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Elements of spin Hurwitz theory: closed algebraic formulas, blobbed topological recursion, and a proof of the Giacchetto–Kramer–Lewański conjecture

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Abstract

In this paper, we discuss the properties of the generating functions of spin Hurwitz numbers. In particular, for spin Hurwitz numbers with arbitrary ramification profiles, we construct the weighed sums which are given by Orlov's hypergeometric solutions of the 2-component BKP hierarchy. We derive the closed algebraic formulas for the correlation functions associated with these tau-functions, and under reasonable analytical assumptions we prove the loop equations (the blobbed topological recursion). Finally, we prove a version of topological recursion for the spin Hurwitz numbers with the spin completed cycles (a generalized version of the Giacchetto–Kramer–Lewański conjecture).

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Contents

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1 Introduction

1.1 Topological recursion and integrability

It is well known that the Chekhov–Eynard–Orantin topological recursion [10] is closely related to integrability. However, the details of a general relationship between the two phenomena remain unclear. Although topological recursion is believed to be a universal property for a huge class of enumerative geometry and physics problems, the proofs of its validity are often model-dependent and technically involved, and at this point, despite the lack of understanding of the general relationship, various universal properties of integrability often help to prove topological recursion.

One of the most general applications of integrability to topological recursion is given by the weighted Hurwitz numbers. The generating functions of the weighted Hurwitz numbers are hypergeometric tau-functions of the 2-component KP (2-KP) hierarchy. The study of topological recursion for the general hypergeometric solutions of the 2-KP hierarchy was initiated in [1, 2] (subsuming a huge list of particular examples known before). Many elements of the general construction including quantum and classical spectral curves were properly identified there. However, the topological recursion was proved only for an infinite-dimensional family of solutions with polynomial weight functions and finite sets of the second times of the 2-KP hierarchy. Topological recursion for the much more general families of the hypergeometric solutions of the 2-KP hierarchy was proved in [4, 5]. The proof there is based on the free field description of the KP hierarchy, more specifically, on the free fermion construction and the boson-fermion correspondence.

These results indicate that the same line of reasoning can be applied to any integrable hierarchy with free fermion description. In this paper, we describe topological recursion for the hypergeometric solutions of the 2-component BKP (2-BKP) hierarchy. A well-known neutral fermion description of the BKP hierarchy allows us to follow the general approach for the 2-KP hierarchy, derived in [4, 5]. Many steps can be repeated without essential changes, but some specifics of the 2-BKP case (mostly important, the built-in oddness of the parametrizations) require extra analysis and lead to new phenomena. We derive the general closed algebraic formulas for the correlation functions in the 2-BKP case and prove the blobbed topological recursion [8], that is, the linear and quadratic loop equations.

1.2 BKP and spin Hurwitz theory

The BKP hierarchy is believed to govern the spin Hurwitz numbers in essentially the same way as the KP hierarchy governs the ordinary Hurwitz numbers [19]. However, the important construction of the *weighted spin Hurwitz numbers* (in the sense of [13]) is still unavailable in the literature. In this paper we show how to construct integrable generating functions of spin Hurwitz numbers for arbitrary ramification profiles and number of the branch points. These generating functions are Orlov's hypergeometric tau-functions of the 2-component BKP hierarchy [20], and the weights associated with the ramifications serve as parameters. It is not clear at the moment how to reduce naturally the number of parameters and to define the direct analogs of weighted Hurwitz numbers in the spin case. To this end, we suggest two possible candidates for the elementary weight functions.

It is well known that the tau-functions of the KP and BKP hierarchies are related to each other by a simple quadratic relation [9]. Following Orlov [20], we describe this relation for the hypergeometric tau-functions. Namely, for any hypergeometric tau-function of the 2-BKP hierarchy we find the corresponding tau-function of the 2-KP hierarchy. It is easy to see that such KP tau-function is not unique. This relation between tau-functions should lead to the non-trivial relations between the spin and ordinary Hurwitz numbers.

1.3 Giacchetto–Kramer–Lewański conjecture and its generalization

Additional input and motivation to study the correlation functions of the corresponding hypergeometric 2-BKP tau-functions comes from a recent work of Giacchetto, Kramer, and Lewański [12]. They study in detail the theory of so-called spin Hurwitz numbers with completed cycles, both single and double, whose elements occur naturally in a number of other works in relation to computation of the volumes of strata in the moduli spaces of holomorphic differentials [11] and Gromov–Witten theory of Kähler surfaces [17, Introduction].

Remarkably, Giacchetto, Kramer, and Lewański propose a conjectural statement on $\mathbb{Z}_2$-equivariant version of topological recursion for the correlation differentials of these numbers, and they prove that the statement on topological recursion is equivalent to an

ELSV-type formula for spin Hurwitz numbers with completed cycles that expresses these numbers in terms of the Chiodo classes twisted by the Witten 2-spin class.

Using the formulas for the correlation functions and loop equations we prove a natural generalization of the Giacchetto–Kramer–Lewański conjecture, that is, a $\mathbb{Z}_2$-equivariant version of topological recursion for the double spin Hurwitz numbers with arbitrary finite linear combinations of the spin completed cycles.

Let us remark that while with the motivation coming from [12] we focus on this particular family of spin Hurwitz numbers, we expect that our modification of the methods of [4, 5] should immediately work for other families of the generating functions of the spin Hurwitz numbers, analogous to the families investigated in [4]. We also expect that the integrable approach to the topological recursion in the BKP case should be as universal as for the KP case. Moreover, without significant modifications, it should also work for other integrable hierarchies described by free fermions.

1.4 Notation

A partition λ is *strict*, if $\lambda_1 > \lambda_2 > \lambda_3 > \cdots > \lambda_{\ell(\lambda)} > \lambda_{\ell(\lambda)+1} = 0$, where $\ell(\lambda)$ is the *length* of the partition. We denote the set of strict partitions, including the empty one, by SP. A partition λ is *odd* if all parts in λ are odd. We denote the set of odd partitions, including the empty one, by OP. For a partition λ by $\lambda(k)$ we denote the number of parts equal to k.

1.5 Organization of the paper

In Sect. 2 we recall the neutral fermion description of the BKP hierarchy. In Sect. 3 we explain how to construct the generating functions of the spin Hurwitz numbers that solve the 2-BKP hierarchy and how these tau-functions can be related to the generating functions of the ordinary Hurwitz numbers. Section 4 is devoted to the correlation functions for the general hypergeometric tau-functions of the 2-BKP hierarchy. In Sect. 5 we prove that these correlation functions, under mild analytic assumptions, satisfy linear and quadratic loop equations. In Sect. 6 we derive explicit expressions for the n-point correlation functions. In Sect. 7 we use these expressions to prove the topological recursion for the spin Hurwitz numbers with the spin completed cycles.

2 Neutral fermions and boson-fermion correspondence

In this section we remind the reader the neutral fermion formalism and boson-fermion correspondence in the framework of the BKP hierarchy. More detailed descriptions can be found in [9, 20–22].

2.1 Neutral fermions

Let ϕ_k, $k \in \mathbb{Z}$, be the neutral free fermions satisfying the canonical anticommutation relations

$$\{\phi_k, \phi_m\} = (-1)^k \delta_{k+m,0}. \tag{2.1}$$

Note that $\phi_0^2 = 1/2$. These relations define the Clifford algebra as an associative algebra.

For the vacuum vector $\big|0\big\rangle$ and the co-vacuum $\big\langle 0\big|$, satisfying

$$\phi_m\big|0\big\rangle = 0, \quad \big\langle 0\big|\phi_{-m} = 0, \quad m < 0 \tag{2.2}$$

the elements $\phi_{k_1}\phi_{k_2}\ldots\phi_{k_m}\big|0\big\rangle$ with $k_1 > k_2 > \cdots > k_m \geq 0$ form a basis of the *neutral fermion Fock space* $\mathcal{F}_B$,

$$\mathcal{F}_B = \operatorname{span}\left\{\phi_{k_1}\phi_{k_2}\ldots\phi_{k_m}\big|0\big\rangle \mid k_1 > k_2 > \cdots > k_m \geq 0\right\}, \tag{2.3}$$

and its dual

$$\mathcal{F}_B^* = \operatorname{span}\left\{\big\langle 0\big|\phi_{k_m}\ldots\phi_{k_2}\phi_{k_1} \mid k_1 < k_2 < \cdots < k_m \leq 0\right\}. \tag{2.4}$$

The space $\mathcal{F}_B$ splits into two subspaces

$$\mathcal{F}_B = \mathcal{F}_B^0 \oplus \mathcal{F}_B^1, \tag{2.5}$$

where $\mathcal{F}_B^0$ and $\mathcal{F}_B^1$ denote the subspaces with even and odd numbers of generators ϕ_k, respectively. The same decomposition exists for $\mathcal{F}_B^*$.

There is a nondegenerate bilinear pairing $\mathcal{F}_B \times \mathcal{F}_B^* \to \mathbb{C}$, and the pairing of $\langle U| \in \mathcal{F}_B^*$ and $\big|V\big\rangle \in \mathcal{F}_B$ is denoted by $\big\langle U|V\big\rangle$ with

$$\big\langle 0|0\big\rangle = 1. \tag{2.6}$$

The *vacuum expectation values* of an element a of the Clifford algebra is a pairing of $\big\langle 0\big|$ and $a\big|0\big\rangle$, which is denoted by $\big\langle 0\big|a\big|0\big\rangle$. It is uniquely defined by the anticommutation relations (2.1), property (2.2), and the following relation:

$$\big\langle 0\big|\phi_0\big|0\big\rangle = 0. \tag{2.7}$$

In particular, if a is an odd element of the Clifford algebra, then $\big\langle 0\big|a\big|0\big\rangle = 0$. It is easy to see that the bases in (2.3) and (2.4) are orthogonal. Let us focus on the space $\mathcal{F}_B^0$ and its dual. The basis can be labelled by strict partitions $\lambda \in \mathrm{SP}$ in the following way:

$$\big|\lambda\big\rangle = \begin{cases} \phi_{\lambda_1}\phi_{\lambda_2}\ldots\phi_{\lambda_{\ell(\lambda)}}\big|0\big\rangle & \text{for } \ell(\lambda) = 0 \mod 2, \\ \sqrt{2}\phi_{\lambda_1}\phi_{\lambda_2}\ldots\phi_{\lambda_{\ell(\lambda)}}\phi_0\big|0\big\rangle & \text{for } \ell(\lambda) = 1 \mod 2, \end{cases} \tag{2.8}$$

and similarly for $\mathcal{F}_B^{0*}$. From the anticommutation relations we have

$$\left\langle\lambda|\mu\right\rangle = (-1)^{|\lambda|}\delta_{\lambda,\mu}. \tag{2.9}$$

It is easy to see that

$$\left\langle 0\middle|\phi_k\phi_m\middle|0\right\rangle = \delta_{k+m,0}H[m], \tag{2.10}$$

where

$$H[m] = \begin{cases} 0 & \text{for} \quad m < 0, \\ \frac{1}{2} & \text{for} \quad m = 0, \\ (-1)^m & \text{for} \quad m > 0. \end{cases} \tag{2.11}$$

Bilinear combinations of neutral fermions $\phi_k\phi_m$ satisfy the commutation relations of the Lie algebra B_∞. Let $\left(E_{i,j}\right)_{k,l} = \delta_{i,k}\delta_{j,l}$ be the standard basis of the matrix units $\left\{E_{i,j}\middle|i, j \in \mathbb{Z}\right\}$. Then $\phi_k\phi_{-m}$ corresponds [9] to

$$F_{k,m} = (-1)^m E_{k,m} - (-1)^k E_{-m,-k} \tag{2.12}$$

with the commutation relations

$$\begin{aligned} &\left[F_{a,b}, F_{c,d}\right] \\ &\quad= (-1)^b\delta_{b,c}F_{a,d} - (-1)^a\delta_{a+c,0}F_{-b,d} + (-1)^b\delta_{b+d,0}F_{c,-a} - (-1)^a\delta_{a,d}F_{c,b}. \end{aligned} \tag{2.13}$$

For the bilinear combinations of neutral fermions we introduce the *normal ordering* by

$$:\!\phi_k\phi_m\!:= \phi_k\phi_m - \left\langle 0\middle|\phi_k\phi_m\middle|0\right\rangle. \tag{2.14}$$

It is skewsymmetric

$$:\!\phi_k\phi_m\!:= -\ :\!\phi_m\phi_k\!:, \tag{2.15}$$

in particular, $:\!\phi_k\phi_k\!:=\ 0$. The normal ordered quadratic combinations of neutral fermions satisfy the commutation relations of a central extension of the algebra B_∞:

$$\begin{aligned} [\ :\!\phi_a\phi_b\!:, :\!\phi_c\phi_d\!:] = {} & (-1)^b\delta_{b+c,0}\ :\!\phi_a\phi_d\!: -(-1)^a\delta_{a+c,0}\ :\!\phi_b\phi_d\!: \\ & + (-1)^b\delta_{b+d,0}\ :\!\phi_c\phi_a\!: -(-)^a\delta_{a+d,0}\ :\!\phi_c\phi_b\!: \\ & + (\delta_{c,b}\delta_{a,d} - \delta_{a-c,0}\delta_{b-d,0})((-1)^a H[b] - (-1)^b H[a]), \end{aligned} \tag{2.16}$$

where $H[a]$ is given by (2.11). The operator $:\phi_k\phi_{-m}:$ corresponds to the projective representation of the Lie algebra B_∞, and will also be denoted by $\hat{F}_{k,m}$.

Let us consider the generating function

$$\phi(z) = \sum_{k\in\mathbb{Z}} \phi_k z^k. \tag{2.17}$$

It satisfies the anti-commutation relation

$$\{\phi(z), \phi(w)\} = \delta(z+w). \tag{2.18}$$

Here we introduce the delta-function

$$\delta(z-w) = \sum_{k\in\mathbb{Z}} \left(\frac{z}{w}\right)^k. \tag{2.19}$$

It satisfies

$$\delta(z-w) f(z) = \delta(z-w) f(w) \tag{2.20}$$

for any formal series $f(z) \in \mathbb{C}[[z, z^{-1}]]$ and can be represented as

$$2\delta(z+w) = \iota_{|z|>|w|} \frac{z-w}{z+w} - \iota_{|w|>|z|} \frac{z-w}{w+z}, \tag{2.21}$$

where $\iota_{|z|>|w|}$ is the operation of Laurent series expansion in the region $|z| > |w|$.

Quadratic combinations of the generating functions $\phi(z)$ generate a Lie algebra with the following commutation relations

$$\begin{aligned} [\phi_1(z_1)\phi(w_1), \phi(z_2)\phi(w_2)] &= \delta(w_1+z_2)\phi(z_1)\phi(w_2) - \delta(z_1+z_2)\phi(w_1)\phi(w_2) \\ &\quad + \delta(w_1+w_2)\phi(z_2)\phi(z_1) - \delta(z_1+w_2)\phi(z_2)\phi(w_1). \end{aligned} \tag{2.22}$$

For the normal ordered operator we have

$$\phi(z)\phi(w) = :\phi(z)\phi(w): + \frac{1}{2}\iota_{|z|>|w|} \frac{z-w}{z+w}. \tag{2.23}$$

2.2 Vertex operators

For $k \in \mathbb{Z}_{\text{odd}}$ we introduce *bosonic* operators

$$J_k = \frac{1}{2} \sum_{m\in\mathbb{Z}} (-1)^{m+1} :\phi_m \phi_{-m-k}: \tag{2.24}$$

satisfying a commutation relation of the Heisenberg algebra

$$[J_k, J_m] = \frac{k}{2}\delta_{k+m,0}. \tag{2.25}$$

From (2.2) we have:

$$J_m\big|0\big\rangle = 0, \quad \big\langle 0\big|J_{-m} = 0, \quad m > 0. \tag{2.26}$$

Let us consider the *vertex operator* for the BKP hierarchy introduced in [9],

$$\widehat{V}_B^{(1)}(z) = \exp\left(\sum_{k\in\mathbb{Z}_{\text{odd}}^+} z^k t_k\right)\exp\left(-2\sum_{k\in\mathbb{Z}_{\text{odd}}^+}\frac{1}{kz^k}\frac{\partial}{\partial t_k}\right). \tag{2.27}$$

These operators satisfy the anticommutation relation

$$\left\{\widehat{V}_B^{(1)}(z), \widehat{V}_B^{(1)}(w)\right\} = 2\delta(z+w) \tag{2.28}$$

similar to the relation (2.18).

It is convenient to introduce generating functions of the bosonic operators:

$$J_+(\mathbf{t}) = \sum_{k\in\mathbb{Z}_{\text{odd}}^+} t_k J_k, \qquad J_-(\mathbf{s}) = \sum_{k\in\mathbb{Z}_{\text{odd}}^+} s_k J_{-k}. \tag{2.29}$$

Then one has

$$\begin{aligned} \widehat{V}_B^{(1)}(z)\big\langle 0\big|e^{J_+(\mathbf{t})} &= 2\big\langle 0\big|\phi_0 e^{J_+(\mathbf{t})}\phi(z), \\ \widehat{V}_B^{(1)}(z)\big\langle 0\big|\phi_0 e^{J_+(\mathbf{t})} &= \big\langle 0\big|e^{J_+(\mathbf{t})}\phi(z). \end{aligned} \tag{2.30}$$

Let us consider a bilinear combination of the vertex operators

$$\widehat{Y}_B(z,w) = \frac{1}{2}\widehat{V}_B^{(1)}(z)\widehat{V}_B^{(1)}(w). \tag{2.31}$$

Using the anti-commutation relation (2.28) it is easy to show that the vertex operators $\widehat{Y}_B(z,w)$ satisfy a commutation relation equivalent to the relation (2.22) for the bilinear combinations $\phi(z)\phi(w)$:

$$\begin{aligned} \left[\widehat{Y}_B(z_1,w_1), \widehat{Y}_B(z_2,w_2)\right] &= \delta(w_1+z_2)\widehat{Y}_B(z_1,w_2) - \delta(z_1+z_2)\widehat{Y}_B(w_1,w_2) \\ &\quad + \delta(w_1+w_2)\widehat{Y}_B(z_2,z_1) - \delta(z_1+w_2)\widehat{Y}_B(z_2,w_1). \end{aligned} \tag{2.32}$$

It is also convenient to consider its regularized version, corresponding to $:\phi(z)\phi(-w):$

$$\widehat{V}_B^{(2)}(z,w) = \widehat{Y}_B(z,w) - \frac{1}{2}\iota_{|z|>|w|}\frac{z-w}{z+w}, \tag{2.33}$$

where for the second term we assume the series expansion in $|z| > |w|$. This expression has no pole at $z = -w$, moreover, it is antisymmetric with respect to the permutation of z and w. These vertex operators can be represented as

$$\widehat{V}_B^{(2)}(z,w) = \frac{1}{2}\frac{z-w}{z+w}\left(e^{\sum_{k\in\mathbb{Z}^+_{\mathrm{odd}}} t_k(z^k+w^k)} e^{-2\sum_{k\in\mathbb{Z}^+_{\mathrm{odd}}}\left(\frac{1}{kz^k}+\frac{1}{kw^k}\right)\frac{\partial}{\partial t_k}} - 1\right). \tag{2.34}$$

From (2.30) it follows that

$$\widehat{V}_B^{(2)}(z,w)\langle 0|e^{J_+(\mathbf{t})} = \langle 0|e^{J_+(\mathbf{t})} :\phi(z)\phi(w):. \tag{2.35}$$

2.3 Boson-fermion correspondence

For the neutral fermions the boson-fermion correspondence describes an isomorphism [22]

$$\sigma_B^i : \quad \mathcal{F}_B^i \simeq B^{(i)} = \mathbb{C}[[t_1, t_3, t_5 \dots]] \tag{2.36}$$

for $i = 0, 1$. Here

$$\sigma_B^i(|i\rangle) = 1, \tag{2.37}$$

where we introduce $|1\rangle = \sqrt{2}\phi_0|0\rangle$, and for both $i = 0, 1$ we have

$$\sigma_B^i J_{-k}(\sigma_B^i)^{-1} = \frac{k}{2}t_k, \quad \sigma_B^i J_k(\sigma_B^i)^{-1} = \frac{\partial}{\partial t_k} \tag{2.38}$$

for $k \in \mathbb{Z}^+_{\mathrm{odd}}$. The boson-fermion correspondence is given by

$$\sigma_B^i(|a\rangle) = \begin{cases} \langle 1|e^{J_+(\mathbf{t})}|a\rangle & \text{for} \quad |a\rangle \in \mathcal{F}_B^1, \\ \langle 0|e^{J_+(\mathbf{t})}|a\rangle & \text{for} \quad |a\rangle \in \mathcal{F}_B^0, \end{cases} \tag{2.39}$$

where $\langle 1| = \sqrt{2}\langle 0|\phi_0$. The boson-fermion correspondence between two different representations of the central extension of the B_∞ algebra is given by

$$\sigma_B^i :\phi(z)\phi(w): (\sigma_B^i)^{-1} = \widehat{V}_B^{(2)}(z,w). \tag{2.40}$$

Below we will work only with $\mathcal{F}_B^0$ component of the fermionic Fock space and its bosonic counterpart.

Relation between the Schur Q-functions and the BKP hierarchy, in particular, is described by the following result of You:

Theorem 2.1 ([22]) *For the states (2.8) the boson-fermion correspondence yields*

$$\sigma_B^0(\left|\lambda\right\rangle) = 2^{-\ell(\lambda)/2} Q_\lambda(\mathbf{t}/2). \tag{2.41}$$

Here Q_λ are the Schur Q-functions (see Section III.8 of [18] for definition and details).

It was shown by Date, Jimbo, Kashiwara, and Miwa [9] that for any *group element* of the central extension of the algebra B_∞,

$$G = \exp\left(\sum_{k,m\in\mathbb{Z}} a_{km} :\phi_k\phi_m:\right), \tag{2.42}$$

the bosonic image of the fermionic state $Ge^{J_-(\mathbf{s})}\left|0\right\rangle$ solves the 2-BKP hierarchy. Namely,

$$\tau(\mathbf{t},\mathbf{s}) = \left\langle 0\right| e^{J_+(\mathbf{t})} G e^{J_-(\mathbf{s})} \left|0\right\rangle \tag{2.43}$$

is a tau-function of 2-BKP hierarchy.

3 Hypergeometric tau-functions and weighted spin Hurwitz numbers

In this section we suggest a way to construct the weighed sums of the spin Hurwitz numbers which solve the 2-BKP hierarchy. There is a certain ambiguity associated to the choice of the weights, and we discuss two natural candidates for the role of the elementary weight functions.

3.1 Hypergeometric tau-functions of 2-BKP hierarchy

Following Orlov [20] we consider a set of parameters $T_n, n \in \mathbb{Z}$, such that $T_n = -T_{-n}$. In particular, $T_0 = 0$. Then

$$\sum_{k\in\mathbb{Z}}(-1)^k T_k :\phi_k\phi_{-k}:= 2\sum_{k\in\mathbb{Z}_+}(-1)^k T_k :\phi_k\phi_{-k}:= 2\sum_{k\in\mathbb{Z}_+}(-1)^k T_k \hat{F}_{k,k}. \tag{3.1}$$

Consider the group element

$$\mathcal{D} = \exp\left(\sum_{k\in\mathbb{Z}}(-1)^k T_k :\phi_k\phi_{-k}:\right), \tag{3.2}$$

then

$$\tau(\mathbf{t},\mathbf{s}) = \left\langle 0\right| e^{J_+(\mathbf{t})} \mathcal{D} e^{J_-(\mathbf{s})} \left|0\right\rangle \tag{3.3}$$

is a tau-function of the 2-component BKP hierarchy symmetric in the variables t_k and s_k. From Theorem 2.1 it follows that this tau-function has an equivalent description [20]

$$\tau(\mathbf{t},\mathbf{s}) = \sum_{\lambda\in \mathrm{SP}} 2^{-\ell(\lambda)} e^{2T_{\lambda_1}+\cdots+2T_{\lambda_{\ell(\lambda)}}} Q_\lambda(\mathbf{t}/2) Q_\lambda(\mathbf{s}/2). \tag{3.4}$$

Here SP is the set of all strict partitions including the empty one. These are the *hypergeometric tau-functions* of the 2-BKP hierarchy.

Consider $T(x)$, an odd function such that $T(k) = T_k$ for $k \in \mathbb{Z}$. For future applications it is natural to introduce the topological expansion parameter $\hbar$. Consider a new *even* function $\overline{\psi}(z)$ such that

$$\overline{\psi}(\hbar(z+1/2)) = T(z+1) - T(z). \tag{3.5}$$

We assume that $\overline{\psi}(z)$ is itself a series in $\hbar^2$, $\overline{\psi}(\hbar^2, z) = \sum_{d=0}^{\infty} \hbar^{2g} \psi_{2d}(z)$, where $\psi_{2d}(z)$ is an even formal power series in z, therefore T_k also depend on $\hbar$. The constant term of this series, $\psi_0 = \overline{\psi}(0,z)$, is also denoted by $\psi = \psi(z)$.

Remark 3.1 Our definition of the parameters T_k corresponds to the doubled parameters of [20] with the inverse sign.

3.2 Spin Hurwitz numbers

Spin Hurwitz numbers, which count the ramified coverings with sign coming from spin structure, were introduces by Eskin, Okounkov, and Pandharipande [11]. Using TQFT, Gunningham [14] found a combinatorial expression for all genera spin Hurwitz numbers, which uses the representation theory of Sergeev's group. In this section we recall this combinatorial expression. We address the reader to [11, 12, 14, 16, 17, 19] for the basic definitions and properties. Different authors use different conventions, our notation is consistent with that of [12].

For any set of variables or parameters r_k and any partition μ let us denote

$$r_\mu = \prod_{j=1}^{\ell(\mu)} r_{\mu_j}. \tag{3.6}$$

Let $\mathrm{OP}(d)$ and $\mathrm{SP}(d)$ be the sets of odd partitions and strict partitions of the size d respectively. Then the Schur Q-functions can be expanded as

$$Q_\lambda = 2^{\frac{\ell(\lambda)-\delta(\lambda)}{2}} \sum_{\mu\in \mathrm{OP}(|\lambda|)} \frac{\zeta_\mu^\lambda}{z_\mu} p_\mu \tag{3.7}$$

with the inverse relation

$$p_\mu = 2^{-\ell(\mu)} \sum_{\lambda\in \mathrm{SP}(|\mu|)} 2^{-\frac{\ell(\lambda)+\delta(\lambda)}{2}} \zeta_\mu^\lambda Q_\lambda. \tag{3.8}$$

Here $p_k = kt_k$ are the independent variables,

$$\delta(\mu) = \begin{cases} 0, & \text{for even } \ell(\mu) \\ 1, & \text{for odd } \ell(\mu) \end{cases} \tag{3.9}$$

and $z_\mu = \prod_k \mu(k)! k^{\mu(k)}$.

The characters of the Sergeev group ζ_μ^ρ satisfy the orthogonality relations

$$\sum_{\mu \in \mathrm{OP}(d)} 2^{-\ell(\mu)-\delta(\sigma)} \frac{\zeta_\mu^\rho \zeta_\mu^\sigma}{z_\mu} = \delta_{\rho,\sigma} \tag{3.10}$$

and

$$\sum_{\lambda \in \mathrm{SP}(d)} 2^{-\ell(\sigma)-\delta(\lambda)} \frac{\zeta_\sigma^\lambda \zeta_\rho^\lambda}{z_\sigma} = \delta_{\rho,\sigma}. \tag{3.11}$$

Let us also introduce the *central characters*

$$f_\mu^\lambda = \frac{2^d d!}{2^{\ell(\mu)} z_\mu \dim V^\lambda} \zeta_\mu^\lambda. \tag{3.12}$$

Here

$$\dim V^\lambda = \zeta_{1^d}^\lambda = 2^{\frac{\delta(\lambda)-\ell(\lambda)}{2}} d! Q_\lambda \Big|_{p_k=\delta_{k,1}} \tag{3.13}$$

is the dimension of the irreducible supermodule associated with the strict partition λ.

Let us consider the disconnected spin Hurwitz numbers for the $\mathbb{C}\mathrm{P}^1$ with the ramifications at k branch points given by odd partitions $\mu_1, \ldots, \mu_k$ with $|\mu_j| = d$. The Gunningham formula [14, 16] describes them in terms of the central characters of the Sergeev group:

$$H_d^\theta(\mu_1, \ldots, \mu_k) = 2^{-d-\sum_{i=1}^k \ell^*(\mu_i)/2} \sum_{\lambda \in \mathrm{SP}(d)} 2^{-\delta(\lambda)} \left(\frac{\dim V^\lambda}{d!} \right)^2 \prod_{j=1}^k f_{\mu_j}^\lambda, \tag{3.14}$$

where $\ell^*(\mu) = |\mu| - \ell(\mu)$ is the *colength* of the partition μ.

3.3 From spin Hurwitz numbers to 2-BKP hierarchy

Let us single out two of the k partitions and denote them by μ and ν. Using Eq. (3.12) we can rewrite the spin Hurwitz numbers (3.14) as follows

$$H_d^\theta(\mu_1,\ldots,\mu_{k-2},\mu,\nu)=2^{-\frac{1}{2}(\ell(\mu)+\ell(\nu)+\sum_{i=1}^{k-2}\ell^*(\mu_i))}\sum_{\lambda\in \mathrm{SP}(d)}2^{-\delta(\lambda)}\frac{\zeta_\mu^\lambda}{z_\mu}\frac{\zeta_\nu^\lambda}{z_\nu}\prod_{j=1}^{k-2}f_{\mu_j}^\lambda. \tag{3.15}$$

Let us introduce $k-2$ families of weights $r_m^{(j)}$ for $1\le j\le k-2$, $m\in\mathbb{Z}_+$, associated with $k-2$ branch points. Then for the spin Hurwitz numbers (3.15) we introduce their weighted combinations

$$H_{d,\mathbf{r}}^\theta(\nu,\mu)=\sum_{\mu_1,\ldots,\mu_{k-2}\in \mathrm{OP}(d)}H_d^\theta(\mu_1,\ldots,\mu_{k-2},\mu,\nu)R_{\mu_1}(r^{(1)})\ldots R_{\mu_{k-2}}(r^{(k-2)}), \tag{3.16}$$

where

$$R_\mu(r)=\hbar^{-\ell^*(\mu)}2^{-\ell(\mu)/2}\sum_{\sigma\in \mathrm{SP}(d)}\frac{\dim V^\sigma}{d!2^{d/2}}2^{-\delta(\sigma)}\zeta_\mu^\sigma r_\sigma. \tag{3.17}$$

Let us stress that the trivial ramifications $\mu_j=1^d$ are allowed in the summation. For the empty partition we put $R_\emptyset=1$. If we compare this expression with the decomposition of the functions $p_\mu(Q_\sigma)$ in the basis of Schur Q-functions (3.8), then using Eq. (3.13) we get

$$R_\mu(r)=\hbar^{-\ell^*(\mu)}2^{-\ell^*(\mu)/2}p_\mu(Q_\sigma(\delta_{k,1})r_\sigma). \tag{3.18}$$

In particular,

$$\begin{aligned}
R_{[1]}(r)&=\frac{1}{2}Q_{[1]}(\delta_{k,1})r_1=r_1,\\
R_{[1,1]}(r)&=\frac{1}{2}Q_{[2]}(\delta_{k,1})r_2=r_2,\\
R_{[3]}(r)&=(2\hbar^2)^{-1}\left(\frac{1}{2}Q_{[3]}(\delta_{k,1})r_3-\frac{1}{2}Q_{[2,1]}(\delta_{k,1})r_2r_1\right)=\frac{1}{3\hbar^2}(r_3-r_2r_1),\\
R_{[1,1,1]}(r)&=\frac{1}{2}Q_{[3]}(\delta_{k,1})r_3+\frac{1}{4}Q_{[2,1]}(\delta_{k,1})r_2r_1=\frac{1}{3}(2r_3+r_2r_1).
\end{aligned} \tag{3.19}$$

Definition (3.17) is justified by the following observation: from the orthogonality relation (3.10) it follows that

$$\sum_{\mu\in \mathrm{OP}(d)}2^{-\ell^*(\mu)/2}f_\mu^\lambda R_\mu(r)=r_\lambda. \tag{3.20}$$

Therefore

$$H^{\theta}_{d,\mathbf{r}}(\nu,\mu)=2^{-\frac{\ell(\mu)+\ell(\nu)}{2}}\sum_{\lambda\in \mathrm{SP}(d)}2^{-\delta(\lambda)}\frac{\zeta^{\lambda}_{\mu}}{z_\mu}\frac{\zeta^{\lambda}_{\nu}}{z_\nu}\prod_{j=1}^{k-2}r_\lambda^{(j)}. \tag{3.21}$$

By the Riemann–Hurwitz formula

$$2-2g=\ell(\mu)+\ell(\nu)-\sum_{i=1}^{k-2}\ell^*(\mu_i), \tag{3.22}$$

where g is the genus of the covering curve. Consider the following generating function

$$\tau(\mathbf{t},\mathbf{s})=\sum_{d=0}^{\infty}\sum_{\mu,\nu\in \mathrm{OP}(d)}\hbar^{2g-2+\ell(\mu)+\ell(\nu)}2^{-\frac{\ell(\mu)+\ell(\nu)}{2}}H^{\theta}_{d,\mathbf{r}}(\nu,\mu)\prod_{j=1}^{\ell(\mu)}\mu_j\prod_{j=1}^{\ell(\nu)}\nu_j t_\mu s_\nu. \tag{3.23}$$

Then for any choice of parameters $r_k^{(j)}$ from (3.21) we have

Theorem 3.1 *The generating function $\tau(\mathbf{t},\mathbf{s})$ is a hypergeometric tau-function of the 2-BKP hierarchy*

$$\tau(\mathbf{t},\mathbf{s})=\sum_{\lambda\in \mathrm{SP}}\frac{Q_\lambda(\mathbf{t}/2)Q_\lambda(\mathbf{s}/2)}{2^{\ell(\mu)}}\prod_{j=1}^{k-2}r_\lambda^{(j)}. \tag{3.24}$$

This tau-function can be identified with (3.4) if one puts

$$T_m=\frac{1}{2}\sum_{j=1}^{k-2}\log r_m^{(j)}. \tag{3.25}$$

Similarly to the case of ordinary Hurwitz numbers, we can consider the limit when the maximal number of the branch points k tends to infinity.

3.4 Weighted spin Hurwitz numbers

In the previous section we have constructed the weighted sums of the spin Hurwitz numbers that lead to the tau-functions of the 2-BKP hierarchy. While working with arbitrary parameters $r_k^{(j)}$ allow us to trace more information about the spin Hurwitz numbers from the properties of the tau-function, similarly to the case of the ordinary Hurwitz numbers [2, 13, 23] we would like to introduce the distinguished weights, parametrized by one parameter c_j, $j=1,\ldots,k-2$ for each of $k-2$ points.

By analogy with the ordinary weighted Hurwitz numbers, see [13] and, more specifically, in [2, Equation (3.1)], one would tend to put $R_\mu(r^{(j)})=c_j^{\ell^*(\mu)}$. For this choice the weighted spin Hurwitz numbers (3.16) would be independent of $\hbar$. However, it is easy to see that R_μ should depend on $\hbar$-this is clear from (3.19). Therefore, for the combinations of the spin Hurwitz numbers defined by (3.16) we need some "completion" of the partitions for the rational weight functions,

$$R_\mu(r^{(j)}) = \sum_{k=0}^{\infty} R_\mu^{(k)} \hbar^{2k}, \tag{3.26}$$

a new effect of spin Hurwitz numbers which is absent in the theory of ordinary weighted Hurwitz numbers.

Let us consider the generating function (3.24) for the case with the maximal number of the branch points $k = 3$. We claim that this tau-function can be considered as a generating function of a spin version of dessins d'enfants. We also put $s_k = \delta_{k,1}\hbar^{-1}$, therefore the non-trivial branching is allowed only at two points. Then the tau-function (3.24) reduces to

$$\tau(\mathbf{t}) = \sum_{\lambda \in \mathrm{SP}} \frac{Q_\lambda(\mathbf{t}/2) Q_\lambda(\delta_{k,1})}{2^{\ell(\mu)+|\lambda|}\, \hbar^{|\lambda|}} r_\lambda, \tag{3.27}$$

where $r_\lambda = r_\lambda^{(1)}$. From the orthogonality relation (3.11) and Eq. (3.23) it follows that

$$\tau(\mathbf{t}) = \sum_{\mu \in \mathrm{OP}} \hbar^{-\ell(\mu)} 2^{-\frac{|\mu|+\ell(\mu)}{2}} \prod_{j=1}^{\ell(\mu)} \mu_j t_\mu \frac{R_\mu(r)}{z_\mu}. \tag{3.28}$$

To relate R_μ to $\overline{\psi}(z)$ we use the results of Sect. 4 below. From Eq. (3.28) it follows that the coefficients $R_\mu(r)$'s are proportional to the coefficients of the correlation functions $W_n^\bullet$, namely

$$W_n^\bullet = \hbar^{-n} \sum_{\mu \in \mathrm{OP}, \ell(\mu)=n} 2^{-\frac{|\mu|+n}{2}} R_\mu(r) \sum_{\sigma \in S_n} X_{\sigma(1)}^{\mu_1} \dots X_{\sigma(n)}^{\mu_n}. \tag{3.29}$$

If we require $R_\mu^{(0)} = c^{\ell^*(\mu)}$, then for $n = 1$ the leading term of Eq. (3.29) reduces to

$$W_{0,1}(x) = \frac{1}{2} \sum_{k \in \mathbb{Z}_{\mathrm{odd}}^+} \left(\frac{c}{\sqrt{2}}\right)^{k-1} X^k. \tag{3.30}$$

After identification of this expression with the general expression for the correlation function given by Proposition 4.4 we find $\psi(z)$,

$$\psi(z) = \frac{1}{2} \log \frac{1 + \sqrt{1 + 2c^2 z^2}}{2}. \tag{3.31}$$

One can identify it with $\overline{\psi}$, however, it is also possible to consider the $\hbar$-deformations.

Another possibility is to consider

$$\overline{\psi}(z) = \frac{1}{2}\log\left(1+\frac{c^2z^2}{2}\right) \tag{3.32}$$

The associated functions R_μ, (3.19) are rational functions of c,

$$\begin{aligned}
R_{[1]}(r) &= 1+\frac{1}{8}c^2\hbar^2,\\
R_{[1,1]}(r) &= \left(1+\frac{1}{8}c^2\hbar^2\right)\left(1+\frac{9}{8}c^2\hbar^2\right),\\
R_{[3]}(r) &= c^2\left(1+\frac{1}{8}c^2\hbar^2\right)\left(1+\frac{9}{8}c^2\hbar^2\right),\\
R_{[1,1,1]}(r) &= \left(1+\frac{1}{8}c^2\hbar^2\right)\left(1+\frac{9}{8}c^2\hbar^2\right)\left(1+\frac{17}{8}c^2\hbar^2\right).
\end{aligned} \tag{3.33}$$

For $c = 1$ the generating function $\tau(\mathbf{t})$ for this choice of parametrization can be identified with the generalized BGW tau-function [3],

$$\tau_{BGW}(\mathbf{t}/2)\big|_{N^2\mapsto -\frac{2}{\hbar^2}} = \tau(\mathbf{t},\delta_{k,1}\hbar^{-1}). \tag{3.34}$$

3.5 From BKP to KP

It is well known that the solutions of the BKP hierarchy are related to the solutions of the KP hierarchy for the particular choice of the variables [9]. Following [20], in this section we consider this relation for the hypergeometric tau-functions of both hierarchies. Namely, we relate any hypergeometric 2-BKP tau-function (3.4) to a hypergeometric tau-function of the 2-KP hierarchy.

Let us consider a 2-component system of neutral fermions $\phi_k^{(a)}$, $a = 1, 2$, satisfying the anti-commutation relations

$$\left\{\phi_j^{(k)},\phi_l^{(m)}\right\} = (-1)^j\delta_{k,m}\delta_{j+l,0}. \tag{3.35}$$

Following [15] we can relate them to the *charged free fermions*

$$\phi_j^{(1)} = \frac{\psi_j+(-1)^j\psi_{-j}^*}{\sqrt{2}}, \qquad \phi_j^{(2)} = \sqrt{-1}\,\frac{\psi_j-(-1)^j\psi_{-j}^*}{\sqrt{2}}. \tag{3.36}$$

for $j\in\mathbb{Z}$. We immediately have

$$\psi_j\psi_j^*+\psi_{-j}^*\psi_{-j} = (-1)^j\left(\phi_j^{(1)}\phi_{-j}^{(1)}+\phi_j^{(1)}\phi_{-j}^{(1)}\right). \tag{3.37}$$

Consider the bosonic operators for the charged fermions

$$J_k^{\mathrm{KP}} = \sum_{j\in\mathbb{Z}} :\psi_j \psi^*_{j+k}: . \tag{3.38}$$

Then for odd k we have

$$J_k^{\mathrm{KP}} = J_k^{(1)} + J_k^{(2)}, \tag{3.39}$$

where the bosonic operators $J_k^{(j)}$ are given in terms of the corresponding neutral fermions by Eq.(2.24).

Let us consider the hypergeometric 2-KP tau-function

$$\tau_{\mathrm{KP}}(\mathbf{t},\mathbf{s}) = \left\langle 0 \middle| e^{J_+^{\mathrm{KP}}(\mathbf{t})} e^{2\sum_{j=1}^\infty T_j (:\psi_j\psi_j^*: - :\psi_{-j}^*\psi_{-j}:)} e^{J_-^{\mathrm{KP}}(\mathbf{s})} \middle| 0 \right\rangle, \tag{3.40}$$

where T_k are some parameters and $J_\pm^{\mathrm{KP}}(\mathbf{t}) = \sum_{k=1}^\infty t_k J_{\pm k}^{\mathrm{KP}}$. Note that the group element in (3.40) is not the most general diagonal group element. However, for this choice of the the group element we have a simple relation between this tau-function and a tau-function of the 2-BKP hierarchy. If all even time variables vanish, $t_{2k} = s_{2k} = 0$ for $k \in \mathbb{Z}_+$, then in terms of the neutral fermions we have

$$\tau_{\mathrm{KP}}(\mathbf{t},\mathbf{s})\big|_{t_{2k}=s_{2k}=0} = \langle 0| e^{J_+^{(1)}(\mathbf{t})+J_+^{(2)}(\mathbf{t})} e^{2\sum_{j=1}^\infty (-1)^j T_j (:\phi_j^{(1)}\phi_{-j}^{(1)}:+:\phi_j^{(2)}\phi_{-j}^{(2)}:)} e^{J_-^{(1)}(\mathbf{s})+J_-^{(2)}(\mathbf{s})} |0\rangle \tag{3.41}$$

Therefore [20]

$$\tau_{\mathrm{KP}}(\mathbf{t},\mathbf{s})\big|_{t_{2k}=s_{2k}=0} = \tau(\mathbf{t},\mathbf{s})^2, \tag{3.42}$$

where

$$\tau(\mathbf{t},\mathbf{s}) = \left\langle 0 \middle| e^{J_+(\mathbf{t})} e^{2\sum_{j=1}^\infty (-1)^j T_j :\phi_j\phi_{-j}:} e^{J_-(\mathbf{s})} \middle| 0 \right\rangle \tag{3.43}$$

is a hypergeometric tau-function of 2-BKP hierarchy (3.4). By definition, it depends only on odd times t_{2k+1} and s_{2k+1}.

Let us compare the expansions of the tau-functions τ_{KP} and τ in terms of the corresponding sets of the Schur functions. For the hypergeometric 2-KP tau-function (3.40) one has

$$\tau_{KP}(\mathbf{t},\mathbf{s}) = \sum_\lambda e^{2\sum_{(i,j)\in\lambda} \overline{\psi}(\hbar(j-i-1/2))} s_\lambda(\mathbf{t}) s_\lambda(\mathbf{s}), \tag{3.44}$$

where s_λ are the ordinary Schur functions and the sum runs over all partitions. These tau-functions are generating functions of the ordinary weighted Hurwitz numbers. For the 2-component BKP tau-function (3.43) we have

$$\tau(\mathbf{t},\mathbf{s}) = \sum_{\lambda\in \mathrm{SP}} e^{2\sum_{(i,j)\in\lambda}\overline{\psi}(\hbar(j-1/2))} \frac{Q_\lambda(\mathbf{t}/2)Q_\lambda(\mathbf{s}/2)}{2^{\ell(\lambda)}}. \tag{3.45}$$

We see that for any hypergeometric tau-function of 2-BKP there exist a hypergeometric tau-function of 2-KP satisfying (3.42). It is easy to see that such 2-KP tau-function is not unique. Let λ' denotes the transpose partition of λ. Then, as it follows i.e. from the Giambelli formula,

$$s_\lambda(\mathbf{t})\big|_{t_{2k}=0} = s_{\lambda'}(\mathbf{t})\big|_{t_{2k}=0} \tag{3.46}$$

and for the Eq. (3.44) we have

$$\begin{aligned}\tau_{KP}(\mathbf{t},\mathbf{s})\big|_{t_{2k}=s_{2k}=0} &= \sum_{\lambda} e^{2\sum_{(i,j)\in\lambda}\overline{\psi}(\hbar(j-i-1/2))} s_{\lambda'}(\mathbf{t})s_{\lambda'}(\mathbf{s})\big|_{t_{2k}=s_{2k}=0}\\ &= \sum_{\lambda} e^{2\sum_{(i,j)\in\lambda'}\overline{\psi}(\hbar(j-i-1/2))} s_{\lambda}(\mathbf{t})s_{\lambda}(\mathbf{s})\big|_{t_{2k}=s_{2k}=0}\\ &= \sum_{\lambda} e^{2\sum_{(i,j)\in\lambda}\overline{\psi}(\hbar(i-j-1/2))} s_{\lambda}(\mathbf{t})s_{\lambda}(\mathbf{s})\big|_{t_{2k}=s_{2k}=0}. \end{aligned}\tag{3.47}$$

Therefore, (3.42) is also satisfied for the tau-function (3.44) with $\overline{\psi}(z)$ substituted by $\overline{\psi}(-z-\hbar)$.

Hence, we can relate any generating function of the spin Hurwitz numbers (3.45) to the generating function of the ordinary Hurwitz numbers (3.44). Moreover, we have at least two different hypergeometric tau-functions of the 2-KP hierarchy, corresponding to a given hypergeometric tau-function of the 2-BKP hierarchy. We expect that this identification should lead to a non-trivial relation between spin and ordinary Hurwitz numbers.

Let us consider a few examples. If $T(x) = ax$, then $\overline{\psi}(z) = a$ is a constant, and the tau-function of the 2-BKP hierarchy is very simple

$$\begin{aligned}\tau(\mathbf{t},\mathbf{s}) &= \sum_{\lambda\in \mathrm{SP}} e^{2a|\lambda|}\frac{Q_\lambda(\mathbf{t}/2)Q_\lambda(\mathbf{s}/2)}{2^{\ell(\lambda)}}\\ &= \exp\left(a\sum_{k\in\mathbb{Z}^+_{\mathrm{odd}}} k t_k s_k\right).\end{aligned}\tag{3.48}$$

More complicated example corresponds to $T(x) = \frac{bx^3}{3} + ax$ for some a and b. It is associated with to $\overline{\psi}(z) = \frac{b}{2\hbar^2}z^2 + \frac{b}{12} + a$. The identity (3.42) for this case with $a = \frac{2}{3}b$ and $\hbar = 1$ was proven by Lee [17]. On the KP side the generating function, considered by Lee, is given by the last line of (3.47).

4 Diagonal group element and *n*-point functions

In this section we prove explicit closed algebraic formulas for the correlation functions $W_{g,n}$.

4.1 Operators $\mathbb{J}_k$

For the diagonal group element (3.2) introduce the operators

$$\mathbb{J}_k = \mathcal{D}^{-1} J_k \mathcal{D}, \tag{4.1}$$

Our first goal is to provide a few explicit formulas for these operators. To this end, we introduce a fermionic operator

$$\begin{aligned}\mathcal{E}(u,a) &= :\phi(a^{-1}e^{u/2})\phi(-a^{-1}e^{-u/2}): \\ &= \sum_{k,m\in\mathbb{Z}} (-1)^m :\phi_{m-k}\phi_{-m}: a^k e^{(m-k/2)u} \\ &= \sum_{k,m\in\mathbb{Z}} (-1)^m \hat{F}_{m-k,m} a^k e^{(m-k/2)u}.\end{aligned} \tag{4.2}$$

Let

$$\mathcal{S}(z) = \frac{e^{z/2}-e^{-z/2}}{z}. \tag{4.3}$$

Then in terms of the bosonic operators (2.24) the operator $\mathcal{E}(u,a)$ is a reparametrization of the operator $\widehat{V}_B^{(2)}$ given by (2.34), and can be represented as

$$\mathcal{E}(u,a) = \frac{1}{2}\frac{1+e^{-u}}{1-e^{-u}}\left(\exp\left(2u\sum_{k\in\mathbb{Z}^+_{\text{odd}}} a^{-k}\mathcal{S}(ku)J_{-k}\right)\exp\left(2u\sum_{k\in\mathbb{Z}^+_{\text{odd}}} a^{k}\mathcal{S}(ku)J_{k}\right)-1\right). \tag{4.4}$$

Proposition 4.1 *The operators $\mathbb{J}_k$ belong to the image of the projective representation of B_∞ for all $k \in \mathbb{Z}_{\text{odd}}$*

$$\mathbb{J}_k = \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^{m+1} e^{T_{k+m}-T_{-k-m}-T_m+T_{-m}}\hat{F}_{m,m+k} \tag{4.5}$$

and

$$\mathbb{J}_k = \frac{1}{2}[a^k]e^{2T(\partial_u+1/2a\partial_a)-2T(\partial_u-1/2a\partial_a)}\mathcal{E}(u,a)\big|_{u=0}. \tag{4.6}$$

Proof From (2.13) we have

$$\left[\sum_{a\in\mathbb{Z}}(-1)^a T_a F_{a,a},\, F_{k,m}\right] = (T_k - T_{-k} - T_m + T_{-m})F_{k,m}. \tag{4.7}$$

Hence

$$\mathcal{D}^{-1}\hat{F}_{k,m}\mathcal{D} = e^{-T_k+T_{-k}+T_m-T_{-m}}\hat{F}_{k,m} \tag{4.8}$$

and

$$\mathbb{J}_k = \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^{m+1}e^{T_{k+m}-T_{-k-m}-T_m+T_{-m}}\hat{F}_{m,m+k}, \tag{4.9}$$

or, equivalently

$$\mathbb{J}_k = \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^{m+k+1}e^{T_m-T_{-m}-T_{m-k}+T_{k-m}}\hat{F}_{m-k,m}. \tag{4.10}$$

Comparing it to (4.2) we get

$$\mathbb{J}_k = \frac{1}{2}[a^k]e^{T(\partial_u+1/2a\partial_a)-T(-\partial_u-1/2a\partial_a)-T(\partial_u-1/2a\partial_a)+T(-\partial_u+1/2a\partial_a)}\mathcal{E}(u,a)\big|_{u=0}. \tag{4.11}$$

□

4.2 Topological expansion

In terms of $\overline{\psi}$ we can rewrite the formula for $\mathbb{J}_k$, $k\in\mathbb{Z}^+_{\text{odd}}$, as

$$\begin{aligned}\mathbb{J}_k &= \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^{m+1}e^{T_{k+m}-T_{-k-m}-T_m+T_{-m}}\hat{F}_{m,m+k}\\ &= \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^{m+1}\exp\left(2\sum_{i=1}^{k}\overline{\psi}(\hbar^2,\hbar(m-\frac{1}{2}+i))\right)\hat{F}_{m,m+k}\end{aligned} \tag{4.12}$$

Let $\phi_k(y) := \exp\left(2\sum_{i=1}^{k}\overline{\psi}(\hbar^2, y+\hbar(-\frac{k}{2}-\frac{1}{2}+i))\right) = \exp\left(2k\frac{\mathcal{S}(k\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}(\hbar^2,y)\right)$. Then

$$\mathbb{J}_k = \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^{m+1}\phi_k(\hbar(m+\frac{k}{2}))\hat{F}_{m,m+k}$$

$$
\begin{aligned}
&= -\frac{1}{2}\sum_{r=0}^{\infty} \partial_y^r \phi_k(y)\Big|_{y=0} \cdot [u^r a^k]\mathcal{E}(\hbar u, -a) \\
&= -\frac{1}{4}\sum_{r=0}^{\infty} \partial_y^r \exp\left(2k\frac{\mathcal{S}(k\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}(\hbar^2, y)\right)\Big|_{y=0} \\
&\quad [u^r a^k]\frac{1+e^{-\hbar u}}{1-e^{-\hbar u}} \exp\left(2\hbar u \sum_{l\in\mathbb{Z}^+_{\text{odd}}} (-a)^{-l}\mathcal{S}(l\hbar u)J_{-l}\right) \\
&\quad \exp\left(2\hbar u \sum_{l\in\mathbb{Z}^+_{\text{odd}}} (-a)^{l}\mathcal{S}(l\hbar u)J_{l}\right) \\
&= \frac{1}{4}\sum_{r=0}^{\infty} \partial_y^r \exp\left(2k\frac{\mathcal{S}(k\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}(\hbar^2, y)\right)\Big|_{y=0} \\
&\quad [u^r a^k]\frac{e^{\hbar u/2}+e^{-\hbar u/2}}{u\hbar\mathcal{S}(u\hbar)} \exp\left(2\hbar u \sum_{l\in\mathbb{Z}^+_{\text{odd}}} a^{-l}\mathcal{S}(l\hbar u)J_{-l}\right) \\
&\quad \exp\left(2\hbar u \sum_{l\in\mathbb{Z}^+_{\text{odd}}} a^{l}\mathcal{S}(l\hbar u)J_{l}\right).
\end{aligned} \tag{4.13}
$$

We also consider an arbitrary series $\overline{y}(\hbar^2, z) = \sum_{d=0}^{\infty} \hbar^{2d} y_d(z)$, where each $y_d(z)$ is an odd formal power series in z. The constant term of this series, $y_0 = \overline{y}(0, z)$, is also denoted by $y = y(z)$. The prime object of our interest in this and the subsequent sections are the $\hbar$-expansions of the (disconnected) n-point functions

$$
H_n^{\bullet} = \sum_{k_1,\dots,k_n\in\mathbb{Z}^+_{\text{odd}}} \frac{\partial^n \tau(\mathbf{t}, \hbar^{-1}\mathbf{s})}{\partial t_{k_1}\dots\partial t_{k_1}}\Big|_{\mathbf{t}=\mathbf{0}} \frac{X_1^{k_1}}{k_1}\cdots\frac{X_1^{k_n}}{k_n} \tag{4.14}
$$

and

$$
W_n^{\bullet} = D_1\cdots D_n H_n^{\bullet}, \tag{4.15}
$$

where $D_i := X_i\partial_{X_i}$. Then from the definition of the operators $\mathbb{J}_m$ we have

$$
H_n^{\bullet} = \sum_{m_1,\dots,m_n\in\mathbb{Z}^+_{\text{odd}}}^{\infty} \frac{X_1^{m_1}\dots X_n^{m_n}}{m_1\dots m_n}\big\langle 0\big|\mathbb{J}_{m_1}\dots\mathbb{J}_{m_n} e^{\sum_{d=0}^{\infty}\hbar^{2d}\sum_{k\in\mathbb{Z}^+_{\text{odd}}}\frac{J_{-k}}{\hbar k}[z^k]y_d(z)}\big|0\big\rangle \tag{4.16}
$$

and

$$W_n^\bullet = \sum_{m_1,\dots,m_n \in \mathbb{Z}^+_{\mathrm{odd}}} X_1^{m_1} \dots X_n^{m_n} \big\langle 0 \big| \mathbb{J}_{m_1} \dots \mathbb{J}_{m_n} e^{\sum_{d=0}^\infty \hbar^{2d} \sum_{k\in\mathbb{Z}^+_{\mathrm{odd}}} \frac{J_{-k}}{\hbar k}[z^k] y_d(z)} \big| 0 \big\rangle. \tag{4.17}$$

Using the inclusion–exclusion formulas, we define the *connected* n-point functions H_n and W_n, $n \geq 1$, and they expand in $\hbar$ as

$$H_n = \sum_{g=0}^\infty H_{g,n} \hbar^{2g-2+n}; \qquad W_n = \sum_{g=0}^\infty W_{g,n} \hbar^{2g-2+n}. \tag{4.18}$$

4.3 Preliminary formulas for $H_n^\bullet$ and $W_n^\bullet$

Denote

$$B(z,w) := \frac{zw}{(z-w)^2} + \frac{zw}{(z+w)^2}. \tag{4.19}$$

Proposition 4.2 *We have the following formula for $H_n^\bullet$:*

$$\begin{aligned} H_n^\bullet = &\sum_{m_1,\dots,m_n \in \mathbb{Z}^+_{\mathrm{odd}}} \prod_{i=1}^n \frac{1}{m_i} X_i^{m_i} \sum_{r_1,\dots,r_n=0}^\infty \prod_{i=1}^n \partial_y^{r_i} \exp\left(2m_i \frac{\mathcal{S}(m_i \hbar \partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}\right)\Big|_{y=0} \\ &[\prod_{i=1}^n u_i^{r_i} z_i^{m_i}] \prod_{i=1}^n \frac{e^{\frac{\hbar u_i}{2}} + e^{-\frac{\hbar u_i}{2}}}{4u_i \hbar \mathcal{S}(u_i\hbar)} e^{u_i \mathcal{S}(\hbar u_i z_i \partial_{z_i}) \overline{y}_i} \\ &\prod_{1\leq i<j\leq n} \left(e^{\hbar^2 u_i u_j \mathcal{S}(\hbar u_i z_i \partial_{z_i}) \mathcal{S}(\hbar u_j z_j \partial_{z_j}) B(z_i,z_j)} - 1\right), \end{aligned} \tag{4.20}$$

where $\overline{\psi} = \overline{\psi}(\hbar^2, y)$ and $\overline{y}_i = \overline{y}(\hbar^2, z_i)$. An analogous formula for $W_n^\bullet$ is obtained by replacing $\prod_{i=1}^n \frac{1}{m_i} X_i^{m_i}$ in Equation (4.20) by $\prod_{i=1}^n X_i^{m_i}$.

Proof This formula should be understood as an expansion in the sector $|z_1| \ll |z_2| \ll \cdots \ll |z_n| \ll 1$, and it comes directly from the commutation of the operators $\mathbb{J}_k$ given in Equation (4.13), once one observes that

$$\big[J_+(z), J_-(w)\big] = \frac{1}{4} B(z,w). \tag{4.21}$$

We refer also to an argument in [5, Section 3.2], which does exactly the same in a bit different situation. ◻

4.4 Closed algebraic formula for $W_{g,n}$

We use a change of variables of exactly the type as in [1], namely,

$$X = ze^{-2\psi(y(z))}. \tag{4.22}$$

Define $D := X\partial_X$ and define Q by $D = Q^{-1}z\partial_z$. In the case we have variables $X_1,\dots,X_n$, we define z_i by $X_i := X(z_i)$, and furthermore we use the notation $D_i := X_i\partial_{X_i}$, $Q_i := z_i/X_i \cdot dX_i/dz_i$, $\overline{y}_i = \overline{y}(\hbar^2, z_i)$, $y_i = y(z_i)$, $\overline{\psi}_i = \psi(\hbar^2, y_i)$, and $\psi_i = \psi(y_i)$.

Theorem 4.1 *For $g \geq 0$, $n \geq 2$, $2g - 2 + n > 0$, we have:*

$$\begin{aligned} W_{g,n} = [\hbar^{2g-2+n}] \sum_{\substack{j_1,\dots,j_n,\\ r_1,\dots,r_n=0}}^{\infty} &\left[\prod_{i=1}^{n} D_i^{j_i}[t_i^{j_i}]\frac{1}{Q_i}e^{-2t_i\psi_i}\partial_{y_i}^{r_i}e^{2t_i\frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i}[u_i^{r_i}]\right] \\ &\prod_{i=1}^{n}\frac{e^{\frac{\hbar u_i}{2}}+e^{-\frac{\hbar u_i}{2}}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{u_i\left(\mathcal{S}(\hbar u_i z_i\partial_{z_i})\overline{y}_i - y_i\right)}\sum_{\gamma\in\Gamma_n} \\ &\prod_{(v_k,v_\ell)\in E_\gamma}\left(e^{\hbar^2 u_k u_\ell \mathcal{S}(\hbar u_k z_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)} - 1\right) \end{aligned} \tag{4.23}$$

Here Γ_n is the set of all connected simple graphs on n vertices $v_1,\dots,v_n$, and E_γ is the set of edges of γ.

Remark 4.1 It is an explicit closed algebraic formula of the same type as in [4, 5]. In particular the sum over $j_1,\dots,j_n,r_1,\dots,r_n$ is finite for every (g,n).

Remark 4.2 Note that y_i and hence ∂_{y_i} are odd in z_i. Note also that $\overline{\psi}_i$ and hence $\frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i$ are even in z_i. Note also that D_i and Q_i are even in z_i. Note also that in the second line the coefficient of $u_i^{r_i}$ for odd r_i is even in z_i and the coefficient of $u_i^{r_i}$ for even r_i is odd in z_i. These observations imply that the right hand side of (4.23) is necessarily odd in $z_1,\dots,z_n$.

Remark 4.3 Note that the structure of the formula suggests that there might be non-trivial poles along the diagonals $z_i = z_j$ and antidiagonals $z_i = -z_j$, but in fact the statement of the theorem in particular implies that these polar parts cancel and the resulting formula is non-singular at the diagonals and antidiagonals. Cf. a discussion in [5, Remark 1.3 and Corollary 4.10].

Proof of Theorem 4.1 Recall the formula for $W_n^\bullet$ in Proposition 4.2. Passing to the connected n-point functions W_n via inclusion–exclusion formula we replace

$$\prod_{1\leq i<j\leq n}\left(e^{\hbar^2 u_i u_j\mathcal{S}(\hbar u_i z_i\partial_{z_i})\mathcal{S}(\hbar u_j z_j\partial_{z_j})B(z_i,z_j)} - 1\right) \tag{4.24}$$

with

$$\sum_{\gamma\in\Gamma_n}\prod_{(v_k,v_\ell)\in E_\gamma}\left(e^{\hbar^2u_ku_\ell\mathcal{S}(\hbar u_kz_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)}-1\right). \tag{4.25}$$

Using the series expansion in $u_1,\dots,u_n$ (cf. [5, Lemma 4.5]), which is applicable only if $n\ge 2$ (hence the restriction on n in the statement of the theorem) we can rewrite the formula as

$$\begin{aligned}W_n = &\sum_{m_1,\dots,m_n\in\mathbb{Z}^+_{\rm odd}}\prod_{i=1}^n X_i^{m_i}\cdot\sum_{r_1,\dots,r_n=0}^{\infty}\prod_{i=1}^n\partial_y^{r_i}e^{2m_i\frac{\mathcal{S}(m_i\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}_i}\\ &\left[\prod_{i=1}^n u_i^{r_i}z_i^{m_i}\right]\prod_{i=1}^n\frac{e^{\frac{\hbar u_i}{2}}+e^{-\frac{\hbar u_i}{2}}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{u_i\left(\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i-y_i\right)}\\ &\sum_{\gamma\in\Gamma_n}\prod_{(v_k,v_\ell)\in E_\gamma}\left(e^{\hbar^2u_ku_\ell\mathcal{S}(\hbar u_kz_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)}-1\right).\end{aligned} \tag{4.26}$$

The next observation that we use is the following. Replace the summation in Equation (4.17) from $\sum_{m_1,\dots,m_n\in\mathbb{Z}^+_{\rm odd}}$ to $\sum_{m_1,\dots,m_n\in\mathbb{Z}_{\rm odd}}$. This replacement changes the disconnected $W_n^\bullet$, but when we pass to the connected ones, this replacement just changes the $W_{0,2}$ by adding to it the singular term $\frac{1}{4}B(X_1,X_2)$ (hence the condition $2g-2+n>0$ in the statement of the theorem), cf. [5, Proposition 4.1]. With this adjustment, we have for $g\ge 0$, $n\ge 2$, $2g-2+n>0$:

$$\begin{aligned}W_{g,n} = [\hbar^{2g-2+n}]&\sum_{m_1,\dots,m_n\in\mathbb{Z}_{\rm odd}}\prod_{i=1}^n X_i^{m_i}[z_i^{m_i}]\cdot\sum_{r_1,\dots,r_n=0}^{\infty}\prod_{i=1}^n\partial_y^{r_i}e^{2m_i\frac{\mathcal{S}(m_i\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}_i}\\ &\left[\prod_{i=1}^n u_i^{r_i}\right]\prod_{i=1}^n\frac{e^{\frac{\hbar u_i}{2}}+e^{-\frac{\hbar u_i}{2}}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{u_i\left(\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i-y_i\right)}\\ &\sum_{\gamma\in\Gamma_n}\prod_{(v_k,v_\ell)\in E_\gamma}\left(e^{\hbar^2u_ku_\ell\mathcal{S}(\hbar u_kz_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)}-1\right).\end{aligned} \tag{4.27}$$

Now note two things. First, $\partial_y^{r_i}\exp\left(2m_i\frac{\mathcal{S}(m_i\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}_i\right)$ is even in z_i for even r_i and odd in z_i for odd r_i. On the other hand, in the expression

$$\begin{aligned}&\left[\prod_{i=1}^n u_i^{r_i}z_i^{m_i}\right]\prod_{i=1}^n\frac{e^{\frac{\hbar u_i}{2}}+e^{-\frac{\hbar u_i}{2}}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{u_i\left(\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i-y_i\right)}\\ &\sum_{\gamma\in\Gamma_n}\prod_{(v_k,v_\ell)\in E_\gamma}\left(e^{\hbar^2u_ku_\ell\mathcal{S}(\hbar u_kz_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)}-1\right)\end{aligned} \tag{4.28}$$

we only have terms with r_i+m_i odd. This means that the whole expression in which we take the coefficients of $[\prod_{i=1}^n z_i^{m_i}]$ is odd in $z_1,\dots,z_n$, and thus we can extend the summation to $m_1, \dots, m_n \in \mathbb{Z}$. Second, we can use the trick that for a polynomial g in m we can replace $\sum X^m g(m)$ by $\sum_{j=o}^{\infty} D^j X^m [t^j] g(t)$. These two ideas allow to rewrite (4.27) as

$$
\begin{aligned}
W_{g,n} = [\hbar^{2g-2+n}] & \sum_{j_1,\dots,j_n=0}^{\infty} D_i^{j_i} [t_i^{j_i}] \sum_{m_1,\dots,m_n \in \mathbb{Z}} \prod_{i=1}^{n} X_i^{m_i} [z_i^{m_i}] e^{2m_i \psi_i} \\
& \sum_{r_1,\dots,r_n=0}^{\infty} \prod_{i=1}^{n} e^{-2t_i \psi_i} \partial_y^{r_i} e^{2m_i \frac{\mathcal{S}(m_i \hbar \partial_y)}{\mathcal{S}(\hbar \partial_y)} \overline{\psi}_i} \\
& [\prod_{i=1}^{n} u_i^{r_i}] \prod_{i=1}^{n} \frac{e^{\frac{\hbar u_i}{2}} + e^{-\frac{\hbar u_i}{2}}}{4 u_i \hbar \mathcal{S}(u_i \hbar)} e^{u_i \left(\mathcal{S}(\hbar u_i z_i \partial_{z_i}) \overline{y}_i - y_i \right)} \\
& \sum_{\gamma \in \Gamma_n} \prod_{(v_k, v_\ell) \in E_\gamma} \left(e^{\hbar^2 u_k u_\ell \mathcal{S}(\hbar u_k z_k \partial_{z_k}) \mathcal{S}(\hbar u_\ell z_\ell \partial_{z_\ell}) B(z_k, z_\ell)} - 1 \right)
\end{aligned}
\tag{4.29}
$$

(here for each $r_1, \dots, r_n$ the second line expands in $\hbar$ with the coefficients that are manifestly polynomial in $t_1, \dots, t_n$). Finally, we apply the Lagrange–Bührmann formula for $X_i = z_i e^{-2\psi_i}$ to Equation (4.29) (cf. [5, Lemma 4.7]) and obtain the statement of the theorem.

4.5 Special cases

In this section we discuss the formulas for $W_{g,n}$ for $(g, n) = (0, 2)$ and $n = 1$, in the variable z related to X by $X = z e^{-2\psi(y(z))}$. We begin with unstable terms $(g, n) = (0, 2)$ and $(0, 1)$.

Proposition 4.3 *For $(g, n) = (0, 2)$ we have:*

$$
W_{0,2} = \frac{1}{4 Q_1 Q_2} B(z_1, z_2) - \frac{1}{4} B(X_1, X_2). \tag{4.30}
$$

Proof Indeed, as we discussed in the proof of Theorem 4.1, the change of summation from $m \in Z^+_{\text{odd}}$ to $m \in Z_{\text{odd}}$ and the commutation rules for $J_+(X_1)$, $J_-(X_2)$ imply that the $(0, 2)$ term gets a correction. We have:

$$
\begin{aligned}
W_{0,2} + \frac{1}{4} B(X_1, X_2) &= \sum_{m_1, m_2 \in \mathbb{Z}_{\text{odd}}} X_1^{m_1} X_2^{m_2} [z_1^{m_1} z_2^{m_2}] e^{2m_1 \psi_1 + 2m_2 \psi_2} \frac{1}{4} B(z_1, z_2) \\
&= \sum_{m_1, m_2 \in \mathbb{Z}} X_1^{m_1} X_2^{m_2} [z_1^{m_1} z_2^{m_2}] e^{2m_1 \psi_1 + 2m_2 \psi_2} \frac{1}{4} B(z_1, z_2) \\
&= \frac{1}{4 Q_1 Q_2} B(z_1, z_2).
\end{aligned}
\tag{4.31}
$$

In the second line, in order to change the summation from $m_1, m_2 \in \mathbb{Z}_{\mathrm{odd}}$ to $m_1, m_2 \in \mathbb{Z}$, we use that $B(z_1, z_2)$ is odd in z_1 and z_2 and $\psi_1 = \psi(y_1)$ (respectively, $\psi_2 = \psi(y_2)$) is even in z_1 (respectively, z_2). ◻

Proposition 4.4 *For $(g, n) = (0, 1)$ we have: $W_{0,1}(X) = y(z)/2$.*

Proof It is a straightforward computation. First, recall that

$$
\begin{aligned}
H_{0,1}(X) &= [\hbar^{-1}] \sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}} \frac{X^m}{m} \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}(\hbar^2,y)}\Big|_{y=0} [u^r z^m] \frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\hbar\mathcal{S}(u\hbar)} e^{u\mathcal{S}(\hbar u z\partial_z)\overline{y}(\hbar^2,z)} \\
&= \sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}} \frac{X^m}{m} \sum_{r=0}^{\infty} \partial_y^r e^{2m\psi(y)}\Big|_{y=0} [u^r z^m] \frac{e^{uy(z)}}{2u}.
\end{aligned} \tag{4.32}
$$

With this equation, in order to compute $W_{0,1}$, we consider its differential. We have:

$$
\begin{aligned}
DW_{0,1}(X) &= \sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\psi(y)}\Big|_{y=0} [u^r z^m] z\partial_z \frac{e^{uy(z)}}{2u} \\
&= \sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\psi(y)}\Big|_{y=0} [u^r z^m] \frac{e^{uy(z)} QDy(z)}{2} \\
&= \sum_{m\in\mathbb{Z}^+_{\mathrm{odd}}} X^m [z^m] e^{2m\psi(y(z)))} \frac{QDy(z)}{2} \\
&= \sum_{m\in\mathbb{Z}} X^m [z^m] e^{2m\psi(y(z)))} \frac{QDy(z)}{2} \\
&= \frac{1}{2} Dy(z).
\end{aligned} \tag{4.33}
$$

Hence $W_{0,1}(X) = y(z)/2$. ◻

4.5.1 Stable terms for $n = 1$

Consider $g \geq 1$, $n = 1$, that is, we consider $W_1 = \sum_{g=0}^{\infty} \hbar^{2g-1} W_{g,1}(X)$.

Proposition 4.5 *Under the change of variables $X = ze^{-2\psi(y(z))}$ we have:*

$$
\begin{aligned}
W_1(X) &= \frac{y}{2\hbar} + \sum_{j=1}^{\infty} D^{j-1}[t^j] e^{-2t\psi + 2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} \frac{Dy}{2\hbar} \\
&+ \sum_{j,r=0}^{\infty} D^j [t^j] \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\hbar\mathcal{S}(u\hbar)} e^{u(\mathcal{S}(\hbar u z\partial_z)\overline{y} - y)} \right).
\end{aligned} \tag{4.34}
$$

Here $y = y(z)$, $\overline{y} = \overline{y}(\hbar^2, z)$, $\psi = \psi(y)$, $\overline{\psi} = \overline{\psi}(\hbar^2, y)$, and $D = X\partial_X$.

Proof By direct commutation of the operators, we have:

$$W_1(X) = \sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}\Big|_{y=0} [u^r z^m] \frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\hbar\mathcal{S}(u\hbar)} e^{u\mathcal{S}(\hbar u z\partial_z)\overline{y}}$$

$$= \sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}\Big|_{y=0} [u^r z^m] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\hbar\mathcal{S}(u\hbar)} e^{u\mathcal{S}(\hbar u z\partial_z)\overline{y}} - \frac{e^{uy}}{2u\hbar} \right)$$

$$+ \sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}\Big|_{y=0} [u^r z^m] \frac{e^{uy}}{2u\hbar} \tag{4.35}$$

The first summand is regular in u, so it can be computed as in the proof of Theorem 4.1:

$$\sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}\Big|_{y=0} [u^r z^m] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\hbar\mathcal{S}(u\hbar)} e^{u\mathcal{S}(\hbar u z\partial_z)\overline{y}} - \frac{e^{uy}}{2u\hbar} \right)$$

$$= \sum_{m\in\mathbb{Z}} X^m [z^m] \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\hbar\mathcal{S}(u\hbar)} e^{-uy + u\mathcal{S}(\hbar u z\partial_z)\overline{y}} - \frac{1}{2u\hbar} \right)$$

$$= \sum_{j,r=0}^{\infty} D^j [t^j] \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\hbar\mathcal{S}(u\hbar)} e^{-uy + u\mathcal{S}(\hbar u z\partial_z)\overline{y}} \right). \tag{4.36}$$

The second summand of (4.35) can be computed by differentiation.

$$D \sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}\Big|_{y=0} [u^r z^m] \frac{e^{uy}}{2u\hbar}$$

$$= \sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}\Big|_{y=0} [u^r z^m] \frac{e^{uy(z)} QDy}{2\hbar}$$

$$= \sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r z^m] \frac{QDy}{2\hbar} = \sum_{m\in\mathbb{Z}} X^m [z^m] \sum_{r=0}^{\infty} \partial_y^r e^{2m\frac{\mathcal{S}(m\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r] \frac{QDy}{2\hbar}$$

$$= \sum_{j=0}^{\infty} D^j [t^j] e^{-2t\psi + 2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} \frac{Dy}{2\hbar} = \frac{Dy}{2\hbar} + D\sum_{j=1}^{\infty} D^{j-1} [t^j] e^{-2t\psi + 2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} \frac{Dy}{2\hbar}. \tag{4.37}$$

Combining these two computations, we obtain the statement of the proposition. □

5 Loop equations

Consider the change of variables $X = ze^{-2\psi(y(z))}$. In this section we make a number of extra assumptions of analytical nature on the coefficients ψ_{2d} and y_{2d} of the $\hbar^2$-

expansions of $\overline{\psi}$ and $\overline{y}$, and with these assumptions we prove the linear and quadratic loop equations for the $W_{g,n}$'s that we computed in the closed form in the previous section (or, more precisely, for the symmetric differentials that we construct from $W_{g,n}$'s).

5.1 Assumptions

Let z be a global affine coordinate on $\mathbb{C}\mathrm{P}^1$. We assume that $\psi'(y(z))$ and $y'(z)$ can be analytically extended to rational functions on $\mathbb{C}\mathrm{P}^1$. These assumptions imply that $X = ze^{-2\psi(y(z))}$ extends to a global function on $\mathbb{C}\mathrm{P}^1$ and $d\log X$ is a rational 1-form in the global coordinate z.

The rational 1-form $d\log X$ has a finite number of zeros, and we assume that all zeros of $d\log X$ are simple. We also assume that the coefficients of the positive degrees of $\hbar$ in the series $\overline{\psi}(\hbar^2, y(z))$ and $\overline{y}(\hbar^2, z)$ are rational functions in z as well, and their singular points are disjoint from the zeros of $d\log X$.

Proposition 5.1 *Under these assumptions the symmetric n-differentials*

$$\omega_{g,n} := 2^{1-g} W_{g,n}(X_1, \dots, X_n) \prod_{i=1}^{n} d\log X_i + \delta_{g,0}\delta_{n,2}\frac{1}{2}B(X_1, X_2) d\log X_1 d\log X_2, \quad g \geq 0, n \geq 1, \quad (5.1)$$

analytically extend to global rational differentials on $(\mathbb{C}\mathrm{P}^1)^n$ *for* $2g - 2 + n > 0$.

Proof This statement follows directly from the structure of formulas given in Equations (4.23) and (4.34). □

Note the factor 2^{1-g}. It is a compensation for the fact that the natural $W_{0,1}$ and $W_{0,2}$ that we obtained in the previous section are twice less than the formulas one might expect from the point of view of the spectral curve topological recursion, see Sect. 7.1.

5.2 Blobbed topological recursion

Let p be a simple zero point of $d\log X$. Let σ denote the deck transformation of X near p.

Definition 5.1 We say that the system of symmetric n-differentials $\{\omega_{g,n}\}_{g\geq 0, n\geq 1}$ satisfies the linear loop equations at p if for any $g \geq 0$, $n \geq 0$,

$$\omega_{g,n+1}(w, z_{[\![n]\!]}) + \omega_{g,n+1}(\sigma(w), z_{[\![n]\!]}) \quad (5.2)$$

is holomorphic at $w \to p$ and vanishes at $w = p$.

We say that the system of symmetric n-differentials $\{\omega_{g,n}\}_{g\geq 0, n\geq 1}$ satisfies the quadratic loop equations at p if for any $g \geq 0$, $n \geq 0$, $(g,n) \neq (1,0)$,

$$\omega_{g-1,n+2}(w,\sigma(w),z_{[\![n]\!]}) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = [\![n]\!]}} \omega_{g_1,n_1+1}(w,z_{I_2})\omega_{g_2,n_2+1}(\sigma(w),z_{I_2}) \quad (5.3)$$

is holomorphic at $w \to p$ and has a zero of order at least two at $w = p$. In the case $(g,n) = (1,0)$ we require the same property, but we remove the singularity from the first summand, that is, we replace $\omega_{0,2}(w,\sigma(w))$ with

$$\begin{aligned} &\left(\omega_{0,2} - \frac{1}{2} B(X_1,X_2) d\log X_1 d\log X_2\right)|_{X_1=X(w),X_2=X(\sigma(w))} \\ &= 2W_{0,2}(X_1,X_2) d\log X_1 d\log X_2|_{X_1=X(w),X_2=X(\sigma(w))}. \end{aligned} \quad (5.4)$$

If all zero points of $d\log X$ are simple and $\omega_{g,n}$'s satisfy the linear and quadratic loop equations at each of them, then we say that the system of symmetric n-differentials $\{\omega_{g,n}\}_{g\geq 0,n\geq 1}$ satisfies the blobbed topological recursion [8].

Theorem 5.1 *Under the analytic assumptions listed in Sect. 5.1 the system of symmetric differentials (5.1) satisfies the blobbed topological recursion.*

5.3 Proof of Theorem 5.1

Consider the connected correlation function defined as

$$\begin{aligned} \mathcal{W}_{g,n} = [\hbar^{2g-2+n}] &\sum_{m_1,\dots,m_n \in \mathbb{Z}_{\mathrm{odd}}} X_1^{m_1} \dots X_n^{m_n} \\ &\left\langle 0 \right| \frac{J_{m_1}}{\hbar m_1} [a^0] \mathcal{E}(\hbar v, a) J_{m_2} \dots J_{m_n} \mathcal{D} e^{\sum_{d=0}^{\infty} \hbar^{2d} \sum_{k\in\mathbb{Z}^+_{\mathrm{odd}}} \frac{J_{-k}}{\hbar k}[z^k] y_d(z)} \left| 0 \right\rangle^{\circ}. \end{aligned} \quad (5.5)$$

(here by $\langle 0 | - | 0 \rangle^{\circ}$ we mean the connected vacuum expectation obtained by inclusion–exclusion formula from the disconnected one).

Lemma 5.1 *For* $\sum_{m\in\mathbb{Z}_{\mathrm{odd}}} J_m X^m/(\hbar m) = \sum_{m\in\mathbb{Z}} J_m X^m/(\hbar m)$ *we have:*

$$\left[\sum_{m\in\mathbb{Z}} \frac{J_m X^m}{\hbar m}, [a^0]\mathcal{E}(\hbar v, a)\right] = v\mathcal{S}(\hbar v X \partial_X)\left(\frac{\mathcal{E}(\hbar v, X) + \mathcal{E}(-\hbar v, X)}{2}\right). \quad (5.6)$$

Proof A straightforward computation using Equations (4.4) and (2.25). □

Remark 5.1 Note that $\mathcal{E}(u,X) = -\mathcal{E}(-u,-X)$. Hence the right hand side of Equation (5.6) is odd in X. Note also that the right hand side of Equation (5.6) is manifestly odd in v.

Our next goal is to compute the coefficients of v^1 and v^3 in (5.6) applied to the covacuum.

Lemma 5.2 *We have:*

$$\langle 0|[v^1]v\mathcal{S}(\hbar vX\partial_X)\left(\frac{\mathcal{E}(\hbar v, X)+\mathcal{E}(-\hbar v, X)}{2}\right) = 2\langle 0|\sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m J_m; \tag{5.7}$$

$$\begin{aligned}\langle 0|[v^3]v\mathcal{S}(\hbar vX\partial_X)\left(\frac{\mathcal{E}(\hbar v, X)+\mathcal{E}(-\hbar v, X)}{2}\right) &= \hbar^2\left(\frac{1}{6}(X\partial_X)^2+\frac{1}{6}\right)\langle 0|\sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m J_m \\ &\quad + \hbar^2\frac{4}{3}\langle 0|\Big(\sum_{m\in\mathbb{Z}^+_{\text{odd}}} X^m J_m\Big)^3\end{aligned} \tag{5.8}$$

Proof A straightforward computation using Equation (4.4). □

Corollary 5.1 *We have:*

$$\begin{aligned}\mathcal{W}_{g,n}(X_{[\![n]\!]}) =& v\Big(2W_{g,n}(X_{[\![n]\!]})\Big) + v^3\bigg(\frac{4}{3}\Big(W_{g-2,n+2}(X_1, X_1, X_1, X_{[\![n]\!]\setminus 1}) \\ &+ 3\sum_{\substack{g_1+g_2=g-1\\ I_1\sqcup I_2=[\![n]\!]\setminus 1}} W_{g_1,n_1+1}(X_1, X_{I_1})W_{g_2,n_2+2}(X_1, X_1, X_{I_2}) \\ &+ \sum_{\substack{g_1+g_2+g_3=g\\ I_1\sqcup I_2\sqcup I_3=[\![n]\!]\setminus 1}} W_{g_1,n_1+1}(X_1, X_{I_1})W_{g_2,n_2+1}(X_1, X_{I_2})W_{g_3,n_3+1}(X_1, X_{I_3})\Big) \\ &+ \left(\frac{1}{6}(X_1\partial_{X_1})^2+\frac{1}{6}\right)W_{g-1,n}(X_{[\![n]\!]})\bigg) + O(v^5),\end{aligned} \tag{5.9}$$

where we have to substitue $W_{0,2}(X_i, X_j) + \frac{1}{4}B(X_i, X_j)$ instead of $W_{0,2}(X_i, X_j)$ in all instances when the arguments are not the same, that is, $i \neq j$.

Now, repeating *mutatis mutandis* the arguments of the proofs of Theorem 4.1 and Propositions 4.3, 4.4, and 4.5, we obtain closed algebraic formulas for $\mathcal{W}_{g,n}$.

Lemma 5.3 *Under the change of variables $X = ze^{-2\psi(y(z))}$ we have:*

$$\begin{aligned}\mathcal{W}_{g,n}(X_{[\![n]\!]}) =& [\hbar^{2g-2+n}]\sum_{\substack{j_1,\dots,j_n,\\ r_1,\dots,r_n=0}}^{\infty}\left[\prod_{i=2}^n D_i^{j_i}[t_i^{j_i}]\frac{1}{Q_i}e^{-2t_i\psi_i}\partial_{y_i}^{r_i}e^{2t_i\frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i}[u_i^{r_i}]\right] \\ &\left[D_1^{j_1}[t_1^{j_1}]\frac{1}{Q_1}e^{-2t_1\psi_1}\partial_{y_1}^{r_1}e^{2t_1\frac{\mathcal{S}(t_1\hbar\partial_{y_1})}{\mathcal{S}(\hbar\partial_{y_1})}\overline{\psi}_1}v\mathcal{S}(t_1\hbar\partial_{y_1})(e^{vy_1}+e^{-vy_1})[u_1^{r_1}]\right] \\ &\prod_{i=1}^n\frac{e^{\frac{\hbar u_i}{2}}+e^{-\frac{\hbar u_i}{2}}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{u_i\left(\mathcal{S}(\hbar u_i z_i\partial_{z_i})\overline{y}_i-y_i\right)}\sum_{\gamma\in\Gamma_n}\prod_{(v_k,v_\ell)\in E_\gamma}\left(e^{\hbar^2u_ku_\ell\mathcal{S}(\hbar u_kz_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)}-1\right)\end{aligned} \tag{5.10}$$

for $n \geq 2$, $(g,n) \neq (0,2)$. Here Γ_n is the set of all connected simple graphs on n vertices $v_1, \ldots, v_n$, and E_γ is the set of edges of γ.

In the case $(g,n) = (0,2)$ we have

$$\mathcal{W}_{0,2} = \frac{v(e^{vy_1} + e^{-vy_1})}{4Q_1Q_2} B(z_1, z_2) \tag{5.11}$$

In the case $(g,n) = (0,1)$ we have

$$\mathcal{W}_{0,1} = \frac{1}{2}(e^{vy_1} - e^{-vy_1}). \tag{5.12}$$

In the case $g \geq 1$, $n = 1$ we have

$$\begin{aligned}\mathcal{W}_{g,1} =& [\hbar^{2g-1}] \sum_{j_1,r_1=0}^{\infty} D_1^{j_1}[t_1^{j_1}] \frac{1}{Q_1} e^{-2t_1\psi_1} \partial_{y_1}^{r_1} e^{2t_1 \frac{\mathcal{S}(t_1\hbar\partial_{y_1})}{\mathcal{S}(\hbar\partial_{y_1})}\overline{\psi}_1} v\mathcal{S}(t_1\hbar\partial_{y_1})(e^{vy_1} + e^{-vy_1}) \\ & [u_1^{r_1}] \frac{e^{\frac{\hbar u_1}{2}} + e^{-\frac{\hbar u_1}{2}}}{4u_1\hbar\mathcal{S}(u_1\hbar)} e^{-u_1y_1 + u_1\mathcal{S}(\hbar u_1 z_1\partial_{z_1})\overline{y}_1} \\ & + [\hbar^{2g-1}] \sum_{j_1=1}^{\infty} D^{j_1-1}[t^{j_1}] e^{-2t_1\psi_1 + 2t_1\frac{\mathcal{S}(t_1\hbar\partial_{y_1})}{\mathcal{S}(\hbar\partial_{y_1})}\overline{\psi}_1} v\mathcal{S}(t_1\hbar\partial_{y_1})(e^{vy_1} + e^{-vy_1}) \frac{Dy}{2}.\end{aligned} \tag{5.13}$$

In all these formulas we use $y_i = y(z_i)$, $\overline{y}_i = \overline{y}(\hbar^2, z_i)$, $\psi_i = \psi(y_i)$, $\overline{\psi}_i = \overline{\psi}(\hbar^2, y_i)$, $Q_i = Q(z_i)$, $X_i = X(z_i)$, $D_i = X_i\partial_{X_i} = Q_i^{-1} z_i \partial_{z_i}$.

Proof First, observe that $[a^0]\mathcal{E}(\hbar v, a)$ can be represented as the coefficient of $[\epsilon^1]$ in the expression

$$e^{\epsilon[a^0]\mathcal{E}(\hbar v, a)} = e^{\epsilon\sum_{k\in\mathbb{Z}}(-1)^k e^{\hbar vk}:\phi_k\phi_{-k}:} = e^{\epsilon\sum_{k\in\mathbb{Z}_+}(-1)^k(e^{\hbar vk} - e^{-\hbar vk}):\phi_k\phi_{-k}:} \tag{5.14}$$

For $T(k) = \frac{1}{2\hbar}(e^{\hbar vk} - e^{-\hbar vk})$ we have $T(k+1) - T(k) = \Delta\overline{\psi}(\hbar^2, \hbar(k+\frac{1}{2}))$, where

$$\Delta\overline{\psi}(\hbar^2, y) = \frac{1}{2\hbar}(e^{\frac{\hbar v}{2}} - e^{-\frac{hv}{2}})(e^{vy} + e^{-vy}) = v + v^3\left(\frac{\hbar^2}{24} + \frac{y^2}{2}\right) + O(v^5). \tag{5.15}$$

Define $\widetilde{\mathbb{J}}_k$ as the conjugation of $\mathbb{J}_k$ with the operator given in (5.14). It is operator of exactly the same type as $\mathbb{J}_k$, we just replace $\overline{\psi}(\hbar^2, y)$ by $\overline{\psi}(\hbar^2, y) + \epsilon\Delta\overline{\psi}(\hbar^2, y)$ in its definition. By (4.13) we have:

$$\widetilde{\mathbb{J}}_k = \frac{1}{4}\sum_{r=0}^{\infty} \partial_y^r \exp\left(2k\frac{\mathcal{S}(k\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}(\overline{\psi}(\hbar^2, y) + \epsilon\Delta\overline{\psi}(\hbar^2, y))\right)\Big|_{y=0}$$
$$[u^r a^k]\frac{e^{\hbar u/2} + e^{-\hbar u/2}}{u\hbar\mathcal{S}(u\hbar)} \exp\left(2\hbar u \sum_{l\in\mathbb{Z}^+_{\text{odd}}} a^{-l}\mathcal{S}(l\hbar u)J_{-l}\right) \exp\left(2\hbar u \sum_{l\in\mathbb{Z}^+_{\text{odd}}} a^{l}\mathcal{S}(l\hbar u)J_{l}\right). \tag{5.16}$$

Let $\mathcal{W}_n = \sum_{g=0}^{\infty} \hbar^{2g-2+n}\mathcal{W}_{g,n}$. Then

$$\mathcal{W}_n(X_{[\![n]\!]}) = [\epsilon^1] \sum_{m_1,\dots,m_n\in\mathbb{Z}^+_{\text{odd}}} \frac{1}{m_1} X_1^{m_1}\dots X_n^{m_n}$$
$$\left\langle 0\middle|\widetilde{\mathbb{J}}_{m_1}\dots\mathbb{J}_{m_n} e^{\sum_{d=0}^{\infty}\hbar^{2d}\sum_{k\in\mathbb{Z}^+_{\text{odd}}}\frac{J_{-k}}{\hbar k}[z^k]y_d(z)}\middle|0\right\rangle^{\circ}. \tag{5.17}$$

By commutation of the operators, we have:

$$\mathcal{W}_n = \sum_{m_1,\dots,m_n\in\mathbb{Z}^+_{\text{odd}}} \prod_{i=1}^{n} X_i^{m_i} \cdot \sum_{r_1,\dots,r_n=0}^{\infty} \frac{1}{m_1}[\epsilon^1]\partial_y^{r_1}$$
$$\exp\left(2m_1\frac{\mathcal{S}(m_1\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}(\overline{\psi} + \epsilon\Delta\overline{\psi}(\hbar^2, y))\right)\Big|_{y=0}$$
$$\prod_{i=2}^{n}\partial_y^{r_i}\exp\left(2m_i\frac{\mathcal{S}(m_i\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}\right)\Big|_{y=0}\left[\prod_{i=1}^{n}u_i^{r_i}z_i^{m_i}\right]\prod_{i=1}^{n}\frac{e^{\frac{\hbar u_i}{2}} + e^{-\frac{\hbar u_i}{2}}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{u_i\mathcal{S}(\hbar u_i z_i\partial_{z_i})\overline{y}_i}$$
$$\sum_{\gamma\in\Gamma_n}\prod_{(v_k,v_\ell)\in E_\gamma}\left(e^{\hbar^2 u_k u_\ell \mathcal{S}(\hbar u_k z_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)} - 1\right), \tag{5.18}$$

where Γ_n is the set of all connected simple graphs on n vertices $v_1,\dots,v_n$, and E_γ is the set of edges of γ.

As in the proof of Theorem 4.1 and Propositions 4.4 and 4.5, we use that for any formal power series $G(y)$ in y and $F(u)$ in u, we have

$$\sum_{r=0}^{\infty}\partial_y^r G(y)|_{y=0}[u^r]F(u) = \sum_{r=0}^{\infty}\partial_y^r G(y)[u^r]e^{-uy}F(u) \tag{5.19}$$

([5, Lemma 4.5]). Applying it to Equation (5.18) in the cases $n \geq 2$, $(g, n) \neq (0, 2)$ (these cases have to be treated separately, it is the same situation as in the proof of Theorem 4.1), we obtain:

$$
\begin{aligned}
\mathcal{W}_n = & \sum_{m_1,\dots,m_n \in \mathbb{Z}^+_{\text{odd}}} \prod_{i=1}^{n} X_i^{m_i} [z_i^{m_i}] e^{2m_i\psi_i} \cdot \sum_{r_1,\dots,r_n=0}^{\infty} \\
& e^{-2m_1\psi_1} \partial_{y_1}^{r_1} \left(\exp\left(2m_1 \frac{\mathcal{S}(m_1\hbar\partial_{y_1})}{\mathcal{S}(\hbar\partial_{y_1})} \overline{\psi}_1 \right) v\mathcal{S}(m_1\hbar\partial_{y_1})(e^{vy_1} + e^{-vy_1}) \right) \\
& \prod_{i=2}^{n} e^{-2m_i\psi_i} \partial_{y_i}^{r_i} \exp\left(2m_i \frac{\mathcal{S}(m_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})} \overline{\psi}_i \right) [\prod_{i=1}^{n} u_i^{r_i}] \prod_{i=1}^{n} \frac{e^{\frac{\hbar u_i}{2}} + e^{-\frac{\hbar u_i}{2}}}{4u_i\hbar\mathcal{S}(u_i\hbar)} e^{u_i\left(\mathcal{S}(\hbar u_i z_i \partial_{z_i})\overline{y}_i - y\right)} \\
& \sum_{\gamma\in\Gamma_n} \prod_{(v_k,v_\ell)\in E_\gamma} \left(e^{\hbar^2 u_k u_\ell \mathcal{S}(\hbar u_k z_k \partial_{z_k}) \mathcal{S}(\hbar u_\ell z_\ell \partial_{z_\ell}) B(z_k,z_\ell)} - 1 \right),
\end{aligned} \tag{5.20}
$$

Starting from this point all further steps just repeat the computations made in the proof of Theorem 4.1. We use three ideas:

- extend the summation to $m_1, \dots, m_n \in \mathbb{Z}$;
- capture the polynomial dependence on $m_1, \dots, m_n$ replacing their entrances by $t_1, \dots, t_n$ and applying $\prod_{i=1}^n \sum_{j_i=1}^\infty D_i^{j_i}[t_i^{j_i}]$;
- apply Lagrange–Bührmann formula for the change of variables.

This completes the proof of Equation (5.10). All other equations stated in the lemma are obtained by small variations of this argument, which repeat the corresponding special cases in the proofs of Propositions 4.3, 4.4, and 4.5. □

Corollary 5.2 *The functions $\mathcal{W}_{g,n}$, $g \geq 0$, $n \geq 1$, are formal power series in v, whose coefficients are rational functions in the variables $z_{[\![n]\!]}$, that near each simple zero point p of $d\log X$ satisfy the property that*

$$
\mathcal{W}_{g,n}(z_1, z_{[\![n]\!]\setminus 1}) + \mathcal{W}_{g,n}(\sigma(z_1), z_{[\![n]\!]\setminus 1}) \tag{5.21}
$$

is holomorphic at $z_1 \to p$. Here σ is the deck transformation of X at p.

Proof This follows directly from the structure of the formulas in Lemma 5.3. We apply $D_1^{j_1}$ to a rational function that has a simple pole at $w \to p$ (coming from the factor $1/Q_1$). A function with at most simple pole automatically satisfies (5.21), and the operator D_1 preserves this property. □

Proof of Theorem 5.1 Fix a zero point p of $d\log X$ (which by assumption is simple) and let σ be the deck transformation of X near this point. For any function $f(z)$ defined in the neighborhood of p we define

$$
\mathsf{S}_z f(z) = f(z) + f(\sigma(z)). \tag{5.22}
$$

Then the linear loop equations at the point p for the symmetric differentials expressed as in Equation (5.1) can be equivalently rewritten as

$$
\mathsf{S}_{z_1} W_{g,n}(z_{[\![n]\!]}) \tag{5.23}
$$

is holomorphic at $z \to p$ for any (g, n). Corollary 5.2 applied to the coefficients of $[v^1]$ in $\mathcal{W}_{g,n}$ implies that it is indeed the case.

Note also that Corollary 5.2 applied to the coefficients of $[v^3]$ in $\mathcal{W}_{g,n}$ implies that $\mathsf{S}_{z_1}[v^3]\mathcal{W}_{g,n}$ is holomorphic at $z \to p$ for any (g,n). Using explicit formula for $[v^3]2^{3-g}\mathcal{W}_{g,n}$ given in Equation (5.9) and the linear loop equations, we conclude that

$$\begin{aligned}
&\mathsf{S}_{z_1}\Big(2^{1-(g-2)}W_{g-2,n+2}(z_1,z_1,z_1,z_{[\![n]\!]\setminus 1})\\
&\quad+3\sum_{\substack{g_1+g_2=g-1\\ I_1\sqcup I_2=[\![n]\!]\setminus 1}} 2^{1-g_1}W_{g_1,n_1+1}(z_1,z_{I_1})2^{1-g_2}W_{g_2,n_2+2}(z_1,z_1,z_{I_2})\\
&\quad+\sum_{\substack{g_1+g_2+g_3=g\\ I_1\sqcup I_2\sqcup I_3=[\![n]\!]\setminus 1}} 2^{1-g_1}W_{g_1,n_1+1}(z_1,z_{I_1})2^{1-g_3}W_{g_2,n_2+1}(z_1,z_{I_2})2^{1-g_3}W_{g_3,n_3+1}(z_1,z_{I_3})\Big)
\end{aligned} \tag{5.24}$$

is holomorphic at $z \to p$ for any (g,n). Here we abuse the notation a little bit since each time we use $2W_{0,2}(z_i,z_j)$ with $i \neq j$, we actually mean $\frac{1}{2}B(z_i,z_j)$.

This particular system of equations is studied in a bit different situation in [7, Lemma 20]. The main difference between our situation and the one studied in [7, Lemma 20] is the choice of B, which is the standard Bergman kernel in [7], but it does not affect the proof in any step. Another difference is the rescaling of $W_{g,n}$ by 2^{1-g} in the definition of $\omega_{g,n}$'s, but both (5.24) and the quadratic loop equations are homogeneous with respect to this rescaling.

So, adjusted in our situation [7, Lemma 20] proves that the holomorphy of the expression given in (5.24) implies the quadratic loop equations for the symmetric differentials $\omega_{g,n}$ given by Equation (5.1), under the condition that y does not vanish at $z = p$. The latter condition is obviously satisfied in our situation. Indeed, the point p satisfies the equation $1 - 2p\psi'(y(p))y'(p) = 0$. On the other hand, ψ' is an odd function in y, so at any point z where $y(z) = 0$, we have $1 - 2z\psi'(y(z))y'(z) = 1$. Therefore, y does not vanish at $z = p$. Hence the symmetric differentials $\omega_{g,n}$ satisfy the quadratic loop equations.

6 Formulas for $H_{g,n}$

In this section we derive expressions for $H_{g,n}$ by integration of the earlier derived expressions for $W_{g,n}$. Since the case of $W_{g,1}$ was a bit special, we firstly perform a separate computation for $H_{g,1}$.

Proposition 6.1 *For $g \geq 1$ we have:*

$$\begin{aligned}
H_{g,1} &= [\hbar^{2g}]\sum_{j=2}^{\infty} D^{j-2}[t^j]e^{-2t\psi+2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}\frac{Dy}{2}\\
&\quad+[\hbar^{2g}]\sum_{j=1}^{\infty} D^{j-1}[t^j]\sum_{r=0}^{\infty}\frac{1}{Q}e^{-2t\psi}\partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}[u^r]\left(\frac{e^{\frac{\hbar u}{2}}+e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)}e^{-uy+u\mathcal{S}(\hbar u z\partial_z)\overline{y}}\right)
\end{aligned}$$

$$+ [\hbar^{2g}] \int_0^z \left(\frac{1}{S(\hbar\partial_y)} \overline{\psi} - \psi \right) y' dz + [\hbar^{2g}] \int_0^z \frac{dz}{2z} (\overline{y} - y) . \tag{6.1}$$

Here, as usual, we use $y = y(z)$, $\overline{y} = \overline{y}(\hbar^2, z)$, $\psi = \psi(y) = \psi(y(z))$, $\overline{\psi} = \overline{\psi}(\hbar^2, y) = \overline{\psi}(\hbar^2, y(z))$, $Q = Q(z)$, $X = X(z)$, $D = X\partial_X = Q^{-1}z\partial_z$.

Proof Recall that for $g \geq 1$

$$W_{g,1} = [\hbar^{2g}] \sum_{j=1}^{\infty} D^{j-1} [t^j] e^{-2t\psi + 2t \frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}} \frac{Dy}{2}$$
$$+ [\hbar^{2g}] \sum_{j,r=0}^{\infty} D^j [t^j] \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t \frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}} [u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)} e^{-uy + u\mathcal{S}(\hbar u z\partial_z)\overline{y}} \right). \tag{6.2}$$

Hence, for $g \geq 1$

$$H_{g,1} = [\hbar^{2g}] \sum_{j=2}^{\infty} D^{j-2} [t^j] e^{-2t\psi + 2t \frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}} \frac{Dy}{2}$$
$$+ [\hbar^{2g}] \sum_{j=1}^{\infty} D^{j-1} [t^j] \sum_{r=0}^{\infty} \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t \frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}} [u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)} e^{-uy + u\mathcal{S}(\hbar u z\partial_z)\overline{y}} \right)$$
$$+ [\hbar^{2g}] \int_0^z dz \frac{Q}{z} [t^1] e^{-2t\psi + 2t \frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}} \frac{Dy}{2}$$
$$+ [\hbar^{2g}] \int_0^z dz \frac{Q}{z} [t^0] \sum_{r=0}^{\infty} \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t \frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}} [u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)} e^{-uy + u\mathcal{S}(\hbar u z\partial_z)\overline{y}} \right) \tag{6.3}$$

(note that the constant term in z of this expression vanishes). The third term here can be computed as

$$[\hbar^{2g}] \int_0^z dz \frac{Q}{z} [t^1] e^{-2t\psi + 2t \frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)} \overline{\psi}} \frac{Dy}{2} = [\hbar^{2g}] \int_0^z dz \left(\frac{1}{S(\hbar\partial_y)} \overline{\psi} - \psi \right) \frac{QDy}{z}$$
$$= [\hbar^{2g}] \int_0^z \left(\frac{1}{S(\hbar\partial_y)} \overline{\psi} - \psi \right) y' dz. \tag{6.4}$$

The fourth term can be computed as

$$
\begin{aligned}
&[\hbar^{2g}]\int_0^z dz\frac{Q}{z}[t^0]\sum_{r=0}^{\infty}\frac{1}{Q}e^{-2t\psi}\,\partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}[u^r]\left(\frac{e^{\frac{\hbar u}{2}}+e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)}e^{-uy+u\mathcal{S}(\hbar uz\partial_z)\overline{y}}\right)\\
&=[\hbar^{2g}]\int_0^z\frac{dz}{z}[u^0]\left(\frac{e^{\frac{\hbar u}{2}}+e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)}e^{-uy+u\mathcal{S}(\hbar uz\partial_z)\overline{y}}\right)\\
&=[\hbar^{2g}]\int_0^z\frac{dz}{2z}\left(\overline{y}-y\right)=[\hbar^{2g}]\int_0^z\frac{dz}{2z}\left(\overline{y}-y\right). \qquad (6.5)
\end{aligned}
$$

Combining these formulas, we obtain the statement of the proposition. □

In the case $n=2$ we have the following formula for $H_{g,2}$.

Proposition 6.2 *In the case $g=0$ we have*

$$
H_{0,2}=\frac{1}{4}\log\frac{(z_1-z_2)(X_1+X_2)}{(z_1+z_2)(X_1-X_2)}. \qquad (6.6)
$$

For $g\geq 0$ we have:

$$
\begin{aligned}
H_{g,2}=&[\hbar^{2g}]\sum_{\substack{j_1,j_2=1\\ r_1,r_2=0}}^{\infty}\left[\prod_{i=1}^{2}D_i^{j_i-1}[t_i^{j_i}]\frac{1}{Q_i}e^{-2t_i\psi_i}\,\partial_{y_i}^{r_i}e^{2t_i\frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i}[u_i^{r_i}]\right.\\
&\left.\frac{e^{\hbar u_i/2}+e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{-u_iy_i+u_i\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i}\right]\left(e^{\hbar^2u_1u_2\mathcal{S}(\hbar u_1z_1\partial_{z_1})\mathcal{S}(\hbar u_2z_2\partial_{z_2})B(z_1,z_2)}-1\right)\\
&+[\hbar^{2g}]\sum_{\substack{j=1\\ r=0}}^{\infty}\left[D_1^{j-1}[t^j]\frac{1}{Q_1}e^{-2t\psi_1}\,\partial_{y_1}^r e^{2t\frac{\mathcal{S}(t\hbar\partial_{y_1})}{\mathcal{S}(\hbar\partial_{y_1})}\overline{\psi}_1}[u^r]\right.\\
&\left.\frac{e^{\hbar u/2}+e^{-\hbar u/2}}{4u\hbar\mathcal{S}(u\hbar)}e^{-uy_1+u\mathcal{S}(\hbar uz_1\partial_{z_1})\overline{y}_1}\right]\frac{1}{2}\hbar u\mathcal{S}(\hbar uz_1\partial_{z_1})\left(\frac{z_1}{z_1-z_2}-\frac{z_1}{z_1+z_2}\right)\\
&+[\hbar^{2g}]\sum_{\substack{j=1\\ r=0}}^{\infty}\left[D_2^{j-1}[t^j]\frac{1}{Q_2}e^{-2t\psi_2}\,\partial_{y_2}^r e^{2t\frac{\mathcal{S}(t\hbar\partial_{y_2})}{\mathcal{S}(\hbar\partial_{y_2})}\overline{\psi}_2}[u^r]\right.\\
&\left.\frac{e^{\hbar u/2}+e^{-\hbar u/2}}{4u\hbar\mathcal{S}(u\hbar)}e^{-uy_2+u\mathcal{S}(\hbar uz_2\partial_{z_2})\overline{y}_2}\right]\frac{1}{2}\hbar u\mathcal{S}(\hbar uz_2\partial_{z_2})\left(\frac{z_2}{z_2-z_1}-\frac{z_2}{z_2+z_1}\right). \qquad (6.7)
\end{aligned}
$$

Here we use the notation $y_i=y(z_i)$, $\overline{y}_i=\overline{y}(\hbar^2,z_i)$, $\psi_i=\psi(y_i)$, $\overline{\psi}_i=\overline{\psi}(\hbar^2,y_i)$, $Q_i=Q(z_i)$, $X_i=X(z_i)$, $D_i=X_i\partial_{X_i}=Q_i^{-1}z_i\partial_{z_i}$ for $i=1,2$.

Proof Note that all formulas above are odd in both arguments, hence they vanish if any of their arguments vanishes. Hence it is enough to check that $D_1D_2H_{g,2}=W_{g,2}$. In the case $g=0$ it is a straightforward computation. For $g\geq 1$ we recall the relevant special case of Equation (4.23):

$$W_{g,2} = [\hbar^{2g}] \sum_{\substack{j_1,j_2\\ r_1,r_2=0}}^{\infty} \left[\prod_{i=1}^{2} D_i^{j_i} [t_i^{j_i}] \frac{1}{Q_i} e^{-2t_i\psi_i} \partial_{y_i}^{r_i} e^{2t_i \frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i} [u_i^{r_i}] \right]$$
$$\prod_{i=1}^{2} \frac{e^{\hbar u_i/2} + e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)} e^{-u_iy_i+u_i\mathcal{S}(\hbar u_i z_i\partial_{z_i})\overline{y}_i} \left(e^{\hbar^2 u_1u_2\mathcal{S}(\hbar u_1z_1\partial_{z_1})\mathcal{S}(\hbar u_2z_2\partial_{z_2})B(z_1,z_2)} - 1 \right). \tag{6.8}$$

Note that

$$\sum_{j,r=0}^{\infty} D^j [t^j] \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r] = \frac{1}{Q}[u^0] + \sum_{r=0}^{\infty}\sum_{j=1}^{\infty} D^j [t^j] \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r] \tag{6.9}$$

The second summand here can be trivially integrated by applying D^{-1}. In order to integrate the cases of application of $\frac{1}{Q}[u^0]$ we observe that for any $\gamma \in \Gamma_n$ the coefficient of u_i^0 in

$$\begin{aligned} &\frac{1}{Q_i}[u_i^0] \frac{e^{\hbar u_1/2} + e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)} e^{-u_iy_i+u_i\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i} \left(e^{\hbar^2 u_iu_k\mathcal{S}(\hbar u_iz_i\partial_{z_i})\mathcal{S}(\hbar u_kz_k\partial_{z_k})B(z_i,z_k)} - 1 \right) \\ &= D_i \frac{1}{2}\hbar u_k \mathcal{S}(\hbar u_k z_k \partial_{z_k}) \Big(\frac{z_k}{z_k - z_i} - \frac{z_k}{z_k + z_i} \Big). \end{aligned} \tag{6.10}$$

(here $k = 2$ if $i = 1$ and $k = 1$ if $i = 2$), which also admits application of D_i^{-1}. In particular, if we apply this term for both variables, we have:

$$\begin{aligned} &\prod_{i=1}^{2} \left[\frac{1}{Q_i}[u_i^0] \frac{e^{\hbar u_i/2} + e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)} e^{-u_iy_i+u_i\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i} \right] \\ &\left(e^{\hbar^2 u_1u_2\mathcal{S}(\hbar u_1z_1\partial_{z_1})\mathcal{S}(\hbar u_2z_2\partial_{z_2})B(z_1,z_2)} - 1 \right) \\ &= \frac{1}{4Q_1Q_2} B(z_1, z_2), \end{aligned} \tag{6.11}$$

so this case doesn't contribute to $[\hbar^{2g}]$, $g \geq 1$. Combining these computations with the application of $D_1^{-1}D_2^{-1}$, we obtain the statement of the proposition. □

Finally, in the general case of $n \geq 3$ we have the following expression for $H_{g,n}$.

Proposition 6.3 *For a $\gamma \in \Gamma_n$ let I_γ denote the subset of vertices of γ of index ≥ 2. Let $K_\gamma \subset E_\gamma$ be the subset of the set of edges that connect a vertex of index* 1 *to another vertex. When we write $(v_i, v_k) \in K_\gamma$, we assume that v_i is the vertex of index* 1 *(and, therefore, $v_k \in I_\gamma$). We have:*

$$
\begin{aligned}
H_{g,n} = [\hbar^{2g-2+n}] \sum_{\gamma\in\Gamma_n} \prod_{i\in I_\gamma} & \left[\sum_{r_i=0}^{\infty} \sum_{j_i=1}^{\infty} D_i^{j_i-1} [t_i^{j_i}] \frac{1}{Q_i} e^{-2t_i\psi_i} \partial_{y_i}^{r_i} e^{2t_i \frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i} [u_i^{r_i}] \right] \\
& \prod_{i\in I_\gamma} \frac{e^{\hbar u_i/2}+e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)} e^{-u_iy_i+u_i(\mathcal{S}(\hbar u_i z_i\partial_{z_i})\overline{y}_i} \\
& \prod_{(v_k,v_\ell)\in E_\gamma\setminus K_\gamma} \left(e^{\hbar^2 u_k u_\ell \mathcal{S}(\hbar u_k z_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)} - 1 \right) \\
& \prod_{(v_i,v_k)\in K_\gamma} \left(\frac{1}{2}\hbar u_k \mathcal{S}(\hbar u_k z_k\partial_{z_k}) \Big(\frac{z_k}{z_k-z_i} - \frac{z_k}{z_k+z_i} \Big) + \right. \\
& \left[\sum_{r_i=0}^{\infty} \sum_{j_i=1}^{\infty} D_i^{j_i-1} [t_i^{j_i}] \frac{1}{Q_i} e^{-2t_i\psi_i} \partial_{y_i}^{r_i} e^{2t_i \frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i} [u_i^{r_i}] \right] \\
& \left. \frac{e^{\hbar u_i/2}+e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)} e^{-u_iy_i+u_i(\mathcal{S}(\hbar u_i z_i\partial_{z_i})\overline{y}_i} \left(e^{\hbar^2 u_i u_k \mathcal{S}(\hbar u_i z_i\partial_{z_i})\mathcal{S}(\hbar u_k z_k\partial_{z_k})B(z_i,z_k)} - 1 \right) \right).
\end{aligned}
\tag{6.12}
$$

Here, as usual, we use $y_i = y(z_i)$, $\overline{y}_i = \overline{y}(\hbar^2, z_i)$, $\psi_i = \psi(y_i)$, $\overline{\psi}_i = \overline{\psi}(\hbar^2, y_i)$, $Q_i = Q(z_i)$, $X_i = X(z_i)$, $D_i = X_i\partial_{X_i} = Q_i^{-1} z_i\partial_{z_i}$.

Proof Note that $H_{g,n}$ as given in Equation (6.12) vanishes if we set any of its variables to zero (since it is odd in each of its variables). So, the only thing that we have to check is that indeed $D_1\cdots D_n H_{g,n} = W_{g,n}$ as given by Equation (4.23). Recall Equation (4.23):

$$
\begin{aligned}
W_{g,n} =& [\hbar^{2g-2+n}] \sum_{\substack{j_1,\dots,j_n,\\ r_1,\dots,r_n=0}}^{\infty} \left[\prod_{i=1}^{n} D_i^{j_i} [t_i^{j_i}] \frac{1}{Q_i} e^{-2t_i\psi_i} \partial_{y_i}^{r_i} e^{2t_i \frac{\mathcal{S}(t_i\hbar\partial_{y_i})}{\mathcal{S}(\hbar\partial_{y_i})}\overline{\psi}_i} [u_i^{r_i}] \right] \\
& \prod_{i=1}^{n} \frac{e^{\hbar u_i/2}+e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)} e^{-u_iy_i+u_i\mathcal{S}(\hbar u_i z_i\partial_{z_i})\overline{y}_i} \\
& \sum_{\gamma\in\Gamma_n} \prod_{(v_k,v_\ell)\in E_\gamma} \left(e^{\hbar^2 u_k u_\ell \mathcal{S}(\hbar u_k z_k\partial_{z_k})\mathcal{S}(\hbar u_\ell z_\ell\partial_{z_\ell})B(z_k,z_\ell)} - 1 \right).
\end{aligned}
\tag{6.13}
$$

Note that

$$
\sum_{j,r=0}^{\infty} D^j [t^j] \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r] = \frac{1}{Q}[u^0] + \sum_{r=0}^{\infty}\sum_{j=1}^{\infty} D^j [t^j] \frac{1}{Q} e^{-2t\psi} \partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}} [u^r].
\tag{6.14}
$$

The second summand here can be trivially integrated by applying D^{-1}. In order to integrate the cases of application of $\frac{1}{Q}[u^0]$ we observe that for any $\gamma \in \Gamma_n$ the coefficient of u_i^0 in

$$\frac{e^{\hbar u_i/2}+e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{-u_iy_i+u_i\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i}\prod_{(v_i,v_k)\in E_\gamma}\left(e^{\hbar^2u_iu_k\mathcal{S}(\hbar u_iz_i\partial_{z_i})\mathcal{S}(\hbar u_kz_k\partial_{z_k})B(z_i,z_k)}-1\right) \quad (6.15)$$

is non-trivial if and only if i has index 1 in γ. Then there is only one edge $(e_i, e_k) \in E_\gamma$ that is attached to the vertex i. In this case,

$$\begin{aligned}&\frac{1}{Q_i}[u_i^0]\frac{e^{\hbar u_i/2}+e^{-\hbar u_i/2}}{4u_i\hbar\mathcal{S}(u_i\hbar)}e^{-u_iy_i+u_i\mathcal{S}(\hbar u_iz_i\partial_{z_i})\overline{y}_i}\left(e^{\hbar^2u_iu_k\mathcal{S}(\hbar u_iz_i\partial_{z_i})\mathcal{S}(\hbar u_kz_k\partial_{z_k})B(z_i,z_k)}-1\right)\\&=\frac{1}{Q_i}\frac{1}{2}\hbar u_k\mathcal{S}(\hbar u_kz_k\partial_{z_k})B(z_i,z_k)=D_i\frac{1}{2}\hbar u_k\mathcal{S}(\hbar u_kz_k\partial_{z_k})\Big(\frac{z_k}{z_k-z_i}-\frac{z_k}{z_k+z_i}\Big),\end{aligned} \quad (6.16)$$

and we can apply D_i^{-1} to the latter expression. This explains the special summands for $(v_i, v_k) \in K_\gamma$ in Equation (6.12) and completes the proof of the proposition. □

7 Topological recursion for spin Hurwitz number with completed cycles

The goal of this Section is to prove a conjecture proposed by Giacchetto, Kramer, and Lewański. In our terms, it concerns the symmetric n-differentials constructed from Orlov's hypergeometric 2-BKP tau-functions for $\overline{\psi} = \frac{1}{2}\mathcal{S}(\hbar\partial_y)y^{2s}$ and $\overline{y} = z$. But in fact we consider a more general situation, with $\overline{\psi} = \frac{1}{2}\mathcal{S}(\hbar\partial_y)P(y)$ and $\overline{y} = y = R(z)$, where P is an arbitrary even polynomial in y and R is an arbitrary odd polynomial in z, since the arguments in this more general situation do not differ from the ones for the Giacchetto–Kramer–Lewański situation.

Remark 7.1 Note that if we put $\overline{\psi}(y) = \frac{1}{2}\mathcal{S}(\hbar\partial_y)P(y)$, then the weight for the KP hypergeometric tau-function (3.44) does not coincide with the deformation, considered in [4]. Therefore, if in the relation (3.42) one of the tau-functions, τ_{KP} or τ, is described by a suitable version of topological recursion, the other one is not described by it.

7.1 Topological recursion in the odd situation

Consider $\mathbb{C}\mathrm{P}^1$ with a fixed global coordinate z, and with two functions, X and y such that $X(-z) = -X(z)$ and $y(-z) = -y(-z)$, with an extra assumption that dX/X is a rational differential with the simple critical points $p_1, \dots, p_N$ (it is clear that N must be even and the set of critical points is invariant under $z \leftrightarrow -z$) and y is holomorphic near the critical points with $dy|_{p_i} \neq 0$. It is not necessary but both convenient and sufficient for our goals to assume that y is meromorphic. Let

$$\mathcal{B}(z_1, z_2) := \frac{1}{2}\left(\frac{1}{(z_1-z_2)^2}+\frac{1}{(z_1+z_2)^2}\right)dz_1dz_2. \quad (7.1)$$

With this input we construct a system of symmetric differentials $\omega_{g,n}$, $g \geq 0, n \geq 1$, given by

$$\omega_{0,1}(z_1) = y(z_1) d\log X(z_1);$$
$$\omega_{0,2}(z_1, z_2) = \mathcal{B}(z_1, z_2); \tag{7.2}$$

and for $2g - 2 + n > 0$ we use the recursion

$$\omega_{g,n}(z_1, \ldots, z_n) := \frac{1}{2} \sum_{i=1}^{N} \operatorname{Res}_{z \to p_i} \frac{\int_z^{\sigma_i(z)} \mathcal{B}(z_1, \cdot)}{\omega_{0,1}(\sigma_i(z_1)) - \omega_{0,1}(z_1)} \Bigg(\omega_{g-1,n+1}(z, \sigma_i(z), z_{[\![n]\!] \setminus \{1\}}) + \sum_{\substack{g_1+g_2=g,\, I_1 \sqcup I_2 = [\![n]\!] \setminus \{1\} \\ (g_1, |I_1|), (g_2, |I_2|) \neq (0,0)}} \omega_{g_1, 1+|I_1|}(z, z_{I_1}) \omega_{g_2, 1+|I_2|}(\sigma_i(z), z_{I_2}) \Bigg), \tag{7.3}$$

where σ_i is the deck transformation of X near p_i, $i = 1, \ldots, N$. We wouldn't go into the discussion of this peculiar version of this topological recursion, as it should be done in a more general equivariant setup.

For our goals it is sufficient to state the following equivalent reformulation of this version of topological recursion, which is completely parallel to [8, Theorem 2.2] and [6, Section 1].

Lemma 7.1 *A system of meromorphic symmetric differentials $\omega_{g,n}$, $2g - 2 + n > 0$ is obtained from the given starting data (that includes the formulas for $\omega_{0,1}$ and $\omega_{0,2}$) by topological recursion (7.3) if and only if*

(1) This system of differentials satisfies the blobbed topological recursion (see Definition 5.1).
(2) For any $g \geq 0, n \geq 1, 2g - 2 + n > 0$

$$\omega_{g,n}(z_{[\![n]\!]}) = \sum_{i_1, \ldots, i_n = 1}^{N} \left(\prod_{j=1}^{n} \operatorname{Res}_{w_j \to p_{i_j}} \int_{p_{i_j}}^{w_i} \mathcal{B}(\cdot, z_j) \right) \omega_{g,n}(w_{[\![n]\!]}) \tag{7.4}$$

(this is the so-called projection property).

Proof The same argument as in [8, Section 2.4]. □

If we represent the symmetric differential $\omega_{g,n}$ as $\omega_{g,n} = 2^{1-g} W_{g,n} \prod_{i=1}^{n} d\log X_i$, $\omega_{0,2} = 2W_{0,2} d\log X_1 d\log X_2 + \mathcal{B}(X_1, X_2)$, where $W_{g,n} = D_1 \cdots D_n H_{g,n}$, $D_i = X_i \partial_{X_i}$, then the linear loop equations in combination with the projection property can be equivalently reformulated in terms of $H_{g,n}$. This reformulation can be directly applied in the odd case that we consider here, and we recall it and prove for the particular n-point functions of spin Hurwitz numbers with completed cycles in the next section, Sect. 7.2.

7.2 Quasi-polynomiality

The goal of this section is to prove some special property of the functions $H_{g,n}$, and, as a corollary, $W_{g,n}$'s that is sometimes called quasi-polynomiality in the literature and in the context of topological recursion is equivalent to a combination of the so-called projection property and the liner loop equations. We refer to [4, Section 3] for a full discussion.

Recall that with $\overline{\psi} = \frac{1}{2}\mathcal{S}(\hbar\partial_y)P(y)$, $P(-y) = P(y)$ is a polynomial, and $\overline{y} = y = R(z)$, $R(-z) = -R(z)$ is a polynomial we have $X = z\exp(-P(R(z)))$. Let $p_1, \dots, p_N \in \mathbb{C}\mathrm{P}^1$ be the critical points of X. Here $N = \deg P \cdot \deg R \in 2\mathbb{Z}$, we assume that all critical points are simple, and the set of critical points is obviously invariant under the involution $z \leftrightarrow -z$.

Define the space Θ_n as the linear span of functions $\prod_{i=1}^{N} f_i(z_i)$, where each $f_i(z_i)$ is a rational function on $\mathbb{C}\mathrm{P}^1$, $f_i(-z_i) = -f_i(z_i)$, f_i has poles only at the points $p_1, \dots, p_N$, and the principal part of f_i at p_k, $k = 1, \dots, N$, is odd with respect to the corresponding deck transformation σ_k of function X near p_k. The last condition can be reformulated as a requirement that for any $k = 1, \dots, N$ the locally defined function $f_i(z_i) + f_i(\sigma_k z_i)$ is holomorphic at $z_i \to p_k$.

Proposition 7.1 *In the case $\overline{\psi} = \frac{1}{2}\mathcal{S}(\hbar\partial_y)P(y)$, $P(-y) = P(y)$ is a polynomial, and $\overline{y} = y = R(z)$, $R(-z) = -R(z)$ is a polynomial, the functions $H_{g,n}$ belong to the space Θ_n, for any $n \geq 1$, $g \geq 0$ such that $2g - 2 + n > 0$.*

Proof In the proof we analyze the formulas obtained in Propositions 6.1, 6.2, and 6.3. It is clear from the structure of the formulas (6.1), (6.7), and (6.12) that with our assumptions $H_{g,n}$ are rational functions in $z_1, \dots, z_n$.

Consider $H_{g,n}$ as a function of z_1, treating the rest of the variables as parameters. From the shape of the formula we see that it might have poles at $z_1 \to \pm z_i$, $i = 2, \dots, n$, $z_1 \to \infty$, and at the zeros of Q. In this case $Q = z\partial_z \log X = 1 + z\partial_z P(R(z))$, and its zeros are exactly $p_1, \dots, p_N$.

From Remark 4.3 it follows that there are no singularities at $z_1 = \pm z_i$, $i = 2, \dots, n$.

In all terms of the formulas (6.1), (6.7), and (6.12) the principal part at $z_1 \to p_k$ is generated by the iterative application of the operator $D_1 = X_1\partial_{X_1} = Q(z_1)^{-1}z_1\partial_{z_1}$ to a function that is either holomorphic at $z_1 \to p_k$ (as in the first summand of (6.1)), or has a simple pole at $z_1 \to p_k$ (as in the second summand of (6.1), where we divide a function holomorphic at $z_1 \to p_k$ by $Q(z_1)$). Holomorphic functions and functions with a simple pole automatically have principal parts at $z_1 \to p_k$ that are odd with respect to the deck transformation at p_k, and the operator D_1 preserves this property (while increasing the order of the pole at p_k).

Let us now check that there is no pole at $z_1 \to \infty$. Note that the terms that really look special, the last two summands in Equation (6.1), vanish with our assumptions (and that is crucially important since for any other choice of $\overline{\psi}$ and $\overline{y}$ with given $\psi = P$ and $y = R$ it wouldn't be the case). To all other terms in the formulas (6.1), (6.7), and (6.12) the same rough estimation of the order of pole is applicable, cf. [4, Lemma 4.6]. We perform it here only for the second summand in Equation (6.1), since in all other cases the analysis is exactly the same. To this end, consider

$$[\hbar^{2g}]\sum_{j=1}^{\infty} D^{j-1}[t^j]\sum_{r=0}^{\infty}\frac{1}{Q}e^{-2t\psi}\partial_y^r e^{2t\frac{\mathcal{S}(t\hbar\partial_y)}{\mathcal{S}(\hbar\partial_y)}\overline{\psi}}[u^r]\left(\frac{e^{\frac{\hbar u}{2}}+e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)}e^{-uy+u\mathcal{S}(\hbar u z\partial_z)\overline{y}}\right)$$

$$= [\hbar^{2g}] \sum_{j=1}^{\infty} D^{j-1} [t^j] \sum_{r=0}^{\infty} \frac{1}{Q} (\partial_y + 2tP'(y))^r e^{2t(\mathcal{S}(t\hbar\partial_y)-1)P(y)}$$
$$[u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)} e^{u(\mathcal{S}(\hbar u z \partial_z)-1)R(z)} \right) \tag{7.5}$$

Note that the operator $D = Q^{-1} z\partial z$ decreases the order of pole at $z \to \infty$ by $\deg Q = \deg P \deg R$. The same holds for the factor Q^{-1} alone. This means that the order of pole in (7.5) at $z \to \infty$ is equal to the order of pole at $z \to \infty$ of

$$[\hbar^{2g}] \sum_{r=0}^{\infty} (\partial_y + 2tP'(y))^r e^{2t(\mathcal{S}(t\hbar\partial_y)-1)P(y)} \Big|'_{t=z^{-\deg P \deg R}}$$
$$[u^r] \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)} e^{u(\mathcal{S}(\hbar u z \partial_z)-1)R(z)} \right), \tag{7.6}$$

where by $|'$ we mean that we only select the terms with $\deg t \geq 1$. With this substitution observe that each application of the operator $\partial_y + 2tP'(y)$ decreases the order at $z \to \infty$ by $\deg R$. Thus the order of pole of (7.6) at $z \to \infty$ is equal to the order of pole at $z \to \infty$ of

$$[\hbar^{2g}] e^{2t(\mathcal{S}(t\hbar\partial_y)-1)P(y)} \left(\frac{e^{\frac{\hbar u}{2}} + e^{-\frac{\hbar u}{2}}}{4u\mathcal{S}(u\hbar)} e^{u(\mathcal{S}(\hbar u z \partial_z)-1)R(z)} \right) \Big|'_{t=z^{-\deg P \deg R}} \Big|''_{u=z^{-\deg R}}, \tag{7.7}$$

where by $|''$ we mean that we only select the terms with $\deg u \geq 0$. The latter expression is manifestly regular at $z \to \infty$.

Finally, extending our arguments to all variables $z_1, \dots, z_n$, we obtain that $H_{g,n}(z_1, \dots, z_n)$, $2g - 2 + n > 0$, is a rational function that in each of its variables has poles only at the points $p_1, \dots, p_N$ with the odd principal parts with respect to the corresponding deck transformations. This immediately implies that $H_{g,n} \in \Theta_n$. □

7.3 Giacchetto–Kramer–Lewański conjecture and its generalization

Consider the n-functions $H_{g,n}$ constructed from Orlov's hypergeometric BKP tau-functions for $\overline{\psi} = \frac{1}{2}\mathcal{S}(\hbar\partial_y)P(y)$ and $\overline{y} = y = R(z)$, where P is an arbitrary even polynomial in y and R is an arbitrary odd polynomial in z. Recall $X = X(z) = z\exp(-P(R(z)))$. Recall that we defined $W_{g,n} = D_1 \cdots D_n H_{g,n}$, and we set

$$\omega_{g,n}(z_{[\![n]\!]}) := 2^{1-g} W_{g,n}(X_{[\![n]\!]}) \prod_{i=1}^{n} \frac{dX_i}{X_i} + \delta_{g,0}\delta_{n,2} \frac{1}{2} B(X_1, X_2) d\log X_1 d\log X_2, \tag{7.8}$$

With this assignment, it follows from Propositions 4.3 and 4.4 that

$$\omega_{0,1}(z) = y\, d \log X \qquad \text{and} \qquad \omega_{0,2}(z_1, z_2) = \mathcal{B}(z_1, z_2). \tag{7.9}$$

For all other $\omega_{g,n}$, $g \geq 0$, $n \geq 1$, $2g - 2 + n > 0$, we have the following theorem

Theorem 7.1 *(Generalized Giacchetto–Kramer–Lewański conjecture) The symmetric n-differentials $\omega_{g,n}$ are obtained by the odd topological recursion (7.3) for the initial data $X = z \exp(-P(R(z)))$ and $y = R(z)$.*

Proof According to Lemma 7.1 we have to check the blobbed topological recursion and the projection property. The blobbed topological recursion follows from Theorem 5.1, which is proved in a much more general situation (it is obvious that the analytic assumptions listed in Sect. 5.1 are satisfied). On the other hand, the linear loop equations and the projection property are equivalent to the statement of Proposition 7.1. □

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