Generalized equations and their solutions in the (S,0)+(0,S) representations of the Lorentz group [1]

Valeriy V. Dvoeglazov
UAF, Universidad Autónoma de Zacatecas
Apartado Postal 636, Suc. 3, Zacatecas 98061 Zac., México

E-mail: valeri@fisica.uaz.edu.mx

Abstract. I present explicit examples of generalizations in relativistic quantum mechanics. First of all, I discuss the generalized spin-1/2 equations for neutrinos. They have been obtained by means of the Gersten-Sakurai method for derivations of arbitrary-spin relativistic equations. Possible physical consequences are discussed. Next, it is easy to check that both Dirac algebraic equation $Det(\hat{p} - m) = 0$ and $Det(\hat{p} + m) = 0$ for $u-$ and $v-$ 4-spinors have solutions with $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$. The same is true for higher-spin equations. Meanwhile, every book considers the equality $p_0 = E_p$ for both $u-$ and $v-$ spinors of the $(1/2, 0) \oplus (0, 1/2)$ representation, thus applying the Dirac-Feynman-Stueckelberg procedure for elimination of the negative-energy solutions. The recent Ziino works (and, independently, the articles of several others) show that the Fock space can be doubled. We re-consider this possibility on the quantum field level for both $S = 1/2$ and higher spin particles. The third example is: we postulate the non-commutativity of 4-momenta, and we derive the mass splitting in the Dirac equation. The applications are discussed.

1. Generalized Neutrino Equations.

A. Gersten [1] proposed a method for derivations of massless equations of arbitrary-spin particles. In fact, his method is related to the van der Waerden-Sakurai [2] procedure for the derivation of the massive Dirac equation. In the present talk I apply this procedure to the spin-1/2 fields. As a result one obtains equations which generalize the well-known Weyl equations. However, these equations are known for a long time [3]. Raspini [4, 5] analized them in detail.

Let us look at the equation (4) of the Gersten paper [1a] for the two-component spinor field function:

$$(E^2 - c^2\vec{\mathbf{p}}^{\,2})I^{(2)}\psi = \left[EI^{(2)} - c\vec{\mathbf{p}}\cdot\vec{\boldsymbol{\sigma}}\right]\left[EI^{(2)} + c\vec{\mathbf{p}}\cdot\vec{\boldsymbol{\sigma}}\right]\psi = 0\,. \tag{1}$$

Actually, this equation is the massless limit of the Klein-Gordon equation for spin-1/2 in the Sakurai book [2]. In the massive case case one should substitute m^2c^4 into the right-hand side of Eq. (1). However, instead of equation (3.25) of [2] one can define the two-component 'right' field function

$$\phi_R = \frac{1}{m_1 c}(i\hbar\frac{\partial}{\partial x_0} - i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\psi, \quad \phi_L = \psi \tag{2}$$

with the different mass parameter m_1. In such a way we come to the system of the first-order

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differential equations:

$$(i\hbar\frac{\partial}{\partial x_0}+i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\phi_R=\frac{m_2^2c}{m_1}\phi_L\,, \tag{3}$$

$$(i\hbar\frac{\partial}{\partial x_0}-i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\phi_L=m_1c\phi_R\,. \tag{4}$$

It can be re-written in the 4-component form:

$$\begin{pmatrix} i\hbar(\partial/\partial x_0) & i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\\ -i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla} & -i\hbar(\partial/\partial x_0)\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}= \tag{5}$$
$$=\frac{c}{2}\begin{pmatrix}(m_2^2/m_1+m_1) & (-m_2^2/m_1+m_1)\\ (-m_2^2/m_1+m_1) & (m_2^2/m_1+m_1)\end{pmatrix}\begin{pmatrix}\psi_A\\ \psi_B\end{pmatrix}$$

for the function $\Psi=column(\psi_A\quad\psi_B)=column(\phi_R+\phi_L\quad\phi_R-\phi_L)$. The generalized equation (5) can be written in the covariant form.

$$\left[i\gamma^\mu\partial_\mu-\frac{m_2^2c}{m_1\hbar}\frac{(1-\gamma^5)}{2}-\frac{m_1c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi=0\,. \tag{6}$$

The standard representation of γ^μ matrices has been used here. You may compare this framework with the spin-1 case [7].

If $m_1=m_2$ we can recover the standard Dirac equation. As noted in [3b] this procedure can be viewed as the simple change of the representation of γ^μ matrices. However, this is true unless $m_2\neq 0$ only. Otherwise, the entries in the transformation matrix become to be singular. However, one can either repeat a similar procedure (the modified Sakurai procedure) starting from the *massless* equation (4) of [1a] or put $m_2=0$ in eq. (6). It is necesary to stress that the term *'massless'* is used in the sense that $p_\mu p^\mu=0$. The *massless equation* is:

$$\left[i\gamma^\mu\partial_\mu-\frac{m_1c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi=0\,. \tag{7}$$

Then, we may have different physical consequences following from (7) comparing with those which follow from the Weyl equation. The mathematical reason of such a possibility of different massless limits is that the corresponding change of representation of γ^μ matrices involves mass parameters m_1 and m_2 themselves.

It is interesting to note that we can also repeat this procedure for other definition (or for even more general definitions):

$$\phi_L=\frac{1}{m_3c}(i\hbar\frac{\partial}{\partial x_0}+i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})\psi,\quad \phi_R=\psi\,. \tag{8}$$

This is due to the fact that the parity properties of the two-component spinor are undefined in the two-component equation. The resulting equation is

$$\left[i\gamma^\mu\partial_\mu-\frac{m_4^2c}{m_3\hbar}\frac{(1+\gamma^5)}{2}-\frac{m_3c}{\hbar}\frac{(1-\gamma^5)}{2}\right]\tilde\Psi=0\,, \tag{9}$$

which gives us yet another equation in the massless limit ($m_4\to 0$):

$$\left[i\gamma^\mu\partial_\mu-\frac{m_3c}{\hbar}\frac{(1-\gamma^5)}{2}\right]\tilde\Psi=0\,, \tag{10}$$

differing in the sign at the γ_5 term.

The above procedure can be generalized to *any* Lorentz group representations, *i. e.*, to any spins. The physical content of the generalized $S = 1/2$ *massless* equations is not the same as that of the Weyl equation. The excellent discussion can be found in [3]. The theory does *not* have chiral invariance. Those authors call the additional parameters as the measures of the degree of chirality. Apart of this, Tokuoka introduced the concept of the gauge transformations for the 4-spinor fields. He also found some strange properties of the anti-commutation relations (see §3 in [3a]). And finally, the equations (7,10) describe *four* states, two of which answer for the positive energy $p_0 = |\mathbf{p}|$, and two others answer for the negative energy $p_0 = -|\mathbf{p}|$.

I just want to add the following remarks to the discussion. The operator of the *chiral-helicity* $\hat{\eta} = (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})$ (in the spinorial representation) used in [3b] does *not* commute, *e.g.*, with the Hamiltonian of the equation (7):

$$[\mathcal{H}, \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}]_- = 2\frac{m_1 c}{\hbar}\frac{1-\gamma^5}{2}(\boldsymbol{\gamma}\cdot\hat{\mathbf{p}})\,. \tag{11}$$

For the eigenstates of the *chiral-helicity* the system of corresponding equations can be read ($\eta =\uparrow,\downarrow$)

$$i\gamma^\mu\partial_\mu\Psi_\eta - \frac{m_1 c}{\hbar}\frac{1+\gamma^5}{2}\Psi_{-\eta} = 0\,. \tag{12}$$

The conjugated eigenstates of the Hamiltonian $|\Psi_\uparrow + \Psi_\downarrow >$ and $|\Psi_\uparrow - \Psi_\downarrow >$ are connected, in fact, by γ^5 transformation $\Psi \to \gamma^5\Psi \sim (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\Psi$. However, the γ^5 transformation is related to the PT ($t \to -t$ only) transformation, which, in its turn, can be interpreted as $E \to -E$, if one accepts the Stueckelberg idea about antiparticles. We associate $|\Psi_\uparrow + \Psi_\downarrow >$ with the positive-energy eigenvalue of the Hamiltonian $p_0 = |\mathbf{p}|$ and $|\Psi_\uparrow - \Psi_\downarrow >$, with the negative-energy eigenvalue of the Hamiltonian ($p_0 = -|\mathbf{p}|$). Thus, the free chiral-helicity massless eigenstates may oscillate one to another with the frequency $\omega = E/\hbar$ (as the massive chiral-helicity eigenstates, see [6a] for details). Moreover, a special kind of interaction which is not symmetric with respect to the chiral-helicity states (for instance, if the left chiral-helicity eigenstates interact with the matter only) may induce changes in the oscillation frequency, like in the Wolfenstein (MSW) formalism.

The conclusion may be similar to that which was achieved before: the dynamical properties of the massless particles (*e. g.*, neutrinos and photons) may differ from those defined by the well-known Weyl and Maxwell equations [6, 8, 9].

2. Negative Energies in the Dirac Equation.

The Dirac equation is:

$$[i\gamma^\mu\partial_\mu - m]\Psi(x) = 0\,. \tag{13}$$

Usually, everybody uses the following definition of the field operator [10] in the pseudo-Euclidean metrics:

$$\Psi(x) = \frac{1}{(2\pi)^3}\sum_h \int \frac{d^3\mathbf{p}}{2E_p}[u_h(\mathbf{p})a_h(\mathbf{p})e^{-ip\cdot x} + v_h(\mathbf{p})b_h^\dagger(\mathbf{p})]e^{+ip\cdot x}]\,, \tag{14}$$

as given *ab initio*. After actions of the Dirac operator at $\exp(\mp ip_\mu x^\mu)$ the 4-spinors ($u-$ and $v-$) satisfy the momentum-space equations: $(\hat{p}-m)u_h(p) = 0$ and $(\hat{p}+m)v_h(p) = 0$, respectively; the h is the polarization index. However, it is easy to prove from the characteristic equations $Det(\hat{p}\mp m) = (p_0^2 - \mathbf{p}^2 - m^2)^2 = 0$ that the solutions should satisfy the energy-momentum relation $p_0 = \pm E_p = \pm\sqrt{\mathbf{p}^2 + m^2}$ in both cases.

Let me remind the general scheme of construction of the field operator, which has been presented in [11]. In the case of the $(1/2,0)\oplus(0,1/2)$ representation we have:

$$\begin{aligned}
&\Psi(x) = \frac{1}{(2\pi)^3}\int d^4p\,\delta(p^2-m^2)e^{-ip\cdot x}\Psi(p) = \\
= \;& \frac{1}{(2\pi)^3}\sum_h \int d^4p\,\delta(p_0^2-E_p^2)e^{-ip\cdot x}u_h(p_0,\mathbf{p})a_h(p_0,\mathbf{p}) = \\
= \;& \frac{1}{(2\pi)^3}\int \frac{d^4p}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)][\theta(p_0)+\theta(-p_0)]e^{-ip\cdot x} \\
\times \;& \sum_h u_h(p)a_h(p) = \frac{1}{(2\pi)^3}\sum_h\int\frac{d^4p}{2E_p}[\delta(p_0-E_p)+\delta(p_0+E_p)]\left[\theta(p_0)u_h(p)a_h(p)e^{-ip\cdot x}+\right. \\
+ \;& \left.\theta(p_0)u_h(-p)a_h(-p)e^{+ip\cdot x}\right] = \frac{1}{(2\pi)^3}\sum_h\int\frac{d^3\mathbf{p}}{2E_p}\theta(p_0)\left[u_h(p)a_h(p)|_{p_0=E_p}e^{-i(E_pt-\mathbf{p}\cdot\mathbf{x})}+\right. \\
+ \;& \left.u_h(-p)a_h(-p)|_{p_0=E_p}e^{+i(E_pt-\mathbf{p}\cdot\mathbf{x})}\right]
\end{aligned} \tag{15}$$

During the calculations above we had to represent $1=\theta(p_0)+\theta(-p_0)$ in order to get positive- and negative-frequency parts. Moreover, during these calculations we did not yet assumed, which equation this field operator (namely, the $u-$ spinor) does satisfy, with negative- or positive- mass?

In general we should transform $u_h(-p)$ to the $v(p)$. The procedure is the following one [12]. In the Dirac case we should assume the following relation in the field operator:

$$\sum_h v_h(p)b_h^\dagger(p) = \sum_h u_h(-p)a_h(-p)\,. \tag{16}$$

We know that [13]

$$\begin{aligned}
\bar u_{(\mu)}(p)u_{(\lambda)}(p) &= +m\delta_{\mu\lambda}\,, & (17)\\
\bar u_{(\mu)}(p)u_{(\lambda)}(-p) &= 0\,, & (18)\\
\bar v_{(\mu)}(p)v_{(\lambda)}(p) &= -m\delta_{\mu\lambda}\,, & (19)\\
\bar v_{(\mu)}(p)u_{(\lambda)}(p) &= 0\,, & (20)
\end{aligned}$$

but we need $\Lambda_{(\mu)(\lambda)}(p)=\bar v_{(\mu)}(p)u_{(\lambda)}(-p)$. By direct calculations, we find

$$-mb^\dagger_{(\mu)}(p) = \sum_\lambda \Lambda_{(\mu)(\lambda)}(p)a_{(\lambda)}(-p)\,. \tag{21}$$

Hence, $\Lambda_{(\mu)(\lambda)} = -im(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}$, $\mathbf{n}=\mathbf{p}/|\mathbf{p}|$, and

$$b^\dagger_{(\mu)}(p) = i\sum_\lambda(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}a_{(\lambda)}(-p)\,. \tag{22}$$

Multiplying (16) by $\bar u_{(\mu)}(-p)$ we obtain

$$a_{(\mu)}(-p) = -i\sum_\lambda(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}b^\dagger_{(\lambda)}(p)\,. \tag{23}$$

The equations are self-consistent. In the $(1,0)\oplus(0,1)$ representation the similar procedure leads to somewhat different situation:

$$a_{(\mu)}(p) = [1-2(\mathbf{S}\cdot\mathbf{n})^2]_{(\mu)(\lambda)}a_{(\lambda)}(-p)\,. \tag{24}$$

This signifies that in order to construct the Sankaranarayanan-Good field operator, it satisfies $[\gamma_{\mu\nu}\partial_\mu\partial_\nu - \frac{(i\partial/\partial t)}{E}m^2]\Psi(x) = 0$, we need additional postulates. For instance, one can try to construct the left- and the right-hand side of the field operator separately each other [15].

First of all to mention, we have, in fact, $u_h(E_p, \mathbf{p})$ and $u_h(-E_p, \mathbf{p})$, and $v_h(E_p, \mathbf{p})$ and $v_h(-E_p, \mathbf{p})$, originally, which may satisfy the equations:

$$\left[E_p(\pm\gamma^0) - \boldsymbol{\gamma}\cdot\mathbf{p} - m\right] u_h(\pm E_p, \mathbf{p}) = 0\,. \tag{25}$$

Due to the properties $U^\dagger\gamma^0 U = -\gamma^0$, $U^\dagger\gamma^i U = +\gamma^i$ with the unitary matrix $U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \gamma^0\gamma^5$ in the Weyl basis, we have

$$\left[E_p\gamma^0 - \boldsymbol{\gamma}\cdot\mathbf{p} - m\right] U^\dagger u_h(-E_p, \mathbf{p}) = 0\,. \tag{26}$$

The properties of the $U-$ matrix are opposite to those of $P^\dagger\gamma^0 P = +\gamma^0$, $P^\dagger\gamma^i P = -\gamma^i$ with the usual $P = \gamma^0$, thus giving $\left[-E_p\gamma^0 + \gamma\cdot\mathbf{p} - m\right] P u_h(-E_p, \mathbf{p}) = -\left[\hat{p} + m\right]\tilde{v}_?(E_p, \mathbf{p}) = 0$. The relations of the $v-$ spinors of the positive energy to $u-$ spinors of the negative energy are frequently forgotten, $\tilde{v}_?(E_p, \mathbf{p}) = \gamma^0 u_h(-E_p, \mathbf{p})$. Thus, unless the unitary transformations do not change the physical content, we have that the negative-energy spinors $\gamma^5\gamma^0 u^-$ (see (26)) satisfy the accustomed "positive-energy" Dirac equation. We should then expect the same physical content. Their explicite forms $\gamma^5\gamma^0 u^-$ are different from the textbook "positive-energy" Dirac spinors. They are the following ones:

$$\tilde{u}(p) = \frac{N}{\sqrt{2m(-E_p + m)}} \begin{pmatrix} -p^+ + m \\ -p_r \\ p^- - m \\ -p_r \end{pmatrix}, \tag{27}$$

$$\tilde{\tilde{u}}(p) = \frac{N}{\sqrt{2m(-E_p + m)}} \begin{pmatrix} -p_l \\ -p^- + m \\ -p_l \\ p^+ - m \end{pmatrix}. \tag{28}$$

$E_p = \sqrt{\mathbf{p}^2 + m^2} > 0$, $p_0 = \pm E_p$, $p^\pm = E \pm p_z$, $p_{r,l} = p_x \pm ip_y$. We use tildes because we do not yet know their polarization properties. Their normalization is to $(-2N^2)$. Next, $\tilde{v}(p) = \gamma^0 u^-$. They are not equal to $v_h(p) = \gamma^5 u_h(p)$. Obviously, they also do not have well-known forms of the usual $v-$ spinors in the Weyl basis, differing by phase factors and in the signs at the mass terms.

One can again prove that the matrix

$$P = e^{i\theta}\gamma^0 = e^{i\theta}\begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \tag{29}$$

can be used in the parity operator as well as in the original Weyl basis. However, if we would take the phase factor to be zero we obtain that while $u_h(p)$ have the eigenvalue $+1$ of the parity, but $(R = (\mathbf{x} \to -\mathbf{x}, \mathbf{p} \to -\mathbf{p}))$

$$PR\tilde{u}(p) = PR\gamma^5\gamma^0 u(-E_p, \mathbf{p}) = -\tilde{u}(p)\,, \tag{30}$$

$$PR\tilde{\tilde{u}}(p) = PR\gamma^5\gamma^0 u(-E_p, \mathbf{p}) = -\tilde{\tilde{u}}(p)\,. \tag{31}$$

In the case of choosing the phase factor $\theta = \pi$ we recover usual parity properties. We again confirmed that the relative (particle-antiparticle) intrinsic parity has physical significance only.

Similar formulations have been presented in Refs. [16], and [17]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [18], who first presented the theory in the 2-dimensional representation of the inversion group in 1956. M. Markov wrote *two* Dirac equations with the opposite signs at the mass term [16] long ago:

$$[i\gamma^\mu \partial_\mu - m]\,\Psi_1(x) = 0\,, \tag{32}$$
$$[i\gamma^\mu \partial_\mu + m]\,\Psi_2(x) = 0\,. \tag{33}$$

In fact, he studied all properties of this relativistic quantum model (while he did not know yet the quantum field theory in 1937). Next, he added and subtracted these equations:

$$i\gamma^\mu \partial_\mu \varphi(x) - m\chi(x) = 0\,, \tag{34}$$
$$i\gamma^\mu \partial_\mu \chi(x) - m\varphi(x) = 0\,. \tag{35}$$

Thus, φ and χ solutions can be presented as some superpositions of the Dirac 4-spinors $u-$ and $v-$. These equations, of course, can be identified with the equations for the Majorana-like $\lambda-$ and $\rho-$, which we presented in Ref. [6]:

$$i\gamma^\mu \partial_\mu \lambda^S(x) - m\rho^A(x) = 0\,, \tag{36}$$
$$i\gamma^\mu \partial_\mu \rho^A(x) - m\lambda^S(x) = 0\,, \tag{37}$$
$$i\gamma^\mu \partial_\mu \lambda^A(x) + m\rho^S(x) = 0\,, \tag{38}$$
$$i\gamma^\mu \partial_\mu \rho^S(x) + m\lambda^A(x) = 0\,. \tag{39}$$

Neither of them can be regarded as the Dirac equation. However, they can be written in the 8-component form as follows:

$$[i\Gamma^\mu \partial_\mu - m]\,\Psi_{(+)}(x) = 0\,, \tag{40}$$
$$[i\Gamma^\mu \partial_\mu + m]\,\Psi_{(-)}(x) = 0\,, \tag{41}$$

with

$$\Psi_{(+)}(x) = \begin{pmatrix}\rho^A(x)\\ \lambda^S(x)\end{pmatrix}, \Psi_{(-)}(x) = \begin{pmatrix}\rho^S(x)\\ \lambda^A(x)\end{pmatrix}, \Gamma^\mu = \begin{pmatrix}0 & \gamma^\mu\\ \gamma^\mu & 0\end{pmatrix}. \tag{42}$$

It is easy to find the corresponding projection operators, and the Feynman-Stueckelberg propagator.

In the previous papers I explained that the connection with the Dirac spinors has been found [6, 19]. For instance,

$$\begin{pmatrix}\lambda^S_\uparrow(\mathbf{p})\\ \lambda^S_\downarrow(\mathbf{p})\\ \lambda^A_\uparrow(\mathbf{p})\\ \lambda^A_\downarrow(\mathbf{p})\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1 & i & -1 & i\\ -i & 1 & -i & -1\\ 1 & -i & -1 & -i\\ i & 1 & i & -1\end{pmatrix}\begin{pmatrix}u_{+1/2}(\mathbf{p})\\ u_{-1/2}(\mathbf{p})\\ v_{+1/2}(\mathbf{p})\\ v_{-1/2}(\mathbf{p})\end{pmatrix}, \tag{43}$$

provided that the 4-spinors have the same physical dimension. Thus, we can see that the two 4-spinor systems are connected by the unitary transformations, and this represents itself the rotation of the spin-parity basis. However, it is usually assumed that the $\lambda-$ and $\rho-$ spinors describe the neutral particles, meanwhile $u-$ and $v-$ spinors describe the charged particles. Kirchbach [19] found the amplitudes for neutrinoless double beta decay ($00\nu\beta$) in this scheme. It is obvious from (43) that there are some additional terms comparing with the standard calculations of those amplitudes. As Markov wrote himself, he was expecting "new physics" from these equations.

Barut and Ziino [17] proposed yet another model. They considered γ^5 operator as the operator of the charge conjugation. Thus, the charge-conjugated Dirac equation has the different sign comparing with the ordinary formulation:

$$[i\gamma^\mu \partial_\mu + m]\Psi^c_{BZ} = 0\,, \tag{44}$$

and the so-defined charge conjugation applies to the whole system, fermion + electromagnetic field, $e \to -e$ in the covariant derivative. The superpositions of the Ψ_{BZ} and Ψ^c_{BZ} also give us the "doubled Dirac equation", as the equations for $\lambda-$ and $\rho-$ spinors. The concept of the doubling of the Fock space has been developed in the Ziino works (cf. [18, 20]) in the framework of the quantum field theory. In their case the self/anti-self charge conjugate states are simultaneously the eigenstates of the chirality. Next, it is interesting to note that we have for the Majorana-like field operators $(a_\eta(\mathbf{p}) = b_\eta(\mathbf{p}))$:

$$\left[\nu^{^{ML}}(x^\mu) + \mathcal{C}\nu^{^{ML}\,\dagger}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p} \tag{45}$$
$$\sum_\eta \left[\begin{pmatrix} i\Theta\phi^{*\,\eta}_{_L}(p^\mu)\\ 0\end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \begin{pmatrix} 0\\ \phi^\eta_{_L}(p^\mu)\end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right]\,,$$

$$\left[\nu^{^{ML}}(x^\mu) - \mathcal{C}\nu^{^{ML}\,\dagger}(x^\mu)\right]/2 = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_p} \tag{46}$$
$$\sum_\eta \left[\begin{pmatrix} 0\\ \phi^\eta_{_L}(p^\mu)\end{pmatrix} a_\eta(p^\mu)e^{-ip\cdot x} + \begin{pmatrix} -i\Theta\phi^{*\,\eta}_{_L}(p^\mu)\\ 0\end{pmatrix} a^\dagger_\eta(p^\mu)e^{ip\cdot x}\right]\,,$$

which, thus, naturally lead to the Ziino-Barut scheme of massive chiral fields, Ref. [17].

Finally, I would like to mention that, in general, in the Weyl basis the $\gamma-$ matrices are *not* Hermitian, $\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0$. So, $\gamma^{i^\dagger} = -\gamma^i$, $i = 1, 2, 3$, the pseudo-Hermitian matrix. The energy-momentum operator $i\partial_\mu$ is obviously Hermitian. So, the question, if the eigenvalues of the Dirac operator $i\gamma^\mu\partial_\mu$ (the mass, in fact) would be always real? The question of the complete system of the eigenvectors of the *non*-Hermitian operator deserve careful consideration [21].

The main points of this Section are: there are "negative-energy solutions" in that is previously considered as "positive-energy solutions" of relativistic wave equations, and vice versa. Their explicit forms have been presented in the case of spin-1/2. Next, the relations to the previous works have been found. For instance, the doubling of the Fock space and the corresponding solutions of the Dirac equation obtained additional mathematical bases. Similar conclusion can be deduced for the higher-spin equations.

3. Non-commutativity in the Dirac Equation.

The non-commutativity [22] exibits interesting peculiarities in the Dirac case. We analized Sakurai-van der Waerden method of derivations of the Dirac (and higher-spins too) equation [23]:

$$(EI^{(4)} + \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)(EI^{(4)} - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\Psi_{(4)} = 0\,. \tag{47}$$

As in the original Dirac work, we have

$$\beta^2 = 1\,,\quad \alpha^i\beta + \beta\alpha^i = 0\,,\quad \alpha^i\alpha^j + \alpha^j\alpha^i = 2\delta^{ij}\,. \tag{48}$$

For instance, their explicite forms can be chosen

$$\alpha^i = \begin{pmatrix}\sigma^i & 0\\ 0 & -\sigma^i\end{pmatrix}\,,\quad \beta = \begin{pmatrix} 0 & 1_{2\times 2}\\ 1_{2\times 2} & 0\end{pmatrix}\,, \tag{49}$$

where σ^i are the ordinary Pauli 2×2 matrices. Obviously, the inverse operators of the Dirac operators of both the positive- and negative- masses exist in the non-commutative case.

We also postulate the non-commutativity relations for the components of 4-momenta: $[E, \mathbf{p}^i]_- = \Theta^{0i} = \theta^i$. Therefore the equation (47) will *not* lead to the well-known equation $E^2 - \mathbf{p}^2 = m^2$. Instead, we have

$$\left\{E^2 - E(\boldsymbol{\alpha}\cdot\mathbf{p}) + (\boldsymbol{\alpha}\cdot\mathbf{p})E - \mathbf{p}^2 - m^2 - i(\boldsymbol{\sigma}\otimes I_{(2)})[\mathbf{p}\times\mathbf{p}]\right\}\Psi_{(4)} = 0\,. \tag{50}$$

For the sake of simplicity, we may assume the last term to be zero. Thus, we come to

$$\left\{E^2 - \mathbf{p}^2 - m^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\right\}\Psi_{(4)} = 0\,. \tag{51}$$

Let us apply the unitary transformation. It is known [24, 6] that one can

$$U_1(\boldsymbol{\sigma}\cdot\mathbf{a})U_1^{-1} = \sigma_3|\mathbf{a}|\,. \tag{52}$$

Some relations for the components $\mathbf{a}$ should be assumed. Moreover, in our case $\boldsymbol{\theta}$ should not depend on E and $\mathbf{p}$. Otherwise, we must take the non-commutativity $[E, \mathbf{p}^i]_-$ into account again. For $\boldsymbol{\alpha}$ matrices we re-write (52) to

$$\mathcal{U}_1(\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\mathcal{U}_1^{-1} = |\boldsymbol{\theta}|\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix} = \alpha_3|\boldsymbol{\theta}|\,. \tag{53}$$

The explicit form of the U_1 matrix is ($a_{r,l} = a_1 \pm i a_2$):

$$\begin{aligned} U_1 &= \frac{1}{\sqrt{2a(a+a_3)}}\begin{pmatrix}a+a_3 & a_l\\ -a_r & a+a_3\end{pmatrix} = \frac{1}{\sqrt{2a(a+a_3)}}\\ &\times\ [a + a_3 + i\sigma_2 a_1 - i\sigma_1 a_2]\,,\\ \mathcal{U}_1 &= \begin{pmatrix}U_1 & 0\\ 0 & U_1\end{pmatrix}\,. \end{aligned} \tag{54}$$

Let us apply the second unitary transformation:

$$\mathcal{U}_2\alpha_3\mathcal{U}_2^\dagger = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}\alpha_3\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}\,. \tag{55}$$

The final equation is

$$[E^2 - \mathbf{p}^2 - m^2 - \gamma^5_{chiral}|\boldsymbol{\theta}|]\Psi'_{(4)} = 0\,. \tag{56}$$

In the physical sense this implies the mass splitting for a Dirac particle over the non-commutative space, $m_{1,2} = \pm\sqrt{m^2 \pm \theta}$. This procedure may be attractive for explanation of the mass creation and the mass splitting for fermions.

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