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COHERENT VS INCOHERENT EFFECTS AND DEBYE LENGTH

G. Franchetti, GSI, Darmstadt, Germany

Abstract

For some class of studies, the space charge is treated as frozen, allowing to capture the dynamics of incoherent phenomena. We explore the possibility that a beam may exhibit non-resonant coherent behavior by developing and studying a one-dimensional model.

INTRODUCTION

The issue of whether space charge effects in a ring can be modeled by a frozen space charge or not becomes, in term of the dynamics, the issue of whether the behavior of the beam in a high-intensity beam is coherent or incoherent. In neutral plasmas, this property is incorporated into the Debye length of λ_D. If a test particle is placed into a neutral plasma having a temperature T and equal positive ion and electron densities n, the excess electric potential set up by an extra charge is effectively screened off in a distance λ_D by charge redistribution in the plasma. This effect is called "Debye shielding" and $\lambda_D = \tilde{v}/\omega$ where $\tilde{v}$ is the thermal velocity of the particles and $\omega = \left[q^2 n/(m\epsilon_0)\right]^{1/2}$ is the plasma frequency. For a particle beam of size a stored in an accelerator Ref. [1] says that if $\lambda_D \gg a$ the screening will be ineffective and single particle behavior will dominate, while if $\lambda_D \ll a$ the collective effects due to the beam self-fields will play an important role. However, a particle beam in an accelerator is formed by particles with the same charge state. Therefore how the Debye mechanism comes to play it is not so evident. To clarify what happens, we construct a simplified model and explore the dynamics.

A ONE DIMENSIONAL MODEL

In order to investigate the role of the Debye length in a particle beam stored in an accelerator, we construct a simple one-dimensional model. We consider a region of space with a focusing electric field $E_z = -\mathcal{K}z$ along the z axis that does not depends on the transverse x and y coordinates. In order to simplify the dynamics, we assume that the particles of charge q and mass m are frozen in planes. Therefore, instead of discussing the dynamics of micro-particles we study the motion of micro-planes. One micro-plane has position z, velocity $\dot{z}$, is normal to the z axis, and has uniform particle surface density n_p.

The force on a charged particle is $F_z = qE_z$, being E_z the composition of the electric fields along z created by all micro-planes and the focusing field. The electric field $E_z(z, z')$ created at z by the micro-plane located at z' is readily obtained from Gauss law as $E_z(z, z') = \text{sign}(z - z')qn_p/(2\epsilon_0)$, with ϵ_0 the vacuum permettivity. As all particles in a micro-plane are subject to the same force, the equation of motion of the micro-plane at z is

$$m\frac{d^2z}{dt^2} = -q\mathcal{K}z + q^2\frac{n_p}{2\epsilon_0}\text{sign}(z - z'). \quad (1)$$

We note two features of this model: 1) the motion of micro-planes is not subject to "collision". In fact, when $z = z'$ there is a discontinuity in the electric field, but not a divergence. Therefore for a large number of micro-planes, this effect may be made arbitrarily small. 2) the acceleration of one plane due to the Coulomb field exerted by another micro-plane does not vanish with the distance. This effect is understood from the infinite extension of the micro-planes. This model makes stronger the coherent response of this system as any plane feels equally the forces of all the micro-planes present into the system.

A continuous beam is formed by many micro-planes, say N_0, with density distribution function $\rho_N(z) = \Delta N(z)/\Delta z$, where $\Delta N(z)$ is the number of micro-planes in $[z, z + \Delta z]$. From Eq. (1) it is straightforward that the Coulomb force on a micro-plane located at z is proportional to $-N_+(z) + N_-(z)$, where $N_-(z) = \int_{-\infty}^{z} \rho_N(z')dz'$ is the number of micro-planes with $z' < z$. *Mutatis mutandi* for $N_+(z)$. As $N_0 = N_+(z) + N_-(z)$ the equation of motion of one plane reads

$$\frac{d^2z}{dt^2} = -k_{z0}z + q^2\frac{n_p}{2m\epsilon_0}[2N_-(z) - N_0]. \quad (2)$$

where in analogy to the beam dynamics in accelerators we define $k_{z0} = (q/m)\mathcal{K}$.

This equation allows computing the evolution of the distribution of micro-planes when their phase space distribution is known. The dynamical coordinates of one micro-plane are $(z, \dot{z})$, and a distribution of micro-planes is identified by a distribution function $f(z, \dot{z})$. For brevity of language we use the term "particle" instead using "micro-plane". As for 2D beams, a special role is played by a stationary particle distribution. This special class of particle distributions has the property that $f(z, \dot{z})$ does not change in time. This happens naturally if all forces acting on one particle are linear in z and if the particle distribution is a function of the invariant $\epsilon_z = \gamma_z z^2 + \beta_z \dot{z}^2$, being β_z, γ_z the optical functions of the system (in the time domain). This means that the particle distribution is $f\left(\frac{\epsilon_z}{E_z}\right)$, with E_z is the beam phase space emittance. The linearity of the forces requires $N_-(z) \propto z$, which is possible only if

$$\int f\left(\frac{\epsilon_z}{E_z}\right)d\dot{z} = \rho_N(z) = \text{constant}. \quad (3)$$

for any z inside the distribution. The function $f()$ satisfying Eq. (3) can be constructed with a "slice by slice" procedure with the result shown in Fig. 1a. The markers show the numerical findings and the red curve is a fit. This particle distribution is also modeled with an acceptable approximation, by transforming the bi-normal distribution (ξ, ϕ) according

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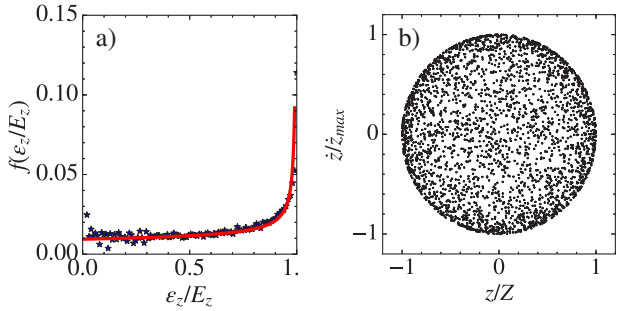

Figure 1: Part a: the function $f(\epsilon_z/E_z)$ is obtained by requiring that the projection is uniform. Part b: particle distribution in the normalized phase space.

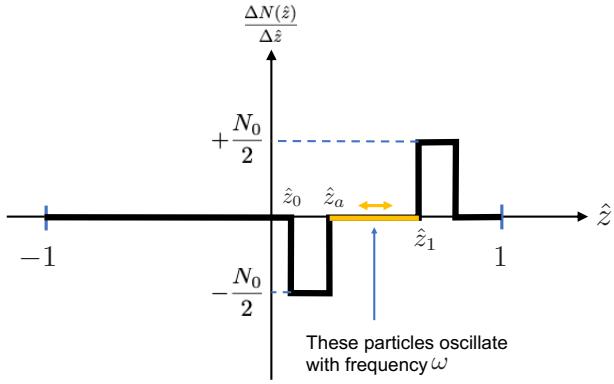

Figure 2: Schematic of the particle density perturbation in the first instants of motion.

to

$$\frac{z}{Z} = \xi F(\xi,\phi) \quad \frac{\dot{z}}{\dot{z}_{max}} = \phi F(\xi,\phi) \tag{4}$$

with $F(\xi,\phi) = 12/\{4^{1/3}[8+(\xi^2+\phi^2)^{3/2}]\}$ and $Z, \dot{z}_{max}$ the maximum extensions of the distribution in phase space. In Fig. 1b we show this particle distribution.

A matched stationary particle distribution located in the interval $[-Z, Z]$ has cumulative number of particles $N_-(z) = N_0(z+Z)/(2Z)$. Hence the equation of motion reads $\frac{d^2z}{dt^2} = -k_{z0}z + \omega^2 z$, with $\omega = (\frac{q^2 n_0}{m\epsilon_0})^{1/2}$ the Debye frequency, and $n_0 = n_p N_0/(2Z)$ the particle density of the stationary distribution. It is now convenient scaling the time to the phase created by the focusing field, namely using the variable $\theta = \sqrt{k_{0z}}t$, and also scaling the particle coordinate with the distribution size $\hat{z} = z/Z$. We find

$$\frac{d^2\hat{z}}{d\theta^2} = -\hat{z} + \frac{\omega^2}{k_{z0}}\hat{z}. \tag{5}$$

The condition of stationary particle distribution matching is obtained using the optical functions from the space charge depressed focusing strength $k_z = k_{z0} - \omega^2$, namely $\beta_z = 1/\sqrt{k_z}$ and $\gamma_z = 1/\beta_z$. Any particle in this system will satisfy the relation $\beta_z \dot{z}^2 + \gamma_z z^2 =$ constant. Therefore for a distribution with size Z we find that the consistent size in the velocity is $\dot{z}_{max} = \sqrt{k_z}Z$, hence using Eqs. (4) we generate the stationary distribution. In Eq. (5) we recognize the incoherent tune depression $Q_{z,inch}/Q_{z0} = \sqrt{1-\omega^2/k_{z0}}$ and for convenience we define the relative Debye "tune" $Q_{z,D}/Q_{z0} = \omega/\sqrt{k_{z0}}$. These two quantities satisfy relation $(Q_{z,D}/Q_{z0})^2 + (Q_{z,inch}/Q_{z0})^2 = 1$.

THE SPACE CHARGE LIMIT

In this model the space charge limit is reached when $Q_{z,inch} = 0$, namely when the Coulomb forces compensate the force of the lattice. In this case, the stationary particle distribution is just uniform in $[-1, 1]$ with each particle having zero velocity. In order to evaluate/investigate the effect of a possible Debye mechanism, we consider a particle distribution at the space charge limit and apply a perturbation to the velocity of the particles in the region $[\hat{z}_0, \hat{z}_1]$. The particle density $\rho_N(\hat{z})$ is now perturbed, hence the cumulative number of particles is $N_-(\hat{z}) = N_{0-}(\hat{z}) + \delta N_-(\hat{z})$, with $N_{0-}(\hat{z})$ corresponding to the stationary distribution, which create $\omega^2 = k_{z0}$. In this notation $\delta N_-(\hat{z})$ can be positive or negative, but as the number of particles is preserved, it is always $\delta N_-(\infty) = 0$. Therefore Eq. (2) reads

$$\frac{d^2\hat{z}}{d\theta^2} = \frac{\omega^2}{k_{z0}}\frac{2}{N_0}\delta N_-(\hat{z}). \tag{6}$$

We next model $\delta N_-(\hat{z})$ in the first part of the motion. We add the velocity $\Delta v > 0$ to all particles in the region $[\hat{z}_0, \hat{z}_1]$ and let the system evolve. Let's call $\hat{z}_a$ the particle initially located at $\hat{z}_0$, which is subject to the perturbation. This beam particle will move with speed $v = \Delta v$ and will leave an empty region behind (Fig. 2). Hence the cumulative perturbation in the first instant of motion reads

$$\delta N_-(\hat{z}_a) = -\frac{N_0}{2}(\hat{z}_a - \hat{z}_0), \tag{7}$$

and the equation of motion of the particle $\hat{z}_a$ is

$$\frac{d^2\hat{z}_a}{d\theta^2} = -\frac{\omega^2}{k_{z0}}(\hat{z}_a - \hat{z}_0). \tag{8}$$

A generic particle with coordinate $\hat{z}_r$ in the yellow region of Fig. 2 is subjected to the same equation of motion as $\delta N_-(\hat{z}_r) = \delta N_-(\hat{z}_a)$. The initial conditions of this particle are $\hat{z}_r = \hat{z}_{r,0}$ and $\frac{d\hat{z}_r}{d\theta} = \frac{d\hat{z}_r}{dt}\frac{dt}{d\theta} = \Delta v/(\sqrt{k_{z0}}Z)$. Therefore particles in the center of the perturbed region will oscillate coherently according to

$$\hat{z}_r = \hat{z}_{r,0} + \frac{\Delta v}{\omega Z}\sin\left(\frac{\omega}{\sqrt{k_{z0}}}\theta\right). \tag{9}$$

This formula shows that the oscillation of $\hat{z}_r$ has amplitude of $\hat{L}_D = \Delta v/(\omega Z)$. It is evident that if $\hat{L}_D > \hat{z}_1 - \hat{z}_0$ any particle in the yellow region cannot follow Eq. (9) because $\delta N_-(\hat{z})$ does not follow Eq. (7) already after a phase advance θ given by $\frac{\omega}{\sqrt{k_{z0}}}\theta = \pi/2$. We show this effect in Fig. 3 where we plot the particle distribution after 5 Debye oscillations. In the part a) the Debye length is $\hat{L}_D = (\hat{z}_1 - \hat{z}_0)/20$ while in the part b) we set $\hat{L}_D = (\hat{z}_1 - \hat{z}_0)/2$. The comparison of the two pictures shows that only if $\hat{L}_D$ is much smaller than the size of the perturbed region the perturbation can survive locally. In both pictures, the initial distribution is colored to

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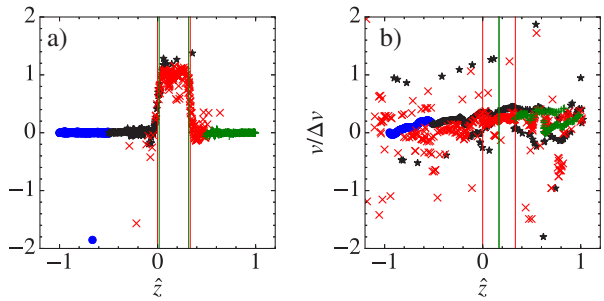

Figure 3: Phase space after 5 Debye oscillations. On the part a) is the case with $\hat{L}_D/(\hat{z}_1 - \hat{z}_0) = 1/20$, and on the part b) the case with $\hat{L}_D/(\hat{z}_1 - \hat{z}_0) = 1/2$.

highlight the dynamics. We see that in Fig. 3a the particles do not excessively diffuse, while in Fig. 3b the red colored particles spread over all the initial distribution length. $\hat{L}_D$ plays the role of the Debye length.

ABOVE THE SPACE CHARGE LIMIT

The scenario discussed in the previous section regards the case in which the charge density of micro-planes is extreme. For a less dense particle beam we do not approach the condition $Q_{z,inch} = 0$ and Eq. (2) acquires the form

$$\frac{d^2\hat{z}}{d\theta^2} = -\frac{k_z}{k_{z0}}\hat{z} + \frac{\omega^2}{k_{z0}}\frac{2}{N_0}\delta N_-(\hat{z}). \qquad (10)$$

This equation shows a dynamics governed by the coexistence of two competing effects: 1) a Debye dynamics characterized by the term with ω^2/k_{z0} which involve the perturbation $\delta N_-(\hat{z})$; 2) a depressed lattice k_z/k_{z0} dynamics, which acts incoherently over all particles.

The form of the perturbation $\delta N_-(\hat{z})$ in the first part of motion is now not equal to Eq. (7), which is an upper bound, namely $|\delta(\hat{z}_a)| < (N_0/2)(\hat{z}_a - \hat{z}_0)$. From Eq. (10) we see that the dominance of the incoherent regime over the coherent one surely happens for $\omega^2 < k_z$, which in terms of optical functions becomes $\omega^2 < \gamma_z/\beta_z$. Multiplying and dividing by the beam rms phase space emittance this condition reads $\tilde{z} < \tilde{v}_z/\omega$, where $\tilde{z}$ is the rms size of the beam, and $\tilde{v}$ is effectively the rms thermal component of the velocity, hence we recognize here the Debye length $\lambda_D = \tilde{v}_z/\omega$. Defining $\Delta Q_z = Q_{z,inch} - Q_{z0}$ as the incoherent space charge tune-shift it is straightforward to show that

$$\frac{\lambda_D}{\tilde{z}} = \frac{1 + \Delta Q_z/Q_{z0}}{[-2\Delta Q_z/Q_{z0} - (\Delta Q_z/Q_{z0})^2]^{1/2}}.$$

At the space charge limit $\Delta Q_z/Q_{z0} = -1$ the Debye length is $\lambda_D = 0$, instead at $\Delta Q_z/Q_{z0} = -0.29$ we find $\lambda_D = \tilde{z}$.

To avoid the complication of modeling the initial perturbation we just do not apply any. We instead create several matched beams each characterized by a specific relative Debye length of $\lambda_D/\tilde{z}$ and study the oscillatory behavior of an ad-hoc test particle. The initial conditions of this particle are $\hat{z} = \hat{z}_{ini} = 0.83, \dot{z} = 0$ and its evolution is computed by solving numerically Eq. (10) with 5000 integration steps per Debye length for a time interval corresponding to 10 oscillations. In Fig. 4 we show the results. In the part a) the red markers show Q_z/Q_{z0} as obtained from the test particle motion. For comparison we also plot $Q_{z,inch}/Q_{z0}$ (solid line), and relative Debye "tune" $Q_{z,D}/Q_{z0}$ (dotted line) as obtained from the theory. In Fig. 4b we show with red markers the average particle position (taken always positive) and with black markers the scaled standard deviation of $z/\tilde{z}$ as function of $\lambda_D/\tilde{z}$. Part a) shows that for $\lambda_D/\tilde{z} \gtrsim 0.1$ the tune

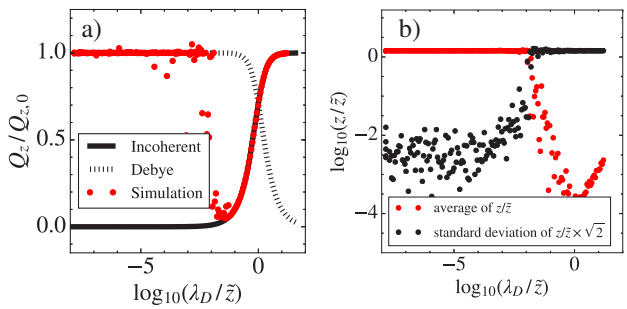

Figure 4: Oscillatory properties of a test particle as function of $\lambda_D/\tilde{z}$.

of the test particle is locked to the incoherent tune as one expects. In the interval $0.01 \lesssim \lambda_D/\tilde{z} \lesssim 0.1$ the test particle tune makes a transition to the pure Debye tune, which surprisingly becomes equal to the un-depressed tune Q_{z0}. Part b) shows that the average center of oscillation remains close to the origin of the system $z = 0$ for large $\lambda_D/\tilde{z}$, but progressively approach $z = z_{ini}$ for $\lambda_D/\tilde{z} \to 0.01$. In this interval, the amplitude of oscillation (black markers) remains locked to the initial amplitude with respect to $z = 0$. In the interval $10^{-4} \lesssim \lambda_D/\tilde{z} \lesssim 0.01$ the amplitude of oscillation shrinks to $z/\tilde{z} \simeq 10^{-2}$. For $\lambda_D/\tilde{z} \lesssim 0.01$ the average position of the center of oscillation grows and becomes locked to the initial particle position.

SUMMARY

With this study, we find evidence that the relative Debye length effectively is an indicator of the incoherent/coherent character of the particle dynamics in this model. For small $\lambda_D/\tilde{z}$ each particle oscillates around its initial position with the Debye frequency although this system is not a neutral plasma.

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REFERENCES

[1] M. Reiser, *Theory and Design of Charged Particle Beams*, Wiley-VCH, 2004.