Dynamical Prequantization, Spectrum-generating algebras and the Classical Kepler and Harmonic Oscillator Problems

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ABSTRACT: The prequantization scheme for the three dimensional classical Kepler and harmonic oscillator problems has been discussed in the light of the work of Souriau and Kostant and via the spectrum-generating algebras associated with the dynamical systems.

1. Introduction

Recently, a surge of activities on the problem of quantization of classical systems has been initiated by Souriau's programme[1] and Kostant's work on quantization and unitary representations[2].

Weyl's Ω-rule: Earlier, Weyl[3] prescribed a remarkable method of constructing phase-space representation of quantum mechanics(i.e., a linear one-to-one map of operators in a Hilbert space into c-number functions). If g(q,p) is a classical observable, then define the Fourier transform as

$$g(q,p) = \iint_{-\infty}^{+\infty} \gamma(\xi,\eta)\, e^{i(\xi\cdot q + \eta\cdot p)}\, d\xi\, d\eta . \quad (1.1)$$

Since the correspondence is linear, the phase-space representation can be completely specified by the operators associated with $\exp(i(\xi q+\eta p))$. He prescribed then the Ω-rule such that

$$e^{i(\xi\cdot q+\eta\cdot p)} \longrightarrow \Omega(\xi,\eta)\, e^{i(\xi\cdot\hat{q} + \eta\hat{p})} . \quad (1.2)$$

Thus, the operator $\hat{g}$ $(\hat{q},\hat{p})$ corresponding to g(q,p) is given by

$$\hat{g}(\hat{q},\hat{p}) = \iint_{-\infty}^{+\infty} \gamma(\xi,\eta)\, \Omega(\xi,\eta)\, e^{i(\xi\cdot\hat{q}+\eta\hat{p})}\, d\xi\, d\eta \quad (1.3)$$

We have the inverse mapping,

$$g(q,p) = \frac{1}{2\pi} \iint_{-\infty}^{+\infty} \mathrm{Tr}\left(\hat{g}\, e^{-i(\xi\hat{q}+\eta\hat{p})}\right) . \Omega(\xi,\eta)^{-1} \times \quad (1.4)$$
$$\times\, e^{i(\xi q + \eta\cdot p)}\, d\xi\, d\eta .$$

$\Omega(\xi,\eta)$ is the boundary value of an entire analytic function of ξ, η and has no zeros for real ξ, η. Further,

$$\Omega^{*}(\xi,\eta) = \Omega(-\xi,-\eta), \quad (1.5)$$

$$\text{and} \quad \Omega(0,0) = 1 . \quad (1.6)$$

(1.5) implies the reality condition for Ω and ensures that the real functions are mapped onto self-adjoint operators and vice-versa. Weyl's Ω-rule, however fails in general since the distribution function could be negative.

In Schrödinger approach, one resorts to formal quantization of classical generalized coordinates X_k and the canonical momenta P_k which are defined locally. If we consider $M=S^1$, the unit circle, multiplication by the angle $X=\theta$, is not an operator in the Hilbert space of periodic functions $f(\theta)= F(\theta+2\pi)$. This elucidates the difficulties with the formal quantization of generalized coordinates and momenta.

Dirac's work involves a map of classical dynamical variables f_i to self-adjoint, irreducible operators $K(f_i)$ with suitable domains in a Hilbert space satisfying

$$[K(f_i), K(f_j)] = i\hbar K([f_i, f_j])$$
$$\text{and } K(1) = I .$$

In all conventional approaches, the solution of the Dirac problem is carried out by quantizing the Heisenberg algebra: (q,p,I) while no apriori guarentee is made for preserving the self-adjointness of the rest of the operator functions $f(q,p)$.

Van Hove's prequantization scheme[4]: Let $\hbar > 0$. A prequantization scheme on a manifold $M(= R_{2n}$, the Euclidean phase-space) is a mapping $f(q,p)$ (the C^{∞} functions of infinitesimal canonical transformations which generate one-parameter subgroups of the Lie pseudo-group of contact transformations) onto the set of self-adjoint operators in a complex, infinite dimensional, separable Hilbert space such that

$$K([g,f]) = \frac{1}{i\hbar}\,[K(g), K(f)],$$
$$K(1) = \mathbb{1}.$$

Van Hove's method of Euclidean prequantization fails, however, in a simple dynamical system like the Kepler problem where the Hamiltonian vector field is not 'complete' since orbits with l=0 reach the point q=0 within a finite lapse of time.

Souriau's scheme:[1] Let (M, Ω) be the symplectic manifold of a classical dynamical system, M= the state space and Ω = the symplectic closed 2-form on M. The diffeomorphism ϕ : M $\rightarrow$ M is a canonical transformation if $\phi^{*}(\Omega) = \Omega$. Let F(M) be C^{∞} real-valued functions on M. For each $f_a \in F(M)$, define the vector field X_{f_a} such that

$$d f_a = X_{f_a} \lrcorner\, \Omega$$
$$[f_a, f_b] = X_{f_a}(f_b), \quad \forall\, f_a, f_b \in F(M).$$

Under the above Poisson bracket relation, the vector space F(M) becomes a Lie algebra.

Considerthe Hamiltonian dynamical system(M, Ω ,H) where H $\in$ F(M) is the Hamiltonian function if it has no critical points(dH =0). The integral curves of the vector field X_H are solutions of the Hamilton's equations and generate the one-parameter group of canonical transformations. Thus, in Souriau's scheme, the vector fields X_H are complete also. The essential feature of Souriau's prequantization scheme is that it enables to construct the contact manifold Ω_{2n+1}, one dimension higher than the phase space Ω_{2n}.

2. Dynamical Prequantization [5,6,7]

We discuss in this section the prequantization scheme for the classical

Kepler and harmonic oscillator problems in the light of the Kostant-Souriau scheme and using the dynamical symmetry associated with the classical mechanical systems.

a) Let (M, Ω) be the symplectic manifold for a classical dynamical system; M= the state space and Ω=the canonical closed 2-form on M.It admits a maximal dynamical symmetry K($\mathbb{K}$, correspondingly is the Lie algebra of infinitesimal canonical transformations on M) acting transitively on each energy surface $M_E \approx K/K_0$, K_0 being the stability subgroup of some point on M_E.This implies that all the orbits of the dynamical system are diffeomorphic to one another and that the Hamiltonian is a certain function of the canonical invariants of K.

b)The vector field X_H on M generates a global action in $\Omega$of R(R= O(2) or U(1) for compact orbits, = O(1,1) for non -compact orbits).This defines the Hamilton group G_H (= R) and the Hamiltonian appears as a function of the single ~~of~~ element of the Lie algebra.

c)There exists a dynamical group(spectrum-generating group) G such that it possesses a global canonical action in Ω and contains K & G_H as subgroups.The compact and non-compact orbits correspond to different open intervals of the energy and correspondingly there exists analytic continuation within the submanifolds(energy surfaces).

Further,the elements of the Lie algebra $\mathbb{G}$ of G satisfy the classical equation of motion,

$$\frac{\partial \mathbb{G}}{\partial t} + [H, \mathbb{G}] = 0 .$$

Note that the elements of $\mathbb{K}$ and $\mathbb{G}$ are independent of time as it should be.

d) We note that the construction of the canonical realisation for G provides directly Souriau's prequantization in the following sense.

The irreducible representation of G (quantal representation) is such that every eigenspace of G_H carries an irreducible representation of the 'symmetry group' K, i.e., the Casimir operators of K commutes with G_H (irreducibility condition).

3. Construction of the spectrum-generating algebras

Let (M, Ω) be the symplectic manifold of a classical dynamical system. Ω = the symplectic closed two-form on M is such that $d\Omega=0$. Let $F(M)=\{f_1, f_2, \ldots, f_n\}$ be the C^∞ functions on M. Ω is called the phasespace for the underlying dynamical system and in the canonical co-ordinates is given by

$$\Omega = \sum_i dp_i \wedge dq_i - dH \wedge dt. \qquad (3.1)$$

For $\forall\, f_a \in F(M)$, we define the covariant and the contravariant vector fields df_a and X_{f_a} respectively as

$$df_a = X_{f_a} \lrcorner\, \Omega = (\partial f_a / \partial p_i)\, dp_i + (\partial f_a / \partial q_i)\, dq_i \qquad (3.2)$$

and $$X_{f_a} = \frac{\partial}{\partial t} + \sum_i \left(\frac{\partial f_a}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial f_a}{\partial q_i} \frac{\partial}{\partial p_i} \right), \qquad (3.3)$$

where $X_{f_a} \lrcorner\, \Omega$ defines the contraction of Ω by X_{f_a}. Let $\{X_{f_a}\} = V(M)$. For $f_a, f_b \in F(M)$, we have

$$X_{f_a}(f_b) = \frac{\partial f_b}{\partial t} + \sum_i \left(\frac{\partial f_a}{\partial p_i} \frac{\partial f_b}{\partial q_i} - \frac{\partial f_a}{\partial q_i} \frac{\partial f_b}{\partial p_i} \right). \qquad (3.4)$$

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Under the Poisson bracket relation (3.4), the real vector space F(M) becomes a Lie algebra. The map $f_a \rightarrow X_{f_a}$ is a Lie algebra homomorphism of F(M) into V(M) on M;

i.e., $$X_{\alpha f + \beta g} = \alpha X_f + \beta X_g,$$

$$X_{[f,g]} = X_f X_g - X_g X_f, \quad f, g \in F(M). \qquad (3.5)$$

Consider the triplet (M, Ω, H), the Hamiltonian dynamical system. Then

$$H \rightarrow X_H = \frac{\partial}{\partial t} + \sum_i \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right) \qquad (3.6)$$

and $$dH = \sum_i \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i \right) + \frac{\partial H}{\partial t} dt. \qquad (3.7)$$

If $H = H(p_i, q_i)$, then $\frac{\partial H}{\partial t} = 0$. Now,

$$X_H(f_a) \quad = \quad \frac{\partial f_a}{\partial t} + [H, f_a] \cdot \tag{3.8}$$

Using the classical equation of motion, we have

$$[H, f_a] + \frac{\partial f_a}{\partial t} = df_a/\, dt = 0. \tag{3.9}$$

Thus, $\mathbb{G} = \{ f_a,\ a = 1,2,3,\ldots\ldots,n : df_a/\, dt = \frac{\partial f_a}{\partial t} + [H, f_a] = 0 \}$ defines the spectrum generating algebra for the given Hamiltonian H. If $[H , f_a] = 0 = X_H(f_a)$, then we obtain the symmetry algebra $\mathbb{K}$ $\mathbb{G} = \{f_a,\ 1 \leq a \leq m < n : \frac{\partial f_a}{\partial t} = 0, [H , f_a] = 0 \}$.

4.(a) Kepler motion [6,8]

We have $H = p^2/\, 2 + V(q)$, $V(q) = V(\, q^2\,) = - 1/q$, (4.1)

where we have used the reduced mass $\mu = 1$ and the coupling parameter $\lambda = 1$. The constants of the motion which have vanishing Poisson bracket with H are given by[8]

$$\begin{aligned} \underline{L} &= \underline{q} \times \underline{p} \ , \\ \underline{f} &= a_1(\, q^2, H , l^2)\, \underline{q} + a_2\, (\, q^2, H , l^2)\, \underline{p} \\ &= (\, \alpha_0'/l^2)\, \underline{L} \times \underline{A} - \alpha_2'\underline{A} \ , \end{aligned} \tag{4.2}$$

where $\underline{A} = \underline{p} \times \underline{L} - \underline{q}/q$(the conventional Lenz-vector),

$$\underline{L} \times \underline{A} = l^2\, (1 - q/l^2\,)\underline{p} + (\underline{p}\cdot\underline{q}\, /\, q\, l^2\,)\, \underline{q} \ , \tag{4.3}$$

$$a_1 = \alpha_0'\, (\, \underline{p}\cdot\underline{q}\, /\, l^2 q\,) - \alpha_2'(2H + 1/q), \tag{4.4}$$

$$a_2 = \alpha_0'(\, 1 - q/l^2\,) + \alpha_2'\, \underline{p}\cdot\underline{q} \ . \tag{4.5}$$

α_0', α_2' are arbitrary constants depending upon H and l^2. The vectors $\underline{A}$ and $\underline{L} \times \underline{A}$ lie on the plane of the orbit and so also $\underline{f}$. For negative energy motions ($E < 0$), we have

$$\begin{aligned} \underline{f}\cdot\underline{f} &= a_1\, q^2 + 2\, a_1 a_2\, \underline{p}\cdot\underline{q} + a_2{}^2\, p^2 \\ &= -\sigma l^2 + \sigma_0(H), \end{aligned} \tag{4.6}$$

where $\sigma_0(H) > l^2 \geq - 1/2H$ and $\sigma = +1$. For $E \gtrless 0$, σ takes values -1 and 0 respectively. Thus, the symmetry algebra $\mathbb{K}$ is spanned by

($\underline{L}$, $\underline{f}$) whose Poisson bracket relations satisfy the Lie algebra isomorphic to O(4) for E 0. Further, the Casimir invariants are given by

$$C_1 = \underline{L}^2 + \underline{f}\,2 = \sigma_0(H),$$
$$C_2 = 0 = \underline{L} \cdot \underline{f} \; .$$

We note that the symmetry algebra K possesses the local canonical action on the energy surface $M_E \approx O(4)/\,O(2)$, i.e., M_E is a homogeneous space with stability subgroup O(2). We consider $\mathbb{C}$ to be such that the commutant of O(4) is O(2) and $\mathbb{C}$ contains O(4) ⊗ O(2) as subgroups.

Let $\mathbb{C}$ = O(4,2) whose elements satisfy the following equal-time Poisson bracket relations:

$$[L_{ab}, L_{cd}]_t = g_{ac} L_{bd} + g_{bd} L_{ac} - g_{ad} L_{bc} - g_{bc} L_{ad},$$

$a,b,c,d = 1,2,3,4,5,6$ and $g_{ii} = g_{44} = - g_{55} = - g_{66} = 1$; $i,j = 1,2,3.$ (4.7)

Let us identify $L_i = \epsilon_{ijk} L_{jk}$, $f_i = L_{i4}$, $M_i = L_{i5}$,

$\Gamma_i = L_{i6}$, $T = L_{45}$, $\Gamma_4 = L_{46}$ and $\Gamma_0 = L_{56}$.

Let $\underline{B} = A_1 \underline{q} + A_2 \underline{p}$. (4.8)

From (4.7) and (4.8), we have

$$[\underline{f} , \underline{B}]_t = S = a_1 A_2 - a_2 A_1 + \text{terms involving } \underline{p} \times \underline{p} ,\ \underline{q} \times \underline{q}$$

and $\underline{q} \times \underline{p}$, (4.9)

and $[S, \underline{f}]_t = \underline{B} = A_1 \underline{q} + A2\, \underline{p}$. (4.10)

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From (4.9) and (4.10), we obtain

$$A_1 = \left(\frac{u}{q}\frac{\partial a_1}{\partial H} + \frac{a_2}{q}\right)\frac{\partial S}{\partial q} + \frac{1}{l}\left(a_1 u + a_2 p^2\right)\frac{\partial S}{\partial l}$$

$$A_2 = \frac{u}{q}\frac{\partial S}{\partial q}\frac{\partial a_2}{\partial H} - 1/_l\left(a_1 q^2 + a_2 u\right)\frac{\partial S}{\partial l},$$

$u = \underline{p}\cdot\underline{q}$. (4.11)

Thus, $S = a_1 A_2 - a_2 A_1$

$$= \left[\frac{u}{q}\left(a_1 \frac{\partial a_2}{\partial H} - a_2 \frac{\partial a_1}{\partial H}\right) - a_2^2/q\right]\frac{\partial S}{\partial q} + \left(l + 1/_{2Hl}\right)\frac{\partial S}{\partial l} \quad (4.12)$$

We consider the following cases:

a) $\alpha_0' = 0, \quad \alpha_2' = \frac{1}{\sqrt{-2H}} \quad , \underline{f} = -\alpha_2' \underline{A}$

b) $\alpha_2' = 0, \quad \alpha_0'/l = 1/\sqrt{-2H} \quad , \underline{f} = 1/l\sqrt{-2H}(\underline{L} \times \underline{A})$.

For case (a) (using (4.12) and equation of motion),we have

$$S = \frac{X(H)}{\sqrt{-2H}}\left(-\frac{1+2Hq}{\sqrt{-2H}}\cos\beta - u\sin\beta\right) + \frac{Y(H)}{\sqrt{-2H}}\left(-\frac{1+2Hq}{\sqrt{-2H}}\sin\beta + u\cos\beta\right),$$

$$\beta = (-2H)^{\frac{1}{2}}(u - 2Ht) . \tag{4.13}$$

(4.13) shows that S is a linear combination of two rotational scalars Γ_4 and T (say) with arbitrary coefficients X(H) and Y(H). Let us put

$$X(H) = Y(H) = (-2H)^{\frac{1}{2}}$$

and $$\Gamma_4 = -\frac{1+2Hq}{\sqrt{-2H}}\cos\beta - u\sin\beta$$

$$T = -\frac{1+2Hq}{\sqrt{-2H}}\sin\beta + u\cos\beta . \tag{4.14}$$

Substituting for S from (4.14) in (4.11) and using the values of a_1 and a_2 , we obtain after some simplification

$$\underline{M} = q\,\underline{p}\cos\beta - (-2H)^{-\frac{1}{2}}(u\,\underline{p} - \underline{q}/q)\sin\beta$$

and $$\underline{\Gamma} = q\,\underline{p}\sin\beta + (-2H)^{-\frac{1}{2}}(u\,\underline{p} - \underline{q}/q)\cos\beta \tag{4.15}$$

Now, the equal-time commutation relation between M and gives us

$$\Gamma_0 = (-2H)^{-\frac{1}{2}}. \tag{4.16}$$

Note that is independent of time.

For case(b),we obtain the following expressions for $\underline{M}$, $\underline{\Gamma}$, Γ_4, T and Γ_0 .

$$\underline{M} = \frac{1}{l}\left[(2+2Hq)\,\underline{q} - uq\,\underline{p}\right]\cos\beta + 1/l\sqrt{-2H}\left[-\frac{u}{q}(1+2Hq)\,\underline{q} + (u^2-q)\,\underline{p}\right]\sin\beta$$

$$\underline{\Gamma} = 1/l\left[(2+2Hq)\,\underline{q} - uq\,\underline{p}\right]\sin\beta - 1/l\sqrt{-2H}\left[-\frac{u}{q}(1+2Hq)\,\underline{q} + (u^2-q)\,\underline{p}\right]\cos\beta . \tag{4.17}$$

The structure for Γ_4, T, Γ_0 remains same as in the case (a).

The Casimir invariants for O(4,2) and $O(4)_{1,2,3,4}$ are given by

$$Q_2 = \tfrac{1}{2} L_{ab} L^{ab} = 0,$$
$$Q_3 = -1/48 \; \epsilon^{abcdef} L_{ab}L_{cd}L_{ef} = 0,$$

$$Q_4 = 1/4\ (\ L_{ab}\ L^{bc}\ L_{cd}\ L^{da} - Q_2^2 - 8\ Q_2\ )\ = 0, \quad (4.18)$$

$$C_1 = \underline{L}^2 + \underline{f}^2 = -1/2H = (\ \Gamma_o\ )^2,$$

$$C_2 = \underline{L}\cdot\underline{f} = 0. \qquad (4.19)$$

We note that the eigenspace $\{\ \Psi_{n l m}\}$ of $O(4)_{1,2,3,4}$ furnishes the eigen-states for Γ_o (irreducibility condition). In case of non-compact orbits i.e, ,for $\mathbb{G}_H = O(1,1)$, $\mathbb{G}$ is still $O(4,2)$; however, the symmetry algebra is $O(3,1)_{1,2,3,5}$ and $\mathbb{G}_H$ is generated by L_{46}.

4.b Harmonic oscillator[5]

We have $H = p^2/2 + q^2/2, \quad m = \omega = 1. \qquad (4.20)$

In this case, $\mathbb{K} = SU(3)$ and is spanned by $\underline{L}$, T_{ij};,where

$$\underline{L} = \underline{q} \times \underline{p}\ ,$$

$$T_{ij} = T_1\ q_i q_j + T_2\ (\ q_i p_j + q_j p_i\ ) + T_3\ p_i p_j - 1/3\ \delta_{ij}\ T_{kk}, \qquad (4.21)$$

where T_1, T_2, T_3 are given by

$$T_1 = \varphi_1 + 2\ \underline{p}\cdot\underline{q}\ \varphi_2 + p^2 \varphi_3\ ,$$

$$T_2 = (\ p^2 - q^2)\ \varphi_2 - \underline{p}\cdot\underline{q}\ \varphi_3\ ,$$

$$T_3 = \varphi_1 - 2\ \underline{p}\cdot\underline{q}\ \varphi_2 + q^2 \varphi_3\ ,$$

$$T_1\ T_3 - T_2^2 = \sigma = +1. \qquad (4.22)$$

φ_1, φ_2, φ_3 are arbitrary functions of H and l^2 and satisfy

$$\varphi_1^2 + 2H\ \varphi_1\varphi_3 + l^2\varphi_3 - 4\ (H^2 - l^2)\ \varphi_2^2 = +1,$$

$$2H\ \varphi_1 + 2l^2\ \varphi_3 = \Phi(H). \qquad (4.23)$$

The Casimir operators C_2, C_3 are given by

$$C_2 = L_i\ L_i + \tfrac{1}{2}\ T_{ij}\ T_{ij} = -\ (-2H\ /\sqrt{3}\ )^2,$$

$$C_3 = (3 L_i L_j - T_{jk} T_{ki}) T_{ij}$$
$$= -\frac{1}{3\sqrt{3}} \Phi^3(H) = (-2H/\sqrt{3})^3 \qquad \text{(special case; } \varphi_2 = \varphi_3 = 0, \varphi_1 = 1\text{)}. \qquad (4.24)$$

The energy surface M_E on which the global canonical action of K is defined is given by

$$M_E \approx SU(3)/SU(2) \approx S^5, E \neq 0,$$
$$\approx SU(3)/SU(3) \approx \{0\}, E = 0, \qquad (4.25)$$

where $K_o = SU(2)$ ($SU(3)$) is the stability subgroup of some point on the homogeneous space(energy surface) for $E > 0$($E = 0$). Thus, the whole phase space Ω_6 is filled by the energy surface according to

$$\Omega_6 = \{0\} \cup \{R \times S^5\} = R_6 . \qquad (4.26)$$

The Hamiltonian flow defines in R_6 a global action of U(1) which together with SU(3) gives the global realisation of SU(3,1).

To construct $\mathfrak{C} = SU(3,1) = \{f_a : \underline{L}, T_{ij}, \underline{P}, \underline{K}, S |$ $df_a/dt = 0 = \frac{\partial f_a}{\partial t} + [H, f_a]\}$ satisfying the Poisson bracket relations;

$$[L_i, L_j] = \epsilon_{ijk} L_k$$
$$[L_i, T_{km}] = \epsilon_{ikl} T_{lm} + \epsilon_{iml} T_{kl},$$
$$[T_{ij}, T_{kl}] = L_m(\epsilon_{ikm}\delta_{jl} + \epsilon_{ilm}\delta_{jk} + \epsilon_{jkm}\delta_{il} + \epsilon_{jlm}\delta_{ik}),$$
$$[L_i, P_j] = \epsilon_{ijk} P_k,$$
$$[L_i, K_j] = \epsilon_{ijk} K_k$$
$$[T_{ij}, P_k] = K_i \delta_{jk} + K_j \delta_{ik},$$
$$[T_{ij}, K_k] = -(P_i \delta_{jk} + P_j \delta_{ik}), \qquad (4.27)$$

we proceed as follows:

$$\text{Let} \quad \underline{P} = f\, \underline{p} + g\, \underline{q}$$
$$\underline{K} = f'\, \underline{p} + g'\, \underline{q} . \qquad (4.28)$$

Using (4.27) and the classica equation of motion, we finally obtain (for the special case : $\varphi_1 = 1, \varphi_2 = \varphi_3 = 0, \Phi(H) = 2H$)

$$T_{ij} = (q_i q_j + p_i p_j) - (2H/3)\ \delta_{ij},$$

$$\underline{P} = \underline{p} \cos t - \underline{q} \sin t,$$

$$\underline{K} = \underline{p} \sin t + \underline{q} \cos t,$$

$$S = 2H.$$

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