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Asymptotic quasi-normal frequencies, horizon area spectra and multi-horizon spacetimes

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Abstract. There exists a long term debate about a possible link of the asymptotic quasinormal (QNM) modes to the black hole thermodynamics. The first conjecture providing a link between the classical asymptotic black hole oscillation frequencies and black hole thermodynamics was formulated more than a decade ago by Hod. The conjecture was later modified by Maggiore. We analyse the behavior of the asymptotic frequencies of the spherically symmetric multi-horizon spacetimes (in particular Reissner-Nordström, Schwarzschild-deSitter, Reissner-Nordström-deSitter spacetime) and provide some suggestions for how to interpret the results following the spirit of the modified Hod's conjecture. The interpretation suggested is in some sense analogical to the Schwarzschild black hole case, but has some new specific features. This paper[1] refers to work done over a longer period of time contained in papers [1, 2, 3, 4] and also to some extent in [5].

1. Introduction

Black holes, when perturbed, show certain characteristic damped oscillations which are called quasi-normal modes (QNMs). Consider (scalar, electromagnetic, gravitational) perturbation of a spherically symmetric black hole spacetime given by a line element

$$-f(r,c_i)\cdot dt^2 + f^{-1}(r,c_i)\cdot dr^2 + r^2\cdot d\Omega^2. \tag{1}$$

By c_i we mean black hole spacetime parameters, such as mass, charge, or eventually in the asymptotically non-flat spacetime the cosmological constant. After decomposing the perturbation into tensor spherical harmonics and furthermore applying the normal modes decomposition, one can reduce the problem for the dynamics of the perturbations into a 1-dimensional stationary Schrödinger-like equation:

$$\frac{d^2\phi_\ell(x)}{dx^2} + \Big[\omega^2 - V(x)\Big]\cdot\phi_\ell(x) = 0. \tag{2}$$

By x we mean the tortoise coordinate defined as

$$\frac{dr}{dx} = f(r),$$

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and $V(x)$ is a Regge-Wheeler/Zerilli potential which depends on the black hole background parameters, the type of the perturbation, the parity of the perturbation and the wave mode number ℓ. The quasi-normal modes (QNMs) are solutions of the equation (2) such that correspond to the purely outgoing wave boundary conditions. (There are some subtleties, due to which one needs to slightly refine this simply sounding definition, but these are not going to be analysed here.) From the definition of the QNMs one can calculate for each perturbation the QNM frequencies ω. They are complex non-real frequencies, (for the $\exp(i\omega t)$ normal mode convention the frequencies have positive imaginary parts). The QNM frequencies depend on the wave-mode number and the black hole parameters. Furthermore, typically they form an infinite discrete set with unbounded imaginary part ω_I, $(\omega = \omega_R + i \cdot \omega_I)$. The physical meaning of the QNM frequencies is that they are the poles of the Green function and due to this fact the QNMs dominate within a specific time scale the time evolution of an *arbitrary* compactly supported black hole perturbation. Thus QNMs can be described as the characteristic black hole oscillations, (or the “sound” of black holes).

2. The highly damped modes and conjectured connection to quantum gravity

The highly damped (asymptotic) frequencies, defined by ℓ fixed and $|\omega_I| \to \infty$, are in the known cases independent on the wave-mode number and depend only on the black hole parameters. For the Schwarzschild black hole they are given as, (κ is the surface gravity of the black hole horizon):

$$\omega_n = (\text{offset}) + i(\text{gap}) \cdot n + O(n^{-1/2}) = 4\ln(3) \cdot \kappa + i\kappa \cdot \left(n + \frac{1}{2}\right) + O(n^{-1/2}). \tag{3}$$

Hod [6] used Bohr’s correspondence principle to conjecture that the asymptotic modes might be linked to the quantum black hole state transition. He linked the mass of the emitted quantum to the real part of asymptotic frequencies as, (everywhere in the paper we use the Planck units):

$$\Delta M = \lim_{n\to\infty} \omega_{nR}.$$

This suggestion plugged into the first law of black hole mechanics gives an equispaced horizon area spectrum with the area quantum $\Delta A = 4\ln(3)$.

Such quantum is of the form $\Delta A = 4\ln(k)$, $k \in \mathbf{N}$, and horizon area spectrum with the quantum of such form was already previously suggested due to the statistical interpretation of entropy [7]. Moreover, in [8] a connection was made between the asymptotic real part of the frequencies and the Immirzi parameter in Loop Quantum Gravity. The same results as in [6] can be obtained also by a different line of thinking: by using Bohr-Sommerfeld quantization condition for the adiabatic invariants and by considering the frequency asymptotic real part $\omega_{\infty R}$ to be the black hole’s characteristic frequency [9]. However, Hod’s conjecture led to many difficulties and as a result Maggiore suggested a modification which removed most of the complications [10]. According to Maggiore’s modification of Hod’s conjecture (or shorter, as we use in this paper, Maggiore’s conjecture) QNMs have to be considered as a collection of damped harmonic oscillators and the mass quantum is connected to:

$$\Delta M = \lim_{n\to\infty} \Delta_{(n,n-1)} \sqrt{\omega_{nR}^2 + \omega_{nI}^2}. \tag{4}$$

Typically it holds that:

$$\lim_{n\to\infty} \Delta_{(n,n-1)} \sqrt{\omega_{nR}^2 + \omega_{nI}^2} = \lim_{n\to\infty} \Delta_{(n,n-1)} \omega_{nI}$$

and thus

$$\Delta M = \lim_{n\to\infty} \Delta_{(n,n-1)}\omega_{nI}.$$

In the case of Schwarzschild black hole this gives an equispaced area spectrum with a quantum of a form originally suggested by Bekenstein [11]:

$$\Delta A = 8\pi.$$

(The non-statistical form the of area quantum is thought not to be a problem, because of the analysis being essentially of a semi-classical nature.)

3. Spherically symmetric multi-horizon black hole spacetimes

For the multi-horizon spherically symmetric black hole spacetimes, the Reissner-Nordström spacetime (R-N), Schwarzschild-deSitter spacetime (S-dS), Reissner-Nordström-deSitter spacetime (R-N-dS), and the scalar, vector, tensor perturbations, the master equations for the asymptotic QNM frequencies were derived in [12, 13, 14, 15] and can be all written in the form:

$$\sum_{A=1}^{M} C_A \cdot \exp\left(\sum_{i=1}^{N} Z_{Ai}\frac{2\pi\omega}{|\kappa_i|}\right) = 0. \tag{5}$$

Here Z_{Ai} is an $M \times N$ matrix, such that takes one of the values $Z_{Ai} = \{0, 1, 2\}$, furthermore N is the number of horizons and κ_i are the surface gravities of the different horizons. The solutions of this formula have in general much more complicated behaviour as the asymptotic QNM frequencies of the Schwarzschild single horizon case (3). The fact that there do not generally exist analytic solutions for the formulas (5) prevented people to derive general consequences of the Maggiore's conjecture for the multi-horizon spherically symmetric cases (apart of some simplifying limits, like for example for the R-N black hole in a small charge limit [16]).

In [3, 4] we analysed the behaviour of the solutions of (5) with the following results: If the ratio of all of the surface gravities is a rational number then all the frequencies split in a finite number of equispaced families, (labeled by a), of the form:

$$\omega_{an} = (\text{offset})_a + in \cdot \ \text{lcm}(|\kappa_1|, |\kappa_2|, ..., |\kappa_N|). \tag{6}$$

Here "lcm" means the least common multiple of the numbers in the bracket, hence

$$\text{lcm}(|\kappa_1|, |\kappa_2|, ..., |\kappa_N|) = p_1 \cdot |\kappa_1| = ... = p_N \cdot |\kappa_N|,$$

where $\{p_1, ..., p_N\}$ is a set of relatively prime integers. If the ratio of two of the surface gravities is irrational, then there does *not* exist an equispaced subsequence in the sequence of asymptotic QNM frequencies. This means, although the master equations (5) cannot be in general analytically solved, one can still extract a lot of key analytic information about the behaviour of the solutions.

Moreover, the analytical insights, (without explicitly analytically knowing the solutions), are enough to prove that for the R-N black hole hold the following statements [2] (they were strictly proven for R-N black hole, but one expects them to hold for all the three cases):

1. In case the ratio of the surface gravities is irrational, the $n \to \infty$ limit for $\Delta_{(n,n-1)}\omega_{nI}$ does *not* exist.
2. For the rational ratio of the surface gravities and the R-N black hole the only case in which the limit $n \to \infty$ of $\Delta_{(n,n-1)}\omega_{nI}$ exists is if all the frequencies are given by families of the form (6) with the same (offset). But this cannot be the case when the ratio of the surface gravities is given by two relatively prime integers whose product is an odd number [2].

The previous considerations seem to suggest, that the modified Hod's conjecture has very little chance to survive the multi-horizon case. However the significantly different behaviour for the cases of rational / irrational ratios of surface gravities and the general splitting of frequencies into families (6) seem to indicate something important. Moreover, it was already observed that surface gravities rational ratios have significant consequences for the multi-horizon spacetime thermodynamics [17]. Based on this observations let us pick the R-N case where the thermodynamical interpretation is straightforward (but keep in mind that all the calculations can be repeated in an exact analogy for the other S-dS, R-N-dS cases) and consider the following [1]: Let us presuppose that both of the horizons in the R-N spacetime, the outer horizon with the area A_+ and the inner Cauchy horizon with the area A_- have equispaced area spectra given as $A_\pm = 8\pi\gamma n_\pm$.

The perturbations are supposed to carry *no* charge, so one expects that only the ADM mass of the black hole will be changed. Thus one can write the change of the areas of the black hole horizons as:

$$\Delta A_\pm = \frac{8\pi \Delta M}{\kappa_\pm}. \tag{7}$$

But $\Delta A_\pm$ can be given only as $\Delta A_\pm = 8\pi\gamma m_\pm$, which implies $\Delta M = \gamma\kappa_\pm m_\pm$. Furthermore, this implies

$$\kappa_+ m_+ = \kappa_- m_- \quad \rightarrow \quad \frac{\kappa_+}{\kappa_-} = \frac{m_-}{m_+}. \tag{8}$$

This means, that if the single ADM mass transitions have to be allowed, the surface gravities ratio must be rational. Furthermore, if one wants the emitted mass quantum to be as small as possible, such that it is still compatible with the quantization of the two horizon areas, one obtains:

$$\Delta M = \gamma \cdot \mathrm{lcm}(\kappa_+, |\kappa_-|). \tag{9}$$

Then modified Hod's conjecture suggests that

$$\lim_{n\to\infty} \Delta_{(n,n-1)}\omega_{nI} = \gamma \cdot \mathrm{lcm}(\kappa_+, |\kappa_-|). \tag{10}$$

This is indeed true if one takes the following interpretation of the frequencies (slight modification of Maggiore's conjecture): the straightforward extension of Maggiore's conjecture to the multi-horizon case is misleading, in fact only the equispaced families carry information about the quantum black hole mass transitions. (Every frequency belongs to one of the families.) Thus one has to first identify the equispaced families and then take the limit in the spacing in the imaginary part of the frequencies within each of the families. Such interpretation then fixes together with the formula (6) the γ parameter to be $\gamma = 1$. This means the area spectra of both of the horizons are given as $8\pi n$.

Note the importance of the surface gravities rational ratio condition in the preceeding argumentation: If the condition would not be fulfilled, then the absence of periodic sub-structures in case of irrational surface gravity ratio, together with the non-existence of the limit $n \to \infty$ of $\Delta_{(n,n-1)}\omega_{nI}$ means, that in such case does not seem to be even a slightly modified way how to understand the highly damped modes via Maggiore's conjecture. The fact that the rational ratios of the surface gravities are by the logic developed in this section guaranteed to exist seems to be another coincidence that "miraculously" fits into the picture. (On the other hand, since rational numbers are dense in the set of real numbers, we leave open the question of how much such rational ratios correspond to some quantum characteristics of surface gravities / horizon temperatures.)

Let us remind here, that the same analysis can be repeated for both S-dS and R-N-dS spacetimes: Assuming that all the horizons have the same equispaced area spectra, the single M parameter transitions lead to the surface gravities rational ratio condition and the QNM

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frequencies given by the formula (6) fix the spectra of all the horizons to be $8\pi n$, (after one considers our generalization/modification of Maggiore's conjecture). Let us also mention that the same area quantization of all the horizons in spherically symmetric multi-horizon spacetimes was obtained by a different (although not completely unrelated) way of thinking by [18].

4. Discussion

To summarize: We suppose that also in the multi-horizon case the *modified* Hod's conjecture provides information about the spacetime horizons spectra, only the way the information is encoded is more tricky than in the single horizon case. (This is hardly anything surprising as the quantization of more than one horizon might play a role in the game.) The QNM frequencies are consistent with each of the horizons being quantized with spectra given as $8\pi n$ (in Planck units). If these conclusions are accepted then still many open questions remain, for example: 1. Can be the results from this paper generalized to an arbitrary static spherically symmetric multi-horizon spacetime? (The results of [19] indicate that the behaviour of the asymptotic frequencies described in this paper holds at least in case of a general two-horizon spacetime, with one black hole horizon and one cosmological de-Sitter horizon.) 2. Could similar interpretation survive in case of charged black holes that occur after a collapse of matter? (In such case the black hole interior is very different to the extremely idealized R-N, R-N-dS cases. For example a weak mass-inflation singularity occurs at the inner horizon, but some results indicate that possibility of crossing the inner horizon might still be considered [20].)

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