

On the maximally symmetric vacua of generic Lovelock gravities*

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Abstract

We survey elementary features of Lovelock gravity and its maximally symmetric vacuum solutions. The latter is solely determined by the real roots of a dimension-dependent polynomial. We also recover the static spherically symmetric (black hole) solutions of Lovelock gravity using Palais' symmetric criticality principle. We show how to linearize the generic field equations of Lovelock models about a given maximally symmetric vacuum, which turns out to factorize into the product of yet another dimension-dependent polynomial and the linearized Einstein tensor about the relevant background. We also describe how to compute conserved charges using linearized field equations along with the relevant background Killing isometries. We further describe and

* This work is dedicated to the memory of Prof. Stanley Deser.

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discuss the special vacua which are defined by the simultaneous vanishing of the aforementioned polynomials.

Keywords: Lovelock gravity, maximally symmetric vacua, exact black hole solutions

1. Introduction

General relativity (GR) has passed numerous observational and experimental tests and is essential for our present (classical) understanding of the Universe. Its quantum formulation is far less developed, although string theory seems to be highly relevant. It is thus of interest to investigate extensions and/or modifications of GR to look for novel phenomena and to study gravity in various dimensions.

In this context, Lovelock models are of great interest since they offer a natural extension of GR to higher dimensions. Lovelock gravities have this flexibility by seamlessly introducing additional curvature terms that encode higher-dimensional information, as required by e.g. superstring theory, without giving up on many of the appealing features of GR. Most notably, they possess field equations that are strikingly similar to those of GR in four dimensions, yet also have unique features in higher dimensions. (See e.g. [1] and the references therein.)

More specifically, Lovelock gravities form a special class of higher curvature gravity models since they include at most second order derivatives of the metric tensor and they only depend on the Riemann tensor but not its derivatives [2]. At the classical level, their Hamiltonian formulation involves only the canonical gravitational degrees of freedom of GR [3]. In a completely classical setting, ‘pure Lovelock gravity’ [4, 5], the Lovelock model $\mathcal{L} = -2\Lambda + \alpha_{[(D-1)/2]} \mathcal{L}_{[(D-1)/2]}$ that is described by the lowest and highest terms in (2.1) and (2.2), is claimed to be the *unique* generalization of GR to higher ($D \geq 5$) dimensions with a ‘kinematic’⁵ static vacuum spacetime [5]. At the quantum level, they do not suffer from ghosts that typically occur in higher curvature theories [6–8].

Perhaps this is a good place to say a few words as to why we have picked this topic as a contribution to the special issue dedicated to Stanley Deser’s memory. He has been a driving force in the development of many of the tools used here and he has extensively utilized these in some of his seminal works that we cite.

The outline of the paper is as follows: in section 2 we briefly describe the key aspects of Lovelock gravity, its generic action in D dimensions, the source-free field equations that follow and its maximally symmetric vacua, which are found by determining the real roots of a polynomial of degree $[(D-1)/2]$. We then give an outline of how the static spherically symmetric (black hole) solutions of Lovelock gravity that are already found in [9, 10] can be easily derived using the so-called Weyl–Palais trick [11, 12] in section 3. Section 4 is devoted to deriving the source-free linearized field equations about a given maximally symmetric vacuum, which in fact factorizes into the product of a polynomial of degree $[(D-1)/2] - 1$ and the linearized Einstein tensor about the given background. The latter polynomial turns out to be related to the derivative of the former. Section 5 briefly presents how conserved gravitational charges are defined using background Killing isometries for generic Lovelock theories, which turns out to be proportional to the usual Einstein charges. In section 6 we study in detail the *special vacua*, that are defined via the simultaneous vanishing of the aforementioned polynomials,

⁵ One containing a nontrivial vacuum solution.

about which nontrivial solutions, such as black holes, black strings or branes, of a generic Lovelock theory might exist. We then finish up with our conclusions.

In appendix A we present the explicit forms of the Euler densities for $D = 6, 8, 10$ dimensions that are used in Lovelock models. Appendix B gives the explicit expression of the Euler density \mathcal{L}_k for a generic $k \geq 1$ when the D -dimensional metric (3.1) and its relevant curvature tensors are substituted directly into \mathcal{L}_k . Appendix C is devoted to the determination of special vacua for the dimensions $8 \leq D \leq 11$.

2. Lovelock gravity

The action of Lovelock theories in $D \geq 3$ spacetime dimensions is given by⁶

$$I = \int d^D x \sqrt{|g|} \mathcal{L}(g^{ab}, R_{abcd}(g)), \quad (2.1)$$

with the Lagrangian

$$\mathcal{L} := -2\Lambda + \sum_{k=1}^{[(D-1)/2]} \alpha_k \mathcal{L}_k, \quad (2.2)$$

where Λ is the bare cosmological constant, $[x]$ denotes the integer part of a real number x , α_k are properly chosen coupling constants of appropriate dimensions with

$$\mathcal{L}_k := \frac{1}{2^k} \delta_{c_1 d_1 \dots c_k d_k}^{a_1 b_1 \dots a_k b_k} R^{c_1 d_1}_{a_1 b_1} R^{c_2 d_2}_{a_2 b_2} \dots R^{c_k d_k}_{a_k b_k}, \quad k \geq 1. \quad (2.3)$$

Here the generalized Kronecker delta is totally antisymmetric in all up and down indices, and is defined as

$$\delta_{c_1 \dots c_k}^{a_1 \dots a_k} = k! \delta_{[c_1 \dots c_k]}^{[a_1 \dots a_k]} = k! \delta_{[c_1 \dots c_k]}^{[a_1 \dots a_k]}, \quad (2.4)$$

and the first few terms⁷ in the Lagrangian (2.2) are

$$\mathcal{L}_1 = R, \quad (2.5)$$

$$\mathcal{L}_2 = R^2 - 4R_{ab}R^{ab} + R^{cd}_{ab}R_{cd}^{ab}. \quad (2.6)$$

When D is even, $\mathcal{L}_{D/2}$ is the Euler density, which gives a topological invariant when integrated over a compact manifold without a boundary. The variation of $\mathcal{L}_{D/2}$ for even D is a total derivative and does not contribute to the equations of motion. Moreover \mathcal{L}_k terms vanish identically for $k > D/2$. So there is a natural difference between even and odd dimensions: as seen from (2.2), the Euler density of the even dimension is added each time one moves up to the next odd dimension, whereas no new term is added when one moves from an odd dimension to the next even dimension.

⁶ Intrinsically the treatment here uses the second-order metric formulation and the connection is Levi-Civita. For a discussion on what happens for the first-order (or Palatini) formulation and other interesting features of Lovelock theories, see the explicatory review [1].

⁷ We give the explicit forms of a few others in the appendix A.

Closely related to Lovelock models is the so-called Chern–Simons (CS) gravity theories which are defined only in odd dimensions and can be seen as a higher dimensional ($D \geq 5$) generalization of the CS description of three-dimensional Einstein gravity as a gauge theory. CS gravities are particular cases of Lovelock models that admit a CS-like action with specific set of coefficients α_k depending on the odd dimension D . In complete analogy, there is also so-called Born-Infeld (BI) gravity theories that exist in even dimensions with a precise set of coefficients depending on the dimension D [13, 14].

The source-free field equations of a general Lovelock theory (2.1) are

$$E^a_b := \Lambda \delta^a_b + \sum_{k=1}^{[(D-1)/2]} \alpha_k \mathcal{G}_k^a_b = 0, \tag{2.7}$$

where

$$\mathcal{G}_k^a_b := -\frac{1}{2^{k+1}} \delta^{a a_1 b_1 \dots a_k b_k}_{b c_1 d_1 \dots c_k d_k} R^{c_1 d_1}_{a_1 b_1} R^{c_2 d_2}_{a_2 b_2} \dots R^{c_k d_k}_{a_k b_k}, \tag{2.8}$$

and it is easy to see that $\mathcal{G}_1^a_b$ reduces to the ordinary Einstein tensor G^a_b . It is also apparent that the diffeomorphism invariance of the action (2.1) leads directly to a generalized Bianchi identity $\nabla_a E^a_b = 0$.

Obviously, Lovelock gravity in D dimensions may have up to $[(D - 1)/2]$ distinct maximally symmetric vacua \bar{g}_{ab} [15, 16], for which the Riemann curvature tensor is⁸

$$\bar{R}^{cd}_{ab} = \lambda \delta^{cd}_{ab}, \quad \bar{R}^a_b = \lambda (D - 1) \delta^a_b, \quad \bar{R} = \lambda D(D - 1), \quad \text{for a real constant } \lambda, \tag{2.9}$$

depending on the relevant couplings α_k . The maximally symmetric spacetime (2.9) is Minkowski when $\lambda = 0$, de Sitter (dS) when $\lambda > 0$ and Anti de Sitter (AdS) when $\lambda < 0$. Since (2.9) gives

$$\bar{\mathcal{G}}_k^a_b = -\frac{1}{2} \lambda^k \frac{(D - 1)!}{(D - (2k + 1))!} \delta^a_b, \tag{2.10}$$

the field equations (2.7) for a maximally symmetric spacetime (2.9) imply

$$\bar{E}^a_b = \Lambda \delta^a_b + \sum_{k=1}^{[(D-1)/2]} \alpha_k \bar{\mathcal{G}}_k^a_b = 0, \tag{2.11}$$

that reduces to a polynomial equation (of degree $[(D - 1)/2]$) for λ

$$p(\lambda) := \Lambda - \frac{1}{2} (D - 1)(D - 2) \sum_{k=1}^{[(D-1)/2]} \omega_k \lambda^k = 0. \tag{2.12}$$

Here we have defined rescaled versions of the parameters α_k in the Lagrangian (2.2)

$$\omega_k := \alpha_k \frac{(D - 3)!}{(D - (2k + 1))!}, \tag{2.13}$$

⁸ We will put a bar on a curvature tensor when working with maximally symmetric spacetimes, thus e.g. $\bar{R}^{cd}_{ab} := R^{cd}_{ab}(\bar{g})$ for the metric \bar{g} of a maximally symmetric spacetime.

for the sake of additional transparency and clarity in what follows, and to rid the ensuing calculations of unwieldy numbers. Note in passing that the parameter in front of the usual Einstein–Hilbert piece (2.5) is $\alpha_1 = \omega_1$ now.

The real roots of the polynomial equation (2.12) determine physically viable maximally symmetric vacua. When $[(D - 1)/2]$ is odd, there is at least one maximally symmetric vacua, but for generic $[(D - 1)/2]$ and ω_k , there is a wide range of possibilities going from the ‘no vacuum case’ to ‘the maximum number, i.e. $[(D - 1)/2]$ distinct vacua case’. In general, one can, at least in principle, study (hyper)surfaces in the parameter space defined by ω_k for which the number of maximally symmetric vacua change [16]. In fact, a detailed classification of vacuum solutions with spherical, planar or hyperbolic spatial geometry can be found in [17], where the conditions on the coupling constants ω_k are also examined to further determine degenerate maximally symmetric vacua.

3. The shortcut to static black hole solutions

An appealing feature of Lovelock gravity models is that they admit explicit static spherically symmetric (black hole) solutions that allow for deformations of the ‘Schwarzschild geometry’. Even though such solutions are well-known [9, 10], here we want to recapitulate an alternative and much shorter way for driving this family using the Weyl–Palais trick [11, 12, 18] of substituting in the Lovelock action (2.1) the gauge-fixed Schwarzschild metrics (see (3.1) below) endowed with the desired spherical symmetries. This trick considerably simplifies the derivation of the relevant field equations and, implicitly upholding Bianchi identities, guarantees the determination of all relevant solutions [12].

The most general D -dimensional static spherically symmetric metric in Schwarzschild coordinates is⁹

$$ds^2 = -a(r) b^2(r) dt^2 + \frac{dr^2}{a(r)} + r^2 d\Omega_{D-2}^2, \tag{3.1}$$

where $d\Omega_{D-2}^2$ is the metric on the unit $(D - 2)$ -dimensional sphere S^{D-2} . The essence of the ‘symmetric criticality principle’ [12] lies in the straightforward substitution of (3.1) and its relevant curvature tensors (which are calculated relatively easily by making use of the ‘composite structure’ of (3.1)) into the Lagrangian (2.2) and its pieces \mathcal{L}_k (2.3) that make up the action (2.1), which gives e.g. via (2.5) and (2.6)¹⁰

$$\mathcal{L}_1 = -\frac{r(b'(3ra' + 2(D - 2)a) + 2rab'') + b(r^2a'' + (D - 2)(2ra' + (D - 3)(a - 1)))}{r^2b}, \tag{3.2}$$

$$\begin{aligned} \mathcal{L}_2 = & \frac{(D - 3)(D - 2)}{r^4b} (2r\{b'[r(5a - 3)a' + 2(D - 4)(a - 1)a] \\ & + 2r(a - 1)ab''\} + b\{a[2r^2a'' - 2(D - 4)(-2ra' + D - 5)] \\ & + 2r[a'(ra' - 2(D - 4)) - ra''] + (D - 5)(D - 4)(a^2 + 1)\}), \end{aligned} \tag{3.3}$$

where a prime indicates derivative with respect to the coordinate r .

⁹ In fact one may take the metric functions a and b to depend also on the t -coordinate, which, however, drops out of the actions in the ensuing discussion and can be shown to lead to Birkhoff’s theorem à la [19, 20].

¹⁰ In appendix B, we give the analogous expressions for the Lagrangians listed in appendix A.

As a nontrivial and illustrative example, consider the $D = 5$ case. Then, the Lovelock action (2.1) reduces to the simpler form

$$I \rightarrow \int_0^\infty dr r^3 b (-2\Lambda + \alpha_1 \mathcal{L}_1 + \alpha_2 \mathcal{L}_2) \Big|_{D=5}, \quad (3.4)$$

by discarding the contributions from the angular integrations. The reduced field equations that follow from the calculus of variations on the two metric functions $a(r)$ and $b(r)$ decouple, and immediately lead to the well-known Boulware–Deser solution [15] with two separate branches of static black holes (3.1)

$$D = 5 : \quad b(r) = 1, \quad a_\pm(r) = 1 + \frac{\alpha_1 r^2}{4\alpha_2} \left(1 \pm \sqrt{1 + \frac{4\Lambda\alpha_2}{3\alpha_1^2} + \frac{m}{r^4}} \right), \quad (3.5)$$

for generic values of the parameters α_1, α_2 and Λ , and an integration constant m that is related to the mass of the black hole(s).

Obviously, the aforementioned static spherically symmetric (black hole) solutions of Lovelock gravity [9, 10] can be derived readily in a similar fashion. To this end, the analog of (3.4) is

$$I \rightarrow \int_0^\infty dr b r^{D-2} \left(-2\Lambda + \sum_{k=1}^{[(D-1)/2]} \alpha_k \mathcal{L}_k \right), \quad (3.6)$$

and one finds, using (B.4), that

$$b r^{D-2} \mathcal{L}_k = \frac{(D-2)!}{(D-2k)!} \left(\left(-2ka(1-a)^{k-1} b' r^{D-2k} + b \left(r^{D-2k} (1-a)^k \right)' \right)' - b' (1-a)^k \left(r^{D-2k} \right)' \right). \quad (3.7)$$

Substituting (3.7) into (3.6) and dropping the total derivative terms, the relevant reduced action reads

$$- \int_0^\infty dr \left(-2\Lambda b r^{D-2} + (D-2) b' \sum_{k=1}^{[(D-1)/2]} \omega_k (1-a)^k r^{D-2k-1} \right). \quad (3.8)$$

after using (2.13). Once again the variation of (3.8) with respect to $a(r)$ sets $b(r) = 1$, without loss of generality, whereas the variation of (3.8) with respect to $b(r)$ gives

$$2\Lambda r^{D-2} - (D-2) \sum_{k=1}^{[(D-1)/2]} \omega_k \left((1-a)^k r^{D-2k-1} \right)' = 0,$$

which is easily integrated to yield

$$\sum_{k=1}^{[(D-1)/2]} \omega_k \left(\frac{(1-a)^k}{r^2} \right)^k = \mathcal{M} r^{1-D} + \frac{2\Lambda}{(D-1)(D-2)}, \quad (3.9)$$

for an integration constant \mathcal{M} . The polynomial for the metric function $a(r)$ (3.9) is precisely the implicit condition on the relevant metric function in [10] after careful identifications. For example, the Boulware–Deser solution [15] reproduced in the previous paragraph (3.5) follows from setting $D = 5$ in (3.9), making use of (2.13) and identifying the parameter m in (3.5) with $4\omega_2\mathcal{M}/\omega_1^2$.

4. Linearized field equations

We now turn to the linearization of the source-free field equations of Lovelock models about a generic maximally symmetric background. Let us assume that there exists a well-defined maximally symmetric vacuum \bar{g}_{ab} for which $E^a_b(\bar{g}) := \bar{E}^a_b = 0$, i.e. that $p(\lambda) = 0$ in (2.12) has at least one real root. Then the source-free field equations $E^a_b(g) = 0$ for a more general metric g_{ab} can be linearized about this vacuum (or ‘background’) \bar{g}_{ab} using the metric fluctuations (or ‘deviations’) h_{ab} , with $h_{ab} := g_{ab} - \bar{g}_{ab}$, provided that the deviation h_{ab} goes to zero ‘sufficiently fast’ as one approaches the background \bar{g}_{ab} that is typically located at ‘the boundary at infinity’ for $\lambda \leq 0$. Indicating the linearized tensorial quantities with a subscript ‘L’, one finds using (2.9) that

$$\begin{aligned} (\mathcal{G}_k^a_b)_L &:= -\frac{1}{2^{k+1}} \delta_{bc_1d_1\dots c_kd_k}^{aa_1b_1\dots a_kb_k} \left((R^{c_1d_1}_{a_1b_1})_L \bar{R}^{c_2d_2}_{a_2b_2} \dots \bar{R}^{c_kd_k}_{a_kb_k} \right. \\ &\quad \left. + \bar{R}^{c_1d_1}_{a_1b_1} (R^{c_2d_2}_{a_2b_2})_L \dots \bar{R}^{c_kd_k}_{a_kb_k} + \dots + \bar{R}^{c_1d_1}_{a_1b_1} \bar{R}^{c_2d_2}_{a_2b_2} \dots (R^{c_kd_k}_{a_kb_k})_L \right), \\ &= -\frac{k}{2^{k+1}} \delta_{bc_1d_1\dots c_kd_k}^{aa_1b_1\dots a_kb_k} (R^{c_1d_1}_{a_1b_1})_L \bar{R}^{c_2d_2}_{a_2b_2} \dots \bar{R}^{c_kd_k}_{a_kb_k}, \\ &= -\frac{k}{4} \frac{(D-3)! \lambda^{k-1}}{(D-(2k+1))!} \delta_{bc_1d_1}^{aa_1b_1} (R^{c_1d_1}_{a_1b_1})_L, \\ &= k \frac{(D-3)! \lambda^{k-1}}{(D-(2k+1))!} (G^a_b)_L, \end{aligned} \quad (4.1)$$

where

$$(R^{cd}_{ab})_L = \bar{R}_{abe}{}^{[c} h^{d]e} + 2 \bar{\nabla}_{[a} \bar{\nabla}^{[d} h_{b]}{}^c], \quad (4.2)$$

$$(R^a_b)_L = \frac{1}{2} (\bar{\nabla}^c \bar{\nabla}^a h_{bc} + \bar{\nabla}_c \bar{\nabla}_b h^{ac} - \bar{\nabla}^a \bar{\nabla}_b h - \bar{\square} h^a_b) - h^{ac} \bar{R}_{bc}, \quad (4.3)$$

$$R_L = \bar{\nabla}_a \bar{\nabla}_b h^{ab} - \bar{\square} h - h^{ab} \bar{R}_{ab}, \quad (4.4)$$

$$(G^a_b)_L := (R^a_b)_L - \frac{1}{2} R_L \delta_b^a. \quad (4.5)$$

Here the raising and lowering of all indices are done with respect to the background \bar{g}_{ab} , $h := \bar{g}^{ab} h_{ab}$, $\bar{\nabla}$ indicates the covariant derivative with respect to the background metric and $\bar{\square} := \bar{\nabla}_a \bar{\nabla}^a$. (See e.g. [21] and the references therein for details.) With these preliminaries, it is now straightforward to arrive at the source-free linearized field equations¹¹

$$(E^a_b)_L := \sum_{k=1}^{[(D-1)/2]} \alpha_k (\mathcal{G}_k^a_b)_L = q(\lambda) (G^a_b)_L = 0, \quad (4.6)$$

¹¹ This is again a verification that source-free Lovelock theories indeed represent a natural generalization of Einstein’s GR with a unitary massless spin-2 field content.

where we have defined a polynomial $q(\lambda)$ (of degree $[(D - 1)/2] - 1$) as

$$q(\lambda) := \sum_{k=1}^{[(D-1)/2]} k\omega_k \lambda^{k-1}. \quad (4.7)$$

Taking the trace of (4.6), we find

$$\left(1 - \frac{D}{2}\right) q(\lambda) R_L = 0, \quad (4.8)$$

so that R_L has to vanish provided $q(\lambda) \neq 0$.

Note in passing that the polynomials $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7) are related to each other as

$$\frac{dp}{d\lambda} + \frac{(D-1)(D-2)}{2} q(\lambda) = 0. \quad (4.9)$$

For the case of CS gravities, when the polynomials $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7) simultaneously share at least one real root, say $\tilde{\lambda}$, then the relevant CS theory about the corresponding maximally symmetric vacuum enjoys a symmetry enhancement from local Lorentz symmetry to AdS (if $\tilde{\lambda} < 0$) with a different number of degrees of freedom and modified dynamics. Similar arguments hold for CS dS (if $\tilde{\lambda} > 0$) and CS Poincaré gravities (if $\tilde{\lambda} = 0$) [14].

It is natural to expect that nontrivial solutions, such as black holes, black strings or branes, of a generic Lovelock theory about a special vacuum, defined via the simultaneous vanishing of the polynomials $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7), be special themselves. At the classical level, they must define peculiar points where the geometric properties of distinct, e.g. black hole, solution sets merge or bifurcate. It is tempting to speculate that there is a holographic interpretation of Lovelock gravity similar to that of ordinary gravity. Then these peculiar points should analogously correspond to branching points for the flows of (conformal) field theories with different characteristics.

5. Conserved charges

A natural question to ask in the context of the linearized theory is about currents and conserved charges. The present section contains such a discussion and briefly depicts an adaption of the covariant generalization of the celebrated ADM mass definition [22] to Lovelock gravity.

Through linearization, the generalized Bianchi identity $\nabla_a E^a_b = 0$ can be utilized for constructing a conserved vector current

$$J^a := (E^a_b)_L \bar{\xi}^b = q(\lambda) J^a_{\text{Ein}}, \quad \text{where} \quad J^a_{\text{Ein}} := (G^a_b)_L \bar{\xi}^b, \quad (5.1)$$

with $\bar{\nabla}_a J^a = 0$, using a background Killing vector $\bar{\xi}^a$, for which $\bar{\nabla}_{(a} \bar{\xi}_{b)} = 0$, via similar arguments as for GR [23]. Since the current J^a_{Ein} naturally leads to a conserved and background gauge invariant gravitational charge¹²

$$Q_{\text{Ein}}(\bar{\xi}) := \oint_{\partial\Sigma} d^{D-2}x \sqrt{|q|} n_{[a} r_{b]} \ell^{ab}, \quad (5.2)$$

¹² See e.g. [21, 23, 24] for details. The charge definition (5.2) implicitly assumes the background to be either Minkowski or AdS, i.e. $\lambda \leq 0$, to avoid the pathologies that arise from the cosmological horizons of dS spaces.

where ℓ_{ab} is the potential two-form of the current $J_{\text{Ein}}^a := \bar{\nabla}_b \ell^{ab}$, and $\ell^{ab} = \ell^{[ab]}$ is explicitly [21, 23, 25]

$$\ell^{ab}(\bar{\xi}) = \bar{\xi}_c \bar{\nabla}^{[a} h^{b]c} + \bar{\xi}^{[b} \bar{\nabla}_c h^{a]c} + h^{c[b} \bar{\nabla}_c \bar{\xi}^{a]} + \bar{\xi}^{[a} \bar{\nabla}^{b]} h + \frac{1}{2} h \bar{\nabla}^{[a} \bar{\xi}^{b]}, \quad (5.3)$$

$$= -3 \delta_{eci}^{abd} \bar{\xi}^e \bar{\nabla}^i h_d{}^c + h^{c[b} \bar{\nabla}_c \bar{\xi}^{a]} + \frac{1}{2} h \bar{\nabla}^{[a} \bar{\xi}^{b]}, \quad (5.4)$$

it immediately follows that the conserved gravitational charge that follows from the current J^a is

$$Q(\bar{\xi}) = q(\lambda) Q_{\text{Ein}}(\bar{\xi}). \quad (5.5)$$

This means that when $q(\lambda) = 0$, all conserved charges $Q(\bar{\xi})$ (5.5), including the mass, vanish.

In this work, we are not interested in the thermodynamics of black holes in Lovelock gravity theories. Here is a number of papers that discuss this issue [26–28]. (See also the papers that cite these works.)

6. Solutions at special vacua

So we see that the real roots of the polynomial equation $p(\lambda) = 0$ (2.12) determine the physically viable vacua, whereas the polynomial $q(\lambda)$ (4.7) is important for determining the dynamics (4.6) and the conserved charges (5.5) of Lovelock theories. Hence it is natural to ask about the simultaneous real roots of both $p(\lambda) = 0$ and $q(\lambda) = 0$ for each dimension $D \geq 3$. If such a real root exists and is denoted by λ_r , then one can write $p(\lambda) = (\lambda - \lambda_r)^2 r(\lambda)$, where the factor $r(\lambda)$ is a polynomial (of degree $[(D - 1)/2] - 2$) since $p(\lambda)$ and $q(\lambda)$ are related to each other by (4.9). That is, the root λ_r is at least a double root of $p(\lambda)$. In this case the conserved charge (5.5) is no longer reliable and one has to resort to other charge definitions, see e.g. [14].

With the ideas presented in the last paragraph of section 4 as an extra motivation, we hereby set out to determine the simultaneous roots of $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7) explicitly to identify the special vacua first. We next find out the static spherically black hole solutions of Lovelock models about these special vacua. Even though static spherically symmetric solutions of $E^a{}_b = 0$ (2.7) in higher dimensions with generic α_k was originally given in [9] and later refined in [10], the static spherically symmetric black hole solutions¹³ we explicitly find here are only implicitly contained in these works and do not follow easily. Hence we examine carefully some of these special black hole solutions.

- The simplest examples are the $D = 3$ and $D = 4$ cases, for which

$$D = 3 : \quad p(\lambda) = \Lambda - \omega_1 \lambda = 0, \quad q(\lambda) = \omega_1 = 0, \quad (6.1)$$

$$D = 4 : \quad p(\lambda) = \Lambda - 3\omega_1 \lambda = 0, \quad q(\lambda) = \omega_1 = 0. \quad (6.2)$$

For either dimension, both $p(\lambda) = 0$ and $q(\lambda) = 0$ cannot be solved nontrivially. The vacuum is simply given by $\lambda = \Lambda/\omega_1$ when $D = 3$, and by $\lambda = \Lambda/(3\omega_1)$ when $D = 4$, both with $\omega_1 \neq 0$.

- The first non-trivial, but simple and illustrative, example occurs when $D = 5$, for which

¹³ See [29] for the asymptotic large r limit of these black hole solutions, and [30] for the behavior of the entropy going as the square of the horizon radius.

$$D = 5 : \quad p(\lambda) = \Lambda - 6\omega_1\lambda - 6\omega_2\lambda^2, \quad q(\lambda) = \omega_1 + 2\omega_2\lambda = 0. \quad (6.3)$$

The first thing to note is that $p(\lambda)$ has real roots only if $3\omega_1^2 + 2\omega_2\Lambda \geq 0$. As already mentioned and rediscovered at the end of section 3, there are two separate branches of static black holes (3.5) in $D = 5$. If one sets the ‘mass parameter’ $m = 0$ in (3.5), one ends up with two maximally symmetric (background) spaces with¹⁴

$$a_{\pm}(r) \Big|_{m=0} = 1 - \lambda_{\pm}r^2, \quad \lambda_{\pm} = -\frac{\omega_1}{2\omega_2} \left(1 \pm \sqrt{1 + \frac{2\omega_2\Lambda}{3\omega_1^2}} \right), \quad (6.4)$$

and λ_{\pm} are the two generic roots of the polynomial $p(\lambda) = 0$ (6.3), provided $3\omega_1^2 + 2\omega_2\Lambda \geq 0$, as already mentioned. Note that the two roots λ_{\pm} merge precisely at the common root of $p(\lambda)$ and $q(\lambda)$, located at $\bar{\lambda} = -\omega_1/(2\omega_2)$, provided $\omega_2 \neq 0$ and $\Lambda = -3\omega_1^2/(2\omega_2)$. On the other hand, if one sets $\Lambda = -3\omega_1^2/(2\omega_2)$ at the very beginning, one ends up with two different spacetimes (3.1), for both of which $b(r) = 1$, but

$$a_1 = 1 + \frac{\omega_1}{2\omega_2} r^2, \quad a_2 = c + \frac{\omega_1}{2\omega_2} r^2, \quad (6.5)$$

for an arbitrary integration constant c . The choice of a_1 (6.5) in (3.1) clearly corresponds to a maximally symmetric spacetime (2.9) with $\lambda = -\omega_1/(2\omega_2)$. However, a_2 (6.5) with $c \neq 1$ represents a geometry given by a direct product $M \times S^3$, where M is a two-dimensional maximally symmetric spacetime (2.9) with $\lambda = -\omega_1/(2\omega_2)$ and $c \neq 1$ is a scale that sets the radius of the sphere S^3 . So in $D = 5$, the simultaneous real root of $p(\lambda)$ and $q(\lambda)$ (6.3) is a bifurcation point in the solution space where a maximally symmetric vacuum and a product space $M \times S^3$ emerge together.

- The discussion for $D = 6$ resembles the one for $D = 5$. To start with, one now has

$$p(\lambda) = \Lambda - 10\omega_1\lambda - 10\omega_2\lambda^2 = 0, \quad q(\lambda) = \omega_1 + 2\omega_2\lambda = 0, \quad (6.6)$$

and $p(\lambda)$ has real roots only if $5\omega_1^2 + 2\omega_2\Lambda \geq 0$. In analogy to the $D = 5$ case, there are again two separate branches of static black holes (3.1)¹⁵

$$D = 6 : \quad b(r) = 1, \quad a_{\pm}(r) = 1 + \frac{\omega_1 r^2}{2\omega_2} \left(1 \pm \sqrt{1 + \frac{2\Lambda\omega_2}{5\omega_1^2} + \frac{\tilde{m}}{r^5}} \right), \quad (6.7)$$

for generic values of the parameters ω_1, ω_2 and Λ , and an integration constant \tilde{m} that is related to the mass of the black hole(s). When one sets the parameter $\tilde{m} = 0$ in (6.7), one again finds two maximally symmetric (background) spaces with

$$a_{\pm}(r) \Big|_{\tilde{m}=0} = 1 - \lambda_{\pm}r^2, \quad \lambda_{\pm} = -\frac{\omega_1}{2\omega_2} \left(1 \pm \sqrt{1 + \frac{2\omega_2\Lambda}{5\omega_1^2}} \right), \quad (6.8)$$

where λ_{\pm} are the two generic roots of the polynomial $p(\lambda) = 0$ (6.6), provided $5\omega_1^2 + 2\omega_2\Lambda \geq 0$, of course. Once again the two roots λ_{\pm} merge at the common root of $p(\lambda)$ and $q(\lambda)$, located

¹⁴ Recall the redefinition (2.13) we made earlier. So $\alpha_1 = \omega_1$ and $\alpha_2 = \omega_2/2$ now.

¹⁵ Recall that the static black hole solutions of generic Lovelock models in various dimensions can be found in [10].

at $\tilde{\lambda} = -\omega_1/(2\omega_2)$, provided $\omega_2 \neq 0$ and $\Lambda = -5\omega_1^2/(2\omega_2)$. However, the discussion is more interesting than the one for $D = 5$ when one seeks out solutions of the form (3.1) by setting $\Lambda = -5\omega_1^2/(2\omega_2)$ in the first place. In that case, apart from the maximally symmetric metric (3.1) with $b(r) = 1$ and $a(r) = 1 - \tilde{\lambda}r^2$, there is also the static black hole (3.1), with $b(r) = 1$ again, but now¹⁶

$$a(r) = 1 - \tilde{\lambda}r^2 + \frac{\tilde{M}}{\sqrt{r}}. \tag{6.9}$$

Since $q(\lambda) = 0$ precisely for $\tilde{\lambda}$, this black hole has *vanishing* conserved charges as argued in section 5 and one must resort to other means to determine e.g. its mass. To this end, a more careful discussion which uses the full Lovelock action (2.1), rather than the linearized field equations $(E^a{}_b)_L$ as in section 5, that also carefully accounts for the boundary terms emerging during the variation of the action is needed. However, that in itself is quite a different enterprise which we leave aside.

- The $D = 7$ case is much more involved. Now the polynomials read

$$p(\lambda) = \Lambda - 15\omega_1\lambda - 15\omega_2\lambda^2 - 15\omega_3\lambda^3 = 0, \quad q(\lambda) = \omega_1 + 2\omega_2\lambda + 3\omega_3\lambda^2 = 0. \tag{6.10}$$

The generic roots of $q(\lambda)$ are

$$\lambda_{\pm} = \frac{-\omega_2 \pm \sqrt{\omega_2^2 - 3\omega_1\omega_3}}{3\omega_3}, \tag{6.11}$$

and these are real when $\omega_3 \leq \omega_2^2/(3\omega_1)$. Each one of these is also a simultaneous root of $p(\lambda)$ if

$$\Lambda(\lambda_{\pm}) = \frac{5}{9\omega_3^2} \left(2\omega_2^3 - 9\omega_1\omega_2\omega_3 \mp 2(\omega_2^2 - 3\omega_1\omega_3)^{3/2} \right). \tag{6.12}$$

Using the generic ω_k ($k = 1, 2, 3$) and either a generic Λ or one of these specially tuned $\Lambda(\lambda_{\pm})$ values, one can indeed find black hole solutions of the form (3.1) (again with $b(r) = 1$) but neither of the corresponding triplets of $a(r)$ are worthy of displaying here given their bulky form.

Sadly, the discussion does not simplify much with $\omega_2 = 0$, for which the roots of $q(\lambda)$ read

$$\lambda_{\pm} = \pm \sqrt{-\frac{\omega_1}{3\omega_3}}, \quad \text{provided } \omega_1/\omega_3 < 0. \tag{6.13}$$

These are also roots of $p(\lambda)$ if the cosmological constant is tuned to

$$\Lambda(\lambda_{\pm}) = \pm 10\omega_1 \sqrt{-\frac{\omega_1}{3\omega_3}}. \tag{6.14}$$

However, the unwieldiness of the relevant triplets of $a(r)$ persists and we refrain from displaying them here for the sake of readability.

¹⁶ Setting $\Lambda = -5\omega_1^2/(2\omega_2)$ in the solutions of [10] is misleading, since it does not recover the two distinct classes we find here.

• For completeness sake, we also carefully study the simultaneous real roots of the polynomials $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7) for the dimensions $8 \leq D \leq 11$. However, the relevant analysis for these cases are rather elaborate so we relegate it to appendix C.

• Finally we would like to examine Lovelock gravities in generic $D \geq 5$ dimensions with the additional requirement that all $\omega_k = 0$ for $k > 2$. This particular choice coincides with the cosmological Einstein-Gauss-Bonnet theory in $D \geq 5$. In that case, $p(\lambda)$ (2.12) is quadratic and $q(\lambda)$ (4.7) is linear in λ :

$$D \geq 5: \quad p(\lambda) = \Lambda - \frac{1}{2}(D-1)(D-2)(\omega_1\lambda + \omega_2\lambda^2) \quad q(\lambda) = \omega_1 + 2\omega_2\lambda, \quad (6.15)$$

and this generalizes the arguments we gave for $D=5$ and $D=6$ above. The polynomial $p(\lambda)$ has real roots only if $(D-1)(D-2)\omega_1^2 + 8\omega_2\Lambda \geq 0$. The first thing to note is that there are two separate branches of static black holes [31]

$$D \geq 5: \quad b(r) = 1, \quad a_{\pm}(r) = 1 + \frac{\omega_1 r^2}{2\omega_2} \left(1 \pm \sqrt{1 + \frac{8\Lambda\omega_2}{(D-2)(D-1)\omega_1^2} + \tilde{M}r^{1-D}} \right), \quad (6.16)$$

for generic values of the parameters ω_1, ω_2 and Λ , and an integration constant \tilde{M} that is related to the mass of the black hole(s). The two maximally symmetric (background) spaces follow when $\tilde{M} = 0$ in (6.16)

$$a_{\pm}(r) \Big|_{\tilde{M}=0} = 1 - \lambda_{\pm} r^2, \quad \lambda_{\pm} = -\frac{\omega_1}{2\omega_2} \left(1 \pm \sqrt{1 + \frac{8\omega_2\Lambda}{(D-2)(D-1)\omega_1^2}} \right), \quad (6.17)$$

provided $(D-1)(D-2)\omega_1^2 + 8\omega_2\Lambda \geq 0$. The two roots λ_{\pm} merge at $\tilde{\lambda} = -\omega_1/(2\omega_2)$, if $\omega_2 \neq 0$ and

$$\Lambda = -\frac{\omega_1^2}{8\omega_2}(D-1)(D-2).$$

The discussion follows the one we gave for $D=6$ now: apart from the maximally symmetric metric (3.1) with $b(r) = 1$ and $a(r) = 1 - \tilde{\lambda}r^2$, there is also the static black hole (3.1) with

$$D \geq 6: \quad b(r) = 1, \quad a(r) = 1 - \tilde{\lambda}r^2 + \hat{M}r^{(5-D)/2}. \quad (6.18)$$

where \hat{M} is again an integration constant. Just like its six-dimensional counterpart (6.9), since $q(\tilde{\lambda}) = 0$, all conserved charges of this black hole vanish as per the prescription of section 5. This case shows that the special loci of simultaneous real roots of $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7) may provide room for ‘massless’ black hole solutions that deserve to be further studied.

7. Summary and discussion

We have studied the basic features of Lovelock gravity, including its generic D -dimensional action, the source-free field equations and its maximally symmetric vacuum solutions. We have recapitulated that such vacua are determined by the real roots of the polynomial $p(\lambda)$ (2.12) which in general is of degree $[(D-1)/2]$. We have also rederived the generic static spherically symmetric black hole solutions of Lovelock gravity using the symmetric criticality principle

of Palais. We have also sketched how the generic field equations of Lovelock gravity can be linearized about a given maximally symmetric vacuum, and shown that the outcome is the product of the linearized Einstein tensor about the relevant background with a second polynomial $q(\lambda)$ (4.7), which is of degree $[(D-1)/2] - 1$ and is related to the derivative of $p(\lambda)$. We have also outlined how conserved gravitational charges can be calculated using linearized field equations and background Killing isometries for a generic Lovelock model. We have shown that the charges are always equal to the product of the usual Einstein charges with the polynomial $q(\lambda)$ (4.7).

We have further studied in detail the determination of the *special vacua*, that are defined as the common real roots of the polynomials $p(\lambda)$ and $q(\lambda)$. The discussion was relatively easy for $3 \leq D \leq 5$. $D=6$ was the simplest interesting case since we were able to identify a nontrivial black hole with vanishing conserved charges at the unique special vacuum, which has a slightly different nature than the analogous Boulware–Deser-like bifurcating black holes already present there. Thus we expect that there exist nontrivial and perhaps unusual solutions at those *special vacua* with possibly interesting features on the holography side in $D \geq 7$ as well. Even though we were able to analytically determine the *special vacua* at $D=7$, the corresponding black hole solutions were hardly encouraging and we have not been able to consolidate our hunch. Unfortunately the discussion gets even more sharply complicated with $D > 7$. It may pay off to resort to numerical methods on top of the analytical approach we have employed so far.

Data availability statement

No new data were created or analysed in this study.

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Appendix A. Other terms in the Lagrangian (2.2)

Here we list the explicit forms of a few other terms in the Lagrangian (2.2):

$$\begin{aligned} \mathcal{L}_3 = & 16R_a^c R^{ab} R_{bc} - 12R_{ab} R^{ab} R + R^3 + 24R^{ab} R^{cd} R_{abcd} + 3RR_{abcd} R^{abcd} \\ & - 24R^{ab} R_a^{cde} R_{bcde} - 8R_a^e R^{abcd} R_{bfde} + 2R_{ab}^{ef} R^{abcd} R_{cdef}, \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \mathcal{L}_4 = & -96R_a^c R^{ab} R_b^d R_{cd} + 48R_{ab} R^{ab} R_{cd} R^{cd} + 64R_a^c R^{ab} R_{bc} R - 24R_{ab} R^{ab} R^2 + R^4 \\ & + 96R^{ab} R^{cd} RR_{abcd} + 6R^2 R_{abcd} R^{abcd} - 96R^{ab} RR_a^{cde} R_{bcde} - 384R_a^c R^{ab} R^{de} R_{bdce} \\ & + 96R^{ab} R^{cd} R_a^{ef} R_{bdef} + 192R^{ab} R^{cd} R_a^e R_{bedf} - 32RR_a^e R^{abcd} R_{bfde} + 8RR_{ab}^{ef} R^{abcd} R_{cdef} \\ & + 192R_a^c R^{ab} R_b^{def} R_{cdef} - 192R^{ab} R^{cd} R_a^e R_{cedf} + 384R^{ab} R_a^{cde} R_b^f R_{chef} \end{aligned}$$

$$\begin{aligned}
 & -24R_{ab}R^{ab}R_{cdef}R^{cdef} - 96R^{ab}R_a{}^cdeR_{bc}{}^f{}^hR_{defh} + 192R^{ab}R_a{}^c{}^dR_c{}^ef{}^hR_{defh} \\
 & -96R_a{}^ef{}^hR^{abcd}R_b{}^hiR_{dhfi} - 96R_{ab}{}^ef{}^hR^{abcd}R_c{}^hiR_{dijh} + 6R_{ab}{}^ef{}^hR^{abcd}R_{cd}{}^hiR_{efhi} \\
 & -48R_{abc}{}^eR^{abcd}R_d{}^f{}^hiR_{efhi} + 48R_a{}^ef{}^hR^{abcd}R_b{}^hiR_{efhi} + 3R_{abcd}R^{abcd}R_{efhi}R^{efhi}. \tag{A.2}
 \end{aligned}$$

$$\begin{aligned}
 \mathcal{L}_5 = & 768R_a{}^cR^{ab}R_b{}^dR_c{}^eR_{de} - 640R_{ab}R^{ab}R_c{}^eR^{cd}R_{de} - 480R_a{}^cR^{ab}R_b{}^dR_{cd}R + 240R_{ab}R^{ab}R_{cd}R^{cd}R \\
 & + 160R_a{}^cR^{ab}R_{bc}R^2 - 40R_{ab}R^{ab}R^3 + R^5 + 240R^{ab}R^{cd}R^2R_{abcd} + 10R^3R_{abcd}R^{abcd} \\
 & - 240R^{ab}R^2R_a{}^cdeR_{bcde} - 1920R_a{}^cR^{ab}R^{de}RR_{bdce} + 480R^{ab}R^{cd}RR_{ac}{}^ef{}^hR_{bdef} \\
 & - 1920R^{ab}R^{cd}R^{ef}R_{ace}{}^hR_{bdfh} + 1920R_a{}^cR^{ab}R_d{}^f{}^deR_{becf} + 960R^{ab}R^{cd}R_{a}{}^ef{}^hR_{bedf} \\
 & - 80R^2R_a{}^ef{}^hR^{abcd}R_{bfde} + 20R^2R_{ab}{}^ef{}^hR^{abcd}R_{cdef} + 960R_a{}^cR^{ab}R_{b}{}^def{}^hR_{cdef} \\
 & + 3840R_a{}^cR^{ab}R_b{}^dR^{ef}R_{cedf} - 960R_{ab}R^{ab}R^{cd}R^{ef}R_{cedf} - 960R^{ab}R^{cd}R_{a}{}^ef{}^hR_{cedf} \\
 & - 1920R_a{}^cR^{ab}R^{de}R_{bd}{}^f{}^hR_{cefh} - 3840R_a{}^cR^{ab}R^{de}R_b{}^f{}^hR_{cefh} + 1920R^{ab}R_{a}{}^cde{}^f{}^hR_b{}^d{}^hR_{chef} \\
 & - 120R_{ab}R^{ab}R_{cdef}R^{cdef} - 3840R^{ab}R^{cd}R^{ef}R_{acb}{}^hR_{defh} - 480R^{ab}R_{a}{}^cde{}^f{}^hR_{bc}{}^f{}^hR_{defh} \\
 & - 1920R_a{}^cR^{ab}R_b{}^dR_c{}^ef{}^hR_{defh} + 960R_{ab}R^{ab}R^{cd}R_c{}^ef{}^hR_{defh} + 960R^{ab}R_{a}{}^c{}^dR_c{}^ef{}^hR_{defh} \\
 & + 3840R^{ab}R_a{}^cde{}^f{}^hR_{ch}{}^jR_{deij} + 3840R_a{}^cR^{ab}R^{de}R_b{}^f{}^hR_{dfeh} + 1920R^{ab}R^{cd}R_a{}^ef{}^hR_{b}{}^hiR_{dijh} \\
 & - 1920R^{ab}R^{cd}R_a{}^ef{}^hR_{b}{}^hiR_{dijh} + 3840R^{ab}R^{cd}R_a{}^ef{}^hR_b{}^hiR_{dhei} - 480R_{a}{}^ef{}^hR^{abcd}R_b{}^hiR_{dhfi} \\
 & - 3840R^{ab}R^{cd}R_a{}^ef{}^hR_b{}^hiR_{dhfi} - 3840R^{ab}R^{cd}R_{ac}{}^ef{}^hR_b{}^hiR_{dijh} - 480R_{ab}{}^ef{}^hR^{abcd}R_c{}^hiR_{dijh} \\
 & - 3840R_a{}^cR^{ab}R_b{}^def{}^hR_c{}^hiR_{dijh} + 320R_{ab}R^{ab}R_c{}^hiR^{cdef}R_{dijh} \\
 & - 768R_a{}^ef{}^hR^{abcd}R_{bf}{}^ijR_{cei}{}^kR_{dkhj} + 160R_a{}^cR^{ab}R_{bc}R_{defh}R^{defh} + 480R^{ab}R^{cd}R_{ac}{}^ef{}^hR_{bd}{}^hiR_{efhi} \\
 & + 30R_{ab}{}^ef{}^hR^{abcd}R_{cd}{}^hiR_{efhi} + 960R_a{}^cR^{ab}R_b{}^def{}^hR_{cd}{}^hiR_{efhi} - 80R_{ab}R^{ab}R_{cd}{}^hiR^{cdef}R_{efhi} \\
 & - 3840R^{ab}R^{cd}R_{ac}{}^ef{}^hR_d{}^f{}^hiR_{efhi} - 240R_{abc}{}^eR^{abcd}R_d{}^f{}^hiR_{efhi} - 1920R_a{}^cR^{ab}R_b{}^d{}^eR_d{}^f{}^hiR_{efhi} \\
 & + 3840R^{ab}R_a{}^cde{}^f{}^hR_{b}{}^f{}^hiR_{chd}{}^jR_{efij} + 1920R^{ab}R_a{}^cde{}^f{}^hR_{b}{}^f{}^hiR_{ch}{}^ijR_{efij} \\
 & + 240R_{a}{}^ef{}^hR^{abcd}R_b{}^hiR_{d}{}^ef{}^hR_{efhi} + 1920R^{ab}R^{cd}R_a{}^ef{}^hR_b{}^hiR_{efhi} - 1920R^{ab}R^{cd}R_a{}^ef{}^hR_b{}^hiR_{eifh} \\
 & + 3840R^{ab}R_a{}^cde{}^f{}^hR_b{}^d{}^f{}^hR_c{}^ijR_{eijh} + 1920R^{ab}R_a{}^cde{}^f{}^hR_{bc}{}^f{}^hR_d{}^ijR_{eijh} \\
 & - 3840R^{ab}R_a{}^c{}^dR_c{}^ef{}^hR_d{}^ijR_{eijh} + 1920R_a{}^ef{}^hR^{abcd}R_{bf}{}^hiR_d{}^jkR_{ekij} \\
 & + 240R^{ab}R^{cd}R_{acbd}R_{efhi}R^{efhi} + 15R_{abcd}R^{abcd}R_{efhi}R^{efhi} + 960R^{ab}R_a{}^cde{}^f{}^hR_b{}^de{}^hR_c{}^hijR_{fij} \\
 & - 480R^{ab}R_a{}^cde{}^f{}^hR_{bc}{}^f{}^hR_{de}{}^ijR_{fij} + 960R^{ab}R_a{}^c{}^dR_c{}^ef{}^hR_{de}{}^ijR_{fij} \\
 & + 1920R^{ab}R_a{}^cde{}^f{}^hR_{bcd}{}^f{}^hR_e{}^hijR_{fij} - 1920R^{ab}R_a{}^c{}^dR_c{}^ef{}^hR_e{}^hijR_{fij} \\
 & - 480R_{ab}{}^ef{}^hR^{abcd}R_c{}^hiR_{d}{}^ijkR_{fijk} - 3840R^{ab}R_a{}^cde{}^f{}^hR_b{}^d{}^hR_c{}^ijR_{fijh} \\
 & + 960R^{ab}R_a{}^cde{}^f{}^hR_b{}^f{}^hiR_c{}^deR_{fjhi} - 1920R_a{}^ef{}^hR^{abcd}R_b{}^hiR_d{}^jkR_{fjhk} \\
 & + 1920R_a{}^ef{}^hR^{abcd}R_b{}^d{}^iR_e{}^jkR_{fjhk} - 1920R_{ab}{}^ef{}^hR^{abcd}R_c{}^hiR_d{}^jkR_{fjhk} \\
 & - 480R_{ab}{}^ef{}^hR^{abcd}R_{cd}{}^hiR_e{}^jkR_{fjkij} + 1920R_{abc}{}^eR^{abcd}R_d{}^f{}^hiR_e{}^jkR_{fjkij} \\
 & - 80R_{abcd}R^{abcd}R_e{}^jkR^{efhi}R_{fjkij} - 240R^{ab}R_a{}^cde{}^f{}^hR_{bcde}R_{fjij}R^{fjij} \\
 & + 24R_{ab}{}^ef{}^hR^{abcd}R_{cd}{}^hiR_{ef}{}^jkR_{hijk} - 480R_{abc}{}^eR^{abcd}R_d{}^f{}^hiR_{ef}{}^jkR_{hijk} \\
 & + 20R_{abcd}R^{abcd}R_{ef}{}^jkR^{efhi}R_{hijk} + 480R_{abc}{}^eR^{abcd}R_d{}^f{}^hR_{ef}{}^ijkR_{hijk} \\
 & + 960R_{ab}{}^ef{}^hR^{abcd}R_c{}^hiR_d{}^f{}^jkR_{hijk} - 384R_a{}^ef{}^hR^{abcd}R_b{}^d{}^iR_e{}^jkR_{hijk}. \tag{A.3}
 \end{aligned}$$

Appendix B. Explicit form of $\mathcal{L}_{k \geq 1}$ for the metric (3.1)

Here we first give the explicit expressions of the Lagrangians listed in appendix A when (3.1) and its relevant curvature tensors are substituted into them:

$$\begin{aligned} \mathcal{L}_3 = & \frac{(D-2)!(1-a)}{(D-6)!r^6b} \left(3r\{b'[r(7a-3)a' + 2(D-6)(a-1)a] + 2r(a-1)ab''\} \right. \\ & + b\{a[3r^2a'' - 2(D-6)(-3ra' + D-7)] + 3r[a'(2ra' - 2(D-6)) - ra''] \\ & \left. + (D-7)(D-6)(a^2 + 1)\} \right), \end{aligned} \tag{B.1}$$

$$\begin{aligned} \mathcal{L}_4 = & \frac{(D-2)!(1-a)^2}{(D-8)!r^8b} \left(4r\{b'[r(9a-3)a' + 2(D-8)(a-1)a] + 2r(a-1)ab''\} \right. \\ & + b\{a[4r^2a'' - 2(D-8)(-4ra' + D-9)] + 4r[a'(3ra' - 2(D-8)) - ra''] \\ & \left. + (D-9)(D-8)(a^2 + 1)\} \right), \end{aligned} \tag{B.2}$$

$$\begin{aligned} \mathcal{L}_5 = & \frac{(D-2)!(1-a)^3}{(D-10)!r^{10}b} \left(5r\{b'[r(11a-3)a' + 2(D-10)(a-1)a] + 2r(a-1)ab''\} \right. \\ & + b\{a[5r^2a'' - 2(D-10)(-5ra' + D-11)] + 5r[a'(4ra' - 2(D-10)) - ra''] \\ & \left. + (D-11)(D-10)(a^2 + 1)\} \right). \end{aligned} \tag{B.3}$$

A quick glance at (3.3), (B.1)–(B.3) reveals the structure of generic¹⁷ \mathcal{L}_k for the D -dimensional metric (3.1):

$$\begin{aligned} \mathcal{L}_{k \geq 1} = & \frac{(D-2)!(1-a)^{k-2}}{(D-2k)!r^{2k}b} \left(kr\{b'[r((2k+1)a-3)a' + 2(D-2k)(a-1)a] + 2r(a-1)ab''\} \right. \\ & + b\{a[kr^2a'' - 2(D-2k)(-kra' + D-2k-1)] + kr[a'((k-1)ra' - 2(D-2k)) - ra''] \\ & \left. + (D-2k-1)(D-2k)(a^2 + 1)\} \right). \end{aligned} \tag{B.4}$$

Appendix C. Simultaneous roots of $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7)

In this appendix we carefully study the simultaneous real roots of $p(\lambda)$ (2.12) and $q(\lambda)$ (4.7) for the dimensions $8 \leq D \leq 11$, given a Lovelock theory (2.2) (with (2.3)) containing the parameters Λ and α_k , or equivalently ω_k (2.13), ($1 \leq k \leq [(D-1)/2]$). Our strategy is to first figure out the conditions that guarantee the reality of the roots of the smaller-degree polynomial $q(\lambda)$, assuming that the coefficient $\alpha_{[(D-1)/2]} \neq 0$, and to later tune up the bare cosmological constant Λ so that the real roots of $q(\lambda)$ are also roots of $p(\lambda)$. Throughout we also assume that α_1 , or identically ω_1 , is always strictly positive, i.e. $\omega_1 > 0$.

After the discussion of the most general case, we also examine the simpler example of the vanishing of all α_k , hence ω_k (2.13), except for ω_1 and the highest relevant k , i.e. the case with

¹⁷ It is not obvious at all but in fact (B.4) also reduces to (3.2) when $k = 1$, hence the subscript $k \geq 1$.

$\omega_1 > 0$, $\omega_{[(D-1)/2]} \neq 0$ and $\omega_k = 0$ for $1 < k < [(D-1)/2]$ for the relevant dimensions $8 \leq D \leq 11$. Note that the latter model corresponds to the cosmological Einstein theory coupled with the highest order Euler density for $8 \leq D \leq 11$.

C.1. $D = 8$

The relevant polynomials are

$$p(\lambda) = \Lambda - 21\omega_1\lambda - 21\omega_2\lambda^2 - 21\omega_3\lambda^3 = 0, \quad q(\lambda) = \omega_1 + 2\omega_2\lambda + 3\omega_3\lambda^2 = 0, \tag{C.1}$$

with the same $q(\lambda)$ (6.10) as in $D = 7$. The generic roots of $q(\lambda)$ are identical to those (6.11) of the $D = 7$ case, which are real for $\omega_3 \leq \omega_2^2/(3\omega_1)$. These are also simultaneous roots of $p(\lambda)$ provided

$$\Lambda(\lambda_{\pm}) = \frac{7}{9\omega_3^2} \left(2\omega_2^3 - 9\omega_1\omega_2\omega_3 \mp 2(\omega_2^2 - 3\omega_1\omega_3)^{3/2} \right),$$

$7/5$ times those (6.12) of the $D = 7$ case. The analogous discussion for what happens when $\omega_2 = 0$ leads to λ_{\pm} identical to those (6.13) of the $D = 7$ case again. These are also roots of $p(\lambda)$ if

$$\Lambda(\lambda_{\pm}) = \pm 14\omega_1 \sqrt{-\frac{\omega_1}{3\omega_3}},$$

which is again $7/5$ times those (6.14) of the $D = 7$ case.

C.2. $D = 9$

Now

$$p(\lambda) = \Lambda - 28\omega_1\lambda - 28\omega_2\lambda^2 - 28\omega_3\lambda^3 - 28\omega_4\lambda^4 = 0, \tag{C.2}$$

$$q(\lambda) = \omega_1 + 2\omega_2\lambda + 3\omega_3\lambda^2 + 4\omega_4\lambda^3 = 0.$$

The procedure for determining the roots of the cubic polynomial $q(\lambda)$ (C.2) is a well-known, e.g. [32], but onerous task. Let us keep the discussion succinct and state the most relevant points. First define the following auxiliary variables:

$$Q := \frac{\omega_2}{6\omega_4} - \frac{\omega_3^2}{16\omega_4^2},$$

$$P := -\frac{\omega_1}{8\omega_4} + \frac{\omega_2\omega_3}{16\omega_4^2} - \frac{\omega_3^3}{64\omega_4^3},$$

$$\Delta := Q^3 + P^2 = \frac{\omega_1^2}{64\omega_4^2} - \frac{\omega_1\omega_2\omega_3}{64\omega_4^3} + \frac{\omega_1\omega_3^3}{256\omega_4^4} + \frac{\omega_2^3}{216\omega_4^3} - \frac{\omega_2^2\omega_3^2}{768\omega_4^4}.$$

$$\Xi := \sqrt[3]{P + \sqrt{\Delta}}, \quad \Upsilon := \sqrt[3]{P - \sqrt{\Delta}}.$$

Then the *formal roots* of the polynomial $q(\lambda)$ (C.2) are given by

$$\begin{aligned} \lambda_1 &= -\frac{\omega_3}{4\omega_4} + (\Xi + \Upsilon), \\ \lambda_2 &= -\frac{\omega_3}{4\omega_4} - \frac{1}{2}(\Xi + \Upsilon) + i\frac{\sqrt{3}}{2}(\Xi - \Upsilon), \\ \lambda_3 &= -\frac{\omega_3}{4\omega_4} - \frac{1}{2}(\Xi + \Upsilon) - i\frac{\sqrt{3}}{2}(\Xi - \Upsilon), \end{aligned}$$

and it follows [32] that when

- i) $\Delta > 0$, there exist one real and a pair of complex conjugate roots of $q(\lambda)$ (C.2);
- ii) $\Delta = 0$, all roots of $q(\lambda)$ (C.2) are real and at least two of them are equal;
- iii) $\Delta < 0$, all roots of $q(\lambda)$ (C.2) are real and unequal.

Depending on each separate case, the *real roots* λ_i , ($i = 1, 2, 3$), can be used in the polynomial $p(\lambda)$ (C.2) to tune up the bare cosmological constant Λ .

The analysis simplifies considerably when one sets $\omega_2 = \omega_3 = 0$, for which

$$Q = 0, \quad P = -\frac{\omega_1}{8\omega_4}, \quad \Delta = P^2 > 0, \quad \Xi = \sqrt[3]{2P}, \quad \Upsilon = 0,$$

and there is only one real root of $q(\lambda)$ at

$$\lambda_r = \sqrt[3]{-\frac{\omega_1}{4\omega_4}}, \tag{C.3}$$

regardless of the sign of ω_4 . This is also a root of $p(\lambda)$ if

$$\Lambda(\lambda_r) = 21\omega_1 \sqrt[3]{-\frac{\omega_1}{4\omega_4}}. \tag{C.4}$$

C.3. $D = 10$

Now

$$p(\lambda) = \Lambda - 36\omega_1\lambda - 36\omega_2\lambda^2 - 36\omega_3\lambda^3 - 36\omega_4\lambda^4 = 0, \tag{C.5}$$

and $q(\lambda)$ is identical to the one (C.2) in $D = 9$, so the discussion closely follows that of the $D = 9$ case, with identical expressions for $Q, P, \Delta, \Xi, \Upsilon$ and the formal roots λ_i , ($i = 1, 2, 3$), as in the previous subsection C.2. Once again the *real* λ_i can be used in the polynomial $p(\lambda)$ (C.5) to tune up the bare constant Λ .

The analogous discussion for what happens when one sets $\omega_2 = \omega_3 = 0$ is also identical to the one in the $D = 9$ case, having the same special root λ_r (C.3) as in $D = 9$, with only the specially tuned-up Λ given by

$$\Lambda(\lambda_r) = 27\omega_1 \sqrt[3]{-\frac{\omega_1}{4\omega_4}} \tag{C.6}$$

instead of (C.4).

C.4. $D = 11$

Finally

$$\begin{aligned} p(\lambda) &= \Lambda - 45\omega_1\lambda - 45\omega_2\lambda^2 - 45\omega_3\lambda^3 - 45\omega_4\lambda^4 - 45\omega_5\lambda^5 = 0, \\ q(\lambda) &= \omega_1 + 2\omega_2\lambda + 3\omega_3\lambda^2 + 4\omega_4\lambda^3 + 5\omega_5\lambda^4 = 0. \end{aligned} \tag{C.7}$$

It is important to understand the nature of the roots of $q(\lambda)$ (C.7). To this end, we follow the discussion in [33] and first define the following parameters with rather unwieldy coefficients

$$\begin{aligned} \Delta &:= 16(2000\omega_1^3\omega_3^3 - 2400\omega_1^2\omega_2\omega_4\omega_5^2 - 1800\omega_1^2\omega_3^2\omega_5^2 + 2160\omega_1^2\omega_3\omega_4^2\omega_5 - 432\omega_1^2\omega_4^4 \\ &\quad + 2700\omega_1\omega_2^2\omega_3\omega_5^2 - 120\omega_1\omega_2^2\omega_4^2\omega_5 - 1800\omega_1\omega_2\omega_3^2\omega_4\omega_5 + 432\omega_1\omega_2\omega_3\omega_4^3 + 405\omega_1\omega_3^4\omega_5 \\ &\quad - 108\omega_1\omega_3^3\omega_4^2 - 675\omega_2^4\omega_5^2 + 540\omega_2^3\omega_3\omega_4\omega_5 - 128\omega_2^3\omega_4^3 - 135\omega_2^2\omega_3^3\omega_5 + 36\omega_2^2\omega_3^2\omega_4^2), \\ P &:= 24(5\omega_3\omega_5 - 2\omega_4^2), \\ R &:= 16(25\omega_2\omega_5^2 - 15\omega_3\omega_4\omega_5 + 4\omega_4^3), \\ \Delta_0 &:= 60\omega_1\omega_5 - 24\omega_2\omega_4 + 9\omega_3^2, \\ S &:= 16(500\omega_1\omega_5^3 - 200\omega_2\omega_4\omega_5^2 - 225\omega_3^2\omega_5^2 + 240\omega_3\omega_4^2\omega_5 - 48\omega_4^4). \end{aligned}$$

Then the nature of the roots are as follows¹⁸:

i) $\Delta < 0$: there are two (distinct) real and a pair of complex conjugate roots.

ii) $\Delta > 0$: when $P < 0$ and $S < 0$, all four roots are real and distinct.

When $P > 0$ or $S > 0$, there are two pairs of complex conjugate roots.

iii) $\Delta = 0$: when $P < 0$ and $S < 0$ and $\Delta_0 \neq 0$, there are one real double and two real simple roots.

When $S > 0$ or ($P > 0$ and ($S \neq 0$ or $R \neq 0$)), there are a real double root and a pair of complex conjugate roots.

When $\Delta_0 = 0$ and $S \neq 0$, there are a triple root and a simple root, all real.

When $S = 0$ and $P < 0$, there are two real double roots.

When $S = 0$ and $P > 0$ and $R = 0$, there are two complex conjugate double roots.

When $S = 0$ and $\Delta_0 = 0$, all four roots are equal to $-\omega_4/(5\omega_5)$.

Further defining [33]

$$\begin{aligned} p &:= \frac{3}{25\omega_5^2} (5\omega_3\omega_5 - 2\omega_4^2), \\ q &:= \frac{50\omega_2\omega_5^2 - 30\omega_3\omega_4\omega_5 + 8\omega_4^3}{125\omega_5^3}, \\ \Delta_1 &:= 54(-20\omega_1\omega_3\omega_5 + 8\omega_1\omega_4^2 + 10\omega_2^2\omega_5 - 4\omega_2\omega_3\omega_4 + \omega_3^3), \\ Q &:= \sqrt[3]{\frac{\Delta_1 + \sqrt{\Delta_1^2 - 4\Delta_0^3}}{2}}, \\ T &:= \frac{1}{2}\sqrt{-\frac{2}{3}p + \frac{1}{15\omega_5}\left(Q + \frac{\Delta_0}{Q}\right)}, \end{aligned}$$

¹⁸ Even though there seems to be uncovered cases, these cannot occur as explained in [33].

where any one of the three cube roots of Q can be used in T [33], the generic roots of the quartic polynomial $p(\lambda)$ (C.7) are [33]

$$\lambda_{1,2} = -\frac{\omega_4}{5\omega_5} - T \pm \frac{1}{2} \sqrt{-4T^2 - 2p + \frac{q}{T}},$$

$$\lambda_{3,4} = -\frac{\omega_4}{5\omega_5} + T \pm \frac{1}{2} \sqrt{-4T^2 - 2p - \frac{q}{T}}.$$

Finally the *real roots* λ_i , ($i = 1, 2, 3, 4$), can be used in the polynomial $p(\lambda)$ (C.7) to tune up the bare cosmological constant Λ .

When one sets $\omega_2 = \omega_3 = \omega_4 = 0$, it follows by examining $\Delta = 32000\omega_1^3\omega_5^3$ and $S = 8000\omega_1\omega_5^3$ that $q(\lambda)$ has no real roots when $\omega_1\omega_5 > 0$, i.e. when $\omega_5 > 0$. When $\omega_5 < 0$, there are two distinct real roots given by

$$\lambda_{\pm} = \pm \sqrt[4]{-\frac{\omega_1}{5\omega_5}}.$$

These are also roots of $p(\lambda)$ if

$$\Lambda(\lambda_{\pm}) = \pm 36\omega_1 \sqrt[4]{-\frac{\omega_1}{5\omega_5}}.$$

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