Letter

Dressing operators in equivariant Gromov–Witten theory of $\mathbb{CP}^1$

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Abstract

Okounkov and Pandharipande proved that the equivariant Toda hierarchy governs the equivariant Gromov–Witten theory of $\mathbb{CP}^1$. A technical clue of their method is a pair of dressing operators on the Fock space of 2D charged free fermion fields. We reformulate these operators as difference operators in the Lax formalism of the 2D Toda hierarchy. This leads to a new explanation to the question of why the equivariant Toda hierarchy emerges in the equivariant Gromov–Witten theory of $\mathbb{CP}^1$. Moreover, the non-equivariant limit of these operators turns out to capture the integrable structure of the non-equivariant Gromov–Witten theory correctly.

Keywords: Gromov–Witten theory, Riemann sphere, equivariant Toda hierarchy, dressing operators, Lax formalism

1. Introduction

The Gromov–Witten theory of the Riemann sphere $\mathbb{CP}^1$ is known to be related to integrable hierarchies of the Toda type. Such a link was first observed by physicists employing random matrix models [1, 2]. This discovery was enhanced to a mathematical statement called the Toda conjecture [3–5]. The Toda conjecture was proved by several methods [6–10] and generalized to $\mathbb{CP}^1$ with orbifold points [11–14]. The relevant integrable hierarchies are the 1D Toda hierarchy, the bigraded Toda hierarchy [15, 16] and a kind of the Kac–Wakimoto hierarchies [17]. Another direction of generalization is the equivariant Gromov–Witten theory of $\mathbb{CP}^1$ [7, 18–20]. The integrable hierarchies emerging therein are the equivariant Toda hierarchy [21] and its bigraded version [19].

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The most exotic among these integrable hierarchies will be the equivariant Toda hierarchy. This integrable hierarchy, like the 1D Toda hierarchy, is a reduction of the 2D Toda hierarchy. Okounkov and Pandharipande [7] encode the Gromov–Witten invariants into the vacuum expectation value of an operator product on the fermionic Fock space of 2D charged free fermion fields. This fermionic expression can be converted into a tau function of the 2D Toda hierarchy, and eventually turns out to be a tau function of the equivariant Toda hierarchy. Milanov [18] starts from a bosonic expression of the Gromov–Witten invariants in the Givental theory [22], and derives the Hirota bilinear equations of the equivariant Toda hierarchy. These results are further generalized to $\mathbb{CP}^1$ with two orbifold points [19, 20].

This paper reconsiders Okounkov and Pandharipande's method for the equivariant Gromov–Witten theory from the perspective of the Lax formalism of the 2D Toda hierarchy. A central role in their method is played by what they call dressing operators[1]. These operators on the fermionic Fock space are used to convert the fermionic expression of the Gromov–Witten invariants into a standard expression of tau functions of the 2D Toda hierarchy. It is, however, not very clear from this expression that the tau function is indeed a tau function of the equivariant Toda hierarchy. We reformulate Okounkov and Pandharipande's dressing operators as difference operators like the Lax operators of the 2D Toda hierarchy. This enables us to explain the relation to the equivariant Toda hierarchy in a more direct manner. Throughout this paper, the ordinary equivariant Toda hierarchy and its orbifold generalizations are treated on an equal footing.

2. Equivariant Toda hierarchy as reduction of 2D Toda hierarchy

Let us recall the Lax formalism of the 2D Toda hierarchy (see the recent review [23]). $\boldsymbol{t} = \{t_k\}_{k=1}^\infty$ and $\bar{\boldsymbol{t}} = \{\bar{t}_k\}_{k=1}^\infty$ are the two sets of time variables. s is the spatial coordinate, which is understood to be a continuous variable throughout this paper. Λ denotes the shift operator

$$\Lambda = e^{\partial_s}, \qquad \partial_s = \partial/\partial s.$$

The Lax operators $L, \bar{L}$ of the 2D Toda hierarchy are difference (or pseudo-difference) operators of the form

$$L = \Lambda + \sum_{n=1}^\infty u_n \Lambda^{1-n}, \qquad \bar{L}^{-1} = \sum_{n=0}^\infty \bar{u}_n \Lambda^{n-1},$$

$$u_n = u_n(s, \boldsymbol{t}, \bar{\boldsymbol{t}}), \qquad \bar{u}_n = \bar{u}_n(s, \boldsymbol{t}, \bar{\boldsymbol{t}}), \qquad \bar{u}_0(s, \boldsymbol{t}, \bar{\boldsymbol{t}}) \neq 0,$$

and satisfy the Lax equations

$$\frac{\partial L}{\partial t_k} = [B_k, L], \qquad \frac{\partial L}{\partial \bar{t}_k} = [\bar{B}_k, L],$$

$$\frac{\partial \bar{L}}{\partial t_k} = [B_k, \bar{L}], \qquad \frac{\partial \bar{L}}{\partial \bar{t}_k} = [\bar{B}_k, \bar{L}],$$

where

$$B_k = (L^k)_{\geqslant 0}, \qquad \bar{B}_k = (\bar{L}^{-k})_{<0}.$$

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[1] This terminology is somewhat confusing, because dressing operators of a different kind are already used in the Lax formalism of the 2D Toda hierarchy.

$(\ )_{\geqslant 0}$ and $(\ )_{<0}$ denote the projection onto the non-negative and negative powers of Λ:

$$\left(\sum_{n\in\mathbb{Z}} a_n \Lambda^n\right)_{\geqslant 0} = \sum_{n\geqslant 0} a_n \Lambda^n, \qquad \left(\sum_{n\in\mathbb{Z}} a_n \Lambda^n\right)_{<0} = \sum_{n<0} a_n \Lambda^n.$$

Let us mention that these equations should be formulated in the $\hbar$-dependent form [24] to accommodate the $\hbar$-expansion (i.e. genus expansion) of the Gromov–Witten theory. To avoid notational complexity, however, we dare not to consider the $\hbar$-dependent form. We can move to the $\hbar$-dependent formulation, at least formally, by rescaling the variables as $t_k \to t_k/\hbar$, $\bar{t}_k \to \bar{t}_k/\hbar$ and $s \to s/\hbar$.

The Lax operators can be expressed in a dressed form as

$$L = W\Lambda W^{-1}, \qquad \bar{L} = \bar{W}\Lambda\bar{W}^{-1}$$

with the dressing operators

$$W = 1 + \sum_{n=1}^{\infty} w_n \Lambda^{-n}, \qquad \bar{W} = \sum_{n=0}^{\infty} \bar{w}_n \Lambda^n,$$

$$w_n = w_n(s,\boldsymbol{t},\bar{\boldsymbol{t}}), \qquad \bar{w}_n = \bar{w}_n(s,\boldsymbol{t},\bar{\boldsymbol{t}}), \qquad \bar{w}_0(s,\boldsymbol{t},\bar{\boldsymbol{t}}) \neq 0.$$

The dressing operators $W, \bar{W}$ satisfy the Sato equations

$$\frac{\partial W}{\partial t_k} = B_k W - W\Lambda^k, \qquad \frac{\partial W}{\partial \bar{t}_k} = \bar{B}_k W,$$

$$\frac{\partial \bar{W}}{\partial t_k} = B_k \bar{W}, \qquad \frac{\partial \bar{W}}{\partial \bar{t}_k} = \bar{B}_k \bar{W} - W\Lambda^{-k}.$$

The logarithm of $L, \bar{L}$ can be defined with the dressing operators as

$$\log L = W \log \Lambda W^{-1} = \partial_s - \frac{\partial W}{\partial s} W^{-1},$$

$$\log \bar{L} = \bar{W} \log \Lambda \bar{W}^{-1} = \partial_s - \frac{\partial \bar{W}}{\partial s} \bar{W}^{-1}.$$

Note that $\log \Lambda = \partial_s$.

Let a, b be positive integers. They are related to the orders of two orbifold points of $\mathbb{CP}^1$. The case of $a = b = 1$ amounts to the ordinary $\mathbb{CP}^1$. The equivariant Toda hierarchy of type (a, b) can be obtained by imposing the reduction condition [5, 19]

$$L^a - \nu \log L = \bar{L}^{-b} - \nu \log \bar{L} - \nu \log Q, \tag{1}$$

where ν and Q are parameters of the reduction. ν is called the equivariant parameter. In the non-equivariant limit as $\nu \to 0$, the reduced system turns into the bigraded Toda hierarchy of type (a, b) [16]. Q is related to the particular solution that we shall consider later on.

(1) implies that both sides become an operator of the form

$$\mathfrak{L} = B_a + \bar{B}_b - \nu \log \Lambda. \tag{2}$$

The Lax equations of the 2D Toda hierarchy turn into the Lax equations

$$\frac{\partial \mathfrak{L}}{\partial t_k} = [B_k, \mathfrak{L}], \qquad \frac{\partial \mathfrak{L}}{\partial \bar{t}_k} = [\bar{B}_k, \mathfrak{L}]$$

for this reduced Lax operator. In the language of tau functions, (1) amounts to the condition that the tau function depends on s in the particular form

$$\tau = Q^{s^2/2} f(\{t_k + \delta_{ka}s/\nu, \bar{t}_k + \delta_{kb}s/\nu\}_{k=1}^{\infty}). \tag{3}$$

3. Dressing operators $V, \bar{V}$

Okounkov and Pandharipande's dressing operators $V, \bar{V}$ [7] are elements of the $GL(\infty)$ group acting on the Fock space of 2D charged free fermion fields[2]. The tau function of the equivariant Gromov–Witten theory, generalized to $\mathbb{CP}^1$ with two orbifold points of orders a, b [20], can be thereby expressed as

$$\tau = \langle s| \exp\left(\sum_{k=1}^{\infty} t_k J_k\right) g \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k J_{-k}\right) |s\rangle, \tag{4}$$

where

$$g = V^{-1} e^{J_a/a} Q^{L_0} e^{J_{-b}/b} \bar{V}^{-1}. \tag{5}$$

J_k's are the generators of the $U(1)$ current algebra of the free fermion system, and L_0 is the zero mode of the Virasoro algebra therein.

Okounkov and Pandharipande's construction of $V, \bar{V}$ relies on the correspondence between fermionic operators and $\mathbb{Z} \times \mathbb{Z}$ matrices. Those matrices can be further represented by difference operators on the discrete space $\mathbb{Z}$, e.g.

$$J_k \longleftrightarrow \Lambda^k, \qquad L_0 \longleftrightarrow H = s - 1/2. \tag{6}$$

Actually, Λ^k and H are meaningful in the continuous space as well. We shall redefine V and $\bar{V}$ as well to be difference operators on the continuous space.

In our reformulation, V and $\bar{V}$ are difference operators of the form[3]

$$V = 1 + \sum_{n=1}^{\infty} v_n \Lambda^{-n}, \qquad \bar{V} = 1 + \sum_{n=1}^{\infty} \bar{v}_n \Lambda^n,$$
$$v_n = v_n(s), \qquad \bar{v}_n = \bar{v}_n(s),$$

and satisfy the following intertwining relations:

$$(\Lambda^a + H - \nu \log \Lambda)V = V(\Lambda^a - \nu \log \Lambda), \tag{7}$$

$$\bar{V}(\Lambda^{-b} + H - \nu \log \Lambda) = (\Lambda^{-b} - \nu \log \Lambda)\bar{V}. \tag{8}$$

These operators can be constructed by power series expansion with respect to ν, see section 5. We shall show in the next section that (7) and (8) lead correctly to a solution of the reduction condition (1).

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[2] These operators are denoted by W, W^* in Okounkov and Pandharipande's notation. We have changed the notation to avoid confusion with the aforementioned dressing operators $W, \bar{W}$ for the Lax operators.

[3] This implies that their fermionic counterparts leave invariant the ground state of the charge-s sector in the Fock space as $\langle s|V = \langle s|$ and $\bar{V}|s\rangle = |s\rangle$.

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As further evidence for the validity of our reformation, let us mention that (7) and (8) in the case of $a = b = 1$ are consistent with Okounkov and Pandharipande's intertwining relations between two operators $\mathcal{A}(z, w)$ and $\tilde{\mathcal{A}}(z, w)$ on the fermionic Fock space. The counterparts of these operators on the difference operator side of the correspondence (6) are

$$\mathcal{A}(z, w) = \left(\frac{\zeta(w)}{w}\right)^z \sum_{k \in \mathbb{Z}} \frac{\zeta(w)^k}{(1 + z)_k} \mathcal{E}_k(w),$$

$$\tilde{\mathcal{A}}(z, w) = \sum_{k \in \mathbb{Z}} \frac{w^k}{(1 + z)_k} \Lambda^k,$$

where

$$\mathcal{E}_k(z) = \mathrm{e}^{z(H+k/2)}\Lambda^k, \qquad \zeta(z) = \mathrm{e}^{z/2} - \mathrm{e}^{-z/2}, \qquad (1 + z)_k = \frac{\Gamma(1 + z + k)}{\Gamma(1 + z)}.$$

For matching with Okounkov and Pandharipande's notation, let us introduce the new parameter $u = 1/\nu$ and rewrite (7), specialized to $a = b = 1$, as

$$(u(\Lambda + H) - \log \Lambda)V = V(u\Lambda - \log \Lambda). \tag{9}$$

Proposition 1. (9) *implies the intertwining relations*

$$\mathcal{A}(m, mu)V = V\tilde{\mathcal{A}}(m, mu), \quad m = 1, 2, \ldots. \tag{10}$$

Proof. Exponentiating both sides of (9), we have the identity

$$\mathrm{e}^{m(u\Lambda + uH - \log \Lambda)}V = V \mathrm{e}^{m(u\Lambda - \log \Lambda)}.$$

The exponential on the right side boils down to

$$\mathrm{e}^{m(u\Lambda - \log \Lambda)} = \mathrm{e}^{mu\Lambda}\Lambda^{-m}.$$

Since $[\log \Lambda, H] = 1$, we can use the Baker–Campbell–Hausdorff formula to compute the exponential on the left side as

$$\begin{aligned} \mathrm{e}^{m(u\Lambda + uH - \log \Lambda)} &= \mathrm{e}^\Lambda \mathrm{e}^{m(uH - \log \Lambda)} \mathrm{e}^{-\Lambda} \\ &= \mathrm{e}^\Lambda \mathrm{e}^{-mu^2/2} \mathrm{e}^{muH}\Lambda^{-m} \mathrm{e}^{-\Lambda} \\ &= \mathrm{e}^\Lambda \mathcal{E}_{-m}(mu)\mathrm{e}^{-\Lambda}. \end{aligned}$$

Therefore

$$\mathrm{e}^\Lambda \mathcal{E}_{-m}(mu)\mathrm{e}^{-\Lambda}V = V \mathrm{e}^{mu\Lambda}\Lambda^{-m}.$$

Since

$$\mathcal{A}(m, mu) = \frac{m!}{(mu)^m} \mathrm{e}^\Lambda \mathcal{E}_{-m}(mu)\mathrm{e}^{-\Lambda},$$

$$\tilde{\mathcal{A}}(m, mu) = \frac{m!}{(mu)^m} \mathrm{e}^{mu\Lambda}\Lambda^{-m},$$

we find that (10) holds. □

(10) takes exactly the same form as the one presented by Okounkov and Pandharipande [7]. We can derive a similar intertwining relation for $\bar{V}$ from (8) in much the same way. Thus Okounkov and Pandharipande's intertwining relations can be, at least partially, recovered in our reformulation of the dressing operators $V, \bar{V}$.

4. Algebraic relation of Lax operators $L, \bar{L}$

The dressing operators $W, \bar{W}$ of the tau function (4) can be captured by the factorization problem (see the review [23])

$$\exp\left(\sum_{k=1}^{\infty} t_k \Lambda^k\right) U \exp\left(-\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right) = W^{-1}\bar{W}, \tag{11}$$

where U is the difference operator

$$U = V^{-1} e^{\Lambda^a/a} Q^H e^{\Lambda^{-b}/b} \bar{V}^{-1} \tag{12}$$

that corresponds to the operator (5) on the fermionic Fock space. This operator satisfies the following intertwining relation, which implies the reduction condition (1) to the equivariant Toda hierarchy.

Proposition 2.

$$(\Lambda^a - \nu \log \Lambda)U = U(\Lambda^{-b} - \nu \log \Lambda - \nu \log Q). \tag{13}$$

Proof. We use (7) and the operator identities

$$e^{\Lambda^a/a} H e^{-\Lambda^a/a} = \Lambda^a + H,$$

$$e^{-\Lambda^{-b}/b} H e^{\Lambda^{-b}/b} = \Lambda^b + H,$$

$$\log \Lambda \, Q^H = Q^H(\log \Lambda + \log Q)$$

to derive (13) from (12) as

$$(\Lambda^a - \nu \log \Lambda)U = V^{-1}(\Lambda^a + H - \nu \log \Lambda)e^{\Lambda^a/a} Q^H e^{\Lambda^{-b}/b} \bar{V}^{-1}$$

$$= V^{-1} e^{\Lambda^a/a}(H - \nu \log \Lambda)Q^H e^{\Lambda^{-b}/b} \bar{V}^{-1}$$

$$= V^{-1} e^{\Lambda^a/a} Q^H(H - \nu \log \Lambda - \nu \log Q)e^{\Lambda^{-b}/b} \bar{V}^{-1}$$

$$= V^{-1} e^{\Lambda^a/a} Q^H e^{\Lambda^{-b}/b}(\Lambda^{-b} + H - \nu \log \Lambda - \nu \log Q)\bar{V}^{-1}$$

$$= U(\Lambda^{-b} - \nu \log \Lambda - \nu \log Q).$$

$$\square$$

Proposition 3. *The Lax operators obtained from the solution of the factorization problem* (11) *satisfy the reduction condition* (1).

Proof. Let us rewrite (11) as

$$U = \exp\left(-\sum_{k=1}^{\infty} t_k \Lambda^k\right) W^{-1}\bar{W} \exp\left(\sum_{k=1}^{\infty} \bar{t}_k \Lambda^{-k}\right)$$

and plug it into (13). After some algebra, we find that

$$W(\Lambda^a - \nu \log \Lambda)W^{-1} = \bar{W}(\Lambda^{-b} - \nu \log \Lambda - \nu \log Q)\bar{W}^{-1}.$$

This is nothing but (1). □

5. Construction of $V, \bar{V}$

We construct the dressing operators $V, \bar{V}$ by the power series expansion

$$V = \sum_{k=0}^{\infty} \nu^k V_k, \qquad \bar{V} = \sum_{k=0}^{\infty} \nu^k \bar{V}_k \qquad (14)$$

with respect to ν. As it turns out below, V_k's become difference operators of the form

$$V_0 = 1 + \sum_{n=1}^{\infty} v_{0n}\Lambda^{-n}, \qquad V_k = \sum_{n=ka}^{\infty} v_{kn}\Lambda^{-n}, \quad k \geqslant 1. \qquad (15)$$

$\bar{V}_k$'s, too, take a similar form. Since $\bar{V}$ can be obtained from the formal adjoint (or transpose) V^* of V as

$$\bar{V} = V^*|_{\nu \to -\nu, \, a \to b}, \qquad (16)$$

we present the construction of V only.

(7) splits into the following set of equations for V_k's:

$$(\Lambda^a + H)V_0 = V_0\Lambda^a, \qquad (17)$$

$$(\Lambda^a + H)V_k - \log \Lambda V_{k-1} = V_k\Lambda^k - V_{k-1} \log \Lambda, \quad k \geqslant 1. \qquad (18)$$

Since V_0 is assumed to be invertible, see (15), we can rewrite (17) and (18) as

$$[\Lambda^a, V_0] + HV_0 = 0, \qquad (19)$$

$$[\Lambda^a, V_0^{-1}V_k] = V_0^{-1}[\log \Lambda, V_{k-1}]. \quad k \geqslant 1. \qquad (20)$$

Proposition 4. *There are difference operators of the form* (15) *with polynomial coefficients* $v_{kn} = v_{kn}(s)$ *that satisfy* (19) *and* (20).

Proof. We first solve (19). This equation can be translated to the difference equations

$$v_{0,n+a}(s + a) - v_{0,n+a}(s) = -Hv_{0n}(s) \qquad (21)$$

for the coefficients v_{0n} of V_0. Starting from $v_{00} = 1$ and $v_{01} = \ldots = v_{0,a-1} = 0$, we can find the v_{0n}'s recursively with the aid of the difference identity

$$(s + a)s(s - a)\ldots(s - (k - 2)a) - s(s - a)\ldots(s - (k - 1)a)$$

$$= kas(s - a)\ldots(s - (k - 2)a) \qquad (22)$$

among the a-step factorial products as follows. Let us examine the difference equation

$$v_{0a}(s + a) - v_{0a}(s) = -Hv_{00}(s) = -s + 1/2$$

at the first stage of the recursion. The identity (22) for $k = 2$ and $k = 1$ gives

$$(s + a)s - s(s - a) = 2as, \qquad (s + a)s - s = a.$$

Hence a polynomial solution of the difference equation can be obtained in the form

$$v_{0a}(s) = -\frac{1}{2a}(s + a)s + \frac{1}{2a}s.$$

Since $v_{01} = \ldots = v_{0,a-1} = 0$, the subsequent a terms $v_{0,a+1}, \ldots, v_{0,2a-1}$ can be chosen to be equal to 0. The next non-trivial stage is the difference equation

$$v_{0,2a}(s + a) - v_{0,2a}(s) = -Hv_{0a}(s).$$

Expanding $-Hv_{0a}(s)$ into a linear combination of $s(s - a)(s - 2a)$, $s(s - a)$ and s, we can apply the identity (22) for $k = 2, 1, 0$ to find a polynomial solution of this equation. Repeating this procedure, we obtain a set of polynomials v_{0a} that satisfy (21). We now turn to (20) and solve these equations step-by-step with respect to k. Suppose that V_{k-1} has been constructed to be a difference operator of the form (15) with polynomial coefficients. (20) consists of the difference equations

$$v'_{k,n+a}(s + a) - v'_{k,n+a}(s) = f_{kn}(s)$$

for the coefficients of

$$V_0^{-1}V_k = \sum_{n=ka}^{\infty} v'_{kn}\Lambda^{-n}, \qquad V_0^{-1}[\log \Lambda, V_{k-1}] = \sum_{n=(k-1)a}^{\infty} f_{kn}\Lambda^{-n}.$$

Since f_{kn} is a polynomial in s, we can find a polynomial $v'_{k,n+a}$ that satisfies this difference equation. Thus $V_0^{-1}V_k$, hence V_k, becomes a difference operator with polynomial coefficients. $\square$

Remark 1. The intertwining relations (7) and (8) do not determine V and $\bar{V}$ uniquely, leaving the gauge freedom

$$V \to V\left(1 + \sum_{n=1}^{\infty} c_n\Lambda^{-n}\right), \qquad \bar{V} \to \left(1 + \sum_{n=1}^{\infty} \bar{c}_n\Lambda^n\right)\bar{V}$$

of multiplying difference operators with constant coefficients $c_n, \bar{c}_n$.

Remark 2. Since the V_k's take the particular form as shown in (15), V itself can be expressed as

$$V = 1 + \sum_{n=1}^{\infty}\sum_{k=0}^{\infty} \nu^k v_{kn}\Lambda^{-n}.$$

The sum over k is actually a finite sum because $v_{kn} = 0$ for $k > n/a$. Therefore the coefficients v_n of V are polynomials in both s and ν. Thus there is no divergence problem for the expansion into powers of ν. This is also the case for $\bar{V}$.

6. Non-equivariant limit

The leading terms V_0, $\bar{V}_0$ in the expansion (14) may be thought of as the non-equivariant limit

$$V_0 = \lim_{\nu \to 0} V, \qquad \bar{V}_0 = \lim_{\nu \to 0} \bar{V}$$

of $V, \bar{V}$. These operators will play a role in the non-equivariant Gromov–Witten theory of $\mathbb{CP}^1$. Similar ideas can be found in the recent papers of Chen and Guo [25] and Alexandrov [26]. Let us consider this issue briefly.

In the non-equivariant limit, the operator U in the factorization problem (11) is replaced by

$$U_0 = V_0^{-1} \, e^{\Lambda^a/a} Q^H \, e^{\Lambda^{-b}/b} \bar{V}_0^{-1}. \tag{23}$$

V_0 and $\bar{V}_0$ satisfy the intertwining relations

$$(\Lambda^a + H)V_0 = V_0 \Lambda^a, \qquad \bar{V}_0(\Lambda^{-b} + H) = \Lambda^{-b} \bar{V}_0. \tag{24}$$

(13) turns into

$$\Lambda^a U_0 = U_0 \Lambda^{-b}, \tag{25}$$

which implies that L and $\bar{L}$ satisfy the reduction condition

$$L^a = \bar{L}^{-b} \tag{26}$$

to the bigraded Toda hierarchy of type (a, b) [16].

Actually, we have the refinement

$$\Lambda^a U_0 = U_0 \Lambda^{-b} = V_0^{-1} \, e^{\Lambda^a/a} H Q^H \, e^{\Lambda^{-b}/b} \bar{V}_0^{-1} \tag{27}$$

of (25), which can be derived in the same way as in the derivation of (13). Exponentiating (27) gives

$$\exp \left(\sum_{k=1}^{\infty} T_k \Lambda^{ka} \right) U_0 = U_0 \, \exp \left(\sum_{k=1}^{\infty} T_k \Lambda^{-kb} \right)$$

$$= V_0^{-1} \, e^{\Lambda^a/a} \, \exp \left(\sum_{k=1}^{\infty} T_k H^k \right) Q^H \, e^{\Lambda^{-b}/b} \bar{V}_0^{-1}. \tag{28}$$

The new variables T_k can be identified with coupling constants of the non-equivariant Gromov–Witten theory [6, 25, 26]. In the language of fermions, (28) amount to relations of the form

$$\langle s | \exp \left(\sum_{k=1}^{\infty} T_k J_{ka} \right) g_0 | s \rangle = \langle s | g_0 \, \exp \left(\sum_{k=1}^{\infty} T_k J_{-kb} \right) | s \rangle$$

$$= \langle s | e^{J_a/a} \, \exp \left(\sum_{k=1}^{\infty} T_k P_k \right) Q^H \, e^{J_{-b}/b} | s \rangle, \tag{29}$$

where

$$g_0 = V_0^{-1} \, e^{J_a/a} Q^H \, e^{J_{-b}/b} \bar{V}_0^{-1} \tag{30}$$

and P_k's are fermion operators that correspond to H^k by the correspondence (6). Note that V_0^{-1} and $\bar{V}_0^{-1}$ disappear in the last line of (29) because $\langle s|V_0^{-1} = \langle s|$ and $\bar{V}_0^{-1}|s\rangle = |s\rangle$. Thus the coupling constants T_k can be identified with part of the time variables of the bigraded Toda hierarchy.

Let us mention that this construction can capture the extended (logarithmic) flows of the 1D/bigraded Toda hierartchy [15, 16] as well. Let $\tilde{T} = \{\tilde{T}_k\}_{k=1}^{\infty}$ be time variables of the extended flows and deform U_0 as

$$U_0(\tilde{T}) = \exp\left(\sum_{k=1}^{\infty} \tilde{T}_k \Lambda^{ka} \log \Lambda\right) U_0 \exp\left(-\sum_{k=1}^{\infty} \tilde{T}_k \Lambda^{-kb} \log \Lambda\right). \quad (31)$$

By the intertwining relation (25) of U_0, the deformed operator $U_0(\tilde{T})$ satisfies the differential equations

$$\frac{\partial U_0(\tilde{T})}{\partial \tilde{T}_k} = \Lambda^{ka}[\log \Lambda, U_0(\tilde{T})] = \Lambda^{ka}\frac{\partial U_0(\tilde{T})}{\partial s} \quad (32)$$

hence turns out to be a genuine difference operator (i.e. does not contain $\log \Lambda$). Thus the factorization problem (11) persists to be meaningful. The associated dressing operators $W, \bar{W}$ satisfy the Sato equations

$$\frac{\partial W}{\partial \tilde{T}_k} = C_k W - W\Lambda^{ka} \log \Lambda, \qquad \frac{\partial \bar{W}}{\partial \tilde{T}_k} = C_k \bar{W} - \bar{W}\Lambda^{-kb} \log \Lambda$$

of the extended flows [15, 16]. The C_k's are defined as

$$C_k = \mathcal{L}^k \log \Lambda - \left(W\Lambda^{ka}W^{-1}\frac{\partial W}{\partial s}W^{-1}\right)_{\geqslant 0} - \left(\bar{W}\Lambda^{-kb}\bar{W}^{-1}\frac{\partial \bar{W}}{\partial s}\bar{W}^{-1}\right)_{<0},$$

where $\mathcal{L}$ denotes the difference operator of finite order defined by both sides of the reduction condition (26).

7. Conclusion

We have reformulated Okounkov and Pandharipande's dressing operators as difference operators that satisfy the intertwining relations (7) and (8). This formulation fits well into the Lax formalism of the 2D Toda hierarchy. These dressing operators are building blocks of the factorization problem (11) that captures Okounkov and Pandharipande's tau function in the Lax formalism. It is a rather immediate consequence of the factorization problem that the Lax operators satisfy the reduction condition (1) to the equivariant Toda hierarchy. We have thus found a new explanation to the question of why the equivariant Toda hierarchy emerges in the Gromov–Witten theory of $\mathbb{CP}^1$.

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Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

ORCID iDs

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References

[1] Eguchi T and Yang S-K 1994 The topological CP^1 model and the large-n matrix integral *Mod. Phys. Lett.* A **9** 2893–902
[2] Eguchi T, Hori K and Yang S-K 1995 Topological σ models and large-N matrix integral *Int. J. Mod. Phys.* A **10** 4203–24
[3] Pandharipande R 2000 The Toda equations and the Gromov–Witten theory of the Riemann sphere *Lett. Math. Phys.* **53** 59–74
[4] Okounkov A 2000 Toda equations for Hurwitz numbers *Math. Res. Lett.* **7** 447–53
[5] Getzler E 2001 The Toda conjecture *Symplectic Geometry and Mirror Symmetry* ed K Fukaya *et al* (Singapore: World Scientific) pp 51–79
[6] Okounkov A and Pandharipande R 2006 Gromov–Witten theory, Hurwitz theory, and completed cycles *Ann. Math.* **163** 517–60
[7] Okounkov A and Pandharipande R 2006 The equivariant Gromov–Witten theory of $\mathbf{P}^1$ *Ann. Math.* **163** 561–605
[8] Zhang Y 2002 On the CP^1 topological sigma model and the Toda lattice hierarchy *J. Geom. Phys.* **40** 215–32
[9] Dubrovin B and Zhang Y 2004 Virasoro symmetries of the extended Toda hierarchy *Commun. Math. Phys.* **250** 161–93
[10] Milanov T 2006 Gromov–Witten theory of $\mathbb{CP}^1$ and integrable hierarchies (arXiv:math-ph/0605001)
[11] Milanov T and Tseng H-H 2008 The spaces of Laurent polynomials, Gromov–Witten theory of $\mathbb{P}^1$-orbifolds, and integrable hierarchies *J. Reine Angew. Math.* **622** 189–235
[12] Carlet G and van de Leur J 2013 Hirota equations for the extended bigraded Toda hierarchy and the total descendent potential of $\mathbb{CP}^1$ orbifolds *J. Phys. A: Math. Theor.* **46** 405205
[13] Milanov T, Shen Y and Tseng H-H 2016 Gromov–Witten theory of fano orbifold curves, gamma integral structures and ADE-Toda hierarchies *Geom. Topol.* **20** 2135–218
[14] Cheng J and Milanov T 2019 Gromov–Witten invariants and the extended D-Toda hierarchy (arXiv:1909.12735)
[15] Carlet G, Dubrovin B and Zhang Y 2004 The extended Toda hierarchy *Moscow Math. J.* **4** 313–32
[16] Carlet G 2006 The extended bigraded Toda hierarchy *J. Phys. A: Math. Gen.* **39** 9411–35
[17] Cheng J and Milaov T 2019 The extended D-Toda hierarchy (arXiv:1909.12735)
[18] Milanov T E 2008 The equivariant Gromov–Witten theory of $\mathbb{CP}^1$ and integrable hierarchies *Int. Math. Res. Not.* **2008** rnn073
[19] Milanov T E and Tseng H-H 2011 Equivariant orbifold structures on the projective line and integrable hierarchies *Adv. Math.* **226** 641–72
[20] Johnson P 2014 Equivariant Gromov–Witten theory of one dimensional stacks *Commun. Math. Phys.* **327** 333–86
[21] Getzler E 2004 The equivariant Toda lattice (arXiv:math/0207025)
[22] Givental A 2001 Gromov–Witten invariants and quantization of quadratic Hamiltonians *Moscow Math. J.* **1** 551–68
[23] Takasaki K 2018 Toda hierarchies and their applications *J. Phys. A: Math. Theor.* **51** 203001
[24] Takasaki K and Takebe T 1995 Integrable hierarchies and dispersionless limit *Rev. Math. Phys.* **7** 743–808

[25] Chen C-Y and Guo S 2019 Quantum curve and bilinear fermionic form for the orbifold Gromov–Witten theory of $\mathbb{P}[r]$ (arXiv:1912.00558)

[26] Alexandrov A 2020 Matrix model for the stationary sector of Gromov–Witten theory of $\mathbf{P}^1$ (arXiv:2001.08556)