11]. Green functions on multi-loop vacuum diagrams were obtained by considering these diagrams as one-loop diagrams with insertions of propagators. The Green function on any vacuum diagram containing a "Hamiltonian circuit" could be found by this method. A natural hope of further generalization is to find the worldline Green function on an arbitrary one-dimensional topology, without the limitation that this topology must be one-loop with insertions. In this chapter, we give such a general method to obtain the Green function for scalar field theory at arbitrary multi-loop level. We show that the electric circuit can be an analog in solving this problem.

On the other hand, Mathews [12] and Bjorken [13], from the second-quantized method (usual field theory), gave a method with the electric circuit analogy to evaluate Feynman diagrams at arbitrary loop level. Their result of the Feynman parameter integral representation of scattering amplitudes was, in principle, the most general result, but because of the limitation of second-quantization, diagrams with the same topology except for different number or placement of the external lines were treated separately, and the analogy between the kinetic quantities on the Feynman diagram and the electric quantities on the circuit was not completely clear.

Fairlie and Nielsen generalized Mathews and Bjorken's circuit analog method to strings, and obtained the Veneziano amplitude and 1-loop string amplitude [14]. Although they didn't use the term "Green function", they in fact obtained the expression of the Green function on the disk and annulus worldsheets of the bosonic string with the help of the 2D electric circuit analogy.

These attempts indicate that the problem of solving for the Green function on a certain topological space and the problem of solving a circuit may be related. Indeed, we show that there is an exact analogy between the two kinds of problems in the 1D case.

In this chapter, we first briefly review the quantization of the free scalar particle in section 2.2. In section 2.3, we present the vertex operators needed for the calculation of the amplitudes in scalar theory. In section 2.4, We show a complete analogy among the problems of finding the Green function, the static electric field and the electric circuit. By applying a general method to solve the electric circuit, we give a compact expression of the Green function,

$$\tilde{G}(\tau,\tau') = -\frac{1}{2}s + \frac{1}{2}\mathbf{v}^{\mathrm{T}}\Omega^{-1}\mathbf{v}$$

where the scalar s, vector $\mathbf{v}$, and matrix Ω are quantities depending only on the topological and geometrical properties of the 1D worldgraph space and the position of the external sources τ and τ'. This expression is similar to that of the bosonic string [5]. In Section 2.5, a calculation of the vacuum bubble amplitude is given to complete the discussion. Section 2.6 summarizes the rules and gives some examples on worldgraphs at the tree, 1-loop and 2-loop levels.

The results in this chapter are obtained through collaboration with Warren Siegel and are published in [15].

2.2 Free particle

We begin our discussion with a brief review of the quantum mechanics of a scalar particle moving without interaction with any potential. To proceed, note that we have at least two ways of writing down a quantum theory: one is to give the classical action and quantize it and the other one is to directly construct the BRST charge for the theory. We will do both ways for the

current case of scalar particle. In the next chapter, when we discuss spinning particle, it will be proved simpler and more general to construct the BRST charge directly.

The classical action for a free particle is

$$S_{\rm cl} = \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2} e^{-1} \dot{X}^{\mu} \dot{X}_{\mu} + \frac{1}{2} e m^2 \right) \tag{2.1}$$

where e is the worldline vielbein, X^{μ} are worldline scalers which represent the space-time coordinates of the particle and m is the mass of the particle. The classical action has a worldline parameterization symmetry, and it can be fixed by the standard BRST procedure, i.e., by introducing the Faddeev-Popov ghosts b and c and the auxiliary field B which enforces the gauge-fixing function $e = 1$. Now the gauge-fixed action reads,

$$S_{\rm gf} = \int d\tau \left(\frac{1}{2} e^{-1} \dot{X}^{\mu} \dot{X}_{\mu} + \frac{1}{2} e m^2 + iB\,(e-1) + eb\dot{c} \right) \tag{2.2}$$

By integrating out B, we get a simpler version of the gauge-fixed action (1.2), which is used for calculating the amplitudes. Following the standard procedure of obtaining the Nother charge, we find the BRST charge to be

$$Q = cH = \frac{1}{2} c \left(p^2 + m^2 \right) \tag{2.3}$$

The complete set of states can be written as $|0\rangle \otimes |k\rangle$ and $c\,|0\rangle \otimes |k\rangle$ among which the following states are physical (namely, they are in the cohomology of the BRST charge Q and have vanishing ghost number),

$$|0\rangle \otimes |k\rangle \qquad k^2 + m^2 = 0 \tag{2.4}$$

On the other hand, one can start from the constraints of the system and directly write down the BRST charge. The contraint for the scalar particle is just

$$p^2 + m^2 = 0 \tag{2.5}$$

So we introduce ghost c and anti-ghost b for this constraint, and the BRST charge is the same as eq. (2.3).

2.3 Vertex operators

As we have mentioned, in worldgraph method, we use vertex operator insertions to represent the external lines. In the present case, we need vertex operators that can represent external scalars and these operators are easily constructed by requiring that they are planar waves and should be BRST invariant. The integrated ones are

$$\mathcal{O} = \int d\tau W(\tau) = \int d\tau \exp\left[ik \cdot X(\tau)\right]$$

and the fixed ones are

$$V(\tau) = cW(\tau) = c \exp\left[ik \cdot X(\tau)\right]$$

It is straightforward to check that these vertex operators are indeed BRST invariant, i.e.,

$$[Q, \mathcal{O}] = \{Q, V\} = 0$$

where Q is defined in (2.3) and the on-shell condition of the external particle, $k^2 + m^2 = 0$, is implicitly used.

2.4 Green function

The only green function we need in the scalar case is the X propagator on the graph, i.e.,

$$\eta^{ab} G_{\mathrm{M}}(\tau, \tau') \equiv \left\langle X^a(\tau) X^b(\tau') \right\rangle \tag{2.6}$$

It satisfies the following differential equation:

$$\ddot{G}_{\mathrm{M}}(\tau, \tau') = -\delta(\tau - \tau') + \rho_{\mathrm{M}} \tag{2.7}$$

where the dots mean derivatives with respect to τ and ρ is a constant, of which the integral over the whole 1D space gives 1, i.e., the inverse of the total

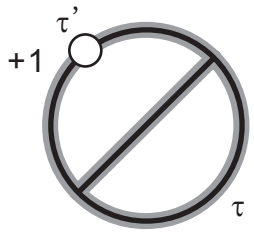

Figure 2.1: Two-loop topological space with a unit positive charge at τ' and a unit negative charge uniformly distributed over the whole space.

volume (length) of the 1D space

$$\rho_{\mathrm{M}} = \frac{1}{\int_{\mathrm{M}} d\tau} \tag{2.8}$$

This constant is required by the compactness of the worldgraph: the differential equation (2.7) will be ill-defined on a compact space if one omits the ρ_{M} term. One will immediately spot the problem if one considers $G_{\mathrm{M}}(\tau, \tau')$ as the electric potential on a compact 1D space and the $\delta(\tau - \tau')$ as a positive electric charge placed at τ. The electric lines will come out of the positive electric charge and end either at negative charge or at infinity, but for compact space there is no infinity and there is no negative charge by definition, so this kind of setup will be not possible.

2.4.1 Electric circuit analogy

We note that the differential equation (2.7) is just the Poisson equation the electric potential should satisfy when there is a unit positive charge at τ' and a constant negative charge density of magnitude ρ over the whole space. This suggests to us to consider the corresponding static electric problem where the Green function is just the electric potential at τ of the above setup.

To demonstrate the general solution of the Poisson equation, we solve for the Green function on the two-loop worldline as an example. Consider the 1D topological space as shown in fig. 2.1 with a unit positive charge at τ' and a unit negative charge uniformly distributed over the whole space. According to

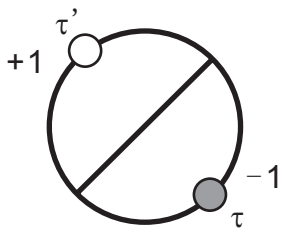

Figure 2.2: Two-loop topological space with a unit positive charge at τ' and a unit negative charge at τ.

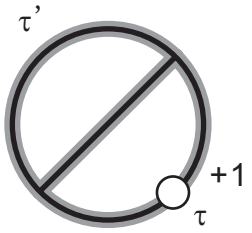

Figure 2.3: Two-loop topological space with a unit positive charge at τ and a unit negative charge uniformly distributed over the whole space.

the above argument, the Green function $G(\tau, \tau')$ is the electric potential at τ. Now we can use Gauss' law and single-valuedness of the potential to write down equations and solve for the expression of the potential. The potential indeed gives a right form for the Green function, but it contains many terms that will be canceled out when calculating the scattering amplitude using equation (1.5), and hence can be further simplified.

Note that the setup of the static electric problem in fig. 2.1 can be regarded as the superposition of the 2 setups in fig. 2.2 and fig. 2.3. Let $G(\tau, \tau')$ denote the potential at τ of the setup shown in 2.1, and $\bar{G}(\tau, \tau')$ denote the potential at τ in fig. 2.2. The potential at τ in fig. 2.3 is then $G(\tau, \tau)$. Thus

$$G(\tau, \tau') = \bar{G}(\tau, \tau') + G(\tau, \tau)$$

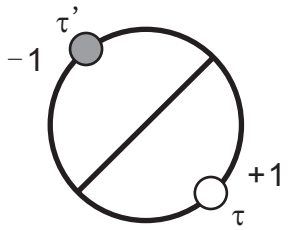

Figure 2.4: Two-loop topological space with a unit negative charge at τ' and a unit positive charge at τ.

We further define $\tilde{G}(\tau,\tau')$ as the symmetric part of $\bar{G}(\tau,\tau')$, i.e.,

$$\begin{aligned}\tilde{G}(\tau,\tau') &\equiv \frac{1}{2}\left[\bar{G}(\tau,\tau')+\bar{G}(\tau',\tau)\right] \\ &= \left[\frac{1}{2}G(\tau,\tau')+\frac{1}{2}G(\tau',\tau)\right]-\frac{1}{2}G(\tau,\tau)-\frac{1}{2}G(\tau',\tau')\end{aligned} \tag{2.9}$$

If we use $\tilde{G}(\tau,\tau')$ instead of $G(\tau,\tau')$ in equation (1.5), the sum can be rewritten as

$$\begin{aligned}-\frac{1}{2}\sum_{i,j}k_i\cdot k_j\tilde{G}(\tau_i,\tau_j) &= -\frac{1}{2}\sum_{i,j}k_i\cdot k_j\left\{\frac{1}{2}\left[G(\tau_i,\tau_j)+G(\tau_j,\tau_i)\right]\right\} \\ &\quad +\frac{1}{2}\sum_{i,j}k_i\cdot k_j\left\{\frac{1}{2}\left[G(\tau_i,\tau_i)+G(\tau_j,\tau_j)\right]\right\} \\ &= -\frac{1}{2}\sum_{i,j}k_i\cdot k_j\left\{\frac{1}{2}\left[G(\tau_i,\tau_j)+G(\tau_j,\tau_i)\right]\right\} \\ &= -\frac{1}{2}\sum_{i,j}k_i\cdot k_jG(\tau_i,\tau_j)\end{aligned}$$

We have used conservation of momentum $\sum k_i=0$ in the second step and rearranged the summands in the third step. This shows that using $\tilde{G}(\tau,\tau')$ in equation (1.5) is equivalent to using $G(\tau,\tau')$. Note that same procedure is usually applied to construct the Green function for bosonic strings [5],

$$\tilde{G}(z,w)=G(z,w)-\frac{1}{2}G(z,z)-\frac{1}{2}G(w,w)$$

which is similar to equation (2.10).

Now we have to look for an electric field analog of $\tilde{G}(\tau,\tau')$. Since $\bar{G}(\tau,\tau')$

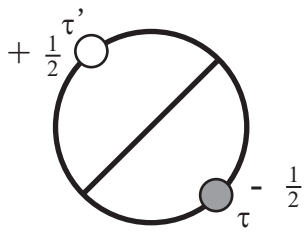

Figure 2.5: Two-loop topological space with a half unit positive charge at τ' and a half unit negative charge at τ.

is the electric potential at τ of the setup shown in fig. 2.2, it can be written as the potential at τ' plus the potential difference from τ to τ'. And $\bar{G}(\tau', \tau)$ is the potential at τ' of the setup shown in fig. 2.4. The sum of $\bar{G}(\tau, \tau')$ and $\bar{G}(\tau', \tau)$ just gives the potential difference from τ to τ' of fig. 2.2 because the potential at τ' of fig. 2.2 cancels the potential at τ' of fig. 2.4. $\tilde{G}(\tau, \tau')$ is half that potential difference, and therefore is just the potential difference from τ to τ' of fig. 2.5.

It is now clear that, to write down the expression of the scattering amplitude, we only have to know the symmetric Green function $\tilde{G}$, and use the following formula:

$$\mathscr{A}(\mathrm{M}, N) = \prod_{a=1}^{\mu} \int_{\mathrm{F}_a} dT_a \, \mathscr{V}_{\mathrm{M}} \prod_{c \notin \mathrm{C}} \int_{\mathrm{M}} d\tau_c \exp\left[- \sum_{i<j} k_i \cdot k_j \tilde{G}(\tau_i, \tau_j) \right] \qquad (2.10)$$

This simplifies the expression of the Green function.

Now that $\tilde{G}(\tau, \tau')$ is the potential difference from τ to τ' in fig. 2.5, we can apply Gauss' law (of 1D space) and single-valuedness of the potential to write down the equations the Green function (electric potential Φ) and its first derivative (electric field E) should satisfy. We assume the direction and value of E as shown in fig. 2.6 and τ' and τ are respectively on T_1 and T_3. According to Gauss' law, we have the following equations:

$$\begin{aligned} a + b &= +\frac{1}{2} \\ -a - c + d &= 0 \\ -b + c + e &= 0 \\ -d - e &= -\frac{1}{2} \end{aligned}$$

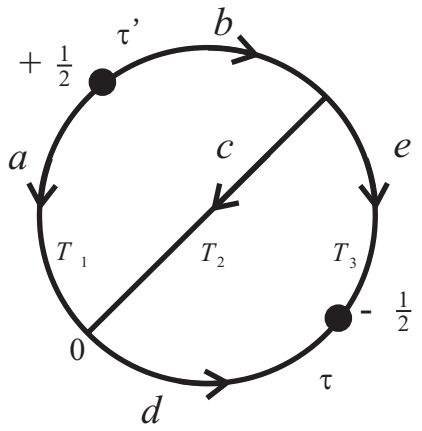

Figure 2.6: Two-loop topological space with a half unit positive charge at τ' and a half unit negative charge at τ. The lengths of the three arcs are T_1, T_2, T_3. τ' and τ are respectively on T_1 and T_3 and denote the lengths from the origin. The magnitudes of the electric field on each part of the space are denoted by a - e and the directions are chosen arbitrarily.

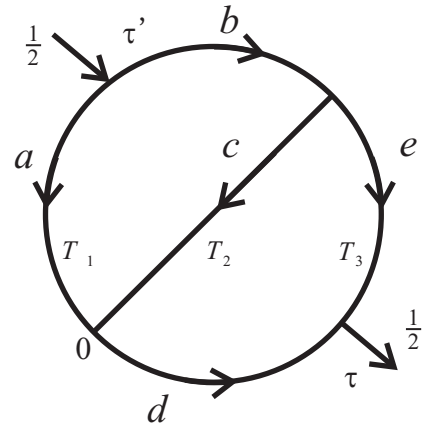

Figure 2.7: Two-loop circuit with a half unit current input at τ' and withdrawn at τ. The resistances of the three arcs are T_1, T_2, T_3. τ' and τ are respectively on T_1 and T_3 and denote the resistance from the origin. The currents on the parts of the circuit are denoted by a - e and the directions are chosen arbitrarily.

The single-valuedness of the potential requires

$$\begin{aligned} cT_2 + b(T_1 - \tau') &= a\tau' \\ cT_2 + d\tau &= e(T_3 - \tau) \end{aligned}$$

If we regard the electric field E as the current I, the electric potential Φ as the voltage V, and the length on the 1D space l as the resistance r, these equations are just the Kirchhoff equations of a circuit of the same shape as the worldline and with half unit current going into the circuit at τ' and half unit current coming out at τ, as shown in fig. 2.7. It is easy to see that this equivalence between 1D static electric field and circuit is valid for an arbitrary 1D topological space. We have the following relations for 1D (note that there is no cross-sectional area in the 1D case):

$$\begin{aligned} \sigma E &= I \\ \rho l &= r \\ \Phi &= V \end{aligned}$$

where σ and ρ $(= 1/\sigma)$ are respectively the 1D conductivity and resistivity, and the relation $El = \Phi$ is equivalent to Ohm's law $Ir = V$. In addition, Gauss' law is equivalent to Kirchhoff's current law, and the single-valuedness of the potential is equivalent to Kirchhoff's voltage law. Thus the two problems are indeed equivalent. The Green function $\tilde{G}(\tau, \tau')$ on a particular 1D topological space M can be understood as the voltage difference from τ to τ' when a half unit current is input into the "circuit" (1D space) at τ' and withdrawn at τ.

Above we have shown that in the 1D case the problem of solving for the first-quantized particle Green function, the potential difference in a static electric field and the voltage difference in a circuit are analogous. Before we solve the circuit problem for the Green function, it is interesting to give a complete analogy among the quantities in these three kinds of problems (Table 2.1).

2.4.2 Solving the circuit

The next thing we have to do is to find a general expression for the Green function, i.e., a method to obtain the voltage difference described above. A

Table 2.1: Analogy among quantities in three kinds of problems

Particle	Static electric field	Electric circuit
Green function $\tilde{G}$	potential difference $\Delta\Phi$	voltage difference ΔV
position x	electric potential Φ	voltage V
momentum p	electric field E	current I
proper time τ	length l	resistance r
action S	energy U	power P
external force F	electric charge Q	emf $\mathscr{E}$

formula has already been given in graph theory (see, e.g., [16]). Here we summarize this result and develop it into a formula which is similar to the known form of the Green function on the 2D worldsheet of bosonic string theory [5].

The voltage difference from τ to τ' can be obtained by the following steps:

(1) Connect τ and τ' by a wire with zero resistance.

(2) The resulting graph has vertices $V = v_1, ..., v_n$ and edges $E = e_1, ...e_{m-1}, e_m$. Denote by e_m the edge (wire) we have just added in. Set the directions of all the edges arbitrarily.

(3) Assign voltage, current and resistance on each edge; they can be written in the form of vectors:

$$\begin{aligned} \mathbf{U} &= (U_1, U_2, ..., U_m)^{\mathrm{T}} \\ \mathbf{I} &= (I_1, I_2, ..., I_m)^{\mathrm{T}} \\ \mathbf{r} &= (r_1, r_2, ..., r_m)^{\mathrm{T}} \end{aligned}$$

Also define the external-force-driven voltage $\mathbf{u}$, which has, in our case, only one non-zero component (the last one). Assume it has magnitude 1:

$$\mathbf{u} = (u_1, u_2, ..., u_m)^{\mathrm{T}} = (0, 0, ..., 1)^{\mathrm{T}} \tag{2.11}$$

(4) Find all the "independent" loops in the graph: There should be $(m - n + 1)$. These loops can be identified by the following method: Choose an arbitrary spanning tree (a connected subgraph that contains all the vertices and is a tree) of the graph. There are always $(m - n + 1)$ chords (the edges

not belonging to the spanning tree). Adding a chord to the spanning tree will generate a one-loop graph. Thus each chord gives a loop in the graph, and all the loops obtained by this way are independent of each other. Therefore there are $(m - n + 1)$ loops. Assign an arbitrary direction to each loop.

(5) Define matrices $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{R}$ as follows:

$$\mathcal{B}_{ij} = \begin{cases} 1 & \text{if } v_i \text{ is the initial vertex of } e_j, \\ -1 & \text{if } v_i \text{ is the terminal vertex of } e_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.12}$$

$$\mathcal{C}_{ij} = \begin{cases} 1 & \text{if } e_i \text{ is in the same direction of loop } l_j, \\ -1 & \text{if } e_i \text{ is in the opposite direction of loop } l_j, \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

$$\mathcal{R}_{ij} = \begin{cases} r_i & i = j, \\ 0 & i \neq j. \end{cases} \tag{2.14}$$

(6) With the above definitions, we can write down Kirchhoff's current law, Kirchhoff's voltage law and Ohm's law in compact forms as follows:

$$\begin{aligned} \mathcal{B}\mathbf{I} &= \mathbf{0} \\ \mathcal{C}^{\mathrm{T}}\mathbf{U} &= \mathbf{0} \\ \mathbf{U} &= \mathcal{R}\mathbf{I} + \mathbf{u} \end{aligned}$$

The solution to the current on each edge is

$$\mathbf{I} = -\mathcal{C}\left(\mathcal{C}^{\mathrm{T}}\mathcal{R}\mathcal{C}\right)^{-1}\mathcal{C}^{\mathrm{T}}\mathbf{u}$$

(7) The total resistance between τ and τ' (excluding the added [last] edge) is then minus the voltage on the last edge divided by the current on that edge, i.e.,

$$R(\tau, \tau') = -\frac{U_m}{I_m} = -\frac{u_m}{I_m} = -\frac{1}{I_m}$$

The Green function is then minus this resistance times the current, one half,

since it is the voltage difference from τ to τ',

$$\tilde{G}(\tau,\tau') = -\frac{1}{2}R(\tau,\tau') = \frac{1}{2I_m} = -\frac{1}{2\mathbf{u}^{\mathrm{T}}\mathcal{C}\left(\mathcal{C}^{\mathrm{T}}\mathcal{R}\mathcal{C}\right)^{-1}\mathcal{C}^{\mathrm{T}}\mathbf{u}} \tag{2.15}$$

where the last step comes from extracting the last component of $\mathbf{I}$ by using the vector $\mathbf{u}$ defined in equation (2.11).

We can further simplify equation (2.15), by considering the physical meaning of each part of the denominator:

(1) $\mathcal{C}^{\mathrm{T}}\mathbf{u}$ and $\mathbf{u}^{\mathrm{T}}\mathcal{C}$: Since $\mathbf{u} = (0,...,0,1)^{\mathrm{T}}$, $\mathcal{C}^{\mathrm{T}}\mathbf{u}$ is an $(m-n+1)$-component column vector whose ith component is the direction of the last (mth) edge on the ith loop (1 for "same", -1 for "opposite", and 0 for "not on the loop"). If we appropriately choose our independent loops, we can achieve that the mth edge is only on the $(m-n+1)$th (last) loop and has the same direction as this loop. This is always achievable in steps: (a) Choose the spanning tree in such a way that the mth edge doesn't belong to the spanning tree, i.e., is a chord. (b) Define the loop generated by adding the mth edge to the spanning tree to be the $(m-n+1)$th loop. (c) Define the direction of the $(m-n+1)$th loop to be the same as the mth edge. By doing so, we find that $\mathcal{C}^{\mathrm{T}}\mathbf{u}$ is just a $(m-n+1)$ component column vector with the last component non-vanishing and of value 1.

$$\mathcal{C}^{\mathrm{T}}\mathbf{u} = (0,...,0,1)^{\mathrm{T}}$$

And $\mathbf{u}^{\mathrm{T}}\mathcal{C}$ is the transpose of $\mathcal{C}^{\mathrm{T}}\mathbf{u}$. Define for convenience

$$\mathbf{P} \equiv \mathcal{C}^{\mathrm{T}}\mathbf{u}$$

Note that $\mathbf{P}^{\mathrm{T}}\mathcal{M}\mathbf{P}$ gives the $[(m-n+1),(m-n+1)]$ component of any matrix $\mathcal{M}$ with dimension $(m-n+1)\times(m-n+1)$.

(2) $\mathcal{C}^{\mathrm{T}}\mathcal{R}\mathcal{C}$: $\mathcal{C}^{\mathrm{T}}\mathcal{R}\mathcal{C}$ is an $(m-n+1)\times(m-n+1)$ matrix. The components can be interpreted as the sum of the "signed" resistances,

$$\left(\mathcal{C}^{\mathrm{T}}\mathcal{R}\mathcal{C}\right)_{ij} = \sum_{k\,\in\,\text{all edges}} f\left(k,i,j\right) r_k$$

where r_k is the resistance on kth edge and f is the "sign":

$$f(n,i,j) = \begin{cases} 1 & n\text{th edge is on loop } i \text{ and } j \text{ with the same orientation,} \\ -1 & n\text{th edge is on loop } i \text{ and } j \text{ with the same orientations,} \\ 0 & n\text{th edge is not on loop } i \text{ and } j. \end{cases} \tag{2.16}$$

We define for convenience

$$\mathcal{M} \equiv \begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{v}^{\mathrm{T}} & s \end{pmatrix} \equiv \mathcal{C}^{\mathrm{T}} \mathcal{R} \mathcal{C} \tag{2.17}$$

where Ω is an $(m-n) \times (m-n)$ matrix, $\mathbf{v}$ is an $(m-n)$-component vector, and s is a scalar.

Now the formula for the Green function (2.15) becomes

$$\tilde{G} = -\frac{1}{2\mathbf{P}^{\mathrm{T}} \mathcal{M}^{-1} \mathbf{P}}$$

Since $\mathbf{P}^{\mathrm{T}} \mathcal{M}^{-1} \mathbf{P}$ is just the $[(m-n+1), (m-n+1)]$ component of $\mathcal{M}^{-1}$, we have

$$\mathbf{P}^{\mathrm{T}} \mathcal{M}^{-1} \mathbf{P} = \frac{\det \Omega}{\det \mathcal{M}}$$

So, we have

$$\tilde{G} = -\frac{\det \mathcal{M}}{2 \det \Omega}$$

Next we evaluate $\det \mathcal{M}$. By the usual matrix algebra (e.g., defining the determinant by a Gaussian integral and doing the "$m-n$" integrals first),

$$\det \mathcal{M} = \det \begin{pmatrix} \Omega & \mathbf{v} \\ \mathbf{v}^{\mathrm{T}} & s \end{pmatrix} = (\det \Omega) \left[s - \mathbf{v}^{\mathrm{T}} \Omega^{-1} \mathbf{v} \right]$$

So we have the following expression for the Green function:

$$\tilde{G} = -\frac{\det \mathcal{M}}{2 \det \Omega} = -\frac{1}{2} s + \frac{1}{2} \mathbf{v}^{\mathrm{T}} \Omega^{-1} \mathbf{v} \tag{2.18}$$

2.5 Vacuum bubbles

To complete this general method of writing down the scattering amplitude, we need to give the expression for the vacuum bubble amplitude with a given topology $\mathscr{V}_{\mathrm{M}}(T_a)$ defined in equation (1.6). This can always be achieved by evaluating the bubble diagram by the second-quantized method, but here we give a derivation by direct calculation in first-quantization. Note that the path integral (1.6) is the sum over all possible momentum configurations of the product of the expectation values of the free evolution operator between two states at the ends of each edge:

$$\begin{aligned}\mathscr{V}_{\mathrm{M}}(T_a) &= \sum_{\{p\}}\left\{\prod_{a=1}^{\mu}\left\langle p_a\left|e^{-T_a\left(p^2+m^2\right)/2}\right|p_a\right\rangle\right\} \\ &= \sum_{\{p\}}\left\{\exp\left[\sum_{a=1}^{\mu}-\frac{1}{2}T_a\left(p_a^2+m^2\right)\right]\right\}\end{aligned}$$

where p_a is the momentum of the particle traveling on the ath edge. The sum over all the configurations of p_a can be written as the the integration over all the possible values of p_a, but they are not independent from each other. Each of the μ p_a's can be written as a linear combination of L k_i's where L is the number of loops of the graph and k_i is the loop momentum on the ith loop. So the amplitude $\mathscr{V}_{\mathrm{M}}(T_a)$ can be written as

$$\begin{aligned}\mathscr{V}_{\mathrm{M}}(T_a) &= \exp\left[-\frac{1}{2}\left(\sum_{a=1}^{\mu}T_a\right)m^2\right] \\ &\quad\times\int\left(\prod_{i}^{L}dk_i\right)\exp\left[-\frac{1}{2}\underline{\sum_{a=1}^{\mu}T_a\left(\sum_{i=1}^{L}g_{ai}k_i\right)^2}\right]\end{aligned} \tag{2.19}$$

where

$$g_{ai}=\begin{cases}1 & \text{edge } a \text{ has the same direction as loop } i,\\ -1 & \text{edge } a \text{ has the opposite direction as loop } i,\\ 0 & \text{edge } a \text{ is not on loop } i.\end{cases}$$

It is easy to see the following points: (a) If edge a is on loop i, there is a $T_a k_i^2$ term in the underlined sum in equation (2.20), and vice versa. (b) If edge

a is on both loop i and loop j, there is a term $2T_a k_i k_j$ in the sum, and vice versa. Further, if on edge a both loops have the same (opposite) direction, there is a positive (negative) sign before the term, and vice versa. Thus if we use the factor $f(a,i,j)$ defined in equation (2.16), we can write the underlined part in equation (2.20) in a compact form, and further in terms of the period matrix according to the definition (2.17), or definition (2.22) below:

$$\begin{aligned}\sum_{a=1}^{\mu} T_a \left(\sum_{i=1}^{L} g_{ai} k_i\right)^2 &= \sum_{a=1}^{\mu}\sum_{i,j=1}^{L} f\left(a,i,j\right) T_a k_i k_j \\ &= \sum_{i,j=1}^{L}\left[\sum_{a=1}^{\mu} f\left(a,i,j\right) T_a\right] k_i k_j = \sum_{i,j=1}^{L} \Omega_{ij} k_i k_j\end{aligned}$$

The amplitude $\mathscr{V}_{\mathrm{M}}\left(T_a\right)$ can then be calculated easily:

$$\begin{aligned}\mathscr{V}_{\mathrm{M}}\left(T_a\right) &= \exp\left[-\frac{1}{2}\left(\sum_{a=1}^{\mu} T_a\right) m^2\right] \int \left(\prod_{i}^{L} dk_i\right) \exp\left[-\frac{1}{2}\sum_{i,j=1}^{L} \Omega_{ij} k_i k_j\right] \\ &= \exp\left[-\frac{1}{2}\left(\sum_{a=1}^{\mu} T_a\right) m^2\right] \left(\det \Omega\right)^{-D/2} \qquad (2.20)\end{aligned}$$

where D is the dimension of the spacetime.

2.6 Amplitudes

2.6.1 General rules

We now summarize our results for general amplitudes. To find the Green function:

(1) Consider the "circuit" (topology M with two more vertices at τ and τ' respectively) as a graph. Assign a number to each edge. Define arbitrarily the direction of each edge. Assign a number to each loop. Define arbitrarily the direction of each loop.

(2) Find the period matrix Ω by the following definition:

$$\Omega_{ij} = \sum_{k \in \text{all edges}} f(k,i,j) r_k \qquad (2.21)$$

where

$$f(n,i,j)=\begin{cases}1 & n\text{th edge is on loop } i \text{ and } j \text{ with the same orientation,}\\ -1 & n\text{th edge is on loop } i \text{ and } j \text{ with the same orientations,}\\ 0 & n\text{th edge is not on loop } i \text{ and } j.\end{cases}$$

Each r_k may be τ, τ', a T_a of the topological space M without external lines, $T_a-\tau$, or $T_a-\tau'$. It is obvious that Ω does not depend on τ and τ', and only depends on the properties of M: To see this, note that the graph of the circuit has just two more vertices (at positions τ and τ') than the graph of M and each of them may separate an edge in the graph of M into two "sub-edges". But these changes on the graph do not affect the period matrix: They neither give new loops nor eliminate loops and the sub-edges will always be on or off a loop simultaneously. Thus the period matrix of the graph of the circuit is just the period matrix of the graph of M. And one only has to calculate it once for one certain topology,

$$\Omega_{ij}=\sum_{a=1}^{\mu} f(a,i,j)T_a \tag{2.22}$$

(3) Find a path (call it "reference path") connecting τ and τ'. Choose its direction arbitrarily. Define the scalar s as the total resistance on the reference path.

(4) Find the vector $\mathbf{v}$ defined as follows:

$$\mathbf{v}_i=\sum_{k\,\in\,\text{all edges}} f(k,i,0)r_k$$

where "0" means the reference path.

(5) The Green function is given by

$$\tilde{G}=-\frac{1}{2}s+\frac{1}{2}\mathbf{v}^{\mathrm{T}}\Omega^{-1}\mathbf{v}$$

The amplitude is then given by

$$\mathscr{A}(\mathrm{M},N)=\prod_{a=1}^{\mu}\int_{\mathrm{F}_a}dT_a\,\mathscr{V}_{\mathrm{M}}\prod_{c\notin\mathrm{C}}\int_{\mathrm{M}}d\tau_c\exp\left[-\sum_{i<j}k_i\cdot k_j\tilde{G}(\tau_i,\tau_j)\right]$$

with the Green function as above, and the vacuum bubble amplitude is

$$\mathscr{V}_{\mathrm{M}}\left(T_a\right)=\exp\left[-\frac{1}{2}\left(\sum_{a=1}^{\mu}T_a\right)m^2\right]\left(\det\Omega\right)^{-D/2}$$

Although we have defined and derived all these quantities s, $\mathbf{v}$ and Ω in terms of the concepts in the circuit problem, it is easy to give the worldline geometric interpretation by noting that the resistance is an analog of the proper time, i.e., the length of the worldline (Table 2.1). s is just the total proper time of the reference path, and the components of $\mathbf{v}$ are the sums of the "signed" proper time on the common edges of the reference path and each loop. Entries of Ω are sums of the "signed" proper time on the common edges of each pair of loops. (All the signs are given by f.) The components of $\mathbf{v}$ and entries of Ω can also be expressed as integrals of the "Abelian differentials" on the loops of the worldline,

$$\begin{aligned} v_i &= \int_{\tau'}^{\tau}\omega_i \\ \Omega_{ij} &= \oint_i \omega_j \end{aligned}$$

where ω_i is the line element on loop i and the second integral is around loop i along its direction. Then our expression for the particle amplitude has a similar structure to the bosonic string amplitude, with the Green function on the 2D worldsheet [5]:

$$\tilde{G}\left(w,z\right)=-2\ln\left|E\left(w,z\right)\right|+2\pi\ \mathrm{Im}\int_w^z\omega\left(\mathrm{Im}\ \Omega\right)^{-1}\mathrm{Im}\int_w^z\omega$$

where E is the prime form, the vector ω contains the basis of the Abelian differentials and the matrix Ω is the period matrix.

2.6.2 Examples

Here we give some examples of obtaining the Green functions on different topologies. For the finite line of length T in fig. 2.8, there is no loop, so there is no period matrix Ω nor vector $\mathbf{v}$. s is just the total resistance between τ

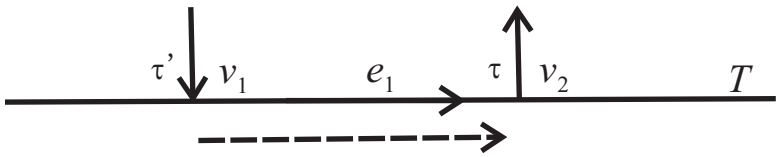

Figure 2.8: The topology of a line with length T. There is no loop and hence no period matrix nor vector $\mathbf{v}$. The only path between τ and τ' is the edge connecting the two vertices e_1, so we choose it as the reference path.

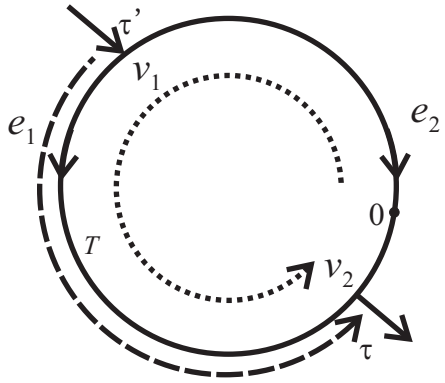

Figure 2.9: The one-loop topology. T is the circumference of the circle. There are two edges (e_1,e_2) and two vertices (v_1,v_2). τ and τ' are the lengths from v_1 and v_2 to the origin though a counterclockwise path. The directions of the edges are chosen arbitrarily and marked in the figure. There is only one loop and it is marked by dotted lines. The reference path is marked by a dashed line.

and τ'. So the Green function is

$$\tilde{G}_{-}(\tau,\tau') = -\frac{1}{2}s = -\frac{1}{2}|\tau-\tau'|$$

and

$$\mathscr{V}_{-}(T) = \exp\left(-\frac{1}{2}Tm^2\right)$$

The amplitude is then given by equation (2.10). One just has to note that there is one modulus T for this case and to fix the residual symmetry: Two of the external lines should be fixed at one end of the line and another two should be fixed at the other end.

For the circle, there is one loop as shown in fig. 2.9. So the period matrix

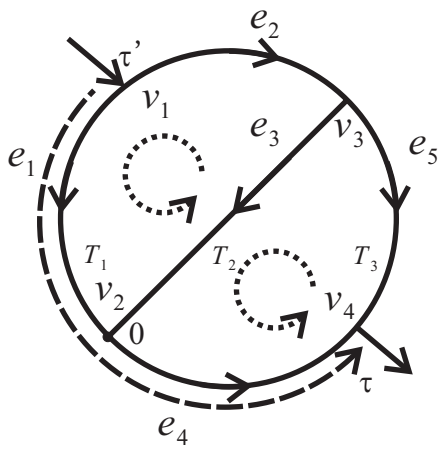

Figure 2.10: A two-loop topology. T_1, T_2 and T_3 are the lengths of the three arcs. τ and τ' are respectively the length on the 3rd and 1st arc from the origin. (If the Green function with τ and τ' on different arcs is needed, simply repeat the steps for this special case.) There are 5 edges (e_1 to e_5) and 4 vertices (v_1 to v_4). The directions of the edges are chosen arbitrarily and marked in the figure. There are two independent loops and they are marked by dotted lines. The reference path is marked by a dashed line.

is 1×1. The only entry of Ω is then

$$\Omega = T$$

And according to the definition, s and $\mathbf{v}$ are

$$s = |\tau - \tau'|$$

$$\mathbf{v} = |\tau - \tau'|$$

Thus the Green function on the circle is

$$\tilde{G}_{\bigcirc}(\tau, \tau') = -\frac{1}{2}s + \frac{1}{2}\mathbf{v}^{\mathrm{T}}\Omega^{-1}\mathbf{v} = -\frac{1}{2}|\tau - \tau'| + \frac{(\tau - \tau')^2}{2T}$$

and the vacuum bubble amplitude is

$$\mathscr{V}_{\bigcirc}(T) = \exp\left(-\frac{1}{2}Tm^2\right) T^{-D/2}$$

For the Green function on the 2-loop graph shown in fig. 2.10, the period

matrix is 2×2 and $\mathbf{v}$ is a two-component vector. Ω, s and $\mathbf{v}$ are

$$\begin{aligned} \Omega &= \begin{pmatrix} T_1 + T_2 & -T_2 \\ -T_2 & T_2 + T_3 \end{pmatrix} \\ s &= \tau' + \tau \\ \mathbf{v} &= (\tau', \tau)^{\mathrm{T}} \end{aligned}$$

So the Green function is, by plugging all the above into equation (2.18),

$$\begin{aligned} \tilde{G}_{\ominus}(\tau, \tau') &= -\frac{1}{2(T_1T_2 + T_2T_3 + T_3T_1)} \\ &\quad \times [T_1(T_2 + T_3 - \tau)\tau + T_1(T_2 + T_3)\tau' \\ &\quad -T_3\tau'^2 + T_2(T_3 - \tau - \tau')(\tau + \tau')] \end{aligned}$$

and the vacuum bubble amplitude is

$$\mathscr{V}_{\ominus}(T_1, T_2, T_3) = \exp\left[-\frac{1}{2}(T_1 + T_2 + T_3)m^2\right](T_1T_2 + T_2T_3 + T_3T_1)^{-D/2}$$

Chapter 3

Yang-Mills

3.1 Introduction

First-quantization has provided an efficient way of calculating Yang-Mills amplitudes. A set of rules for writing down 1-loop Yang-Mills amplitudes was first derived by Bern and Kosower from evaluating heterotic string amplitudes in the infinite string-tension limit [6]. Later an alternative derivation of the same rules (but only for the 1-loop effective action) from first-quantization of the particle was given by Strassler [8]. It is worth noting that Bern and Kosower's string approach naturally produces whole non-abelian amplitudes including both one-particle-irreducible (1PI) and non-1PI amplitudes since such a distinction was not present at the string level. On the other hand Strassler's first-quantized particle approach is based on a path integral representation of the effective action, and therefore is intrinsically more convenient for calculating the effective action itself, or 1PI amplitudes. In addition, the generalization of these first-quantized rules to multi-loop amplitudes has not been clear. A generalization of Bern and Kosower's approach seems difficult, because the calculation of multi-loop string amplitudes is very complicated. On the other hand, although there has been successful generalization of Strassler's approach to derive 2-loop Yang-Mills amplitudes from first-quantization of the particle with the sewing method [17], a more concise representation for Yang-Mills amplitudes that is easier for generalization to all loop levels will be preferred, if it exists. In fact, such a representation has not yet been found even for Yang-Mills tree amplitudes. This is partially because the vacuum, ghost mea-

sure and Green function needed for the calculation of trees and multi-loops have not been clarified. Although there are already many ways to compute Yang-Mills tree amplitudes, it is important to clarify how first-quantization of the particle works at tree level for the purpose of generalizing this method to both non-1PI cases and multi-loop level. This is the main purpose of this chapter. To derive the first-quantized rules for trees, we start from theories of free relativistic spinning particles, which were first developed by Brink et al. [18] and many others [19]. In these theories the spin degree of freedom is encoded in the worldline supersymmetry. More precisely, the BRST quantization of the particle action with N-extended worldline supersymmetry shows that the cohomology is of a spin-$\frac{N}{2}$ particle.

In this chapter we study the $N = 2$ theory, which describes a spin-1 particle. We derive the vertex operator for background gauge fields via the usual BRST quantization method, thus ensuring background gauge invariance. (The coupling of background vector fields to the spin-$\frac{1}{2}$ particle was formulated in [18]. It was used to calculate effective actions in [20].) We proceed to show how the correct amplitudes can be derived. In the usual worldline approach, all interactions are derived by coupling external fields to the 1-dimensional worldline or loop. For higher-point tree graphs this approach usually requires sewing lower-point tree graphs to the worldline. This does not fit the picture of perturbation in the first-quantized theory since one would normally anticipate a formalism that can calculate any amplitude without calculating the lower-point amplitudes first: The knowledge of Green functions and vertex operators should be sufficient. Here we propose an alternative ("worldgraph") approach that includes 1D topological spaces that are not strictly 1D manifolds: They are not always locally R^1, but only fail to be so at a finite number of points. Taking these spaces into account we derive a set of rules for computing amplitudes that can be extended to all possible graphs.

We organize this chapter as follows: In section 3.2 we give a brief review of a general formalism to describe free spinning particles with arbitrary spin. In section 3.3 we focus on the spin-1 particle: introducing background Yang-Mills interaction to the theory and deriving the vertex operator for the external Yang-Mills fields. In section 3.4 we derive for several special cases the fermionic Green functions needed for the calculation of Yang-Mills tree amplitudes. In

section 3.5, we give the simplest non-trivial vacuum amplitude, i.e., the one-loop vacuum bubble. In section 3.6, we present the calculation of 3 and 4-point trees, and one-loop amplitudes. We obtain these amplitudes through two different routes, namely, the worldline approach and the worldgraph approach.

The results in this chapter are obtained through collaboration with Yu-tin Huang and Warren Siegel and are published in [21].

3.2 Free particle

A classical relativistic massless spin-1 particle can be described by the following action

$$\begin{aligned} S_{\rm cl} &= \int_{-\infty}^{\infty} d\tau \left(\frac{1}{2} e^{-1} \dot{X}^\mu \dot{X}_\mu - \frac{i}{2} \left(\bar{\psi} \cdot \dot{\psi} + \psi \cdot \dot{\bar{\psi}} \right) \right. \\ &\quad \left. -ie^{-1} \left(\chi \bar{\psi} \cdot \dot{X} + \bar{\chi} \psi \cdot \dot{X} \right) + \frac{1}{2} f \left(\bar{\psi} \cdot \psi - \psi \cdot \bar{\psi} \right) \right) \end{aligned} \tag{3.1}$$

where e and X^μ have the same meaning as in the case of scalar particles, the complex fermionic field χ is the super-vielbein and ψ^μ are worldline fermionic fields which will introduce spin of the particle in space-time. In fact this action is the action for worldline scalar and spinor coupling to $N = 2$ worldline supergravity (with local super-reparametrization symmetry).

Generally a relativistic spin-s particle can be described by coupling worldline scalars and spinors to $N = 2s$ worldline supergravity. The action for spin-s particle is a straight-forward extension of eq. (3.1) and can be found in [19]. Note that this action reduces to the action for scalar particles when $s = 0$.

We can quantize action (3.1) by the Faddeev-Popov method, but it will be simpler to directly construct the BRST charge, so we follow the latter approach.

We begin with the free BRST charge for arbitrary spin. A useful method for deriving gauge invariant actions is the OSp(1,1|2) formalism [22], where one starts with the light-cone SO(D−2) linearly realized by the physical states, and adds two bosonic coordinates to restore Lorentz covariance and two fermionic coordinates to cancel the additional degrees of freedom. Thus the SO(D − 2)

representation is extended to $\mathrm{OSp}(\mathrm{D}-1,1|2)$, and the non-linearly realized $\mathrm{SO}(\mathrm{D}-1,1)$ of the physical states is extended to $\mathrm{OSp}(\mathrm{D},2|2)$. The action then uses only the subgroup $\mathrm{SO}(\mathrm{D}-1,1)\otimes\mathrm{OSp}(1,1|2)$, where the $\mathrm{OSp}(1,1|2)$ is a symmetry of the unphysical (orthogonal) directions under which the physical states should be singlets (in the cohomology). We use $(A,B...)$ for $\mathrm{OSp}(\mathrm{D},2|2)$ indices, $(a,b...)$ for the $\mathrm{SO}(\mathrm{D}-1,1)$ part and $(+,-)$, $(\oplus,\ominus)$ for the bosonic and fermionic indices of $\mathrm{OSp}(1,1|2)$ respectively. The easiest way is then to begin with linear generators J^{AB} of $\mathrm{OSp}(\mathrm{D},2|2)$, use the gauge symmetry to gauge away the $+$ direction of $\mathrm{OSp}(1,1|2)$ and use equations of motion to fix the $-$ direction. Then the kinetic operator of the action is simply the delta function of the $\mathrm{OSp}(1,1|2)$ generators (now non-trivial due to solving the equation of motion).

One can further simplify things by utilizing only a subset of the generators of $\mathrm{OSp}(1,1|2)$. (This is analogous to the method of finding $\mathrm{SU}(2)$ singlets by looking at states annihilated by J_3 and J_-.) In the end one is left with the group $\mathrm{IGL}(1)$ with generators $J^{\oplus\ominus}$ and $J^{\oplus-}$. Relabeling $c=x^{\oplus}$ and $b=\partial_{\oplus}$,

$$J=iJ^{\oplus\ominus}+1=cb+iS^{\oplus\ominus},\quad Q=J^{\oplus-}=\tfrac{1}{2}c\partial^2+S^{\oplus a}\partial_a+S^{\oplus\oplus}b \tag{3.2}$$

J will be the ghost number and Q the BRST charge. One is then left with the task of finding different representation for S^{AB} satisfying the algebra

$$\left[S_{AB},S^{CD}\right\}=-\delta_{[A}{}^{[C}S_{B\}}{}^{D\}}$$

There may be more than one representation corresponding to the same spin. It is easy to build massless spin-$\frac{1}{2}$ representations using gamma matrices

$$\text{spin-}\tfrac{1}{2}:\qquad S_{AB}=-\tfrac{1}{2}[\gamma_A.\gamma_B\},\quad \{\gamma_A,\gamma_B]=-\eta_{AB}$$

and spin-1 using ket-bra

$$\text{spin-1}:\qquad S_{AB}=|_{[A}\rangle\langle_{B\}}|,\quad \langle_A|_B\rangle=\eta_{AB}$$

All higher spins can be built out of these two. For a review of the $\mathrm{OSp}(1,1|2)$ formalism see [23].

For our purpose we use first-quantized fields (i.e. fields on a worldline) to

form representations. It is known that the free relativistic spin-$\frac{N}{2}$ particle can be described by a first-quantized action with N-extended worldline supersymmetry [18]. For example, for spin $\frac{1}{2}$ we use $N = 1$ worldline fields ψ^A where ψ^a are fermionic fields and $\psi^{\oplus} = i\gamma$, $\psi^{\ominus} = i\beta$ are the bosonic ghosts for SUSY. We summarize this representation as follows

$$\begin{aligned} S^{ab} &= \tfrac{1}{2}\left[\psi^a, \psi^b\right] = \psi^a\psi^b \\ S^{\oplus a} &= \tfrac{i}{2}\left\{\gamma, \psi^a\right\} = i\gamma\psi^a \\ S^{\oplus\oplus} &= \tfrac{1}{2}\left\{\gamma, \gamma\right\} = \gamma^2 \end{aligned} \tag{3.3}$$

and

$$\begin{gathered} \{\psi^a, \psi^b\} = \eta^{ab} \\ [\gamma, \psi^a] = 0 \\ [\gamma, \gamma] = 0 \end{gathered}$$

In the following we focus on the $N = 2$ spinning particle representation for massless vector states. Now, due to $N = 2$ there are a pair of worldline spinors $\left(\psi^a, \bar{\psi}^b\right)$ and similarly bosonic ghosts $\left(\gamma, \bar{\gamma}, \beta, \bar{\beta}\right)$. The spin operators are then:

$$\begin{aligned} S^{ab} &= \bar{\psi}^a\psi^b - \bar{\psi}^b\psi^a \\ S^{\oplus a} &= i\gamma\bar{\psi}^a + i\bar{\gamma}\psi^a \\ S^{\oplus\oplus} &= 2\gamma\bar{\gamma} \end{aligned} \tag{3.4}$$

with the following (anti-)commutation relations for the fields:

$$\{\bar{\psi}^a, \psi^b\} = \eta^{ab}$$

$$\{\bar{\psi}^a, \bar{\psi}^b\} = \{\psi^a, \psi^b\} = [\gamma, \beta] = [\bar{\gamma}, \bar{\beta}] = 0$$

$$[\gamma, \bar{\beta}] = [\bar{\gamma}, \beta] = \{b, c\} = 1$$

3.3 Vertex operators

Interaction with external fields is introduced by covariantizing all the derivatives in the free BRST charge and adding a term proportional to $iF_{ab}S^{ab}$, which is the only term allowed by dimension analysis and Lorentz symmetry.

The relative coefficient can be fixed by requiring the new interacting BRST charge $Q_{\rm I}$ to be nilpotent. In general the result is:

$$Q_{\rm I} = \tfrac{1}{2} c \left(\nabla^2 + i F_{ab} S^{ab} \right) + S^{\oplus a} \nabla_a + S^{\oplus\oplus} b \tag{3.5}$$

where we use the following convention for the covariant derivative and the field strength:

$$\nabla_a \equiv \partial_a + i A_a$$

$$i F_{ab} \equiv [\nabla_a, \nabla_b]$$

The nilpotency of $Q_{\rm I}$ can be used to derive vertex operators that are Q closed. If we define the vertex operator as

$$V = Q_{\rm I} - Q$$

Then

$$Q_{\rm I}^2 = 0 \Rightarrow \{Q, V\} + V^2 = 0$$

In the linearized limit, which is relevant for asymptotic states, we take only the part of V that is linear in background fields (denoted by V_0). Then one has

$$\{Q, V_0\} = 0$$

There will be an additional U(1) symmetry in the $N = 2$ model. The vector states should be U(1) singlets and can be picked out by multiplying the original $Q_{\rm I}$ in eq. (3.5) with an additional δ function (a U(1) projector).

$$Q'_{\rm I} = \delta \left(J_{\rm U(1)} \right) Q_{\rm I}$$

$J_{\rm U(1)}$ is the U(1) current:

$$J_{\rm U(1)} = \tfrac{1}{2} \left(\psi \cdot \bar{\psi} - \bar{\psi} \cdot \psi \right) - \gamma \bar{\beta} + \bar{\gamma} \beta = -\bar{\psi} \cdot \psi + \tfrac{D}{2} - \gamma \bar{\beta} + \bar{\gamma} \beta$$

where D is the spacetime dimension and $\bar{\psi}^a$, $\bar{\gamma}$, $\bar{\beta}$ have U(1) charge -1, and their complex conjugates have $+1$. This U(1) constraint is important in that it ensures that $Q_{\rm I}$ for the $N = 2$ model is indeed nilpotent. We will show this is the case.

Before choosing any specific representation, we have

$$\begin{aligned} Q_{\rm I}'^2 &= \delta\left(J_{\rm U(1)}\right) Q_{\rm I}^2 = \delta\left(J_{\rm U(1)}\right) \tfrac{1}{2}\left\{Q_{\rm I}, Q_{\rm I}\right\} \\ &= \delta\left(J_{\rm U(1)}\right) \tfrac{1}{2}\left\{-icS^{\oplus a}[\nabla^b, F_{ab}]\right. \\ &\qquad \left. -icS^{\oplus c}S^{ab}\left[\nabla_c, F_{ab}\right] + iS^{\oplus\oplus}S^{ab}F_{ab} + iS^{\oplus a}S^{\oplus b}F_{ab}\right\} \end{aligned} \tag{3.6}$$

To understand how the projector works for the $N = 2$ model, consider normal ordering with respect to the following scalar vacuum:

$$(\gamma,\, \beta,\, \psi,\, b)\,|0\rangle = 0$$

This vacuum has U(1) charge +1 . A general normal-ordered operator with ≥ 2 barred fields on the left (unbarred fields are on the right), acting on any state built from the above vacuum, will either vanish or have negative U(1) charge. Therefore normal-ordered operators with ≥ 2 barred fields will be projected out by $\delta(J_{\rm U(1)})$. Actually this property can be made true for any vacuum: One just needs to shift the current by a constant in the projection operator.

With this in mind we have the following:

$$\begin{aligned} \delta\left(J_{\rm U(1)}\right) S^{\oplus a}S^{\oplus b} &= \delta\left(J_{\rm U(1)}\right)\left(i\gamma\bar{\psi}^a + i\bar{\gamma}\psi^a\right)\left(i\gamma\bar{\psi}^b + i\bar{\gamma}\psi^b\right) \\ &= -\delta\left(J_{\rm U(1)}\right)\bar{\gamma}\gamma\eta^{ab} = -\delta\left(J_{\rm U(1)}\right)\frac{1}{2}S^{\oplus\oplus}\eta^{ab} \\ \delta\left(J_{\rm U(1)}\right) S^{\oplus\oplus}S^{ab} &= \delta\left(J_{\rm U(1)}\right) 2\gamma\bar{\gamma}\left(\bar{\psi}^a\psi^b - \bar{\psi}^b\psi^a\right) = 0 \\ \delta\left(J_{\rm U(1)}\right) S^{\oplus c}S^{ab} &= \delta\left(J_{\rm U(1)}\right)\left(i\gamma\bar{\psi}^c + i\bar{\gamma}\psi^c\right)\left(\bar{\psi}^a\psi^b - \bar{\psi}^b\psi^a\right) \\ &= \delta\left(J_{\rm U(1)}\right)\left(i\bar{\gamma}\psi^b\eta^{ac} - i\bar{\gamma}\psi^a\eta^{bc}\right) \end{aligned} \tag{3.7}$$

Note that one could arrive at the same algebra for the spin operators if one were to use the spin-1 ket-bra representation introduced in the previous section; thus one again sees that the U(1) projector acts as picking out vector states. In fact the nilpotency of the BRST charge can be checked more easily using the ket-bra representation; however, for completeness we plug the above result into our previous calculation for $Q_I^{2\prime}$. We have

$$\delta\left(J_{\rm U(1)}\right) Q_{\rm I}^2 = c\delta\left(J_{\rm U(1)}\right)\left(\bar{\psi}^a\gamma - \bar{\gamma}\psi^a\right)\left[\nabla^b, F_{ab}\right]$$

which is proportional to the equation of motion satisfied by the asymptotic states. So we have proved that $\delta\left(J_{\mathrm{U}(1)}\right) Q_{\mathrm{I}}^2 = 0$.

The vertex operator is then easily obtained by considering Q_{I} as an expansion of Q,

$$\begin{aligned} V_0 &= [Q_{\mathrm{I}} - Q]_{\text{linear in } A} \\ &\equiv cW_{\mathrm{I}} + W_{\mathrm{II}} \\ &= \tfrac{1}{2} c \left[2iA \cdot \partial + i \left(\partial_a A_b - \partial_b A_a\right) S^{ab}\right] + iA_a S^{\oplus a} \\ &= -\epsilon_a \left[c\left(i\dot{X}^a + \bar{\psi}^b \psi^a k_b - \bar{\psi}^a \psi^b k_b\right) + \left(\gamma \bar{\psi}^a + \bar{\gamma} \psi^a\right)\right] \exp\left[ik \cdot X\left(\tau\right)\right] \end{aligned} \tag{3.8}$$

This vertex operator satisfies

$$\{Q, V_0\} = 0$$

The integrated vertex can be derived by noting:

$$[Q, W_{\mathrm{I}}] = \partial V_0 \;\rightarrow\; \left[Q, \int W_{\mathrm{I}}\right] = 0$$

More complicated vertex operators are needed for the usual worldline formalism. We will discuss in detail how these operators arise in section V. In the worldgraph formalism linearized vertex operators derived above will be sufficient.

We conclude this section by give some remarks on the vacuum. When calculating amplitudes, the vacuum with which one chooses to work dictates the form of vertex operator and insertions one needs. In string theory, different choices of vacuum are called different pictures. The scalar vacuum discussed above is defined by the expectation value

$$\langle 0|c|0\rangle \sim 1$$

The conformal vacuum of string theory

$$\langle 0|ccc|0\rangle \sim 1$$

does not exist in particle theory since there aren't that many zero modes to

saturate at tree level. On the other hand one could also treat the worldline SUSY ghosts' zero modes, which would require additional insertions. These are defined by the vacuum

$$(\bar{\beta},\, \beta,\, \psi,\, b)\left|\hat{0}\right\rangle = 0 \Rightarrow \left\langle \hat{0}\right|c\delta(\gamma)\delta(\bar{\gamma})\left|\hat{0}\right\rangle \sim 1$$

which has U(1) charge 2 and is thus not a physical vacuum.

To use the vertex operator we found above, we need to find a U(1) neutral vacuum $\left|\tilde{0}\right\rangle$ that is in the cohomology of the free BRST charge Q. It is related to the previous vacuum through the following relation:

$$\left|\tilde{0}\right\rangle = \bar{\beta}\left|0\right\rangle = \delta\left(\frac{1}{2}\gamma^2\right)\left|\hat{0}\right\rangle,$$

which leads to

$$\left\langle \tilde{0}\right|\gamma c\bar{\gamma}\left|\tilde{0}\right\rangle \sim 1$$

This vacuum can be understood as the Yang-Mills ghost. It has ghost number -1 and lies in the cohomology only at zero momentum, indicating a constant field. Therefore it corresponds to the global part of the gauge symmetry: Gauge parameters satisfying $Q\Lambda = 0$ have no effect on the gauge transformations in the free theory, $\delta\phi = Q\Lambda$. In principal one could proceed to compute amplitudes in the available vacua mentioned above; however, due to its U(1) neutral property, the Yang-Mills ghost vacuum should be the easiest to extend to higher loops, since it would be easier to enforce U(1) neutrality.

3.4 Green functions

In section 2.4, we have already derived the Green function for X fields on any worldgraph. In this section we consider the Green function for the fermions ψ and $\bar{\psi}$.

3.4.1 Worldline

We start with the simplest case - the $\psi\bar{\psi}$ propagator on an infinite worldline. Using the Yang-Mills ghost vacuum $\left|\tilde{0}\right\rangle$ we defined in the last subsection, this

propagator is easily evaluated,

$$\eta^{ab} G_F(\tau, \tau') \equiv \left\langle \psi^a(\tau)\, \bar{\psi}^b(\tau') \right\rangle = \eta^{ab} \Theta(\tau - \tau')$$

where Θ is a step function which is zero if the argument is negative and one if the argument is positive. Note that the fermionic Green function does not have the naive relation with the bosonic Green function

$$G_F \neq -\dot{G}_{\mathrm{B}} = \frac{1}{2}\mathrm{sign}(\tau - \tau')$$

It differs by a constant $\frac{1}{2}$. This is due to different boundary conditions: The vacuum we choose, which is at $t = -\infty$, is defined to be annihilated by ψ^a; therefore on a time ordered line the expectation can be non-vanishing only if ψ is at later time then $\bar{\psi}$.

Later, when we discuss the amplitudes, we will present two different-looking but equivalent approaches. One is called the worldline approach and the other is called the worldgraph approach. In worldline approach, a worldgraph is always decomposed into worldlines and thus knowing the worldline propagator is sufficient. However, it will be nicer if we can avoid this decomposition and directly use the Green function on a particular worldgraph. We have shown that this is possible for the scalar amplitudes by giving a general formula of the Green function on arbitrary worldgraph, but it is not known whether the general formula for the $\psi\bar{\psi}$ Green function on arbitrary worldgraph exists. In the following subsections we will study a few special cases.

3.4.2 Three-point tree

As mentioned previously, it is desirable even for tree graphs to develop a formalism that does not require an identification of a worldline to which external states are attached. Intuitively such a formalism would require one to simply identify 1D topological spaces that connect the external lines. This idea is very similar to string theory calculations and goes back as far as 1974 [24]. The main challenge for this "worldgraph approach" (following [25]) is the definition of Green functions on these non-differentiable topological spaces (non-differentiable because at interacting points it is not locally R^1). Previously,

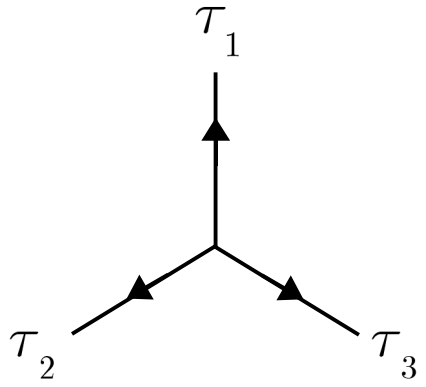

Figure 3.1: The topological space for a three-point interaction.

for multi-loops such Green functions have been derived by a combination of one-loop Green functions and insertions: See [26] for review. Recently in [15] a more straightforward way to derive multi-loop Green functions was developed for scalar particles using the electric circuit analog. (A similar approach was used in [25].) Since fermion Green functions are related to bosons through a derivative (up to additional terms due to choice of vacuum or boundary conditions), what remains is to consistently define derivatives on these 1D topological spaces. We will use tree graphs as our testing ground.

Consider the three-point amplitude: One has only one graph, fig. 3.1. The arrows indicate the direction in which each τ_i is increasing. For scalar fields it was shown [15] that the appropriate Green function is proportional to the distance between two insertions; for the 3-point graph this is taken to be $-\frac{1}{2}(\tau_i + \tau_j)$.

To define derivatives, one notes that they are worldline vectors and therefore must be conserved at each interaction point. This leads to the conclusion that if we denote the worldgraph derivative on each line as $D(\tau_i)$, for the three-point graph they must satisfy:

$$D_{\tau_1} + D_{\tau_2} + D_{\tau_3} = 0 \tag{3.9}$$

This can be solved by defining the worldgraph derivatives as follows:

$$
\begin{aligned}
D_{\tau_1} &= \partial_{\tau_2} - \partial_{\tau_3} \\
D_{\tau_2} &= \partial_{\tau_3} - \partial_{\tau_1} \\
D_{\tau_3} &= \partial_{\tau_1} - \partial_{\tau_2}
\end{aligned}
\tag{3.10}
$$

There is another solution which corresponds to (counter-)clockwise orientation. The choice of orientation can be fixed by matching it with the color ordering. Since the derivative is a local operator, its definition will not be altered if the three-point graph is connected to other pieces to form larger graphs. The fermionic Green function then follows from the bosonic by taking ψ as a worldline scalar and $\bar{\psi}$ as a worldline vector:

$$
G_F(\tau_i, \tau_j) \equiv \langle \bar{\psi}(\tau_i)\psi(\tau_j)\rangle = 2\langle D_{\tau_i} X(\tau_i) X(\tau_j)\rangle
$$

Armed with these two Green functions we can show how the three-point amplitude works.

Now the worldline derivatives in $W_{\rm I}$ are replaced by worldgraph derivatives defined in eq. (3.10) and they give:

$$
\left\langle i\epsilon_1 \cdot D_{\tau_1} X(\tau_1) e^{i\left[\sum_{i=1}^3 k_i \cdot X(\tau_i)\right]} \right\rangle = -(\epsilon_1 \cdot k_3)
$$

$$
\left\langle i\epsilon_2 \cdot D_{\tau_2} X(\tau_2) e^{i\left[\sum_{i=1}^3 k_i \cdot X(\tau_i)\right]} \right\rangle = -(\epsilon_2 \cdot k_1)
$$

$$
\left\langle i\epsilon_3 \cdot D_{\tau_3} X(\tau_3) e^{i\left[\sum_{i=1}^3 k_i \cdot X(\tau_i)\right]} \right\rangle = -(\epsilon_3 \cdot k_2)
$$

The fermionic Green functions are (with $F_{ij} \equiv \langle \bar{\psi}(\tau_i)\psi(\tau_j)\rangle$):

$$
\begin{aligned}
F_{12} = -1, \quad F_{23} = -1, \quad F_{31} = -1 \\
F_{21} = +1, \quad F_{32} = +1, \quad F_{13} = +1
\end{aligned}
\tag{3.11}
$$

3.4.3 Four-point tree

Now we consider the four-point tree. As in [15] the bosonic Green function is still $-\frac{1}{2}L$, where L is the length between two fields. Thus it is the same as in the three-point case, except that when the two fields sit on opposite ends of the modulus, one needs to add the value of the modulus T. The worldgraph

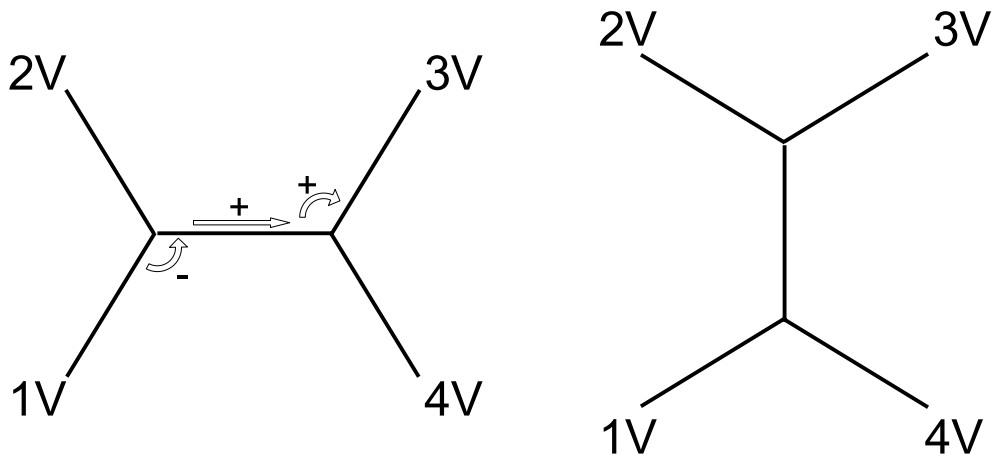

Figure 3.2: The two graphs for the four-point interaction

derivatives still act the same way, since the definition is local, irrespective of other parts of the graph. This gives the following result for the s-channel graph:

$$\left\langle i\epsilon_1 \cdot D_{\tau_1} X(\tau_1) e^{i\left[\sum_{i=1}^4 k_i \cdot X(\tau_i)\right]} \right\rangle = -(\epsilon_1 \cdot k_2)$$

$$\left\langle i\epsilon_2 \cdot D_{\tau_2} X(\tau_2) e^{i\left[\sum_{i=1}^4 k_i \cdot X(\tau_i)\right]} \right\rangle = +(\epsilon_2 \cdot k_1)$$

$$\left\langle i\epsilon_3 \cdot D_{\tau_3} X(\tau_3) e^{i\left[\sum_{i=1}^4 k_i \cdot X(\tau_i)\right]} \right\rangle = -(\epsilon_3 \cdot k_4)$$

$$\left\langle i\epsilon_4 \cdot D_{\tau_4} X(\tau_4) e^{i\left[\sum_{i=1}^4 k_i \cdot X(\tau_i)\right]} \right\rangle = +(\epsilon_4 \cdot k_3)$$

The fermionic Green functions are again more subtle. There are two types, that for bc ghosts and that for the $\bar{\psi}\psi$. First one notes that on the modulus, which is a worldline, both Green functions should be a step function, as explained in the previously. This is sufficient for the b, c ghosts. For $\bar{\psi}\psi$, since they can contract with each other on the same three-point graph or contract across the modulus, one must take the combined result: For contractions on the same three-point graph the rules are just as eq. (3.11), while for contraction across the modulus one multiplies the two Green functions on the two vertices with one from the modulus. For example, in the s-channel graph fig. 3.2:

$$\left\langle \bar{\psi}(\tau_1)\psi(\tau_3) \right\rangle = \left\langle \bar{\psi}(\tau_1)\psi(\tau_T) \right\rangle \left\langle \bar{\psi}(\tau_T)\psi(\tau_3) \right\rangle \Theta(T) = -1$$

As one can see, the contraction across the modulus is broken down as if there were a pair $\bar{\psi}\psi$ on each end of the modulus, contracting with the vertices separately, and a final step function due to the fact that the modulus is a worldline. (We choose the left time to be earlier.) We now list all the relevant Green functions for the s-channel graph. The Green functions for the bc ghosts are

$$\langle c(\tau_1)b(T)\rangle = 1, \quad \langle c(\tau_2)b(T)\rangle = 1, \quad \langle c(\tau_3)b(T)\rangle = 0, \quad \langle c(\tau_4)b(T)\rangle = 0$$

and the Green functions for the $\bar{\psi}\psi$ are (recall that we have defined $F_{ij} \equiv \langle\bar{\psi}(\tau_i)\psi(\tau_j)\rangle$)

$$\begin{array}{llll} F_{12} = +1, & F_{21} = -1, & F_{34} = +1, & F_{43} = -1 \\ F_{23} = +1, & F_{32} = 0, & F_{14} = +1, & F_{41} = 0 \\ F_{13} = -1, & F_{31} = 0, & F_{24} = -1, & F_{42} = 0 \end{array}$$

3.4.4 One-loop

We now discuss the fermionic Green function on a loop needed in worldgraph approach. In this case we need to insert a U(1) projector in the loop to pick out all the U(1) neutral states. That is, one inserts:

$$\begin{aligned} \delta\left[J_{\mathrm{U}(1)}\right] &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp\left[i\frac{\theta}{T}\int_0^T d\tau J_{\mathrm{U}(1)}\right] \\ &= \frac{1}{2\pi}\int_0^{2\pi} d\theta \exp\left[i\frac{\theta}{T}\int_0^T d\tau(-\bar{\psi}\cdot\psi + \tfrac{D}{2} - \gamma\bar{\beta} + \bar{\gamma}\beta)\right] \end{aligned}$$

Similar approaches have been taken in [8] and [27]. In [8], $i\theta$ is interpreted as a mass to be taken to infinity at the end, and together with GSO-like projection kills all U(1) non-neutral states. For us the U(1) projector naturally gets rid of all unwanted states. Furthermore the worldline ghosts were not taken into account in [8]; therefore they need to include the effect of Faddeev-Popov ghosts by adding covariant scalars to the action. This is sufficient for one loop, since they couple in the same way, yet will no longer be true for higher loops. Here we've (and also [27]) included all the worldline ghosts; thus the Faddeev-Popov ghosts are naturally included. In [27] gauge fixing the U(1) gauge field on a loop leads to a modulus, which is equivalent to θ in our U(1)

projector insertion. The two views are analogous.

The inclusion of a U(1) projector amounts to additional quadratic terms in the action which will modify the Green function and introduce an additional θ-dependent term to the measure. Here we give a brief discussion of its effect. The kinetic operator for the SUSY partners and SUSY ghosts is now:

$$\partial_\tau + i\frac{\theta}{T}$$

The θ term can be absorbed by redefining the U(1) charged fields,

$$\Psi' = e^{i\theta\tau/T}\Psi \quad \bar{\Psi}' = e^{-i\theta\tau/T}\bar{\Psi}$$

where $\Psi = (\psi^a, \gamma, \beta)$. Then the integration over θ is really integrating over all possible boundary conditions since:

$$\Psi'(T) = e^{i\theta}\Psi'(0)$$

Without loss of generality, we choose the periodic boundary condition for the original fields Ψ.

The fermionic Green function will be modified to

$$G_F(\tau,\tau') = \frac{e^{-\frac{i\theta(\tau-\tau')}{T}}}{2i\sin\frac{\theta}{2}}\left[e^{i\frac{\theta}{2}}\Theta(\tau-\tau') + e^{-i\frac{\theta}{2}}\Theta(\tau'-\tau)\right]$$

which satisfies the periodic boundary condition and differential equation

$$\left(\partial_\tau + i\frac{\theta}{T}\right)G_F(\tau,\tau') = \delta(\tau-\tau')$$

3.5 Vacuum bubble

The vacuum bubble amplitude is the path integral without vertex operator insertions. In the present case, it is the product of two parts. One is the path integral over the coordinate scalars (i.e. X) and the coordinate ghosts and the other is the path integral over the SUSY partners (i.e. ψ and $\bar{\psi}$) and SUSY ghosts. The first part is exactly the same as the vacuum bubble amplitude in scalar case, while the second part is new and we now discuss.

3.5.1 Trees

For a tree level worldgraph (including the worldline), this path integral is always trivially 1. In the worldline approach, since we always decompose a worldgraph into the worldlines of particles, we only need to know this vacuum bubble. In the worldgraph approach, this is the appropriate vacuum bubble for all tree amplitudes.

3.5.2 One-loop

To use worldgraph approach for one-loop amplitudes, we need to know the 1-loop vacuum bubble. It is can be computed through mode expansion on a circle with periodic boundary condition:

$$\mathrm{Det}\left(\partial_\tau + i\frac{\theta}{T}\right)^{D-2} = \left[2i\sin\left(\tfrac{\theta}{2}\right)\right]^{D-2}$$

where D comes from the $\psi\bar{\psi}$ integration and -2 comes from SUSY ghosts.

3.6 Amplitudes

Now we are ready to calculate Yang-Mills amplitudes. As we have mentioned there are two equivalent routes to proceed from now. One (the worldline approach) is to always consider an arbitrary worldgraph as being constructed from worldlines of the particles and one uses only the Green function of $\psi\bar{\psi}$ on a line, and the other (the worldgraph approach) is to consider the worldgraph as a 1D topological space and one uses Green function of $\psi\bar{\psi}$ on different topology. In both approaches, rules of calculating amplitudes beyond 1-loop level are not known when this thesis is written.

3.6.1 Worldline approach

In the worldline approach, one starts by choosing a specific worldline, and then inserts relevant vertex operators for external states. For YM theory, where the worldline state is the same as external states, namely a vector, the choice for worldline is less obvious. Previous work on the worldline formalism was geared

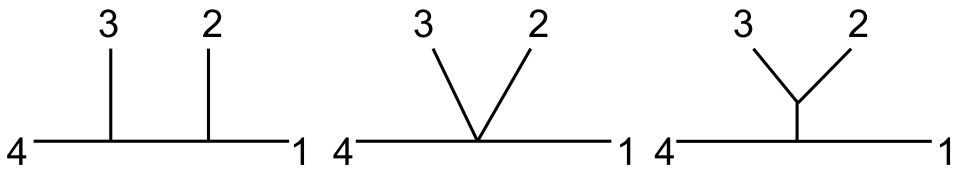

Figure 3.3: Three diagrams to be calculated if one chooses to connect lines 1 and 4 as the worldline. The second diagram needs a pinch operator, and the third diagram needs a vertex operator representing the tree attached to the worldline.

toward the calculation of one-loop amplitudes, where the loop itself provides a natural candidate for the worldline. This advantage is not present for tree or higher-loop amplitudes. Furthermore, for higher-point tree graphs, calculating the amplitude from the worldline requires sewing lower-point tree amplitudes to the worldline. This is unsatisfactory from the viewpoint of first-quantized perturbation theory.

In general, to calculate an n-point tree-level partial amplitude in the worldline approach:

1. Choose a specific color ordering (e.g., 12...n). Label external lines counter-clockwise.

2. Draw a worldline between any two of the external lines (e.g., line 1 and line n) and connect all other external lines to this worldline in the following three ways: (a) Use the linearized vertex operator V_0 defined in section 3.3. (b) Use a vertex operator that is quadratic in background fields ("pinching"). This quadratic vertex operator ("pinch operator") can be derived from eq.(3.8) by extending the field strength to contain the non-abelian terms and takes the form

$$v^{(ij)} = \epsilon_{ia}\epsilon_{jb}c\left(\bar{\psi}^b\psi^a - \bar{\psi}^a\psi^b\right)e^{i(k_i+k_j)\cdot X}$$

(c) Have the external lines first form a lower-point tree graph and then connect to the worldline through either of the two vertex operators mentioned previously. This corresponds to replacing $A^a = \epsilon^a e^{ik\cdot X}$ with the non-linear part of the solution to the field equations that the background field satisfies. For example, for a four-point tree amplitude there are the three graphs shown in fig. 3.3, representing the three different ways external fields can attach to the worldline.

For lower-point graphs it is possible to choose the worldline in such a way that only linear vertex operators are required. We will show this in our actual computation for the four-point amplitude.

3. For each of the diagrams from above, insert three fixed vertex operators (respectively fixed at $\tau = \infty, 0, -\infty$). Two of them represent the initial and final external states that were connected to form the worldline, while the remaining one can be any of the operators described above. For example, one has:

$$V^{(n)}(\infty)V^{(2)}(0)V^{(1)}(-\infty) \text{ or } V^{(n)}(\infty)v^{(32)}(0)V^{(1)}(-\infty)$$

where the superscript (i) represents the momentum and polarization vector of the external line i.

4. Insert the remaining vertex operators as the integrated ones, e.g.,

$$\int W_{\rm I}^{(i)} \text{ or } \int v^{(ij)}$$

with the integration regions so chosen that the diagram is kept planar.

5. Evaluate the expectation value with respect to the Yang-Mills ghost vacuum.

The fact that one needs to calculate lower-point tree graphs for a general tree graph is unsatisfactory, since one should be able to calculate an arbitrary-point amplitude without the knowledge of its lower-point counterparts. This was less of a problem in the previous one-loop calculations, since one can claim that the method was really for 1PI graphs, and therefore sewing is necessary to calculate graphs that are not 1PI. It is more desirable to be able to calculate any amplitude with the knowledge of just the vertex operators and Green functions. This will be the aim of the worldgraph approach, which we leave to the next subsection. We first proceed to show how to calculate 3- and 4-point trees, and one-loop amplitudes, by the worldline approach.

In the 3-point case, we connect line 1 and line 3 as the worldline. The three vertex operators are respectively fixed at $\tau_C \to \infty$, $\tau_B = 0$ and $\tau_A \to -\infty$. Note that we need one c ghost to saturate the zero-mode and give a non-

vanishing expectation value:

$$\begin{aligned} A_3 &= \left\langle V^{(3)}(\tau_C)\, V^{(2)}(\tau_B)\, V^{(1)}(\tau_A)\right\rangle \\ &= \left\langle \left[cW_{\rm I}^{(3)}(\tau_C)\right]\left[W_{\rm II}^{(2)}(\tau_B)\right]\left[W_{\rm II}^{(1)}(\tau_A)\right]\right\rangle \\ &+ \left\langle \left[W_{\rm II}^{(3)}(\tau_C)\right]\left[cW_{\rm I}^{(2)}(\tau_B)\right]\left[W_{\rm II}^{(1)}(\tau_A)\right]\right\rangle \\ &+ \left\langle \left[W_{\rm II}^{(3)}(\tau_C)\right]\left[W_{\rm II}^{(2)}(\tau_B)\right]\left[cW_{\rm I}^{(1)}(\tau_A)\right]\right\rangle \end{aligned} \tag{3.12}$$

The first term and the third term vanish due to $\epsilon\cdot\dot{X}$ in $W_{\rm I}$ contracting with the $e^{ik\cdot X}$ in the other two $W_{\rm II}$'s, which are proportional to $\epsilon_3\cdot k_3$ and $\epsilon_1\cdot k_1$ respectively, and vanish in the Lorenz gauge. The remaining term becomes

$$\begin{aligned} A_3 &= \left\langle \left[W_{\rm II}^{(3)}(\tau_C)\right]\left[cW_{\rm I}^{(2)}(\tau_B)\right]\left[W_{\rm II}^{(1)}(\tau_A)\right]\right\rangle \\ &= -\epsilon_{3a}\epsilon_{2c}\epsilon_{1d}\Big\langle \left[\left(\gamma\bar{\psi}^a+\bar{\gamma}\psi^a\right)e^{ik_3\cdot X}\right]_{\tau_C} \\ &\qquad \times c\left[\left(k_1^c+(\bar{\psi}^b\psi^c-\bar{\psi}^c\psi^b)k_{2b}\right)e^{ik_2\cdot X}\right]_{\tau_B}\left[\left(\gamma\bar{\psi}^d+\bar{\gamma}\psi^d\right)e^{ik_1\cdot X}\right]_{\tau_A}\Big\rangle \\ &= -\left[(\epsilon_3\cdot\epsilon_1)(\epsilon_2\cdot k_3)+(\epsilon_1\cdot\epsilon_2)(\epsilon_3\cdot k_1)+(\epsilon_2\cdot\epsilon_3)(\epsilon_1\cdot k_2)\right] \end{aligned} \tag{3.13}$$

As usual (see, e.g., [15]), the contractions among the exponentials give an overall factor of $e^{-\sum_{A\le i<j\le C} k_i\cdot k_j G_B(\tau_i-\tau_j)}$ in the final result, but this factor equals 1 if we go on-shell.

For the 4-point amplitude (with color-ordering 1234), one can calculate the three diagrams in fig. 3.3, but as we have mentioned, one can simplify the calculation by choosing a worldline between line 1 and line 3. In this case, there is only one diagram to be calculated (fig. 3.4), and there is only one integrated vertex operator — line 4. We fix the other three as $\tau_{\rm D}\to\infty$, $\tau_{\rm C}=0$ and $\tau_{\rm A}\to-\infty$, and the integrated vertex has integration region $\tau_{\rm D}\ge\tau_{\rm B}\ge\tau_{\rm A}$. We then have:

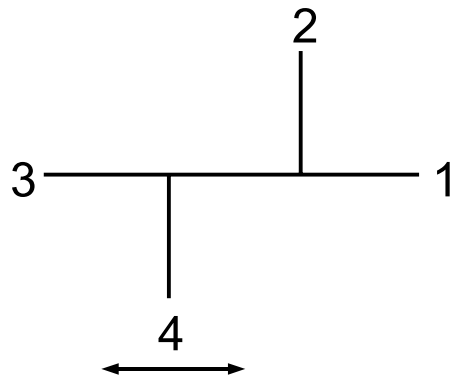

Figure 3.4: If one chooses to connect line 1 and line 3 as the worldline, there is only one diagram to be calculated. There is no need for pinch or more complicated operators. Note that line 4 is the integrated vertex and the integration region can be from $-\infty$ to $+\infty$, still keeping the graph planar.

$$
\begin{aligned}
A_4 &= \left\langle \left[V^{(3)}(\tau_D)\right]\left[V^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[V^{(1)}(\tau_A)\right]\right\rangle \qquad (3.14)\\
&= \left\langle \left[cW_{\mathrm{I}}^{(3)}(\tau_D)\right]\left[W_{\mathrm{II}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle\\
&\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[cW_{\mathrm{I}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle\\
&\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[W_{\mathrm{II}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[cW_{\mathrm{I}}^{(1)}(\tau_A)\right]\right\rangle
\end{aligned}
$$

The first and third term again vanish, for the same reason as in the three-point case. The remaining term can be written in two parts by separating the integration region:

$$
\begin{aligned}
A_4 &= A_{4s} + A_{4t} \qquad (3.15)\\
&= \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[cW_{\mathrm{I}}^{(4)}(\tau_C)\right]\left[\int_{\tau_A}^{\tau_C} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle\\
&\quad + \left\langle \left[W_{\mathrm{II}}^{(3)}(\tau_D)\right]\left[\int_{\tau_C}^{\tau_D} W_{\mathrm{I}}^{(2)}(\tau_B)\,d\tau_B\right]\left[cW_{\mathrm{I}}^{(4)}(\tau_C)\right]\left[W_{\mathrm{II}}^{(1)}(\tau_A)\right]\right\rangle
\end{aligned}
$$

Actually one can see these two terms as representing the s-channel and t-

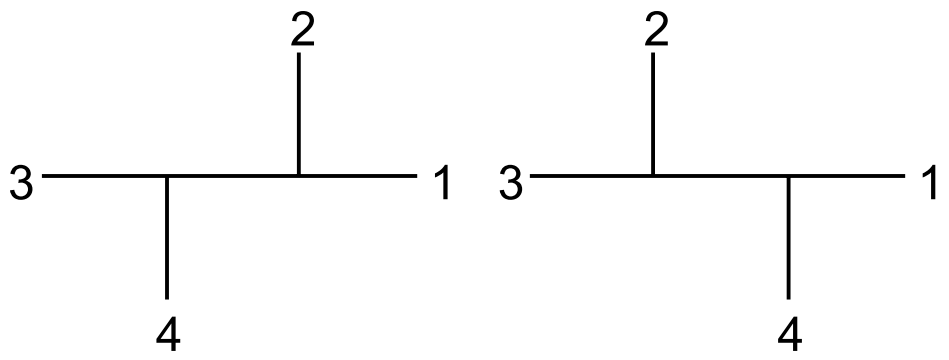

Figure 3.5: Two integration regions. The integrated vertex sits at 2.

channel graphs from the second-quantized approach (see fig. 3.5). The τ's are time ordered according to the order they appear on the worldline. The results are:

$$A_{4s} = -\frac{2}{s}\left[\begin{array}{l} -\frac{s}{4}\left(\epsilon_1\cdot\epsilon_3\right)\left(\epsilon_2\cdot\epsilon_4\right) - \frac{u}{2}\left(\epsilon_1\cdot\epsilon_2\right)\left(\epsilon_4\cdot\epsilon_3\right) \\ +\left(\epsilon_2\cdot k_1\right)\left(\epsilon_4\cdot k_3\right)\left(\epsilon_1\cdot\epsilon_3\right) + \left(\epsilon_1\cdot k_2\right)\left(\epsilon_3\cdot k_4\right)\left(\epsilon_2\cdot\epsilon_4\right) \\ +\left(\epsilon_1\cdot k_3\right)\left(\epsilon_2\cdot k_4\right)\left(\epsilon_3\cdot\epsilon_4\right) + \left(\epsilon_4\cdot k_2\right)\left(\epsilon_3\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_2\right) \\ -\left(\epsilon_1\cdot k_2\right)\left(\epsilon_4\cdot k_3\right)\left(\epsilon_2\cdot\epsilon_3\right) - \left(\epsilon_3\cdot k_4\right)\left(\epsilon_2\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_4\right) \\ -\left(\epsilon_1\cdot k_4\right)\left(\epsilon_2\cdot k_3\right)\left(\epsilon_3\cdot\epsilon_4\right) - \left(\epsilon_3\cdot k_2\right)\left(\epsilon_4\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_2\right) \end{array}\right]$$

$$A_{4t} = -\frac{2}{t}\left[\begin{array}{l} -\frac{t}{4}\left(\epsilon_1\cdot\epsilon_3\right)\left(\epsilon_2\cdot\epsilon_4\right) - \frac{u}{2}\left(\epsilon_1\cdot\epsilon_4\right)\left(\epsilon_2\cdot\epsilon_3\right) \\ +\left(\epsilon_1\cdot k_4\right)\left(\epsilon_3\cdot k_2\right)\left(\epsilon_2\cdot\epsilon_4\right) + \left(\epsilon_2\cdot k_3\right)\left(\epsilon_4\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_3\right) \\ +\left(\epsilon_1\cdot k_3\right)\left(\epsilon_4\cdot k_2\right)\left(\epsilon_2\cdot\epsilon_3\right) + \left(\epsilon_2\cdot k_4\right)\left(\epsilon_3\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_4\right) \\ -\left(\epsilon_1\cdot k_2\right)\left(\epsilon_4\cdot k_3\right)\left(\epsilon_2\cdot\epsilon_3\right) - \left(\epsilon_2\cdot k_1\right)\left(\epsilon_3\cdot k_4\right)\left(\epsilon_1\cdot\epsilon_4\right) \\ -\left(\epsilon_1\cdot k_4\right)\left(\epsilon_2\cdot k_3\right)\left(\epsilon_3\cdot\epsilon_4\right) - \left(\epsilon_3\cdot k_2\right)\left(\epsilon_4\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_2\right) \end{array}\right]$$

The sum of the above two parts is exactly the 4-point Yang-Mills tree amplitude. Note that we don't need the pinch operator in this calculation. This is because there cannot be a pinch operator representing line 2 and line 4, since they are not adjacent in the color ordering.

It is straightforward to generalize this method to the calculation of 1-loop 1PI diagrams. The new feature in this case is that one must ensure U(1) neutrality inside the loop. One can think of the diagram as connecting both ends of a tree diagram, and only sum over U(1) neutral states. The U(1)

neutral states are written as:

$$|A,p\rangle = \begin{cases} |a,p\rangle = \gamma\bar{\psi}^a \left|\tilde{0}\right\rangle \otimes |p\rangle \\ |\text{ghost},p\rangle = \left|\tilde{0}\right\rangle \otimes |p\rangle \\ |\text{antighost},p\rangle = \gamma\bar{\gamma} \left|\tilde{0}\right\rangle \otimes |p\rangle \end{cases}$$

where p is the momentum of the state, and the last two states are the Faddeev-Popov ghosts for background gauge fixing. The general expression for the amplitude of n-point 1-loop 1PI diagrams is then

$$\begin{aligned} A_n^{1-\text{loop}} &= \sum_{A,p} \int_0^\infty dT \left\langle A,p\right| V^{(n)}\left(\tau_n\right) \prod_{i=1}^{n-1} \int_{\tau_{i-1}\leq\tau_i\leq\tau_{i+1}} d\tau_i W_{\text{I}}^{(i)}\left(\tau_i\right) \left|A,p\right\rangle \\ &\quad + \text{diagrams with pinch operators} \end{aligned} \tag{3.16}$$

where we define $\tau_0 = 0$ and fix $\tau_n = T$. Note that at one-loop we don't have the freedom to choose worldline (it should always be the loop), so one cannot avoid using the pinch operators.

We postpone an actual calculation to the next subsection, where we will calculate the two-point contribution to the effective action by using the world-graph approach.

3.6.2 Worldgraph approach

For the three-point tree graph fig. 3.1 we start with:

$$\begin{aligned} A_3 &= \left\langle V^{(3)}\left(\tau_3\right) V^{(2)}\left(\tau_2\right) V^{(1)}\left(\tau_1\right)\right\rangle \\ &= \left\langle \left[cW_{\text{I}}^{(3)}\left(\tau_3\right)\right] \left[W_{\text{II}}^{(2)}\left(\tau_2\right)\right] \left[W_{\text{II}}^{(1)}\left(\tau_1\right)\right]\right\rangle \\ &\quad + \left\langle \left[W_{\text{II}}^{(3)}\left(\tau_3\right)\right] \left[cW_{\text{I}}^{(2)}\left(\tau_2\right)\right] \left[W_{\text{II}}^{(1)}\left(\tau_1\right)\right]\right\rangle \\ &\quad + \left\langle \left[W_{\text{II}}^{(3)}\left(\tau_3\right)\right] \left[W_{\text{II}}^{(2)}\left(\tau_2\right)\right] \left[cW_{\text{I}}^{(1)}\left(\tau_1\right)\right]\right\rangle \end{aligned} \tag{3.17}$$

Using the above one can compute eq. (3.17). The first term becomes:

$$\begin{aligned} A_{3-1} &= \langle [cW_{\rm I}(\tau_3)]\,[W_{\rm II}(\tau_2)]\,[W_{\rm II}(\tau_1)]\rangle \\ &= -\epsilon_{3a}\epsilon_{2c}\epsilon_{1d}\langle c[iDX^a + (\bar\psi^b\psi^a - \bar\psi^a\psi^b)k_{3b}]_{\tau_3} \\ &\quad \times[\gamma\bar\psi^c + \bar\gamma\psi^c]_{\tau_2}[\gamma\bar\psi^d + \bar\gamma\psi^d]_{\tau_1} e^{ik_1\cdot X_{\tau_1}} e^{ik_2\cdot X_{\tau_2}} e^{ik_3\cdot X_{\tau_3}}\rangle \\ &= -\epsilon_{2c}\epsilon_{1d}\langle[-(\epsilon_3\cdot k_2) + (\bar\psi^b\psi^a - \bar\psi^a\psi^b)\epsilon_{3a}k_{3b}]_{\tau_3} \\ &\quad \times[\bar\psi^c(\tau_2)\psi^d(\tau_1) + \psi^c(\tau_2)\bar\psi^d(\tau_1)]\rangle \\ &= 2(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_1) + 2(\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_1) + 2(\epsilon_1\cdot k_3)(\epsilon_2\cdot\epsilon_3) \end{aligned} \tag{3.18}$$

A similar derivation gives the second and third terms:

$$\begin{aligned} A_{3-2} &= \langle [W_{\rm II}(\tau_3)]\,[cW_{\rm I}(\tau_2)]\,[W_{\rm II}(\tau_1)]\rangle \\ &= 2(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_1) + 2(\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_1) + 2(\epsilon_1\cdot k_3)(\epsilon_2\cdot\epsilon_3) \\ A_{3-3} &= \langle [W_{\rm II}(\tau_3)]\,[W_{\rm II}(\tau_2)]\,[cW_{\rm I}(\tau_1)]\rangle \\ &= 2(\epsilon_3\cdot k_2)(\epsilon_2\cdot\epsilon_1) + 2(\epsilon_2\cdot k_1)(\epsilon_3\cdot\epsilon_1) + 2(\epsilon_1\cdot k_3)(\epsilon_2\cdot\epsilon_3) \end{aligned} \tag{3.19}$$

Note that the three terms are the same, which respects the symmetry of the graph.

For the 4-point amplitude we have two graphs (s channel and t channel, see fig. 3.2) constructed by connecting two three-point worldgraphs on a worldline. The worldline in the middle is actually a modulus of the theory, and one must insert a b ghost. We focus on the s-channel graph; the t-channel graph can later be derived by exchanging the external momenta and polarizations in the s-channel amplitude. We wish to derive

$$\begin{aligned} A_{4s} &= \int_0^\infty dT\,\left\langle V^{(4)}(\tau_4)V^{(3)}(\tau_3)b(T)V^{(2)}(\tau_2)V^{(1)}(\tau_1)\right\rangle \\ &= \int_0^\infty dT \left[\begin{array}{l} \left\langle W_{\rm II}^{(4)}(\tau_4)cW_{\rm I}^{(3)}(\tau_3)b(T)cW_{\rm I}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ +\left\langle cW_{\rm I}^{(4)}(\tau_4)cW_{\rm I}^{(3)}(\tau_3)b(T)W_{\rm II}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ +\left\langle cW_{\rm I}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)b(T)cW_{\rm I}^{(2)}(\tau_2)W_{\rm II}^{(1)}(\tau_1)\right\rangle \\ +\left\langle cW_{\rm I}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)b(T)W_{\rm II}^{(2)}(\tau_2)cW_{\rm I}^{(1)}(\tau_1)\right\rangle \\ +\left\langle W_{\rm II}^{(4)}(\tau_4)cW_{\rm I}^{(3)}(\tau_3)b(T)W_{\rm II}^{(2)}(\tau_2)cW_{\rm I}^{(1)}(\tau_1)\right\rangle \\ +\left\langle W_{\rm II}^{(4)}(\tau_4)W_{\rm II}^{(3)}(\tau_3)b(T)cW_{\rm I}^{(2)}(\tau_2)cW_{\rm I}^{(1)}(\tau_1)\right\rangle \end{array}\right] \end{aligned} \tag{3.20}$$

Equipped with the Green functions one can compute eq. (3.20). We do the bc contractions first. Each term has two such contractions; using the above Green functions we see that the second and last terms cancel. We then have:

$$A_{4s} = \int_0^\infty dT \left[\begin{array}{l} \left\langle cW_{\rm II}^{(4)}(\tau_4) W_{\rm I}^{(3)}(\tau_3) W_{\rm I}^{(2)}(\tau_2) W_{\rm II}^{(1)}(\tau_1) \right\rangle \\ - \left\langle cW_{\rm I}^{(4)}(\tau_4) W_{\rm II}^{(3)}(\tau_3) W_{\rm I}^{(2)}(\tau_2) W_{\rm II}^{(1)}(\tau_1) \right\rangle \\ + \left\langle cW_{\rm I}^{(4)}(\tau_4) W_{\rm II}^{(3)}(\tau_3) W_{\rm II}^{(2)}(\tau_2) W_{\rm I}^{(1)}(\tau_1) \right\rangle \\ - \left\langle cW_{\rm II}^{(4)}(\tau_4) W_{\rm I}^{(3)}(\tau_3) W_{\rm II}^{(2)}(\tau_2) W_{\rm I}^{(1)}(\tau_1) \right\rangle \end{array} \right]$$

Expanding out all possible contractions and implementing the Green functions and noting that

$$\begin{aligned} \langle DX(\tau_1) DX(\tau_3) \rangle &= -2\delta(T), \quad \langle DX(\tau_2) DX(\tau_4) \rangle = -2\delta(T) \\ \langle DX(\tau_1) DX(\tau_4) \rangle &= +2\delta(T), \quad \langle DX(\tau_2) DX(\tau_3) \rangle = +2\delta(T) \end{aligned}$$

With these Green functions in hand we arrive at the following s-channel amplitude:

$$A_{4s} = \frac{8}{s} \left[\begin{array}{l} + \frac{s}{4} (\epsilon_1 \cdot \epsilon_4) (\epsilon_2 \cdot \epsilon_3) - \frac{s}{4} (\epsilon_2 \cdot \epsilon_4) (\epsilon_1 \cdot \epsilon_3) - (\frac{s}{4} + \frac{u}{2}) (\epsilon_1 \cdot \epsilon_2) (\epsilon_4 \cdot \epsilon_3) \\ + (\epsilon_2 \cdot k_1) (\epsilon_4 \cdot k_3) (\epsilon_1 \cdot \epsilon_3) + (\epsilon_1 \cdot k_2) (\epsilon_3 \cdot k_4) (\epsilon_2 \cdot \epsilon_4) \\ + (\epsilon_1 \cdot k_3) (\epsilon_2 \cdot k_4) (\epsilon_3 \cdot \epsilon_4) + (\epsilon_4 \cdot k_2) (\epsilon_3 \cdot k_1) (\epsilon_1 \cdot \epsilon_2) \\ - (\epsilon_1 \cdot k_2) (\epsilon_4 \cdot k_3) (\epsilon_2 \cdot \epsilon_3) - (\epsilon_3 \cdot k_4) (\epsilon_2 \cdot k_1) (\epsilon_1 \cdot \epsilon_4) \\ - (\epsilon_1 \cdot k_4) (\epsilon_2 \cdot k_3) (\epsilon_3 \cdot \epsilon_4) - (\epsilon_3 \cdot k_2) (\epsilon_4 \cdot k_1) (\epsilon_1 \cdot \epsilon_2) \end{array} \right]$$

A similar calculation can be done for the t-channel graph, and the result is simply changing the labeling of all momenta and polarizations in the s-channel result according to:

$$\begin{array}{ccc} s & \to & t \\ 1 & \to & 4 \\ 2 & \to & 1 \\ 3 & \to & 2 \\ 4 & \to & 3 \end{array}$$

We arrive at:

$$A_{4t} = \frac{8}{t}\left[\begin{array}{l} +\frac{t}{4}\left(\epsilon_4\cdot\epsilon_3\right)\left(\epsilon_2\cdot\epsilon_1\right) - \frac{t}{4}\left(\epsilon_2\cdot\epsilon_4\right)\left(\epsilon_1\cdot\epsilon_3\right) - \left(\frac{t}{4}+\frac{u}{2}\right)\left(\epsilon_1\cdot\epsilon_4\right)\left(\epsilon_2\cdot\epsilon_3\right) \\ +\left(\epsilon_1\cdot k_4\right)\left(\epsilon_3\cdot k_2\right)\left(\epsilon_2\cdot\epsilon_4\right) + \left(\epsilon_2\cdot k_3\right)\left(\epsilon_4\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_3\right) \\ +\left(\epsilon_1\cdot k_3\right)\left(\epsilon_4\cdot k_2\right)\left(\epsilon_2\cdot\epsilon_3\right) + \left(\epsilon_2\cdot k_4\right)\left(\epsilon_3\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_4\right) \\ -\left(\epsilon_1\cdot k_2\right)\left(\epsilon_4\cdot k_3\right)\left(\epsilon_2\cdot\epsilon_3\right) - \left(\epsilon_2\cdot k_1\right)\left(\epsilon_3\cdot k_4\right)\left(\epsilon_1\cdot\epsilon_4\right) \\ -\left(\epsilon_1\cdot k_4\right)\left(\epsilon_2\cdot k_3\right)\left(\epsilon_3\cdot\epsilon_4\right) - \left(\epsilon_3\cdot k_2\right)\left(\epsilon_4\cdot k_1\right)\left(\epsilon_1\cdot\epsilon_2\right) \end{array}\right]$$

Adding the two channels again gives the complete 4-point amplitude.

Our final example is to demonstrate a worldgraph calculation of the two-point contribution to the effective action. At one loop there are two zero-modes, one modulus (the circumference of the loop) and one Killing vector. The proper insertions for the vacuum are:

$$\left\langle \tilde{0}|bc|\tilde{0}\right\rangle \sim 1$$

In general, the n-point 1-loop 1PI amplitude can thus be written as

$$\begin{aligned} A_n^{1-\text{loop}} &= g^n \int_0^\infty \frac{1}{T^{D/2}} dT \left\langle \delta\left[J_{\mathrm{U}(1)}\right] bV^{(n)}\left(\tau_n\right) \prod_{i=1}^{n-1} \int_{\tau_{i-1}\le\tau_i\le\tau_{i+1}} d\tau_i W_{\mathrm{I}}^{(i)}\left(\tau_i\right) \right\rangle \\ &\quad +\text{diagrams with pinch operators} \\ &= g^n \frac{1}{2\pi} \int_0^\infty \frac{1}{T^{D/2}} dT \int_0^{2\pi} d\theta \left[2i\sin\left(\tfrac{\theta}{2}\right)\right]^{D-2} \\ &\quad \times \left\langle W_{\mathrm{I}}^{(n)}\left(\tau_n\right) \prod_{i=1}^{n-1} \int_{\tau_{i-1}\le\tau_i\le\tau_{i+1}} d\tau_i W_{\mathrm{I}}^{(i)}\left(\tau_i\right) \right\rangle \\ &\quad +\text{diagrams with pinch operators} \end{aligned} \tag{3.21}$$

We've added the coupling constant g, but omitted group theory factors, such as a trace and a factor N_c of the number of colors for the planar contribution. The XX contraction should be calculated by the 1-loop bosonic Green function:

$$\left\langle X^a\left(\tau\right) X^b\left(\tau'\right)\right\rangle = \eta^{ab} G_B\left(\tau-\tau'\right) = \eta^{ab}\left[-\frac{1}{2}\left|\tau-\tau'\right| + \frac{\left(\tau-\tau'\right)^2}{2T}\right]$$

Thus the two-point contribution to the effective action is (including the usual $-$ sign for the action, $\frac{1}{2}$ for permutations, and group theory factor for

this case)

$$
\begin{aligned}
\Gamma_2^{1-\text{loop}} &= -\frac{g^2 N_c}{4\pi}\int_0^\infty \frac{1}{T^{D/2}}dT \int_0^{2\pi} d\theta \left[2i\sin\left(\tfrac{\theta}{2}\right)\right]^{D-2} \\
&\quad \times \int_0^T d\tau \left\langle W_{\rm I}^{(2)}(T)\, W_{\rm I}^{(1)}(\tau)\right\rangle \\
&= -g^2 N_c \int_0^\infty \frac{1}{T^{D/2}}dT \\
&\quad \times \int_0^T d\tau \left[\begin{array}{l} \left(\delta(T-\tau)-\frac{1}{T}\right)(\epsilon_1\cdot\epsilon_2) \\ +\left(\frac{1}{2}-\frac{\tau}{T}\right)^2 (\epsilon_2\cdot k_1)(\epsilon_1\cdot k_2) \\ -(\epsilon_2\cdot k_1)(\epsilon_1\cdot k_2)+(k_1\cdot k_2)(\epsilon_1\cdot\epsilon_2) \end{array}\right] \\
&\quad \times \exp\left[\tfrac{1}{2}k_1\cdot k_2\left(T-\tau-\tfrac{(T-\tau)^2}{T}\right)\right] \\
&= -g^2 N_c \left(\frac{k_1^2}{2}\right)^{-\epsilon}\left(1-\frac{1}{12}\right)\Gamma(\epsilon) \\
&\quad \times[(\epsilon_1\cdot\epsilon_2)(k_1\cdot k_2)-(\epsilon_1\cdot k_2)(\epsilon_2\cdot k_1)] \\
&= -\frac{11}{24}\text{tr}\left\{F_1^{ab}\left[\frac{1}{\epsilon}-\log\left(\frac{1}{2}k_1^2\right)\right]F_{2ab}\right\} \qquad (3.22)
\end{aligned}
$$

In the final line we have used dimensional regularization $D = 4 - 2\epsilon$, and dropped the term with the δ function, which gives the tadpole contribution. Modified minimal subtraction was used, with the conventions of ref. [28]. Note that the $-\frac{1}{12}$ piece comes from the scalar graph while the 1 comes from terms with the fermion Green function. The diagram with pinch operator does not contribute in this case.

Chapter 4

Outlook

4.1 Multiloop Yang-Mills

It is naturally desirable to see how the first-quantized method works for multiloop amplitudes of Yang-Mills theory. In this case, one will probably need to insert extra operators other than the vertex operators, which are similar to the so-called picture changing operators in string theory. Along the direction of the worldline approach, one has to decompose the multiloop worldgraph into several worldlines and calculate the amplitudes on these worldlines separately (using only the worldline Green function). Along the direction of the worldgraph approach, one has to solve for the various fermionic Green function on worldgraphs with different topologies. The form of the Green function can be more and more complicated as the number of loops increase. Whether there exists a general form of the fermionic Green function, just like the case of bosonic Green function, is an interesting open question.

4.2 Gravity

The worldgraph method can also be applied to study the gravity amplitudes. The first step is to study the spectrum of the $N = 4$ spinning particle and derive the vertex operator. Then the Green function and vacuum bubble will be straightforward generalizations from the Yang-Mills case.

4.3 Super-Yang-Mills

In principle, the worldgraph method can be naturally applied to calculate the super-Yang-Mills amplitudes. However, there are some obstacles to overcome. The covariant quantization of superparticles will introduce a infinite pyramid of ghosts and anti-ghosts, and the appropriate vertex operators in various pictures are not clearly known yet. Further, the vertex operators will probably depend on the whole ghost pyramid (i.e. infinite number of fields), in addition to knowing the Green function of each of these fields, one also needs to figure out a compact formula for performing contractions between two infinite sums of fields.

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