

THE SECOND FUNDAMENTAL THEOREM OF INVARIANT THEORY FOR THE ORTHOSYMPLECTIC SUPERGROUP

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Abstract. The first fundamental theorem of invariant theory for the orthosymplectic supergroup scheme $\mathrm{OSp}(m|2n)$ states that there is a full functor from the Brauer category with parameter $m - 2n$ to the category of tensor representations of $\mathrm{OSp}(m|2n)$. This has recently been proved using algebraic supergeometry to relate the problem to the invariant theory of the general linear supergroup. In this work, we use the same circle of ideas to prove the second fundamental theorem for the orthosymplectic supergroup. Specifically, we give a linear description of the kernel of the surjective homomorphism from the Brauer algebra to endomorphisms of tensor space, which commute with the orthosymplectic supergroup. The main result has a clear and succinct formulation in terms of Brauer diagrams. Our proof includes, as special cases, new proofs of the corresponding second fundamental theorems for the classical orthogonal and symplectic groups, as well as their quantum analogues, which are independent of the Capelli identities. The results of this paper have led to the result that the map from the Brauer algebra $\mathcal{B}_r(m - 2n)$ to endomorphisms of $V^{\otimes r}$ is an isomorphism if and only if $r < (m + 1)(n + 1)$.

CONTENTS

1	Introduction	53
1.1	Superalgebraic geometry	54
1.2	Linear superalgebra and the group scheme $\mathrm{GL}(V)$	54
1.3	The orthosymplectic supergroup $\mathrm{OSp}(V)$	55
2	Invariant theory for $\mathrm{GL}(V)$	56
2.1	Symmetric group action	56
2.2	Invariant theory for $\mathrm{GL}(V)$ -the first fundamental theorem	56

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2.3	Invariant theory for $GL(V)$ -the second fundamental theorem	57
2.4	The SFT for $GL(V)$ -second formulation	59
3	Invariant theory for the orthosymplectic supergroup	59
3.1	The Brauer category and FFT for $OSp(V)$	59
4	From invariants of $OSp(V)$ to invariants of $GL(V)$.	62
4.1	Extension of functions	62
4.2	Construction of invariants of $GL(V)$ from those of $OSp(V)$	64
4.3	Diagrammatic interpretation	65
5	The second fundamental theorem for $OSp(V)$	66
5.1	The main theorem	67
5.2	Interpretation in the first formulation	69
6	Application to the classical groups	70
6.1	The orthogonal case: $G = O(m, \mathbb{C})$	70
6.2	The symplectic case	71
	Appendix A. Superpolarization	72
	References	74

§1. Introduction

This paper is a sequel to [19] and [6], in which we proved the first fundamental theorem (FFT) of invariant theory for the orthosymplectic group superscheme $G = OSp(m|2n, \mathbb{C})$. The FFT may be interpreted either in terms of the group superscheme $G = OSp(m|2n, \mathbb{C})$ or in terms of the points $G(\Lambda)$ of G over the infinite dimensional Grassmann algebra Λ , which form an actual group of matrices over Λ . In this work, we adopt the scheme-theoretic point of view of [6]. The FFT provides a set of generators for the invariants of G on $V^{\otimes r}$, where V is the “natural” representation of G on the superspace V of superdimension $(m|2n)$; this theorem is equivalent to the statement that there is a surjection $\mathcal{B}_r(m-2n) \rightarrow \text{End}_G(V^{\otimes r})$, where $\mathcal{B}_r(m-2n)$ is the r -string Brauer algebra with parameter $m-2n$. In this paper, we give a linear description of all relations among these generators; this is the second fundamental theorem (SFT) of invariant theory.

The thrust of the geometric approach we use is to reduce the questions concerning $OSp(m|2n, \mathbb{C})$ to similar questions concerning the action of $GL(m|2n, \mathbb{C})$ on a larger space. In this work, we show that this reduction may be performed so as to throw light on the SFT for $OSp(m|2n, \mathbb{C})$ using a combination of geometric and diagrammatic methods.

In this section, we briefly recall the results of [6, 19] in a form convenient for the present exposition.

1.1 Superalgebraic geometry

We refer to [6, Section 2] for the basic facts concerning superschemes. We shall work in the category of superschemes over the complex field \mathbb{C} . Each such superscheme S gives rise to a functor on the category of (super) commutative \mathbb{C} -superalgebras

$$S : A \mapsto S(A) := \text{Hom}(\text{Spec}(A), S),$$

which takes the superalgebra A to the set of A -points of S . This is the functor of points associated with S with which we identify S . We shall use the term “function” on a superscheme S to mean a global section of the structure sheaf $\mathcal{O} = \mathcal{O}_S$ of S .

If $W_{\mathbb{C}}$ is a finite dimensional complex superspace, one associates to it the superscheme $W = \text{Spec}(S(W_{\mathbb{C}}^*))$, whose corresponding functor of points \mathcal{W} takes a supercommutative superalgebra A to $\mathcal{W}(A) := (W \otimes_{\mathbb{C}} A)_{\bar{0}}$. In the special case $W_{\mathbb{C}} = \mathbb{C}$ (even), the corresponding superscheme is denoted as \mathbb{G}_a . When $W_{\mathbb{C}}$ has an extra structure, for example, if it is a \mathbb{C} -algebra, the scheme W will inherit that structure. Thus, \mathbb{G}_a is an additive group.

Given an action of an affine supergroup G over \mathbb{C} on a (complex) supervector space W , for any supercommutative superalgebra A , $G(A)$ acts on $W \otimes_{\mathbb{C}} A$.

DEFINITION 1.1. A vector $w \in W$ is said to be invariant under G if for any A , $w \otimes 1$ is fixed by each element of $G(A)$. Denote the space of invariants by W^G . This is a vector subspace of W .

Equivalently, in the language of Hopf algebras, if $F(G)$ is the coordinate superalgebra of G (so $G = \text{Spec}(F(G))$), we have the coaction $W \rightarrow W \otimes F(G)$ as well as the map $w \mapsto w \otimes 1 \in W \otimes F(G)$. Then $W^G = \text{Ker}(W \rightrightarrows W \otimes F(G))$.

If the functor of points associated with the fixed point space W^G is written \mathcal{W}^G , then for any supercommutative superalgebra A , we have $\mathcal{W}^G(A) = ((W \otimes_{\mathbb{C}} A)^{G(A)})_{\bar{0}}$.

1.2 Linear superalgebra and the group scheme $\text{GL}(V)$

See [19, Sections 2.1, 2.2]. We denote by $V = (V)_{\bar{0}} \oplus (V)_{\bar{1}}$ a $\mathbb{Z}/2\mathbb{Z}$ -graded complex vector space of superdimension $\text{sdim } V = (m|2n)$ so that

$\dim(V)_{\bar{0}} = m$ and $\dim(V)_{\bar{1}} = 2n$. The supersymmetric algebra $S(V^*)$ is the supercommutative super \mathbb{C} -algebra defined in [19, Section 3.2], and the scheme \mathcal{V} corresponding to V is $\mathcal{V} := \text{Spec}(S(V^*))$. Its associated functor of points is described above.

As a special case, we have the affine superscheme $\mathcal{E}nd(V)$ of endomorphisms of V , which is associated with the supervector space $\text{End}(V) := \text{End}_{\mathbb{C}}(V)$; it has the open subscheme $\text{GL}(V)$ (see [6, 2.10]) whose associated functor of points takes the supercommutative superalgebra A to $\text{Aut}_A(V \otimes_{\mathbb{C}} A)$.

1.3 The orthosymplectic supergroup $\text{OSp}(V)$

Consider a nondegenerate even bilinear form

$$(1.1) \quad B := (-, -) : V \times V \longrightarrow \mathbb{C},$$

which is supersymmetric, that is, $(u, v) = (-1)^{[u][v]}(v, u)$ for all $u, v \in V$. This implies that the form is symmetric on $(V)_{\bar{0}} \times (V)_{\bar{0}}$, skew symmetric on $(V)_{\bar{1}} \times (V)_{\bar{1}}$, and satisfies $(V_{\bar{0}}, V_{\bar{1}}) = 0 = (V_{\bar{1}}, V_{\bar{0}})$. Also, by nondegeneracy, $\dim(V)_{\bar{1}} = 2n$ must be even. We call this a nondegenerate *supersymmetric form* on V . Let $\eta = \begin{pmatrix} I_m & 0 \\ 0 & J_n \end{pmatrix}$, where I_m is the identity matrix of size $m \times m$ and J_n is a skew symmetric matrix of size $2n \times 2n$ given by $J_n = \text{diag}(\sigma, \dots, \sigma)$ with $\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then there exists an ordered homogeneous basis $\mathcal{E} = (e_1, e_2, \dots, e_{m+2n})$ of V such that

$$(1.2) \quad (e_a, e_b) = \eta_{ab}, \quad \text{for all } a, b.$$

As in [6, Example 2.6], the form $(-, -)$ gives rise to a bilinear morphism of functors

$$(1.3) \quad B : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{G}_a,$$

which corresponds (cf. [6, Section 2.7]) to a quadratic form $q \in S^2(V)_{\bar{0}}$, which is related to B via

$$(1.4) \quad \begin{aligned} q(x) &= \frac{1}{2}B(x, x) & \text{and} \\ B(x, y) &= q(x + y) - q(x) - q(y). \end{aligned}$$

DEFINITION 1.2. [6, 2.11] The orthosymplectic group superscheme $G = \text{OSp}(V)$ is defined as the subscheme of $\text{GL}(V)$ which preserves the form $B(-, -)$ or, equivalently, which preserves the quadratic form q . Explicitly, $G = \{g \in \text{GL}(V) \mid q \circ g^{-1} = q\}$.

An A -point of the group scheme corresponding to $\mathrm{OSp}(V)$ is an automorphism of $V \otimes_{\mathbb{C}} A$ which fixes the A -bilinear form induced on $V \otimes_{\mathbb{C}} A$ by $B(-, -)$.

§2. Invariant theory for $\mathrm{GL}(V)$

In this section, we state the two fundamental theorems of invariant theory for the supergroup $\mathrm{GL}(m|\ell)$ in a form convenient for use in our context.

2.1 Symmetric group action

In this section, we take V to be a \mathbb{C} -superspace with $\mathrm{sdim}(V) = (m|\ell)$ and, as above, write \mathcal{V} for the corresponding (super) scheme whose associated functor of points is described above. Given any two \mathbb{Z}_2 -graded \mathbb{C} -vector spaces, V, W , we may form the tensor product $V \otimes_{\mathbb{C}} W$ and $\mathrm{Hom}_{\mathbb{C}}(V, W)$. These are \mathbb{Z}_2 -graded in the usual way. Thus, $\mathrm{GL}(V)$ acts on the superspaces $V^{\otimes r}$ for $r = 1, 2, \dots$. We have a superpermutation $\tau : V \otimes W \longrightarrow W \otimes V$, given by

$$(2.1) \quad \tau(v \otimes w) = (-1)^{[v][w]} w \otimes v,$$

for homogeneous $v \in V, w \in W$, and extended linearly. When $V = W$, $\tau \in \mathrm{End}_{\mathrm{GL}(V)}(V \otimes V) = (\mathrm{End}_{\mathbb{C}}(V \otimes V))^{\mathrm{GL}(V)}$.

Bearing in mind the definition of invariants in Definition 1.1, for any r , we have a homomorphism of \mathbb{C} -algebras

$$(2.2) \quad \varpi_r : \mathbb{C}\mathrm{Sym}_r \longrightarrow \mathrm{End}(V^{\otimes r})^{\mathrm{GL}(V)},$$

in which the simple transpositions in Sym_r are mapped to the endomorphisms τ of (2.1), acting on the appropriate factors of the product.

Since our strategy for studying $\mathrm{OSp}(V)$ is to reduce to the case of $\mathrm{GL}(V)$, we next state the fundamental theorems for $\mathrm{GL}(V)$.

2.2 Invariant theory for $\mathrm{GL}(V)$: the FFT

The FFT for $\mathrm{GL}(V)$ may be stated as follows.

THEOREM 2.1. [4, Theorems 3.3 and 3.7] *The map ϖ_r of (2.2) is surjective for all r .*

This theorem can be proved in similar manner to the classical case using the superpolarisation result in Appendix A.

Now for any pair of complex vector spaces V, W , the canonical isomorphism

$$(2.3) \quad \mathrm{Hom}_{\mathbb{C}}(V, W) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{C}}(V \otimes_{\mathbb{C}} W^*, \mathbb{C})$$

of supervector spaces is $\mathrm{GL}(V)$ -equivariant. Hence, we have the following reformulation of Theorem 2.1. First, observe that the element

$$(2.4) \quad \lambda_0 : (v_1, \dots, v_r, \phi_1, \dots, \phi_r) \mapsto \phi_1(v_1)\phi_2(v_2) \cdots \phi_r(v_r)$$

lies in $(\mathrm{Hom}(V^{\otimes r} \otimes (V^*)^{\otimes r}, \mathbb{C}))^{\mathrm{GL}(V)}$; that is, it is invariant according to Definition 1.1.

Next, note that Sym_r acts on $\mathrm{Hom}(V^{\otimes r} \otimes (V^*)^{\otimes s}, \mathbb{C})$ via the map $\lambda \mapsto \pi(\lambda) := \lambda \circ (\varpi_r(\pi^{-1}) \times \mathrm{id}_{(V^*)^{\otimes s}})$ (for $\lambda \in \mathrm{Hom}(V^{\otimes r} \otimes (V^*)^{\otimes s}, \mathbb{C})$ and $\pi \in \mathrm{Sym}_r$).

For $\pi \in \mathrm{Sym}_r$, let $\delta_\pi = \pi(\lambda_0) \in \mathrm{Hom}(V^{\otimes r} \otimes (V^*)^{\otimes r}, \mathbb{C})$. Thus, for homogeneous $v_1, \dots, v_r, \phi_1, \dots, \phi_r$,

$$(2.5) \quad \delta_\pi(v_1, \dots, v_r, \phi_1, \dots, \phi_r) = \pm \prod_{i=1}^r \phi_i(v_{\pi(i)}).$$

COROLLARY 2.2. *We have*

$$(2.6) \quad \mathrm{Hom}(V^{\otimes r} \otimes (V^*)^{\otimes s}, \mathbb{C})^{\mathrm{GL}(V)} = \begin{cases} 0 & \text{if } r \neq s \\ \mathrm{span} \{ \delta_\pi \mid \pi \in \mathrm{Sym}_r \} & \text{if } r = s. \end{cases}$$

The first statement is trivial since the center of $\mathrm{GL}(V)$ is just \mathbb{C} and $z \in \mathbb{C}$ acts on $\mathrm{Hom}(V^{\otimes r} \otimes (V^*)^{\otimes s}, \mathbb{C})$ as multiplication by z^{r-s} . The second statement is easily seen to be equivalent to Theorem 2.1.

2.3 Invariant theory for $\mathrm{GL}(V)$: the SFT

The SFT for $\mathrm{GL}(V)$ describes the kernel of the surjective homomorphism ϖ_r of Theorem 2.1.

The following result is an easy consequence of [4, Theorem 3.20].

THEOREM 2.3. *Let V be a super \mathbb{C} -vector space with $\mathrm{sdim}(V) = m|\ell$. If $r < (m+1)(\ell+1)$, then the surjective (by Theorem 2.1) homomorphism*

$$\varpi_r : \mathbb{C}\mathrm{Sym}_r \longrightarrow (\mathrm{End}_{\mathbb{C}}(V))^{\otimes r}$$

of superalgebras is an isomorphism. If $r \geq (m+1)(\ell+1)$, then the kernel of ϖ_r is the (two-sided) ideal of ASym_r generated by the Young symmetrizer of the partition with $m+1$ rows and $\ell+1$ columns.

The kernel is therefore generated by an idempotent which is explicitly described as follows. Consider the $(m+1) \times (\ell+1)$ array of integers below, which form a standard tableau.

1	2	...	$\ell + 1$
$\ell + 2$	$\ell + 3$...	$2\ell + 2$
...
...
$m\ell + m + 1$	$m\ell + m + 2$...	$m\ell + m + \ell + 1$

Let R and C be the subgroups of $\text{Sym}_{m\ell+m+\ell+1}$ (regarded as the subgroup of Sym_r which permutes the first $(m+1)(\ell+1)$ numbers) which stabilize the rows and columns of the array, respectively. Thus,

$$R = \text{Sym}\{1, 2, \dots, \ell + 1\} \times \text{Sym}\{\ell + 2, \ell + 3, \dots, 2\ell + 2\} \times \dots \times \text{Sym}\{m\ell + m + 1, m\ell + m + 2, \dots, m\ell + m + \ell + 1\},$$

while

$$C = \text{Sym}\{1, \ell + 2, \dots, m\ell + m + 1\} \times \text{Sym}\{2, \ell + 3, \dots, m\ell + m + 2\} \times \dots \times \text{Sym}\{\ell + 1, 2\ell + 2, \dots, m\ell + m + \ell + 1\},$$

where $\text{Sym}\{X\}$ denotes the group of permutations of the finite set X .

Then in the group ring $A\text{Sym}_{m\ell+m+\ell+1} \subseteq A\text{Sym}_r$, let $e = e(m, \ell)$ be the (even) element defined by

$$(2.7) \quad e(m, \ell) = \left(\sum_{\pi \in R} \pi \right) \left(\sum_{\sigma \in C} \varepsilon(\sigma) \sigma \right) = \alpha^+(R) \alpha^-(C),$$

where ε is the sign character of Sym_r , and for any subset $H \subseteq \text{Sym}_r$, we write $\alpha^+(H)$ (resp. $\alpha^-(H)$) for the element $\sum_{h \in H} h$ (resp. $\sum_{h \in H} \varepsilon(h)h$) of $\mathbb{C}\text{Sym}_r$.

It is known that $(|R|!|C|!)^{-1}e(m, \ell)$ is a primitive idempotent in the group algebra $\mathbb{C}\text{Sym}_{m\ell+m+\ell+1}$. It is also well known that $\mathbb{C}\text{Sym}_r = \bigoplus_{\mu} I(\mu)$, where μ runs over the partitions of r , and $I(\mu)$ is a simple ideal of $\mathbb{C}\text{Sym}_r$ for each μ . In this notation, the ideal $I(m, \ell)$ of $\mathbb{C}\text{Sym}_r$ which is generated by $e(m, \ell)$ is the sum of the $I(\mu)$ over those partitions μ that contain an $(m+1) \times (\ell+1)$ rectangle.

COROLLARY 2.4. *If $r < (m+1)(\ell+1)$, then $\text{Ker}(\varpi_r) = 0$. Otherwise, $\text{Ker}(\varpi_r) = I(m, \ell) := \bigoplus_{\mu} I(\mu)$ over those partitions μ of r that contain a rectangle of size $(m+1) \times (\ell+1)$.*

2.4 The SFT for $GL(V)$: second formulation

We shall now translate the FFT for $GL(V)$ into the setting of Corollary 2.2. This is easily achieved by noting the commutativity of the following diagram (2.4):

$$\begin{array}{ccc}
 \mathbb{C}\text{Sym}_r & \xrightarrow{\omega_r} & \text{End}(V^{\otimes r}) \\
 & \searrow \delta & \swarrow \phi_{r,r} \sim \\
 & \text{Hom}(V^{\otimes r} \otimes (V^*)^{\otimes r}, \mathbb{C}) &
 \end{array}$$

Here, ϖ_r is as in (2.2), $\phi_{r,r}$ is a special case of the map in (2.3), and δ is the map taking $\pi \in \text{Sym}_r$ to the $\delta_\pi = \pi(\lambda_0)$ of (2.5).

Since $\phi_{r,r}$ is an isomorphism, it follows that $\text{Ker}(\varpi_r) = \text{Ker}(\delta)$. The following statement is therefore evident.

COROLLARY 2.5. If δ_π ($\pi \in \text{Sym}_r$) is as in (2.5), then for any elements $a_\pi \in \mathbb{C}$, $\sum_{\pi \in \text{Sym}_r} a_\pi \delta_\pi = 0$ if and only if $\sum_{\pi \in \text{Sym}_r} a_\pi \pi$ lies in the ideal $I(m, \ell)$ of $\mathbb{C}\text{Sym}_r$. In particular, $\text{Ker}(\delta) = 0$ if $r < (m+1)(\ell+1)$.

§3. Invariant theory for the orthosymplectic supergroup

In this section, we take $\ell = 2n$ and G to be the orthosymplectic group scheme $G = \mathrm{OSp}(V) \subseteq \mathrm{GL}(V)$ as in Definition 1.2. Clearly, G acts on $V^{\otimes r}$, and $\mathrm{End}_G(V^{\otimes r}) = \mathrm{End}(V^{\otimes r})^G$ is a finite dimensional associative algebra which contains $\mathrm{End}(V^{\otimes r})^{\mathrm{GL}(V)}$. One formulation of the FFT for $\mathrm{OSp}(V)$, which is proved in [6, 19], describes this algebra. The most convenient formulation of the FFT for $\mathrm{OSp}(V)$ is through the Brauer category.

3.1 The Brauer category and FFT for $\mathrm{OSp}(V)$

Let $\mathcal{B} = \mathcal{B}(m-2n)$ be the Brauer category over \mathbb{C} with parameter $m-2n$ (see [18]). This has objects $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ and morphisms $\text{Hom}_{\mathcal{B}}(s, t) := \mathcal{B}_s^t$, the linear span of all Brauer diagrams from s to t ($s, t \in \mathbb{N}$). Recall that the category is generated by the four morphisms depicted below:



These will be denoted, respectively, by I , X , A , and U , and they belong to \mathcal{B}_1^1 , \mathcal{B}_2^2 , \mathcal{B}_2^0 , and \mathcal{B}_0^2 , respectively.

Let $(e_a), (e_a^*)$ ($a = 1, 2, \dots, m + 2n$) be a pair of \mathbb{C} -bases of V , which are dual with respect to the form B in the sense that $B(e_a^*, e_b) = \delta_{ab}$ (the Kronecker delta) for all a, b . The element $c_0 = \sum_a e_a \otimes e_a^* \in V \otimes V$ is then independent of the basis chosen and is G -invariant.

The FFT for $\mathrm{OSp}(V)$ may now be stated as follows (cf. [18, Theorem 4.8]).

THEOREM 3.1. *Let $V = \mathbb{C}^{m|2n}$ and $G = \mathrm{OSp}(V)$, and let \mathcal{T}_G be the full subcategory of G -modules with objects $V^{\otimes r}$ ($r = 0, 1, 2, \dots$). Then there is a full functor $F : \mathcal{B}(m - 2n) \rightarrow \mathcal{T}_G$, defined by $F(r) = V^{\otimes r}$, $F(I) = \mathrm{id}_V$, $F(X) = \tau \in \mathrm{End}(V \otimes V)$, and $F(A) = B(-, -) : v \otimes V \rightarrow \mathbb{C}$, and $F(U) = \gamma$ is the map defined by $\gamma(1) = c_0$.*

REMARK 3.2. Recall that since $-\mathrm{id}_V \in \mathrm{OSp}(V)$ and acts on $V^{\otimes r}$ as $(-1)^r$, it follows that $\mathrm{Hom}_G(V^{\otimes t}, V^{\otimes s}) = 0$ if $s \not\equiv t \pmod{2}$ because any element $\alpha \in \mathrm{Hom}_G(V^{\otimes t}, V^{\otimes s})$ would satisfy $\alpha = -\alpha$ since it commutes with $-\mathrm{id}_V$.

Thus, to say that F is full means that it induces surjections on the respective Hom spaces, that is, it means that the maps $F_t^s : \mathcal{B}_t^s \rightarrow \mathrm{Hom}_G(V^{\otimes t}, V^{\otimes s})$ are surjective for all $t, s \in \mathbb{N}$ such that $s \equiv t \pmod{2}$. Since the Hom spaces are (linearly) isomorphic for each t, s such that $s + t = 2r$ ($r \in \mathbb{N}$ constant), this surjectivity is equivalent to the corresponding statement for any particular pair t, s with $t + s = 2r$; common choices are $s = t = r$ and $t = 2r, s = 0$, and we give those formulations now. Taking $s = t = r$, we obtain the following statement.

THEOREM 3.3. [19, Theorem 5.4] *Let $\mathcal{B}_r(m - 2n)$ be the Brauer algebra on r strings with parameter $m - 2n$ and generators s_i, e_i for $i = 1, \dots, r - 1$ (see [19, Section 5.2]). Then there is a surjective homomorphism $\mathcal{B}_r(m - 2n) \rightarrow \mathrm{End}_G(V^{\otimes r})$ such that $s_i \mapsto \varpi_r(s_i)$ and $e_i \mapsto \gamma_i$ for all i , where $\gamma_i = \underbrace{\mathrm{id}_V \otimes \dots \otimes \mathrm{id}_V}_{i-1} \otimes \gamma \otimes \underbrace{\mathrm{id}_V \otimes \dots \otimes \mathrm{id}_V}_{r-i-1}$, and $\gamma \in \mathrm{End}_G(V^{\otimes 2})$ takes $v \otimes w$ to $B(v, w)c_0$, that is, $\gamma = F(U) \circ F(A)$.*

To state the FFT for the case $t = 2r, s = 0$, recall that a typical diagram in \mathcal{B}_{2r}^0 is of the form depicted in Figure 1.

The diagram D may be obtained from D_0 , depicted in Figure 2, by composition with a permutation.

Indeed, any permutation $\pi : 2r \rightarrow 2r$ may be thought of as a diagram like the one depicted in Figure 3.

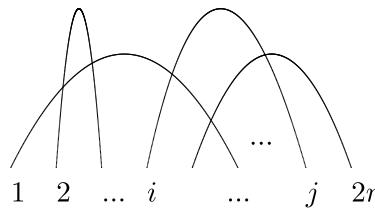


Figure 1.
Diagram $D : 2r \rightarrow 0$.



Figure 2.
Diagram $D_0 : 2r \rightarrow 0$.

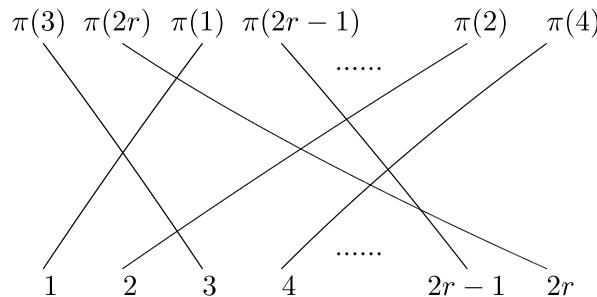


Figure 3.
Permutation $\pi : 2r \rightarrow 2r$.

The right action of Sym_{2r} on diagrams $2r \rightarrow 0$ is then given in terms of diagrams by composition of the diagrams in Figures 1 and 3. It is then clear that any diagram $D : 2r \rightarrow 0$ may be obtained from the diagram D_0 depicted in Figure 2 by composition with a permutation $\pi \in \text{Sym}_{2r}$ (see Figure 4).

To identify the linear maps defined by the diagrams above, define $\kappa_0 : V^{\otimes 2r} \rightarrow \mathbb{C}$ by

$$(3.1) \quad \kappa_0(v_1 \otimes \cdots \otimes v_{2r}) = B(v_1, v_2)B(v_3, v_4) \cdots B(v_{2r-1}, v_{2r}).$$

For each permutation $\pi \in \text{Sym}_{2r}$, define

$$(3.2) \quad \kappa_\pi(v_1 \otimes \cdots \otimes v_{2r}) = \kappa_0 \circ \varpi_{2r}(\pi^{-1})(v_1 \otimes \cdots \otimes v_{2r}).$$

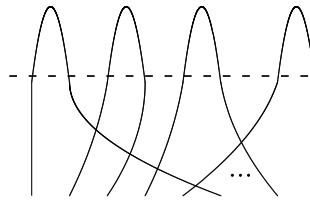


Figure 4.

Diagram D in Figure 1 as composition $D_0\pi_D$.

It is then easily verified that given a diagram $D \in \mathcal{B}_{2r}^0$, if $D = D_0 \circ \pi$ (for $\pi \in \text{Sym}_{2r}$), then although π is not uniquely defined by this property, κ_π depends only on D . We therefore write $\kappa_\pi = \kappa_D$, and we have the following formulation of the FFT.

THEOREM 3.4. *Let $V = \mathbb{C}^{m|2n}$ and $G = \text{OSp}(V)$. Then for each integer r , we have a surjective map $\kappa: \mathcal{B}_{2r}^0 \longrightarrow \text{Hom}_G(V^{\otimes 2r}, \mathbb{C})$ which takes the diagram D in Figure 1 to κ_D .*

DEFINITION 3.5. Define the subgroup $C \subseteq \text{Sym}_{2r}$ as the stabilizer of κ_0 in Sym_{2r} . That is, $C := \{\pi \in \text{Sym}_{2r} \mid \kappa_\pi = \kappa_0\}$.

The idempotent $e(C) \in \mathbb{C}\text{Sym}_{2r}$ is defined by $e(C) = |C|^{-1} \sum_{\pi \in C} \pi$.

It is clear that $|C| = 2^r r!$ so that $|\text{Sym}_{2r}/C| = \prod_{i=1}^r (2i-1) = |\mathcal{B}_{2r}^0|$.

§4. From invariants of $\text{OSp}(V)$ to invariants of $\text{GL}(V)$

In this section, we recall the basic results of [6, 19] to describe diagrammatically an explicit injective map from invariants of $\text{OSp}(V)$ to invariants of $\text{GL}(V)$, which will permit the use of the SFT for $\text{GL}(V)$ to deduce our result for $\text{OSp}(V)$. We use the language and formulation of [6].

REMARK 4.1. In particular, for any finite dimensional complex vector space $U_{\mathbb{C}}$, we will denote by U the corresponding affine superscheme. Specifically, for vector spaces $V_{\mathbb{C}}, W_{\mathbb{C}}$, $\text{Hom}(V, W)$ denotes the affine scheme corresponding to $\text{Hom}_{\mathbb{C}}(V_{\mathbb{C}}, W_{\mathbb{C}})$.

4.1 Extension of functions

We recall the setup of [6, Section 3]. In particular, we denote by \overline{Q} the scheme of quadratic forms on V and by Q the open dense subscheme of nondegenerate forms. The nondegenerate quadratic form $q \in Q$ is given, and $\text{OSp}(V)$ is its isotropy group under the transitive action of $\text{GL}(V)$ on Q , which is given by $(g, q') \in \text{GL}(V) \times Q \mapsto q' \circ g^{-1}$.

Let us write E for the superscheme $E = \mathcal{E}nd(V)$. We then have the epimorphism $u : E \rightarrow \overline{Q}$ given by $u(T) = q \circ T$. Clearly, $u^{-1}(Q) = \mathrm{GL}(V)$, and we have the commutative square

$$(4.1) \quad \begin{array}{ccc} \mathrm{GL}(V) & \hookrightarrow & E \\ \downarrow u_0 & & \downarrow u \\ Q & \hookrightarrow & \overline{Q}, \end{array}$$

where the embedding of $\mathrm{GL}(V)$ in E is given by $g \mapsto g^{-1}$ and $u_0(g) = q \circ g^{-1}$.

If f is any $\mathrm{OSp}(V)$ -invariant function on $V^{\otimes r}$, one defines a corresponding $\mathrm{GL}(V)$ -invariant function F on $V^{\otimes r} \times Q$ by

$$(4.2) \quad F(v_1 \otimes \cdots \otimes v_r, q') = F(v_1 \otimes \cdots \otimes v_r, g(q)) := f(g^{-1}v_1, \dots, g^{-1}v_r),$$

where $q' = g(q) = q \circ g^{-1}$ ($g \in \mathrm{GL}(V)$) is an arbitrary form in Q . Note that since $g \in \mathrm{GL}(V)$ is determined up to right multiplication by $x \in \mathrm{OSp}(V)$, the definition in (4.2) depends only on $q' = g(q)$ and not on g since f is $\mathrm{OSp}(V)$ -invariant. Moreover, the $\mathrm{GL}(V)$ invariance of F is immediate from the definition.

The following result is [6, Theorem 3.4].

THEOREM 4.2. *Let f be a linear function on $V^{\otimes r}$ which is $\mathrm{OSp}(V)$ -invariant and let F be the corresponding $\mathrm{GL}(V)$ -invariant function on $V^{\otimes r} \times Q$ defined in (4.2). Then there is a unique extension \overline{F} of F from $V^{\otimes r} \times Q$ to $V^{\otimes r} \times \overline{Q}$.*

REMARK 4.3. For each $(v_1 \otimes \cdots \otimes v_r, q', g) \in V^{\otimes r} \times Q \times \mathrm{GL}(V)$, it is evident that

$$F(v_1 \otimes \cdots \otimes v_r, q' \circ g) = F(gv_1 \otimes \cdots \otimes gv_r, q').$$

Since $V^{\otimes r} \times Q \times \mathrm{GL}(V)$ is schematically dense in $V^{\otimes r} \times \overline{Q} \times E$, it follows that

$$(4.3) \quad \overline{F}(v_1 \otimes \cdots \otimes v_r, q' \circ T) = \overline{F}(Tv_1 \otimes \cdots \otimes Tv_r, q')$$

for all $(v_1 \otimes \cdots \otimes v_r, q', T) \in V^{\otimes r} \times \overline{Q} \times E$. In particular, \overline{F} is $\mathrm{GL}(V)$ -invariant.

Note that by Remark 3.2, if f in Theorem 4.2 is nonzero, it follows that r is even, and we write $r = 2d$ below.

4.2 Construction of invariants of $\mathrm{GL}(V)$ from those of $\mathrm{OSp}(V)$

In this section, we shall prove the following result. Notation is as in Remark 4.1.

THEOREM 4.4. *Maintain the notation and formalism of Sections 1 and 2. Given a nondegenerate orthosymplectic form on V , there is a canonical injective map*

$$h : \mathrm{Hom}(V^{\otimes r}, \mathbb{C})^{\mathrm{OSp}(V)} \longrightarrow \mathrm{Hom}(V^{\otimes r} \times (V^*)^{\otimes r}, \mathbb{C})^{\mathrm{GL}(V)}.$$

Proof. Suppose that $L \in \mathrm{Hom}(V^{\otimes r}, \mathbb{C})^{\mathrm{OSp}(V)}$ and $L \neq 0$. Then $r = 2d$ is even and L is an even linear function on $V^{\otimes 2d}$ which satisfies $L(gv_1 \otimes \cdots \otimes gv_{2d}) = L(v_1 \otimes \cdots \otimes v_{2d})$ for $v_i \in V$ and $g \in \mathrm{OSp}(V)$. Using the construction above as well as Theorem 4.2, we obtain a $\mathrm{GL}(V)$ -invariant function F'_L on $V^{\otimes 2d} \times \overline{Q}$, which satisfies (4.2) and (4.3). Moreover, by [6, 3.5(iii)], F'_L is homogeneous of degree d in \overline{Q} . A supervariant of the usual polarization argument (cf. Proposition A.1) shows that there is, therefore, a unique multilinear function F_L on $V^{\otimes 2d} \times \overline{Q}^{\otimes d}$ such that

$$F'_L(v_1 \otimes \cdots \otimes v_r, q') = F_L(v_1 \otimes \cdots \otimes v_r, q' \otimes q' \otimes \cdots \otimes q').$$

Now there is a well-known canonical isomorphism $\overline{Q} \simeq S^2(V^*)$, where S^2 denotes the symmetric square. Further, the canonical decomposition

$$V^* \otimes V^* = S^2(V^*) \oplus \wedge^2(V^*)$$

is $\mathrm{GL}(V)$ -invariant, whence $S^2(V^*)$ is canonically a quotient of $(V^*)^{\otimes 2}$ so that functions on $\overline{Q}^{\otimes d}$ may be canonically lifted to $(V^*)^{2d}$. It follows that F_L may be lifted to a $\mathrm{GL}(V)$ -invariant multilinear function, which we call H_L , on $V^{2d} \times (V^*)^{2d}$. Write $h(L) = H_L$. This map is canonical in the sense that no choices are made after q is specified.

The injectivity of h follows from the fact that

$$\begin{aligned} H_L(v_1 \otimes \cdots \otimes v_r \otimes q \otimes \cdots \otimes q) &= F_L(v_1 \otimes \cdots \otimes v_r, q \otimes \cdots \otimes q) \\ &= F'_L(v_1 \otimes \cdots \otimes v_r, q) \\ &= L(v_1 \otimes \cdots \otimes v_r), \end{aligned}$$

from which it is apparent that L is determined by H_L . □

4.3 Diagrammatic interpretation

In this section, we identify the construction of the previous section with the diagrammatic description in Theorems 3.3 and 3.4.

We begin with the passage from the function F_L to H_L in the proof of Theorem 4.4. This depends on lifting multilinear functions from $\overline{Q}^d \simeq (S^2(V^*))^{\otimes d}$ to $((V^*)^{\otimes 2})^{\otimes d} \simeq (V^*)^{\otimes 2d}$. It follows from the construction of $h(L)$, where $L \in (\text{Hom}(V^{\otimes 2d}, \mathbb{C}))^{\text{Osp}(V)}$, that the value $H_L(v_1 \otimes \cdots \otimes v_{2d} \otimes \phi_1 \otimes \cdots \otimes \phi_{2d})$ depends only on the image of $(\phi_1 \otimes \cdots \otimes \phi_{2d})$ under the projection $(V^*)^{\otimes 2d} \rightarrow S^d(S^2(V^*))$, which, by the polarization lemma A.1, is spanned by elements of the form $(q' \otimes q' \otimes \cdots \otimes q')$, where $q' \in \overline{Q} \simeq S^2(V^*)$.

Bearing in mind the above remarks, the following result is clear.

LEMMA 4.5. *The canonical projection $p : (V^*)^{\otimes 2d} \rightarrow S^d(S^2(V^*))$ is realized by the action of the idempotent $\varpi_{2d}(e(C))$, where $e(C)$ is given by $e(C) = |C|^{-1} \sum_{\sigma \in C} \sigma$ and C is defined in Definition 3.5. We have*

$$(4.4) \quad \begin{aligned} & H_L(v_1 \otimes \cdots \otimes v_{2d} \otimes \phi_1 \otimes \cdots \otimes \phi_{2d}) \\ &= F_L(v_1 \otimes \cdots \otimes v_{2d}, \varpi_{2d}(e(C))(\phi_1 \otimes \cdots \otimes \phi_{2d})). \end{aligned}$$

Moreover, since functions are, by definition, even sections of the structure sheaf of the relevant scheme, we have

$$(4.5) \quad \begin{aligned} & F_L(v_1 \otimes \cdots \otimes v_{2d}, \varpi_{2d}(e(C))(\phi_1, \dots, \phi_{2d})) \\ &= F_L(\varpi_{2d}(e(C))(v_1 \otimes \cdots \otimes v_{2d}), \phi_1 \otimes \cdots \otimes \phi_{2d}) \end{aligned}$$

so that

$$(4.6) \quad \begin{aligned} & H_L(v_1 \otimes \cdots \otimes v_{2d} \otimes \phi_1 \otimes \cdots \otimes \phi_{2d}) \\ &= F_L(\varpi_{2d}(e(C))(v_1 \otimes \cdots \otimes v_{2d}), \phi_1 \otimes \cdots \otimes \phi_{2d}). \end{aligned}$$

There are actions of Sym_{2d} on both $\text{Hom}(V^{\otimes 2d}, \mathbb{C})$ and on $\text{Hom}(V^{\otimes 2d} \times (V^*)^{2d}, \mathbb{C})$ given (for $\pi \in \text{Sym}_{2d}$) by composition with $\varpi_{2d}(\pi^{-1})$ and $\varpi_{2d}(\pi^{-1}) \times \text{id}_{(V^*)^{2d}}$, respectively. Evaluation at $(v_1 \otimes \cdots \otimes v_{2d}, q \circ T \otimes q \circ T \otimes \cdots \otimes q \circ T)$ (with $v_i \in V$ and $T \in \text{End}(V)$) shows, taking into account Proposition A.1, the following.

LEMMA 4.6. *In the notation of Theorem 4.4, we have, for $\pi \in \text{Sym}_{2d}$ and $L \in (\text{Hom}(V^{2d}, \mathbb{C}))^{\text{Osp}(V)}$,*

$$h(\pi L) = \pi h(L).$$

This relation may be written as $h(\kappa_0 \circ \varpi_{2d}(\pi)) = \lambda_0 \circ e(C) \circ (\varpi_{2d}(\pi) \times \text{id}_{(V^*)^{2d}})$.

LEMMA 4.7. Let $\kappa_\pi = \kappa_0 \circ \varpi_{2d}(\pi^{-1})$ be as in (3.2). Then with λ_0 as in (2.4), we have

$$(4.7) \quad h(\kappa_\pi) = \lambda_0 \circ e(C) \circ (\varpi_{2d}(\pi) \times \text{id}_{(V^*)^{2d}}).$$

Proof. It follows from Lemma 4.5 and (4.6) that $h(\kappa_0) = \lambda_0 \circ e(C)$. Hence, by Lemma 4.6, $h(\kappa_\pi) = h(\kappa_0 \circ \varpi_{2d}(\pi^{-1})) = h(\kappa_0) \circ (\varpi_{2d}(\pi^{-1}) \times \text{id}_{(V^*)^{2d}}) = \lambda_0 \circ e(C) \circ (\varpi_{2d}(\pi^{-1}) \times \text{id}_{(V^*)^{2d}})$. \square

This last statement has an attractive interpretation in terms of diagrams. In Proposition 4.8, it is understood that $\mathbb{C}\text{Sym}_{2d} \subset \mathcal{B}_{2d}^{2d}$, and if \mathcal{I}_{2d} is the ideal of \mathcal{B}_{2d}^{2d} consisting of the span of the nonmonic diagrams, then $\mathcal{B}_{2d}^{2d} = \mathbb{C}\text{Sym}_{2d} \oplus \mathcal{I}_{2d}$. Thus, the map χ takes \mathcal{B}_{2d}^0 into $\mathbb{C}\text{Sym}_{2d} \subset \mathcal{B}_{2d}^{2d}$.

PROPOSITION 4.8. *The following diagram is commutative.*

$$(4.8) \quad \begin{array}{ccc} \mathcal{B}_{2d}^0 & \xrightarrow{\kappa} & (\text{Hom}_{\mathbb{C}}(V^{\otimes 2d}, \mathbb{C}))^{\text{OSp}(V)} \\ \downarrow \chi & & \downarrow h \\ \mathcal{B}_{2d}^{2d} & \xrightarrow{\delta} & (\text{Hom}_{\mathbb{C}}(V^{\otimes 2d} \otimes (V^*)^{\otimes 2d}, \mathbb{C}))^{\text{GL}(V)} \end{array}$$

Here, κ is the map defined in Theorem 3.4, δ is defined on $\mathbb{C}\text{Sym}_{2d}$ in Section 2.4 and extended to \mathcal{B}_{2d}^{2d} by defining it as zero on \mathcal{I}_{2d} , h is the injective map defined in Theorem 4.4, and χ is defined as follows. For any diagram $D = D_0 \circ \pi_D \in \mathcal{B}_{2d}^0$, $\chi(D) = e(C)\pi_D \in \mathbb{C}\text{Sym}_{2d} \subseteq \mathcal{B}_{2d}^{2d}$, where $e(C)$ is the idempotent defined in Definition 3.5.

Proposition 4.8 is a restatement of Lemma 4.7. In terms of diagrams, the diagram D depicted in Figure 4 is taken to the element of \mathcal{B}_{2d}^{2d} depicted in Figure 5.

§5. The SFT for $\text{OSp}(V)$

In this section, we continue with the notation above. V is a complex vector space with $\text{sdim}(V) = (m|2n)$. If $B(-, -) \in \text{Hom}(V^2, \mathbb{C})$ is the given nondegenerate orthosymplectic form on V , $G = \text{OSp}(V)$ is its isometry group scheme, as explained in Section 2. The space $(\text{Hom}(V^{2d}, \mathbb{C}))^{\text{OSp}(V)}$

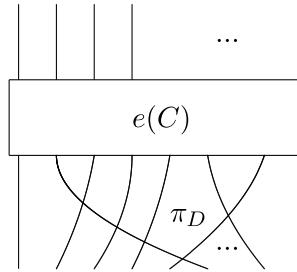


Figure 5.
The element $\chi(D)$ of \mathcal{B}_{2d}^{2d} .

of invariant multilinear maps has the universal scheme-theoretic meaning explained in Definition 1.1.

The FFT for $\mathrm{OSp}(V)$ asserts that the map κ in (4.8) (see Proposition 4.8) is surjective. The SFT for $\mathrm{OSp}(V)$ describes the kernel of κ .

5.1 The main theorem

The following theorem is the main result of this paper.

THEOREM 5.1. *Let κ be the map in the commutative diagram (4.8). Then*

$$\mathrm{Ker}(\kappa) = D_0 \circ I(m, n),$$

where $I(m, n)$ is the ideal of CSym_{2d} which is the sum of the two-sided ideals corresponding to partitions which contain an $(m+1) \times (2n+1)$ rectangle and D_0 is the diagram in \mathcal{B}_{2d}^0 depicted in Figure 3.

Proof. Since the map h in (4.8) is injective, it follows that $\mathrm{Ker}(\kappa) = \mathrm{Ker}(\delta \circ \chi)$. Hence, $b \in \mathcal{B}_{2d}^0$ lies in $\mathrm{Ker}(\kappa)$ if and only if $\chi(b) \in \mathrm{Ker}(\delta)$, and by Corollary 2.5, $\mathrm{Ker}(\delta) = \mathcal{I}_{2d} + I(m, n)$, where \mathcal{I}_{2d} is the ideal generated by the nonmonic diagrams in \mathcal{B}_{2d}^{2d} . Moreover, δ factors through $\mathcal{B}_{2d}^{2d}/\mathcal{I}_{2d} \simeq \mathrm{CSym}_{2d}$, and the kernel of the restriction of δ to CSym_{2d} is $I(m, n)$.

But we claim that $\chi^{-1}(I(m, n)) = D_0 \circ I(m, n)$, for if $x \in I(m, n)$, then by Proposition 4.8, $\chi(D_0 \circ x) = e(C)x \in I(m, n)$, while conversely if $D_0 \circ x \in \chi^{-1}(I(m, n))$ for some $x \in \mathrm{CSym}_{2d}$, then $\chi(D_0 \circ x) = e(C)x \in I(m, n)$. But then $D_0 \circ x = D_0 \circ e(C)x \in D_0 \circ I(m, n)$, proving the assertion. The theorem follows. \square

A typical element of the kernel of κ is shown in Figure 6.

The following statement concerns the range of values of d for which $\mathrm{Ker}(\kappa) \neq 0$.

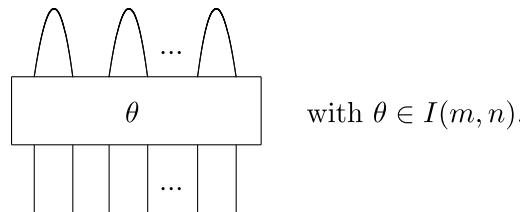


Figure 6.
Elements of $\text{Ker}(\kappa)$.

COROLLARY 5.2. *With notation as in Theorem 5.1, the kernel of κ is zero if $2d < (m+1)(2n+1)$. Further, $\text{Ker}(\kappa) \neq 0$ for $d \geq (m+1)(2n+1)$.*

Proof. The first assertion is immediate from Theorem 5.1. The second follows from the fact that

$$I(m, n) = \mathbb{U}^{(m+1)(2n+1)}(D \circ (I(m, n) \otimes 1^{\otimes(m+1)(2n+1)})),$$

where $D : 2(m+1)(2n+1) \rightarrow 0$ is the diagram with arcs $(i, 2(m+1)(2n+1) - i + 1)$, $i = 1, \dots, (m+1)(2n+1)$ and \mathbb{U} is the map described in the first paragraph of Section 5.2. The argument of $\mathbb{U}^{(m+1)(2n+1)}$ on the right side is therefore a nonzero element of $\text{Ker}(\kappa)$. \square

Note that as the discussion of the classical orthogonal and symplectic cases below shows, Corollary 5.2 is not optimal. Here is a nonclassical example.

EXAMPLE 5.3. We take $m = 1 = n$; thus, we are in the case of $\text{OSp}(1|2)$. In this case, the smallest value of d for which $\text{Ker}(\kappa)$ could be nonzero is $d = 3$ ($2d = (m+1)(2n+1) = 2 \times 3 = 6$). However, a straightforward calculation shows that in this case (i.e., $d = 3$), we have $D_0 I(1, 2) = 0$ in B_6^0 . Thus, here $\text{Ker}(\kappa) = 0$ when $2d = (m+1)(2n+1)$. In fact, in this case, one sees that $\text{Ker}(\kappa) = 0$ if and only if $d \leq 3$. For this, one must verify that $\text{Ker}(\kappa) \neq 0$ when $d = 4$, that is, that $D_0 I(1, 2) \neq 0$ in B_8^0 .

REMARK 5.4. The available evidence seems to point to the statement that the smallest value of d for which $\text{Ker}(\kappa) \neq 0$ is $d = (m+1)(n+1)$ (i.e., $2d = (m+1)(2n+1) + m+1$). This result has now been proved by Zhang in [29, Theorem 5.11].

Note also that in [7], it is proved that κ is injective for some small values of d .

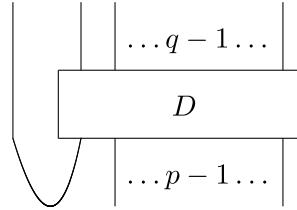


Figure 7.
 $\mathbb{U}(D)$ for $D \in \mathcal{B}_p^q$.

5.2 Interpretation in the first formulation

Theorem 5.1 provides the SFT for $\mathrm{OSp}(V)$ in its second formulation; that is, it describes the linear structure of $\mathrm{Hom}_{\mathbb{C}}(V^{\otimes 2d}, \mathbb{C})^{\mathrm{OSp}(V)}$. It may be useful to point out that it may be reinterpreted in terms of the Brauer algebra action $\eta_r : \mathcal{B}_r(m-2n) \rightarrow \mathrm{End}_{\mathrm{OSp}(V)}(V^{\otimes r})$ (see [19, Corollary 5.7]). For this, we note that in the Brauer category [19], for nonnegative integers p, q with $p > 0$, there is an isomorphism $\mathbb{U} : \mathcal{B}_p^q \xrightarrow{\sim} \mathcal{B}_{p-1}^{q+1}$. It is denoted by \mathbb{U}_{p-1}^1 in [18, Corollary 2.8] and is depicted in Figure 7.

One therefore has an isomorphism $\mathbb{U}^d : \mathcal{B}_{2d}^0 \rightarrow \mathcal{B}_d^d$, and this leads to the following extension of the diagram (4.2).

$$\begin{array}{ccc}
 \mathcal{B}_d(m-2n) & \xrightarrow{\eta} & (\mathrm{End}_{\mathbb{C}}(V^{\otimes d}))^{\mathrm{OSp}(V)} \\
 \mathbb{U}^d \uparrow \cong & & \downarrow \cong \\
 \mathcal{B}_{2d}^0 & \xrightarrow{\kappa} & (\mathrm{Hom}_{\mathbb{C}}(V^{\otimes 2d}, \mathbb{C}))^{\mathrm{OSp}(V)} \\
 \chi \downarrow & & \downarrow h \\
 \mathbb{C}\mathrm{Sym}_{2d} \subseteq \mathcal{B}_{2d}^{2d} & \xrightarrow[\delta]{} & (\mathrm{Hom}_{\mathbb{C}}(V^{\otimes 2d} \times (V^*)^{\otimes 2d}, \mathbb{C}))^{\mathrm{GL}(V)}.
 \end{array} \tag{5.1}$$

The top map $\eta : \mathcal{B}_d(m-2n) \rightarrow \mathrm{End}_{\mathrm{OSp}(V)}(V^{\otimes d})$ is precisely the map discussed in the paper [18]. Theorem 5.1 has the following evident consequence.

COROLLARY 5.5. *The kernel of the surjective algebra homomorphism $\eta : \mathcal{B}_d(m-2n) := \mathcal{B}_d^d \rightarrow \mathrm{End}_G(V^{\otimes d})$ is $\mathbb{U}^d(D_0 I(m, n))$.*

REMARK 5.6. In [17] and [18], we showed that in the classical cases when $m = 0$ or when $n = 0$, the kernel is actually generated by a single idempotent in the associative algebra $\mathcal{B}_d(m-2n)$, but we do not have a similar result in the general case (cf. [29]).

§6. Application to the classical groups

In this final section, we shall show how our main theorem provides a new proof of the SFT in the classical cases of the orthogonal and symplectic groups over \mathbb{C} (see, e.g., [9, 21]). We therefore take V as above and apply Theorem 5.1 in those respective cases.

6.1 The orthogonal case: $G = O(m, \mathbb{C})$

Here, we take $V = V_0$ to be purely even and the form $B(-, -)$ to be symmetric. To apply our result, observe that the ideal $I(m, n) = I(m, 0)$ of $\mathbb{C}\text{Sym}_{2d}$ is spanned (over \mathbb{C}) by elements of the form $\pi\alpha^-(m+1)\pi'$, where $\pi, \pi' \in \text{Sym}_{2d}$ and for ℓ such that $1 \leq \ell \leq 2d$, $\alpha^-(\ell) = \sum_{\sigma \in \text{Sym}_\ell} \varepsilon(\sigma)\sigma$, where ε is the alternating character of Sym_{2d} and $\text{Sym}_\ell \subseteq \text{Sym}_{2d}$ is the subgroup which permutes the first ℓ symbols.

Theorem 5.1 asserts that $\text{Ker}(\kappa : \mathcal{B}_{2d}^0(\mathbb{C}) \longrightarrow \text{Hom}_{\mathbb{C}}(V^{\otimes 2d}, \mathbb{C})^{\text{O}\text{Sp}(V)})$ is spanned by the elements $D_0\pi\alpha^-(m+1)\pi'$. The next lemma shows that if d is small, the kernel is zero.

LEMMA 6.1. *If $d \leq m$, then each element $D_0\pi\alpha^-(m+1)\pi'$ of \mathcal{B}_{2d}^0 is zero.*

Proof. Let $\pi \in \text{Sym}_{2d}$ and consider the diagram $D = D_0\pi$. If $d \leq m$, then $d < m+1$, and it follows that at least one arc of D has both ends in $\{1, 2, \dots, m+1\}$ (if each arc with an end in $\{1, 2, \dots, m+1\}$ had an end outside $\{1, 2, \dots, m+1\}$, we would have $2d \geq 2(m+1)$). Suppose this arc is from i to j , where $1 \leq i < j \leq m+1$, and write (ij) for the transposition in Sym_{2d} which interchanges i and j .

Then by the rules for multiplying Brauer diagrams, we have $D(ij) = D$, whence $D_0\pi(ij)\alpha^-(m+1) = D_0\pi\alpha^-(m+1)$.

But by the alternating property of $\alpha^-(m+1)$, we have $(ij)\alpha^-(m+1) = -\alpha^-(m+1)$, whence $D_0\pi(ij)\alpha^-(m+1) = -D_0\pi\alpha^-(m+1)$. The lemma follows. \square

Figure 8 depicts an example of a linear combination of diagrams which is zero by Lemma 6.1.

COROLLARY 6.2. *If $d \leq m$, then $\text{Ker}(\kappa) = 0$, and $\kappa : \mathcal{B}_{2d}^0(\mathbb{C}) \longrightarrow \text{Hom}_{\mathbb{C}}(V^{\otimes 2d}, \mathbb{C})^{\text{O}(V)}$ is an isomorphism.*

Our main theorem now has the following interpretation in the present case (cf. [18, Theorem 3.4]). Recall that diagrams $D \in \mathcal{B}_{2d}^0$ are in canonical bijection with partitions of $\{1, \dots, 2d\}$ into pairs.

THEOREM 6.3. *Let $V = V_0 = \mathbb{C}^m$ and $G = O(m, \mathbb{C})$. If $d \leq m$, then $\text{Ker}(\kappa) = 0$. Assume that $d \geq m+1$.*

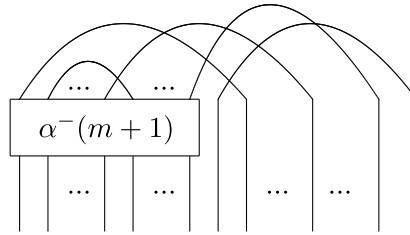


Figure 8.
Zero element in \mathcal{B}_{2d}^0 .

Define a set of linear relations among the κ_D as follows. Let S, S' be two disjoint subsets of $\{1, \dots, 2d\}$ as depicted in Figure 9 such that $|S| = |S'| = m + 1$, and for $\sigma \in \text{Sym}(\{S\})$, let $D(\pi, \pi'; \sigma)$ be the diagram in \mathcal{B}_{2d}^0 which pairs $\sigma(s_i)$ with s'_i , where the $s_i \in S$, $s'_i \in S'$ are written in increasing order, and the points in $\{1, \dots, 2d\} \setminus (S \cup S')$ are paired as they are in each diagram occurring in $D_0 \pi \alpha^-(m+1) \pi'$.

Then for each $\pi, \pi' \in \text{Sym}_{2d}$,

$$(6.1) \quad \sum_{\sigma \in \text{Sym}_{m+1}} \varepsilon(\sigma) \kappa_{D(\pi, \pi', \sigma)} = 0,$$

and all relations among the κ_D are linear consequences of these relations.

Proof. The first statement is Corollary 6.2.

By Theorem 5.1, all relations among the κ_D are consequences of the relations $\kappa(D_0 \pi \alpha^-(m+1) \pi') = 0$, for $\pi, \pi' \in \text{Sym}_{2d}$.

Now the argument of the proof of Lemma 6.1 shows that if two of the points in the set $S := \{\pi^{-1}(1), \pi^{-1}(2), \dots, \pi^{-1}(m+1)\}$ are paired by D_0 then $D_0 \pi \alpha^-(m+1) \pi' = 0$ in \mathcal{B}_{2d}^0 , and so to obtain a nonzero element of the kernel, we may assume that no two points in S are paired by D_0 and, hence, that the set S' of points which are paired by D_0 with a point of S is disjoint from S .

In this case, the relation $\kappa(D_0 \pi \alpha^-(m+1) \pi') = 0$ is easily seen to translate into (6.1), and the result is now clear. \square

6.2 The symplectic case

In this case, we take $V = V_{\bar{1}}$. The form $B(-, -)$ is then skew, and $G = \text{Sp}(2n, \mathbb{C})$. The ideal $I(m, n) = I(0, n)$ is, in this case, spanned (over \mathbb{C}) by elements of the form $\pi \alpha^+(2n+1) \pi'$, where $\pi, \pi' \in \text{Sym}_{2d}$ and for

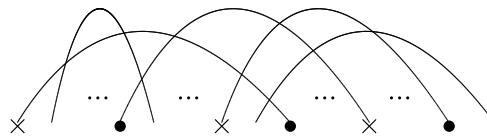


Figure 9.

The set S consists of \bullet 's and S' of \times 's.

each integer ℓ with $1 \leq \ell \leq 2d$, $\alpha^+(\ell) = \sum_{\sigma \in \text{Sym}_\ell} \sigma$, where as above, $\text{Sym}_\ell \subseteq \text{Sym}_{2d}$ is the subgroup which permutes the first ℓ symbols.

In this case, the analogue of Lemma 6.1 is the trivial observation that there are no nonzero elements of the kernel unless $2d \geq 2n + 1$, that is, $d \geq n + 1$.

Taking into account the fact that for a symplectic form $(v, v) = 0$ for all $v \in V$, we obtain the following form of the main theorem for the symplectic case.

THEOREM 6.4. *Let $V = V_{\bar{1}} = \mathbb{C}^{2n}$, and $G = \text{Sp}(2n, \mathbb{C})$. If $d \leq n$, then $\text{Ker}(\kappa) = 0$. Assume that $d \geq n + 1$.*

Define a set of linear relations among the κ_D as follows. Let S be a subset of $\{1, \dots, 2d\}$ such that $|S| = 2n + 1$. For any diagram $D \in \mathcal{B}_{2d}^0$ and $\sigma \in \text{Sym}(\{S\})$, let $D(S, \sigma)$ be the diagram $D(S, \sigma) = D\sigma$, where σ is regarded as an element of $\text{Sym}_{2d} \subset \mathcal{B}_{2d}^{2d}$ which fixes the points outside S .

Then

$$(6.2) \quad \sum_{\sigma \in \text{Sym}(\{S\})} \kappa_{D(S, \sigma)} = 0,$$

and all relations among the κ_D are linear consequences of these relations.

This is clear from Theorem 5.1.

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Appendix A. Superpolarization

PROPOSITION A.1. *Let $M_{\mathbb{C}}$ be a $\mathbb{Z}/2\mathbb{Z}$ -graded complex vector space with $\text{sdim}(M_{\mathbb{C}}) = (k|\ell)$, let A be a supercommutative super \mathbb{C} -algebra, and let $M = M_{\mathbb{C}} \otimes_{\mathbb{C}} A$, $\mathbb{Z}/2\mathbb{Z}$ -graded in the usual way. Let $B \subseteq M_{\bar{0}}$ be a subset satisfying the conditions*

- (i) $AB = M$; that is, B generates M as A -module.

(ii) For any two distinct elements $b, c \in B$, there are infinitely many complex numbers λ such that $b + \lambda c \in B$.

Let $T^r(M) = M \otimes_A M \otimes_A \cdots \otimes_A M$ (r factors), and let $S^r(M)$ be the supersymmetric part of $T^r(M)$ (cf. [19, Section 3]).

Then for each integer $r \geq 0$, the space $(S^r(M))_{\bar{0}}$ is the $A_{\bar{0}}$ -span of the elements $\{b \otimes b \otimes \cdots \otimes b \mid b \in B\}$.

Proof. This will be by induction on r , the case $r = 1$ being trivial, since by the condition (i), $A_{\bar{0}}B = M_{\bar{0}}$ so that $M_{\bar{0}}$ is the $A_{\bar{0}}$ -span of B (and $M_{\bar{1}}$ is the $A_{\bar{1}}$ -span of B).

Now it follows from [19, Lemma 3.7] that $S^r(M) = \alpha^+(r)T^r(M)$, where $\alpha^+(r) = \sum_{\sigma \in \text{Sym}_r} \sigma$, with Sym_r acting on $T^r(M)$ in the usual way through ϖ_r (see (2.2)). Denote by $\Sigma(r)$ the $A_{\bar{0}}$ -span of $\{b \otimes b \otimes \cdots \otimes b \mid b \in B\}$. Then by the last remark, it suffices to show that for any element $m_1 \otimes m_2 \otimes \cdots \otimes m_r \in T^r(M)_{\bar{0}}$, we have

$$(A1) \quad \alpha^+(r)(m_1 \otimes m_2 \otimes \cdots \otimes m_r) \in \Sigma(r).$$

We shall prove (A1) by induction on r . Assume (A1) for a smaller number of factors; we consider separately two cases.

Case 1: at least one of the factors m_i lies in $M_{\bar{0}}$. In this case, since for any element $\sigma \in \text{Sym}_r$ we have $\alpha^+(r)\sigma = \alpha^+(r)$, we may clearly assume that $m_r \in M_{\bar{0}}$. Then $m_1 \otimes m_2 \otimes \cdots \otimes m_{r-1} \in T^{r-1}(M)_{\bar{0}}$, and by induction, we have $\alpha^+(r-1)(m_1 \otimes m_2 \otimes \cdots \otimes m_{r-1}) \in \Sigma(r-1)$.

But from the coset decomposition of Sym_r with respect to Sym_{r-1} , we have

$$\alpha^+(r) = \left(1 + \sum_{i=1}^{r-1} (i, r) \right) \alpha^+(r-1),$$

where (i, j) denotes the transposition of i and j in Sym_r and Sym_{r-1} is the subgroup of Sym_r which permutes $\{1, \dots, r-1\}$. The last two observations imply that $\alpha^+(r)(m_1 \otimes m_2 \otimes \cdots \otimes m_r)$ is a $A_{\bar{0}}$ -linear combination of elements of the form

$$(A2) \quad \left(1 + \sum_{i=1}^{r-1} (i, r) \right) (b \otimes b \otimes \cdots \otimes b \otimes c),$$

where $b, c \in B$.

Now denote by $s_{i,j}(b, c)$ the sum of all tensors of the form $\cdots \otimes b \otimes \cdots \otimes c \otimes \cdots \in T^r(M)$, where all factors are either b or c , and there are i factors equal to b and j factors equal to c . Then the element in (A2) is $s_{1,r-1}(b, c)$, and it will suffice to show that if $b, c \in B$, then $s_{i,j}(b, c) \in \Sigma(r)$ for all i, j such that $i + j = r$. To see this last point, suppose $\lambda \in \mathbb{C}$ is such that $b + \lambda c \in B$. Then $\Sigma(r) \ni (b + \lambda c) \otimes (b + \lambda c) \otimes \cdots \otimes (b + \lambda c) = \sum_{j=0}^r \lambda^j s_{i,j}(b, c)$. By taking $r + 1$ distinct values λ for which this relation holds, we obtain, by the invertibility of the van der Monde matrix, an equation for each element $s_{i,j}(b, c)$ as a \mathbb{C} -linear combination of the elements $(b + \lambda c) \otimes (b + \lambda c) \otimes \cdots \otimes (b + \lambda c) \in \Sigma(r)$. Thus, $s_{r-1,1}(b, c) \in \Sigma(r)$, and the proof in Case 1 is complete.

Case 2: each factor $m_i \in M_{\bar{1}}$. In this case, we must have r even. Since $\alpha^+(r)$ is linear in each variable m_i and since $M_{\bar{1}} = A_{\bar{1}}B$, we may assume that for $i = 1, 2, \dots, r$, we have $m_i = \lambda_i b_i$, where $\lambda_i \in A_{\bar{1}}$ and $b_i \in B$. Then

$$\begin{aligned} \alpha^+(r)(\lambda_1 b_1 \otimes \cdots \otimes \lambda_r b_r) &= \sum_{\sigma \in \text{Sym}_r} \sigma(\lambda_1 b_1 \otimes \cdots \otimes \lambda_r b_r) \\ &= \sum_{\sigma \in \text{Sym}_r} \varepsilon(\sigma) \lambda_{\sigma(1)} \lambda_{\sigma(2)} \cdots \lambda_{\sigma(r)} b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(r)} \\ &= \sum_{\sigma \in \text{Sym}_r} \varepsilon(\sigma)^2 \lambda_1 \lambda_2 \cdots \lambda_r b_{\sigma(1)} \otimes \cdots \otimes b_{\sigma(r)} \\ &= \lambda_1 \lambda_2 \cdots \lambda_r \alpha^+(r)(b_1 \otimes \cdots \otimes b_r), \end{aligned}$$

where ε is the alternating character of Sym_r . But since r is even, $\lambda_1 \cdots \lambda_r \in A_{\bar{0}}$; finally, observe that by Case 1, $\alpha^+(r)(b_1 \otimes \cdots \otimes b_r) \in \Sigma(r)$, and the proof is complete. \square

COROLLARY A.2. *Let $F \in S^d(M^*)$ be a function of degree d on the affine superspace M . Then there exists a function $\tilde{F} \in \text{Hom}(M^d, \mathbb{G}_a)$ such that $F(m) = \tilde{F}(m, \dots, m)$.*

Proof. Take $B = M(A)_{\bar{0}}$ in Proposition A.1. \square

REFERENCES

- [1] M. Atiyah, R. Bott and V. K. Patodi, *On the heat equation and the index theorem*, Invent. Math. **19** (1973), 279–330.
- [2] A. Baha Balantekin and I. Bars, *Representations of supergroups*, J. Math. Phys. **22**(8) (1981), 1810–1818.

- [3] A. Berele, *Invariant theory for matrices over the Grassmann algebra*, Adv. Math. **237** (2013), 33–61.
- [4] A. Berele and A. Regev, *Hook Young diagrams with applications to combinatorics and to representations of Lie superalgebras*, Adv. Math. **64** (1987), 118–175.
- [5] P. Deligne and J. W. Morgan, “Notes on supersymmetry (following Joseph Bernstein)”, in *Quantum Fields and Strings: A Course for Mathematicians*, Vol. 1, 2 (Princeton, NJ, 1996/1997), American Mathematical Society, Providence, RI, 1999, 41–97.
- [6] P. Deligne, G. I. Lehrer and R. B. Zhang, *The first fundamental theorem of invariant theory for the orthosymplectic super group*, Adv. Math. **327** (2018), 4–24.
- [7] M. Ehrig and C. Stroppel, *Schur–Weyl duality for the Brauer algebra and the orthosymplectic Lie superalgebra*, Math. Z. **284**(1–2) (2016), 595–613.
- [8] P. H. Dondi and P. D. Jarvis, *Diagram and superfield techniques in the classical superalgebras*, J. Phys. **A** **14**(3) (1981), 547–563.
- [9] R. Goodman and N. R. Wallach, *Representations and Invariants of the Classical Groups*, Cambridge University Press, Cambridge, 2003, third corrected printing.
- [10] J. J. Graham and G. I. Lehrer, *Cellular algebras*, Invent. Math. **123** (1996), 1–34.
- [11] J. J. Graham and G. I. Lehrer, *The representation theory of affine Temperley–Lieb algebras*, Enseign. Math. (2) **44**(3–4) (1998), 173–218.
- [12] J. J. Graham and G. I. Lehrer, *Diagram algebras, Hecke algebras and decomposition numbers at roots of unity*, Ann. Sci. Éc. Norm. Supér. (4) **36** (2003), 479–524.
- [13] J. J. Graham and G. I. Lehrer, “Cellular algebras and diagram algebras in representation theory”, *Representation Theory of Algebraic Groups and Quantum Groups*, Advance Studies in Pure Mathematics **40**, 141–173. Math. Soc. Japan, Tokyo, 2004.
- [14] V. Kac, *Lie superalgebras*, Adv. Math. **26**(1) (1977), 8–96.
- [15] G. I. Lehrer and R. B. Zhang, *Strongly multiplicity free modules for Lie algebras and quantum groups*, J. Algebra **306** (2006), 138–174.
- [16] G. I. Lehrer and R. B. Zhang, “A Temperley–Lieb analogue for the BMW algebra”, in *Representation Theory of Algebraic Groups and Quantum Groups*, Progress in Mathematics **284**, Birkhäuser/Springer, New York, 2010, 155–190.
- [17] G. I. Lehrer and R. B. Zhang, *The second fundamental theorem of invariant theory for the orthogonal group*, Ann. of Math. (2) **176** (2012), 2031–2054.
- [18] G. I. Lehrer and R. B. Zhang, *The Brauer category and invariant theory*, J. Eur. Math. Soc. **17** (2015), 2311–2351.
- [19] G. I. Lehrer and R. B. Zhang, *The first fundamental theorem of invariant theory for the orthosymplectic supergroup*, Comm. Math. Phys. **349** (2017), 661–702.
- [20] G. I. Lehrer, H. Zhang and R. B. Zhang, *A quantum analogue of the first fundamental theorem of invariant theory*, Comm. Math. Phys. **301** (2011), 131–174.
- [21] C. Procesi, *Lie Groups. An Approach through Invariants and Representations*, Universitext, Springer, New York, 2007, xxiv+596 pp.
- [22] A. Salam and J. Strathdee, *Super-gauge transformations*, Nuclear Phys. **B76** (1974), 477–482.
- [23] M. Scheunert, *The Theory of Lie Superalgebras; An Introduction*, Lecture Notes in Mathematics **716**, Springer, Berlin–Heidelberg–New York, 1979.
- [24] M. Scheunert and R. B. Zhang, *The general linear supergroup and its Hopf superalgebra of regular functions*, J. Algebra **254**(1) (2002), 44–83.
- [25] A. Sergeev, *An analogue of the classical theory of invariants for Lie superalgebras*, Funktsional. Anal. i Prilozhen. **26**(3) (1992), 88–90; (Russian) translation in Funct. Anal. Appl. **26**(3) (1992), 223–225.

- [26] A. Sergeev, *An analog of the classical invariant theory for Lie superalgebras. I, II*, Michigan Math. J. **49**(1) (2001), 113–146; 147–168.
- [27] V. S. Varadarajan, *Supersymmetry for Mathematicians: An Introduction*, Courant Lecture Notes in Mathematics **11**, New York University, Courant Institute of Mathematical Sciences, New York, 2004, American Mathematical Society, Providence, RI.
- [28] B. De Witt, “*Supermanifolds*”, in *Cambridge Monographs on Mathematical Physics*, 2nd ed., Cambridge University Press, Cambridge, 1992.
- [29] Y. Zhang, *On the second fundamental theorem of invariant theory for the orthosymplectic supergroup*, J. Algebra **501** (2018), 394–434.

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