

Twisted kink dynamics in multiflavor chiral Gross–Neveu model

Michael Thies*

Institut für Theoretische Physik, Universität Erlangen-Nürnberg, D-91058, Erlangen, Germany

E-mail: michael.thies@gravity.fau.de

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Abstract

The Gross–Neveu model with $U_L(N_f) \times U_R(N_f)$ chiral symmetry is reconsidered in the large N_c limit. The known analytical solution for the time dependent interaction of any number of twisted kinks and breathers is cast into a more revealing form. The (x, t) -dependent factors are isolated from constant coefficients and twist matrices. These latter generalize the twist phases of the single flavor model. The crucial tool is an identity for the inverse of a sum of two square matrices, derived from the known formula for the determinant of such a sum.

Keywords: multiflavor Gross–Neveu model, twisted kinks, inverse of a sum of matrices

1. Introduction

Exactly solvable model problems play a central role in teaching as well as in ‘intellectual body building’ (John Negele). This is well documented in textbooks on basic subjects like classical mechanics, electrodynamics, thermodynamics, or quantum mechanics. In advanced subjects like quantum field theory, it becomes increasingly difficult to identify such problems. Here one has to compromise, for example by resorting to a lower number of dimensions. In $1 + 1$ dimensions in particular, a number of quantum field theories are accessible by analytical means, either exactly or at least in certain limits. A famous example is the Gross–Neveu (GN) model featuring self-interacting Dirac fermions with point interactions in $1 + 1$ dimensions [1].

* Author to whom any correspondence should be addressed.



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Different variants of this model are distinguished by their symmetries. In the case of continuous chiral symmetry, they are referred to as chiral GN models or 2D Nambu–Jona-Lasinio (NJL) models [2]. The model of interest here belongs to this category and possesses the non-Abelian chiral symmetry group $U_L(N_f) \times U_R(N_f)$. Its Lagrangian reads

$$\begin{aligned} \mathcal{L} = & \sum_{k=1}^{N_c} \sum_{\alpha=1}^{N_f} \bar{\psi}_{k,\alpha} i\bar{\partial}\psi_{k,\alpha} + \frac{g^2}{4} \sum_{a=0}^{N_f^2-1} \left[\left(\sum_{k=1}^{N_c} \sum_{\alpha=1}^{N_f} \bar{\psi}_{k,\alpha} (\lambda^a)_{\alpha\beta} \psi_{k,\beta} \right)^2 \right. \\ & \left. + \left(\sum_{k=1}^{N_c} \sum_{\alpha=1}^{N_f} \bar{\psi}_{k,\alpha} i\gamma_5 (\lambda^a)_{\alpha\beta} \psi_{k,\beta} \right)^2 \right] \\ = & \bar{\psi} i\bar{\partial}\psi + \frac{g^2}{4} \sum_{a=0}^{N_f^2-1} [(\bar{\psi} \lambda^a \psi)^2 + (\bar{\psi} i\gamma_5 \lambda^a \psi)^2]. \end{aligned} \quad (1)$$

The upper line shows explicitly the way in which ‘color’ indices (k) and ‘flavor’ indices (α, β) are contracted. Note that both of these refer to flavor in the present context, but color does not enter into the four-fermion interaction vertices. The lower line is the conventional short-hand notation in this context where indices are suppressed whenever possible. The λ^a denote the $N_f^2 - 1$ $SU(N_f)$ generators in the fundamental representation with the usual normalization, supplemented by $\lambda^0 = \sqrt{2/N_f}$ (proportional to a unit matrix) to account for $U(1)$,

$$\begin{aligned} \text{Tr } \lambda^a \lambda^b = 2\delta_{ab}, \quad a, b = 0 \dots N_f^2 - 1 \\ \sum_{a=0}^{N_f^2-1} \lambda_{\alpha\beta}^a \lambda_{\gamma\delta}^a = \lambda_{\alpha\beta}^a \lambda_{\gamma\delta}^a = 2\delta_{\alpha\delta} \delta_{\beta\gamma}. \end{aligned} \quad (2)$$

The prefactor $g^2/4$ in (1) has been chosen such as to match the convention of the standard chiral GN model with $U_L(1) \times U_R(1)$ chiral symmetry,

$$\mathcal{L} = \bar{\psi} i\bar{\partial}\psi + \frac{g^2}{2} [(\bar{\psi} \psi)^2 + (\bar{\psi} i\gamma_5 \psi)^2] \quad (N_f = 1). \quad (3)$$

In $3 + 1$ dimensions, Lagrangian (1) is well known from the $SU(3)$ -flavor version of the NJL model [3]. There one usually adds a term which breaks $U_A(1)$ (the ‘t Hooft determinant), but we see no reason to do so in $1 + 1$ dimensions. In order to validate a semiclassical approach, we shall only consider the ‘t Hooft limit $N_c \rightarrow \infty, N_c g^2 = \text{constant}$, in the present work. We do not include a bare fermion mass term, as this would prevent us from treating the problem analytically. The basic semiclassical tools for a fermionic theory are the Hartree–Fock (HF) approach for static problems and the time dependent Hartree–Fock (TDHF) approach for dynamical problems.

Studies of GN type field theories fall roughly into two categories: thermodynamics and phase diagrams, or solitonic bound states and their interactions. The issue which has perhaps received most attention in nuclear and particle physics are the phase diagrams as a function of temperature and chemical potentials. In condensed matter physics on the other hand where closely related models arise in the context of quasi-one dimensional systems (polymers, superconductors, trapped ions), the focus has typically been on soliton spectra and dynamics. In this context, soliton refers to the behavior of the mean field, related to the fermionic single particle wave functions by self-consistency. This type of bound state is a toy model for composite, relativistic objects, mimicking hadrons in real life.

In strong interaction physics and quantum chromodynamics (QCD), the method of choice has become the lattice Monte Carlo calculation in Euclidean space. Unfortunately, neither of the issues just mentioned can be fully handled by this method. In the case of dense matter, the sign problem is still a serious obstacle against working at finite chemical potential. Time dependent problems like hadron scattering can only indirectly be dealt with, for instance by calculating scattering lengths with Lüscher's method [4]. These limitations make it desirable to gain some experience with interacting relativistic bound states of massless fermions in a reliable way, complementary to numerical studies of lattice QCD. GN type models offer exact, analytical solutions for both phase diagrams and bound state dynamics, if only in $1 + 1$ dimensions.

In the present work, we reconsider the problem of soliton dynamics in the multiflavor chiral GN model (1). To put our study into perspective, let us briefly recall the state of the art of solving time dependent problems in GN models.

Static solitons have been found early on [5] and have thoroughly been studied since then [6]. The subject of soliton dynamics in the original GN model with discrete chiral symmetry also starts with reference [5] where the first time dependent mean field solution was found, the breather. This is a collectively excited soliton, vibrating in its rest frame. The authors guess its form by analogy with the sine-Gordon breather. They point out that it should be related to kink–antikink scattering by analytic continuation. This idea was taken up again in reference [7] where kink–antikink scattering was solved in detail. The generalization to any number of colliding kinks followed soon afterwards [8]. The twisted kink in the one-flavor chiral GN model was discovered by Shei using inverse scattering theory [9]. It is a ‘chord soliton’ in the sense that its mean field traces out a straight line between two different vacua on the chiral circle. Bound states of an arbitrary number of twisted kinks were first found in reference [10] and generalized to time dependent scattering and breather phenomena in reference [11]. Compact analytical formulas are given for any number and complexity of solitons or breathers. More recently, these works have been further extended to the multiflavor case, first by using methods akin to inverse scattering theory in condensed matter physics [12]. It turns out that the formalism of reference [13] from the particle physics side can also be generalized rather easily to N_f flavors, see reference [14] for the case $N_f = 2$. As a matter of fact, the restriction to $N_f = 2$ is unnecessary, as the formalism is practically independent of N_f . The fact that the N_f flavor model can still be solved exactly is non-trivial. The static case was explored in more depth in reference [15], a paper which has some overlap with the present one.

Summarizing, we already have at our disposal all the tools needed to compute bound and scattering states of twisted solitons or breathers. Exact expressions are available for both mean field and spinors in closed analytical form. Self-consistency has been established quite generally. If the only goal was to present figures or animations of specific collision events, this would be sufficient. However, in view of the pedagogical thrust of such studies, one would like to better understand what is going on, in particular concerning the role of flavor degrees of freedom. Reference [15] has already provided us with useful additional insights for the static case, for instance for widely spaced solitons and the vacua in between. Here we propose to extend this study to the most general time dependent solution, trying to disentangle flavor from the other degrees of freedom as much as possible.

This paper is organized as follows. In section 2, we collect some basic facts about the chiral GN model with N_f flavors. Section 3 briefly reviews the analytical mean field solution for any number of solitons and/or breathers. We then proceed from there and transform the result for mean field and spinors into a more illuminating form. To this end, we first had to derive an exact expression for the inverse of a sum of two matrices which may also be of interest for other purposes, see section 4. In section 5 we apply it to the case where there are no breathers

but only solitons (twisted kinks or bound states thereof). The general case including breathers is the subject of section 6. This is followed by illustrative examples covering the single twisted kink, section 7, scattering or bound states of two kinks, section 8, and the breather, section 9. Finally, section 10 contains a short summary and conclusions. The proof of a mathematical identity is relegated to the [appendix](#).

2. Basic facts about the model

Lagrangian (1) has an obvious $U(N_c)$ symmetry and a somewhat less obvious $U_L(N_f) \times U_R(N_f)$ chiral symmetry. The latter becomes manifest once we decompose the spinors into left- and right-handed chiralities,

$$\psi = \frac{1 + \gamma_5}{2} \psi_R + \frac{1 - \gamma_5}{2} \psi_L. \quad (4)$$

Using the representation

$$\gamma^0 = \sigma_1, \quad \gamma^1 = i\sigma_2, \quad \gamma_5 = \gamma^0 \gamma^1 = -\sigma_3 \quad (5)$$

of the Dirac matrices together with light cone coordinates

$$z = x - t, \quad \bar{z} = x + t, \quad \partial_0 = \bar{\partial} - \partial, \quad \partial_1 = \bar{\partial} + \partial, \quad (6)$$

we find

$$\mathcal{L} = 2i\psi_R^\dagger \bar{\partial} \psi_R - 2i\psi_L^\dagger \partial \psi_L + 2g^2(\psi_{L,\alpha}^\dagger \psi_{R,\beta})(\psi_{R,\beta}^\dagger \psi_{L,\alpha}). \quad (7)$$

Again, color indices within bilinears are contracted to singlets. As $U(N_f)$ acts only on the flavor indices, the interaction term is manifestly $U_L(N_f) \times U_R(N_f)$ chirally invariant, as is the free Lagrangian for massless Dirac fermions. The Euler–Lagrange equations

$$\begin{aligned} 2i\partial\psi_{L,\alpha} &= 2g^2(\psi_{L,\alpha} \psi_{R,\beta}^\dagger) \psi_{R,\beta}, \\ 2i\bar{\partial}\psi_{R,\alpha} &= -2g^2(\psi_{R,\alpha} \psi_{L,\beta}^\dagger) \psi_{L,\beta}, \end{aligned} \quad (8)$$

lead directly to the TDHF equations in the large N_c limit,

$$\begin{aligned} 2i\partial\psi_L &= -\Delta^\dagger \psi_R, \\ 2i\bar{\partial}\psi_R &= \Delta \psi_L. \end{aligned} \quad (9)$$

Here, the mean field Δ is a color singlet, but a $N_f \times N_f$ matrix in flavor space. The self-consistency condition reads

$$\Delta_{\alpha\beta} = -2N_c g^2 \langle \psi_{R,\alpha} \psi_{L,\beta}^\dagger \rangle, \quad (10)$$

where we have replaced bilinears by expectation values and pulled out the factor N_c by performing trivial color sums. Due to the dyadic structure of Δ , it is now evident that both the TDHF equation and self-consistency are preserved under global chiral $U_L(N_f) \times U_R(N_f)$ transformations,

$$\psi_L \rightarrow U_L \psi_L, \quad \psi_R \rightarrow U_R \psi_R, \quad \Delta \rightarrow U_R \Delta U_L^\dagger. \quad (11)$$

This observation can be used to simplify the vacuum problem. Due to spontaneous symmetry breakdown, the (homogeneous) vacuum is characterized by a constant matrix Δ . Using the freedom of performing global chiral transformations, we map it onto the unit matrix times a scale factor,

$$\Delta_{\alpha\beta} = m \delta_{\alpha\beta}. \quad (12)$$

The constant m plays the role of dynamical fermion mass. The HF vacuum energy density can be evaluated as the sum over single particle energies plus a double counting correction,

$$\begin{aligned} \mathcal{E}_{\text{vac}} &= \mathcal{E}_{\text{sp}}(m) + \mathcal{E}_{\text{dc}}(m), \\ \mathcal{E}_{\text{sp}}(m) &= -N_c N_f \int_{-\Lambda/2}^{\Lambda/2} \frac{dk}{2\pi} \sqrt{k^2 + m^2} \\ &= -N_c N_f \left[\frac{\Lambda^2}{8\pi} - \frac{m^2}{4\pi} \left(\ln \frac{m^2}{\Lambda^2} - 1 \right) \right], \\ \mathcal{E}_{\text{dc}}(m) &= N_f \frac{m^2}{2g^2}. \end{aligned} \quad (13)$$

The gap equation follows by minimizing \mathcal{E}_{vac} with respect to m ,

$$0 = 1 + \frac{N_c g^2}{2\pi} \ln \frac{m^2}{\Lambda^2}, \quad (14)$$

and is indistinguishable from that of the one-flavor model. The renormalized vacuum energy density

$$\mathcal{E}_{\text{vac}} = -N_c N_f \frac{m^2}{4\pi} \quad (15)$$

is just N_f times the known energy density of the one-flavor model. The fermion mass m which arises from dimensional transmutation can be set equal to 1 by choice of units. This is what we shall do in the present work. The vacuum manifold then coincides with the group $U(N_f)$, as can be seen by applying all global chiral transformations to $\Delta = 1$. For one flavor, it reduces to $U(1)$, i.e. the familiar chiral circle.

3. Reminder of the multisoliton solution

In the large N_c limit, it is possible to solve bound state and scattering problems in model (1) explicitly. This holds for any number of solitons and/or breathers as well as for any N_f . Originally, a general solution has been found for the one-flavor model. In reference [12] (in condensed matter physics) and [14] (in particle physics), this solution was subsequently generalized to several flavors. In reference [14] in particular, the $N_f = 2$ case has been treated in some detail. Since the generalization to arbitrary N_f is trivial, we infer the solution from this work and present only the necessary definitions and results, referring to [14] for detailed proofs. The upcoming sections of the present work may then be regarded as an elaboration on this general solution.

The starting point for attacking N soliton problems is an N -dimensional vector e with components [13]

$$e_i = e^{i(\zeta_i^* \bar{z} - z/\zeta_i^*)/2}. \quad (16)$$

The ζ_i are complex numbers ($\text{Im } \zeta_i > 0$) characterizing the pole positions of the TDHF continuum wave functions in the complex ζ plane. Here, ζ is the spectral parameter related to light cone momentum and energy ('uniformizing parameter' in condensed matter language),

$$k = \frac{1}{2} \left(\zeta - \frac{1}{\zeta} \right), \quad E = -\frac{1}{2} \left(\zeta + \frac{1}{\zeta} \right). \quad (17)$$

Note that

$$k_\mu x^\mu = -\frac{1}{2} \left(\zeta \bar{z} - \frac{z}{\zeta} \right), \quad (18)$$

so that e_i is recognized as a plane wave evaluated at a complex spectral parameter corresponding to a bound state pole. The crucial step when going from one to N_f flavors consists in dressing each e_i with a flavor vector \vec{p}_i

$$e_i \rightarrow e_i \vec{p}_i, \quad e_{i,\alpha} = e_i p_{i,\alpha} \quad (\text{no } i\text{-sum}). \quad (19)$$

Here, \vec{p}_i is a N_f -component, constant, complex vector with components $p_{i,\alpha}$. Its precise meaning will be clarified later on. Since a common real factor multiplying \vec{p}_i can always be absorbed in the soliton positions, we can assume that these vectors are normalized ($\vec{p}_i^\dagger \vec{p}_i = 1$) without loss of generality. However they are in general neither orthogonal nor even linearly independent. This is obvious since the number of solitons may exceed the number of flavors. We continue using Greek indices for flavor and suppress the indices $i = 1 \dots N$ referring to the bound state poles whenever possible. Then we have to modify the results of references [13] as follows: continuum TDHF spinors are now $2N_f$ -component objects

$$\psi_{\zeta,\alpha} = \frac{1}{\sqrt{1 + \zeta^2}} \begin{pmatrix} \zeta \chi_{1,\alpha} \\ -\chi_{2,\alpha} \end{pmatrix} e^{i(\zeta \bar{z} - z/\zeta)/2}. \quad (20)$$

The following ansatz for the $\chi_{i,\alpha}$ is motivated by the assumed pole structure of the continuum spinors (N poles, corresponding to N bound states),

$$\begin{aligned} \chi_{1,\alpha} &= \left(\delta_{\alpha\beta} + i \sum_{i=1}^N \frac{1}{\zeta - \zeta_i} \varphi_{1,i,\alpha} e_{i,\beta}^* \right) q_\beta, \\ \chi_{2,\alpha} &= \left(\delta_{\alpha\beta} - i \sum_{i=1}^N \frac{\zeta}{\zeta - \zeta_i} \varphi_{2,i,\alpha} e_{i,\beta}^* \right) q_\beta. \end{aligned} \quad (21)$$

The q_β are the amplitudes of the flavor components of the incoming plane wave

$$\psi_{\zeta,\alpha}|_{\text{in}} = \frac{1}{\sqrt{1 + \zeta^2}} \begin{pmatrix} \zeta \\ -1 \end{pmatrix} e^{i(\zeta \bar{z} - z/\zeta)/2} q_\alpha. \quad (22)$$

The vacuum at $x \rightarrow -\infty$ will always be chosen as $\Delta_{\text{vac}} = 1$. When summing over all continuum states, the \vec{q} should be chosen in all flavor directions (one component 1, all the others 0) to account for incoming waves in the different flavor channels. The $\varphi_{1,i,\alpha}, \varphi_{2,i,\alpha}$ introduced

in equation (21) are closely related to the components of bound state spinors. They can be evaluated by linear algebra as follows: define a Hermitian $N \times N$ matrix B ,

$$B_{ij} = i \frac{e_{i,\beta} e_{j,\beta}^*}{\zeta_j - \zeta_i^*} = i \frac{e_i e_j^*}{\zeta_j - \zeta_i^*} \vec{p}_j^\dagger \vec{p}_i. \quad (23)$$

The $\varphi_{1,i,\alpha}, \varphi_{2,i,\alpha}$ then satisfy the following system of linear, algebraic equations

$$\begin{aligned} (\omega + B)\varphi_{1,\alpha} &= e_\alpha, \\ (\omega + B)\varphi_{2,\alpha} &= -f_\alpha, \end{aligned} \quad (24)$$

where $f_{i,\alpha} = e_{i,\alpha}/\zeta_i^*$. Like in the one flavor case, a constant, Hermitian $N \times N$ matrix ω encoding further information about the soliton configuration (geometry, initial conditions, breather frequency and amplitude) has been introduced. The dimension of the linear system (24) does not increase with the number of flavors, but depends only on the total number of bound state poles. What is new as compared to the one-flavor case is the factor $\vec{p}_j^\dagger \vec{p}_i$ in B_{ij} and the fact that one gets a pair of linear equations for each flavor component α . The most important result for the following is the expression for the mean field, now a $N_f \times N_f$ matrix

$$\Delta_{\alpha\beta} = \delta_{\alpha\beta} + i e_\beta^\dagger \frac{1}{\omega + B} f_\alpha. \quad (25)$$

Orthonormal bound states can be constructed as in the one-flavor case by linear combinations of the φ_i ,

$$\hat{\varphi}_i = \sum_j C_{ij} \varphi_j, \quad \int dx \hat{\varphi}_{i,\alpha}^\dagger \hat{\varphi}_{j,\alpha} = \delta_{ij}. \quad (26)$$

The resulting condition coincides with the one in the one-flavor case,

$$2C\omega^{-1}C^\dagger = 1. \quad (27)$$

A central ingredient of the TDHF calculation is the self-consistency condition. We introduce two diagonal $N \times N$ matrices

$$\begin{aligned} M_{ij} &= -i\delta_{ij} \ln(-\zeta_i^*), \\ N_{ij} &= 4\pi\delta_{ij}\nu_i, \end{aligned} \quad (28)$$

where ν_i is the occupation fraction of bound state i . The self-consistency condition then assumes the form

$$\omega M^\dagger + M\omega = C^\dagger NC \quad (29)$$

independently of the number of flavors.

The following observations carry over from the one-flavor to the multiflavor models. Owing to chiral symmetry, model (1) gives rise to N_f^2 conserved vector and axial vector Noether currents

$$\begin{aligned} \partial^\mu j_\mu^a &= \partial^\mu (\bar{\psi} \gamma_\mu \lambda^a \psi) = 0 \\ \partial^\mu j_{5,\mu}^a &= \partial^\mu (\bar{\psi} \gamma_\mu \gamma_5 \lambda^a \psi) = 0. \end{aligned} \quad (30)$$

In $1 + 1$ dimensions, vector and axial vector currents are not independent, but satisfy

$$j_5^{a,0} = j^{a,1}, \quad j_5^{a,1} = j^{a,0}. \quad (31)$$

Adding and subtracting the conservation laws (30) and introducing light cone coordinates (6), one finds

$$\bar{\partial}(\psi_R^\dagger \lambda^a \psi_R) = 0, \quad \partial(\psi_L^\dagger \lambda^a \psi_L) = 0. \quad (32)$$

If we take the expectation value of these equations in an arbitrary state, we conclude that the right-handed density $\rho_R^a = \langle \psi_R^\dagger \lambda^a \psi_R \rangle$ depends only on $z = x - t$, the left-handed density $\rho_L^a = \langle \psi_L^\dagger \lambda^a \psi_L \rangle$ only on $\bar{z} = x + t$, i.e. they can only move with the speed of light to the right or to the left (or be constant). In a localized, massive state like a solitonic bound state or breather, these densities must therefore vanish identically. Hence we anticipate that all densities and current densities must vanish inside an arbitrary soliton or multisoliton state, at least in the strict thermodynamic and chiral limit. This should hold for left- and right-handed fermions separately, or, equivalently, for charge and current densities. This can indeed be verified by a detailed computation (see reference [14]) and holds for all flavor currents including the fermion current (the $a = 0$ component).

The formulas given in the present section are sufficient for computing the space-time evolution of any multisoliton event. However, it turns out that one can convert the result into a more transparent and useful form, notably in the absence of breathers. Reference [15] has already dealt with the static case in a similar spirit. The goal of the following sections is to simplify the time dependent case as well. Thus we shall start from equation (25) for Δ and transform it to a more instructive expression. The key problem here is how to invert the matrix $(\omega + B)$. Once this has been achieved, the formulas for the spinors can be simplified as well.

4. Inverting a sum of two matrices

Let us go back to equation (25) for the mean field,

$$\Delta_{\alpha\beta} = \delta_{\alpha\beta} + i e_\beta^\dagger \frac{1}{\omega + B} f_\alpha, \quad (33)$$

with

$$\begin{aligned} B_{ij} &= i \frac{e_i e_j^*}{\zeta_j - \zeta_i^*} \sigma_{ji}, \quad \sigma_{ji} = \vec{p}_j^\dagger \vec{p}_i, \\ e_{i,\alpha} &= e_i p_{i,\alpha}, \quad f_{i,\alpha} = (\zeta_i^*)^{-1} e_{i,\alpha}. \end{aligned} \quad (34)$$

The constant matrix ω is diagonal for the scattering case (solitons, bound states) and off diagonal for problems involving breathers. Although expression (33) is exact, it is not yet very transparent. Space-time-, flavor- and parameter dependences are inextricably entangled. We therefore pull out the (x, t) dependent factors from B by introducing a diagonal $N \times N$ matrix E as follows

$$\begin{aligned} B &= ESE^\dagger, \\ E_{ij} &= \delta_{ij} e_i, \\ S_{ij} &= \frac{i}{\zeta_j - \zeta_i^*} \sigma_{ji}, \quad S^\dagger = S. \end{aligned} \quad (35)$$

Thus

$$\omega + B = E(v + S)E^\dagger, \quad v = E^{-1}\omega(E^\dagger)^{-1}. \quad (36)$$

Inserting this expression into (33) yields

$$\Delta_{\alpha\beta} = \delta_{\alpha\beta} + i e_\beta^\dagger (E^\dagger)^{-1} \frac{1}{v + S} E^{-1} f_\alpha. \quad (37)$$

In the denominator, the (x, t) dependence has now been shifted to the matrix v . The diagonal matrices $E^{-1}, (E^\dagger)^{-1}$ acting on the vectors $f_\alpha, e_\beta^\dagger$ cancel the (x, t) dependence of these vertex functions. Inserting e_β^\dagger and f_α , Δ is expressed in flavor space via a sum over dyadics $\vec{p}_j \vec{p}_i^\dagger$,

$$\Delta = 1 + i \sum_{i,j} \left(\frac{1}{v + S} \right)_{ij} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_i^\dagger. \quad (38)$$

The coefficients of $\vec{p}_j \vec{p}_i^\dagger$ require inverting the sum of a Hermitian, space-time dependent matrix v and a Hermitian, constant matrix S . The explicit flavor dependence is through the dyadics $\vec{p}_j \vec{p}_i^\dagger$. In addition, there is an implicit flavor dependence through S which still depends on the flavor scalars $\sigma_{ij} = \vec{p}_i^\dagger \vec{p}_j$.

Along the same lines, we rewrite the expressions for the spinors in terms of the inverse of the matrix $(v + S)$. Using once again arrows for vectors in flavor space, we find

$$\begin{aligned} \vec{\varphi}_{1,i} &= \sum_j \frac{1}{e_i^*} \left(\frac{1}{v + S} \right)_{ij} \vec{p}_j, \\ \vec{\varphi}_{2,i} &= - \sum_j \frac{1}{e_i^*} \left(\frac{1}{v + S} \right)_{ij} \frac{1}{\zeta_j^*} \vec{p}_j, \\ \vec{\chi}_{1,\vec{q}} &= \left(1 + i \sum_{i,j} \frac{1}{\zeta - \zeta_i} \left(\frac{1}{v + S} \right)_{ij} \vec{p}_j \vec{p}_i^\dagger \right) \cdot \vec{q}, \\ \vec{\chi}_{2,\vec{q}} &= \left(1 + i \sum_{i,j} \frac{\zeta}{\zeta - \zeta_i} \left(\frac{1}{v + S} \right)_{ij} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_i^\dagger \right) \cdot \vec{q}. \end{aligned} \quad (39)$$

The full continuum spinors become

$$\vec{\psi}_{\zeta,\vec{q}} = \frac{1}{\sqrt{1 + \zeta^2}} \begin{pmatrix} \zeta \vec{\chi}_{1,\vec{q}} \\ -\vec{\chi}_{2,\vec{q}} \end{pmatrix} e^{i(\zeta \vec{z} - z/\zeta)/2}. \quad (40)$$

Common to all expressions (38) and (39) is the appearance of the inverse matrix $(v + S)^{-1}$. Since all the space-time dependence is now in v , this raises the question about a useful expression where one can keep track of v and S separately also in $1/(v + S)$. Since we could not find an appropriate formula in the literature, we first derive a general expression for the inverse of a sum of two square matrices.

We start from a known expression for the determinant of a sum of matrices [16]. It reads as follows:

$$\det(A + B) = \det A + \det B + \sum_{r=1}^{N-1} \sum_{\alpha, \beta} (-1)^{s(\alpha) + s(\beta)} \det A[\alpha|\beta] \det B(\alpha|\beta). \quad (41)$$

A and B are N -square matrices. The outer sum is over integers r from 1 to $N - 1$. For a particular r , the inner sum is over all strictly increasing integer sequences α and β of length r chosen from $1, \dots, N - 1$. $A[\alpha|\beta]$ (square brackets) is the r -square submatrix of A lying in the rows α and columns β . $B(\alpha|\beta)$ (round brackets) is the $(N - r)$ -square submatrix of B lying in rows complementary to α and columns complementary to β . $s(\alpha)$ is the sum of all integers in α .

Before continuing, we introduce the operation of ‘inflating’ a submatrix to the original matrix size by adding rows and columns filled with zero’s. Given a submatrix $B(\alpha|\beta)$ as just described (round brackets), we denote by $\widetilde{B}(\alpha|\beta)$ the N -square matrix obtained by filling rows β and columns α with zero’s (note the interchange of rows and columns as compared to the definition of $B(\alpha|\beta)$). Since this definition is unfamiliar, we illustrate it with a simple example. Starting from the 4×4 matrix

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & b_{23} & b_{24} \\ b_{31} & b_{32} & b_{33} & b_{34} \\ b_{41} & b_{42} & b_{43} & b_{44} \end{pmatrix}, \quad (42)$$

we extract the submatrix

$$B(13|34) = \begin{pmatrix} b_{21} & b_{22} \\ b_{41} & b_{42} \end{pmatrix} \quad (43)$$

and ‘inflate’ it to

$$\widetilde{B}(13|34) = \begin{pmatrix} 0 & b_{21} & 0 & b_{22} \\ 0 & b_{41} & 0 & b_{42} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (44)$$

Below we shall also find it convenient to work with inflated submatrices with square brackets. Since submatrices with round brackets and square brackets are trivially related, we can easily infer the correct definition of $\widetilde{B}[\alpha|\beta]$ (square brackets) from the definition of $\widetilde{B}(\alpha|\beta)$ (round brackets). In our example,

$$B[13|34] = B(24|12) = \begin{pmatrix} b_{13} & b_{14} \\ b_{33} & b_{34} \end{pmatrix}. \quad (45)$$

Hence

$$\widetilde{B}[13|34] = \widetilde{B}(24|12) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ b_{13} & 0 & b_{14} & 0 \\ b_{33} & 0 & b_{34} & 0 \end{pmatrix}. \quad (46)$$

Thus the correct definition of $\widetilde{B}[\alpha|\beta]$ is as follows: take the submatrix $B[\alpha|\beta]$ and add rows complementary to β and columns complementary to α filled with zero’s.

We are interested in the inverse matrix $(A + B)^{-1}$, i.e.

$$\frac{1}{A + B} = \frac{\text{adj}(A + B)}{\det(A + B)}. \quad (47)$$

Here, adj denotes the classical adjoint (or adjugate) matrix, the transpose of the cofactor matrix. The denominator is taken care of by equation (41). For the numerator, we find a similar equation

$$\text{adj}(A + B) = \text{adj } B + \sum_{r=1}^{N-1} \sum_{\alpha, \beta} (-1)^{s(\alpha) + s(\beta)} \det A[\alpha | \beta] \widetilde{\text{adj } B(\alpha | \beta)}, \quad (48)$$

where one of the \det -factors in the sum has been replaced by the tilde-symbol, now applied to the adjugate matrix. Incidentally, the $r = N - 1$ term in (48) is equal to $\text{adj } A$.

Actually, equation (48) is a consequence of (41). To show this, consider the (i, j) -matrix element of equation (48). Left-hand side,

$$(\text{lhs})_{ij} = [\text{adj}(A + B)]_{ij} = (-1)^{i+j} \det [A(j|i) + B(j|i)]. \quad (49)$$

The determinant of the sum $A(j|i) + B(j|i)$ in turn can again be evaluated with the help of equation (41) for $N \rightarrow N - 1$. In order to apply the formula literally, the row and column indices should run from 1 to $N - 1$. Now the indices run from 1 to N with j missing in the row indices and i missing in the column indices. This does not affect the determinants of submatrices in (41), but it does affect the phase factor. The correct result is

$$\begin{aligned} \det [A(j|i) + B(j|i)] &= \det A(j|i) + \det B(j|i) \\ &+ \sum_{r=1}^{N-2} \sum_{\bar{\alpha}_j, \bar{\beta}_i} (-1)^{s(\bar{\alpha}_j) + s(\bar{\beta}_i) + n(\bar{\alpha}_j) + n(\bar{\beta}_i)} \det A[\bar{\alpha}_j | \bar{\beta}_i] \det B(j, \bar{\alpha}_j | i, \bar{\beta}_i). \end{aligned} \quad (50)$$

The $\bar{\alpha}_j, \bar{\beta}_i$ are defined like the α, β above except that $\bar{\alpha}_j$ does not contain the index j , $\bar{\beta}_i$ does not contain the index i . Their maximal length is thereby reduced to $N - 2$. Furthermore, $n(\bar{\alpha}_j)$ is the number of elements of the sequence $\bar{\alpha}_j$ that are $> j$, $n(\bar{\beta}_i)$ the number of elements of β_i that are $> i$. These modifications of the phase factor are necessary because the labeling of rows and columns in $A(j|i), B(j|i)$ is not the standard one.

On the right-hand side of (48), we split the index set α into sets α_j containing j and sets $\bar{\alpha}_j$ not containing j . Similarly for β, β_i (containing i) and $\bar{\beta}_i$ (not containing i). The summation over α, β then gives rise to 4 terms

$$\sum_{\alpha, \beta} \rightarrow \sum_{\alpha_j, \beta_i} + \sum_{\alpha_j, \bar{\beta}_i} + \sum_{\bar{\alpha}_j, \beta_i} + \sum_{\bar{\alpha}_j, \bar{\beta}_i}. \quad (51)$$

If we take the (i, j) -matrix element of $\widetilde{\text{adj}}(B(\alpha | \beta))$ in equation (48), only the last term in (51) contributes since the rows filled with zero's are given by β , the columns filled with zero's by α . To get a non-zero row index i , β should not contain i , hence $\bar{\beta}_i$ is needed. To get a non-zero column index j , α should not contain j , hence $\bar{\alpha}_j$ is needed. The (i, j) -matrix element of $\widetilde{\text{adj}} B(\bar{\alpha}_j | \bar{\beta}_i)$ in (48) is given by

$$\left[\widetilde{\text{adj } B(\bar{\alpha}_j | \bar{\beta}_i)} \right]_{ij} = (-1)^{i+j+n(\bar{\alpha}_j)+n(\bar{\beta}_i)} \det B(j, \bar{\alpha}_j | i, \bar{\beta}_i). \quad (52)$$

The extra phase factor is again due to the non-standard labeling of rows and columns in the submatrices. This comes about as follows. When filling the rows labeled by $\bar{\beta}_i$ with zero's, the row index increases by the number of elements of $\bar{\beta}_i$ that are $< i$, i.e. $r - n(\bar{\beta}_i)$. When filling the columns labeled by $\bar{\alpha}_j$ with zero's, the column index increases by the number of elements of $\bar{\alpha}_j$ that are $< j$, i.e. $r - n(\bar{\alpha}_j)$. This is the reason behind the phase factor in (51). In addition, we pick up a term from $\text{adj } B$ in (48). Putting everything together, the (i, j) -matrix element of the right-hand side of (48) becomes identical to the left-hand side as given in equations (49) and (50).

Summarizing, we write down the full expression for the inverse of a sum of two matrices, expressing the right-hand side by submatrices. To clarify the idea behind this expression, we multiply A by a formal parameter ϵ . The terms of the sum then go like ϵ^r . Thus one can think of the formula as representing an exact expression for $1/(\epsilon A + B)$ as a rational function in ϵ , to be contrasted to the power series expansion familiar from perturbation theory,

$$\frac{1}{\epsilon A + B} = \frac{\text{adj } B + \sum_{r=1}^{N-2} \epsilon^r \sum_{\alpha, \beta} (-1)^{s(\alpha)+s(\beta)} \det A[\alpha | \beta] \widetilde{\text{adj } B(\alpha | \beta)} + \epsilon^{N-1} \text{adj } A}{\det B + \sum_{r=1}^{N-1} \epsilon^r \sum_{\alpha, \beta} (-1)^{s(\alpha)+s(\beta)} \det A[\alpha | \beta] \det B(\alpha | \beta) + \epsilon^N \det A}. \quad (53)$$

The numerator (denominator) is a polynomial in ϵ of degree $N - 1(N)$. The coefficients are given explicitly in terms of determinants and adjoints of submatrices of A and B .

An important special case for our purpose is the case where A is diagonal. This covers all multisoliton interactions without breathers. Let us simplify the notation and formulas for this particular case. Assume that

$$A_{ij} = \delta_{ij} A_i. \quad (54)$$

Then in both equations (41) and (48) only $\alpha = \beta$ appears, so that the phase factor drops out. Using the simplified notation

$$A[\alpha | \alpha] = A[\alpha], \quad B(\alpha | \alpha) = B(\alpha), \quad (55)$$

we get

$$\begin{aligned} \det(A + B) &= \det A + \det B + \sum_{\alpha \in K_{N-1}} \det A[\alpha] \det B(\alpha), \\ \text{adj}(A + B) &= \text{adj } B + \sum_{\alpha \in K_{N-1}} \det A[\alpha] \widetilde{\text{adj } B(\alpha)}, \end{aligned} \quad (56)$$

where K_n denotes the set of all sequences of the type α as defined above with length $r = 1, \dots, n$. Moreover, the determinants of A and its submatrices simplify to

$$\det A = \prod_{i=1}^N A_i, \quad \det A[\alpha] = \prod_{i \in \alpha} A_i. \quad (57)$$

5. Multisoliton dynamics without breathers

If we disregard breathers for the moment, the matrix ω is diagonal,

$$\omega_{ij} = \delta_{ij}\omega_i. \quad (58)$$

According to (36), v is also diagonal,

$$v_{ij} = \delta_{ij}v_i, \quad v_i = \frac{\omega_i}{|e_i|^2} = \frac{1}{V_i}, \quad (59)$$

where the V_i are the basic profile functions of the solitons familiar from the one-flavor case [13]. Their explicit form will be given below when we discuss examples of few soliton problems. Recall that the mean field Δ in the single flavor case could be represented as a ratio of two multivariate polynomials in the V_i . Our goal is a corresponding expression for the multiflavor case. Since Δ is a flavor matrix, we expect the coefficients in the numerator to be flavor matrices as well. In fact, equation (56) is exactly what is needed for this purpose. If we replace v_i by $1/V_i$, the formula gives us directly the numerator and denominator of Δ as polynomials in the V_i . For the N soliton problem, matrix inversion needs to be done only for constant square matrices $S(\alpha)$ of dimension N and lower. The space-time dependence is contained in the monomials of V_i 's. By contrast, the original expression for Δ , equation (33), requires matrix inversion for every (x, t) . Besides, as we shall see shortly, the structure of the result is more transparent and can be used to gain further analytical insights.

Let us introduce the V_i at this stage. If we multiply numerator and denominator of $(v + S)^{-1}$ by $\det V = 1/\det v = V_1 \dots V_N$, we find that $\det V[\alpha]$ now multiplies the determinant or the adjoint of the matrix S where the rows and columns in α are kept, rather than being removed ($S[\alpha]$ in the notation of equation (55)). Thus

$$(v + S)^{-1} = \frac{\sum_{\alpha \in K_N} \det V[\alpha] \widetilde{\text{adj } S[\alpha]}}{1 + \sum_{\alpha \in K_N} \det V[\alpha] \det S[\alpha]}. \quad (60)$$

As pointed out below equation (46), the tilde above $\text{adj } S[\alpha]$ instructs us to fill all rows and columns complementary to the set α with zero's. Unlike equation (56), the sum over α includes the term of length $r = N$, with $V[1 \dots N] = V$, $S[1 \dots N] = S$. This is allowed here since there are no submatrices with round brackets of type $B(\alpha)$ which would be ill-defined for $r = N$. Also note that the term in the numerator with α of length 1 equals V . After inserting (60) into expression (38) for Δ , we arrive at the final result for the mean field,

$$\Delta = \frac{1 + \sum_{\alpha \in K_N} \xi_\alpha [V]_\alpha U_\alpha}{1 + \sum_{\alpha \in K_N} \xi_\alpha [V]_\alpha} \quad (61)$$

with the shorthand notation

$$\begin{aligned} \xi_\alpha &= \det S[\alpha], \\ [V]_\alpha &= \det V[\alpha] = \prod_{i \in \alpha} V_i, \\ U_\alpha &= 1 + i \sum_{i,j} \left(\widetilde{S[\alpha]^{-1}} \right)_{ij} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_i^\dagger. \end{aligned} \quad (62)$$

We have replaced $\text{adj } S[\alpha]$ by $S[\alpha]^{-1}$ times $\det S[\alpha] = \xi_\alpha$ in U_α .

The result (61) and (62) is very simple indeed. Every single term in the numerator has the same structure, except that the indices are constrained to the sequence α . One can check that this result reduces to the known expression in the one-flavor case. To this end, treat \vec{p}_i as a one-component object with $p_i = 1$ for all i . Since all $\sigma_{ij} = 1$, the matrix $S[\alpha]$ becomes a Cauchy matrix for which the determinant and the inverse matrix are explicitly calculable [17]. In the multiflavor case, the matrices U_α are unitary, as we will now check. For notational simplicity, consider $U := U_{1\dots N}$, the asymptotic vacuum at $x \rightarrow \infty$. The other U_α 's can be handled similarly by merely restricting and relabeling the soliton indices. We start from

$$\begin{aligned} U &= 1 + i \sum_{n,j} (S^{-1})_{nj} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_n^\dagger, \\ U^\dagger &= 1 - i \sum_{i,m} (S^{-1})_{im} \frac{1}{\zeta_i} \vec{p}_m \vec{p}_i^\dagger, \end{aligned} \quad (63)$$

where we have used the hermiticity of S . Computing UU^\dagger yields a 1, two terms linear and one term quadratic in the dyadics. The quadratic term just cancels the sum of the linear terms. To show this, use

$$(\vec{p}_j \vec{p}_n^\dagger)(\vec{p}_m \vec{p}_i^\dagger) = \sigma_{nm} \vec{p}_j \vec{p}_i^\dagger, \quad (64)$$

express σ_{nm} by S_{nm} , equation (35), and verify the identity

$$(S^{-1})_{nj}(S^{-1})_{im}\sigma_{nm} = -i(S^{-1})_{ij}(\zeta_i - \zeta_j^*). \quad (65)$$

Along the same lines, we could insert our expression for $(v + S)^{-1}$ into the spinors, equation (39), in the compact form

$$\left(\frac{1}{v + S} \right)_{ij} = \frac{\sum_{\alpha \in K_N} \xi_\alpha [V]_\alpha \widetilde{(S[\alpha]^{-1})}_{ij}}{1 + \sum_{\alpha \in K_N} \xi_\alpha [V]_\alpha}. \quad (66)$$

Since the gain of insight is less obvious than in the case of Δ , we shall not write down the resulting expressions here.

The U_α 's have a simple physics interpretation. This becomes particularly clear if we assume that the solitons are all well separated, either by choice of bound state configuration (see [15]) or at a certain time during a scattering process. Let us assume that the positions of the solitons are ordered such that $x_i \ll x_j \ll x_k \dots$. In between two neighboring solitons, the mean field must reduce to that of a twisted vacuum, characterized by a locally constant matrix $\in U(N_f)$. Proceeding from $x \rightarrow -\infty$ towards $x \rightarrow \infty$, the vacua are ordered as $1, U_i, U_{ij}, U_{ijk}, \dots U_{12\dots N}$. The fact that all permutations can occur is responsible for the proliferation of U_α 's in Δ , necessary to account for all possible orderings. In the one-flavor case, the V_i are the same as here but the U_α go over into the familiar twist factors. This follows from the remarkable identity for the Cauchy matrix S ,

$$U_\alpha \rightarrow 1 + i \sum_{i,j} (S[\alpha]^{-1})_{ij} \frac{1}{\zeta_j^*} = \prod_{k \in \alpha} \frac{\zeta_k}{\zeta_k^*} \in U(1) \quad (N_f = 1). \quad (67)$$

The prefactors ξ_α also depend on N_f , since the matrix S contains the scalar products $\vec{p}_i^\dagger \vec{p}_j$. Thus we can think of the U_α either as twist matrices, or as possible vacua far away from the

solitons. If two solitons are close together, the mean field in between is no longer related to a vacuum, but becomes (x, t) -dependent. However, expression (61) remains valid.

The picture of widely separated solitons which emerges involves a sequence of solitons connecting the vacua $1 \rightarrow U_1 \rightarrow U_{12} \rightarrow U_{123} \dots$ and all permutations. The intrinsic form of a single kink connects $1 \rightarrow U_i$. By a chiral transformation, we may identify intrinsic solitons with the sequence $1 \rightarrow U_1, 1 \rightarrow U_{12}U_1^\dagger, 1 \rightarrow U_{123}U_{12}^\dagger$ etc. This corresponds to the decomposition of the vacua according to

$$\begin{aligned} U_{12} &= (U_{12}U_1^\dagger)U_1, \\ U_{123} &= (U_{123}U_{12}^\dagger)(U_{12}U_1^\dagger)U_1, \end{aligned} \quad (68)$$

etc. Each factor is an elementary twist matrix. It is of some interest to evaluate these elementary twist matrices, since they clearly show how a soliton is influenced by the flavor structure of the other twisted kinks. It is sufficient to compute UU_k^\dagger for this purpose, where $U = U_{12\dots N}$ is the highest U and U_k differs from U by the missing k th row and column. All other products in (68) can be obtained by restricting and renaming the indices. Here we only give the result, referring to the [appendix](#) for the derivation. Let

$$\begin{aligned} U &= 1 + i \sum_{i,j} (S^{-1})_{ij} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_i^\dagger, \\ U_k^\dagger &= 1 - i \sum_{n,m} (\widetilde{S}_k^{-1})_{nm} \frac{1}{\zeta_n^*} \vec{p}_m \vec{p}_n^\dagger. \end{aligned} \quad (69)$$

where S_k is the matrix obtained from S by deleting the k th row and column. Then we find

$$UU_k^\dagger = 1 + \kappa_k \vec{Q}_k \vec{Q}_k^\dagger, \quad \kappa_k = \frac{(\zeta_k - \zeta_k^*)}{\zeta_k^*} \quad (\text{no } k\text{-sum}) \quad (70)$$

with

$$\begin{aligned} \vec{Q}_k &= \vec{K}_k (\vec{K}_k^\dagger \vec{K}_k)^{-1/2}, \\ \vec{K}_k &= \sum_j (S^{-1})_{kj} \frac{1}{\zeta_j^*} \vec{p}_j. \end{aligned} \quad (71)$$

The normalization factor entering (71) is given by

$$\vec{K}_k^\dagger \vec{K}_k = -i \left(\frac{1}{\zeta_k^*} - \frac{1}{\zeta_k} \right) (S^{-1})_{kk} \quad (\text{no } k\text{-sum}). \quad (72)$$

These results will be used again in the applications in sections 8 and 9.

6. General case including breathers

In the general case, the matrix ω is non-diagonal. This describes breathers, solitons that are time dependent in their rest frame. The simplest breather requires a 2×2 block submatrix in ω . Additional diagonal elements then describe solitons in interaction with the breather and each other. Larger block submatrices would correspond to more complex breathers built out of more than two twisted kink constituents. In the preceding chapter, an important step was going from the diagonal matrix v to the inverse diagonal matrix V . This enabled us to exhibit

the flavor structure and the possible intermediate vacua in a clear fashion. Here we generalize this procedure to non-diagonal matrices v, V .

Recall that

$$v = E^{-1} \omega (E^\dagger)^{-1}, \quad v_{ij} = \frac{\omega_{ij}}{e_i e_j^*} \quad (73)$$

and introduce the inverse of the matrix v as

$$V = v^{-1} = E^\dagger \omega^{-1} E, \quad V_{ij} = e_i^* (\omega^{-1})_{ij} e_j. \quad (74)$$

Our starting point is the identity for the inverse of a sum of two matrices (see section 4),

$$\begin{aligned} \frac{1}{v + S} &= \frac{\text{adj}(v + S)}{\det(v + S)}, \\ \text{adj}(v + S) &= \text{adj } S + \sum_{r=1}^{N-1} \sum_{\alpha, \beta} (-1)^{s(\alpha)+s(\beta)} \det v[\alpha|\beta] \widetilde{\text{adj } S(\alpha|\beta)}, \\ \det(v + S) &= \det v + \det S + \sum_{r=1}^{N-1} \sum_{\alpha, \beta} (-1)^{s(\alpha)+s(\beta)} \det v[\alpha|\beta] \det S(\alpha|\beta). \end{aligned} \quad (75)$$

(75)

Using Jacobi's complementary minor formula, we can express $\det v[\alpha|\beta]$ by the determinant of a submatrix of $V = v^{-1}$ as follows

$$\det v[\alpha|\beta] = (-1)^{s(\alpha)+s(\beta)} \frac{\det V(\beta|\alpha)}{\det V}. \quad (76)$$

Replacing $\det v$ by $1/\det V$ and expanding numerator and denominator in (75) by $\det V$, we get

$$\frac{1}{V^{-1} + S} = \frac{\sum_{r=1}^{N-1} \sum_{\alpha, \beta} \det V(\beta|\alpha) \widetilde{\text{adj } S(\alpha|\beta)} + \det V \text{adj } S}{1 + \sum_{r=1}^{N-1} \sum_{\alpha, \beta} \det V(\beta|\alpha) \det S(\alpha|\beta) + \det V \det S}. \quad (77)$$

As in the diagonal case, it is more convenient to switch notation from $V(\beta|\alpha)$ where rows β and columns α are missing to $V[\beta'|\alpha']$ where the complementary rows β' and complementary columns α' are kept. Since both S and V in (77) now involve round brackets, we can switch both matrices to square brackets and sum over the complementary sequences α', β'

$$V(\beta|\alpha) = V[\beta'|\alpha'], \quad S(\alpha|\beta) = S[\alpha'|\beta']. \quad (78)$$

The range of r from 1 to $N - 1$ does not change under this transition. Renaming the summation indices α', β' into α, β at the end to ease the notation, we find

$$\frac{1}{V^{-1} + S} = \frac{\sum_{r=1}^N \sum_{\alpha, \beta} \det V[\beta|\alpha] \widetilde{\text{adj } S[\alpha|\beta]}}{1 + \sum_{r=1}^N \sum_{\alpha, \beta} \det V[\beta|\alpha] \det S[\alpha|\beta]}. \quad (79)$$

This is the generalization of (60) to non-diagonal V . We now insert this expression into Δ , equation (38), treating the $r = N$ terms in the sums separately for notational reasons, with the result

$$\begin{aligned}\Delta &= \frac{\mathcal{N}}{\mathcal{D}}, \\ \mathcal{N} &= 1 + \sum_{r=1}^{N-1} \sum_{\alpha, \beta} \xi_{\alpha \beta} U_{\alpha \beta} \det V[\beta | \alpha] + \xi U \det V, \\ \mathcal{D} &= 1 + \sum_{r=1}^{N-1} \sum_{\alpha, \beta} \xi_{\alpha \beta} \det V[\beta | \alpha] + \xi \det V.\end{aligned}\tag{80}$$

Generalizing the ξ_α in section 5 we have defined

$$\xi_{\alpha \beta} = \det S[\alpha | \beta], \quad \xi = \det S.\tag{81}$$

The twist matrices $U_{\alpha \beta}$ in flavor space are the generalization of U_α in equation (62),

$$\begin{aligned}U_{\alpha \beta} &= 1 + i \sum_{i,j} \left(\widetilde{S[\alpha | \beta]}^{-1} \right)_{ij} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_i^\dagger, \\ U &= 1 + i \sum_{i,j} (S^{-1})_{ij} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_i^\dagger.\end{aligned}\tag{82}$$

The matrix U in the last line is the asymptotic vacuum at $x \rightarrow \infty$, independently of whether ω is diagonal or non-diagonal. Unlike the U_α in the preceding section, the $U_{\alpha \beta}$ are no longer unitary but satisfy the generalised unitarity relation

$$U_{\alpha \beta} U_{\beta \alpha}^\dagger = 1.\tag{83}$$

The proof is similar to the proof that $UU^\dagger = 1$ following equation (63). Start from

$$\begin{aligned}U_{\alpha \beta} &= 1 + i \sum_{n,j} \left(\widetilde{S[\alpha | \beta]}^{-1} \right)_{nj} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_n^\dagger, \\ U_{\beta \alpha}^\dagger &= 1 - i \sum_{i,m} \left(\widetilde{S[\beta | \alpha]}^{-1} \right)_{mi}^* \frac{1}{\zeta_i} \vec{p}_m \vec{p}_i^\dagger.\end{aligned}\tag{84}$$

Notice that with this choice of dummy indices, we must have $n \in \beta, j \in \alpha, i \in \beta, m \in \alpha$ due to the definition of the tilde symbol. The product $U_{\alpha \beta} U_{\beta \alpha}^\dagger$ yields 1, two terms linear in the dyadics and a term quadratic in the dyadics. This last term reads

$$U_{\alpha \beta} U_{\beta \alpha}^\dagger \Big|_{\text{quad}} = \sum_{i,j} \frac{1}{\zeta_i \zeta_j^*} \vec{p}_j \vec{p}_i^\dagger \sum_{n,m} \sigma_{nm} \left(\widetilde{S[\alpha | \beta]}^{-1} \right)_{nj} \left(\widetilde{S[\beta | \alpha]}^{-1} \right)_{mi}^*\tag{85}$$

where

$$\sigma_{nm} = \vec{p}_n^\dagger \vec{p}_m = -i(\zeta_n - \zeta_m^*) S_{mn}.\tag{86}$$

In the term containing ζ_m^* , we perform the summation over n as follows

$$\sum_n S_{mn} (\widetilde{S[\alpha|\beta]^{-1}})_{nj} = \delta_{mj}. \quad (87)$$

This term cancels the term linear in the dyadics contained in $U_{\beta\alpha}^\dagger$. In the term containing ζ_n , we use again (86) together with the hermiticity of S to get

$$\sum_m S_{nm}^* (\widetilde{S[\beta|\alpha]^{-1}})_{mi}^* = \delta_{ni}. \quad (88)$$

The resulting term cancels the term linear in the dyadics contained in $U_{\alpha\beta}$. This proves the assertion (83). Finally, products of V_i 's in the diagonal case (62) are replaced by determinants of submatrices of V depending only on the e_i, e_i^* and ω_{ij} .

This is the end of the formal part of the present work. The first step has been to transform the original expressions (25) for the mean field and (20), (21) and (24) for the spinors into (38)–(40) where the flavor structure has been exposed. A central element of all these expressions is the inverse matrix $(v + S)^{-1}$. In order to separate space-time dependence from the other dependencies, we derived a closed expression for the inverse of a sum of two square matrices, starting from a well-known formula for the determinant of a sum of matrices. The final result for the mean field Δ is surprisingly simple and given in (61) and (62) for solitons only (diagonal ω) and in (80)–(82) for the general case including breathers (non-diagonal ω). In the following sections, we shall use these results as starting point to illustrate the formalism with simple examples.

7. Example I: single twisted kink

We first have to understand thoroughly a single twisted kink, the basic building block of all more complex configurations. For one soliton, $v + S$ is a 1×1 ‘matrix’ so that there is no issue of matrix inversion. Nevertheless, the mean field Δ is a $N_f \times N_f$ matrix and we focus on the role of the flavor vector \vec{p}_1 . We start from

$$\left(\frac{1}{v + S} \right)_{11} = \frac{V_1}{1 + S_{11}V_1}, \quad S_{11} = \frac{i}{\zeta_1 - \zeta_1^*}. \quad (89)$$

Upon defining

$$\tilde{V}_1 = S_{11}V_1, \quad (90)$$

we find a very simple expression for Δ ,

$$\Delta = \frac{1 + U_1 \tilde{V}_1}{1 + \tilde{V}_1} \quad (91)$$

with

$$U_1 = 1 + \kappa_1 \vec{p}_1 \vec{p}_1^\dagger, \quad \kappa_1 = \frac{\zeta_1 - \zeta_1^*}{\zeta_1^*}. \quad (92)$$

We recall from the one-flavor model that

$$\zeta_1 = -\frac{e^{-i\theta_1}}{\eta_1}, \quad \eta_1 = \sqrt{\frac{1 + v_1}{1 - v_1}}, \quad (93)$$

where v_1 is the velocity of the soliton. The function \tilde{V}_1 is independent of N_f and given by

$$\tilde{V}_1 = \frac{S_{11}}{\omega_{11}} e^{2 \sin \theta_1 x'} \quad (94)$$

with x' the boosted form of x ,

$$x' = \frac{x - v_1 t}{\sqrt{1 - v_1^2}}. \quad (95)$$

The twisted kink interpolates between the vacua $\Delta = 1$ at $x \rightarrow -\infty$ and $\Delta = U_1$ at $x \rightarrow \infty$, the twist matrix U_1 replacing the phase factor $e^{-2i\theta_1}$ in the one-flavor case,

$$U_1 = 1 + (e^{-2i\theta_1} - 1) \vec{p}_1 \vec{p}_1^\dagger = \exp \left\{ -2i\theta_1 \vec{p}_1 \vec{p}_1^\dagger \right\}. \quad (96)$$

Equivalently, U_1 may be written in the canonical form of a $U(N_f)$ group element

$$U_1 = \exp \left\{ -2i\theta_1 n^a \lambda^a \right\}, \quad n^a = \vec{p}_1^\dagger \lambda^a \vec{p}_1, \quad n^a n^a = 1. \quad (97)$$

Thus the twist angle has the same interpretation as in the one-flavor case, namely as a ‘rotation angle’. The novel vectors \vec{p}_1 (in the fundamental representation) serve to define the ‘rotation axis’ n^a , a unit vector in the adjoint representation. If there is only one soliton, we are free to choose a frame in which this axis is pointing into the 0-direction. Then everything reduces to the $U(1)$ case.

Due to the close relationship with the single flavor case, we refrain from discussing the continuum spinors. It is of some interest though to look at the normalized bound state spinors with Dirac components

$$\vec{\varphi}_{1,1} = -\zeta_1^* \vec{\varphi}_{2,1} = \frac{1}{\sqrt{2} S_{11}} \frac{1}{e_1^*} \frac{\tilde{V}_1}{1 + \tilde{V}_1} \vec{p}_1. \quad (98)$$

In the one flavor case, the total fermion charge of the bound state is given by the occupation fraction,

$$Q = \nu_1 = \frac{\theta_1}{\pi}, \quad (99)$$

where we have used the self-consistency condition. The analogous calculation in the multiflavor case yields the flavor charge $Q^a = \langle \vec{\varphi}^\dagger \lambda^a \vec{\varphi} \rangle$ in the bound state to be

$$Q^a = \nu_1 \vec{p}_1^\dagger \lambda^a \vec{p}_1 = \frac{\theta_1}{\pi} n^a. \quad (100)$$

This gives yet another physical interpretation of \vec{p}_1 , namely determining the direction of the flavor vector (a generalization of the isospin vector) associated with the bound orbit. We have already mentioned that the full charge density vanishes due to a cancellation between continuum and bound states, a consequence of chiral symmetry and current conservation. In the one flavor case, it was recently pointed out that the total charge of a twisted kink is infrared sensitive and needs some regularization, either by a small bare fermion mass [18] or a finite box [19]. The conclusion was that the charge is spread out over the whole space in the thermodynamic and chiral limit and hence becomes invisible. Nevertheless, a regularized integrated charge can be defined consistently and agrees with the charge of the fermions in the bound state. For a single soliton, the same arguments could be applied here as well, giving a more direct physical meaning to the vector n^a as flavor vector of the twisted kink as a whole.

8. Example II: scattering and bound state of two twisted kinks

Equations (61) and (62) yield the following mean field Δ for two twisted kinks,

$$\Delta = \frac{1 + \xi_1 V_1 U_1 + \xi_2 V_2 U_2 + \xi_{12} V_1 V_2 U_{12}}{1 + \xi_1 V_1 + \xi_2 V_2 + \xi_{12} V_1 V_2}. \quad (101)$$

Introducing

$$\tilde{V}_i = \xi_i V_i, \quad \tilde{\xi}_{12} = \frac{\xi_{12}}{\xi_1 \xi_2}, \quad (102)$$

this goes over into

$$\Delta = \frac{1 + \tilde{V}_1 U_1 + \tilde{V}_2 U_2 + \tilde{\xi}_{12} \tilde{V}_1 \tilde{V}_2 U_{12}}{1 + \tilde{V}_1 + \tilde{V}_2 + \tilde{\xi}_{12} \tilde{V}_1 \tilde{V}_2}. \quad (103)$$

Using equations (35) and (62), we evaluate $\tilde{\xi}_{12}$,

$$\begin{aligned} \xi_i &= S_{ii} = \frac{i}{\zeta_i - \zeta_i^*}, \\ \tilde{\xi}_{12} &= 1 - \frac{(\zeta_1 - \zeta_1^*)(\zeta_2 - \zeta_2^*)}{(\zeta_2 - \zeta_1^*)(\zeta_1 - \zeta_2^*)} \sigma_{12} \sigma_{21}. \end{aligned} \quad (104)$$

The twist factors entering the numerator of Δ are

$$\begin{aligned} U_i &= 1 + \kappa_i \vec{p}_i \vec{p}_i^\dagger, \quad \kappa_i = \frac{\zeta_i - \zeta_i^*}{\zeta_i^*}, \\ U_{12} &= 1 - \frac{\vec{p}_1 \vec{p}_1^\dagger}{(\zeta_2 - \zeta_2^*) \zeta_1^* \det S} - \frac{\vec{p}_2 \vec{p}_2^\dagger}{(\zeta_1 - \zeta_1^*) \zeta_2^* \det S} \\ &\quad + \frac{\sigma_{12} \vec{p}_1 \vec{p}_2^\dagger}{(\zeta_1 - \zeta_2^*) \zeta_1^* \det S} + \frac{\sigma_{21} \vec{p}_2 \vec{p}_1^\dagger}{(\zeta_2 - \zeta_1^*) \zeta_2^* \det S}, \\ \det S &= -\frac{1}{(\zeta_1 - \zeta_1^*)(\zeta_2 - \zeta_2^*)} \tilde{\xi}_{12}. \end{aligned} \quad (105)$$

The leftmost solitons (incoming kink I or outgoing kink II) are characterized by intrinsic flavor vectors \vec{p}_1, \vec{p}_2 . The rightmost solitons (outgoing kink I or incoming kink II) have the intrinsic twist matrices

$$\begin{aligned} U_{12} U_2^\dagger &= 1 + \kappa_1 \vec{Q}_1 \vec{Q}_1^\dagger, \\ U_{12} U_1^\dagger &= 1 + \kappa_2 \vec{Q}_2 \vec{Q}_2^\dagger, \end{aligned} \quad (106)$$

with the intrinsic flavor vectors

$$\begin{aligned}
\vec{Q}_1 &= \mathcal{N}_1 [\zeta_2^*(\zeta_2 - \zeta_1^*)\vec{p}_1 - \zeta_1^*(\zeta_2 - \zeta_2^*)\sigma_{21}\vec{p}_2], \\
\vec{Q}_2 &= \mathcal{N}_2 [\zeta_1^*(\zeta_1 - \zeta_2^*)\vec{p}_2 - \zeta_2^*(\zeta_1 - \zeta_1^*)\sigma_{12}\vec{p}_1], \\
\mathcal{N}_1^{-2} &= \frac{\zeta_2\zeta_2^*}{\zeta_1\zeta_1^*}\mathcal{N}_2^{-2} = (\zeta_1 - \zeta_2^*)(\zeta_2 - \zeta_1^*)(\zeta_1 - \zeta_1^*)(\zeta_2 - \zeta_2^*)\zeta_2\zeta_2^* \det S.
\end{aligned} \tag{107}$$

Twisted kinks are now characterized by a twist angle and a twist axis. In the one-flavor case, the axis is frozen and the angle is conserved during the collision. The only observable of a two-soliton scattering event is then the time delay. For many flavors, the twist angles are still conserved, but the twist axes are rotated during the collision. There are now two observables, the time delay and the change of orientation of the twist axis. Both of these depend on the relative orientation of the two flavor axes of the colliding solitons. If we choose $\vec{p}_1 = \vec{p}_2$, the flavor vectors are aligned and everything is concentrated in a single flavor component. Then we are back at the one-flavor case, including the time delay. In the other extreme, choosing \vec{p}_1, \vec{p}_2 or, equivalently, the flavor vectors n_1^a, n_2^a to be orthogonal, the two kinks decouple. There is no scattering at all and the solitons cross each other without interaction. The novel feature of the multiflavor model is the fact that we can control the strength of the interaction between two solitons with the help of the parameter $\sigma_{12} = \vec{p}_1^\dagger \vec{p}_2$. To see how the theory interpolates between the extreme cases $\sigma_{12} = 1$ and $\sigma_{12} = 0$ just discussed, we determine how the scattering observables depend on this parameter. To this end, we first extract the asymptotic form of the incoming and outgoing solitons from the full expression for Δ , equation (103),

$$\begin{aligned}
\Delta_{\text{in}}^{\text{I}} &= \lim_{\tilde{V}_2 \rightarrow 0} \Delta = \frac{1 + \tilde{V}_1 U_1}{1 + \tilde{V}_1}, \\
\Delta_{\text{out}}^{\text{I}} &= \lim_{\tilde{V}_2 \rightarrow \infty} \Delta = \left(\frac{1 + \tilde{\xi}_{12} \tilde{V}_1 U_{12} U_2^\dagger}{1 + \tilde{\xi}_{12} \tilde{V}_1} \right) U_2, \\
\Delta_{\text{in}}^{\text{II}} &= \lim_{\tilde{V}_1 \rightarrow \infty} \Delta = \left(\frac{1 + \tilde{\xi}_{12} \tilde{V}_2 U_{12} U_1^\dagger}{1 + \tilde{\xi}_{12} \tilde{V}_2} \right) U_1, \\
\Delta_{\text{out}}^{\text{II}} &= \lim_{\tilde{V}_1 \rightarrow 0} \Delta = \frac{1 + \tilde{V}_2 U_2}{1 + \tilde{V}_2}.
\end{aligned} \tag{108}$$

In the 2nd and 3rd line we have exhibited the intrinsic form of the twisted kinks. During the collision, the intrinsic flavor vector of soliton I changes from \vec{p}_1 to \vec{Q}_1 , that of soliton II from \vec{Q}_2 to \vec{p}_2 , see also figure 1. The change in flavor direction is characterized by

$$(\vec{p}_i^\dagger \lambda^a \vec{p}_i)(\vec{Q}_i^\dagger \lambda^a \vec{Q}_i) = |\vec{p}_i^\dagger \vec{Q}_i|^2, \quad (i = 1, 2). \tag{109}$$

The time delay can be found by equating

$$V_i(x, t - (\Delta t)_i) = \tilde{\xi}_{12} V_i(x, t). \tag{110}$$

Consider two solitons with twist angles θ_1, θ_2 and equal and opposite velocity $\pm v$. Introducing the parameter $\lambda = |\sigma_{12}|^2$, we find the time delay

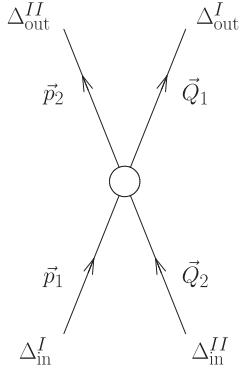


Figure 1. Soliton–soliton scattering. Indicated are the asymptotic mean fields (Δ) and intrinsic flavor vectors (\vec{p} , \vec{Q}).

$$\begin{aligned} \sin \theta_1(\Delta t)_1 = -\sin \theta_2(\Delta t)_2 &= \frac{\sqrt{1-v^2}}{2v} \ln \frac{1+v^2 - (1-v^2)[\lambda C_- + (1-\lambda)C_+]}{1+v^2 - C_+(1-v^2)} \\ &\approx \frac{\sqrt{1-v^2}}{2v} \frac{(1-v^2)(C_+ - C_-)}{1+v^2 - (1-v^2)C_+} \lambda + O(\lambda^2) \end{aligned} \quad (111)$$

where

$$C_{\pm} = \cos(\theta_1 \pm \theta_2). \quad (112)$$

The change in intrinsic flavor spin orientation of soliton I is

$$\begin{aligned} |\vec{p}_1^\dagger \vec{Q}_1|^2 - 1 &= -\frac{2 \sin^2 \theta_2 \lambda (1-\lambda)(1-v)^2}{1+v^2 + (1-v^2)[C_+(1-\lambda) + C_-\lambda]} \\ &\approx -\frac{2 \sin^2 \theta_2 \lambda (1-v)^2}{1+v^2 + (1-v^2)C_+} + O(\lambda^2). \end{aligned} \quad (113)$$

For soliton II, we get the same result except for the substitution $v \rightarrow -v$.

Equations (111)–(113) confirm that λ governs the interaction strength between the two solitons, here exhibited in the observables time delay and flavor spin rotation. As expected, the scattering observables vanish in the limit $\lambda \rightarrow 0$. In the opposite limit $\lambda \rightarrow 1$, the orientation of the axis is not changed and the time delay reduces to what is known from the one-flavor model.

Finally, we mention that by choosing $v = 0$ in the above expressions, we can specialize the scattering problem to the bound state of two twisted kinks. The parameter $\lambda = |\sigma_{12}|^2$ again allows us to interpolate between the one-flavor bound state ($\lambda = 1$) and a pair of non interacting single twisted kinks ($\lambda = 0$).

9. Example III: breathers

Finally, we turn to the twisted breather. A breather at rest can be generated by choosing $\eta_1 = \eta_2 = 1$ and a non-diagonal matrix ω . An example for a two-soliton breather has been discussed in the case $N_f = 2$ before [14], and we have reproduced these results to test our

present formalism for non-diagonal matrix ω . Up to translations in space and time, ω can be chosen as

$$\omega = \begin{pmatrix} \sec \chi & \tan \chi \\ \tan \chi & \sec \chi \end{pmatrix}, \quad \det \omega = 1. \quad (114)$$

The formalism of section 6 then yields the following expression for the mean field Δ

$$\begin{aligned} \Delta &= \frac{\mathcal{N}_b}{\mathcal{D}_b} \\ \mathcal{N}_b &= 1 + \sec \chi (\xi_{1,1} U_{1,1} V_1 + \xi_{2,2} U_{2,2} V_2) \\ &\quad - \tan \chi (\xi_{1,2} U_{1,2} K_{21} + \xi_{2,1} U_{2,1} K_{12}) + \xi_{12,12} U_{12,12} V_1 V_2 \\ \mathcal{D}_b &= 1 + \sec \chi (\xi_{1,1} V_1 + \xi_{2,2} V_2) - \tan \chi (\xi_{1,2} K_{21} + \xi_{2,1} K_{12}) + \xi_{12,12} V_1 V_2 \end{aligned} \quad (115)$$

with

$$V_i = |e_i|^2, \quad K_{ij} = e_i^* e_j. \quad (116)$$

The characteristic novel feature of the breather are the K_{ij} yielding oscillations with the frequency $\Omega = \cos \theta_1 - \cos \theta_2$. The $\xi_{\alpha\beta}$ and $U_{\alpha\beta}$ can now easily be constructed from submatrices of S , see equations (81) and (82). In this way we recover the results from reference [14]. If we choose \vec{p}_1 and \vec{p}_2 to be parallel, we come back to the known one-flavor breather. Choosing \vec{p}_1 and \vec{p}_2 to be orthogonal one finds that the diagonal components Δ_{11}, Δ_{22} are static, whereas the off-diagonal components Δ_{12}, Δ_{21} oscillate with the same frequency as the one-flavor breather. For any other choice of the angle between \vec{p}_1, \vec{p}_2 , all components of Δ start to oscillate with the same frequency but different phases. Thus, unlike in the soliton case, the interaction does not disappear if the flavor vectors are orthogonal. It is mediated by the off-diagonal matrix elements of ω .

The formalism enables us to go beyond this type of complexity. By way of example, we sketch how one would analyse scattering between a breather and a soliton. This shows that one gets some analytical insight even without carrying out the full, tedious calculation to the end. We focus on the asymptotics of the scattering event to get an idea how the breather and the soliton are affected by a collision.

To this end, we consider a three soliton configuration with the ω matrix

$$\omega = \begin{pmatrix} \sec \chi & \tan \chi & 0 \\ \tan \chi & \sec \chi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \det \omega = 1. \quad (117)$$

Evaluating the determinants of submatrices of V , one finds that the formalism predicts already quite a number of terms to account for the breather, the soliton and their interaction,

$$\begin{aligned} \Delta &= \frac{\mathcal{N}}{\mathcal{D}}, \\ \mathcal{N} &= \mathcal{N}_b + \xi_{3,3} U_{3,3} V_3 + \sec \chi (\xi_{13,13} U_{13,13} V_1 + \xi_{23,23} U_{23,23} V_2) V_3 \\ &\quad - \tan \chi (\xi_{13,23} U_{13,23} K_{21} + \xi_{23,13} U_{23,13} K_{12}) V_3 + \xi U V_1 V_2 V_3, \\ \mathcal{D} &= \mathcal{D}_b + \xi_{3,3} V_3 + \sec \chi (\xi_{13,13} V_1 + \xi_{23,23} V_2) V_3 \\ &\quad - \tan \chi (\xi_{13,23} K_{21} + \xi_{23,13} K_{12}) V_3 + \xi V_1 V_2 V_3. \end{aligned} \quad (118)$$

We recognize the breather pieces $\mathcal{N}_b, \mathcal{D}_b$ from equation (115) which we did not spell out again, terms $\sim V_3$ involving only the soliton and interaction terms. As is always the case, the denominator can be obtained from the numerator by setting all $U_{\alpha\beta}$ equal to 1. It would now be straightforward to evaluate all the coefficients and twist factors. However, it is perhaps more instructive to extract the asymptotics. In the real world, this would contain all scattering observables. We have to treat V_1, V_2, K_{12}, K_{21} as being of the same order and consider as in equation (108)

$$\begin{aligned}
\Delta_{b,in} &= \lim_{V_3 \rightarrow 0} \Delta = \frac{\mathcal{N}_b}{\mathcal{D}_b}, \\
\Delta_{b,out} &= \lim_{V_3 \rightarrow \infty} \Delta = \left(\frac{\mathcal{N}_{b'}}{\mathcal{D}_{b'}} \right) U_{3,3}, \\
\mathcal{N}_{b'} &= 1 + \sec \chi \left(\xi_{13,13} \xi_{3,3}^{-1} U_{13,13} U_{3,3}^\dagger V_1 + \xi_{23,23} \xi_{3,3}^{-1} U_{23,23} U_{3,3}^\dagger V_2 \right) \\
&\quad - \tan \chi \left(\xi_{13,23} \xi_{3,3}^{-1} U_{13,23} U_{3,3}^\dagger K_{21} + \xi_{23,13} \xi_{3,3}^{-1} U_{23,13} U_{3,3}^\dagger K_{12} \right) + \xi \xi_{3,3}^{-1} U U_{3,3}^\dagger V_1 V_2, \\
\mathcal{D}_{b'} &= 1 + \sec \chi \left(\xi_{13,13} \xi_{3,3}^{-1} V_1 + \xi_{23,23} \xi_{3,3}^{-1} V_2 \right) \\
&\quad - \tan \chi \left(\xi_{13,23} \xi_{3,3}^{-1} K_{21} + \xi_{23,13} \xi_{3,3}^{-1} K_{12} \right) + \xi \xi_{3,3}^{-1} V_1 V_2, \\
\Delta_{s,in} &= \lim_{V_1 \rightarrow \infty} \Delta = \left(\frac{1 + \xi \xi_{12,12}^{-1} V_3 U U_{12,12}^\dagger}{1 + \xi \xi_{12,12}^{-1} V_3} \right) U_{12,12}, \\
\Delta_{s,out} &= \lim_{V_1 \rightarrow 0} \Delta = \frac{1 + \xi_{3,3} V_3 U_{3,3}}{1 + \xi_{3,3} V_3}.
\end{aligned} \tag{119}$$

The subscripts b and s refer to breather and soliton, respectively.

How are soliton and breather affected during the scattering event? For the soliton, we can again read off the time delay and the change in flavor orientation. For the soliton–soliton collision discussed in section 7, the time delay was proportional to $\ln \tilde{\xi}_{12}$ where $\tilde{\xi}_{12} = \xi_{12} \xi_1^{-1} \xi_2^{-1}$. Now the same formula applies except that the argument of the logarithm is $\xi \xi_{12,12}^{-1} \xi_{3,3}^{-1}$. The intrinsic flavor vector changes from the vector defined by $U U_{12,12}^\dagger$ to \vec{p}_3 . For the breather, things are more complicated since the different building blocks are affected differently, so that the whole internal structure changes. For the time delays, the relevant arguments of the logarithms are

$$\begin{aligned}
V_1 &: \frac{\xi_{13,13}}{\xi_{1,1} \xi_{3,3}}, \quad V_2 : \frac{\xi_{23,23}}{\xi_{2,2} \xi_{3,3}}, \quad V_1 V_2 : \frac{\xi}{\xi_{12,12} \xi_{3,3}} \\
K_{21} &: \frac{\xi_{13,23}}{\xi_{1,2} \xi_{3,3}}, \quad K_{12} : \frac{\xi_{23,13}}{\xi_{2,1} \xi_{3,3}}.
\end{aligned} \tag{120}$$

All the $\xi_{\alpha\beta}$ can be evaluated as determinants of submatrices of S . More interesting is perhaps the observation that the results are so regular that one can guess the general principle behind them, even for more complicated collisions. Since different parts of the breather suffer different time delays, the structure changes. This is also true for the changes in internal flavor direction which are also different for different components of the breather.

10. Summary and conclusions

The chiral GN model with $U_L(N_f) \times U_R(N_f)$ symmetry is probably one of the most complicated quantum field theories that one can still solve analytically, at least in the large N_c limit where semiclassical methods become exact. This has incited us to reconsider the question of soliton dynamics in this model. We were able to build our study on existing results from both condensed matter and particle theory. Although the most important steps in solving the problem had already been done, we felt that the analytical results for the interaction of solitons and breathers had not yet been cast into a sufficiently intuitive form. In the present work, we have therefore reformulated the general solution. The main idea was to manipulate the formal expression for the mean field in such a way that the (x, t) dependent factors are neatly separated from constant coefficients and twist matrices. Explicit expressions for the coefficients and twist matrices could be derived. We were guided by previous results for scattering of N solitons in the single flavor model, simpler than the general expressions. At the core of this problem was the necessity to invert a sum of two square matrices with dimension given by the number of solitons involved. Starting from a well-known formula for the determinant of a sum of two matrices, we derived the corresponding formula for the inverse of a sum and applied it to the problem at hand. If one disregards breathers, the result is particularly simple. The mean field is represented as a quotient of two multivariate polynomials in the basic exponential soliton functions V_i , much like in the one flavor case. The novel feature as compared to a single flavor is the fact that the twist factors in the numerator now become unitary $N_f \times N_f$ matrices, as opposed to phase factors (or $U(1)$ elements) before. Both the coefficients and the twist matrices are given by compact, simple formulas involving submatrices of a constant matrix S . This procedure can be extended in the presence of breathers where things get more involved. Here, submatrices of S appear where different rows and columns are kept. The V_i have to be replaced by determinants of block matrices characteristic for breathers. The example of breather-soliton scattering has been used to illustrate the advantage of the new formulation, for instance in extracting the asymptotics of the scattering process in a simple manner.

As an outlook, we would like to come back to the formula for inverting a sum of two matrices. From a pragmatic point of view, if one is only interested in the (numerical or analytical) result, this is of little help. The expression on the right-hand side of equation (53) is obviously more complicated than the original problem. However we have seen that it has merits for organizing the analytical result, disentangling the separate contributions from the two matrices in the inverse sum. In our application, everything seems to fall into place. It will be interesting to see whether this has other applications in physics as well.

Acknowledgment

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Data availability statement

No new data were created or analysed in this study.

Appendix. Derivation of equations (70)–(72)

$U = U_{1\dots N}$ denotes the unitary matrix with the largest number of indices. U_k is the unitary matrix for the $N - 1$ soliton term where the index k is missing. We want to derive an expression for UU_k^\dagger , an intrinsic soliton. Our starting point is

$$\begin{aligned} U &= 1 + i \sum_{i,j} (S^{-1})_{ij} \frac{1}{\zeta_j^*} \vec{p}_j \vec{p}_i^\dagger, \\ U_k^\dagger &= 1 - i \sum_{n,m} \left(\widetilde{S}_k^{-1} \right)_{nm} \frac{1}{\zeta_n} \vec{p}_m \vec{p}_n^\dagger. \end{aligned} \quad (121)$$

We evaluate

$$\begin{aligned} UU_k^\dagger - 1 &= (U - 1) + (U_k^\dagger - 1) + (U - 1)(U_k^\dagger - 1) \\ &= \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3. \end{aligned} \quad (122)$$

We only need to compute \mathcal{A}_3 ,

$$\begin{aligned} \mathcal{A}_3 &= \sum_{j,n} Y_{jn}^k \frac{1}{\zeta_j^* \zeta_n} \vec{p}_j \vec{p}_n^\dagger, \\ Y_{jn}^k &= \sum_{i,m} (S^{-1})_{ij} \left(\widetilde{S}_k^{-1} \right)_{nm} \sigma_{im}. \end{aligned} \quad (123)$$

Expressing σ_{im} through S_{mi} , find

$$\begin{aligned} Y_{jn}^k &= -i \sum_i \zeta_i (S^{-1})_{ij} X_{ni}^k + i \sum_m \zeta_m^* \left(\widetilde{S}_k^{-1} \right)_{nm} \tilde{X}_{mj}, \\ X_{ni}^k &= \sum_m \left(\widetilde{S}_k^{-1} \right)_{nm} S_{mi}, \\ \tilde{X}_{mj} &= \sum_i S_{mi} (S^{-1})_{ij}, \end{aligned} \quad (124)$$

where

$$\begin{aligned} X_{ni}^k &= \delta_{ni} \quad \text{for } n \neq k, i \neq k, \\ X_{ni}^k &= 0 \quad \text{for } n = k, \\ X_{nk}^k &= -\frac{(\text{adj } S)_{nk}}{\det(S_k)}, \\ \tilde{X}_{mj} &= \delta_{mj}. \end{aligned} \quad (125)$$

This yields

$$Y_{jn} = (1 - \delta_{nk}) \left[i \zeta_j^* \left(\widetilde{S}_k^{-1} \right)_{nj} - i \zeta_n (S^{-1})_{nj} + i \zeta_k \frac{(S^{-1})_{kj} (\text{adj } S)_{nk}}{\det(S_k)} \right]. \quad (126)$$

Insert this result into \mathcal{A}_3 , equation (123). The first term cancels \mathcal{A}_2 . The second term partially cancels \mathcal{A}_1 , leaving the $n = k$ term. This combines with the 3rd term to the simple final result

$$UU_k^\dagger - 1 = i \sum_{j,n} \frac{(S^{-1})_{kj}(S^{-1})_{nk} \det S}{\det S_k} \frac{1}{\zeta_j^* \zeta_n} \vec{p}_j \vec{p}_n^\dagger. \quad (127)$$

This factorizes indeed,

$$\begin{aligned} UU_k^\dagger - 1 &= i \frac{\zeta_k \det S}{\det S_k} \vec{K}_k \vec{K}_k^\dagger, \\ \vec{K}_k &= \sum_j (S^{-1})_{kj} \frac{1}{\zeta_j^*} \vec{p}_j, \\ \vec{K}_k^\dagger &= \sum_i (S^{-1})_{ik} \frac{1}{\zeta_i} \vec{p}_i^\dagger. \end{aligned} \quad (128)$$

The norm is

$$\begin{aligned} \vec{K}_k^\dagger \vec{K}_k &= \sum_{i,j} \frac{1}{\zeta_i \zeta_j^*} (S^{-1})_{ik} (S^{-1})_{kj} \sigma_{ij} \\ &= -i \left(\frac{1}{\zeta_k^*} - \frac{1}{\zeta_k} \right) (S^{-1})_{kk} \quad (\text{no } k\text{-sum}). \end{aligned} \quad (129)$$

Using $(S^{-1})_{kk} = \det S_k / \det S$ and introducing normalized vectors $\vec{Q}_k = \vec{K}_k (\vec{K}_k^\dagger \vec{K}_k)^{-1/2}$, we finally arrive at the standard form

$$UU_k^\dagger = 1 + \kappa_k \vec{Q}_k \vec{Q}_k^\dagger, \quad \kappa_k = \frac{(\zeta_k - \zeta_k^*)}{\zeta_k^*}. \quad (130)$$

ORCID iDs

Michael Thies  <https://orcid.org/0000-0002-8965-7476>

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