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Dedication

Para aquellas personas generosas
que me han dado más
de lo que podría soñar merecer

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Resumen

En esta tesis estudiamos la cancelación de anomalías perturbativas en teoría M en una variedad con frontera. Esto se hace para grupos de Lie clásicos y excepcionales. El objetivo de este esfuerzo es determinar si existen M9-branas no supersimétricas como soluciones de teoría M. Con estos resultados esperamos contribuir a la exploración del paisaje no supersimétrico de la Teoría de Cuerdas. Además, existe un gran interés por entender si dicha teoría es en realidad universal en el sentido que cualquier teoría de gravedad cuántica aparentemente consistente debe ser parte de su conocido paisaje. En esta dirección, también consideramos teorías $F(\overline{R})$ de Hořava-Lifshitz bajo la lupa del programa del Pantano.

En relación con esto, es de nuestro conocimiento que en el pasado se mostró que el polinomio de anomalía de una teoría de gauge supersimétrica acoplada a gravedad en una variedad spin de dimensión diez, que consiste de un supermultiplete vectorial de E_8 más un supermultiplete de gravedad, cancela el flujo de anomalía de los acoplamientos topológicos de teoría M a dicha frontera. Nos referimos a esta teoría en la frontera como una M9-brana con teoría de volumen de mundo E_8. Nuestra tarea es investigar si pueden haber otras soluciones novedosas de M9-branas con una teoría de volumen de mundo G. Este análisis depende de la descripción de anomalías en términos de la teoría del índice de operadores de Dirac definidos en doce dimensiones. Esto también requiere el cálculo de índices o, equivalentemente, de invariantes de Casimir para representaciones arbitrarias del álgebra de Lie de G. Con esto, expresamos caracteres en la representación fundamental de cada grupo G. Luego, resolvemos las restricciones para una búsqueda de nuevas M9-branas, lo que nos conduce a considerar materia en diferentes representaciones del grupo gauge G. De esto se sigue que cualquier nueva solución que encontremos debe ser sin supersimetría. Finalmente, consideramos las reglas de descomposición de la representación adjunta de E_8 en representaciones de sus subgrupos para dar una interpretación de nuestros resultados.

Por otra parte, consideramos teorías $F(\overline{R})$ de Hořava-Lifshitz y estudiamos su consistencia con la conjetura de de Sitter del Pantano. Esto se hace considerando el hecho de que los criterios del Pantano deben ser aplicables a cualquier teoría de gravedad cuántica aparentemente consistente. Para esto, nos enfocamos en construir una teoría de gravedad más un campo escalar canónicamente acoplados. Esto se hace sin introducir ningún contenido de materia, solo usando la geometría del problema. Eventualmente, aplicamos la conjetura de de Sitter al potencial obtenido y esto se traduce en desigualdades para los parámetros de la teoría en consideración, para las cules discutimos su interpretación.

Abstract

This thesis studies the cancellation of perturbative anomalies in M-theory on a manifold with a boundary. This is done for both classical and exceptional Lie groups G. This endeavor aims to determine whether non-supersymmetric M9-brane solutions exist in M-theory. We expect to contribute to the exploration of the non-supersymmetric String Theory Landscape with our results. Furthermore, there is a great interest in understanding whether String Theory is actually universal in the sense that any apparently consistent theory of quantum gravity must be within the whole String Theory Landscape. In this direction, we consider $F(\overline{R})$ theories of Hořava-Lifshitz under the scrutiny of the Swampland program.

To begin with, it has been shown in the past that the anomaly polynomial of a supersymmetric gauge theory coupled to gravity on a ten-dimensional spin manifold, consisting of an E_8 supervector multiplet plus a supergravity multiplet, can cancel the anomaly inflow of the topological couplings of M-theory to such a boundary. We will refer to this as an M9-brane with E_8 worldvolume theory. Our task is to investigate whether there can be other new M9-brane solutions with a G worldvolume theory. This analysis relies on the description of perturbative anomalies in terms of the index theory of Dirac operators defined in twelve dimensions. This also requires the calculation of indices or equivalently Casimir invariants of arbitrary representations of the Lie algebra of G. With this, we manage to express characters in the fundamental representation of any group G. Subsequently, we solve the constraints to search for new M9-branes which leads us to consider matter in different representations of the gauge group G. From this follows that any new solutions are expected to be non-supersymmetric. Finally, we consider the branching rules of the adjoint representation of E_8 under its subgroups to give an interpretation of our findings.

On the other hand, we consider $F(\overline{R})$ theories of Hořava-Lifshitz and study their consistency with the de Sitter conjecture of the Swampland program. This is done by considering the fact that the Swampland criteria should apply to any seemingly consistent theory of quantum gravity. For this, we focus on constructing a theory of gravity canonically coupled to a scalar field. This is done without introducing any matter content, uniquely using the geometry of the problem. Eventually, we apply the de Sitter conjecture to the obtained potential, which is translated into inequalities for the parameters of the theory under consideration, and then we discuss their interpretation.

Chapter 1

Introduction

String Theory born out of an endeavour to comprehend strong nuclear interactions within the S-Matrix theory program (see [1] for a brief account of the early days of string theory), and it has since evolved to become the leading approach in the quest for the unification of all fundamental forces in nature.[1] Perhaps, the first impetus came from the necessity of gravity for the well-definiteness of the theory, thus unification might sound reasonable. Although, the first significant development was the realization that the five 10-dimensional (10d) supersymmetric string theories were free of quantum inconsistencies (this will be the main topic of this Thesis). By the same time of this discovery, a lot of effort was dedicated to connect string theory to the real world via compactification of the six extra dimensions, mainly on Calabi-Yau manifolds [3]. This program is still ongoing and gave rise to the concept of the Landscape of string theory, basically all possible theories coming from a string theory construction (for reviews on the phenomenological side of string theory, see [4] for connections with the Standard Model, while for the cosmological side, see e.g. [5, 6]). Then, around the mid 1995s a second string theory revolution came about.[2] This was built upon the recognition of a set of dualities playing a crucial role in the comprehension of strong-weak coupling limits of string theory/M-theory [8]. This has been a brief account of decades of intense work in string theory, thus it is not intended to provide a historical overview and much less a complete review of the state of affairs. Instead, it brings us to the actual and one of the most active research program in string theory, that is, the Swampland program [9]. For reviews on this, see [10–12], also see [13–15]. The basic idea of this program is to determine whether the Landscape of Effective Field Theories (EFTs) weakly coupled to Einstein gravity actually represent a set of candidate theories that can be consistently completed in the UV regime, and whether these can be realised through string theory constructions, which is actually the belief of the Swampland community. This question has been translated into the endeavour of identifying a set of principles known as Swampland Conjectures (SCs) mainly using the string theory Landscape as a laboratory. Among the SCs, some have received significant support and have been tested in various contexts. These include, for instance: the No global symmetries conjecture [16], the Weak Gravity conjecture [17], and the Distance conjecture [18]. More recently, significant effort have been invested to the Emergence conjecture [19,20] and the Species Scale conjecture [21–24]. Other conjectures are the Cobordism conjecture [25],

[1] In an attempt to draw connections between string theory and Loop Quantum Gravity, see [2] and references therein.

[2] Around the same time, [7] came up with the insight that D-branes serve as sources for the various gauge potentials in the string theory spectrum, a discovery that was also crucial for constructing the duality framework.

Non-Susy Anti-de Sitter (AdS) conjecture [26], the de Sitter (dS) conjecture [26–28] (we will spend some time testing this conjecture on $F(\overline{R})$ Hořava-Lifshitz theories), and some other criteria, but we refer the reader to the reviews mentioned before for those. The important point to be mentioned is the fact that these conjectures are all interconnected, hence this might lead to a few criteria allowing to rule out EFTs that, once we coupled to gravity, cannot be UV completed, and we say that those theories are in the Swampland.

Maybe, a good way to summarize this ongoing effort is by the String Universality principle [29] or String Lamppost principle [30] which is the statement that any EFT coupled to gravity with a UV completion must be realized by string theory. This was tested in 10d, with supersymmetry playing an important role in [31], for six dimensional theories in [32], whereas for d $= 7, 8, 9$ see [30]. It is important to remark that the converse in not necessarily true. In fact, the reason behind the Swampland criteria is to draw a boundary between the *true* Landscape and the Swampland. Therefore, from this follows that, if we already have a consistent UV Quantum Gravity theory , not necessarily coming from a string theory construction, the SCs must be applicable. In this sense, we apply the dS conjecture to $F(\overline{R})$ Hořava-Lifshitz (HL) theories [33] (see [34] and reference therein for a review of HL theories) where a scalar field and a potential could be obtained in terms of the curvature, and finally by making a conformal transformation from the Jordan to the Einstein frame we get a model of gravity plus a scalar that we can bring to a canonically coupled system taking some particular limits relating the constant parameters of the HL theories under consideration. By testing this model under the dS conjecture we are able to interpret it as a set of inequalities for the parameters of the HL theories, consistent with the different regimes of the models we studied. On the other hand, without taking the particular limit just mentioned, we cannot obtain a gravity plus scalar field system canonically coupled. Even though, we propose that the dS conjecture is still applicable. This case is more involved, but again we can test it very explicitly and translate it into a set of inequalities for the parameters of the theory. Even more, we can connect our results with the standard $f(R)$ theories also analyzed in [33].

On the other hand, in the same spirit of the Swampland program, we studied string theory/M-theory universality in 10d/11d by using perturbative anomaly cancellation relaxing the constraints of supersymmetry. This is in contrast to Refs. [30–32] already mentioned where supersymmetry was crucial. However, we stress that anomaly cancellation was instrumental for the conclusions reached in those references. In the following, we will give an introduction to symmetries and anomalies to state and clarify terminology used in the literature as well as to introduce the modern viewpoint of symmetries and anomalies that will be used in the main subject of this thesis.

Let us start by saying that the best way we have to understand our real world is by doing perturbative calculations, once we loose control of the usual expansions we are used to in Quantum Field Theory (QFT), we get into trouble. The most relevant example of this is Quantum Chromedynamics (QCD) where a full understanding of its confining region is missing. In order to pursue this perturbative approach we usually start by writing down a *classical* action principle S subject to well-established classical symmetries like Lorentz invariance or Poincaré symmetry. In QFT this is usually done in flat spacetimes M. However, in theories like General Relativity (GR), where the spacetime can have a nontrivial topology, we require an improvement to invariance under general coordinate transformations. Often, it happens that one formulation could be accompanied with some hidden symmetries that in other but equivalent formulation are not manifest. These classical symmetries could be global or spacetime-dependent symmetries but, before we go into that, let us stress here that they play an important role as a guiding principle to put forward this program of understand our quantum world. In this sense, one of the key observations was the realization that the classical symmetries we start with, could be broken by quantum effects, and this has the name of an anomaly. At first glance this should be disturbing, however, by now we are used to concepts such as the spontaneous breakdown of a symmetry or the breakdown of a classical global or gauge symmetry by radiative corrections. The main focus of this thesis is about gauge anomalies. But, we will first describe global symmetries and their anomalies.

Let us emphasize the difference between a global symmetry of a theory $\mathcal{T}$ and a gauge redundancy of our description of $\mathcal{T}$. A global symmetry is a symmetry that acts nontrivially on observables, quantities that we already measure. In other words, a global symmetry acts nontrivially on the Hilbert space $\mathcal{H}_{\mathcal{T}}$ of $\mathcal{T}$. Let us focus on *ordinary*, continuous internal global symmetries associated with a compact Lie group G for simplicity [35,36]. The main feature of a global symmetry in a Lagrangian description of $\mathcal{T}$ is a conserved current J and a conserved charge Q associated with a parameter ϵ for Abelian or ϵ^a for a non-Abelian symmetry, where a account for the number of generators. Q is conserved in the sense that it commutes with the Hamiltonian, $[Q, H] = 0$, from which follows that for a state $|\mathrm{a}\rangle$ of H with energy $\mathcal{E}_{\mathrm{a}}$, the state $Q\,|\mathrm{a}\rangle$ is also a state of H with energy $\mathcal{E}_{\mathrm{a}}$. In this sense one can rearrange the Hilbert space of states $\mathcal{H}_{\mathcal{T}}$ into unitary irreducible representations of the algebra of the charge or symmetry operators. In field theory[3], the space of charged objects under the symmetry consists of localized operators, defined on a spacetime point and the symmetry operators are defined in $d-1$ slices of the spacetime. This description of global symmetries can as well be extended to discrete symmetries where the notion of a symmetry operator is well defined. Even more, this notion of a global symmetry can be extended from 0-form ordinary fields to p-form fields[4] where, there is also a conserved $(d-p-1)$-current [37]. We can also define symmetry defects supported on a $(d-p-1)$ submanifold of the spacetime. The distinctive feature of this generalization is the fact that the charged objects can now be extended on the spacetime, namely they are supported on p-dimensional cycles of the spacetime manifold. Another characteristic of global symmetries with $p > 0$ is that they have to be Abelian by topological arguments, this means that symmetry operators necessarily commute in contrast to the ordinary ones. With this we can move to the quantum theory and compute correlation functions as well as study selection rules of a theory with this structure[5] of symmetries. For more details, one can consult some of the many reviews on this active field of research, some of them with applications to different branches of physics are e.g [41–45]. For a different ongoing debate on global symmetries when gravity is part of the game the reader may consult, e.g. [46,47] and reference therein.

Often, one starts with a free field theory description regarded the first paragraph of this introduction. However, this is usually not enough to get into a real system, it is needed to introduce interactions. This can be done, for instance, by coupling a conserved current to a external field keeping in mind the original symmetries of the theory. The basic example, for our purposes, is to consider $\mathcal{T}$ as a massless, free fermion field theory with an internal $U(1)$ global symmetry. As we just described, we can introduce a external vector coupling

[3] In this brief account of symmetries, we will always assume Euclidean signature, then there is no subtleties with symmetry operator along the time direction. Therefore, the words symmetry operator or symmetry defect means the same thing.

[4] See Appendix A

[5] It should be emphasized that this concept of symmetries has received generalizations through various directions, we address the reader to some of the reviews and reference therein for details e.g. [38–40].

to the free theory and ask for the symmetries of this new setting. We play the game of finding conserved currents by Noether's theorem for continuous symmetries. It turns out that this came with a profound revelation. In the free field theory the global $U(1)$ symmetry comes with a variation schematically as follows

$$\delta S \sim \int \mathrm{d}\epsilon(x) \wedge *J \,, \tag{1.1}$$

while, once we introduced the vector coupling, we find that the same procedure that leads to (1.1) leaves the action S completely invariant as long as, again schematically

$$\partial_\mu \to D_\mu = \partial_\mu - iA_\mu \,, \qquad A_\mu \to A_\mu + \partial_\mu \epsilon(x) \,, \qquad \epsilon : M \to U(1) \,, \tag{1.2}$$

$$\overline{\psi} \mathrm{i} \gamma^\mu D_\mu \psi \,, \qquad \psi \to e^{\mathrm{i}\epsilon(x)} \psi \,, \tag{1.3}$$

the ordinary derivative is redefined as the covariant derivative and A_μ is the external vector potential that coupled to J, ϵ is a map from the spacetime to the group $U(1)$, and the massless Dirac action now becomes into (1.3). At this stage, A is a nondynamical vector field. And the procedure just described is refereed to as the gauging of the $U(1)$ global symmetry. The same procedure can be extended to any Lie group G, continuous or discrete, with the main difference that A becomes into a matrix-valued field taking values in the Lie algebra of G, and the holonomy, respectively. We can study the quantum theory[6] of this new setup. For the purpose of this work this is conveniently done under the approach of the path-integral quantization in Euclidean signature as

$$Z[A] = \int D\overline{\psi} D\psi \exp\bigl(-S[A, \overline{\psi}, \psi]\bigr) \,, \tag{1.4}$$

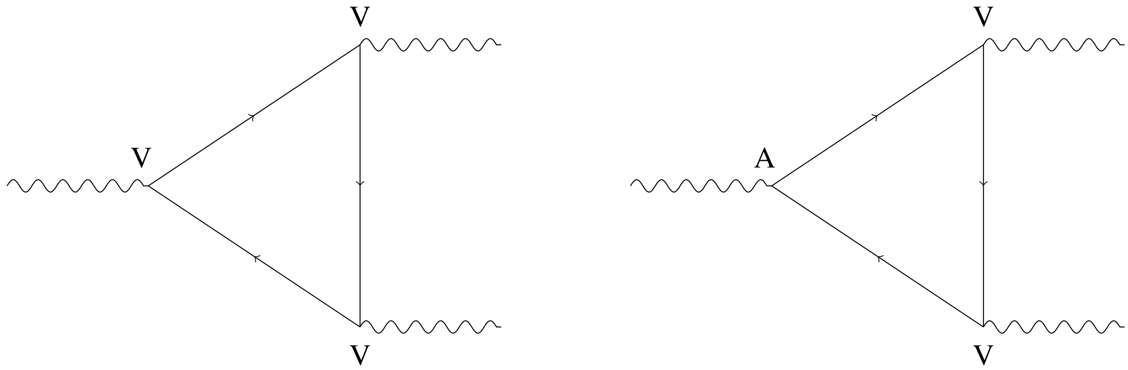
where $Z[A]$ is the partition function as a function of the external background field A, $D\psi$ is the integral measure weighted by the Dirac action coupled to the external potential. More details of this will be reviewed in Chapter 3. Here, we briefly mention the Feynman diagrammatic approach in order to explore quantum consequences.

The object containing tree-level corrections as well as loop corrections is the quantum effective action, usually denoted as Γ, which can be defined as the Legendre transformed of the generating functional W of connected diagrams. It turns out that for vector theories, namely theories $\mathcal{T}_{\mathrm{V}}$ which allow a mass term consistent with all the symmetries of $\mathcal{T}_{\mathrm{V}}$, admit a regularization of loop or quantum corrections that is also consistent with all its symmetries, for instance via Pauli-Villars regularization becoming Γ well defined. In the process of obtaining final answers one does not break any symmetry. Therefore, vector theories, such as Dirac theory, preserve classical symmetries in the quantum theory. See Figure 1.1a for a one-loop Feynman diagram of a 4d theory. However, it was also noted that for free, massless fermion theories there is also a chiral global symmetry

$$\psi \to e^{\mathrm{i}\,\epsilon \gamma_{d+1}} \psi \,, \tag{1.5}$$

where γ_{d+1} is the chirality operator, which for 4d is the usual γ_5 matrix, see Appendix B. By Noether's theorem there is also a classically conserved chiral current. This is such that, instead of having only vector current couplings, one can have a situation such as that shown in Figure 1.1b with one axial current insertion and two vector current insertions for

[6]For the quantization of the free Dirac fermion theory one can see a standard QFT book, e.g. [48]



(a) Fermion triangle with three vector current insertions.

(b) Fermion triangle with two vector current insertions and one axial current.

Figure 1.1: Triangle one-loop Feynman diagrams with wavy lines describing vector bosons for a 4d theory.

example, with fermions running throughout the loop, again for a 4d theory. It turns out that for this particular loop, there is no way to regularize such that preserving the vector currents one also preserves the axial current conservation. As a result, quantum effects lead to the breakdown of the axial current in the presence of vector currents coupled to gauge vector fields [49, 50]. Consequently, what we have found is an obstruction to the gauging of global symmetries in the presence of axial couplings. Although, this is no that bad as it may sound since, it says that we are not allowed to gauge a global symmetry in the presence of axial couplings. In fact, this is the ABJ anomaly, as it is known in honor to Adler, Bell and Jackiw, whom were the first in computing it. This anomaly actually led to the resolution of the π^0 decay problem. Even more, due to G. 't Hooft [51] there is a more powerful approach to the obstruction problem of gauging a global symmetry known as 't Hooft anomaly matching due to its behaviour under renormalization group flow relating IR with UV physics, see [38] for a review and reference therein on this subject.

What we have described up to this point is quite general in the sense that it does not depend on the dimension of the spacetime. Yet, what we have shown in Figure 1.1 are the one-loop corrections of a 4d fermion theory. Nonetheless, this can be generalized in various ways. First of all, this can be extended to theories with a non-Abelian G-symmetry. Furthermore, it can be extended to an axial coupling with the insertion of two energy-momentum tensors instead of two vector currents with fermions also running through the triangle in Figure 1.1b and this is connected to chiral coupling in the presence of gravity [52]. On the other hand, this can also be generalized to higher dimensions. Generically, for a $d = 2n-2$ dimensional theory, it was proved that anomalies of classical symmetries are associated with one-loop diagrams with n vertices with *chiral fermions* running in the loop. The important case for us is in 10d, where the potentially anomalous one-loop diagrams are known as hexagon diagrams. It fact it was proved that only one-loop corrections are important for the computation of anomalies due to axial couplings [53]. Other diagrams can contribute but they are determined by the n-point loop diagrams. Thus, this is why we only emphasize those loop diagrams. In this situation, the effective action Γ becomes ill defined under gauge or diffeomorphism transformations. In fact, for diffeomorphisms is equivalent to consider local Lorentz transformations of the frames (see Appendix B) up to local counterterms. Hence, we will usually refers to diffeomorphisms although, this can be formulated in terms of gauge Lorentz transformations for the purpose

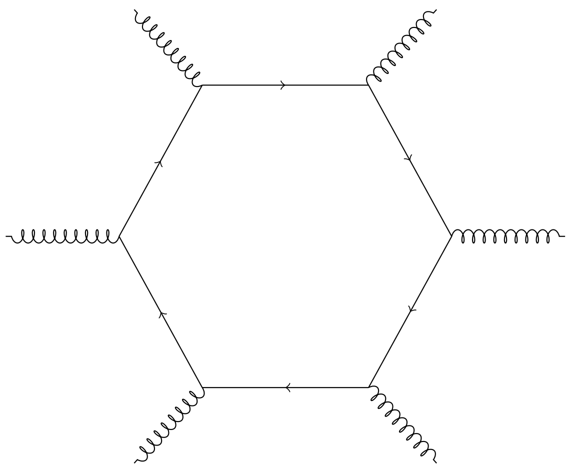


Figure 1.2: Hexagon diagram with external gravitons represented by coiled lines with chiral fermions running in the loop. The external lines can be replaced by gauge vector couplings.

of anomalies.

We can also extend this analysis to genuine gauge theories,[7] the gauge vector couplings are to be considered dynamical fields. We have to include a kinetic term for this field into the action S. In terms of Feynman diagrams we have to think about the propagator of this field as well. In terms of the path integral we can integrate out this field. However, this comes along with important subtleties since, gauge theories are characterized by a redundancy or gauge equivalence, often called gauge symmetry. Though, strictly speaking is not a symmetry in contrast to the global symmetries described above. In this sense, gauge theories are properly described by the theory of fibre bundles, see Appendix B for an rough introduction to this ideas and references for details. Properly, the gauge symmetry we are talking about in physics contexts, is related to the fact that on a general topological space M one cannot choose a global differentiable structure, correspondingly one cannot define a global fibre structure. Instead, we cover M with coordinates charts and this requires gluing consistency conditions over nonempty overlap of charts. For fibre bundles this is translated into the fact that the gauge potential A has to be thought of as a connection on the bundle subject to the gauge equivalence given by

$$A \to A^g = g^{-1} A g + g^{-1} \mathrm{d} g \,, \tag{1.6}$$

for a set of local gauge transformations

$$g : M \to G \,, \tag{1.7}$$

where G is the fibre of the bundle over M, whose transitions functions over patches also take values in G. Note that this reduces to (1.2) for G the Abelian group $U(1)$. This is the best way we have to describe our real world, namely by means of introducing a redundancy in our description. This redundancy always comes along with an much bigger Hilbert space than the true Hilbert space, among the subtleties we already alluded. However, we can use gauge symmetry to find the correct Hilbert space. Unlike global symmetries, gauge symmetry is not a symmetry in the sense that this does not act on the physical Hilbert

[7] Since gravity can be formulated in a similar fashion as usual G-gauge theories by the orthonormal frame bundle, always we say gauge theory we are, for the discussion of anomalies, also taking into account gravity.

space transforming one state into another as global symmetries actually do. Instead, one can introduce an equivalence relation [54, 55] under gauge symmetry to identify states of the bigger Hilbert space so that the physical space is constructed of equivalence classes of sates under the gauge symmetry. As a consequence, gauge symmetry is essential to the well-definiteness of a physical theory. The failure in preserving this redundancy is translated into a disaster. For instance, locality is lost as we will see from non local terms appearing in the presence of an anomaly given by the loops shown in the figures 1.1b and 1.2. We also lose unitarity of the quantum theory due to negative norm states within the unphysical Hilbert space. As we said, all becomes a complete chaos. The first example one encounters where some of these features come up is $U(1)$ Maxwell theory in the field theory context. This is the reason why we must guarantee gauge invariance before we move to the quantization of a gauge theory. Thus, the question in this context is how we go over the chiral anomaly described above in the presence of dynamical gauge theory.

Perhaps, the simple way to see this is as follows [56]. Introduce chirality operators in terms of the chiral matrix γ_{d+1} as

$$P_\pm = \frac{1 \pm \gamma_{d+1}}{2}\,, \tag{1.8}$$

where $P_+ = P_L$ is a projection operator to the space of left-handed or positive chirality fermions $\mathcal{H}_+$ and $P_- = P_R$ is the projector to the space of right-handed or negative chirality fermions $\mathcal{H}_-$. From the hermitian properties of the chirality operator and the anti-commutation relation obeyed by the gamma matrices and the chiral matrix, one can show that for chiral fermions a mass term is forbidden. Consequently, for the Dirac equation introduced before in (1.3) we can also show that the Dirac operator $\gamma^\mu D_\mu = \not{D}$ is such that

$$\not{D} : \mathcal{H}_\pm \to \mathcal{H}_\mp\,, \tag{1.9}$$

it maps the (infinite) positive chiral vector space into the negative chiral vector space and vice versa. On the other hand, at the formal level, the chiral partition function $Z[A]$ $-$ afeter integrating out chiral fermions $-$ is proportional to $\det(\mathrm{i}\not{D}P_L)$. But, we already saw that $\not{D}P_L$ maps positive chirality fermions to negative chirality fermions. Unlike linear operators or endomorphisms, there is no obvious way to define a determinant[8] of such operators, hence we can suspect the presence of an anomaly from this observation. Indeed, from this is also easy to see that the operator $(\mathrm{i}\not{D}P_L)^\dagger(\mathrm{i}\not{D}P_L)$ has a well-defined determinant $|\det(\mathrm{i}\not{D}P_L)|^2$, up the a phase, that is, the anomaly. This phase ambiguity is associated to one-loop Feynman diagrams with gauge vector or gravitational couplings with chiral fermions running throughout the loop. And this is interpreted as the failure to regulate these loops in the gauge invariant consistent way.

Another way to see the one-loop anomaly is by looking at the Jacobian of a chiral transformation of the path-integral measure [61]. This computation has been done to a grate amount of detail, for instance, in [56, 62, 63], we refer the reader to those reference. The important point to stress is the fact that anomalies are UV effects, manifested itself as a phase ambiguity in the chiral partition function, before integrating out gauge fields,

[8] In fact, in a more mathematical side this determinant is thought of as a section of a line bundle, known as the determinant line bundle [57] whose base space is the parameter space of gauge and metric fields [58]. This formalizes the phase ambiguity of the chiral partition function into the holonomy of the determinant line bundle [59] around a loop in the parameter space. Roughly, this holonomy is generally capture by the eta invariant [60] evaluated on a closed $(d+1)$-manifold perfectly matching with the bordism classification of anomalies briefly mention in a moment. See also chapter 3.1.

hence it is not necessary to deal with all the subtleties of gauge theory quantization, even for gravity it is not clear how to sum up over different topologies in order to integrate out gravity from the path integral. Hence, for the purpose of anomalies we can safely avoid this discussion and focus only on the phase ambiguity. As we saw, anomalies are UV effects, but they have been also interpreted as IR effects, which also makes possible to compute anomalies in terms of the topological structure involved in the proper definition of a gauge theory [64, 65] and this will be the path we will follow in this thesis. For a review on this trajectory, see [66]. Also, this mixing of the UV and IR for symmetries and anomalies [51] make also possible to study anomalies of string theory by using its different low-energy limits consisting of massless modes with well-understood low-energy lagrangian descriptions, see e.g. [67–70]. The relevant loop diagrams are those such as that shown in Figure 1.2 with pure gravitational coupling, or with pure gauge vector coupling, or mixed gauge and gravitational [52, 66].

Before we move to the second and main portion of this thesis, that is, string/M-theory anomaly cancellation, let us briefly remark the modern characterization of anomalies in Field Theory. First of all, we have shown that for chiral theories the determinant may suffer from phase ambiguities capture by one-loop diagrams at the diagrammatic level and, in the failure of the path-integral measure to be invariant under chiral transformations by functional methods. At the perturbative level, anomalies are associated with linear transformation that can be continuously deformed to the identity of the symmetry group G. However, among the maps $g : M \to G$ there could be transformations that cannot be smoothly deformed to the identity that can give rise to anomalies as well. In the field theory context this was first discovered in [71] for an $SU(2)$ gauge theory coupled to an odd number of Weyl fermion doublets, transforming in the fundamental of $SU(2)$, under a nontrivial transformation classified by 4-th homotopy group of $SU(2)$, see Chapter 3. Eventually, this was also applied to string theory, notably in [72, 73]. These anomalies involving transformations considering the global structure of the symmetry group were nicknamed global anomalies for obvious reasons. However, we will avoid this terminology here and, instead we will call nonperturbative anomalies. Therefore, we have two sources of anomalies in continuous gauge symmetries. This is in contrast to discrete gauge symmetries where only nonperturbative anomalies are relevant [74, 75] for anomaly cancellation. Consequently, we need systematic procedures to analyze these sources of anomalies to guarantee a well-behaved quantum gauge theory.

One important step in this direction started with the observation that perturbative anomalies are closely related to Index theory of Dirac operators [76–80]. Indeed, this leads to a very precise description of perturbative anomalies in terms of a nonlocal functional up to the fact that one can add local counterterms allowing to change from the consistent to the covariant anomaly [81] without changing the essence of a phase ambiguity in the chiral partition function. Remarkably, this local functional can be computed from a $(d + 2)$ characteristic polynomial translating perturbative anomaly computations into geometric and topological information [82, 83]. On the other hand, we have to deal with nonperturbative anomalies. These anomalies are more subtle, correspondingly led to more sophisticated arguments in order to be detected, for instance the mapping tori construction [71]. Another approach was proposed by [84] (see [85] for the motivation on this approach, see also [86] for a proper treatment of the procedure) considering the embedding[9] of symmetry groups. Nonetheless, the main point is the fact that this pro-

[9]As we said, for discrete symmetries does not make sense to talk about perturbative anomalies, however we can study them by the embedding $\mathbb{Z}_n \to U(1)$, see [74, 75].

cedure mainly focus on G-gauge theories where the symmetry elements were classified by homotopy groups of G without providing a systematic description of nonperturbative anomalies in presence of G-gauge theory as well as gravity, in contrast to the perturbative case.[10] Motivated by arguments in Condensed Matter Physics [94,95], it was conjectured that anomalies of a field theory are classified by bordism theory. Roughly speaking (see Appendix C for some of the concepts here), perturbative and nonperturbative anomalies are capture by $(d+1)$ and $(d+2)$ bordism groups (for the first time using a bordism argument in the study of anomalies, see [96]), respectively, and classes of these bordism groups are detected by a field theory known as Invertible Field Theory (IFT) [97–99]. For a review on this perspective, see [100]. For chiral fermions the IFT corresponds to the Atiyah-Patodi-Singer η-invariant [101–103] and this provides a unified description of perturbative and nonperturbative anomalies for G-gauge theories as well as gravity, as we will see in Chapter 3, see also Appendix C. Moreover, one of the first considerations involving the eta invariant for anomalies can be found in [73,104] and for a reduced sample of recent applications and computations of bordism groups, see [105–128].

The aim of this thesis is to study cancellation of perturbative anomalies. This is because the first step in the assessment of anomalies corresponds to the perturbative side. Once we have convince ourself that there is no perturbative anomalies, for instance by summing up all the contributions of the anomalous degrees of freedom, as happens in the Standard Model [55] or Type IIB [52] superstring; or by a more sophisticated perturbative anomaly cancellation method such as the Green-Schwarz mechanism [129,130], we can go through the nonperturbative side. However, this side of the story of anomalies is out of the scope of this thesis, we hope to return to this in future work (see the end of Chapter 3 for some comments of this). We will consider the 11d low-energy limit of M-theory[11] which has a topological interaction [132], given schematically as follows[12]

$$\int \frac{1}{6} C \wedge C \wedge G - C \wedge I_8(R) \,. \tag{1.10}$$

This low-energy limit of M-theory is quantum mechanically well defined in the sense that is free of anomalies[13]. However, we wonder whether this topological coupling makes sense on a manifold X with boundary, namely whether there can be boundary or edge modes transforming in representations of a symmetry group G such that the anomalous boundary contribution coming from (1.10) is cancelled with anomalies of the edge modes. In other words, we are interested in the anomaly inflow [133] of (1.10) to the boundary (see [59,60, 134] for a modern discussion on inflow capturing nonperturbative effects of chiral fermions and [135] for chiral p-forms, see also [55,136,137] for a discussion on the inflow on brane couplings). This task was carried out by Hořava and Witten in the mid-90s [138,139] for the exceptional group E_8. They found that having boundary modes transforming in the adjoint representation of E_8 plus gravity modes coming from the 11d gravitino, the perturbative anomaly of these modes compensate the anomaly of the topological coupling (1.10) on a manifold with boundary, see [140] for a review. This result turned out to be consistent

[10]For a modern understanding of the method of Elitzur and Nair [84] see [87] and for applications of this new understanding, see [88,89], for the homotopy application of this, see [90–93].

[11]This connection of an 11d with the 10d low-energy limits of string theory was established in [73] by using the 11d supergravity of Cremmer-Julia-Scherk [131].

[12]For the definition of each term in equation (1.10), see Chapter 4.

[13]This theory is defined in an odd dimensional spacetime X, so this theory is free of perturbative anomalies. However, it is possible to have nonperturbative anomalies, anomalies not seen by perturbative methods. We will review this in Chapter 3 and we use it to review anomalies of M-theory in Chapter 4.

with supersymmetry [68, 141, 142] adding to the string/M-theory duality frame [8]. We will investigate whether a similar anomaly cancellation mechanism can occur for any Lie group G with chiral edge modes transforming in representations of G. The main difficulty for arbitrary G is the computation of indices or equivalently Casimir invariants of Lie algebra representations (see Appendix A). Particularly, for exceptional algebras we can carry out the algorithm developed in Chapter 5 very explicitly due to some known result about indices [143]. Also, the number of representations to be considered is of dimension less than the adjoint of E_8 due to chirality reasons, this will be clear from chapter 5. For the classical Lie algebras, many of them are ruled out by dimensionality arguments. For the relevant cases of the $SU(n)$ algebra A_{n-1} we compute the indices at all orders, for the anti-symmetric representations $[2], [3], [4]$ an for the symmetric representations $(2), (3)$, of $SU(n)$ such that with this information it is relatively easy to extend our results for the exceptional case. For the classical algebras B_{2n+1}, C_{2n} and D_{2n}, we show that the only relevant cases that might deserve some attention are for $2 \leq n \leq 6$, see [144] for the details of classical algebras.

One similar example of this in less dimensions, maybe more known, is the quantum Hall effect, where in a two dimensional boundary lives edge modes transforming in a complex representation which have an anomaly. This anomaly compensate the anomaly of a 3d Chern-Coupling for the electromagnetic potential A, defined on a 3d manifold with boundary. A set of lecture note on the quantum Hall effect can be found in [145], also see [146].

An important feature of our search if the fact that any hope of find the appropriate anomaly cancellation mechanism the bulk-boudary setup considered in this work has to involve other representations than the adjoint of a symmetry group, thus if we are able to find a solution, it has to be nonsupersymmetric since gauge boson can only be in the adjoint representation of a group. Whether a possibly new solution is consistent under other checks such as tachyon free solutions, or whether can be related to already known solutions is another question. However, for the explicit examples we have worked out, we are able to relate them to the known E_8 case by studying the branching rules of the adjoint representation of E_8 under subgroups.

The remainder of this thesis is structured as follows. In Chapter 2 we briefly mention the bosonic string, and with little more detail we review the quantization of the supersymmetric string to see how its massless spectrum appears. We briefly mention one of the known nonsupersymmetric string as well. We move to Chapter 3 where we review anomalies of a gauge theory in the path-integral approach. We briefly review the modern viewpoint on anomalies as well as anomaly cancellation mechanisms. In chapter 4, we review perturbative anomaly cancellation in the supersymmetric and one of the nonsupersymmetric strings as well as for M-theory in a useful way for Chapter 5. There, we developed an algorithm to check whether there can be boundary modes solutions charged under a gauge group G. This modes are related to M-theory on a manifold with boundary by inflow so that the bulk-boundary system is free of perturbative anomalies. In an unrelated chapter to anomalies we discuss $F(\overline{R})$ Hořava-Lifshitz theories under the dS Swampland conjecture, it is done in Chapter 6. We conclude in Chapter 7.

In three Appendices we developed material for the rest of this work. Appendix A goes through indices computations of representations of Lie groups. Appendix B we review the geometrical structure of bundles, and briefly we introduce some of the characteristic classes relevant for anomaly calculations. In Appendix C we give a short discussion of the bordism classification of anomalies.

Chapter 2

Review of superstring and M-theory

2.1 Bosonic string

In trying to learn string theory one starts studying a bosonic string. To do this, we use our intuition from the analysis of the classical dynamics of a point particle. This is usually done by embedding the point particle worldline into a flat space or sometimes one starts more generally, in a spacetime with a nontrivial curvature, then the geodesic equations of motion are to be found. In the case of a string, we write down a classical action in terms of the square root of an infinitesimal area element, the Nambu-Goto action, usually in a flat spacetime since, for the string, things become more involved in a background curved spacetime. Nevertheless, we can go perturbatively through the effects of nontrivial background spaces. Thus, we will stick to a flat background in this short discussion. Once we have an action, as a next step one focuses on the symmetries of the classical theory by including both global and gauge symmetries; Poincaré and parameterization invariance. This is important because global symmetries lead to conserved quantities whereas, gauge symmetries lead to redundancies which can facilitate the study of the quantum theory through clever choices of gauge fixing. After this, we determine equations of motion and try to solve them to learn useful things about the classical theory. Eventually, with this information, we go to the quantization of the theory nonetheless, when we move to the quantum theory we do not know how to quantize the bosonic string under this formulation due to the presence of square roots independently of the quantization process we choose.

To overcome this problem, we can start again studying the point particle in a different but equivalent formulation by introducing a kind of auxiliary field whose equation of motion can be used to show the equivalence of both formulations of the point particle classical dynamics. One important feature of this auxiliary field is the lack of a kinetic term in the action such that this is not a dynamical field but in a particular gauge turns out to be the mass-shell condition. In the case of the string, a similar idea goes over, allowing us to write down an equivalent action through an auxiliary, non-dynamical field that turns out to be the 2d worldsheet metric. This action is known as the string sigma model or Polyakov action. Notably, this equivalent formulation has a new (gauge) symmetry in addition to the symmetries already mentioned in the previous paragraph. It turns out that this new symmetry, a Weyl symmetry of the worldsheet metric is basically the whole point of the string worldsheet theory. It is important to mention that Weyl and reparameterization invariance symmetry allows us to gauge-fixed the classical theory in such a way that we are left with a theory that can be viewed as a two-dimensional scalar-free field theory from the perspective of the worldsheet along with simplified constraints coming from the equations of motion of the auxiliary field. This is illuminating because, in principle, we know how to quantize a scalar-free field theory. However, we need to keep in mind the gauge-fixing conditions and constraints (known as Virasoro constraints) coming from the worldsheet metric once we move on to the quantum theory. If one is unable to preserve gauge symmetries at the quantum level, this signals a sick theory, for example, unitarity is completely lost. Therefore, it is an important point to guarantee that the classical symmetries are also preserved at the quantum level and this has important consequences for the bosonic string like fixing the dimensionality of the background spacetime to 26 dimensions.

There are various approaches to quantization but as a first step, we can go through the different stages of canonical quantization starting with the promotion of classical variables to operator-valued variables, and finally determine the Hilbert (more properly, the Fock space) space of the open and closed string. This comes with important difficulties since it is not direct to obtain the physical spectrum of the theory due to the appearance of negative norm states as well as other ambiguities we are used to like regularization of infinities and normal-ordering ambiguities. The Virasoro algebra constructed from generators of Weyl symmetry, related to the constraint equations plays an important role in canonical quantization to determine the true physical spectrum of the theory.

Another approach to quantization is light-cone gauge quantization which deals with the negative norm states exploiting a sort of residual conformal symmetry giving an additional gauge-fixing condition such that we manage to build the true mass spectrum of the bosonic string through the classical solutions of the equations of motion and solving the Virasoro constraints. Although, this way of quantization breaks the manifestly Lorentz invariance of the classical theory, so once we get the quantum physical spectrum we must be sure that Lorentz invariance is also in the quantum theory. We will not pursue any further details in either canonical or light-cone quantization for the bosonic string. However, we stress a couple of important points related to the mass spectrum of the bosonic theory for subsequent discussions.

As we mentioned, there are certain symmetries at the classical level that could be broken by quantum effects. Weyl symmetry in the bosonic string, for example, is broken and one way to see this is by a central extension of the classical Virasoro algebra by a central charge $c = d$. Nonetheless, Weyl symmetry plays a key role in quantization, in fact, due to its nature of being a gauge symmetry its quantum breakdown is a complete disaster. Fortunately, in the canonical quantization for example, one can make sense of the quantum theory provided that the dimension of spacetime $d = 26$, thus the consistency of the free bosonic string fixes the dimension of spacetime. On the other hand, it is believed that Lorentz symmetry is equally important in any relativistic theory, thus we need to make serious efforts to maintain it at the quantum string level. If we look at the mass spectrum of the open string in the light-cone gauge quantization, the first non-tachyonic, excited particle state corresponds to

$$a^i_{-1} \left|0, k\right\rangle \ , \qquad i = 1, \dots d-2 \, , \tag{2.1}$$

where a^i_{-1} is only a creation operator related to the vibration modes of the string, $|0, k\rangle$ the vacuum of the string which is annihilated by any annihilation operator a^i_n with $n > 0$, and k account for the momentum (of the center of mass of the string). State (2.1) is a vector and, in principle, can perfectly furnish a vector representation but, the Lorentz group is $SO(d = 26)$, so it seems impossible to fit this into a representation of $SO(d = 26)$; an anomaly? However, if one studies more carefully representations on one-particle states one realizes that massive states furnish a representation for the *little group* $SO(d-1)$ whereas massless states define a representation for the little group $SO(d-2)$. Therefore, the first excited state of the bosonic string corresponds to a vector field whose quantum can be associated with massless particles transforming in the fundamental representation $\mathbf{24}$ of the little group for the open string. The same reasoning applies to the closed string up to the fact that the first excited state is

$$\tilde{a}^i_{-1} a^j_{-1} |0, k\rangle \ , \qquad i, j = 1, \dots d-2 \, , \tag{2.2}$$

where $\tilde{a}^i_{-1}$ and a^i_{-1} are creation operators for independent left- and right-moving oscillation of the closed bosonic string, $|0, k\rangle$ is again the vacuum as before. Now, to be consistent with Lorentz symmetry, (2.2) corresponds to massless state transforming in a tensor representation $\mathbf{24} \otimes \mathbf{24}$. By standard tools in representation theory, we can decompose that tensor product into a traceless symmetric tensor representation h_{ij}, an anti-symmetric b_{ij}, and a scalar ϕ representation. The remarkable thing is that one of these states can be identified with the quantum of a gravitational field in a d-dimensional spacetime $G_{\mu\nu}$, namely the graviton since its polarization states are exactly the same as the number of components $(d-2)(d-1)/2$ of h_{ij}. This also comes with a spacetime anti-symmetric field, or two-form, known as the Neveu-Schwarz field, and a scalar ϕ, the dilaton. In other words, we can associate a spacetime massless field to each of the massless oscillation modes of the bosonic string. We could continue analyzing the massive spectrum of the theory in the same way as before only that consistency with Lorentz symmetry requires that these states are accommodated into representations of $SO(d-1)$ instead of $SO(d)$.

Unfortunately, the bosonic string comes with its own problems regarding the fact that gravity is not enough in order to pursue a unified description of all the forces and particles of our physical world, namely, there seems no obvious at all how to get gauge bosons or fermions from the bosonic string spectrum, essential for the Standard Model of particle physics. Moreover, we have avoided talking about a negative mass state in the bosonic spectrum which implies the presence of an unstable state, namely the vacuum of the theory is located in the maximum of a potential, and up to now it is not known if this will eventually decay to some stable state. Last, a twenty-six-dimensional spacetime seems awkward at first glance, particularly considering our everyday experience with a four-dimensional world. It turns out that it is possible to address some of these problems if one is able to write down a 2-dimensional supersymmetric version of the bosonic string action. Our next task shall be to revise this with some detail, in particular, we focus on obtaining the massless spectrum of the supersymmetric string for future discussion.

To anyone interested in more details on the bosonic string we refer to the huge amount of literature available, among the books covering this material are [67–70,147–152]. There are also a lot of lecture notes available on the web, a reduced sample is [153–157].

2.2 Supersymmetric string

Here, we will explore how the quantization of a superstring leads to supersymmetric massless spectra as oscillation modes of the supersymmetric string. These modes can be identified with the quantum of fields in the background spacetime. Finally, the dynamics of these fields in the low-energy limit, essentially correspond to supergravity theories. Our primary focus will be on the fermionic sector, as the main topic of this work is anomalies instead of phenomenology, for these see e.g. [158–160] and reference therein. For a more complete treatment of this section see e.g. [67–70, 148, 149]

It is possible to write down a supersymmetric action for the superstring in a particular gauge, the superconformal gauge, after we used the gauge redundancies − reparameterization invariance and super-Weyl symmetry in the supersymmetric case − to gauge-fixed some of the degrees of freedom of a local supersymmetric theory coupled to gravity, reducing the action to the so-called Neveu-Schwarz-Ramond superstring action [161, 162]

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\left(\partial_a X\cdot\partial_b X - \overline{\psi}\cdot\partial\!\!\!/\psi\right)\,, \tag{2.3}$$

where $X^\mu = X^\mu(\tau,\sigma)$ refers to d bosonic fields (also known as wolrdsheet embedding functions), $\psi^\mu = \psi^\mu(\tau,\sigma)$ refers to d Majorana-Weyl doublets, and $A\cdot B = A^\mu B_\mu$ is the usual contraction in an Einstein sum which will constantly be used in this work, and α' is only a constant. This equation is incomplete without the equations of motions associated with its global symmetries; translation invariance and global supersymmetry.

Very generally, from Noether's theorem, we know that for any global symmetry, there exists a conservation law, roughly given by

$$\delta\,S \sim \int d^d x J^\mu_a \partial_\mu \epsilon^a\,, \tag{2.4}$$

where ϵ^a is an infinitesimal parameter associated with the symmetry transformations, J^μ_a is some function of the fields. We have promoted the parameter to be spacetime dependent, $\epsilon^a = \epsilon^a(x)$, in equation (2.4). Still, it is clear that the variation vanishes if the parameter is restricted to a global symmetry, independent of the coordinates, as it must be. Also, note that if the equations of motion are satisfied for a field configuration under an arbitrary variation, then from $\delta\,S = 0$ follows that

$$\partial_\mu J^\mu_a = 0\,, \tag{2.5}$$

and this is the statement of classic current conservation. If we consider translation symmetries of the worldsheet coordinates $\delta\sigma^a = \epsilon^a(\sigma)$ by doing them coordinate dependent then we are dealing with diffeomorphisms of the worldsheet and for a theory coupled to dynamical 2d gravity, before going to the equation (2.3) in our case of study, we find that the conserved current corresponds to the stress-energy momentum tensor

$$T_{ab} \sim \frac{\delta S}{\delta h^{ab}} = 0\,. \tag{2.6}$$

Equation (2.3) is also invariant under global supersymmetry transformation given by

$$\delta X^\mu = \bar\epsilon\psi^\mu\,,\qquad \delta\psi^\mu = \rho^a\partial_a X^\mu\epsilon\,,\qquad \delta\overline{\psi}^\mu = -(\partial_a X^\mu)\bar\epsilon\rho^a\,, \tag{2.7}$$

where ϵ is the supersymmetry parameter corresponding to a Majorana spinor. By making it dependent on the worldsheet coordinates we can compute its conserved current which, up to a total derivative, is given by

$$J^a = \rho^b\rho^a\psi_\mu\partial_b X^\mu = 0 \tag{2.8}$$

Note that the constraint equations (2.6) and (2.8) are equal to zero. This is because, in the general formulation of the superstring action coupled with dynamical gravity and consistent with local supersymmetry, those equations correspond to the equations of motion of the zweibein field e^i_a and to its superpartner χ^a_α (a is a local Lorentz index, while α is a spinor index), respectively. In fact, from Weyl invariance, one can show that T_{ab} is traceless and $\rho_a J^a = 0$ by using $\rho_a \rho^b \rho^a = 0$. We need to keep in mind all these details to find the classical solutions and more importantly when we turn the classical superstring into a consistent quantum superstring.

Equation (2.3) looks so gorgeous if one thinks about a string moving through a flat background at the same time it oscillates. This is the magic of gauge symmetries, but we have to pay a price, all the gauge redundancies we have exploited must be preserved at the quantum level. Moreover, we have global symmetries whose conservation laws play a role in constraining the classical and more importantly the quantum solutions. This discussion has been useful to appreciate the power of symmetries. Nonetheless, it is more illuminating if we choose to work in light-cone worldsheet coordinates $\sigma^{\pm} = (\sigma^1 \pm \sigma^2)/2$ such that equation (2.3) becomes into

$$S = \frac{1}{\pi\alpha'} \int d^2\sigma \left(\partial_+ X \cdot \partial_- X + \frac{i}{2}\, \psi_+ \cdot \partial_- \psi_+ + \frac{i}{2}\, \psi_- \cdot \partial_+ \psi_- \right) , \tag{2.9}$$

where $\partial_{\pm} = \frac{\partial}{\partial \sigma^{\pm}}$, and ψ^{μ}_A with $A = \pm$ are the chiral components of ψ^{μ}. Thus, the equations of motion are

$$\partial_+ \partial_- X^{\mu}(\sigma) = 0 \,, \qquad \partial_- \psi^{\mu}_+(\sigma) = 0 \,, \qquad \partial_+ \psi^{\mu}_-(\sigma) = 0 \,, \tag{2.10}$$

whose general solutions are given by $X^{\mu}(\sigma) = X^{\mu}_L(\sigma^+) + X^{\mu}_R(\sigma^-)$, a linear combination of left- and right-movers, $\psi^{\mu}_L = \psi^{\mu}_+(\sigma^+)$ and $\psi^{\mu}_R = \psi^{\mu}_-(\sigma^-)$ are positives and negatives chiral states corresponding to left- and right-moving modes; respectively. To determine explicit solutions we must distinguish between a closed and an open string. The closed string is topologically made by the identification $\sigma^1 \sim \sigma^1 + 2\pi$ and its dynamics describe a cylinder embedded into spacetime, thus solutions must be supplemented with

$$X^{\mu}(\sigma^0, \sigma^1) = X^{\mu}(\sigma^0, \sigma^1 + 2\pi) \,. \tag{2.11}$$

Remarkably, when we consider fermions a new ingredient enters the game since fermions are not representations of a rotation group $SO(d)$. Instead, they furnish a representation of its double cover, the $\mathrm{Spin}(d)$ group. In other words, an oriented *structure* on the circle, identified with the closed string is not enough to define fermions and we need to lift orientation to a spin structure. This thesis has to do with measuring possible obstructions to define chiral fermions coupled to tangent and fiber bundles, the following chapters and appendices go through the details of this statement. For now, it is enough to say that the circle admits two different spin structures which in stringy settings are usually identified with Ramond (or periodic) and Neveou-Schwarz (or anti-periodic) spin structures. As both chirality modes $\psi_{\pm}$ can be viewed as independent variables in the closed string, we can impose both the periodic (R) and the anti-periodic (NS) spin structures independently such that

$$\psi_R(\sigma^0, \sigma^1) = \pm \psi_R(\sigma^0, \sigma^1 + 2\pi) \,, \quad \psi_L(\sigma^0, \sigma^1) = \pm \psi_L(\sigma^0, \sigma^1 + 2\pi) \,, \tag{2.12}$$

thus, depending on how fermions behave under the identification $\sigma^1 \sim \sigma^1 + 2\pi$ we have four different choices of spin or boundary conditions, namely R-R, R-NS, NS-R, NS-NS, where this notation means that we impose periodic boundary conditions over left-modes and right modes, periodic boundary conditions over left-modes and anti-periodic boundary conditions over right-modes, and so on and so forth. The superstring is much richer than the bosonic string and we have to figure out what each of these sectors corresponds to. Additionally, by using this information we can determine the explicit solutions for the bosonic and fermionic modes by ensuring that they obey the equations of motion as well as the boundary conditions (2.11) and (2.12). On the other hand, we could also consider open strings with appropriate boundary conditions in order to solve the classical theory, then we can proceed to the quantization. However, we focus here only on the closed string. For the purpose of this review, the quantization of the open string with Neuman boundary conditions is not that different. For details and richness of open strings see e.g. [150,151]

We continue the discussion with the closed string. We give its classical solutions without providing any proof but the interested reader can go into the details by looking at the references at the beginning of this section. For the closed string, we have that the bosonic solutions are

$$X_L^\mu(\sigma^+) = \frac{1}{2}(x^\mu + x_0^\mu) + \frac{1}{2}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z},\, n\neq 0} \frac{1}{n}\tilde{\alpha}_n^\mu \, e^{-in\sigma^+} , \tag{2.13}$$

$$X_R^\mu(\sigma^-) = \frac{1}{2}(x^\mu - x_0^\mu) + \frac{1}{2}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z},\, n\neq 0} \frac{1}{n}\alpha_n^\mu \, e^{-in\sigma^-} , \tag{2.14}$$

where x^μ, p^μ corresponds to the center of mass position and momentum of the string, α_n^μ, $\tilde{\alpha}_n^\mu$ are right- and left-moving constant oscillation modes of the closed string, and x_0^μ is only a constant. The general solution is given by the sum, $X^\mu(\sigma) = X_L^\mu(\sigma^+) + X_R^\mu(\sigma^-)$. Normalizations of the above equations are conventional and it is also important to note that the reality conditions on $X^\mu(\sigma)$ implies that $(\alpha_n^\mu)^* = \alpha_{-n}^\mu$ and $(\tilde{\alpha}_n^\mu)^* = \tilde{\alpha}_{-n}^\mu$, where $(\cdots)^*$ means complex conjugation. Note that, if we define a $n = 0$ mode has to be real.

While, the fermionic solutions with periodic (R) boundary conditions are given by

$$\psi_L^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-in\sigma^+} , \qquad \widetilde{\mathrm{R}} , \tag{2.15}$$

$$\psi_R^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^-} , \qquad \mathrm{R} , \tag{2.16}$$

for left- and right-movers independently. With anti-periodic (NS) boundary conditions fermionic solutions are given by

$$\psi_L^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{r\in\frac{1}{2}+\mathbb{Z}} \tilde{b}_r^\mu \, e^{-ir\sigma^+} , \qquad \widetilde{\mathrm{NS}} , \tag{2.17}$$

$$\psi_R^\mu = \sqrt{\frac{\alpha'}{2}} \sum_{r\in\frac{1}{2}+\mathbb{Z}} b_r^\mu \, e^{-ir\sigma^-} , \qquad \mathrm{NS} . \tag{2.18}$$

where d_n^μ, $\tilde{d}_n^\mu$, b_r^μ, $\tilde{b}_r^\mu$, are constant oscillation modes for the fermionic side. The bosonic and fermionic solutions presented here correspond to classical solutions of the equations of motion (2.10) subject to the boundary conditions already discussed for the closed string. However, these need to be supplemented with constraint equations, the super-Virasoro constraints associated with the energy-momentum tensor, which when expressed in terms of light-cone worldsheet coordinates take the following form

$$T_{++} = \partial_+ X \cdot \partial_+ X + \frac{i}{2}\,\psi_+ \cdot \partial_+ \psi_+ = 0\,, \tag{2.19}$$

$$T_{--} = \partial_- X \cdot \partial_- X + \frac{i}{2}\,\psi_- \cdot \partial_- \psi_- = 0\,. \tag{2.20}$$

In addition, we have to include the constraints associated with supersymmetry, namely the vanishing of the supercurrent. According to equation (2.8) we can express the supercurrent constraints in light-cone coordinates as follows

$$J_+ = \psi_+ \cdot \partial_+ \psi_+ = 0\,, \qquad J_- = \psi_- \cdot \partial_- \psi_- = 0\,, \tag{2.21}$$

where we have computed J^0 and J^1 and then we have appropriately combined them to get those two independent constraints.

It should be noted that we have avoided writing down the energy-momentum tensor explicitly in coordinates σ^a, $a = 0, 1$ of the worldsheet because what is really useful is the form presented in equations (2.19) and (2.20), as we move to the quantum theory. To see this, it is more common to express the energy-momentum tensor in terms of the so-called super-Virasoro constraints, that is, in terms of generators of the super-Virasoro algebra that arise by noting that we can average the independent components of the stress tensor over the closed string as follows

$$\mathrm{L}_n = \frac{1}{2\pi\alpha'}\int_0^{2\pi} d\sigma\, T_{++}\, e^{in\sigma^+}\,, \tag{2.22}$$

and this can be separated into two parts, one bosonic and the other fermionic, $L_n = L_n^X + L_n^\psi$ given by

$$\widetilde{\mathrm{L}}_n^X = \frac{1}{2}\sum_{m\in\mathbb{Z}} \tilde{\alpha}_m \cdot \tilde{\alpha}_{n-m}\,, \tag{2.23}$$

$$\widetilde{\mathrm{L}}_n^\psi = \frac{1}{2}\sum_{m\in\mathbb{Z}} m\, \tilde{d}_m \cdot \tilde{d}_{n-m} \quad \mathrm{R}\,, \qquad \widetilde{\mathrm{L}}_n^\psi = \frac{1}{2}\sum_{r\in\frac{1}{2}+\mathbb{Z}} r\, \tilde{b}_r \cdot \tilde{b}_{n-r} \quad \mathrm{NS}\,. \tag{2.24}$$

where we have used $\partial_+ X^\mu = \sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^\mu e^{-in\sigma^+}$ by defining $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ into equation (2.19) and similarly for the fermionic part we have used (2.17) and computed $\partial_+\psi_+$ to evaluate (2.22). In the same way, there is a set of constraints for the closed string concerning right-movers, L_n^X, $\mathrm{L}_n^{\psi,\mathrm{R}}$, $\mathrm{L}_n^{\psi,\mathrm{NS}}$ which are basically given by equations (2.23) and (2.24) with no tilde over the oscillation modes. From this, it follows that we can also average the supercurrent along the closed string such that

$$\mathrm{F}_n = \frac{1}{\sqrt{2}\pi\alpha'}\int_0^{2\pi} d\sigma J_+ e^{in\sigma^+}\,, \qquad \mathrm{G}_r = \frac{1}{\sqrt{2}\pi\alpha'}\int_0^{2\pi} d\sigma J_- e^{ir\sigma^-}\,. \tag{2.25}$$

By using this with the supercurrents given in (2.21) and results already computed, we obtain for the left-movers

$$\widetilde{\mathrm{F}}_n = \sum_{m\in\mathbb{Z}} \tilde{\alpha}_{-m} \cdot \tilde{d}_{m+n} \quad \mathrm{R}\,, \qquad \widetilde{\mathrm{G}}_r = \sum_{m\in\mathbb{Z}} \tilde{\alpha}_{-m} \cdot \tilde{b}_{m+r} \qquad \mathrm{NS}\,. \tag{2.26}$$

We have similar expressions for the right movers for both the Ramond and Neveu-Schwarz sectors. From equations (2.22) to (2.25) follows that the Virasoro operators L_n, $\widetilde{\mathrm{L}}_n$ as well as the supercurrent operators F_n, $\widetilde{\mathrm{F}}_n$, G_r, $\widetilde{\mathrm{G}}_r$ must vanish in correspondence with the energy-momentum and supercurrent constraints.

We have all the ingredients to move on to the quantum theory. The goal is to quantize the superstring consistently with all the classical symmetries already discussed. At the same time, we will determine the energy spectrum of the superstring, particularly the massless spectrum.

As we already mentioned, a good way to start this adventure is by doing what we are used to, canonical quantization. To begin with, promote all classical variables to operator-valued fields. These fields must obey equal-time commutation or anti-commutation relations depending on their nature. More precisely, depending on the spin-statistics of our variables we have to deal with $[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')]$, where $X^\mu(\tau,\sigma)$ are the field solutions given before and $P^\mu(\tau,\sigma')$ its conjugate momentum, and for fermionic fields we deal with $\{\psi^\mu(\tau,\sigma), \psi^\nu(\tau,\sigma')\}$. We can work out the commutator and after an easy but long computation we can find that

$$[X^\mu(\tau,\sigma), P^\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\eta^{\mu\nu}\,, \tag{2.27}$$

it is translated into commutation relations for the center of mass variables and the Fourier modes, namely[1]

$$[x^\mu, p^\nu] = i\eta^{\mu\nu}\,, \qquad [\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta^{\mu\nu}\delta_{n,-m}\,, \tag{2.28}$$

and other commutations relations are zero. Note that, if we define $a_n = \alpha_n/\sqrt{n}$ and $a^\dagger_n = \alpha_{-n}/\sqrt{n}$ then, we find well-known commutation relations for a free harmonic oscillator

$$[a_n, a^\dagger_m] = \delta_{nm}\,, \qquad [\tilde{a}_n, \tilde{a}^\dagger_m] = \delta_{nm}. \tag{2.29}$$

In other words, the oscillation modes of the string have to do with creation and annihilation operators which is somehow expected because this is what happens when we quantize a scalar free field theory. For fermionic fields, we do similar computations leading to the following anticommutation relations for the fermionic Fourier modes

$$\{d^\mu_n, d^\nu_m\} = \eta^{\mu\nu}\delta_{n,-m}\,, \qquad \{b^\mu_r, b^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}\,, \tag{2.30}$$

$$\{\tilde{d}^\mu_n, \tilde{d}^\nu_m\} = \eta^{\mu\nu}\delta_{n,-m}\,, \qquad \{\tilde{b}^\mu_r, \tilde{b}^\nu_s\} = \eta^{\mu\nu}\delta_{r,-s}\,, \tag{2.31}$$

There is an important issue when we play this game of quantization related to ordering ambiguities of the quantum operators essentially due to nontrivial (anti)commutation relations. If we look at the constraint equations we note that, for instance, L^X_0 is proportional to $\sum_{n\in\mathbb{Z}} \alpha_n \cdot \alpha_{-n}$ or to $\sum_{n\in\mathbb{Z}} \alpha_{-n} \cdot \alpha_n$ and one choice or the other is different for a constant contribution as we can see from (2.28). Fortunately, we are also used to this kind of trouble. The way out is by choosing a prescription of quantization, such a scheme will be the normal ordering prescription where any annihilation operator is moved to the right of any creation operator. This is usually denoted by $:\mathcal{O}_i:$ for normal-ordered operators

[1] To get these commutation relations we have made use of the Fourier relation

$$\delta(\sigma-\sigma') = \frac{1}{2\pi}\sum_{n=-\infty}^{+\infty} e^{-in(\sigma-\sigma')}$$

$\mathcal{O}_i$. However, we avoid using this notation by hoping that, it is clear from our discussion what we are doing. Obviously, this is not for free and we need to introduce some normal-ordering constant measuring the arbitrariness in our chosen prescription. We stress that this problem will rear its head only for the super-Virasoro constraints $L_{n=0}$, $\widetilde{L}_{n=0}$. We need to keep in mind this to construct the physically sensible quantum Hilbert space of the superstring.

For a moment, we look only at the bosonic string. As usual, we will choose the vacuum state in such a way that any annihilation operator kills that state: $a^\mu_n\,|0,p\rangle = 0$, $\tilde{a}^\mu_n\,|0,p\rangle = 0$, $n > 0$ where, only for this time in what follows, we have explicitly shown the momentum mode of the string $\hat{p}\,|0,p\rangle = p\,|0,p\rangle$, where $\hat{p}$ must be understood as the momentum operator but, we avoid this notation from our discussion since is not that important. By using the creation operators we construct the tower of states associated with $a_{-n} = a^\dagger_n$, $\tilde{a}_{-n} = \tilde{a}^\dagger_n$. For instance, consider $a^\dagger_n$, then for $n > 0$ we have

$$|n_1, \tilde{n}_1, \cdots m_2, \tilde{m}_2 \cdots\rangle = (a^{\mu_1\,\dagger}_n)^{n_1}(\tilde{a}^{\nu_1\,\dagger}_n)^{n_1} \cdots (a^{\mu_2\,\dagger}_m)^{m_2}(\tilde{a}^{\nu_2\,\dagger}_m)^{m_2} \cdots |0\rangle \tag{2.32}$$

and we interpret this as a multi-particle state which is allowed according to the statistical properties of the creation operators. The reason why the two operators appear in (2.32) is because of a condition known as the level-matching condition, we will see this more explicitly in a moment.

On the other hand, following the statistical properties of fermions then, we find that fermionic operators have only two possible occupation numbers since $(d^\dagger_n \cdot d_n)(d^\dagger_n \cdot d_n) = d^\dagger_n \cdot d_n$, it follows that the eigenvalues of $d^\dagger_n \cdot d_n$ are basically 0 or 1.

If we try to go further with the canonical quantization procedure we find that some of the states have a negative norm, thus the Hilbert space we have constructed is too big to be physically sensible as we can see from the state $a^{0\,\dagger}_n\,|0\rangle$ and the inner product

$$\langle 0|a^0_n a^{0\,\dagger}_n|0\rangle = -1\,. \tag{2.33}$$

These negative norm states are sometimes called ghosts but this is confusing because there is another notion of ghosts in gauge theory quantization which is not physical as well, but they are quite useful to properly quantize gauge theories. We will not use that terminology here, although the problem we face here is not completely alien to that discussion since we have not said anything about the super-Virasoro and super-current constraints which are very useful to get rid of the negative norm states. Before continuing, let us first point out that we have tried to make all the warnings as explicitly as possible in this discussion of the quantum superstring since dealing with those warnings is the main task of any approach to quantization.

In order to go any further, we note here that there is a more convenient way of quantization known as covariant quantization with an extra warning. This is due to the fact that it breaks the manifest Lorentz invariance but, of course, it has to be there at the end of a hard day. In contrast, it has the advantage that we can throw all the negative norm states away by noting that there remains a set of residual gauge symmetries when we are working in worldsheet light-cone coordinates given by a combination of conformal and diffeomorphisms transformations so that $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$. This set of transformations leaves the gauge-fixed worldsheet metric invariant. Therefore, if we are able to choose an appropriate gauge-fixing condition, we may successfully get to the end of our day.

To start with, we introduce light-cone spacetime coordinates

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{d-1}), \qquad X^i, \quad i = 1, \cdots d-2\,, \tag{2.34}$$

and by standard change of coordinates, we can find the new metric accordingly

$$\eta_{+-} = \eta_{-+} = -1\,, \qquad \eta_{ij} = \delta_{ij}\,, \tag{2.35}$$

thus, leading to $ds^2 = -2dX^+dX^- + \delta_{ij}dX^idX^j$. Be aware of this to raise and lower indices, and finally note that a contraction $X_\mu Y^\mu = -2X^+Y^- + X^iY^j$.

We can consider the expression for X^+, given by

$$X^+(\tau,\sigma) = x^+ + \alpha' p^+\tau + i\sqrt{\frac{\alpha'}{2}}\sum_{n\in\mathbb{Z},\,n\neq 0}\frac{1}{n}\left(\alpha_n^+ e^{-in\sigma^-} + \tilde{\alpha}_n^+ e^{-in\sigma^+}\right)\,. \tag{2.36}$$

The clever gauge we mentioned above related to the residual gauge symmetries consists of gauging away the oscillator modes α_n^+, $\tilde{\alpha}_n^+$, we set these modes to zero, leaving us with

$$X^+(\tau,\sigma) = x^+ + \alpha' p^+\tau\,, \tag{2.37}$$

and by looking at the super-Virasoro constraint equations we can determine that

$$\partial_\pm X^- = \frac{1}{\alpha' p^+}(\partial_\pm X^i)^2\,. \tag{2.38}$$

Therefore, we have found that there is no α^- to be quantized because according to our discussion, it is given by the transverse modes, only modes α_n^i, $i = 1,\cdots d-2$ are physically relevant; they are the physical degrees of freedom.

A similar discussion goes through the fermion fields defining $\psi^\pm = (\psi^0 \pm \psi^{d-1})/\sqrt{2}$, ψ^i, $i = 1\,\cdots\,, d-2$, then remembering that we have superconformal and supersymmetry transformations to play with, we can project out the modes $d_n^+ = 0 = \tilde{d}_n^+$ as well as $b_r^+ = 0 = \tilde{b}_r^+$ leaving us with the transverse modes d_n^i, $\tilde{d}_n^i$, $b_n^i\,\tilde{b}_n^i$. The upshot of this discussion is about the degrees of freedom we have to quantize, namely only transverse modes. Thus, in principle, we are getting closer to the physical Hilbert space since we have projected out negative norm states. It is left to apply the constraints. To do this, suppose that $|\mathrm{phy}\rangle$ is a physical state of the Hilbert space of the superstring, then we impose the constraints discussed above, in the form of operator equations

$$(L_0 - a)\,|\mathrm{phy}\rangle = 0\,, \quad (\tilde{L}_0 - a)\,|\mathrm{phy}\rangle = 0\,, \quad L_n\,|\mathrm{phy}\rangle = 0 = \tilde{L}_n\,|\mathrm{phy}\rangle\,, \quad n > 0\,, \tag{2.39}$$

$$F_n\,|\mathrm{phy}\rangle = 0 = \widetilde{F}_n\,|\mathrm{phy}\rangle\,, \qquad n \geq 0\,, \qquad G_r\,|\mathrm{phy}\rangle = 0 = \widetilde{G}_r\,|\mathrm{phy}\rangle\,, \qquad r > 0\,, \tag{2.40}$$

where a is only the normal-ordering constant already mentioned. Note that it suffices to impose these constraints for positive integers otherwise, it will be too strong.

There is a quick way to see what is going on with these constraints and the spectrum of the superstring. However, we warn that it is not the most rigorous way to do it.

We will focus on L_0^X, $L_0^{\psi,\,\mathrm{R}}$, $L_0^{\psi,\,\mathrm{NS}}$ while $\widetilde{L}_0^X$, $\widetilde{L}_0^{\psi,\,\mathrm{R}}$, $\widetilde{L}_0^{\psi,\,\mathrm{NS}}$ can be worked out in a similar fashion. To avoid cluttering with the notation we go through the computations separately, first in the Ramond (R) sector, and then to the Neveu-Schwarz (NS) sector. Consequently, we set $L_0^X = L_0$, $L_0^{\psi,\,\mathrm{R}} = L_0^{\mathrm{R}}$ and $L_0^{\psi,\,\mathrm{NS}} = L_0^{\mathrm{NS}}$, and the same apply for tilde operators. Let us start with (2.23), we thus obtain that[2]

$$
\begin{aligned}
\mathrm{L}_0 &= \frac{1}{2}\sum_{n\in\mathbb{Z}} \alpha_{-n}\cdot\alpha_n = \alpha' p^2 + \frac{1}{2}\sum_{n\neq 0,\, n\in\mathbb{Z}} \alpha^i_{-n}\alpha^i_n \\
&= \alpha' p^2 + \mathrm{N} - \frac{d-2}{24} = 0\,,
\end{aligned}
\tag{2.41}
$$

where we have defined $\mathrm{N} = \sum_{n>0} \alpha^i_{-n}\alpha^i_n$, and we have also used the commutations relations (2.28) restricted to the transverse modes. Note that here is where the normal-ordering ambiguity rears its head with a constant in terms of d, by now, the arbitrary dimension of the background spacetime. In the case of the R sector we obtain

$$
\mathrm{L}_0^{\mathrm{R}} = \frac{1}{2}\sum_{n\in\mathbb{Z}} n d_{-n}\cdot d_n \tag{2.42}
$$

$$
= \mathrm{N}_{\mathrm{R}} + \frac{d-2}{24} = 0\,, \tag{2.43}
$$

where we have defined $\mathrm{N}_{\mathrm{R}} = \sum_{n>0} n d^i_{-n} d^i_n$, also we have made similar arrangements as in (2.42) and used the anticommutation relation (2.30). For the NS we get

$$
\mathrm{L}_0^{\mathrm{NS}} = \sum_{r>\frac{1}{2}} r b^i_{-r} b^i_r - \frac{d-2}{2}\sum_{r>\frac{1}{2}} r \tag{2.44}
$$

$$
= \mathrm{N}_{\mathrm{NS}} - \frac{d-2}{48}\,, \tag{2.45}
$$

where we have defined $\mathrm{N}_{\mathrm{NS}} = \sum_{r>\frac{1}{2}} r b^i_{-r} b^i_r$. The operators N we have defined can be thought of as number operators for the different sectors. Next, from the constraints follow that $(L_0 - \tilde{L}_0)\,|\mathrm{phy}\rangle = 0$ on any physical state and this is the level-matching condition at the quantum level we mentioned above. In order to determine the energy spectrum of the superstring we need one thing more, that is, the Hamiltonian. Roughly speaking, we look at its definition average along the closed string, and that gives us $H \sim L_0 + \widetilde{L}_0$, and this is known as the mass-shell condition. These conditions determine the mass spectrum of the superstring. However, we have to be careful because left-moving and right-moving fermions can be in two different sectors and this gives rise to sector mentioned before and written from now as RR sector, the RNS sector, the NSR sector, and finally to the NSNS sector. By analyzing the mass-shell condition and the level-matching condition we obtain

- The Ramond-Ramond sector:

$$
\mathrm{L}_0 + \mathrm{L}_0^{\mathrm{R}} + \widetilde{\mathrm{L}}_0 + \widetilde{\mathrm{L}}_0^{\mathrm{R}} = 0 \to \alpha' M^2 = \mathrm{N} + \mathrm{N}_{\mathrm{R}} + \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{R}}\,, \tag{2.46}
$$

$$
\mathrm{L}_0 + \mathrm{L}_0^{\mathrm{R}} - \widetilde{\mathrm{L}}_0 - \widetilde{\mathrm{L}}_0^{\mathrm{R}} = 0 \to \mathrm{N} + \mathrm{N}_{\mathrm{R}} = \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{R}}\,, \tag{2.47}
$$

  where M is the mass of the corresponding state by using $\alpha_0 \sim p$ and $p^2 = M^2$. The massless state corresponds to the ground state in this sector, namely $\alpha_n\,|0\rangle = 0$ and

[2] An important but weird result that is very useful to regularize infinite sums is

$$
\sum_{n>0} n = -\frac{1}{12}\,, \qquad \sum_{r>\frac{1}{2}} r = \frac{1}{24}
$$

$d_n |0\rangle = 0$, the same also follows for tilde operators. Note that the second equation above seems to be a true level-matching condition for left- and right-movers, we create the same number of each one by applying the creation operators.

On the other hand, note that the algebra of the Ramond oscillation zero modes d_0^μ is given by $\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}$, so by setting $d_0^\mu = \Gamma^\mu/\sqrt{2}$, we get the Clifford algebra[3]

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu} , \tag{2.48}$$

thus, this means that acting with d_0^μ on the Ramond vacuum is equivalent to acting with gamma matrices. In other words, the Ramond vacuum corresponds to a representation of the Clifford algebra whose dimension is $2^{\frac{10}{2}} = 32$, therefore, we have found Dirac spinors of 32 components. However, the proper vacuum we have to look at is the tensor product

$$|0\rangle_{\mathrm{R}} \otimes |\widetilde{0}\rangle_{\mathrm{R}} , \tag{2.49}$$

and we know from the representation theory of the Lorentz group that the product of two spinor representations can be decomposed into vector representation with a bosonic behaviour. The upshot is that we obtain bosons from the RR sector. Also, note that by applying the creation operators d_{-n}^i, $\tilde{d}_{-n}^i$ we get fermions but its tensor product will again correspond to bosons.

- The Ramond-Neveu-Schwarz sector

$$\alpha' M^2 = \mathrm{N} + \mathrm{N_R} + \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{NS}} - \frac{d-2}{16} , \tag{2.50}$$

$$\mathrm{N} + \mathrm{N_R} = \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{NS}} - \frac{d-2}{16} . \tag{2.51}$$

It seems we get into trouble, the lowest energy state that would have to correspond to a solution of the RNS vacuum is a negative mass state, again we have found a tachyon. For a moment, consider the first excited state which corresponds to $\mathrm{N} = \mathrm{N_R} = \widetilde{\mathrm{N}} = 0$ and $\widetilde{\mathrm{N}}_{\mathrm{NS}} = \frac{1}{2}$. We find that the first excited state is massless provided that the dimension of the background spacetime is $d = 10$, therefore consistent with Lorentz invariance. This dimension is usually known as the critical dimension of the superstring. However, one may worry whether this is consistent with the other sectors. Surprisingly, the answer is yes. But, even worse, we associate the oscillation modes of the string with the quantum states of fields in the background spacetime. Thus, this suggests that the supersymmetry of the worldsheet must somehow be translated into supersymmetry in the background spacetime. So, the number of difficulties found in the construction of a consistent quantum superstring is enormous, it is tempting to give up. We won't do it. In fact, we can see what the first excited state corresponds to by acting with $b_{-1/2}^i$ into the NS vacuum

$$b_{-1/2}^i |0\rangle_{\mathrm{NS}} . \tag{2.52}$$

This is a vector with $d-2$ polarizations, so consistency with Lorentz invariance demands that this is a representation of the little group $SO(8)$, we have a vector boson in spacetime. However, the proper thing we need to examine is the tensor product[4]

$$|0\rangle_{\mathrm{R}} \otimes |\widetilde{0}\rangle_{\mathrm{NS}} , \tag{2.53}$$

[3] See Appendix B

[4] We still need to discuss how we are going to address the negative mass states.

where the R vacuum is a spinor, and this does not change by acting with the vector-like creation operators as we already discussed. In contrast, by acting with b^i_{-n} any number of times on the NS vacuum, we obtain tensor representations of the Lorentz group. Therefore, we find that the RNS sector produces spacetime spinors.

- The Neveu-Schwarz-Ramond sector

$$\alpha' M^2 = \mathrm{N} + \mathrm{N}_{\mathrm{NS}} + \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{R}} - \frac{d-2}{16}\,, \tag{2.54}$$

$$\mathrm{N} + \mathrm{N}_{\mathrm{NS}} = \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{R}} + \frac{d-2}{16}\,. \tag{2.55}$$

We can apply a similar discussion to the NSR sector. In the end, the analysis boils down to the statement that the NSR sector describes spacetime fermions.

- The Neveu-Schwarz-Neveu-Schwarz sector

$$\alpha' M^2 = = \mathrm{N} + \mathrm{N}_{\mathrm{NS}} + \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{NS}} - \frac{d-2}{8}\,, \tag{2.56}$$

$$\mathrm{N} + \mathrm{N}_{\mathrm{NS}} = \widetilde{\mathrm{N}} + \widetilde{\mathrm{N}}_{\mathrm{NS}}\,. \tag{2.57}$$

Note, the NSNS sector has a ground state with negative mass. However, if we consider the first excited states corresponding to $(b^i_{-1/2}\tilde{b}^j_{-1/2})(|0\rangle_{\mathrm{NS}} \otimes |\tilde{0}\rangle_{\mathrm{NS}})$, we obtain the first massless states, provided that $d = 10$, as we found before. Again, using a similar reasoning as before, this has to be a tensor representation of the little group $SO(8)$ in order to be massless.

From the previous analysis, it is not clear how to obtain the normal ordering constant, but it can be related to the critical dimension. Indeed, in each sector, it is given by

$$a_{\mathrm{R}} = \widetilde{a}_{\mathrm{R}} = 0\,, \tag{2.58}$$

$$a_{\mathrm{NS}} = \widetilde{a}_{\mathrm{NS}} = \frac{1}{2}\,. \tag{2.59}$$

Also, we have already pointed out other problems with the superstring energy spectrum and these constants are one of them. For instance, we need to check if the number of spacetime bosons is equal to the number of spacetime fermions, which, a priori, there is no reason to believe will be the case. The quickest way to see this is by looking at the RNS sector. Firstly, the NS vacuum corresponds to a scalar of negative mass, but if we look at the R sector, there is no fermion state with similar characteristics of negative mass. We can go further since the Ramond ground state corresponds to a massless Dirac spinor of 32 components. On the other hand, the first excited massless vector boson of the NS sector gives rise to 8 polarization states. This is a problem because, in principle, any hope to have spacetime supersymmetry requires, among others, the same number of bosons and fermions degrees of freedom. We can do similar observations for the NSR sector. Additionally, we might reasonably wonder whether there is a kind of degeneracy at each of the mass levels to be fixed. There is a way to deal with these problems, and that way of solving each obstacle is known as the GSO projection [163, 164]. It turns out that from the new viewpoint to symmetries and anomalies by the classification[5] of anomalies via invertible phases, it is possible to make a systematic study of the GSO projection at the level of the worldsheet, see [165, 166].

[5] See Appendix C

First of all, it projects out the negative mass states of the NS sector, implying that the first excited state becomes into the ground state of the NS sector, corresponding to 8 vector boson degrees of freedom. On the other hand, in the R sector the first massless state is a fermion with 32 complex components. However, it is possible to impose a reality condition over spinors, a Majorana condition, thus we are left with 32 real components. In addition, we have completely forgotten the supercurrent constraints. It turns out that one of those constraints, $F_0 \left|\text{phy}\right\rangle = 0$, corresponds to a generalized Dirac equation, in fact, it is known as the Dirac-Ramond equation. Nevertheless, for massless states, it is only the Dirac equation $\Gamma^\mu \partial_\mu \left|\text{phy}\right\rangle = 0$. Therefore, this allows us to reduce the fermions degrees of freedom to 16 real components. Finally, in even dimension, we can define a matrix $\Gamma_{11} = \Gamma_0 \Gamma_2 \cdots \Gamma_9$ which must be thought of as the 10-dimensional version of the Dirac matrix γ_5 in four dimensions. Consequently, a fermionic physical state satisfies a chirality condition $\Gamma_{11} \left|\text{phy}\right\rangle = \pm \left|\text{phy}\right\rangle$. This is such that we can define a projection operator $P_\pm = \frac{1}{2}(1 \pm \Gamma_{11})$ projecting out one chirality or the other reducing the fermions degrees of freedom to 8, this is known as the Weyl condition. This turns out to be consistent with the GSO projection, indeed, to construct the GSO projectors we need the chirality matrix. In conclusion, we have projected out the negative mass states and the counting of bosons and fermions matches. Even more surprisingly, this GSO projection still works for massive states where a Weyl condition is in tension with massive fermions in the sense that a massive Dirac operator does not commute with the chirality operator. However, as we mentioned, a systematic study of the GSO projection comes from the theory of superconductors in Condensed Matter Physics. We invite the interested reader to revise the literature. We will not say more on this, instead, we will try to extract the massless spectrum of the five supersymmetric strings, again, keeping in mind all the warnings we have tried to make explicit in our discussion. The spectrum of the five superstrings turns out to be spacetime supersymmetric but we will not try to prove that important feature of these theories. We will be content to proceed by only verifying the match between bosons and fermions.

The rough idea of the GSO projection operators goes as follows. One can define the operators

$$\mathrm{P}_{\mathrm{NS}} = (-1)^{1+\sum_{r>1/2} b^i_{-r} b^i_r} , \qquad \mathrm{P}_{\mathrm{R}} = \Gamma_{11}(-1)^{\sum_{n>1} d^i_{-n} d^i_n} . \tag{2.60}$$

One for the NS sector and the other for the R sector. The claim is that if a state in the NSNS sector has an odd number of b^i_{-r} and odd number of $\tilde{b}^i_{-r}$ survive the projection, otherwise we can safely throw it away. In other words, we keep states of positive P_{NS} parity. For example, $\mathrm{P}_{\mathrm{NS}} \left|0\right\rangle_{\mathrm{NS}} = -\left|0\right\rangle_{\mathrm{NS}}$, thus this state is projected out and corresponds to the annoying negative mass state. Whereas, the first excited state $\tilde{b}^i_{-1/2} b^j_{-1/2} \left|0\right\rangle_{\mathrm{NS}}$ survives the projection. On the other hand, in an R sector, involving the projection P_{R} we have two possibilities depending on the chirality of the ground state, from which excited states are built. Therefore, creation operators can build upon a P_{R}-positive ground states $\left|0^+\right\rangle$ or a P_{R}-negative ground states $\left|0^+\right\rangle$. For definiteness, we can choose left- and right-moving states P_{R}-positive such that the massless spectrum is given by

$$\left|\widetilde{0}^+\right\rangle_{\mathrm{R}} \otimes \left|0^+\right\rangle_{\mathrm{R}} , \quad \widetilde{b}^i_{-1/2} \left|\widetilde{0}\right\rangle_{\mathrm{NS}} \otimes \left|0^+\right\rangle_{\mathrm{R}} , \quad \left|\widetilde{0}^+\right\rangle_{\mathrm{R}} \otimes b^i_{-1/2} \left|0\right\rangle_{\mathrm{NS}} , \quad \widetilde{b}^i_{-1/2} \left|\widetilde{0}\right\rangle_{\mathrm{NS}} \otimes b^i_{-1/2} \left|0\right\rangle_{\mathrm{NS}} , \tag{2.61}$$

and this corresponds to one of the 10-dimensional superstring theories known as Type IIB. This massless spectrum is described by $\mathcal{N} = 2$ supergravity in the low-energy limit, while massive states are far enough from the massless level to be taken into account, and for anomaly purposes we do not care about massive fields. However, they are important for the UV completion of the theory if string theory can be thought of as a consistent theory of gravity, see. Roughly speaking, $\mathcal{N}=2$ corresponds to the number of supersymmetries which is linked to the presence of two gravitinos in its massless spectrum. From its construction, it has a spacetime chiral fermion spectrum. Each of these massless spectra furnishes a representation of the $SO(8)$ group. We will not go into the details of the representation theory of this group but let us mention a couple of points. The first state in (2.61) transforms in the tensor product representation $\widetilde{\mathbf{8}}_s \otimes \mathbf{8_s}$, each of one has a spinor index, thus are representations of the double cover of $SO(8)$, namely $\mathrm{Spin}(8)$, while objects with vector indices transform in representations of $SO(8)$. The second corresponds to $\widetilde{\mathbf{8}}_\mathrm{v} \otimes \mathbf{8_s}$, the third to $\widetilde{\mathbf{8}}_s \otimes \mathbf{8}_\mathrm{v}$ and finally $\widetilde{\mathbf{8}}_\mathrm{v} \otimes \mathbf{8}_\mathrm{v}$ which can be decomposed as follows

$$\begin{aligned} \widetilde{\mathbf{8}}_s \otimes \mathbf{8_s} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35}_+ \, . \\ \widetilde{\mathbf{8}}_\mathrm{v} \otimes \mathbf{8_s} &= \mathbf{8_s} \oplus \mathbf{56_s} \, , \\ \widetilde{\mathbf{8}}_s \otimes \mathbf{8}_\mathrm{v} &= \mathbf{8_s} \oplus \mathbf{56_s} \, , \\ \widetilde{\mathbf{8}}_\mathrm{v} \otimes \mathbf{8}_\mathrm{v} &= \mathbf{1} \oplus \mathbf{28} \oplus \mathbf{35} \, . \end{aligned} \tag{2.62}$$

This amounts to the following field content

$$\begin{aligned} \text{Bosons} &\to \{\phi, g_{MN}, B_{MN}, C, C_{MN}, C^+_{MNPQ}\} \, , \\ \text{Fermions} &\to \{\Psi^\mu_1, \Psi^\mu_2, \psi_1, \psi_2\} \end{aligned} \tag{2.63}$$

A dilaton, a graviton, the Neveu-Schwarz or Kalb-Ramond field which is a 2-form coming from the NSNS sector; a 0-form, a 2-form and self-dual 4-form coming from the RR sector; the RNS and NSR sector give us two positive chirality gravitinos, which are spin-$\frac{3}{2}$ left-handed spinor with a spin and a vector index, and two right-handed spin-$\frac{1}{2}$ spinor fields, dilatinos. Thus, Type IIB is a chiral theory, thus anomalies like in four-dimensional chiral fermions theories are potentially dangerous. However, we must mention that, in contrast to the four-dimensional case, for the Type IIB theory, our concern will be with parity-violating gravitational couplings since there is no gauge vector bosons.[6]

The other inequivalent possibility corresponds to choosing different chirality for the R ground states such that the massless spectrum is given by

$$|\widetilde{0}^-\rangle_\mathrm{R} \otimes |0^+\rangle_\mathrm{R} \, , \quad \widetilde{b}^i_{-1/2}\,|\widetilde{0}\rangle_\mathrm{NS} \otimes |0^+\rangle_\mathrm{R} \, , \quad |\widetilde{0}^-\rangle_\mathrm{R} \otimes b^i_{-1/2}\,|0\rangle_\mathrm{NS} \, , \quad \widetilde{b}^i_{-1/2}\,|\widetilde{0}\rangle_\mathrm{NS} \otimes b^i_{-1/2}\,|0\rangle_\mathrm{NS} \, , \tag{2.64}$$

We can do a similar analysis to get the massless spectrum corresponding to a nonchiral theory involving the representation $\mathbf{8_s}$ that describes one chirality, and the representation $\mathbf{8_c}$ describing the other chirality, as well as the vector representation $\mathbf{8}_\mathrm{v}$, leading to Type IIA superstring. In the low-energy limit, this theory is described by $\mathcal{N}=2$ supergravity. At least for this work, this theory will not play any role, we will not say more on this.

Up to this point, we have not said anything about open strings, in part because the classical and quantum analysis is not that different from the closed string. There are subtleties with boundary terms, therefore, there is an issue with possible boundary conditions, apart from that, the analysis does not change too much. The interesting case arises when we allow mixing between Neumann and Dirichlet boundary conditions due to the possibility of having open strings ending on extended objects, the well-known D-branes [7]. The important point is that we can attach degrees of freedom to the ends of the open string,

[6] This is not entirely true since Type IIB has a global S-duality symmetry given by $\mathsf{SL}_2(\mathbb{C})$ has to be gauge for reasons that we will not discuss here. But interested readers on this amazing subject should view [118].

or charges, known as Chan-Paton factors. For oriented strings, these charges transform under the fundamental representation of $U(N)$ on one end, and under the conjugate representation on the other. At the level of the massless open string spectrum, this adds two extra labels to the vector states $\alpha^i_{-1}\,|0, a, \bar{b}\rangle$ transforming under the fundamental and anti-fundamental and by representation theory arguments this can be associated to the massless gauge bosons[7] of $U(N)$ since $\mathbf{N} \otimes \overline{\mathbf{N}} = \mathbf{1} \oplus \mathbf{Adj}$. Keeping in mind this, we can go through the detailed quantization of the open string, and obtain the full massless spectrum as we did for the closed string. However, something interesting happens when we consider unoriented strings. This refers to strings whose quantum spectrum is invariant under a kind of worldsheet parity operation. One can realize this operation over the mode expansion of the fields, determine how the oscillation modes behave, and then translate that information to the quantum states. It turns out that the massless vector bosons we already mentioned, now transform under the adjoint representation of either $SO(N)$ or $Sp(N)$.

Therefore, we have obtained gauge bosons at the level of 10-dimensional strings, which are absent in the discussion of the closed string. However, note the following. At the level of the previous discussion, there seems to be no way to decide if the two groups are consistent quantum mechanically, namely, there are chiral massless fermions in the R sector of the open string which are the superpartners of the gauge bosons, as is also required by supersymmetry, which couple to each other via a covariant derivative, as in any gauge theory, and this gauge coupling could be parity-violating, thus it is fair to ask if this is well defined for both, either, or neither of the two gauge groups above. We will see more and check this later. By now, we continue observing that if an open string can end somewhere, it seems reasonable to have a situation where the open string ends join together to form a closed string. Actually, a consistent system of open superstrings requires closed superstrings, but this requires introducing a kind of parity operation on the closed string also, giving rise to a theory of closed unoriented superstring, such that it projects out one of the gravitinos. As we saw, Type IIB or IIA contain two gravitinos, while the open superstring massless spectrum contains only one of them, and this is the rationale of this orientation operation also on closed strings. On the other hand, consistency of the closed worldsheet parity operation with the chirality conventions we choose in the closed massless superstring spectrum, already discussed, demands to focus on unoriented closed Type IIB superstring.

In the end, this discussion boils down to the $\mathcal{N} = 1$ Type I superstring theory with G gauge bosons, that is, a massless spectrum given by a combination of a parity invariant Type IIB closed superstring massless spectrum and a parity invariant open superstring massless spectrum

$$\text{Bosons} \to \{\phi, g_{MN}, A_{MN}, A^a_M\}\,, \tag{2.65}$$

$$\text{Fermions} \to \{\Psi^M, \psi, \lambda^a\}\,, \tag{2.66}$$

consisting of a scalar field, the dilaton; a 10-dimensional metric, the graviton; an anti-symmetric potential, a 2-form; a gauge vector boson, a Lie-algebra valued 2-form; a spin $3/2$ spinor field, a left-handed gravitino; a spin $1/2$ fermion, a right-handed dilatino; and a spin $1/2$ spinor, a left-handed dilatino.

We stress again that at this point there is no way to decide the proper G-gauge group. In fact, the choice of the appropriate gauge group G was accompanied by what is now

[7] A more precise description is given through vector bundles over the worldvolume D-brane and by introducing an equivalence was realised that D-branes are classified by K-theory [167]. However, one often thinks of D-branes as the sources for RR p-form fields [7].

known as the first superstring revolution. It is a highly non-trivial result.

There are two more supersymmetric strings known as Heterotic closed superstrings [168, 169], but their construction is sophisticated machinery. Let us not explain it here, it would take too much space to go through any detail of that construction. The important point of those theories is about the massless content of states since these heterotic superstrings allow for gauge bosons. These two supersymmetric heterotic strings have a chiral spectrum and, in principle, one could have an arbitrary gauge group G. However, consistency with $\mathcal{N}=1$ supersymmetry, for example, requires that the gauge groups have to be $SO(32)$ or $E_8 \times E_8$. At the level of the worldsheet theory is not completely clear that this has to be the case, but spacetime supersymmetry, as well as chiral anomaly cancellation, is consistent only with those groups as we will see later. For future reference, we write down the massless content of the heterotic strings where the index a can, in principle, transform under any simple classical Lie gauge group

$$\begin{aligned} \text{Bosons} &\to \{\phi, g_{MN}, B_{MN}, A_M^a\}\,, \\ \text{Fermions} &\to \{\Psi^M, \psi, \lambda^a\}\,. \end{aligned} \tag{2.67}$$

The important point to note is the chiral massless content of fermions. In particular, the super-Yang-Mills multiplet consists of gauginos of one chirality, not both. Therefore, these theories are expected to suffer from quantum anomalies due to parity-violating gauge and gravitational couplings. This concern also applies to Type I superstring.

2.3 Eleven-dimensional supergravity

There is one theory in eleven dimensions known as M-theory whose low-energy limit has a very simple massless content consisting of a metric, a 3-form potential, and a Majorana spin-$3/2$ spinor. The energy is described by 11-dimensional supergravity [131]. It not entirely clear what M-theory is. However, it is extremely relevant in the framework of string dualities [8]. It turns out that in testing the web of string dualities, eleven-dimensional supergravity is not enough to describe the low-energy limit of M-theory. Indeed, this perfectly matches with the fact that M-theory allows for M-brane solutions given by an M2-brane and its magnetic dual, an M5-brane. We will see how this requires refining the eleven-dimensional supergravity to guarantee the quantum consistency of the low-energy limit of M-theory from an anomaly point of view [132].

Throughout the discussion, it is noted that the low-energy description of the five superstrings corresponds to 10-dimensional theories, concretely, to 10-dimensional supergravity theories. This has given rise to an enormous amount of research in trying to make contact with the four-dimensional world around us and serious efforts are still ongoing to reach that goal. This is the subject of string phenomenology. The more recent subject of intense research also connected to phenomenology is the Swampland program [9]. By now there is also an increasing interest in understanding the nonsupersymmetric corner of string theory, see [136] and reference therein. Also see [125] for a seek of nonsupersymmetric solutions in string theory by using pure topological arguments via the cobordism conjecture [25]. In our case, we will use perturbative anomaly cancellation to explore this nonsupersymmetric corner, then we will briefly mention the $SO(16) \times SO(16)$ nonsupersymmetric string [66, 170] where perturbative anomaly cancellation played an important role in its discovery.

2.4 Non-supersymmetric strings

As we saw above the 10-dimensional superstrings are highly constrained by supersymmetry. However, it turns out that there are known 10-dimensional nonsupersymmetric strings that are tachyon free. They are listed below, but we will say a few words only of the heterotic, which is somehow connected to our later work.

- The Sugimoto model [171]
- The Sagnotti model [172, 173]
- The heterotic model [66, 170]

The heterotic model is obtained from the heterotic models already mentioned above, $SO(32)$ or $E_8 \times E_8$. For instance, one can use the E_8 model and define a kind of projection operation in terms of the left and the right fermion operator $(-1)^{F_L}$ and $(-1)^{F_R}$ and the lattice model defined in the construction of the $E_8 \times E_8$ theory. This projection operator projects out any source of purely gravitational anomalies, particularly the gravitino. Then we are left only with charged chiral fermions. As a result, this model is free of tachyon instabilities. What is even more surprising is the fact that the perturbative anomaly of this model is cancelled, rendering the theory anomaly free and tachyon free. We will check anomaly cancellation later on in the text.

In the next chapter, we will review what we mean by an anomaly in a field theory and we will present a new formulation of this concept in the case of gauge theories.

Chapter 3

Symmetries and Anomalies

3.1 Anomalies: A modern takeaway

An anomaly in a symmetry of a physical system, particularly in a QFT with a lagrangian formulation, can have several, but related meanings. Then, very generally, by an anomaly we mean, a classical symmetry is violated by quantum effects. However, this is a bit ambiguous. For example, there could be a global symmetry at the classical level which is eventually gauged by coupling a classical conserved current to a *background* vector or gauge field. As a consequence, we define a covariant derivative, and the theory is now invariant under a local symmetry instead of a symmetry independent of the spacetime coordinates. It is stressed that we consider the background vector field as a nondynamical field. Next, we could move to the quantum theory and ask for the consistency of this new setting. It turns out that, under certain situations, we can find obstructions in this process at the quantum level, namely, there could be certain Fyenmann diagrams that can give rise to anomalous effects. More precisely, if we want to regularize one-loop Feynman diagrams, it is impossible to do it in a completely gauge invariant way (see Introduction 1). This is what happens for chiral fermion theories, and more generally for any chiral theory as we will see later. The chiral anomaly discovered by Adler [49], and separately by Bell and Jackiw [50] is an example of this, which was an extremely important result to explain an *anomolous* behaviour in decaying process[1]. More generally, this is actually related to what is recently known as a 't Hooft anomaly [51] in the literature. It is worth pointing out that this does not represent a catastrophe, only means that the symmetry is not gaugeable.

You may think now, what if we decide to lift the background field to a dynamical one. Let us discuss this point using path integral quantization of a chiral fermion system coupled to a dynamical gauge field associated with some continuous symmetry group G, mainly because this work is about anomalies by looking at the behaviour of the partition function[2] after we introduced a gauge coupling. This essentially means that we could integrate out this field of the path integral. First, we integrate out fermions, leading to what is usually

[1] A couple of words about the statement that the word anomaly is overexploited. Up to the knowledge of the author, Particle physicists used the word anomaly when they find or look at particle decaying processes and find mismatches between theory and experiment which is not alien because that was happening when [49,50] discovered the chiral anomaly, eventually accounting for π^0 decay.

[2] Maybe, a more precise way to say this is by talking about the quantum effective action. Look at equation (3.1)

called the fermion path integral

$$Z[A] = \int DA\, \mathrm{Det}(\mathrm{i}\,\mathcal{D}) \exp(-S[A]) = \int DA\, \exp(-S_{\text{eff}}[A]) \tag{3.1}$$

where we have assumed a well-defined eigenvalue problem for the Dirac operator $\mathrm{i}\,\mathcal{D}$ as usual in Euclidean signature. Clearly, if this is plagued with anomalous Fyenmann diagrams, it does not make any sense to try to compute (3.1) by integrating out the gauge field A. In four dimensions, the anomalous diagrams correspond to one-loop triangle diagrams, where one of the vertices may have a parity-violating gauge coupling and two-vector coupling, for example. On the other hand, the quantization of a dynamical gauge theory requires gauge invariance at any level, the fact that one can introduce an equivalence relation or a redundancy in more physical terms, in the space of gauge fields is essential to define the quantum theory. What is more, this redundancy has consequences for the Hilbert space of the theory, so breaking down gauge invariance is absolutely dangerous in contrast to the case when A is only a probe background field. We saw something of this in Chapter 2. For example, unitarity is lost among other problems. Our discussion here has been about the well-known *gauge principle*. This analysis is usually done in flat spacetime in any advanced course of QFT with A a gauge field lifting the usual derivative to a covariant derivative, hence leading to the fundamental property of gauge covariance. However, we have to generalize this to any spacetime, consequently, global properties of the nontrivial spacetime require that A has to be seen as a connection on a principal bundle and this is the proper way we have to think about gauge theories. See Appendix B for an introduction to these concepts and the notion of bundles.

As a consequence, real problems come about when we move to gauge theories since there is no way to escape without assuring that our gauge theory is anomaly-free if we intend to use it to describe our world, as actually happens. Things become even harder if we consider gravity. Since this can also be formulated as a gauge theory, one may wonder whether the coupling of gravity to a chiral theory could lead to anomalies [52]. Actually, this notion of symmetries and anomalies goes further in the sense that to consistently define a general theory $\mathcal{T}$ on a d-manifold M one needs to be able to assure all the *data* [100] in order to guarantee that $\mathcal{T}$ is well defined, for example orientability of M. However, all the choices that are required could exhibit *topological* obstructions and one needs to deal with these issues to properly quantize the theory $\mathcal{T}$ which, in the end, is the final goal. Some of these data correspond to

- M must be (un)orientable and admit a (s)pin structure
- M must admit fiber bundles
- the absence of *both* perturbative and nonperturbative anomalies

To guarantee this is a highly nontrivial question. Fortunately, in the mathematical literature there exist ways to measure the obstructions to assure the needed data, mainly in terms of characteristic classes, see Appendix B.

The first point involves the tangent bundle structure of a manifold, more physically, the background metric or spin connection if $\mathcal{T}$ refers to fermions. We need to figure out how the topology of this tangent structure affects physics and that, basically, amounts to measuring its nontriviality along the base M. The second point is related to the first one, involving a continuous gauge symmetry group G and a G-connection on a fiber G-bundle P_G. It is remarkable that certain configurations of fields in the path-integral capture this information and, even more, it can be translated into well-defined mathematical objects as we will see. The third point has to do with the fact that the partition function $Z_{\mathcal{T}}[A]$[3] under a *general* transformation, coordinate transformations or gauge transformations, could happen that the fermion path integral

$$Z_{\mathcal{T}'}[A] \neq Z_{\mathcal{T}}[A]\,. \tag{3.2}$$

However, $\mathcal{T}'$ and $\mathcal{T}$ are equivalent theories by definition, hence physics must be independent of choosing one theory or the other. It has been understood a long time ago that the ambiguity in equation (3.2) manifests itself as a phase ambiguity and this is, roughly, related to the third point as follows [58] (see also [59, 60]). Let $\mathcal{Q}/\mathcal{G}$ be the space of gauge and/or spin connections $\mathcal{Q}$, modulo an identification by gauge or diffeomorphism transformations $\mathcal{G}$, connected or not with the identity. Generally, a transformation will be denoted as g. The space $\mathcal{Q}/\mathcal{G}$ can be naturally viewed as the base space of a line bundle, $\mathcal{L} \to \mathcal{Q}/\mathcal{G}$. Note that the fermion path integral is a section of this bundle, hence if the line bundle $\mathcal{L}$ is trivial, then $Z_{\mathcal{T}}[A]$ is globally well defined, namely $Z_{\mathcal{T}}[A] \in \mathbb{C}$. More generally, for nontrivial bundles, we could consider parallel transport along curves among nearby patches (connection) or along closed curves $A_s = (1-s)A + sA^g$, with $s \in [0,1]$ (holonomy) in the base $\mathcal{Q}/\mathcal{G}$. Therefore, it is natural to ask for the connection and holonomy on this line bundle $\mathcal{L}$. It turns out that perturbative anomalies corresponding to transformations g connected with the identity of $\mathcal{G}$ are related to the connection and nonperturbative anomalies[4] associated with transformations disconnected from the identity, to the holonomy. Consequently, the partition function is interpreted as a section of this line bundle instead of a complex function. As we will see, there are well-defined mathematical objects detecting both perturbative and nonperturbative anomalies.

This is the main topic of this thesis. We will review very quickly the formulation of a gauge theory anomaly as is done in any Advance QFT book. It is again pointed out that this will be done by focusing on the path integral formulation instead of a Feynmann diagram discussion. Eventually, we will spend some time reformulating it in a more modern fashion.

3.2 Old fashion

Let $\mathcal{L}(\psi)$ be a lagrangian density, ψ be the fields of a physical theory, for reviews on this section see [56, 145], also [174, 175]. Then, one can write down an action S and study the classical dynamics of that system. We mainly focus on the symmetries. To begin with, consider the following transformation in the fields $\delta\psi = \epsilon\, T(\psi)$ where ϵ is an infinitesimal constant parameter, particularly for a continuous symmetry. If this is a symmetry of $\mathcal{L}(\psi)$ then its variation $\delta\mathcal{L}(\psi)$ vanishes. In fact, it is allowed that the lagrangian varies up to a boundary term but this does not modify the upcoming discussion. Then, recalling what Noether theorem says, we can find a conserved current and a conserved charge by making the parameter of the transformation spacetime dependent, then varying the lagrangian we

[3] Always we write, A refers to, the choice of manifold, structure, background fields, and all the data we need to define $\mathcal{T}$

[4] Note that this way of looking at anomalies is perfectly useful to deal with anomalies on discrete symmetries where does not make any sense to talk about perturbative anomalies. A discrete symmetry includes for example the choice of orientation.

find[5]

$$\delta\mathcal{L} = \mathrm{d}\,\epsilon \wedge *J\,, \tag{3.3}$$

where $J = J(\psi)$ is a functional of the fields, a current, and $*$ is the Hodge star. This corresponds to a change in the action given by $\delta S[\psi] = \int \mathrm{d}\,\epsilon \wedge *J$. By considering an on-shell field configuration, $\delta S[\psi] = 0$ for an arbitrary variation $\delta\psi$, it follows the current conservation law $\mathrm{d}{*}J = 0$, which is basically Noether's theorem. There is also a conserved charge given by $Q[\Sigma] = \int_\Sigma *J$, for a codimention one manifold Σ. Since our focus is on fermions fields, we can see the four-dimensional free massless Dirac Lagrangian as an example

$$\mathcal{L}(\psi) = \overline{\psi}\,\mathrm{i}\gamma^\mu\partial_\mu\psi \tag{3.4}$$

where $\overline{\psi}$ and ψ can be considered as independent variables in Euclidean formulation. From that, one can determine that is classically invariant under phase rotations $\psi \to e^{i\alpha}\psi$ with conserved 1-form current $J = J_\mu \mathrm{d}x^\mu$ with $J_\mu = \overline{\psi}\gamma_\mu\psi$, such that $\mathrm{d} * J = 0$ by using the equations of motion. In determining J, we consider the parameter $\alpha = \alpha(x)$ as spacetime dependent.

Next, one could introduce a coupling between the current and a one-form $A = A_\mu \mathrm{d}x^\mu$, where A_μ is only a vector, to the Dirac Lagrangian as follows

$$q\,A \wedge *J\,, \tag{3.5}$$

where q can be considered as a coupling constant. By doing this one immediately observes that by considering the transformation $\psi \to e^{i\alpha(x)}\psi$ with a local phase rotation, the Dirac Lagrangian with the coupling (3.5), preserves local invariance provided that A_μ transforms nontrivially, $A_\mu \to A_\mu + \partial_\mu\alpha(x)$. This can more properly be stated with the reformulation of the usual derivative in terms of the so-called covariant derivative $D = \mathrm{d} - iA$. This is the gauge principle we mentioned in the introduction to this chapter for abelian theories, that is, the group of continuous transformations is abelian. This basically corresponds to the coupling of fermions to the electromagnetic field and to one of the greatest achievements of the last century. Obviously, when A is considered as a dynamical field, the story above is not complete without the kinetic term of A given by

$$\frac{1}{2q}\,F \wedge *F\,, \tag{3.6}$$

where $F = \mathrm{d}A$ is the field strength associated to the electromagnetic potential A. This discussion can be generalized to non-Abelian groups as well, and these are known as Yang-Mills theories. The main difference is that the field A is a Lie-algebra valued field as well as F and the potential A is subject to the gauge equivalence given by

$$A \to A^g = g^{-1}Ag + g^{-1}\mathrm{d}g\,. \tag{3.7}$$

We will not say more about these theories here, but we return to this later. However, we point out that this is the most accurate way we have to describe the fundamental forces of nature. This is the fundamental reason to spend a lot of time studying these theories.

On the other hand, continuing with Abelian theories, one might also consider parity transformations by using the γ_5 matrix. This is a symmetry of (3.4) whose conserved current is $J^\mu_{\mathrm{A}} = \bar{\psi}\gamma^\mu\gamma_5\psi$, where the subscript A is for axial, under a transformation

[5] For a brief introduction to differential forms and conventions see Appendix B

$U = e^{i\alpha\gamma_5}$. The conservation law is $\partial_\mu J^\mu_{\rm A} = 2im\bar{\psi}\gamma_5\psi$ which must be zero because parity is a symmetry only for massless fermions as in (3.4). We emphasize the symbol m for mass in the axial current conservation because the absence of a mass term for fermions is behind the issue of an anomaly as we will see. This is fine classically but, what if we move to the quantum regime?

As a next step, we need to extend the previous analysis to the quantum level and the strategy will be the Euclidean partition function[6]

$$Z[\mathsf{J}] = \int D\psi \exp\left(-S[\psi] + \int d^dx\, \mathsf{J}\, \psi\right) , \tag{3.8}$$

where $D\psi$ is the integration measure and J are sources for ψ. We now consider a transformation in the fields given by $\psi \to \psi' = \psi + \epsilon(x)T(\psi)$, where ϵ is an arbitrary parameter as before, that depends upon the spacetime coordinates. In the corresponding new variables, the partition function is

$$Z[\mathsf{J}] = \int D\psi' \exp\left(-S[\psi'] + \int d^dx\, \mathsf{J}\, \psi'\right) . \tag{3.9}$$

We have essentially made a change of variables thus, the partition function is unchanged under this, namely, we are considering a different field configuration but infinitesimally related to the original one. Hence, under the above transformation, the measure of integration may change as

$$D\psi' = D\psi \;\det\, M \approx [D\psi](1 + \delta\mathcal{A})\,, \tag{3.10}$$

where, we can think of $\delta\mathcal{A}$ as the infinitesimal version of the Jacobian associated with the transformation $\delta\psi$, then, by using $S[\psi'] - S[\psi] = \delta S[\psi]$, we obtain

$$\langle -\delta\, S[\psi] + \int \mathsf{J}\, T(\psi) + \mathcal{A}\rangle = 0\,, \tag{3.11}$$

where the angular brackets denote the functional average $\langle\mathcal{O}\rangle = \mathcal{N}\int D\psi\, \mathcal{O} \exp(-S'[\psi])$ for $\mathcal{O}$ any operator of the theory. Equation (3.11) corresponds to a set of identities known as Ward-Takahashi identities. It turns out that this can be transformed into a set of relations among Green's functions through the source J. However, the important observation for us is the fact that the measure is not necessarily invariant under a redefinition of the field variables. This is essentially the key observation to figure out if a classical symmetry can be broken by quantum effects. To see this, set the source to zero, and then from the previous discussion we know that $\delta S[\psi]$ is proportional to the classical current conservation equation. However, at the quantum level, we obtain

$$\langle \mathrm{d} * J\rangle = \mathcal{A}\,, \tag{3.12}$$

thus, the classical current conservation has been violated by purely quantum effects. More generally we may work out identities among correlation functions by using the source J as usual in QFT. However, it is reasonable to wonder whether local counterterms coming from sources, for example, could be introduced in order to cancel the path-integral measure contribution $\mathcal{A}$. It turns out that this is not possible and what we will consider as genuine

[6] Since we shall focus mainly on chiral fermions theories, from now we suppress the subscript $\mathcal{T}$ of the partition function stressed in the introduction.

anomalies are terms that cannot be cancelled by any local counterterm.[7] In other words, the anomaly is a purely topological contribution coming from the nontriviality of the geometrical structure mentioned before and, as we will see, there are formal mathematical expressions measuring this information in terms of Index of Dirac operators [176].

One specific example is the Abelian anomaly [49, 50]. Consider massless fermions coupled to a $U(1)$ background gauge field A_μ as we did before, then we have to analyze the partition function

$$Z[A] = \int D\psi D\bar{\psi}\, \exp\bigl(-S[\psi,\bar{\psi}]\bigr)\,, \tag{3.13}$$

$$S[\psi,\bar{\psi}] = \int d^4x \bar{\psi} i \not{D} \psi\,, \tag{3.14}$$

where $\not{D} = \gamma^\mu D_\mu$ with $D_\mu = \partial_\mu - \mathrm{i}A_\mu$ the Dirac operator. The classical action has the chiral $U(1)$ symmetry

$$\psi \to e^{-i\gamma_5\alpha}\psi\,, \qquad \bar{\psi} \to e^{-i\gamma_5\alpha}\bar{\psi}\,, \tag{3.15}$$

where the parameter α is independent of the spacetime coordinates. Now consider a spacetime-dependent parameter. Thus, if we repeat what we have just discussed and after a careful computation,[8] one finds that the quantum conservation of the chiral current breaks down by a term given by

$$\mathcal{A} = \frac{1}{4\pi^2}\, F \wedge F\,. \tag{3.16}$$

There are various ways to compute this, for example, the Fujikawa's method, or the diagrammatic approach computing $\langle 0|T J_5^\mu(x_1) J^\nu(x_2) J^\rho(x_3)|0\rangle$. Each of these methods has its own advantages showing different ways to understand what is going on. But, the conclusion is equivalent, a classical symmetry is not necessarily a symmetry in the quantum realm. In other words, it is impossible to regularize in a completely gauge invariant way contributions coming from certain loop diagrams. Even though, in 4d, quantum field theorists are used to infinities, there is no doubt that regularization of those infinities is a cumbersome procedure. Fortunately, there is an approach connecting with the mathematical theory of $\mathrm{indices}$ of Dirac operators. Roughly, one can make the connection as follows.

In Euclidean signature $\mathrm{i}\not{D}$ is a hermitian operator, then one can study an eigenvalue problem

$$\mathrm{i}\not{D}\,\varphi_k = \lambda_k\,\varphi\,, \qquad \langle\varphi_k|\varphi_l\rangle = \int d^4x \varphi^*(x)\varphi(x) = \delta_{kl} \tag{3.17}$$

where we have introduced a complete orthonormal basis $\{\varphi_k\}$ where λ_k are the eigenvalues of $\mathrm{i}\not{D}$. Now, employing the chirality operator, one can define two sets, the set of positive n_+ and negative n_- chirality eigenvalues, since $\{\gamma_5, \mathrm{i}\not{D}\} = 0$. In computing the index one observes that the nonzero eigenvalues come in pairs of opposite chirality and they cancel each other from a certain regularized sum. For the zero modes, however, this distinction

[7]More precisely, the anomalous phase contribution − the anomaly − is independent of the scheme of regularization we use to deal with the infinities associated to loops. However, we can manage to cancel it in some cases with mild modification of our theory such as the Green-Schwarz method, as we will see later. This is usually known as the anomaly cancellation conditions.

[8]This is a very long computation, besides we have to deal with infinities, as usual in QFT thus, we have to introduce some scheme of regularization but the anomaly is always present.

is not relevant. The index of the Dirac operator is defined as the difference between the number of positive and negative chirality zero modes[9]

$$n_+ - n_- = \mathrm{Index}(\mathrm{i}\not{D})\,, \tag{3.18}$$

and the anomaly in equation (3.16) its connected to the Index by

$$\mathcal{A} = -2\,\mathrm{Index}(\mathrm{i}\not{D})\,, \tag{3.19}$$

and this is clearly connected with the statement of failure of quantum current conservation. Additionally, the index of a Dirac operator is a very formal expression in Mathematics but, there is indeed a well-understood procedure to connect this with the failure of invariance of the effective action $Z[A]$ under (infinitesimal) gauge transformations as well. We will see more of this in the next subsection, as well as in the appendix B. But it is important to say that this result has given a deep relationship between various areas of Mathematics with deep consequences in Physics, perhaps, more notably in String Theory research.

There are various things to be mentioned from the previous discussion. Firstly, we have gauged a global symmetry by coupling it to a fixed vector boson, but in trying to preserve vector current conservation (and Bose symmetry), the axial current conservation is destroyed by quantum effects. Hence, the gauging procedure is not allowed. However, by looking at this phenomenon more carefully we can learn very important lessons as we have already mentioned before. Also, we can extend this analysis to non-Abelian gauge theories, where we find that the anomaly is now given by

$$\mathcal{A} = \frac{1}{4\pi^2}\,\mathrm{tr}_{\mathsf{r}}(F\wedge F)\,, \tag{3.20}$$

where tr_{r} is related to the trace of generators of the Lie algebra, and F is now a matrix-valued field strength taken values in the algebra, see Appendix A for more details.

A related observation here is about a gravitational anomaly. To couple fermions to gravity we need to introduce a *frame* that permits us to lift the general group of coordinates transformation to a subgroup containing spinor representations, see Appendix B for a brief discussion on this. Once we have achieved this, by doing similar computations to what we have done in the $U(1)$ chiral anomaly, we obtain

$$\mathcal{A} = \frac{1}{48}\,p_1(R)\,, \tag{3.21}$$

where $p_1(R)$ is the first Pontrjagin class given in terms of the Riemann curvature two-form R. From one-loop diagrams this is related to a triangle with the insertion an axial current and the insertion of the stress-energy tensor on the vertices leading to the fact that it is not possible to maintain current conservation at the same time we preserve the conservation of the stress-energy tensor.

A second point is related to the fact that we could extend the fixed gauge boson to a dynamical gauge field. Actually, we have to treat genuine gauge theories and consider possible gauge anomalies. Nature demands to do so. Also, an important observation of the previous discussion is that anomalies could only be expected in chiral theories, namely gauge theories coupled to Weyl fermions. Theories that allow a mass term like those constructed with Dirac fermions can always be regularized in a completely gauge invariant

[9] See Appendix B for the definition of $\mathrm{Indices}$ of Dirac operators and our conventions for these expressions.

way by using Pauli-Villars regularization, for instance, preserving all the symmetries. In fact, these theories are known as vector theories and are not part of our work.

A third point related to the previous paragraph that we need to discuss is the following [134]. Assuming that a fermion ψ is transforming in some complex representation $(\sigma(g))^a_b$ of a symmetry group G, we may define the complex conjugate representation such that $\widetilde{\psi}$ transforms as the complex conjugate ψ under all the symmetries. The action of $\widetilde{\psi}$ is the complex conjugate of ψ, from which follows that the partition function of $\psi \oplus \widetilde{\psi}$ corresponds to $Z_\psi \, Z_{\widetilde{\psi}} = |Z_\psi|^2$. In fact, this is related to the fact that we can introduce a mass term consistent with all the symmetries and Fermi statistic, $h^{ab}\varepsilon^{\alpha\beta}\psi_{\alpha a}\widetilde{\psi}_{\beta b}$, where $\varepsilon^{\alpha\beta}$ is an anti-symmetric bilinear form and h^{ab} is a symmetric bilinear form, thus from the previous paragraph follows that $|Z_\psi|^2$ is anomaly free. Correspondingly, we find that the anomaly corresponds to a phase since $|Z_\psi|$ is well defined. This also leads us to conclude that fermions in a real representation can at most suffer from an anomaly given by a sign ambiguity.

Let us see now what the source of the anomaly is in a gauge theory for concreteness, but the same kind of analysis applies to gravity, using the index description and the formal machinery of bundles which is the proper framework to study gauge theories is complete generality.

Let A be *connection* for a G-group in a principal G- bundle $P_G \to M$, where M is the d dimensional spacetime manifold where the physical system is defined, and P_G is a fiber bundle with G a Lie group. Let us consider the fermion partition function $Z[A]$ after integrating out the chiral fermions. Thus, the traditional statement of an anomaly is the lack of invariance under a gauge transformation given by a nonlocal phase factor involving a $(d+1)$-manifold X such that M is the boundary of X, so that

$$\begin{aligned} Z[A] \neq Z[A^g] = Z[A]\text{exp}\left(-2\pi i \int_X \mathcal{A}_{d+1}\right) \,, \\ A \to A^g = g^{-1}Ag + g^{-1}dg \,, \qquad g : M \to G \,, \end{aligned} \tag{3.22}$$

where the phase $\mathcal{A}_{d+1}$ is a $(d+1)$-functional of the connection A. However, we note here that g could be an element connected with the identity of G or an element disconnected from the identity component of G, and this distinction is crucial in understanding that there are two sources of anomaly. We focus first on infinitesimal transformations, which are transformations connected with the identity of G. So, a gauge transformation is a function $g : M \to G$ whose infinitesimal form implies that

$$\delta_\upsilon A = D\upsilon \ , \qquad \delta_\upsilon F = [\upsilon, F] \ , \tag{3.23}$$

where D is the covariant derivative as before, $\upsilon = \upsilon(x)$ is an infinitesimal parameter also taking values in the Lie algebra of G. This allows us to determine the variation of the partition function in terms of a certain $(d+1)$-form [82,177,178] that will be determined in a moment

$$Z[A + \delta_\upsilon A] \neq Z[A] \,, \qquad \delta_\upsilon A = d\upsilon + [A, \upsilon] \,, \tag{3.24}$$

$$Z[A + \delta_\upsilon A] = Z[A]\text{exp}\left(-2\pi i \int_X \mathcal{A}_{d+1}(\upsilon, A)\right) \tag{3.25}$$

These anomalies are called local or perturbative anomalies essentially because they can be seen by perturbative methods [53]. On the other hand, some transformations cannot be deformed to the identity and are not visible at the perturbative level [71]. Therefore, they are more subtle to be determined. This kind of anomaly is known as a global[10] or non-perturbative anomalies. The first example of this was the $SU(2)$ nonperturbative anomaly determined by Witten [71] by studying the fermion partition function of a chiral fermion coupled to $SU(2)$ gauge theory via a connection A_μ valued in the fundamental representation under gauge transformations[11] g classified by a homotopy group, $\pi_4(SU(2)) = \mathbb{Z}_2$. The Dirac action is written as usual

$$\begin{aligned} S[\psi, \overline{\psi}, A] &= \int d^4x\, \overline{\psi}\, \mathrm{i} \left(\partial_\mu (\gamma^\mu \otimes 1) + A^a_\mu (\gamma^\mu \otimes \sigma^a) \right) P_L \psi\,, \\ &= \int d^4x\, \overline{\psi}\, \mathrm{i} \not{D} P_L \psi\,, \end{aligned} \tag{3.26}$$

where $P_L = (1+\gamma^5)/2$ is a chiral projection operator over the space of left-handed Weyl fermions.[12] We have explicitly written that the ψ field furnishes a representation of the Clifford algebra as well as of the $SU(2)$ algebra. This is such that its tensor product defines a real representation, hence $\mathrm{i}\not{D}$ can be seen as a real, anti-symmetric matrix. To determine whether or not there is an anomaly in this theory is more subtle. The basic idea begins by noting that the chiral Dirac operator is a real, anti-symmetric operator, and then it can be put into a diagonal matrix of 2×2 blocks by orthogonal transformations. Each block is composed of conjugate off-diagonal eigenvalues. Then, by studying the spectral flow of these eigenvalues under the g transformation was determined that $Z[A^g] = -Z[A]$, so the effective action is ill defined due to a sign ambiguity as expected for real fermions. More generally, any gauge theory with an odd number of $SU(2)$ doublets has an anomaly determined by the so-called $\mathrm{Mod}\, 2\, \mathrm{Index}$. This might be thought of as an index of the $SU(2)$ Dirac operator where the relevant part to be computed for the anomaly is the $\mathrm{Index}(\mathrm{i}\not{D})$ mod 2. This result suggested that one can study nonperturbative anomalies of any G-gauge theory by looking at the nontrivial homotopy groups of G. However, this seems to miss something, namely, what happens if we couple our gauge theory to gravity. Indeed, by studying a more subtle twisted structure defined by the tangent and fiber bundle structure, Ref. [179] found a new $SU(2)$ nonperturbative anomaly where the spin structure is not required to consistently define fermions on an orientable manifold M. Generally, one could ask whether there is any mathematical object like $\pi_d(G)$ that captures information of the principal bundle as well as the tangent bundle. In addition to these observations, the anomaly in the transformation $g \in [g]$ assumed that the spacetime is topologically a 4-sphere but this is not always true, we would like to consider more general spaces (see [111] for an illuminating discussion on this). It turns out that these observations give rise to a new formulation of the story of anomalies.

In order to elaborate a bit more on the gravity side already mentioned in the previous paragraph. One can consider diffeomorphism transformations of the fermion partition function

$$\delta_{\text{diff}} Z[M]\,, \tag{3.27}$$

[10]Recall that to avoid confusion with the discussion of the previous subsection about global symmetries we use the term nonpertubative instead of global.

[11]The element g has to be thought of as a representative of homotopy class $[g] \in \pi_4(SU(2))$. Roughly speaking, the equivalence relation between two representative elements $g, g' \in [g]$ is defined by continuous deformations of g into g' under a map $H : I \times I \to G$ such that $H(s,0) = g(s)$ and $H(s,1) = g'(s)$, where $s \in I = [0,1]$. Thus, we say that $g \sim g'$. whose equivalence class is denoted by $[g]$.

[12]Recall that the breakdown of chirality is the signal of an anomaly, otherwise there is no anomaly.

where we have written M instead of A to emphasize that we are going to consider diffeomorphisms of M. However, there is an equivalent formulation of this which leads to determining gravitational anomalies in a closely related way as we did for G-gauge theories above [52, 83]. This is achieved by introducing the orthonormal frame bundle as it is explained in Appendix B. The upshot of this is that instead of studying diffeomorphism transformations, we consider local Lorentz gauge transformations L and fermions couple to gravity via a connection known as the spin connection [64, 65]. This implies extending the covariant derivative to include this new piece so that the Dirac operator becomes schematically into $\not{D} = \gamma^\mu D_\mu = e^\mu{}_\alpha \gamma^\alpha (\partial_\mu + \mathrm{i} A^a_\mu T^a + \mathrm{i}\omega_{\mu,\alpha\beta}\sigma^{\alpha\beta})$ where $\omega_{\mu,\alpha\beta}$ is the 1-form spin connection which is anti-symmetric in the index α and β, and $\sigma^{\alpha\beta} = [\gamma^\alpha, \gamma^\beta]/8$ are the generators of Lorentz symmetry. The infinitesimal version of local Lorentz transformations under $L^\alpha{}_\beta = \delta^\alpha{}_\beta + \vartheta^\alpha{}_\beta$, with $\vartheta^\alpha{}_\beta = \vartheta^\alpha{}_\beta(x)$ an infinitesimal spacetime dependent parameter such that $\vartheta_{\alpha\beta}$ anti-symmetric, corresponds to

$$\delta_\vartheta \omega = D\vartheta \,, \qquad \delta_\vartheta R = [\vartheta, R] \,. \tag{3.28}$$

Note the similarity with (3.23) when we look at gravity in the orthonormal tangent frame formulation. This is essentially a gauge theory, correspondingly, preservation of local frame rotations $\delta_\vartheta e^\alpha = \vartheta^\alpha{}_\beta e^\beta$, under the action of the group of local Lorentz transformation is fundamental to incorporate gravitational effects in a physical system. Besides, as we said, the analysis of anomalies can be done in a similar fashion as we did only for G-gauge theories.

On the other hand, for theories in odd dimensions, although there are not perturbative anomalies, there is no reason to expect the absence of nonperturbative anomalies. One example of this was first found in [180, 181] by Redlich and it is known as a "global" parity anomaly for a doublet fermion (in the fundamental) of $SU(2)$. Other important systems are those with discrete symmetries [95, 106]. This is relevant for the Condensed Matter side [146], for example, in the study of 3d superconductors and topological insulators with time-reversal symmetry. This has been also useful in string theory [165, 166]. A more sophisticated example is M-theory in 11d where this theory can be formulated in a manifold with a less known tangent structure known as Pin^+ structure, due to a parity symmetry of this theory [107, 182] (see also [183, 184] for previous work on a spin manifold). With the methods briefly discussed above it seems no obvious at all how to treat these examples. Although we will not consider these more exotic cases in this work, we mention it as a justification for a more systematic description of anomalies. We present a brief discussion on this in the next subsection.

3.3 New fashion

As was described above the anomaly of a chiral fermion theory is the noninvariance of the chiral partition function under gauge/diffeomorphism transformations. However, instead of seeing this as a failure in preserving a gauge/diffeomorphism transformation, the new viewpoint on anomalies consists in formulating our theory in a completely gauge invariant fashion. Whereas, the anomalous phase[13] is determined by a very special theory living in a $(d+1)$-manifold X with boundary M, $\partial X = M$. This special theory is known as an Invertible Field Theory (IFT) meaning that its Hilbert space is one-dimensional, therefore, this is essentially a phase, the anomalous phase [185]. For chiral fermions, the idea is

[13] See Appendix C for a brief discussion on this matter.

developed starting with massive fermions on X assuming that all the field theory data needed to define the boundary theory that one is interested in extends to X. For, instance if one wants a chiral fermion theory in the boundary M coupled to gravity is needed that the spin structure of M, also extends to X [60] (see also [59]). The same is assumed if the fermions are coupled to a connection of a principal G-bundle. This corresponds to the data needed to properly define a field theory mentioned before in this chapter. Then, we consider the familiar Dirac Lagrangian (note that this is also gauge invariant) [60]

$$\mathcal{L} = \bar{\Psi}(\mathrm{i}\,\mathcal{D}_X - m)\Psi\,, \tag{3.29}$$

where $\mathcal{D}_X = \mathrm{i}\not{D}$ is the Dirac operator on the bulk manifold X coupled to a principal bundle as well as gravity. Since X is a manifold with boundary, then we have to impose certain boundary conditions. By imposing certain local boundary conditions it is possible to have edge mode localized on the boundary M.[14] The upshot of this analysis is that by solving the equation of motion coming from (3.29) along with appropriate local boundary conditions, in the large mass limit $m \to \infty$, we can obtain a boundary mode obeying the following

$$(\gamma_{d+1} - 1)\,\Psi|_{\partial X} = (\gamma_{d+1} - 1)\,\psi = 0\,, \qquad \mathcal{D}_M\,\psi = 0\,, \tag{3.30}$$

where γ_{d+1} must be thought of as a chirality operator constructed from the boundary gamma matrices and the gamma matrix along the perpendicular direction to the boundary which obeys the properties of a usual chirality operator, and $\mathcal{D}_M\,\psi = 0$ is the Dirac equation of a chiral massless fermion mode on the boundary. Particularly, we have an anti-commutation relation between the boundary Dirac operator and γ_{d+1}

$$\gamma_{d+1}\,\mathcal{D}_M + \mathcal{D}_M\,\gamma_{d+1} = 0\,, \tag{3.31}$$

which is one of the basic properties satisfied by a chirality operator. The important point to note is that the bulk theory on a closed manifold is clearly free of anomalies. However, on a manifold with boundary, the situation has to be treated carefully. We will focus on the partition function of this setup. It turns out that the Euclidean path integral is convenient to prepare states. Thus, we can define a state $|X\rangle$ by the bulk partition function belonging to the boundary Hilbert space $\mathcal{H}(\partial X)$. We can think of the boundary condition imposed before also defining a state on the boundary Hilbert space denoted by $|\mathsf{L}\rangle$. Hence, we would like to interpret the partition function of the boundary mode as given by

$$Z_\psi \sim \langle \mathsf{L}|X\rangle = Z(X, \mathsf{L})\,. \tag{3.32}$$

Note that the right-hand side of this expression is completely gauge invariant. Hence, this would be a completely gauge invariant definition of the chiral partition function. However, there are problems with this proposal. On the one hand, we assumed that all the structure needed to define the theory $\mathcal{T}_\Psi$ extends to the bulk X, and as we will emphasize later this is a big assumption. On the other hand, the right-hand side has the problem that it depends on what bulk we choose. Certainly, we could have chosen another bulk X' and, in principle, we could do what we did using X. Thus, it is compulsory to measure that ambiguity. One of the deep observations in describing this bulk-boundary setup is the fact that the Hilbert space of certain bulk theories is almost trivially gapped in the sense that is one dimensional in the large mass gap limit, in fact, in this limit the ground state can be completely isolated

[14] The subject of Dirac operators on a manifold with boundary is very subtle and we will not say anything about it, see e.g. [186].

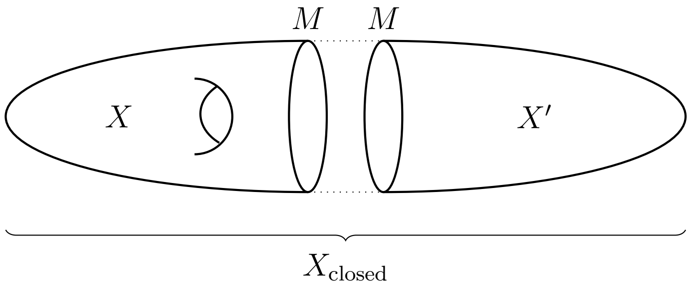


Figure 3.1: A closed manifold under the gluing along a common boundary

and defined a projection operator in terms only of ground states $P_{|\Omega\rangle} = |\Omega\rangle\langle\Omega|$. One example on this kind of theory is (3.29). Therefore, with this observation and considering the setup of Figure 3.1 we can compute the ratio

$$\frac{Z(X,\mathsf{L})}{Z(X',\mathsf{L})} = Z(X_{\text{closed}})\,, \tag{3.33}$$

where we have used (3.32), and $P_{|\Omega\rangle}$. So the ambiguity in the choice of bulk is detected by the partition function of the bulk theory, the IFT, evaluated on a closed $(d+1)$-manifold X_{closed} obtained by gluing X and X' along the common boundary M. We interpret this as given the anomaly of the chiral theory on the boundary M, so the anomalous phase is a partition function evaluated on a closed manifold X_{closed}. This is an important and highly non-trivial claim that captures the nontrivial information of the geometrical structure as we will see. For chiral fermions, it was proven that the anomaly theory determining the right-hand side of (3.33) is given by the Atiyah-Patodi-Singer (APS) η-invariant [59,60]

$$Z[X_{\text{closed}}] = \exp(2\pi\mathrm{i}\,\eta(X_{\text{closed}}))\,, \tag{3.34}$$

by computing the regularized path integral of the bulk field Ψ with a Pauli-Villars regulator of mass m_0. Formally, this is given by

$$\frac{\det(\mathrm{i}\mathcal{D}-m)}{\det(\mathrm{i}\mathcal{D}+m_0)}\,, \tag{3.35}$$

then, by taking $m = m_0$ and $m_0 \to \infty$ was found the anomaly theory of chiral fermions in terms of the η-invariant. This a rather formal object defined by [101–103]

$$\eta = \frac{1}{2}\left(\sum_i \text{sign}(\lambda_i) + \text{dimKer}\,\mathcal{D}_X\right)_{\text{reg}}\,, \tag{3.36}$$

where the λ_i's correspond to the eigenvalues of the appropriate Dirac operator on the bulk manifold X, and reg means we have to introduce some regularization scheme. What is more, this general description is also valid to any dimension. Therefore, the well-defined partition function of a chiral theory is said to be given by

$$Z[M] = Z_{\text{boundary}}[M]Z_{\text{bulk}}[X]\,, \tag{3.37}$$

where $Z_{\text{boundary}}[M]$ is given by the determinant of the chiral boundary Dirac operator (of course, to get a finite number we have to regularize this quantity), while $Z_{\text{bulk}}[X]$ gives us the anomaly by evaluating the anomaly theory on a closed $(d+1)$-manifold, for fermions this corresponds to the η-invariant. A similar analysis can be applied to chiral p-form fields [187].

In summary, the quantum anomaly of a d-dimensional theory $\mathcal{T}$ is given by an invertible field theory or, in the Condensed Matter language, by a symmetry protected topological phase h defined on $(d+1)$ dimensions whose partition function evaluated on a closed $(d+1)$-manifold reproduces the anomalous phase as (see the short review of [188])

$$Z[X] = \exp(2\pi h(X)) \tag{3.38}$$

whereas, as X has boundary $\mathcal{T}$ appears as a boundary of the invertible topological phase. Notice that it can be at most a phase when evaluated on a closed manifold X, since it is natural to think the Hilbert space $\mathcal{H}(\partial X) = \mathcal{H}(\varnothing)$ with a canonical isomorphism $\mathcal{H}(\varnothing) \simeq \mathbb{C}$ suggesting also that $Z[X]$ can just be regarded as a complex number. See Appendix C for more discussion on this matter.

For Weyl fermions the invertible phase is given by the APS η-invariant, and the anomaly is given by evaluating this η-invariant on closed $(d+1)$-manifolds [105, 118, 134] see also [189]. Moreover, this gives a unified description of perturbative and nonperturbative anomalies where the last is exclusively associated with the evaluation of the η-invariant on closed manifolds. However, this can be done only for a handful of spaces, basically for spaces known as Lens spaces in the Mathematical literature, see [189]. For this reason, there are tricky methods to determine global anomalies with no explicit evaluation of the η-invariant, see e.g. [87, 105, 179, 190].

To see that the description of anomalies in terms of the η-invariant also contains perturbative anomalies, notice the fact that if X_{closed} is itself a boundary we could have a $(d+2)$-manifold Y such that $X = \partial Y$. It turns out that there is a generalized index theorem, the APS index theorem [101–103] that allows us to connect this abstract discussion to the perturbative anomalies. The APS index theorem states that for a manifold Y with boundary X, the $\mathrm{Index}(\mathcal{D}_Y)$ of a Dirac operator on Y is related to an index density[15] $\mathcal{I}_{d+2}$ and the η-invariant of the Dirac operator on X as follows

$$\mathrm{Index}(\mathcal{D}_Y) = \int_Y \mathcal{I}_{d+2} + \eta(\mathcal{D}_X)\,. \tag{3.39}$$

Note that one operator is defined on $d+1$ dimensions and the other in $d+2$ which means that the field theory data on M must be extended to X as well as to Y. To find out whether this is possible or not corresponds to an important question of this new reformulation of anomalies. Yet, we leave this observation for a brief discussion on the Appendix C. For the purpose of perturbative anomalies, from (3.38) with $h = \eta$, and (3.39) follow that

$$Z[Y] = \exp\left(-2\pi \mathrm{i} \int_Y \mathcal{I}_{d+2}\right), \tag{3.40}$$

where we have used the fact that the Index is an integer. It is worth mentioning that $\mathcal{T}$ may consist of different fermions species, hence the index density must include all the contributions. We will discuss more on perturbative anomalies in the next section. With this, we finish the discussion of the modern point of view on quantum anomalies. In the next subsection, we will connect the formal expression $\mathcal{I}_{d+2}$ to the anomalous phase $\mathcal{A}_{d+1}$ of the perturbative anomalies.

[15] This density is given in terms of characteristic classes measuring the nontriviality of the geometrical structure involved in the definition of $\mathcal{T}$, see Appendix B for the specific definition.

3.4 Anomaly inflow

This is a huge topic with many far-reaching consequences in different areas of physics [133]. However, we will restrict to the idea that if we put a (topological) theory on a manifold with boundary, under a gauge/local Lorentz transformation, it could happen that this induced an anomalous boundary term [175]. The observation of anomaly inflow is about the possibility of having anomalous boundary degrees of freedom such that its anomaly cancels precisely the anomaly of the bulk theory, this will be the subject of Chapter 5. One of the most striking results of this idea is the Hořava-Witten E_8 M-brane reviewed by the end of Chapter 4. We also mention here that this idea has also been very powerful in M-theory/String Theory in the presence of D-branes deducing whether or not these objects support anomalous degrees of freedom [191–195].

A very important example of Anomaly Inflow is the Wess-Zumino descent procedure [81–83], which allows us to obtain the true perturbative anomaly of a theory under gauge and/or diffeomorphism transformations connected to the identity from the rather formal mathematical object $\mathcal{I}_{d+2}$ already introduced in the definition of the index of a Dirac operator, that is, the index density that captures information of the tangential and fiber structure over a manifold. However, our theory $\mathcal{T}$ may consist of various fermions species then, the perturbative anomaly of $\mathcal{T}$ is given by what we will call the anomaly polynomial $\mathcal{P}_{d+2}$. This is determined by the index densities of the $(d+2)$-dimensional Dirac operators involved in $\mathcal{T}$. Generically, the index density of a spin-$1/2$ Weyl fermion charged under some representation $\sigma(g) = \mathsf{r}$ of gauge group G (we need to be more specific depending on the Dirac operator we are considering) is given by [76]

$$\mathrm{Index}(\mathcal{D}) = \int_Y \mathcal{I}_{d+2}\,, \qquad \mathcal{I}_{d+2}(R,F) = \left[\hat{A}(R)\mathrm{ch}_{\mathsf{r}}(F)\right]\Big|_{d+2} \tag{3.41}$$

where the $\hat{\mathrm{A}}(R)$ is the A-roof or Dirac genus given in terms of traces of powers of the Riemann curvature two-form R, i.e. $\mathrm{tr}\,R^n$, and $\mathrm{ch}_{\mathsf{r}}(F) = \mathrm{tr}_{\mathsf{r}} \exp\left(\frac{\mathsf{i}}{2\pi}F\right)$ is the Chern character in terms of the field strength associated to a connection A with tr_{r} a trace evaluated in some representation r of the Lie algebra of G.[16] The subscript $|_{d+2}$ means that only terms of order $d+2$ are to be considered in that formal expression of the anomaly polynomial. We have also made explicit in (3.41) the dependence in the curvature 2-form R, related to the tangent bundle structure, and the field strength F, related to the principal bundle structure. Notice that the anomaly polynomial is a closed form, and therefore locally exact by Poincaré lemma. From this, the descent procedure tells us that

$$\mathcal{I}_{d+2} = \mathrm{d}I^{(0)}_{d+1}\,, \qquad \delta_{\upsilon,\vartheta} I^{(0)}_{d+1} = \mathcal{A}^{(1)}_{d+1}\,, \tag{3.42}$$

where $I^{(0)}_{d+1}$ is a polynomial in terms of the gauge and spin connection and its curvatures, from which one can make the identification $I^{(0)}_{d+1} = \mathcal{A}_{d+1}$. It turns out that the variation of $I^{(0)}_{d+1}$ with respect to infinitesimal local gauge and Lorentz transformation then $\delta_{\upsilon,\vartheta} I^{(0)}_{d+1} = \mathcal{A}^{(1)}_{d+1} = \mathcal{A}_{d+1}(\upsilon,\vartheta,\cdots)$, where the superscript (0) meaning no dependence on the gauge parameters, while (1) means linear dependence on the gauge parameters, leading to

$$\delta_{\upsilon,\vartheta}\Gamma[M] = \delta_{\upsilon,\vartheta}\int_X \mathcal{A} = \int_X \delta_{\upsilon,\vartheta} I^{(0)}_{d+1} = \int_X \mathcal{A}_{d+1}(\upsilon,\vartheta,A,\omega) \tag{3.43}$$

[16] We always write tr when the trace is evaluated in the fundamental representation.

where we focus only on the phase of the anomalous partition function denoted here as Γ, the effective action, to avoid confusion with the chiral fermion partition function. It can be also shown that $\mathcal{A}^{(1)}_{d+1}$ is a closed differential form, thus it is also locally exact

$$\mathcal{A}^{(1)}_{d+1} = \mathrm{d}I^{(1)}_{d}\,, \tag{3.44}$$

so that the d-dimensional anomaly can be expressed as

$$\delta_{\upsilon,\vartheta}\Gamma[M] = \int_{\partial X} I^{(1)}_{d}\,, \tag{3.45}$$

$I^{(1)}_{d}$ is the anomaly we would expect to see in the d-dimensional theory and, in this way, we get what we introduced before as the anomalous phase in (3.22) from the rather formal index density This anomaly is known as the consistent anomaly meaning that $\delta_{\upsilon_1}(\delta_{\upsilon_1}\Gamma) - \delta_{\upsilon_2}(\delta_{\upsilon_2}\Gamma) = \delta_{\upsilon}\Gamma$ for $[\upsilon_1, \upsilon_2] = \upsilon$. This is known as the Wess-Zumino consistency condition [196]. However, notice that we could modify $\mathcal{A}$ by the shift $I^{(0)}_{d+1} \to I^{(0)}_{d+1} + \mathrm{d}\chi_d$, implying that $\mathcal{A}$ is not uniquely determined consequently, we could add local counterterm to the anomalous phase since $\Gamma[M] \to \Gamma[M] + \int_{\partial X}\chi_d$ where we have used Stokes' theorem. These local counterterms do not cancel the anomaly only allow us to redefine it [81].

One important observation of the previous discussion is the assumption that we can go from a d-dimensional manifold up to a $d+2$-dimensional manifold to compute the perturbative anomaly. One could ask whether or not this is always allowed. It turns out that this is the crucial observation to the new formulation of anomalies, namely the obstruction to lift all the field theory data from d to $d+2$ dimensions, and this is described by a mathematical theory known as bordism theory, see [100] for a review. We will briefly elaborate on that point in the Appendix C. Although this is not that important for this thesis, it is important for work in progress.

3.5 Green-Schwarz anomaly cancellation mechanism

In many of the relevant cases in M-theory/string theory, anomaly inflow is not enough to cancel anomalies of chiral degrees of freedom. The extra key ingredient needed to cancel anomalies, particularly perturbative anomalies, is the Green-Schwarz anomaly cancellation method [129, 130] (see [197] for a nonperturbative formulation of the Green-Schwarz method, also see [198, Appendix G] for a modern perspective on this). The basic idea of this approach tells us that if we have an anomaly polynomial $\mathcal{P}_{d+2}(R,F)$ determining the anomaly of a chiral theory defined on a $(d=10)$-manifold M as we discussed above, then if we manage to factorize $\mathcal{P}_{d+2}(R,F)$ in such way that

$$\mathcal{P}_{d+2}(R,F) = W_4(R,F)\,P_{d-2}(R,F)\,, \tag{3.46}$$

then, it is possible to find a mechanism to cancel-31 a perturbative anomaly measure by $\mathcal{P}_{d+2}(R,F)$. This is a very nontrivial result. As we can see from Appendix A, there are in general many obstructions to achieving that factorization of the anomaly polynomial due to non-trivial Casimir invariants relating higher order traces of the Lie algebra valued field strength F over arbitrary representations to traces over the fundamental representation.

Additionally, this procedure requires a topological coupling in the d-dimensional theory, roughly, given by

$$-\int_M B_2 \wedge P_{d-2}\,, \tag{3.47}$$

where B_2 is a 2-form, and $P_{d-2} = P_{d-2}(R, F)$. In Chapter 2 we saw that the massless spectrum of the 10d superstring theories contains fields of this type, in principle, we could introduce a coupling like (3.47). The question, instead, is whether this term makes any sense in the perturbative formulation of the superstring requiring it. Suppose for now that it is indeed an allowed term. As a consequence, that term (3.47) is such that it contributes an anomaly, surprisingly, an anomaly that cancels the anomaly of the chiral fermions (3.46). To see this, according to our previous discussion, there must be an anomaly theory $h(X)$ capturing the anomaly of the coupling (3.47) such that (3.47) appears as a boundary phase. Since B_2 is a 2-form potential, there is a field strength H_3 associated to it, $H_3 \sim \mathrm{d}B_2$, then we define the anomaly theory as follows

$$h_{H_3}(X) = \int_X H_3 \wedge P_{d-2}\,. \tag{3.48}$$

such that $M = \partial X$. If we focus only on perturbative anomalies, we can assume that X itself is the boundary of $(d+2)$-manifold Y. In turns out that looking carefully at the perturbation theory it was observed that the B_2-field is actually noninvariant under gauge and gravitational interactions, tree diagrams are exchanging B_2 fields such that they also contribute to the anomalous loops diagrams with chiral fermions running throughout the loop [129, 130] (see also [68, 70]). Therefore, consistency of the d-dimensional theory requires that the Bianchi identity of the H_3-field has to be modified accordingly

$$dH_3 = W_4\,, \tag{3.49}$$

where $H_3 = \mathrm{d}B_2 - \Theta_{\mathrm{CS}}(A) - \Theta_{\mathrm{CS}}(\omega)$ is given in terms of the B-field, as usual, but also in terms of the Chern-Simons 3-form constructed from the gauge connection and the spin connection, and $W_4 = W_4(R, F)$ is a 4-form class in terms of gauge and gravitational contributions. Even more remarkable is the fact that this procedure is consistent with supersymmetry in a limited set of higher dimensional theories, indeed it is also required by supersymmetry [141, 142]. Therefore, at the perturbative level, we can see the anomaly theory (3.48) as

$$h_{H_3}(Y) = \int_Y W_4 \wedge P_{d-2}\,, \tag{3.50}$$

and this becomes the d-dimensional theory free of perturbative anomalies because the total perturbative anomaly trivializes

$$\exp(2\pi \mathrm{i}\, h) = \exp(2\pi \mathrm{i}\, h_{\mathrm{fermions}}) \exp(2\pi \mathrm{i}\, h_{H_3}) = 1\,. \tag{3.51}$$

However, notice that integration of (3.49) on a closed cocycle, the class W_4 must vanish. We briefly mention in Appendix B that characteristic classes measure nontrivial information of bundles. In the mathematical literature, it is well-established that a necessary and sufficient condition for the existence of certain structures is measured by the computation of characteristic classes [199]. For instance, the orientation of a real vector bundle over some base manifold M, namely the fact that the transition functions take values in the orthonormal group $SO(d)$ obeying a cocycle condition all over the patches that cover the base, can be measured by the first Stiefel-Whitney class $w_1(M) \in H^1(M, \mathbb{Z}_2)$, where $H^i(M, \mathbb{A})$ refers to the i-th cohomology group of M with coefficients into $\mathbb{A}$ where this can be e.g $\mathbb{A} = \mathbb{R}, \mathbb{Z}, \mathbb{Z}/2\mathbb{Z}$, reals, integers and mod 2 integers, respectively. Therefore, trivialization of $w_1(M)$ is said to be a necessary and sufficient condition for the existence of orientation. On the other hand, trivialization of the second Stiefel-Whitney class $w_2(M)$

is associated with the existence of a spin structure. It turns out that trivialization of the W_4 class is associated with a twisted string structure, and only string structure when the principal bundle under consideration is a trivial bundle, see [136] and reference therein. This requires that the tangent structure of the base manifold M, where the theory $\mathcal{T}$ we are interested in lives, has to admit a string instead of spin structure. Moreover, this lifting is a consequence of the Green-Schwarz anomaly cancellation mechanism for perturbative anomalies. After this, we are left with the question of nonperturbative anomalies once the perturbative has been cancelled. See [136, 200, 201] for steps in this direction.

Nevertheless, we want to emphasize the following issue in this regard. As we already saw, theories requiring the Green-Schwarz anomaly cancellation mechanism also demands that the tangent structure must be lifted to a (twisted) string structure on the boundary. One of the ten-dimensional superstring theories requiring the Green-Schwarz method corresponds to the $E_8 \times E_8$. Additionally, this theory can be seen in the strong coupling limit as a boundary solution of a $d = 11$-dimensional theory [138, 139], i.e. (the low-energy limit) of M-theory, as we will review later. However, the tangent structure of the 11d manifold where M-theory is defined does not require to be a string structure. Thus, the bordism classification of anomalies as discussed in Appendix C does not seem to be enough to deal with this setting in order to study its nonperturbative anomalies. Perhaps, this will be seen more clearly in the next chapter. We also mention that the proper framework to study this setup can be found in[17] Ref. [198, Appendix G] where the correct bordism groups to be computed seem to be *relative* bordism groups. However, this issue is beyond the scope of this work, we hope to return to this soon.

In the next chapter, we study the cancellation of perturbative anomalies in the chiral supersymmetric string theories, Type IIB, Type I, the heterotic strings, M-theory and one of the $SO(16) \times SO(16)$ nonsupersymmetric string.

[17]The author thanks to A. Debray for correspondence on these matters.

Chapter 4

Anomaly cancellation in superstring and M-theory

Anomalies could be found in chiral theories, Weyl fermions[1] and (anti)self-dual scalar fields. In 10d-dimensional superstring theories we saw that Type IIA is a nonchiral theory. So, it is safe to leave this theory out of the anomaly analysis. However, Type IIB is chiral with a spectrum only coupled to gravity thus, we need to be sure that any anomaly coming from its chiral content of fields vanishes for each field or contributes an amount such that cancels each other, then rendering the theory anomaly free. The cases of Type I and the Heterotic strings are even more involved due to gauge degrees of freedom. Hence, there can be gauge and gravitational anomalies for these theories at the perturbative level. The aim here is to review the cancellation of anomalies in these supersymmetric theories. We will also review the cancellation of anomalies for the matter content of the heterotic nonsupersymmetric string $SO(16) \times SO(16)$ to see that the Green-Schwarz mechanism also works in this case. This is standard material but, we will present it in a convenient way for future reference

4.1 Supersymmetric strings

4.1.1 Type IIB

Type IIB has a chiral spectrum of two left-handed gravitinos, two right-handed dilatinos, and one self-dual 4-form. By using the index densities of a spin-$1/2$ chiral fermion (for now, assume that the field content is of positive chirality), a spin-$3/2$, and a self-dual 4-form, given by (see Appendix B)

$$\begin{aligned}
\mathcal{I}_{12}^{1/2}(R) &= \frac{1}{967680}(-31\,p_1^3 + 44\,p_1\,p_2 - 16\,p_3)\,, \\
\mathcal{I}_{12}^{3/2}(R) &= \frac{1}{967680}(225\,p_1^3 - 1620\,p_1\,p_2 + 7920\,p_3)\,, \\
\mathcal{I}_{12}^{\mathrm{sd}}(R) &= \frac{1}{967680}(-256\,p_1^3 + 1664\,p_1\,p_2 - 7936\,p_3)\,,
\end{aligned} \tag{4.1}$$

we can solve a linear system of equation such that $n_{\frac{1}{2}}\mathcal{I}_{12}^{1/2} + n_{\frac{3}{2}}\mathcal{I}_{12}^{3/2} + n_{\mathrm{sd}}\mathcal{I}_{12}^{\mathrm{sd}} = 0$, where each n_* account for the multiplicity of each $*$-specie. By solving this linear system, we

[1] Maybe subject to a reality condition as might happen in dimension $d = 2$ mod 8.

find that the matter content is actually that of Type IIB, namely, the anomaly polynomial

$$\mathcal{P}_{12}(R) = -2\,\frac{\mathcal{I}^{1/2}_{12}(R)}{2} + 2\,\frac{\mathcal{I}^{3/2}_{12}(R)}{2} + \mathcal{I}^{\rm sd}_{12}(R)\,, \tag{4.2}$$

is so that it indeed vanishes, where the minus sign in front of $\mathcal{I}^{1/2}_{12}$ account for the chirality convention we have chosen regarding the indices of the Dirac operators, the multiplication by two account for the two species of each fermion and due to the Majorana or reality condition obeyed for each of the fermion species, we have to divide by two, i.e.

$$n_{\frac{1}{2}} = -2\,, \qquad n_{\frac{3}{2}} = 2\,, \qquad n_{\rm sd} = 1\,, \tag{4.3}$$

giving us the spectrum of the Type IIB superstring. The quantum consistency of Type IIB was established in [52]. In turn, this result inspired confidence in string theory and its low-energy limits although, Type IIB was not favoured to achieve phenomenology. Therefore, also in Ref. [52] studied the quantum consistency of Type I with the extra ingredient that this theory has within its spectrum gauge degrees of freedom. However, these results were a source of concern in the community because, in this case, the anomaly polynomial did not directly vanish as it does in Type IIB. Yet, soon after, a method to cancel the anomaly polynomial of Type I was proposed, rendering it an anomaly-free theory. This is the Green-Schwarz method [129, 130], which was also eventually applied to heterotic strings, rendering all the supersymmetric low-energy limits of string theory anomaly-free. For our purposes, we will review this with some detail for the $E_8 \times E_8$ heterotic string. Finally, we make some comments on the $SO(32)$ theories and the nonsupersymmetric $SO(16) \times SO(16)$.

4.1.2 $E_8 \times E_8$ heterotic string

We will move to the heterotic string where chiral-violating gauge couplings are also present thus, becoming the anomaly analysis more involved. Particularly, we focus on the heterotic $E_8 \times E_8$ superstring since, the $SO(32)$ heterotic as well as the Type I superstring with $SO(32)$ gauge group have similar anomaly analysis. Yet, we comment on this by the end of this analysis.

We will review here perturbative anomalies of $E_8 \times E_8$ heterotic string theory closely following Refs. [138, 139]. This will be useful for subsequent work. The anomaly polynomial of the 10d $E_8 \times E_8$ heterotic string consists of a pure gravitational contribution $\mathcal{I}^{3/2+1/2}_{12}(R) = \mathcal{I}^{\rm Grav}_{12}(R)$ coming from the left-handed spin-$3/2$ gravitino and the right-handed spin-$1/2$ dilatino which is given explicitly by a twelve dimensional polynomial

$$\begin{aligned} \mathcal{I}^{\rm Grav}_{12}(R) &= \frac{1}{2}\hat{A}(R)(\mathrm{ch}\,(R) - 4)\bigg|_{12} \\ &= \frac{1}{967680}\left(128\,p_1^3 - 832\,p_1 p_2 + 3968\,p_3\right)\,. \end{aligned} \tag{4.4}$$

There is another contribution coming from charged matter under some representation of the symmetry group with gauge field in each factor of the product $E_8 \times E_8$ represented by $(\mathsf{248}, \mathsf{1}) \oplus (\mathsf{1}, \mathsf{248})$. This corresponds to left-handed spin-$1/2$ gauginos charged under the adjoint representation a of each E_8. The index density can be written as (keep in mind that the matter content obeys a reality condition)

$$\begin{aligned}\mathcal{I}_{12}^{E_8\times E_8}(R,F_1,F_2) &= \mathcal{I}_{12}(R,F_1)+\mathcal{I}_{12}(R,F_2)\\ &= \frac{1}{2}\hat{A}(R)(\mathrm{ch}_{\mathsf{a}}(F_1)+\mathrm{ch}_{\mathsf{a}}(F_2))\Big|_{12}\end{aligned} \tag{4.5}$$

where one uses the fact that $\mathrm{tr}_{\mathsf{r}}\,F^{2n}=\mathrm{tr}_{\mathsf{r}}\,F_1^{2n}+\mathrm{tr}_{\mathsf{r}}\,F_2^{2n}$ since we can think of $F=\left(\begin{smallmatrix}F_1 & 0\\ 0 & F_2\end{smallmatrix}\right)$ and the Chern character will be denoted as $\mathrm{ch}_{m,\mathsf{r}}(F_i)=\mathrm{ch}_{m,\mathsf{r}}^{(i)}$ with $i=1,2$ for each factor. With this and the information from Appendix B, we thus find that

$$\begin{aligned}\mathcal{I}_{12}^{E_8\times E_8}(R,F_i) =&\frac{1}{2}\mathrm{ch}_{6,\mathsf{a}}^{(i)}-\frac{1}{48}p_1\mathrm{ch}_{4,\mathsf{a}}^{(i)}+\frac{1}{11520}(7p_1-4p_2)\mathrm{ch}_{2,\mathsf{a}}^{(i)}\\ &+\frac{\mathrm{ch}_{0,\mathsf{a}}^{(i)}}{2}\frac{1}{967680}(31\,p_1-44\,p_1p_2+16\,p_3)\,.\end{aligned} \tag{4.6}$$

The total anomaly polynomial is given by the sum of the index densities (pure gravitational) (4.4) plus (gauge, gravity and gauge-gravity part) (4.6) as

$$\mathcal{P}_{12}^{E_8\times E_8}(R,F_1,F_2)=\mathcal{I}_{12}^{\mathrm{Grav}}(R)+\sum_{i=1}^{2}\mathcal{I}_{12}^{E_8\times E_8}(R,F_i)\,. \tag{4.7}$$

Note that we can define

$$\mathcal{P}_{12}^{E_8}(R,F)=\frac{1}{2}\mathcal{I}_{12}^{\mathrm{Grav}}(R)+\mathcal{I}_{12}^{E_8}(R,F)\,, \tag{4.8}$$

the anomaly polynomial of one E_8 factor, then the total anomaly polynomial becomes into

$$\mathcal{P}_{12}^{E_8\times E_8}(R,F_1,F_2)=\sum_{i=1}^{2}\mathcal{P}_{12}^{E_8}(R,F_i)\,. \tag{4.9}$$

Therefore, we will focus only on $\mathcal{P}_{12}^{E_8}(R,F)$ of the total anomaly polynomial in what follows. Eventually, we will join all the pieces to determine the whole anomaly polynomial in the factorized way we are looking for.

By using tools developed in Appendix A we can show that the Chern characters involved in $\mathcal{P}_{12}^{E_8}(R,F)$ with the matrix-valued field strength F in the adjoint representation a of E_8, namely the representation $\mathsf{a}=248$ allows a factorization which will eventually help to find an anomaly-cancellation mechanism; the Green-Schwarz method mentioned before [129, 130]. Besides, by noting that the group of rotations $SO(16)\subset E_8$, is a subgroup of E_8, it follows that $248\to 120+128$, i.e. the adjoint $\mathsf{a}=248$ decomposes into the adjoint and spinorial representation of $SO(16)$ under the embedding $SO(16)\subset E_8$. Thus, with this observation, we can work out all the Chern characters so that we can write them down in the fundamental representation 16 of $SO(16)$ as follows

$$\begin{aligned}\mathrm{ch}_{6,248} &= \frac{1}{24}\left(\mathrm{ch}_{2,16}\right)^3\,,\\ \mathrm{ch}_{4,248} &= \frac{3}{2}\left(\mathrm{ch}_{2,16}\right)^2\,,\\ \mathrm{ch}_{2,248} &= 30\,\mathrm{ch}_{2,16}\,,\\ \mathrm{ch}_{0,248} &= \dim\mathsf{a}\,.\end{aligned} \tag{4.10}$$

Noting that for real representations $\mathrm{ch}_{2,\mathsf{r}}(F) = -c_{2,\mathsf{r}}(F)$, the Chern character is minus the second Chern class then, with equations (4.4) and (4.6) along with (4.10), we find that[2] the anomaly polynomial $\mathcal{P}_{12}^{E_8}(R,F)$ is given by

$$\mathcal{P}_{12}^{E_8}(R,F) = -\frac{1}{96}(2\,c_{2,16}+p_1)\left(\frac{1}{4}(2\,c_{2,16}+p_1)^2 + \frac{1}{8}p_1^2 - \frac{1}{2}p_2\right). \tag{4.11}$$

It should be stressed that the appealing form of (4.11) is possible since E_8 gauge group has a real representation of dimension 248 that allows the cancellation of a third-order Pontrjagin class p_3 accompanied with a numerical coefficient proportional to $(\dim(\mathsf{a}) - 248)$ times p_3. Any attempt of factorization is going to fail in the presence of this term. Another obstruction comes from the sixth Chern character but, as we see from Appendix A a few amount of Lie algebras admit a factorization so that

$$\mathrm{ch}_6 = \mathrm{x}_\mathsf{r}\,\mathrm{ch}_2\,\mathrm{ch}_4 + \mathrm{y}_\mathsf{r}\,\mathrm{ch}_2^3\,. \tag{4.12}$$

It is clear from (4.10) that E_8 group allows such a factorization. All these facts ultimately lead to (4.11). Now, we can determine the anomaly polynomial of the $E_8 \times E_8$ heterotic theory combining (4.11) for each of the components of E_8, thus we get the total anomaly polynomial

$$\mathcal{P}_{12}^{E_8\times E_8}(R,F_1,F_2) = -\frac{1}{24}\frac{c_{2,\mathbf{16}}^{(1)} + c_{2,\mathbf{16}}^{(2)} + p_1}{2}\times \\ \left((c_{2,\mathbf{16}}^{(1)})^2 + (c_{2,\mathbf{16}}^{(2)})^2 - c_{2,\mathbf{16}}^{(1)}\,c_{2,\mathbf{16}}^{(2)} + \frac{1}{2}\,(c_{2,\mathbf{16}}^{(1)} + c_{2,\mathbf{16}}^{(2)})\,p_1 + \frac{3}{8}\,p_1^2 - \frac{1}{2}\,p_2\right), \tag{4.13}$$

where each of the superscript (i), $i = 1,2$ stands for each of the two E_8 groups as stated before. This has exactly the factorized form demanded by the Green-Schwarz method, $\mathcal{P}_{12}^{E_8\times E_8}(R,F_1,F_2) = W_4(R,F)\,P_8(R,F)$ reviewed before, and this implies a Bianchi identity for the field strength H_3 given by

$$\mathrm{d}H_3 = \frac{c_{2,\mathbf{16}}^{(1)} + c_{2,\mathbf{16}}^{(2)} + p_1}{2}\,, \tag{4.14}$$

and the coupling in the effective action schematically given by

$$\int B_2 \wedge P_8(R,F)\,, \tag{4.15}$$

where B_2 is the 2-form potential which is not gauge invariant, but its lack of invariance is what is needed to cancel perturbative anomalies in the $E_8 \times E_8$ heterotic model, and $P_8(R,F)$ is given by the bottom line in equation (4.13). It should be mentioned that this analysis is consistent with supersymmetry for fermions in the adjoint representation of a gauge group G [141,142], see also [68]. Finally, the cancellation of perturbative anomalies is achieved because

$$\exp(2\pi\mathrm{i}\,h_{\mathrm{Total}}) = \exp(2\pi\mathrm{i}\,h_{\mathrm{fermions}})\exp(2\pi\mathrm{i}\,h_{H_3}) = 1\,. \tag{4.16}$$

It is obviously left open the question of global anomaly cancellation in $E_8 \times E_8$ heterotic theory. It turns out that this was solved recently using Topological Modular Forms in [198, 200, 201], see also [136, 202] for string bordism computations and its anomaly.

[2]Behind this computation there is a group theory normalization factor so that an $SU(2)$ instanton is normalized to one, see the next table for our normalization

$SU(n)$	$SO(2n+1)$	$Sp(2n)$	$SO(2n)$	E_6	E_7	E_8	F_4	G_2
1	2	1	2	6	12	60	6	2

4.1.3 $SO(32)$ string

Here we briefly elaborate on perturbative anomaly cancellation of the $SO(32)$ Type I and heterotic string. As we mention in Chapter 2, it is possible to have other superstrings with groups $Sp(N)$ or $SO(N)$ though, we have to check perturbative anomaly cancellation in order to check its quantum consistency. Those groups have a sixth-order trace that at first sight does not have a factorization like (4.10). Nonetheless, something amazing happens due to the following identities [68]

$$\begin{aligned} \mathrm{tr}_{\mathsf{a}}\, F^6 &= (N \pm 32)\,\mathrm{tr}\, F^6 + 15\,(\mathrm{tr}\, F^4)(\mathrm{tr}\, F^2)\,, \\ \mathrm{tr}_{\mathsf{a}}\, F^4 &= (N \pm 8)\,\mathrm{tr}\, F^4 + 3\,(\mathrm{tr}\, F^2)^2\,, \\ \mathrm{tr}_{\mathsf{a}}\, F^2 &= (N \pm 2)\,\mathrm{tr}\, F^2\,, \end{aligned} \tag{4.17}$$

where the plus sign is for trace identities for $Sp(N)$ while the minus sign is for $SO(32)$, relating the traces in the adjoint representation to the fundamental of each group. We have restricted to the adjoint representation for consistency with supersymmetry. Notice that if we choose $N = 32$ then we can get rid of the sixth-order trace only for $SO(32)$, so a factorization may become available in that case. Hence, $Sp(32)$ seems to be ruled out by consistency with anomaly cancellation, namely in the presence of the six-order trace and the restriction of only the adjoint representation we cannot apply the Green-Schwarz method to cancel perturbative anomalies. Moreover, the adjoint representation of $SO(32)$ has $\dim \mathsf{a} = 496$ that is exactly what we also need to cancel the third Pontrjagin class[3]

$$\frac{\dim(\mathsf{a}) - 496}{120960}\, p_3\,. \tag{4.18}$$

We throw the two dangerous terms away due to important group theoretical properties of $SO(32)$ gauge group. This allows us to work out an anomaly polynomial for the $SO(32)$ gauge group with the kind of factorization we are looking for. Even more, this is as well consistent with supersymmetry. We write down the anomaly polynomial for completeness[4] [130]

$$\mathcal{P}_{12}^{SO(32)}(R,F) = -\frac{1}{24}\frac{c_{2,\mathbf{32}} + p_1}{2} \times \left(2\,c_{2,\mathbf{32}}^2 - 4\,c_{4,\mathbf{32}} + \frac{1}{2}\,c_{2,\mathbf{32}}\,p_1 + \frac{3}{8}\,p_1^2 - \frac{1}{2}\,p_2\right)\,. \tag{4.19}$$

The Green-Schwarz coupling follows like in the previous case of $E_8 \times E_8$. With this, we finish the anomaly analysis of the 10d superstring theories. We move to the anomaly analysis on M-theory because these two things will be the main subject of Chapter 5.

4.2 M-theory

The low-energy limit of M-theory was originally considered as eleven-dimensional supergravity describing the classical dynamics of a supergravity multiplet, a graviton, a 3-form potential in the bosonic sector plus the spin-3/2 left-handed gravitino in the fermionic sector [131]. We will not write down explicitly the full action because it is not relevant to

[3] This term also appears for the E_8 anomaly polynomial with a contribution proportional to $(\dim(\mathsf{a}) - 248)\, p_3$, which clearly cancel for E_8.

[4] To be precise, $SO(32)$ has to be identified with its double cover $\mathrm{Spin}(32)/\mathbb{Z}_2$ due to the presence of fermions.

this thesis but, the full action can be found e.g. in [149]. Only the topological terms are important and these are given in equation (4.20).

This 11d theory is related to the 10d superstrings as we mentioned in Chapter 2. For instance, by compactifying this effective action on a circle S^1 was found that this corresponds to Type IIA supergravity, whereas compactification on the orbifold $S^1/\mathbb{Z}_2$ led to $E_8 \times E_8$ supergravity in the zero-size radius limit of S^1. Nonetheless, examining 1-loop effects in Type IIA [203,204], or by considering the existence of M2-brane/M5-brane solutions of M-theory was revealed that the classical 11d supergravity description of M-theory required modifications [132], see also [205]. These modifications involved introducing an additional coupling to ensure the proper topological interaction terms for M-theory. This corresponds to the already known Chern-Simons term of the 11d supergravity and a new term similar to a Green-Schwarz-like interaction

$$S_{\text{CS}} = \frac{2\pi}{6} \int_X C \wedge G \wedge G \,, \qquad S_{\text{GS}} = -2\pi \int_Y C \wedge I_8(R) \,, \tag{4.20}$$

where X is an eleven-dimensional manifold, C is the 3-form potential with G its 4-form field strength, and $I_8(R)$ is a characteristic class given in terms of Pontrjagin classes

$$I_8 = \frac{1}{8} \left[p_2 - \frac{1}{4} p_1^2 \right] \,. \tag{4.21}$$

With this at hand, we are going to check that M-theory is consistent quantum mechanically [132]. To begin with, we should mention here that the most simple way how we think about field strengths is by obeying a kind of Dirac flux quantization like in free Maxwell theory (see [206] for a modern discussion of the duality symmetry of electromagnetism) with the 2-form of the electromagnetic field F as

$$F \to \frac{1}{2\pi} \int_{S^2} F \in \mathbb{Z} \,, \tag{4.22}$$

on a closed S^2 cycle. The proper way to see this is by considering the vector potential A as a connection in a $U(1)$ fiber bundle and the quantization is associated to its nontrivial topology. It is natural to generalize this for p-form potentials of any degree in p. Therefore, we may state that a $(p+1)$-form field strength obeys a flux quantization like $\int_\gamma G$ for a $(p+1)$-cycles of a manifold X. It turns out that this expectation usually does not hold in string/M-theory due to the nontrivial topology of the bundle structure involved, as we will see for G. We will think about this as an obstruction measure in one dimension higher as we just did for anomalies [187,207].

From our recent understanding of anomalies, we might represent the topological couplings of M-theory on a one-dimension higher manifold without care about the unphysical manifold Y since anomaly cancellation should guarantee that any result will be independent of Y. The anomaly theory will be denoted as $h_{C_3}(Y)$. The M-theory topological coupling lifted to Y is given schematically by

$$\int_Y G \wedge G \wedge G - \int_Y G \wedge I_8 \,, \tag{4.23}$$

therefore, it should be true that its anomaly theory has to be such that $\exp(2\pi\mathrm{i}\, h_{C_3}(Y)) = 1$ for anomaly-free eleven-dimensional topological terms. This is almost true up to a mod 2 ambiguity that is ultimately cancelled with a mod 2 ambiguity coming from the eleven-dimensional real gravitino. To see this [132], we will use what is known as E_8 theory

[183–185,208]. Let $P_{E_8} \to Y$ be an E_8-bundle on Y. This is topologically characterized[5] by an integer class $c_2(F)$ which is an element of $H^4(Y, \mathbb{Z})$ ($c_2(F)$ can be thought of as the pullback of a class c_2 in $H^4(BE_8, \mathbb{Z})$ via the classifying map, see Appendix B) where F is the curvature two-form of the E_8-bundle over Y. On the other hand, C can be considered as the source of an M2-brane, then it can be coupled to the M2-brane worldvolume. Such a coupling is not necessarily well defined since there may be anomalous degrees of freedom on the M2-brane worldvolume theory $\mathcal{W}$. Again, by anomaly cancellation, we must take the total partition function of the M2-brane coupled to C as

$$Z_{M2} = Z_{M2}(\mathcal{W})\, e^{2\pi \mathrm{i}\eta(\mathcal{D}_N)}\, e^{2\pi i \int_N G} \tag{4.24}$$

where N is a one-dimension higher manifold with boundary so that $\partial N = \mathcal{W}$, and $\mathcal{W}$ is the worldvolume theory where the anomalous chiral degrees of freedom live and whose anomaly is generally capture by an anomaly theory h, in this case given by the η-invariant of the chiral modes in terms of the Dirac operator lifted to N and coupled to gravity only.[6] From this, it follows that the quantization law of G is shifted by an anomaly in the worldvolume theory, then [132, 187]

$$\int_N G = \frac{1}{2}\int_N \lambda \mod \mathbb{Z}\,, \tag{4.25}$$

where λ is the canonical class of a spin manifold[7] given by $p_1/2$, one half the first Pontrjagin class, which is the pullback of the generator in $H^4(B\mathrm{Spin}, \mathbb{Z})$. This is also known as the canonical class of a spin manifold. In other words, this tells us that the η-invariant captures a pure gravitational anomaly of the chiral modes on the M2-brane worldvolume theory.

The key observation comes from the fact that if we choose the characteristic class $c_2(F)$ (for E_8 this is an integer) so that

$$G = \frac{1}{2}\lambda + c_{2,\mathsf{r}}(F)\,, \tag{4.26}$$

we find that for E_8 gauge theory with $\mathsf{r} = 248$, the adjoint representation of E_8, then

$$\frac{1}{6}\int_Y G \wedge \left(G \wedge G - \frac{1}{8}\left[p_2(R) - \frac{1}{4}p_1^2(R)\right]\right) = -\int_Y \mathcal{P}_{12}^{E_8}(R, F) \tag{4.27}$$

where $\int_Y \mathcal{P}_{12}^{E_8}(R, F)$ is given in equation (4.11).

On the one hand, we also know that this is nothing more than the sum of indices of the twelve-dimensional Dirac operators of the gravity multiplet and one E_8 Yang-Mills multiplet of the $E_8 \times E_8$ heterotic string. Therefore,

$$\int_Y \mathcal{P}_{12}^{E_8}(R, F) = \frac{1}{2}\,\mathrm{Index}(\mathcal{D}_{12}^{P_{E_8}}) + \frac{1}{4}\mathrm{Index}(\mathcal{D}_{12}^{\mathrm{Grav}})\,, \tag{4.28}$$

[5] This is due to homotopy classification of bundles and the fact that topologically the E_8 group is equivalent to some spaces known as Eilenberg-MacLane spaces $K(G, n)$ with G and arbitrary Abelian group, see [96] for a thorough discussion on this issue.

[6] There is a subtlety here because the tangent bundle of the eleven-dimensional M-theory TX restricted to the M2-brane worldvolume can be written as the direct sum $TX|_{\mathcal{W}} = T\mathcal{W} \oplus N$, where N is the normal bundle. Therefore, fermions can be coupled to the tangent bundle $T\mathcal{W}$ as well to the normal bundle N. Considering this, [193] solved an issue of M-theory raised in [191] for the magnetic dual M5-brane due to the diffeomorphisms of the normal bundle. We leave this subtlety out of our discussion.

[7] See the discussion at the end of Chapter 3 where the trivialization of this class led to a string structure.

where we have added an extra one-half in the last term on the right-hand side of equation (4.28) to the already one-half present accounting for the Majorana condition. This is due to the same reason of the extra $1/2$ in the gravitational part in the definition of equation (4.8) in the discussion of $E_8 \times E_8$ anomaly cancellation.

On the other hand, one can show that the 12-dimensional indices are even integers essentially due to a kind of Kramers degeneracy [73]. Hence, it follows form (4.28) that $\exp(2\pi i\, h_{C_3}(Y)) = (-1)^n$ with $n \in \mathbb{Z}$ by using E_8 gauge theory and our knowledge of anomalies, where

$$h_{C_3} = -\int_Y \mathcal{P}_{12}^{E_8}(R,F)\,. \tag{4.29}$$

We have determined the mod 2 ambiguity of the topological term of M-theory. As an aside, we note that equation (4.29) suggests a kind of anomaly inflow, namely the bulk topological term of M-theory will make sense on a manifold with boundary provided that in the boundary lives nontrivial boundary mode contributing an anomaly equal to the anomaly of the bulk term but with minus sing as occurs in equation (4.29). This is the key observation in the Hořava and Witten wall and the main topic of Chapter 5 although, the order of events was not in this manner.

This anomaly has to be cancelled for the well-definiteness of M-theory. Thus, we are left with the task of checking whether the eleven-dimensional gravitino contributes the correct amount. We already know that a real fermion can at least contribute to anomalies with a sign ambiguity. Thus, we might expect the correct answer. Indeed, it turns out that the Majorana gravitino of the eleven-dimensional theory has a gravitational anomaly given by

$$\begin{aligned}\mathcal{I}_{12}^{3/2}(R) &= \frac{1}{2}\hat{A}(R)(\mathrm{ch}\,(R)-2)\bigg|_{12}\,,\\ &= \frac{1}{967680}\left(97\,p_1^3 - 788\,p_1p_2 + 3952\,p_3\right)\,.\end{aligned} \tag{4.30}$$

where integration of this index density is related to the $\mathrm{Index}(\mathcal{D}_{12}^{TY\otimes S^+})$ of a 12d Dirac operator coupled to the tangent bundle. Recall that this is related to the lift of the eleven-dimensional Dirac operator we are interested in. So, this is properly constructed to account for the anomaly of the 11d Dirac operator of the spin-$3/2$ gravitino field. This has a sign ambiguity which is exactly what we need to cancel the ambiguity of the topological couplings of M-theory. Therefore, M-theory is also an anomaly-free theory, namely

$$\exp(2\pi \mathrm{i}\, h_{\mathrm{Total}}) = \exp(2\pi \mathrm{i}\, h_{\mathrm{gravitino}})\exp(2\pi \mathrm{i}\, h_{C_3}) = 1\,. \tag{4.31}$$

What we have reviewed is based upon the analisys of Hořava and Witten. They observed that M-theory on a manifold with boundary or equivalently compactified on an orbifold $S^1/\mathbb{Z}_2$ leads to a configuration where the two ten-dimensional fixed planes associated to the fixed points of the orbifold action must be such that on each of them lives an E_8 vector multiplet. This is required by the anomaly inflow from the M-theory bulk to the boundary, as we already mentioned, and therefore this system is free of perturbative anomalies. This is correlated with the fact that $E_8 \times E_8$ heterotic string at the strong coupling limit determines an extra dimension in terms of the radious r of the circle S^1 or equivalently the interval $l = \pi r$ and the string coupling constant g_s. It follows that when the size of the interval goes to zero we recover the $E_8 \times E_8$ setup as shown pictorially on Figure 4.1 The aim of this thesis is to thoroughly study a similar setting but generalize that construction

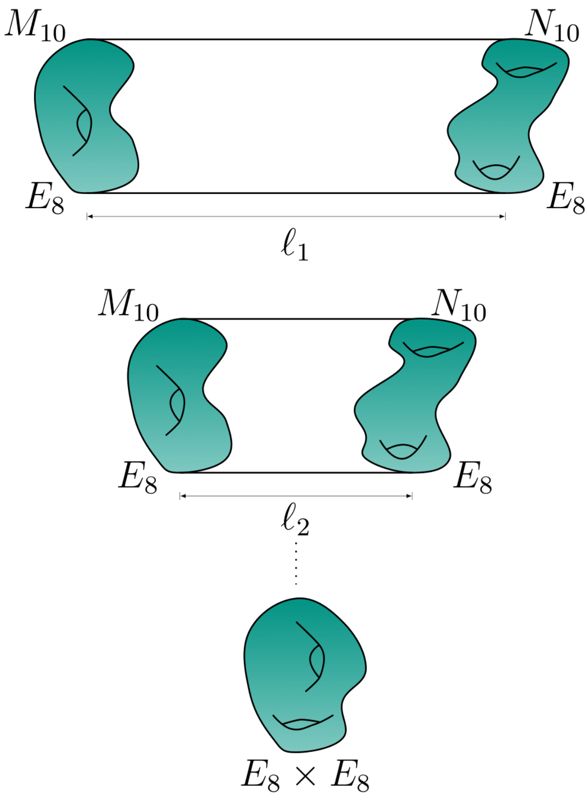


Figure 4.1: This picture shows the strong coupling limit of $E_8 \times E_8$ heterotic string which corresponds to M-theory on a manifold with two boundary components M_{10} and N_{10} with each vector multiplet on each boundary. Whereas the gravitational anomaly corresponds to one-half on each component. These boundary components are associated with the fixed point of the $\mathbb{Z}_2$ action in the orbifold compactification of M-theory. In the zero-size radius limit this configuration boils down to the low-energy limit of $E_8 \times E_8$ heterotic string.

to any Lie group. In Chapter 5 we will particularly study a similar problem with the exceptional groups $G = G_2, F_4, E_6, E_7$ and we will make some analysis for the classical Lie groups, for more details see [144]. As we emphasized by the end of Chapter 3, this is a meaningful question because, before any nonperturbative analysis of anomalies, this has to be preceded by a thorough exploration of perturbative anomalies, which is our goal.

Before undertaking that project, we shall review how perturbative anomaly cancellation works in the nonsupersymmetric $SO(16) \times SO(16)$ theory in the next section.

4.3 Nonsupersymmetric strings

4.3.1 $SO(16) \times SO(16)$ string

Anomaly cancellation for the first two nonsupersymmetric models mentioned earlier in Chapter 2 depends on a more elaborated version of the Green-Schwarz method, but the main idea is behind their cancellation of perturbative anomalies. We refer the interested reader to the literature [171–173] and for an analysis of nonperturbative anomalies see [136].

For the case of the $SO(16) \times SO(16)$ we have to compute the anomaly polynomial

$$\mathcal{I}_{12}(R, F) = \hat{A}(R)\mathrm{ch}_{\mathsf{r}}(F)\Big|_{12} , \tag{4.32}$$

for its chiral matter content consisting of a positive chirality Majorana-Weyl fermion transforming in the fundamental of each $SO(16)$ factor represented by $(\mathsf{16}, \mathsf{16})$, plus negative chirality Majorana-Weyl fermions transforming in the spinor representation of each factor $(\mathsf{128}, \mathsf{1}) \oplus (\mathsf{1}, \mathsf{128})$ [66,170].

To check the cancellation of perturbative anomalies, we have to do a similar computation as in the case of $E_8 \times E_8$ for the gauge part. Notice that there is no pure gravitational contribution coming from fermions. We also have to use the factorized Chern characters in the equation (4.10) to show that the anomaly polynomial is given by [55]

$$\mathcal{P}_{12}^{SO(16)^2}(R, F_1, F_2) = -\frac{1}{24}\frac{c_{2,\mathbf{16}}^{(1)} + c_{2,\mathbf{16}}^{(2)} + p_1}{2}\times \left((c_{2,\mathbf{16}}^{(1)})^2 + (c_{2,\mathbf{16}}^{(2)})^2 + c_{2,\mathbf{16}}^{(1)}\, c_{2,\mathbf{16}}^{(2)} - 4\,(c_{2,\mathbf{16}}^{(1)} + c_{2,\mathbf{16}}^{(2)}) \right) , \quad (4.33)$$

where we have summed each of the index densities contributions coming from $\mathcal{I}_{12}^{(\mathsf{16},\mathsf{16})}$, $\mathcal{I}_{12}^{(\mathsf{128},\mathsf{1})}$ and $\mathcal{I}_{12}^{(\mathsf{1},\mathsf{128})}$ using equation (4.6) for the respective representations, and each of the superscript (i), $i = 1, 2$ differentiate between the group factors $SO(16)$. The Green-Schwarz method for anomaly cancellation is carried out as before using the B_2 field in the gravity multiplet. This closes our presentation of anomaly cancellation in M-theory/string theory. This has been done in a manner that one can easily keep track of the calculations of Chapter 5.

As a result, in the next chapter, we will carry out a search of chiral spectra containing the superpartners of the gravity multiplet, a left-handed Majorana-Weyl gravitino, and a right-handed Majorana-Wyel dilatino. In contrast to the supersymmetric heterotic theories, we will allow matter restricted to be left-handed Majorana-Weyl fermions charged under any representation of a gauge group. The aim is to arrange a given matter content for anomaly cancellation by the Green-Schwarz method as we did in the E_8. Eventually, if we achieve to find any content leading to factorization like in the E_8, we will be able to connect this with the topological terms of M-theory.

Chapter 5

Searching for new M-theory Hořava-Witten boundary walls

In this chapter, we will consider a similar setup as in the Hořava-Witten wall. However, we consider any Lie group instead of using the E_8 gauge theory. Furthermore, we will mostly focus on exceptional algebras. One of the greatest achievements of the result of Hořava-Witten was to hold anomaly cancellation as well as supersymmetry on its bulk-boundary setup, namely the boundary modes are completely supersymmetric involving a supergravity multiplet and a super Yang-Mills multiplet with charged matter in the adjoint representation of E_8. We will relax this condition allowing matter in different representations of a gauge group G. We write down a very general anomaly polynomial in 12 dimensions, then we set and solve the conditions needed to look for the more restricted Green-Schwarz factorization of the anomaly polynomial in order to interpret any possible solution as a nonsupersymmetric boundary condition of M-theory. This analysis is based on Ref. [144]. This section relies heavily on the material developed in Appendix A, we suggest going through that material first as well as we have done in previous Chapters.

5.1 Anomaly Theory

Instead of doing what has been done for E_8 gauge theory, we take F, the curvature two-form of a G-bundle, as a matrix-valued form taking values in the Lie algebra of G, and traces are evaluated in the different representations we are going to consider. In particular, the notation F_{r_i} means that F is a matrix-valued 2-form where the dimension of the matrix generators $T^a_{\mathsf{r}_i}$ involved in the connection $A = A^a T^a_{\mathsf{r}_i}$, is the dimension of the representation r_i used to represent each of the generator of the Lie algebra with $a = 1, \cdots, \mathrm{dim}G$. Note that we have only emphasized the Lie-algebra index, but remind you that there is also a spacetime vector index. Using the anomaly polynomial of a spin-$3/2$ left-handed fermion, a spin-$1/2$ right-handed fermion, and spin-$1/2$ left-handed fermions transforming under r_i, all obeying a Majorana-Weyl condition, we can write down the following general anomaly polynomial

$$
\begin{aligned}
\mathcal{P}_{12}^{G}(R,F_{\mathsf{r}_i}) = {} & \frac{1}{2}\sum_{\mathsf{r}_i}\mathrm{ch}_{6,\mathsf{r}_i} - \frac{1}{48}p_1\sum_{\mathsf{r}_i}\mathrm{ch}_{4,\mathsf{r}_i} + \frac{1}{11520}(7p_1^2-4p_2)\sum_{\mathsf{r}_i}\mathrm{ch}_{2,\mathsf{r}_i} \\
& + \frac{1}{2}\frac{1}{967680}\left[\left(128-31\sum_{\mathsf{r}_i}\mathrm{ch}_{0,\mathsf{r}_i}\right)p_1^3\right. \\
& \qquad + \left(44\sum_{\mathsf{r}_i}\mathrm{ch}_{0,\mathsf{r}_i}-832\right)p_1\,p_2 \\
& \qquad \left. + \left(3968-16\sum_{\mathsf{r}_i}\mathrm{ch}_{0,\mathsf{r}_i}\right)p_3\right],
\end{aligned}
\tag{5.1}
$$

where we have made use of properties of the Chern characters (see e.g. [64]) to express the sum $\sum_{\mathsf{r}_i}\mathrm{ch}_{n,\mathsf{r}_i} := \sum_{\mathsf{r}_i}\mathrm{ch}_{n,\mathsf{r}_i}(F)$ over representations of G, with $n\in\mathbb{Z}$ representing the degree of the Chern character, see also Appendix B. For clarity, we have established $\mathcal{P}_{12}^{G}(R,F_{\mathsf{r}_i})$ as the sum of the index densities involved

$$
\mathcal{P}_{12}^{G}(R,F_{\mathsf{r}_i}) = \frac{1}{2}\mathcal{I}_{12}^{\mathrm{Grav}}(R) + \sum_{\mathsf{r}_i}\mathcal{I}_{12}^{G}(R,F_{\mathsf{r}_i})\,,
\tag{5.2}
$$

where the subscript r_i in the field strength F has been located in the Chern character. But, the meaning is the same, namely that subscript is primarily concerned with traces over generators of the Lie algebra taking values in a given representation.

Note that we have to deal with three terms to try to achieve the factorization we are looking for. The first obstruction we will consider is in the pure gravitational part, that is, p_3. Hence, for the gauge part, we shall consider a general situation described by the following arrangement of representations

$$
\underbrace{(\mathsf{r}_1)\oplus\cdots\oplus(\mathsf{r}_1)}_{n_{\mathsf{r}_1}\ \text{times}}\oplus\underbrace{(\mathsf{r}_2)\oplus\cdots\oplus(\mathsf{r}_2)}_{n_{\mathsf{r}_2}\ \text{times}}\oplus\underbrace{(\mathsf{r}_3)\oplus\cdots\oplus(\mathsf{r}_3)}_{n_{\mathsf{r}_3}\ \text{times}}\oplus\cdots
\tag{5.3}
$$

where r_1 has multiplicity n_{r_1}, r_2 has multiplicity n_{r_2} and so on and so forth. Thus, we can show that $\sum_{\mathsf{r}_i}\mathrm{ch}_{0,\mathsf{r}_i} = \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{dim}(\mathsf{r}_i)$, where we use the fact that $\mathrm{ch}_{0,\mathsf{r}_i} = \mathrm{dim}(\mathsf{r}_i)$. From this we set $\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{dim}(\mathsf{r}_i) - 248 = 0$ to throw the p_3 Pontrjagin class away from equation (5.1). On the other hand, we will use equations (A.1), (A.2) and (A.3) to rewrite equation (5.1) as follows

$$
\begin{aligned}
\mathcal{P}_{12}^{G}(R,F_{\mathsf{r}_i}) = {} & \frac{1}{2}\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{u}_{\mathsf{r}_i}\mathrm{ch}_6 + \frac{1}{180}\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{w}_{\mathsf{r}_i}\mathrm{ch}_2^3 - \frac{1}{288}p_1\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{y}_{\mathsf{r}_i}\mathrm{ch}_2^2 \\
& + \frac{1}{11520}(7p_1^2-4p_2)\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{z}_{\mathsf{r}_i}\mathrm{ch}_2 + \frac{1}{192}p_1p_2 - \frac{1}{256}p_1^3
\end{aligned}
\tag{5.4}
$$

where n_{r_i} is an integer introduced to account for the multiplicity of the representation r_i, and we have also used the fact that exceptional Lie algebras have no nontrivial fourth-order Casimir invariants. Note that, we have written all the Chern characters in the fundamental representation of the group G. Constants $\mathrm{u}_{\mathsf{r}_i}$ $\mathrm{w}_{\mathsf{r}_i}$ $\mathrm{y}_{\mathsf{r}_i}$ and $\mathrm{z}_{\mathsf{r}_i}$ are the group theoretical proportionality constants that allow us to rewrite (5.1) in the form of equation (5.4). Their values depend on the specific representations for each G. Essentially, those constants are given in terms of indices or equivalently Casimir invariants, see Table A.1. Importantly, we are left with $\mathrm{ch}_{6,\mathsf{r}_i}$, a sixth-order Chern character or equivalently a sixth-order trace, and we find that these terms have nontrivial sixth-order Casimir invariants associated with each representation for the group we are considering. However, notice that if we are able to guarantee that

$$\sum_{\mathsf{r}_i} n_{\mathsf{r}_i} \mathrm{u}_{\mathsf{r}_i} = 0 \,, \tag{5.5}$$

then, this would allow us to kill the unfactorizable sixth-order Chern class. Correspondingly, we have translated the problem of killing the main obstructions into a linear system of equations

$$\begin{aligned} n_{\mathsf{r}_1}\mathsf{r}_1 + n_{\mathsf{r}_2}\mathsf{r}_2 + \cdots &= 248 \,, \\ n_{\mathsf{r}_1}\mathrm{u}_{\mathsf{r}_1} + n_{\mathsf{r}_2}\mathrm{u}_{\mathsf{r}_2} + \cdots &= 0 \,, \end{aligned} \tag{5.6}$$

where this system corresponds to an underdetermined system that has more unknowns than equations, therefore, it may have infinitely many solutions or may not have any solution at all. Generically, the above system can be represented in matrix notation as

$$\mathcal{O}\, n = s \,, \tag{5.7}$$

where $\mathcal{O}$ will be a matrix operator whose entries are given by the representations and its sixth-order indices of the group representations of G under consideration, namely it has known values. On the other hand, n will be the vector of unknowns that we wish to find such that the vector s is the solution. To begin with this searching, let $n^{E_8} = (248, 0, 0, \cdots)$ represent the corresponding solution for the E_8 gauge theory that we already review in Chapter 4. Additionally, it is well known that the E_8 group has no nontrivial sixth-order indices, leading to all the $\mathrm{u}_{\mathsf{r}_i}$ entries being zero. Consequently, the most straightforward solution is obtained. Now, suppose n^G represents another solution for an arbitrary gauge group G. Then $n^{E_8} + n^G$ is also a solution, provided that $\mathcal{O}\, n^G = 0$ for G being any other group. Therefore, we can regard the space of solutions as the kernel of the operator $\mathcal{O}$, where any linear combination of vectors in the kernel is also a solution. It is important to note that, at this stage, our discussion imposes no restrictions on the number of representations we could employ. In other words, the vector n^G could have positive and negative entries, meaning that chiral gauge content is built with positive and negative chirality fermions. Therefore, (5.6) may involve infinitely many representations, which may not contribute significantly to the search for nonsupersymmetric boundary conditions of M-theory.

Therefore, in order to proceed further, we observe from the first equation in (5.6) that restricting ourselves to a chiral spectrum, for concreteness, positive chirality, implies that all representations involved in our search must have dimension less than 248, hence limiting the number of representations to a finite set as we will see. This is in agreement with the choice made in Chapter 4, regarding the chirality of the spectrum. Under this assumption, we will apply the algorithm we have developed to any group G, in particular for exceptional groups. This will guarantee to finding a finite set of solutions. Some comments regarding classical Lie groups are in order. Generally, for these classical algebras, one has complex representations, slightly changing the anomaly polynomial (5.1). Moreover, due to the presence of fourth-order traces that are not present for exceptional algebras and A_1 and A_2, the factorization we are pursuing will correspondingly modify the linear system of equations (5.6). This adds more indices to be computed as well as more equations to be solved. Additionally, we should mention that for A_n algebras with $n > 1$ there can be nontrivial third-order traces in contrast to other algebras. Therefore, the analysis is more involved although, it can be carried out in a similar fashion as before [144].

As a result of the preceding discussion, we are left with the following anomaly polynomial

$$\begin{aligned}\mathcal{P}_{12}^{G}(R,F) =& \frac{1}{180}\sum_{\mathrm{r}_i} n_{\mathrm{r}_i}\mathrm{w}_{\mathrm{r}_i}\mathrm{ch}_2^3 - \frac{1}{288}p_1\sum_{\mathrm{r}_i} n_{\mathrm{r}_i}\mathrm{y}_{\mathrm{r}_i}\mathrm{ch}_2^2 \\ &+ \frac{1}{11520}(7p_1^2 - 4p_2)\sum_{\mathrm{r}_i} n_{\mathrm{r}_i}\mathrm{z}_{\mathrm{r}_i}\mathrm{ch}_2 + \frac{1}{192}p_1p_2 - \frac{1}{256}p_1^3\,,\end{aligned} \tag{5.8}$$

where all the trace-identity constants can be computed following [209,210] as we did for the explicit examples we analyzed in Appendix A. Obviously, all this examination is not enough; it remains to determine whether (5.8) factorizes for a given solution n^G. To that end, let us apply our algorithm to as many groups as possible, fulfilling all the conditions we have already discussed up to this point.

As we already said, we must consider the restriction on the charged spectrum to be of positive chirality, as this limits the number of representations to be considered. Additionally, it restricts the group representations we will analyze. For E_8 equation (5.8) straightforwardly reproduces what we already computed in Chapter 4. On the other hand, we will see that the group G_2 has various representations of dimension less than the adjoint of E_8, and remarkably, all are real representations. This, in turn, ensures that traces of odd powers in the curvature field strength in Chern characters vanish, eliminating the need to worry about those terms consistent with (5.8). This suggests that there might be various solutions for the linear system of equations (5.6). If this is the case, we have to determine whether the factorization we are seeking is allowed by the corresponding solutions. On the other hand, we also analyze the group F_4, which offers few representations of dimension less than the adjoint representation 248 of E_8 to work with, thus limiting our findings in this case. Similarly, E_6 and E_7 also have a limited number of representations to work with. All of these aspects will be explicitly explored shortly.

It is well worth mentioning that the Green-Schwarz anomaly factorization we are looking for is certainly more restricted than the standard method [129,130] schematically W_4P_8. This was an important observation in [138,139] where P_8 was still further factorized. This factorization also noted by [204] ultimately leads to the connection with M-theory on a manifold with boundary. Nevertheless, we should certainly expect to find solutions obeying a standard factorization. It turns out that this is the case, as we will see later. As a consequence, we have to ask whether these solutions have some place in the string universality principle. in other words, it seems plausible to ask whether these theories could belong to the ten-dimensional nonsupersymmetric corner of the moduli space of string theory. It turns out that an important ingredient of the worldsheet realization of the low-energy limits of heterotic strings is a two-dimensional Superconformal Field Theory (SCFT) with left- and right-central charges given by $(c_L, c_R) = (16, 0)$. In Ref. [211] was stated that there is no other nonsupersymmetric ten-dimensional heterotic strings than the already known by classifying CFTs with central charge ≤ 16. It seems reasonable to explore these two directions. We hope to return to these observations in future work.

5.2 Exceptional algebras

In this section, we explicitly apply the algorithm developed above. Let us begin with the exceptional algebras where there are no fourth-order Casimir invariants, namely we can express them in terms of products of second-order traces. After that, we proceed to study classical Lie algebras.

5.2.1 G_2 group

We will start analyzing the group G_2 for the list of representations shown in Table A.1 where are also shown the trace-index proportionality constants which are the inputs of the algorithm developed before. We will first determine all possible solutions under the restrictions imposed in the previous section. Then, we choose one of the solutions and calculate all the ingredients needed to find out whether the anomaly polynomial can be factorized to establish a connection with M-theory.

When we apply the algorithm described above we find the following set of solutions

$$\begin{aligned}\{(n_1 = 52, n_7 = 26, n_{14} = 1), \quad (n_1 = 6, n_7 = 13, n_{14} = 5, n_{27} = 3),\\ (n_1 = 152, n_{14} = 3, n_{27} = 2), \quad (n_1 = 8, n_{14} = 1, n_{27} = 6, n_{64} = 1),\\ (n_1 = 102, n_7 = 13, n_{14} = 2, n_{27} = 1), \quad (n_1 = 56, n_{14} = 6, n_{27} = 4),\\ (n_1 = 1, n_{14} = 3, n_{64} = 2, n_{77} = 1)\}\,.\end{aligned}$$

Notice that the set of solutions is finite and limited to seven solutions solving the linear system of equations subject to the constraints we discussed. We next have to check whether or not these solutions lead to a factorized twelve-dimensional anomaly polynomial. To this end, we present the details for one of the solutions and for the other, we collect our results in two different tables for reasons that will become clear later. We take the solution given by

$$(n_1 = 1, n_{14} = 3, n_{64} = 2, n_{77} = 1)\,, \tag{5.9}$$

where the corresponding group theoretical constants are collected in Table 5.1. Next, we

r	$\mathrm{u_r}$	$\mathrm{w_r}$	$\mathrm{y_r}$	$\mathrm{z_r}$	
1	–	–	–	–	$n_1 = 1$
14	-26	$\frac{15}{4}$	$\frac{5}{2}$	4	$n_{14} = 3$
64	-208	75	38	32	$n_{64} = 2$
77	494	$\frac{315}{4}$	$\frac{121}{2}$	44	$n_{77} = 1$
$\sum n_r \ell_0(\mathrm{r}) = 248$	$\sum n_r \mathrm{u_r} = 0$	$\sum n_r \mathrm{w_r} = 240$	$\sum n_r \mathrm{y_r} = 144$	$\sum n_r \mathrm{z_r} = 120$	

Table 5.1: Multiplets of representations solving the constraints to factorize (5.8) and the indices of traces for G_2.

need to plug the results in the last row of Table 5.1 into equation (5.8). By doing that, we find that the anomaly polynomial associated with the solution (5.9) is given by

$$\mathcal{P}_{12}^{G_2}(R,F) = -\frac{1}{96}(8\,c_{2,7}+p_1)\left(\frac{1}{4}(8\,c_{2,7}+p_1)^2+\frac{1}{8}\,p_1^2-\frac{1}{2}\,p_2\right)\,, \tag{5.10}$$

where the second Chern class $c_{2,7}$ has been written in the fundamental of G_2. As a result of this analysis, we have found that the matter content in (5.9) along with a pure gravitational contribution lead to an anomaly polynomial that can be related to the anomaly of the the topological interactions of M-theory, since

$$h_{C_3} = -\int_Y \mathcal{P}_{12}^{G_2}(R,F)\,, \tag{5.11}$$

provided that the 4-form field strength G_4 of theory is chosen as $G = \frac{1}{4}\,p_1 + 2\,c_{2,7}$ as we did for the E_8 gauge theory. Therefore, this suggests that we have found a new boundary condition of M-theory with gauge group G_2. Before we discuss whether this is a new solution we summarize the anomaly polynomials for the G_2 solutions in two tables. Table 5.2 shows the anomaly polynomial of the different solutions of G_2 with the property that the corresponding anomaly polynomial factorizes in the manner demanded by a connection with M-theory. We also provide in that table the corresponding value of the 4-form field strength to establish such a connection. In Table 5.3 we provide the anomaly

G_2 gauge group		
	$(n_1=52, n_7=26, n_{14}=1)$	
$\mathcal{P}_{G_2}(R,F) =$	$-\frac{1}{96}(2\,c_{2,7}+p_1)\left(\frac{1}{4}(2\,c_{2,7}+p_1)^2+\frac{1}{8}\,p_1^2-\frac{1}{2}\,p_2\right)$	$G=\frac{1}{4}\,p_1+\frac{1}{2}\,c_{2,7}$
	$(n_1=6, n_7=13, n_{14}=5, n_{27}=3)$	
$\mathcal{P}_{G_2}(R,F) =$	$-\frac{1}{96}(4\,c_{2,7}+p_1)\left(\frac{1}{4}(4\,c_{2,7}+p_1)^2+\frac{1}{8}\,p_1^2-\frac{1}{2}\,p_2\right)$	$G=\frac{1}{4}\,p_1+c_{2,7}$
	$(n_1=8, n_{14}=1, n_{27}=6, n_{64}=1)$	
$\mathcal{P}_{G_2}(R,F) =$	$-\frac{1}{96}(6\,c_{2,7}+p_1)\left(\frac{1}{4}(6\,c_{2,7}+p_1)^2+\frac{1}{8}\,p_1^2-\frac{1}{2}\,p_2\right)$	$G=\frac{1}{4}\,p_1+\frac{3}{2}\,c_{2,7}$
	$(n_1=1, n_{14}=3, n_{64}=2, n_{77}=1)$	
$\mathcal{P}_{G_2}(R,F) =$	$-\frac{1}{96}(8\,c_{2,7}+p_1)\left(\frac{1}{4}(8\,c_{2,7}+p_1)^2+\frac{1}{8}\,p_1^2-\frac{1}{2}\,p_2\right)$	$G=\frac{1}{4}\,p_1+2\,c_{2,7}$

Table 5.2: Combination of representations and its multiplicity solving the linear system of equations (5.6) related to branching rules of E_8 along with the anomaly polynomial associated with each solution. On the third column, it is given the corresponding 4-form of the theory.

polynomials for the rest of the solutions we found for G_2. Notice that those anomaly polynomials do not factorize as we would expect to establish a relation with M-theory. However, there are two solutions that interestingly have an anomaly polynomial with the minimal factorization required by the Green-Schwarz method although, this is not enough for our aim in this thesis. As we mentioned before, it would be interesting to further study those solutions.

We do have to figure out whether or not the solutions of Table 5.2 are new boundary walls of M-theory on a manifold with boundary. We make the following observations to address this question. We will focus on the branching rules [212] of the E_8 group involving the group G_2. Through [212] the branching $E_8 \supset G_2 \times F_4$, it is found that $\mathbf{248} \to (\mathbf{14}, \mathbf{1}) \oplus (\mathbf{7}, \mathbf{26}) \oplus (\mathbf{1}, \mathbf{52})$ which clearly explains the first spectrum just found in Table 5.2. By looking, for instance, at the following chain $E_8 \supset E_7 \otimes SU(2)$ and

G_2 gauge group	
	$(n_{\mathbf{1}} = 152, n_{\mathbf{14}} = 3, n_{\mathbf{27}} = 2)$
$\mathcal{P}_{G_2}(R,F) =$	$-\frac{1}{768}(2\,c_{2,7} + p_1)\left(\frac{2}{5}(10\,c_{2,7} + p_1)^2 + \frac{13}{5}\,p_1^2 - 4\,p_2\right)$
	$(n_{\mathbf{1}} = 56, n_{\mathbf{14}} = 6, n_{\mathbf{27}} = 4)$
$\mathcal{P}_{G_2}(R,F) =$	$-\frac{1}{768}\left(160\,c_{2,7}^3 + 112\,c_{2,7}^2\,p_1 + 28\,c_{2,7}\,p_1^2 - 16\,c_{2,7}\,p_2 - 4\,p_1\,p_2 + 3\,p_1^3\right)$
	$(n_{\mathbf{1}} = 102, n_{\mathbf{7}} = 13, n_{\mathbf{14}} = 2, n_{\mathbf{27}} = 1)$
$\mathcal{P}_{G_2}(R,F) =$	$-\frac{1}{768}(2\,c_{2,7} + p_1)\left(\frac{2}{3}(6\,c_{2,7} + p_1)^2 + \frac{7}{3}\,p_1^2 - 4\,p_2\right)$

Table 5.3: Combinations of representations solving (5.6) which are not directly related to branching rules of E_8 with its respective anomaly polynomial. Consequently, we saw that the anomaly polynomials do not factorize as we would naively expect.

$E_7 \supset SU(2) \times G_2$, we can determine the second spectrum since

$$\begin{aligned} \mathbf{248} &\to (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{56}, \mathbf{2}) \oplus (\mathbf{133}, \mathbf{1})\,, \\ \mathbf{1} &\to (\mathbf{1}, \mathbf{1})\,, \\ \mathbf{56} &\to (\mathbf{4}, \mathbf{7}) \oplus (\mathbf{2}, \mathbf{14})\,, \\ \mathbf{133} &\to (\mathbf{5}, \mathbf{7}) \oplus (\mathbf{3}, \mathbf{27}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{14})\,, \end{aligned} \tag{5.12}$$

which produces the combination of representations in the spectrum of the second row in Table 5.2.

The third solution is obtained through the branching $E_8 \supset E_6 \otimes SU(3)$ and $E_6 \supset G_2$ by looking at the branching of the adjoint representation of E_8 under this chain of embeddings.

Whereas the last row in Table 5.2 can be found if we look at the set of embeddings

$$G_2 \to SO(14) \to SO(16) \to E_8\,. \tag{5.13}$$

After a careful study of all the branching rules of the E_8 group, we find that all of them lead to one of the solutions shown in Table 5.2, maybe leading to the same solution more than twice but no other than those there. Whereas, solutions in Table 5.3, we were not able to trace back to E_8 branching rules. Under these observations, it follows that the factorized anomaly polynomials in Table 5.2 inherit the structure of the E_8 anomaly polynomial reviewed in Chapter 4 and can be shown properly if we look carefully at the branchings just discussed, as we did using $SO(16)$ in the analysis of E_8 gauge theory in the previous Chapter. It turns out that Table 5.3 cannot be traced back to branching rules, hence the anomaly polynomials do not factorize as we are looking for. In other words, these cases do not inherit any structure from the underlying E_8 gauge theory.

We will make the same analysis for the other exceptional groups by applying the same algorithm as before.

5.2.2 F_4 group

We continue our discussion with F_4. In this case, we have only three representations available to apply our algorithm. The solution we have determined is provided in Table 5.4. This solution can be spelled out by one of the branchings we have already mentioned for G_2, namely $E_8 \supset G_2 \times F_4$. Therefore, this does not constitute anything new. We should once more emphasize that the restriction of considering only one chirality limits the results we obtain. Ultimately, perturbative anomalies arise from having a chiral spectrum. Once again, the identity with M-theory is established with the four-form provided in Table 5.4.

F_4 gauge group		
	$(n_1 = 14, n_{26} = 7, n_{52} = 1)$	
$\mathcal{P}_{F_4}(R,F)$	$-\frac{1}{96}\left(\frac{2}{3}c_{2,26}+p_1\right)\left(\frac{1}{4}\left(\frac{2}{3}c_{2,26}+p_1\right)^2+\frac{1}{8}p_1^2-\frac{1}{2}p_2\right)$	$G=\frac{1}{4}p_1+\frac{1}{6}c_{2,26}$

Table 5.4: The spectrum for F_4 is unique and directly related with E_8. The corresponding anomaly polynomial in the fundamental of F_4 is also provided. Additionally, the M-theory four-form is given with the proper coefficients.

5.2.3 E_6 group

Next, we discuss E_6 gauge theory. Firstly, it turns out that the fundamental representation of the E_6 group is complex, so we can consider its complex conjugate representation and then apply our algorithm. On the other hand, the number of representations is limited, we do not expect anything new. In fact, what we have determined is given in Table 5.5 and it is related to the branching $E_8 \supset E_6 \times SU(3)$.

Note that the anomaly polynomial of F_4 is the same as in this case. This is expected since $27 \to 26 \oplus 1$ under $E_6 \supset F_4$.

E_6 gauge group		
	$(n_1 = 8, n_{27} = 3, n_{\overline{27}} = 3, n_{78} = 1)$	
$\mathcal{P}_{E_6}(R, F)$	$-\frac{1}{96}\left(\frac{2}{3}\,c_{2,27} + p_1\right)\left(\frac{1}{4}\left(\frac{2}{3}\,c_{2,27} + p_1\right)^2 + \frac{1}{8}\,p_1^2 - \frac{1}{2}\,p_2\right)$	$G = \frac{1}{4}p_1 + \frac{1}{6}c_{2,26}$

Table 5.5: Combinations of representations of E_6 with its respective anomaly polynomial directly related to E_8.

5.2.4 E_7 group

The final group is E_7 for which we have only two nontrivial representations to apply the algorithm thus the spectrum we found is the expected coming from the branching under $E_7 \times SU(2) \subset E_8$.

E_7 gauge group		
	$(n_1 = 3, n_{56} = 2, n_{133} = 1)$	
$\mathcal{P}_{E_6}(R, F)$	$-\frac{1}{96}\left(\frac{1}{3}\,c_{2,27} + p_1\right)\left(\frac{1}{4}\left(\frac{1}{3}\,c_{2,27} + p_1\right)^2 + \frac{1}{8}\,p_1^2 - \frac{1}{2}\,p_2\right)$	$G = \frac{1}{4}p_1 + \frac{1}{12}c_{2,26}$

Table 5.6: Solution of E_6 with its respective anomaly polynomials connected with E_8.

We have arrived at the conclusion that the unique exceptional gauge group providing a boundary solution of M-theory is the E_8 group. The other cases providing a factorized anomaly polynomial only acquire the structure of the E_8 case. In other words, there in no nonsupersymmetric heterotic branes with exotic matter than the E_8 M9-branes found by Hořava and Witten [138, 139] using exceptional groups with charged chiral matter. Furthermore, we have found solutions for G_2 that factorize as in the standard Green-Schwarz method and other solutions that do not factorize at all.

In the next section, we deal with classical Lie algebras. W will rule out many of them by dimensionality reasons and by using their indices. Eventually, we are left with a limited number of groups that it is well worth analyzing but the details can be found in [144]. The reason is that the main challenge in those cases is the computation of a large number of indices for a considerable number of representations. However, the algorithm to search for new M9-branes in M-theory with exotic matter in representations of a classical Lie group is the same as the one we worked out at the beginning of this Chapter.

5.3 Classical Lie groups

Here, we focus on the classical Lie algebras. Due to our chirality assumption, we can rule out many of the higher-order groups and deal only with algebras of certain rank n as we will see. Regarding this point, it is more convenient to rewrite Eq. (5.1) in terms of traces of powers of R and F curvatures due to the presence of many more group-theoretic coefficients of the different combinations of fundamental traces appearing in the expansion of a trace of an arbitrary representation.

The classical algebras A_n, B_n, C_n, D_n for $n > 2$ have nontrivial quartic Casimir invariants in general, which implies that there can be fourth-order Chern characters such that the four-order traces can be decomposed as follow

$$\mathrm{tr}_{\mathsf{r}}\, F^4 = \mathrm{x}_{\mathsf{r}}\, \mathrm{tr}\, F^4 + \mathrm{y}_{\mathsf{r}}\, (\mathrm{tr}\, F^2)^2 \tag{5.14}$$

for an arbitrary representation r of the corresponding algebra in terms of the fundamental representation. This new situation contributes to the anomaly polynomial as follows

$$\frac{1}{288}\, p_1 \sum_{\mathsf{r}_i} n_{\mathsf{r}_i} \mathrm{x}_{\mathsf{r}_i} \mathrm{ch}_4\,. \tag{5.15}$$

where $\mathrm{x}_{\mathsf{r}_i}$ is directly related to the coefficients in (5.14) and n_{r_i} accounts for the multiplicity of r_i. Also, in general, we have the following sixth-order trace identity

$$\mathrm{tr}_{\mathsf{r}}\, F^6 = \mathrm{u}_{\mathsf{r}}\, \mathrm{tr}\, F^6 + \mathrm{v}_{\mathsf{r}}\, (\mathrm{tr}\, F^4)(\mathrm{tr}\, F^2) + \mathrm{w}_{\mathsf{r}}\, (\mathrm{tr}\, F^2)^3 + \mathrm{a}_{\mathsf{r}}\, (\mathrm{tr}\, F^3)^2 \tag{5.16}$$

where the last term on the right-hand side of (5.16) is nonvanishing only for the algebra $A_n\ (n \geq 2)$. Also, it can easily be shown that the term $\mathrm{tr}_{\mathsf{r}}\, F^3$ has a nonvanishing third-order Casimir invariant only for representations r which are complex. Usually, this coefficient is known as the anomaly coefficient in the physics literature, see e.g. [62].

More precisely, we can work out a general anomaly polynomial in terms of explicit traces. We will first focus on the pure gauge and mixed gauge-gravitational anomaly. By using the explicit trace identities for the Chern character and Pontrjgin classes, we obtain

$$\begin{aligned} &-\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{u}_{\mathsf{r}_i}\mathrm{tr}\, F^6 - \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{v}_{\mathsf{r}_i}\mathrm{tr}\, F^4\, \mathrm{tr}\, F^2 - \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{w}_{\mathsf{r}_i}(\mathrm{tr}\, F^2)^3 - \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{a}_{\mathsf{r}_i}(\mathrm{tr}\, F^3)^2 \\ &\quad + \frac{5}{8}\,\mathrm{tr}\, R^2 \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{x}_{\mathsf{r}_i}\mathrm{tr}\, F^4 + \frac{5}{8}\,\mathrm{tr}\, R^2 \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{y}_{\mathsf{r}_i}(\mathrm{tr}\, F^2)^2 \\ &\quad - \frac{5}{64}\,(\mathrm{tr}\, R^2)^2 \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{z}_{\mathsf{r}_i}\mathrm{tr}\, F^2 - \frac{1}{16}\,\mathrm{tr}\, R^4 \sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\mathrm{z}_{\mathsf{r}_i}\mathrm{tr}\, F^2\,, \end{aligned} \tag{5.17}$$

up to some overall constant[1] that is not relevant for the analysis of the coefficients appearing on each trace $\mathrm{tr}F^n$ or products of them. These coefficients can be conveniently represented in a vector

$$(\mathrm{u}_{\mathsf{r}_i}, \mathrm{v}_{\mathsf{r}_i}, \mathrm{w}_{\mathsf{r}_i}, \mathrm{a}_{\mathsf{r}_i}, \mathrm{x}_{\mathsf{r}_i}, \mathrm{y}_{\mathsf{r}_i}, \mathrm{z}_{\mathsf{r}_i})\,. \tag{5.18}$$

Notice that some of these coefficients are not important for the factorization problem we are dealing with. In principle, we can get rid of some of them such that we leave with the vector which has to be zero, i.e.

$$\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}(\mathrm{u}_{\mathsf{r}_i}, \mathrm{v}_{\mathsf{r}_i}, \mathrm{a}_{\mathsf{r}_i}, \mathrm{x}_{\mathsf{r}_i}) = \mathbf{0}\,, \tag{5.19}$$

[1] In order to throw the third Pontrjagin class p_3 away from the anomaly polynomial already avoiding any possibility of factorization, we fix $\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\dim(\mathsf{r}_i) - 248 = 0$, which implies that the pure gravitational part reduces to

$$+\frac{15}{64}\,(\mathrm{tr}\, R^2)^3 + \frac{15}{16}\,\mathrm{tr}\, R^2\, \mathrm{tr}\, R^4\,,$$

in terms of explicit traces. Also taking into account the overall constant of 1440 alluded in the text.

supplemented with the condition $\sum_{\mathsf{r}_i} n_{\mathsf{r}_i}\dim(\mathsf{r}_i) - 248 = 0$. Note also that the anomaly coefficient $\mathrm{a}_{\mathsf{r}_i}$ vanishes for the algebras B_n, C_n, and D_n. We should also emphasize that finding a set of solutions satisfying these two conditions does not guarantee a factorized anomaly polynomial as we are looking for.

5.3.1 Algebras B_n, C_n and D_n

Firstly, we will discuss $B_n\,(n \geq 3)$, $C_n\,(n \geq 2)$, and $D_n\,(n \geq 4)$ algebras. We do in this way because for $A_n\,(n \geq 2)$ there is third-order invariants [213]. Secondly, $A_{n=1}$ is special since it does have neither four-order nor sixth-order Casimir invariants, thus it seems that there could be a chance to search for an arrangement of representations such that the anomaly polynomial could factorize. We return to these cases in a moment.

D_n : The first observation we make is that for $D_n,\ n \geq 11$ we have only the trivial and the fundamental representation to play the same game we have been already playing to cancel terms representing an obstruction to the factorization we want. Therefore, the term $\mathrm{tr}_{\mathsf{f}}\, F^4 = \mathrm{x}_{\mathsf{f}}\,\mathrm{tr}\, F^4$, where x_{f} could be normalized to be one, will appear in the anomaly polynomial already avoiding a factorization since there is no way to cancel it. In addition, there is no obvious way to get rid of $\mathrm{tr}_{\mathsf{r}}\, F^6 = \mathrm{u}_{\mathsf{r}}\,\mathrm{tr}\, F^6$. which is in principle nontrivial as well. We have the same situation for algebras $C_n,\ n \geq 12$ and $B_n,\ n \geq 12$. Note that for the purpose of looking for boundary conditions of M-theory $SO(32)$ was discarded as was pointed out in [138] basically by this reason and by supersymmetry.

For $4 \leq n \leq 11$ the representations we have at our disposal are basically the fundamental, the adjoint, and the symmetric representations which have positive group-theoretical constant x_{r} thus the anomaly polynomial comes with a nontrivial four-order trace avoiding a sort of M-theory identity. For the adjoint and the symmetric representations of B_n and D_n algebras, we can compute the indices, so that [68, 214]

$$\mathrm{tr}_{\mathsf{a}}\, F^4 = (2n-8)\,\mathrm{tr}\, F^4 + 3\,(\mathrm{tr}\, F^2)^2\,, \tag{5.20}$$

$$\mathrm{tr}_{\mathsf{sym}}\, F^4 = (2n+8)\,\mathrm{tr}\, F^4 + 3\,(\mathrm{tr}\, F^2)^2\,. \tag{5.21}$$

On the other hand, for $4 \leq n \leq 8$ the spin representation, among others, enters the analysis and it is also known that the four-order trace is related to the fundamental via [68, 214]

$$4\,\mathrm{tr}_{\mathsf{spin}}\, F^4 = -2^{(\frac{2n-5}{2})}\,\mathrm{tr}\, F^4 + 3\,\cdot 2^{(\frac{2n-9}{2})}\,(\mathrm{tr}\, F^2)^2 \tag{5.22}$$

Notice that, $SO(16)$ is directly connected to E_8 via the branching rule through the adjoint and the spin representations where the proportionality constants of $\mathrm{tr}\, F^4$ are such that they vanish for 120 + 128. In this way, the low dimensional cases could deserve more attention where we will have fourth-order and sixth-order Casimir invariants. For instance, Ref. [215] has computed quartic Casimir operators of $SO(9)$, $SO(7)$, $SO(5)$ and $SO(10)$ and $SO(8)$ that could be investigated and according to their results there could be negative eigenvalues for the Casimir operators. Nevertheless, by looking at the branchings of E_8 only the groups $SO(7)$, $SO(14)$, and $SO(16)$ are expected to lead to similar results as the preceding sections. Therefore, this is a more direct way to rule out possible solutions with the $B_n,\ D_n$ algebras. Something similar happens for C_n algebra.

5.3.2 Algebra A_n

Finally, we look at the algebra A_{n-1}, i.e. $SU(n)$ for which algebras of rank $n \geq 15$ are already ruled out by the dimensionality of representations. For algebras $n < 14$

the number of representations increases as we go down in the rank. But, looking at the branching rules [212] of E_8 there is a couple of examples that deserve some attention. Otherwise, we do not expect to find anything new. However, for a thorough analysis, we just realized that using material developed in [214], we can just compute the indices for representations $[2]$, $[3]$, $[4]$, where the notation $[k]$ means that

$$[k] = \frac{n!}{k!(n-k)!}\,, \tag{5.23}$$

as well as for the symmetric representations

$$(2) = \frac{1}{2}n(n+1)\,, \qquad (3) = \frac{1}{6}n(n+1)(n+2)\,. \tag{5.24}$$

As we said, the Chern character obeys very useful properties under direct sum and tensor product of representations that follow from its definition (see Appendix B), namely

$$\mathrm{ch}_{\mathsf{r}_1\oplus\mathsf{r}_2}(F) = \mathrm{ch}_{\mathsf{r}_1}(F) + \mathrm{ch}_{\mathsf{r}_2}(F)\,, \tag{5.25}$$

$$\mathrm{ch}_{\mathsf{r}_1\otimes\mathsf{r}_2}(F) = \mathrm{ch}_{\mathsf{r}_1}(F)\,\mathrm{ch}_{\mathsf{r}_2}(F)\,. \tag{5.26}$$

These properties of the Chern character are particularly useful for representations of $SU(n)$ since this will allow us to evaluate Chern characters of symmetric and anti-symmetric representations. Using the following identities [214]

$$\sum_{m=0}^{\infty} t^m \mathrm{ch}_{[m]}(F) = \det\left(1 + t\,\exp\left(\mathrm{i}\frac{F}{2\pi}\right)\right)\,, \tag{5.27}$$

$$\sum_{m=0}^{\infty} t^m \mathrm{ch}_{(m)}(F) = \det\left(1 - t\,\exp\left(\mathrm{i}\frac{F}{2\pi}\right)\right)^{-1}\,, \tag{5.28}$$

where $[m]$ and (m) denote (anti-)symmetrized representations where the Chern character is evaluated. These are irreducible representations for $SU(n)$ using as reference representation its fundamental vector representation. For instance, the $[m]$ can be thought of as an element of the exterior algebra $\Lambda(V) = \oplus_m \Lambda^m(V)$ (another way to think about this is by symmetrization by Young tableaux). Therefore, the right-hand side of (5.27) and (5.28) is valued in the fundamental of $SU(n)$. This allows us to determine the corresponding Chern characters as follows

$$\det\left(1 + t\,\exp\left(\mathrm{i}\frac{F}{2\pi}\right)\right) = \prod_{k=1}^{\infty} \exp\left[-\frac{(-t)^k}{k}\mathrm{ch}(k\,F)\right]\,, \tag{5.29}$$

$$\det\left(1 - t\,\exp\left(\mathrm{i}\frac{F}{2\pi}\right)\right)^{-1} = \prod_{k=1}^{\infty} \exp\left[\frac{t^k}{k}\mathrm{ch}(k\,F)\right]\,, \tag{5.30}$$

where $\mathrm{ch}(F) = \mathrm{tr}(\mathrm{i}\frac{F}{2\pi})$ denotes the Chern character evaluated in the fundamental or defining representation. With this, one can show that

$$\mathrm{ch}_{[2]}(F) = \frac{1}{2}\mathrm{ch}^2(F) - \frac{1}{2}\mathrm{ch}(2F)\,, \tag{5.31}$$

$$\mathrm{ch}_{[3]}(F) = \frac{1}{6}\mathrm{ch}^3(F) - \frac{1}{2}\mathrm{ch}(2F)\,\mathrm{ch}(F) + \frac{1}{3}\mathrm{ch}(3F)\,, \tag{5.32}$$

$$\mathrm{ch}_{[4]}(F) = \frac{1}{24}\mathrm{ch}^4(F) - \frac{1}{4}\mathrm{ch}^2(F)\,\mathrm{ch}(2F) + \frac{1}{8}\mathrm{ch}^2(2F) + \frac{1}{3}\mathrm{ch}(F)\,\mathrm{ch}(3F) - \frac{1}{4}\mathrm{ch}(4F)\,. \tag{5.33}$$

From this, one can obtain basic trace identities with the corresponding index coefficients (up to some normalization) plus the product of basic traces as well as the dimension of the representation $[m]$. In fact, we have computed the indices for the representations $[2]$, $[3]$, $[4]$, (2) and (3) for the classical Lie algebra A_{n-1}, which are useful for our purposes. They are given by

$$\begin{aligned} I_k([2]) &= n - 2^{k-1} \\ I_k([3]) &= \frac{1}{2}(n^2 - (1+2^k)n + 2\cdot 3^{k-1}) \\ I_k((2)) &= n + 2^{k-1} \\ I_k((3)) &= \frac{1}{2}(n^2 + (1+2^k)n + 2\cdot 3^{k-1}) \\ I_k([4]) &= \frac{1}{6}(n^3 - 3(2^{k-1}+1)n^2 + 2(3\cdot 2^{k-2} + 3^k + 1)n - 6\cdot 4^{k-1}) \,, \end{aligned} \tag{5.34}$$

and for each of the representations, we have obtained the following trace identities

$$\mathrm{tr}_{[2],(2)}F^6 = (n \mp 2^5)\mathrm{tr}F^6 + 15\,\mathrm{tr}F^4\,\mathrm{tr}F^2 - 10\,(\mathrm{tr}F^3)^2 \,, \tag{5.35}$$
$$\mathrm{tr}_{[2],(2)}F^4 = (n \mp 2^3)\mathrm{tr}F^4 + 3\,(\mathrm{tr}F^2)^2 \,, \tag{5.36}$$
$$\mathrm{tr}_{[2],(2)}F^2 = (n \mp 2)\mathrm{tr}F^2 \,, \tag{5.37}$$

where the upper sign is for the antisymmetric $[2]$ and the lower sign is for the symmetric (2) representations. Whereas, for the representations $[3]$ and (3), we have determined that

$$\begin{aligned} \mathrm{tr}_{[3],(3)}F^6 = \frac{1}{2}(n^2 \mp 65n + 486)\mathrm{tr}F^6 + 15(n \mp 10)\mathrm{tr}F^4\,\mathrm{tr}F^2 \\ + 10(n \mp 8)(\mathrm{tr}F^3)^2 + 15(\mathrm{tr}F^2)^3 \,, \end{aligned} \tag{5.38}$$

$$\mathrm{tr}_{[3],(3)}F^4 = \frac{1}{2}(n^2 \mp 17n + 54)\mathrm{tr}F^4 + 3(n \mp 4)(\mathrm{tr}F^2)^2 \,, \tag{5.39}$$

$$\mathrm{tr}_{[3],(3)}F^2 = \frac{1}{2}(n^2 \mp 5n + 6)\mathrm{tr}F^2 \,. \tag{5.40}$$

Finally, for the representation $[4]$, we find that

$$\begin{aligned} \mathrm{tr}_{[4]}F^6 = \frac{1}{6}(n^3 - 99n^2 + 1556n - 6144)\mathrm{tr}F^6 + \frac{15}{2}(n^2 - 21n + 92)\mathrm{tr}F^4\,\mathrm{tr}F^2 \\ + 5(n^2 - 17n + 68)(\mathrm{tr}F^3)^2 + 15(n-6)(\mathrm{tr}F^2)^3 \,, \end{aligned} \tag{5.41}$$

$$\mathrm{tr}_{[4]}F^4 = \frac{1}{6}(n^3 - 27n^2 + 188n - 384)\mathrm{tr}F^4 + \frac{3}{2}(n^2 - 9n + 20)(\mathrm{tr}F^2)^2 \,, \tag{5.42}$$

$$\mathrm{tr}_{[4]}F^2 = \frac{1}{6}(n^3 - 9n^2 + 26n - 24)\mathrm{tr}F^2 \,. \tag{5.43}$$

However, for algebras with $n \leq 9$ there are more representations that the already worked out. Fortunately for us, properties of the Chern character come to the rescue since we can use the fact that the tensor product of representations is the direct sum of, in general, reducible representations, namely

$$\mathsf{r}_1 \otimes \mathsf{r}_2 = \oplus_i n_i \, \mathsf{r}_1 \tag{5.44}$$

under which

$$\mathrm{ch}_{\mathsf{r}_1}(F)\,\mathrm{ch}_{\mathsf{r}_2}(F) = \sum_i n_i \mathrm{ch}_{\mathsf{r}_i}(F). \tag{5.45}$$

Thus, by computing the tensor product of two known characters for representations r_1 and r_2 we can compute the character of an unknown representation by subtracting what we already know from the right-hand side of (5.45).

Consider the next simple example. Let $\mathsf{16}\otimes\mathsf{16}$ be the tensor product of the fundamental representation of $SU(16)$ which can be decomposed as follows

$$\mathsf{16} \otimes \mathsf{16} = \mathsf{120} \oplus \mathsf{136}, \tag{5.46}$$

By using (5.45) andand (5.34) for the representation $[2] = \mathsf{120}$ we find that

$$\mathrm{ch}_{\mathsf{136}=[3]}(F) = 136 + \frac{18}{2!}\mathrm{tr}(F)^2 + \frac{20}{3!}\mathrm{tr}(F)^3 + \frac{24}{4!}\mathrm{tr}(F)^4 + \frac{32}{5!}\mathrm{tr}(F)^5 + \frac{48}{6!}\mathrm{tr}(F)^6 + \cdots \tag{5.47}$$

where $\cdots$ represents higher-order traces and products of lower ones. Note also the agreement with equations in (5.34) for the representation $[3] = \mathsf{136}$. Note also that for lower order $SU(n)$ algebras one has to re-express higher-order traces in terms of lower-order ones. To see how to do this one can look at Ref. [214].

It is also useful to know the trace identities for the adjoint representation of $SU(n)$ given by (up to sixth order)

$$\begin{aligned}
\mathrm{tr}_{\mathsf{a}}\, F^6 &= 2n\,\mathrm{tr}\, F^6 + 30\,(\mathrm{tr}\, F^4)(\mathrm{tr}\, F^2) - 20(\mathrm{tr}\, F^3)^2\,, \\
\mathrm{tr}_{\mathsf{a}}\, F^4 &= 2n\,\mathrm{tr}\, F^4 + 6\,(\mathrm{tr}\, F^2)^2\,, \\
\mathrm{tr}_{\mathsf{a}}\, F^2 &= 2n\,\mathrm{tr}\, F^2\,,
\end{aligned} \tag{5.48}$$

As we said, the main challenge is the computation of indices, knowing that, the next step is the application of our algorithm for an anomaly polynomial, with some extra subtleties, as we did for the exceptional algebras. For the details, see [144] as well as for the remaining cases of the algebras B_n, C_n, and D_n.

Chapter 6

The dS conjecture and Hořava-Lifshitz theories

6.1 Anti de Sitter Conjecture

One of the most challenging tasks in string theory has been the construction of a 4d de Sitter (dS) space, in contrast to well-understood Anti-de Sitter (AdS) solutions [216]. There are two prominent dS solutions, one is known as KKLT [217] and the other is the Large Volume Scenario (LVS) [218, 219]. This search for a solution with a potential whose minima is a positive value plays a role in trying to explain the actual stage of our cosmological universe being connected to a cosmological constant responsible for its accelerated expansion. However, this fact of having such a few number of dS solutions makes it plausible to ask whether string theory may host (metastable) dS vacua within its Landscape of solutions. Indeed, this observation has led to the Swampland community to propose a lower bound on the derivative of potentials (more generally the gradient), establishing that [27]

$$\frac{|V_\phi|}{V} \geq c\,, \tag{6.1}$$

the scalar potential of an EFT coupled to gravity must satisfy a bound on its derivatives with respect to scalar fields, where c is a constant of order one. This was mainly based on observations on string theory constructions. Nevertheless, by more careful considerations of the entropy of dS space and the Distance conjecture, the above proposal was refined to the statement that [26, 28] the scalar potential of a theory coupled to gravity must satisfy either

$$\frac{|V_\phi|}{V} \geq c\,, \tag{6.2}$$

or

$$\frac{V_{\phi\phi}}{V} \leq -c'\,, \tag{6.3}$$

where c and c' are constants of order one. Often, string theory EFT constructions come with more than one scalar, so the left-hand side must be changed by the minimum eigenvalue of the Hessian of the potential.

In principle, we could apply these general statements to a theory coupled with gravity, hence this chapter aims to test these conjectures firstly with standard $f(R)$ theories and secondly with $F(\overline{R})$ Hořa-Lifshitz (HL) theories [33]. We found it interesting to test the dS conjectures on these models since, if this conjecture is ultimately true, it will be fundamental to rule out apparently consistent quantum theories of gravity. Otherwise, we have to find them a place in the Landscape of string theory.

6.1.1 Standard $f(R)$ theories

To start with, let us consider standard $f(R)$ theories [220], which are a generalization of General Relativity (GR), in the frame known as the Jordan frame

$$S = \frac{1}{2}\int d^4x\sqrt{-g}f(R)\,, \tag{6.4}$$

where $f(R)$ is some well-behaved function of the curvature scalar R without any matter component. Under a Taylor expansion of the f function around a small curvature, GR corresponds to the linear order in the expansion. Notice that we could consider the following equivalent action [221,222]

$$S = \frac{1}{2}\int d^4x\sqrt{-g}[\psi R - U]\,, \tag{6.5}$$

where $\psi = \frac{df}{dR} = f_R$ is just an auxiliary field and $U = R\psi - f(R)$. By considering a conformal transformation as follows $\widetilde{g}_{\mu\nu} = e^{2\varphi}g_{\mu\nu}$ with $\varphi = \frac{1}{2}\ln f_R$, we can obtain GR plus a canonically coupled scalar field whose action is now described in the Einstein frame by

$$S = \int d^4x\sqrt{-\widetilde{g}}\left[\frac{\widetilde{R}}{2} - \frac{1}{2}\widetilde{g}^{\mu\nu}\widetilde{\nabla}_\mu\phi\widetilde{\nabla}_\nu\phi - V(\phi)\right]\,, \tag{6.6}$$

where ϕ is a scalar field in terms of the curvature

$$\phi = \sqrt{\frac{3}{2}}\ln f_R\,, \tag{6.7}$$

with a potential given by

$$V(\phi) = \frac{Rf_R - f}{2f_R^2}\,. \tag{6.8}$$

We emphasize that the scalar field we are looking at is in terms of the Riemann curvature, we have not included any matter coupling. However, it is expected that the dS conjecture also applies to this kind of system because what we obtained in (6.4) is GR plus a scalar field along with a potential. Hence, we will consider this system under the dS Swampland conjecture, and after studying this problem, we will find that we can restrict the parameters of the theory, and correspondingly its solutions. In addition, following the dS conjecture (6.2) we will restrict the derivative of the potential to positive regions, and then we impose that

$$Rf_R - f > 0\,, \tag{6.9}$$

and from (6.7), we also note that $f_R > 0$. Under these considerations, we can apply the dS conjectures to the potential (6.8). From (6.2) we get the following

$$|2f - Rf_R| \geq \sqrt{\frac{3}{2}}c(Rf_R - f), \tag{6.10}$$

and (6.3) leads to

$$\frac{f_R^2 + f_{RR}(Rf_R - 4f)}{3f_{RR}(Rf_R - f)} < -\widetilde{c}\,. \tag{6.11}$$

One could consider an arbitrary function $f(R)$ in the above equation. However, if look for a setup where the conjecture (6.2) applies with no dependence on the curvature R and thus, valid for any curvature R, it is proposed that

$$\frac{|2f - Rf_R|}{Rf_R - f} = A\,, \tag{6.12}$$

where A is any constant that should be positive in order to achieve (6.9). Let us remark that this is an ansatz and therefore it is not the only way to reach (6.10), but it is the only way to fulfil it independently of R. For the case of $2f > Rf_R$, this allows us to determine that

$$f(R) = \beta R^{\alpha}\,, \tag{6.13}$$

where $\alpha = \frac{A+2}{A+1}$ with $1 < \alpha < 2$. To avoid problematic issues with this $f(R)$ function, it is imposed that $f_R > 0$ and $f_{RR} > 0$ [221]. Both conditions are consistent with $\beta > 0$. With this, we can substitute this form of the $f(R)$ into the first statement of the dS conjecture, and we get the following inequality for the α parameter

$$\alpha < \frac{2 + \sqrt{\frac{3}{2}}c}{1 + \sqrt{\frac{3}{2}}c}\,. \tag{6.14}$$

By using the fact that c is an order one constant, this leads to $\alpha \lesssim 1.45$. Therefore, we have found that $f(R) = \beta R^{1+0.45}$, which is saying that we should not be far from standard GR. Nevertheless, this does not exactly correspond to GR plus a scalar field. Scenarios such as this with $f(R) = R^{1+\epsilon}$, where ϵ is a small number were considered in [223]. It has been found that the conjectures were compatible with the region of ϵ of phenomenological interest. Importantly, the ansatz (6.12) and consistency with the dS conjecture have led to a function that could be within a region of phenomenological applications.

By considering the case $2f < Rf_R$, we obtain the same form but in this case $\alpha = \frac{A-2}{A-1}$ and thus $\alpha > 2$. This form can also fulfil the two conditions $f_R > 0$ and $f_{RR} > 0$ for $\beta > 0$. However, in this range of values α, the first dS conjecture can not be achieved.

On the other hand, substituting the form (6.13) into (6.3) we obtain that the second statement of the dS conjecture corresponds to

$$\frac{(\alpha - 2)^2}{3(\alpha - 1)^2} < -c'\,, \tag{6.15}$$

which is independent of R but cannot be satisfied for any value of α. Therefore, we have found that a power law f function is not compatible with the second statement of the dS conjecture. Recall that the refined dS conjecture is an either statement, therefore there is not a contradiction.

In the following, we shall consider the $F(\overline{R})$ Hořava-Lifshitz theories and the application of the dS conjectures to find out what kind of conclusions we can draw from that analysis.

6.1.2 $F(\overline{R})$ Hořava-Lifshitz theories

Hořava-Lifshitz theories have a natural formulation in terms of the ADM decomposition of spacetime where the metric is written in the general form

$$ds^2 = -N^2 dt^2 + g^{(3)}_{ij}(dx^i + N^i dt)(dx^i + N^i dt)\,, \tag{6.16}$$

where N is the lapse function, N^i the shift functions and $g_{ij}^{(3)}$ the three-metric. We point out that the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric has this form with vanishing shift functions and it is always possible to choose the lapse function to be equal to one. In this formulation, the Hořava-Lifshitz action [224–228] describing gravity can be expressed in terms of an extrinsic curvature denoted as K_{ij}, a parameter λ or coupling constant, and a potential term for gravity denoted as $\mathcal{L}(g_{ij}^{(3)})$. This term is generally written in terms of the scalar curvature $R^{(3)}$ of the three metric $g_{ij}^{(3)}$ and seven constants accompanying higher spatial derivative terms written in terms of the Ricci scalar of the spatial three-metric. In our investigation, we will consider an FLRW universe and for that case, the specific form of the potential does not matter, it always vanishes thus, for our purpose we can ignore it. In the ADM formulation, it is natural to ask for a Hamiltonian constraint. For HL theories this constraint can have a global nature determined by the time-dependent lapse function. We say that in this case, we are in the projectable version of the theory. Other versions appear when the lapse function may also depend on the spatial variables. The upshot is that the equation of motion can be modified by a boundary term due to the global nature of the Hamiltonian constraint, implying that in one version of the theory may or may not be and this is important later. For now, however, we emphasize that this theory has also been generalize to an $F(\overline{R})$ theory whose general form is

$$S_{F(\overline{R})} = \int d^4x \sqrt{g^{(3)}} N F(\bar{R}) \,, \tag{6.17}$$

where now $\bar{R}$ can be understood as a generalization of R which includes the new terms of spatial derivatives which are proposed to have the form [229, 230]

$$\overline{R} = K^{ij}K_{ij} - \lambda K^2 + 2\mu \nabla_\rho (n^\rho \nabla_\sigma n^\sigma - n^\sigma \nabla_\sigma n^\rho) + \mathcal{L}(g_{ij}^{(3)}) \,, \tag{6.18}$$

where μ is a constant. The term containing μ is usually omitted in the standard $f(R)$ theory since it turns out to be a total derivative term. However, it is necessary for these theories. As we can see for the above definitions, the limit $\lambda \to 1$, $\mu \to 1$ of this theory will lead to the standard $f(R)$ theory used in the previous section. It is also pointed out that this general theory has been used along with the FLRW metric resulting in interesting cosmological scenarios, see e.g. [229–231].

As we said, we will also consider a flat FLRW metric which can be expressed as

$$ds^2 = -dt^2 + a^2(t)\left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] \,, \tag{6.19}$$

from which follows that (6.18) can be rewritten as

$$\overline{R} = (3 - 9\lambda + 18\mu)H^2 + 6\mu \frac{d}{dt}(H) \,, \tag{6.20}$$

where $H = \frac{1}{a}\frac{da}{dt}$ is the Hubble parameter. On the other hand, considering only gravity, the first equation of motion obtained through the Hamiltonian constraint is [229]

$$F(\overline{R}) - 6\left[(1 - 3\lambda + 3\mu)H^2 + \mu\dot{H}\right] F'(\overline{R}) + 6\mu H \frac{dF'(\overline{R})}{dt} - \frac{C}{a^3} = 0 \,, \tag{6.21}$$

where $\dot{H} = \frac{dH}{dt}$, $F' = \frac{dF}{d\overline{R}}$ and where $C \neq 0$ is an integration constant in the projectable version of the theory, while the non-projectable version of the theory $C = 0$ and this is the only difference between both versions that will be relevant for us. While, variation with respect to $g_{ij}^{(3)}$, its equation of motion yields [229]

$$F(\overline{R})-2(1-3\lambda+3\mu)(\dot{H}+3H^2)F'(\overline{R})+2(3\lambda-1)H\frac{dF'(\overline{R})}{dt}+2\mu\frac{d^2F'(\overline{R})}{dt^2}=0\,. \quad (6.22)$$

This set of equations looks more involved than the standard $f(R)$ case, nonetheless, in the limit $\mu, \lambda \to 1$ this theory can be recovered. In addition, this set of equations will play an important role in subsequent discussions.

Before, let us consider the $F(\overline{R})$ action (6.17) which can be rewritten as in the $f(R)$ case but now in terms of the three-metric as

$$S=\frac{1}{2}\int d^4x\sqrt{g^{(3)}}N\left[\bar{R}F'(\bar{R})-U\right], \quad (6.23)$$

with $U=\bar{R}F'(\bar{R})-F(\bar{R})$. Now, by performing a conformal transformation in the three-metric, $g_{ij}^{(3)}=e^{-\bar{\phi}}\widetilde{g}_{ij}^{(3)}$, and choosing $N=1$ and N^i as we already did for FLRW, we can obtain an action in the Einstein frame as [230,232]

$$S=\frac{1}{2}\int d^4x\sqrt{\widetilde{g}^{(3)}}\left[\widetilde{K}^{ij}\widetilde{K}_{ij}-\lambda\widetilde{K}^2+\left(-\frac{1}{2}+\frac{3\lambda}{2}-\frac{3\mu}{2}\right)\widetilde{g}^{(3)ij}\dot{\widetilde{g}}_{ij}^{(3)}\dot{\bar{\phi}}\right.$$
$$\left.+\left(\frac{3}{4}-\frac{9\lambda}{4}+\frac{9\mu}{2}\right)\dot{\bar{\phi}}^2-\mathcal{L}(e^{-\bar{\phi}}g_{ij}^{(3)})-2V(\bar{\phi})\right], \quad (6.24)$$

with

$$\bar{\phi}=\frac{2}{3}\ln F'(\overline{R})\,, \qquad V(\bar{\phi})=\frac{\overline{R}F'-F}{2F'}\,. \quad (6.25)$$

Notice that if we choose $\mu=\lambda-\frac{1}{3}$, then

$$\phi=\alpha\bar{\phi}=\frac{2\alpha}{3}\ln F'(\overline{R})\,, \qquad \alpha=\frac{\sqrt{3(3\lambda-1)}}{2}\,. \quad (6.26)$$

Hence, this allows us to recast (6.24) into the following form

$$S=\int d^4x\sqrt{\widetilde{g}^{(3)}}\left[\frac{\widetilde{K}^{ij}\widetilde{K}_{ij}-\lambda\widetilde{K}^2-\mathcal{L}(e^{-\bar{\phi}}g_{ij}^{(3)})}{2}+\frac{\dot{\phi}^2}{2}-V(\phi)\right]. \quad (6.27)$$

Thus, in the Einstein frame, we have obtained an action of gravity plus a canonically coupled scalar field, so this system can be tested with the dS conjecture and this is done momentarily. We might also consider the general case given by (6.24) although, this is not in the canonically normalized way we would wish. Thus, the dS conjectures, which is our concern, could not be applicable. However, we note that we also get a potential for that general case, which is in principle what we need to test the conjectures, Therefore, it is proposed that that case can also be analyzed. Even though, the details are not presented here, they can be consulted in [33].

Hence, we are already in the position to apply the dS conjecture (6.2) to the limiting case we have obtained by choosing $\mu=\lambda-\frac{1}{3}$, from which we find that

$$\frac{|V_\phi|}{V}=\frac{3}{2\alpha}\frac{|F|}{\overline{R}F'-F}>c\,, \quad (6.28)$$

and as before, we work in the positive region of the potential. On the other hand, the second dS conjecture takes the general form given by

$$\frac{V_{\phi\phi}}{V} = \frac{9}{4\alpha^2}\frac{(F')^2 - F''F}{F''(\overline{R}F' - F)} < -c' . \tag{6.29}$$

We may proceed as in the standard $f(R)$ case, noting that the conjectures could be made independent of $\overline{R}$ by considering a power law function. But, rather than that, we will use the equations of motion for the $F(\overline{R})$ theory in the Jordan frame for our choice of $\mu = \lambda - \frac{1}{3}$ as well as an ansatz for the scale factor or equivalently for the Hubble parameter of cosmological interest, and then we construct the $F(\overline{R})$. Finally, we will check if this function can fulfill the conjectures at the same time that leads to cosmological interesting solutions.

Non-projectable case

Let us start with the non-projectable case, that is, $C = 0$. The simplest ansatz of cosmological interest one could consider is a constant Hubble parameter. However, for the limiting case in which we are working, this choice leads to a vanishing F function. Therefore, we instead propose to study an accelerated expanding universe in the form of a power law with the time parameter, that is

$$a(t) = t^n , \tag{6.30}$$

where n is a constant that will be considered positive. Thus the Hubble parameter is $H = n/t$. With this ansatz using (6.20) we obtain that $\overline{R}$ is related to the time variable as

$$\overline{R} = \frac{(3\lambda - 1)n(3n-2)}{t^2}. \tag{6.31}$$

We also note from (6.26) that we must always have $3\lambda > 1$ so that the scalar field is properly defined. We will generally be interested in values of n that describe accelerating expanding universes and therefore from the above equation we further note that considering positive values for $\overline{R}$ we obtain the condition $3n - 2 > 0$. Therefore, $\overline{R}$ grows inversely with time, so the behaviour is similar to what we could expect of the curvature.

After we choose the limiting or reduced case, $\mu = \lambda - \frac{1}{3}$, equations (6.21) and (6.22) can be combined into the following second-order differential equation in time derivative (we also used the non-projectable condition)

$$t^2 \frac{d^2}{dt^2} F'(\overline{R}) - 3nF'(\overline{R}) = 0 , \tag{6.32}$$

which can be solved, giving us the following solution

$$F'(\overline{R}) = c_1 t^{\alpha_+} + c_2 t^{\alpha_-} , \tag{6.33}$$

where c_1 and c_2 are integration constants and

$$\alpha_\pm = \frac{1}{2}\left[1 \pm \sqrt{1+12n}\right] . \tag{6.34}$$

Using (6.31) we can find the form of F as a function of $\overline{R}$, we obtain

$$F(\overline{R}) = -\frac{A_1}{\beta_+ \overline{R}^{\beta_+}} + \frac{A_2}{\beta_-}\overline{R}^{\beta_-} , \tag{6.35}$$

where we have defined the positive constants

$$\beta_+ = \frac{\alpha_+}{2} - 1 = \frac{1}{4}\left[\sqrt{1+12n} - 3\right] \geq 0, \quad \beta_- = 1 - \frac{\alpha_-}{2} = \frac{1}{4}\left[\sqrt{1+12n} + 3\right] > \frac{3}{2}. \tag{6.36}$$

The last inequality follows from the condition $3n - 2 > 0$ in both cases and we have also defined

$$A_1 = c_1 \left[(3\lambda - 1)n(3n-2)\right]^{\beta_+ + 1}, \quad A_2 = \frac{c_2}{\left[(3\lambda - 1)n(3n-2)\right]^{\beta_- - 1}}. \tag{6.37}$$

This general solution contains two terms of powers of $\overline{R}$ and thus we expect that it can fulfill the dS conjecture for any of the terms taken independently. The condition to have a non-negative potential for the scalar field in the Einstein frame in this case takes the form

$$\overline{R}F' - F = \frac{A_1}{\overline{R}^{\beta_+}}\left(1 + \frac{1}{\beta_+}\right) + A_2 \overline{R}^{\beta_-}\left(1 - \frac{1}{\beta_-}\right) > 0. \tag{6.38}$$

Moreover, from (6.36) we obtain that

$$1 + \frac{1}{\beta_+} = \frac{\sqrt{1+12n} + 1}{\sqrt{1+12n} - 3} > 0, \quad 1 - \frac{1}{\beta_-} = \frac{\sqrt{1+12n} - 1}{\sqrt{1+12n} + 3} > 0. \tag{6.39}$$

Thus, the condition (6.38) can be easily fulfilled by taking positive values for the integration constants c_1 and c_2. In this case, we obtain

$$|F| = \left| -\frac{A_1}{\beta_+ \overline{R}^{\beta_+}} + \frac{A_2}{\beta_-}\overline{R}^{\beta_-} \right|. \tag{6.40}$$

Since there is a minus sign in the first term the F function cannot have a definite sign for all values of $\bar{R}$, and thus we cannot fulfill the dS conjecture for all values of $\bar{R}$ if we consider both terms at the same time as we anticipated. Thus let us consider each term separately.

Considering first the positive power factor on $F(\overline{R})$, namely $c_1 = 0$ and $c_2 > 0$, then the dS conjecture (6.2) gives rise to

$$\frac{1}{3} < \lambda < \frac{1}{3} + \frac{16}{c^2(\sqrt{1+12n} - 1)^2}. \tag{6.41}$$

In the other case, if we consider the negative power factor on $F(\overline{R})$ by choosing $c_1 > 0$ and $c_2 = 0$ the conjecture leads to

$$\frac{1}{3} < \lambda < \frac{1}{3} + \frac{16}{c^2(\sqrt{1+12n} + 1)^2}. \tag{6.42}$$

These two cases are consistent with the positive-region condition for the potential as can be checked from $\overline{R}F' - F > 0$. Thus, the dS conjecture leads in both cases to an inequality for the HL parameter λ. Further, note that in the limit $n \to \infty$, we get that $\lambda = 1/3$. Thus in order to satisfy the conjecture independently of $\overline{R}$ and in order to have a fast expansion, we obtain that λ must be bigger but close to $1/3$ and thus away from its IR limit value $\lambda = 1$. Also, in both cases the first dS conjecture leads to a region of validity for the λ parameter and since neither β_+ nor β_- in (6.35) depend on λ, the form of the $F(\bar{R})$ function is not constrained by the conjecture, it only depends on n, and thus we have the

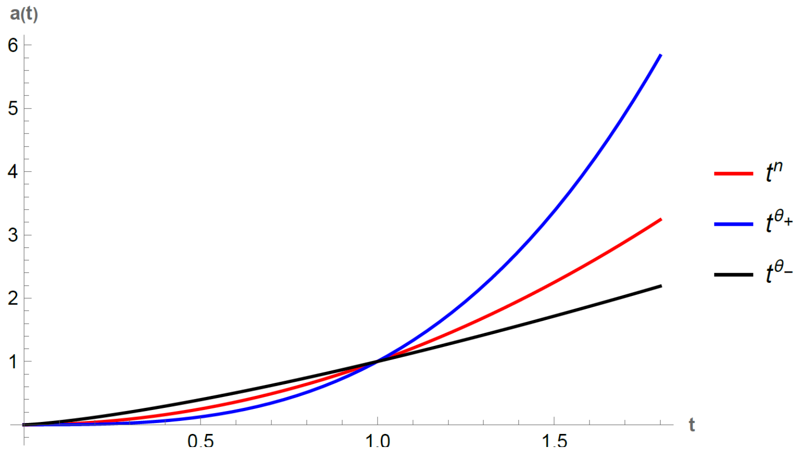


Figure 6.1: Profile of the scale factor in the Jordan frame (red curve) and in the Einstein frame for negative power of the curvature in F (blue curve) and with positive power (black curve).

freedom to choose any positive values of interest for these terms. This is in contrast to the standard $f(R)$ case.

On the other hand, we can apply (6.3), the second statement of the dS conjecture. In both cases, either $c_1 = 0$ or $c_2 = 0$, we obtain that it is never satisfied. Thus, the $F(\overline{R})$ function we have constructed is not compatible with the second statement of the dS conjecture. Therefore, we conclude that for the power-law expansion, we have considered the first statement of dS conjectures can be fulfilled for each term on the solution of F independently of $\bar{R}$, leading to an inequality for the HL parameter λ which is in agreement with the difficulties of achieving this scenario in GR, namely it leads us to the opposite of the infrared limit, that is, to the UV limit $\lambda \to 1/3$.

One can also consider the behaviour of the scale factor, and therefore of the corresponding expansion of the universe for the profiles of the curvature found before, namely for the negative or positive power of $\overline{R}$. By the conformal transformation relating both frames, the Jordan and Einstein frame, we find that the scale factor can be expressed as

$$\widetilde{a}(t) = t^n \left[c_1 t^{\frac{1}{2}\left(1+\sqrt{1+12n}\right)} + c_2 t^{\frac{1}{2}\left(1-\sqrt{1+12n}\right)} \right]^{1/3}, \tag{6.43}$$

where each term can be studied separately. The behaviour of this is shown in Figure 6.1 for $n = 2$ where it shows that for $t > 1$ the expansion is faster for negative power of the curvature in F than for negative power, even faster than what we see in the Jordan frame. For $t < 1$ this is flipped as we can see from the Figure 6.1.

Projectable case

We can consider the projectable case which is characterized by a nonvanishing constant C. Instead of (6.32), thus, the equation we are led is

$$\frac{d^2}{dt^2} F'(\bar{R}) - \frac{3n}{t^2} F'(\bar{R}) + \frac{3C}{2(3\lambda - 1)t^{3n}} = 0. \tag{6.44}$$

by using also the power-law scale factor in the temporal coordinate. By a similar analysis as before, we obtain a set of inequalities by using the first statement of the dS conjecture given by [33]

$$\lambda < \frac{1}{3} + \frac{16}{c^2(\sqrt{1+12n}-1)^2}, \tag{6.45}$$

$$\lambda < \frac{1}{3} + \frac{4}{c^2(3n-2)^2}. \tag{6.46}$$

From this, one can deduce that the most restricted condition is (6.46) for most of the range of values of n. while for $n < (2+\sqrt{2})/2$ it is most restricted (6.46). Thus, the dS conjecture is satisfied for (6.46) for most of the values of n for every value of the curvature. On the other hand, one could also check the second statement of the conjecture. In the end, we find that it cannot be analyzed for all values of the curvature in general. It is also possible to study the behaviour of the scale factor and in this case, one finds that the constant C increases the scale factor, making the expansion faster.

Chapter 7

Conclusions

In this thesis we have mainly focus on the cancellation of perturbative anomalies associated to infinitesimal local gauge transformations, connected with the identity of a gauge group. In particular, we have considered the topological terms of M-theory on a manifold with boundary. In this setup, these terms are ill defined given a contribution on the boundary. The question we have answered is whether there can be nontrivial boundary modes such that they contribute an anomaly proportional to the bulk anomaly. By doing this we have reproduced the E_8 M9-brane worlvolume theory of Hořava and Witten. We have also explicitly determined that any M9-brane of M-theory with a G worldvolume theory, with $G = G_2, F_4, E_6$, and E_7, exceptional groups, inherits the restricted Green-Schwarz anomaly polynomial factorization from E_8. Otherwise, the anomaly polynomial associated to the nontrivial edge modes does not factorize as we were searching for an M9-brane interpretation. We consider this as a nontrivial result because this firmly establishes that there is no other M-theory boundary solutions than the already known E_8 brane. Some preliminary results seems to suggest that this is also the case for classical Lie algebras. For this aim, we have to extend the computation of indices for the relevant representations to be considered subject to the assumed chirality condition in this work. This indices are usually found for a limited amount of groups and for traces of fourth-order only. In this thesis, we have computed explicitly the indices for the representations $[2], [3], [4]$ and for (2) and (3) at all orders for the Lie algebra A_{n-1} (Lie algebra of $SU(n)$ group). This is the main challenge for the searching of new M9-branes solutions.

We remark that the kinematic restrictions we have studied are very general in the sense that they do not rely upon any constraint imposed by supersymmetry. Nevertheless, the algorithm we have developed contains as a solution the supersymmetric case. The unique fundamental assumption of our search is a chiral matter content whose anomaly polynomial allows a Green-Schwarz factorization. This is highly restrictive, but it is the unique and best mechanism we have to cancel anomalies in string theory/M-theory. Indeed, this method is so fundamental that have been extended to non-perturbative anomaly cancellation [197] (see also [198]). However, as we discussed in Chapter 3 perturbative anomaly cancellation is not the unique kinematic restriction coming from the study of anomalies. In addition, we have to consider nonperturbative (or global) anomalies associated to the global structure of the diffeomorphism group (or Lorentz) as well as the G gauge group. In this direction, another important result of a our work is that we have limited that more difficult task of nonperturbative anomalies to the E_8 gauge group only. That is, for that task it seems that we have to compute eleven-dimensional *relative* bordism groups [198] for a boundary structure $\zeta_1 = \mathrm{string} \times E_8$ and a bulk structure $\zeta_2 = \mathrm{spin}$ (or a more general structure, denoted as m^c [182, 185] due to the flux quatization condition obeyed by the 4-form G), namely $\Omega^{\zeta_1/\zeta_2}(\mathrm{pt})$ which is a highly nontrivial aim, out of the scope of this thesis (but see the discussion at the end of Chapter 3).

Furthermore, on the one hand, it is well worth to mention that we have found anomaly polynomials for the group G_2 that did not factorize as in the restricted Green-Schwarz mechanism we considered in Chapter 5. However, some of the solutions in Table 5.3 obey the standard Green-Schwarz method of anomaly cancellation as we reviewed in Chapter 3 which is sufficient for the quantum consistency of a theory under perturbative anomalies. This might suggest new nonsupersymmetric anomaly-free theories in ten dimensions. On the other hand, in Ref. [89] was established that the list of non-supersymmetric ten-dimensional heterotic supergravities was already completed from the 1980s [66, 170, 233, 234]. This was donde by considering the worldsheet realization of heterotic strings via a chiral (0,1) fermionic QFT with central charge equal to 16 (the supersymmetric cases are related to bosonic chiral QFT with central charge equal to 16). This approach is indeed more powerful than that developed here because it does not care about the smoothness of the background, where this is a crucial ingredient to use the index description of anomalies. Therefore, this apparently consistent theories with perturbative anomaly cancellation that we have found must be ruled out somehow. One possibility is by considering nonperturbative anomalies. This requires the calculation of the string bordism group $\Omega_{11}^{\text{string}}(BG_2)$ to determine whether or not there is an anomaly encoded in this group, see [136]. Other possibility could be a more sophisticated dynamical argument. We hope to make progress in either of this directions in the future.

On the other hand, we want to add that our presentation of the anomaly polynomials associated with the anomalous degrees of freedom we considered along the whole text has been in terms of characteristic classes. Although, this is not usual in the physics literature, it is convenient to appreciate the structure encoded in a characteristic class as we discuss in Chapter 3. It is more common to work with the trace representation of the classes in terms of linear combinations of $\mathrm{tr}(F)^n$ and $\mathrm{tr}(R)^n$ by means of its representation in cohomology by a characteristic polynomial in the curvatures. We present here the characteristic polynomial for the Chern class which is heavily used in the text in case the reader wishes to recover what is usually found in standard textbooks, for example [68, 70]. In a compact way, that characteristic polynomial is given by

$$\sum_j c_{j,\mathsf{r}} s^j = \det\left(1 + \frac{\mathrm{i}F}{2\pi}s\right) , \tag{7.1}$$

where s is only a parameter. By a Taylor expansion of the determinant, we can find that

$$\sum_j c_{j,\mathsf{r}} s^j = 1 + \mathrm{i}\frac{\mathrm{tr}_{\mathsf{r}} F}{2\pi} s + \frac{\mathrm{tr}_{\mathsf{r}} F^2 - (\mathrm{tr}_{\mathsf{r}} F)^2}{2(2\pi)^2} s^2 + \cdots \tag{7.2}$$

Expanding this to the appropriate order you can plug this into the anomaly polynomials in the main text as well as what we have given for the Pontrjagin[1] classes in equation (B.27)

[1]For completeness, one can determine the set of equations given in (B.27) using (define $\frac{R}{2\pi} \to R$)

$$\sum_j p_{j,\mathsf{r}} s^j = \det\left(1 + \frac{R}{2\pi}s\right)$$

one can recover usual presentations of anomaly polynomials. This cohomology representation of characteristic classes is known as Chern-Weil theory. We hope that this serve as a motivation to learn this and other mathematical tools heavily used in our modern viewpoint of anomalies as well as other branches of Mathematical Physics.

In an unrelated chapter to symmetries and anomalies, we have presented the results of applying the de Sitter (dS) conjectures to $F(\overline{R})$ theories of Hořava-Lifshitz in a background with FLRW metric. We basically focus on the case where by doing a conformal transformation from the Jordan frame to the Einstein frame, we obtain a theory of gravity plus a canonically coupled scalar field. This is done by appropriately choosing the parameters of the theory. Although, the more general case can also be studied [33]. We should mention that this field comes from geometrical information instead of introducing some explicit matter field. Also, by this manipulation we get a potential for this theory, thus we can apply the dS conjecture to this setting. Before, we solve the equations of motion of the $F(\overline{R})$ theory in the Jordan frame using a power-law ansatz for the Hubble parameter instead of arbitrarily choosing some particular form. With this, we find F functions as power-law in terms of the curvature R. In the end, by applying the dS conjectures we are able to determine a set of inequalities for the parameters of the theory suggesting that in the limit of a increasingly faster expansion, the parameter that controls the UV description of the Hořava-Lifshitz theory has to approach to the respective UV-value, $\lambda \to 1/3$. This is presented for the non-projectable and projectable version of the theory. On the other hand, we also present the behaviour of the scale factor in the Einstein frame by doing a conformal transformation. This has been depicted in Figuere 6.1 where is shown the profile of the expansion. To sum up, we have found that the (first statement) of the dS conjecture is consistent with a power-law F functions and the UV behaviour of this theories. The second statement of the dS conjecture is applied, but this is never fulfilled.

Appendix A

Group theory representation

A.1 Needed of Representation Theory of Lie Groups

In this appendix, we review what we need from group theory. In particular, we will spend some time on the representation theory of groups and how to measure the algebraic properties of the representation space. This plays an important role in studying anomalies of a theory where the physical content is in a given representation.

The anomaly polynomial density $[\hat{A}(R)\mathrm{ch}_{\mathsf{r}}(F)]|_{2n+2}$ involves traces of matrices in some representation r of a continuous symmetry group G, a classical or exceptional Lie group for our purposes. In order to find mechanisms to cancel anomalies encoded in the anomaly polynomial one needs to deal with higher-order traces of the generators of the Lie algebras $\mathcal{L}(G)$ represented in some representation. It is important to study if these traces admit some kind of factorization. For a ten-dimensional system, we encounter sixth-order traces of the matrix-valued two-form field strength F. Thus, if we look for a factorization mechanism we have to explore group theory identities relating traces of different orders. To this end, we are going to review the material developed in [209,210]. Another useful reference for the computation of indices is [214].

Let r be an arbitrary irreducible representation and f be an irreducible reference representation of a Lie algebra $\mathcal{L}(G)$ and for simplicity we will focus here on exceptional Lie algebras. Therefore, we need to determine the coefficients in the trace identities relating higher-order traces in an arbitrary representation to traces in the reference representation and products of lower ones as follows

$$\mathrm{tr}_{\mathsf{r}}\, F^6 = \mathrm{u}_{\mathsf{r}}\, \mathrm{tr}\, F^6 + \mathrm{v}_{\mathsf{r}}\, (\mathrm{tr}\, F^4)(\mathrm{tr}\, F^2) + \mathrm{w}_{\mathsf{r}}\, (\mathrm{tr}\, F^2)^3 \tag{A.1}$$

$$\mathrm{tr}_{\mathsf{r}}\, F^4 = \mathrm{x}_{\mathsf{r}}\, \mathrm{tr}\, F^4 + \mathrm{y}_{\mathsf{r}}\, (\mathrm{tr}\, F^2)^2\,, \tag{A.2}$$

$$\mathrm{tr}_{\mathsf{r}}\, F^2 = \mathrm{z}_{\mathsf{r}}\, \mathrm{tr}\, F^2\,, \tag{A.3}$$

which are related to eigenvalues of Casimir invariants of r and correspondingly to $2n^{\mathrm{th}}$ (modified) order index, which are defined as

$$\ell_{2n}(\mathsf{r}) = \sum_{w\in\Delta(\mathsf{r})} (w,w)^{2n}\,, \tag{A.4}$$

where $\Delta(\mathsf{r})$ is the weight system of the representation r and the sum is over all the weights w, $(\cdot,\cdot)$ stands for a nondegenerate, symmetric bilinear product.

It turns out that for r and f it is satisfied the relation

$$\begin{aligned}\mathrm{tr}_{\mathsf{r}}\, F^2 &= \frac{\ell_2(\mathsf{r})}{\ell_2(\mathsf{f})}\mathrm{tr}_{\mathsf{f}}\, F^2\,,\\ &= \frac{\ell_2(\mathsf{r})}{\ell_2(\mathsf{f})}\, \mathrm{z}_{\mathsf{f}}\, \mathrm{tr}\, F^2\,,\end{aligned} \tag{A.5}$$

where in the second equality we have used Eq. (A.3) for the reference representation so that the second-order trace in the representation r can be written in the reference representation with coefficient given by

$$\mathrm{z}_{\mathsf{r}} = \frac{\ell_2(\mathsf{r})}{\ell_2(\mathsf{f})}\, \mathrm{z}_{\mathsf{f}}\,. \tag{A.6}$$

Note that if the reference representation is restricted to be the fundamental or defining representation then $\mathrm{z}_{\mathsf{f}} = 1$ and the second-order index can be normalized so that $\ell_2(\mathsf{r})/\ell_2(\mathsf{f}) = 1$ for $\mathsf{r} = \mathsf{f}$. In fact, the second-order index is related to the second-order Casimir invariant as follows (up to some normalization). The second-order index is given by

$$\ell_2(\mathsf{r}) = \frac{\dim \mathsf{r}}{\dim \mathsf{a}}\, I_2(\mathsf{r})\,, \tag{A.7}$$

where $I_2(\mathsf{r})$ is the second-order Casimir invariant. For the coefficients of fourth-order traces in the representation r one has a relation given by

$$\frac{\mathrm{tr}_{\mathsf{r}}\, F^4 - \mathrm{k}(\mathsf{r})\, \mathrm{tr}_{\mathsf{r}}\, (F^2)^2}{\mathrm{tr}_{\mathsf{f}}\, F^4 - \mathrm{k}(\mathsf{f})\, \mathrm{tr}_{\mathsf{f}}\, (F^2)^2} = \frac{\bar{\ell}_4(\mathsf{r})}{\bar{\ell}_4(\mathsf{f})}\,, \quad \text{with} \quad \mathrm{k}(\mathsf{r}) = \frac{3}{2+\dim \mathsf{a}}\left(\frac{\dim \mathsf{a}}{\dim \mathsf{r}} - \frac{1}{6}\frac{\ell_2(\mathsf{a})}{\ell_2(\mathsf{r})}\right)\,, \tag{A.8}$$

where $\bar{\ell}_4(\mathsf{r})$ is the modified fourth-order index given in terms of the fourth- and second-order index as defined in Eq. (A.4) and the rank r of the algebra, i.e.

$$\bar{\ell}_4(\mathsf{r}) = \ell_4(\mathsf{r}) - \frac{(r+2)}{r}\frac{\dim \mathsf{a}}{2+\dim \mathsf{a}}\left(\frac{\ell_2(\mathsf{r})}{\dim \mathsf{r}} - \frac{1}{6}\frac{\ell_2(\mathsf{a})}{\dim \mathsf{a}}\right)\ell_2(\mathsf{r})\,. \tag{A.9}$$

It was determined that $\bar{\ell}_4(\mathsf{r})$ is more fundamental since it is related to *genuine* fourth-order Casimir invariants J_4 in the sense that if $\bar{\ell}_4(\mathsf{r})$ vanishes then any other fourth-order Casimir invariant is given by the square of second-order Casimir invariant. There are no independent fourth-order Casimir invariants. This happens for exceptional and the $A_{r=1,2}$ algebras [210] leading to the fact that for this algebras

$$\begin{aligned}\mathrm{tr}_{\mathsf{r}}\, F^4 &= \mathrm{k}(\mathsf{r})\, \mathrm{tr}_{\mathsf{r}}\, (F^2)^2\,,\\ &= \mathrm{z}_{\mathsf{r}}^2\, \mathrm{k}(\mathsf{r})\, (\mathrm{tr}\, F^2)^2\,,\end{aligned} \tag{A.10}$$

so that $\mathrm{y}_{\mathsf{r}} = \mathrm{z}_{\mathsf{r}}^2\, \mathrm{k}(\mathsf{r})$ given a way to compute other coefficient in Eq. (A.2). For any other algebra, it is found from Eq. (A.8) that

$$\mathrm{x}_{\mathsf{r}} = \frac{\bar{\ell}_4(\mathsf{r})}{\bar{\ell}_4(\mathsf{f})}\, \mathrm{x}_{\mathsf{f}}\,, \tag{A.11}$$

$$\mathrm{y}_{\mathsf{r}} = \frac{\bar{\ell}_4(\mathsf{r})}{\bar{\ell}_4(\mathsf{f})}\, \mathrm{y}_{\mathsf{f}} + \mathrm{k}(\mathsf{r})\, \mathrm{z}_{\mathsf{r}}^2 - \frac{\bar{\ell}_4(\mathsf{r})}{\bar{\ell}_4(\mathsf{f})}\, \mathrm{k}(\mathsf{f})\, \mathrm{z}_{\mathsf{f}}^2\,, \tag{A.12}$$

meaning that for other algebras than the exceptional ones and the $A_{r=1,2}$, we have to compute modified indices or equivalently modified Casimir invariants.

To go to sixth-order trace identities firstly set

$$\mathrm{S}(\mathsf{r}) = \frac{1}{8+\dim \mathsf{a}}\left[\frac{\dim \mathsf{a}}{\dim \mathsf{r}} - \frac{1}{3}\frac{\ell_2(\mathsf{a})}{\ell_2(\mathsf{r})} + \frac{1}{30}\frac{\bar{\ell}_4(\mathsf{a})}{\bar{\ell}_4(\mathsf{r})}\right], \tag{A.13}$$

$$\mathrm{T}(\mathsf{r}) = \frac{1}{4+\dim \mathsf{a}}\left[\left(\frac{\dim \mathsf{a}}{\dim \mathsf{r}}\right)^2 - \frac{1}{2}\frac{\dim \mathsf{a}}{\dim \mathsf{r}}\frac{\ell_2(\mathsf{a})}{\ell_2(\mathsf{r})} + \frac{1}{12}\left(\frac{\ell_2(\mathsf{a})}{\ell_2(\mathsf{r})}\right)^2\right] \tag{A.14}$$

So far we have been quite general writing down formulas to determine the index coefficients in Eqs. (A.1), (A.2) and (A.3) in the sense that they apply to any Lie algebra (up to some exceptions), the reference representation can also be considered arbitrary. In the following to avoid writing down many long equations we will restrict the reference representation to be the fundamental [209]. Therefore, from the sixth-order trace identity of [209] we have a relation where $\bar{\ell}_6(\mathsf{r})$ is again related to modified sixth-order Casimir invariants. Two special cases are the A_1 and E_8 algebras since there are no fundamental sixth- and fourth-order Casimir invariants thus, we can show that their proportionality coefficients reduce to

$$\mathrm{tr}_\mathsf{r} F^6 = \frac{15\,\mathrm{T}(\mathsf{r})}{2+\dim \mathsf{a}}(\mathrm{tr}_\mathsf{r} F^2)^3, \tag{A.15}$$

$$= \frac{15\,\mathrm{T}(\mathsf{r})}{2+\dim \mathsf{a}}\,\mathrm{z}_\mathsf{r}^3\,(\mathrm{tr}\, F^2)^3, \tag{A.16}$$

where $\mathrm{T}(\mathsf{r})$ is defined in equation (A.14). From the above equation, we can extract the w_r coefficient in terms of the second-order index. However, this is not the case for other exceptional algebras meaning that they have genuine sixth-order Casimir invariants but not four-order, i.e. they satisfy the identity $\mathrm{tr}_\mathsf{r} F^4 - \mathrm{K}(\mathsf{r})\,(\mathrm{tr}_\mathsf{r} F^2)^2 = 0$. Thus, we find out that the only relevant coefficients are

$$\mathrm{u}_\mathsf{r} = \frac{\bar{\ell}_6(\mathsf{r})}{\bar{\ell}_6(\mathsf{f})}, \tag{A.17}$$

$$\mathrm{w}_\mathsf{r} = \frac{15}{2+\dim \mathsf{a}}\left(\mathrm{z}_\mathsf{r}^3\,\mathrm{T}(\mathsf{r}) - \frac{\bar{\ell}_6(\mathsf{r})}{\bar{\ell}_6(\mathsf{f})}\,\mathrm{T}(\mathsf{f})\right), \tag{A.18}$$

where

$$\bar{\ell}_6(\mathsf{r}_i) = \ell_6(\mathsf{r}_i) - \frac{\mathrm{T}(\mathsf{r}_i)}{2+\dim \mathsf{r}_i}\frac{(r+2)(r+4)}{r^2}(\ell_2(\mathsf{r}_i))^2 \tag{A.19}$$

After using the material reviewed above we get the following group-theoretical constant u_r, w_r, y_r and z_r for exceptional algebras

G	r	u	w	y	z
G_2	7	1	–	$\frac{1}{4}$	1
	14	-26	$\frac{15}{4}$	$\frac{5}{2}$	4
	27	39	$\frac{15}{4}$	$\frac{27}{4}$	9
	64	-208	75	38	32
	77	494	$\frac{315}{4}$	$\frac{121}{2}$	44
	$77'$	-1235	$\frac{1275}{4}$	$\frac{385}{4}$	55
	182	3666	$\frac{2925}{4}$	$\frac{663}{2}$	156
	189	-456	735	270	144
F_4	26	1	–	$\frac{1}{12}$	1
	52	-7	$\frac{5}{36}$	$\frac{5}{12}$	3
E_6	27	1	–	$\frac{1}{12}$	1
	78	-6	$\frac{5}{36}$	$\frac{1}{2}$	4
E_7	56	1	–	$\frac{1}{24}$	1
	133	-2	$\frac{5}{288}$	$\frac{1}{6}$	3

Table A.1: Representations less than 248 for exceptional algebras with the coefficients or group-theoretical constant for higher-order traces.

Appendix B

Bundles, connections and characteristic classes

This appendix reviews the main idea of gauge theories from a more mathematical point of view since this offers a more convenient way to calculate anomalies of theories coupled to gauge bosons and gravity. We will start by introducing the language of differential geometry, particularly differential forms. For a more complete account see, for example [64,65].

B.1 Tangent and fiber bundle structures

Let us describe how the essence of bundles comes up without being fully mathematically precise and much less, complete in our presentation. To do this, consider the following picture that already appears in Nakahara's book but here extended to three open sets U_i, U_j, and U_k in the open covering $\{U_i\}$ of M, including the coordinate parameterization of a point $p \in U_i \cap U_j \cap U_k$ in the bottom of the figure. In the upper part, we show the local trivializations associated with the maps $\phi_{i,p}^{-1} : \pi^{-1}(U_i) \to U_i \times F$ where F is called the fiber (if F is itself a manifold) and is the same at each point of $p \in M$. The whole of that picture defines a bundle with base manifold M (bottom), with fiber F (upper) and total space E (whole), this information is denoted as $\pi : E \to M$, with π a projection map and is called a fiber bundle. That image shows a local description, but the aim of introducing the open sets above is that we can move through the entire space provided that some compatibility constraints are satisfied along the whole of M or E depending on what we are interested in.

First, we will consider the base manifold and its differentiable structure. Through the notion of directional derivative along a curve passing through p, it is not that difficult to show that partial derivatives define a basis for the tangent space T_pM defined by all the vectors tangent to the point $p \in M$. This can be done with the coordinate parameterization via $x^\mu(p)$ using the chart U_k or $y^\nu(p)$ using the chart U_j and these two parameterizations may be connected by the following relation between one base $\{\frac{\partial}{\partial x^\mu}\} = \{\partial_\mu\}$ and the other

$$\frac{\partial}{\partial x^\mu} = \frac{\partial y^\nu}{\partial x^\mu}\frac{\partial}{\partial y^\nu}\,, \tag{B.1}$$

where the coefficients $e^\nu_\mu(p) = \frac{\partial y^\nu}{\partial x^\mu}\Big|_p$, belonging to the *structure* group of general linear transformations $GL(d,\mathbb{R})$, can be interpreted as coordinate transformations changing from one basis to the other, namely for $V \in T_pM$, then

$$V = V^\mu \frac{\partial}{\partial x^\mu} = \widetilde{V}^\nu \frac{\partial}{\partial y^\nu} \,, \qquad \widetilde{V}^\nu = e^\nu_\mu(p) V^\mu \,. \tag{B.2}$$

Thus, if we are able to guarantee this coordinate reparameterization along the entire space M by gluing together different charts (U_i, φ_i), then this allows us to have globally well-defined differentiable objects on M. In the picture, this is captured in the transition function $\psi_{jk}(p) : \varphi_k(U_j \cap U_k) \to \varphi_j(U_j \cap U_k)$ with $\psi_{jk}(p) = \varphi_j \circ \varphi_k^{-1}$ demanding that they are infinitely differentiable. More general objects are (r, s)-tensors, which can be thought of as multilinear maps with r upper indices entries and s lower indices entries and correspond to globally defined objects. One distinguished $(0, 2)$-tensor is the metric g which is symmetric in the two indices. The metric allows us to introduce a basis $\{e_\alpha\}$ via the coordinate transformation $e_\alpha = e^\mu_\alpha(p)\partial_\mu$ also introducing a notion of orthonormality such that at each point

$$e^\mu_\alpha \, e^\nu_\beta \, g_{\mu\nu} = \delta_{\alpha\beta} \,, \tag{B.3}$$

and even more, a notion of orientability by requiring $\det e^\mu_\alpha > 0$. We have lifted the structure group from $GL(d, \mathbb{R})$ to $SO(d)$ locally. This is a huge step which is not always achievable globally. We will say some words on this later.

We describe very briefly another set of distinguished tensors. A differential form, which is a completely anti-symmetric tensor, also known as a p-form such that

$$\omega_{\mu_1 \cdots \mu_p} = \omega_{[\mu_1 \cdots \mu_p]} \,, \tag{B.4}$$

whose complete anti-symmetry is encoded in the wedge product operation, then a p-form will be expressed as

$$\omega_p = \frac{1}{p!} \omega_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \,, \tag{B.5}$$

where the $p!$ normalization is usual and avoids overcounting and $dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$ might be thought as a certain basis of a group. Roughly, if one dx^{μ_n} jumps through an even number of $dx^{\mu_{m_i}} \wedge \cdots \wedge dx^{\mu_{m_j}}$ there is no change in sign, while for jumping through an odd number there is a minus sign. The precise way to define this is via the symmetric group S_p. From this property of p-forms, it follows that

$$\omega_p \wedge \eta_q = (-1)^{pq} \eta_q \wedge \omega_p \,. \tag{B.6}$$

It is possible to define differentiation over p-forms via a map that lifts by one the degree of a p-form

$$d\,\omega_p = \frac{1}{p!} \frac{\partial\, \omega_{\mu_1 \cdots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p} \,, \tag{B.7}$$

given us a $(p + 1)$-form, from which follows that $(d\omega)_{\mu_1 \mu_2 \cdots \mu_{p+1}} = (p + 1)\partial_{[\mu_1} \omega_{\mu_2 \cdots \mu_{p+1}]}$. It is also possible to define the integration of p-forms over a p-dimensional (sub)manifold. If one assume that there is a $p + 1$-dimensional manifold with boundary $\partial\,\Sigma$, there is a generalization of Stoke's theorem by

$$\int_{\partial\,\Sigma} \omega_p = \int_\Sigma d\,\omega_p \,, \tag{B.8}$$

and this is one of the most beautiful results of the theory of p-forms. In fact, that piece of information about vector calculus that one needs to learn to appreciate electromagnetism is beautifully encoded in this theory of differential forms.

To define a well-defined volume form, also known as top form, we need the metric

$$\mathrm{vol(M)} = \sqrt{g}\, dx^1 \cdots dx^d \,, \tag{B.9}$$

this is the integration measure. This is sometimes associated with the Levi-Civita density since this becomes the Levi-Civita anti-symmetric symbol $\epsilon_{\mu_1 \cdots \mu_d}$ into a tensor, invariant under general coordinates transformations.

There is one more operation that is possible to be defined only for manifolds with metric, it is called the Hodge star. It can also be thought of as a map, where a p-form is mapped to a $(d-p)$-form defined as

$$*(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}) = \frac{\sqrt{g}}{(d-p)!} \epsilon^{\mu_1 \cdots \mu_p}{}_{\nu_{p+1} \cdots \nu_d} dx^{\nu_{p+1}} \wedge \cdots \wedge dx^{\nu_d} \,, \tag{B.10}$$

and we choose the convention that $\epsilon_{1,\cdots,d} = 1$. Note that $\mathrm{vol}(M) = *1$ is a compact expression for the volume form. This will allow us to write expressions in a compact way in the main text. Finally, note that $*(*\omega) = (-1)^{p(d-p)}\omega$. As an example of this notation consider the action of a real scalar field

$$\frac{1}{2}\int_M \mathrm{d}\phi \wedge *\mathrm{d}\phi = \frac{1}{2}\int_M \mathrm{d}^d x \sqrt{g}\, g^{\mu\nu}\, \partial_\mu \phi\, \partial_\nu \phi \,. \tag{B.11}$$

Other actions, like a Yang-Mills action, can be worked out with this rough description of differential p-form theory.

To set conventions, we take γ-matrices to satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \,, \tag{B.12}$$

where $g^{\mu\nu}$ is the Euclidean metric already introduced. In even dimension $d = 2n$ we define the chirality operator to be

$$\gamma_{d+1} = \mathrm{i}^{-n} \gamma^1 \gamma^2 \cdots \gamma^{2n} \,. \tag{B.13}$$

This prepares the stage for a more general description of the tangent space in terms of the tangent bundle which is basically the collection of all the tangent spaces at each point $p \in M$. We already mentioned this notion of a bundle but here, we focus on the local information and then how to extend this to the total space. Above, we introduced this notion of a local trivialization by the diffeomorphism $\phi_{i,p}^{-1} : \pi^{-1}(U_i) \to U_i \times F$, which means that some subset $\pi^{-1}(U_i)$ is mapped onto product $U_i \times F$. Correspondingly, if we look at the triple intersection, the three maps must somehow be related. There must be a transition function such that $t_{ji}(p) = \phi_{j,p}^{-1} \circ \phi_{i,p} : F \to F$ which takes values in a group G, the structure group, acting on the fibres coordinates such that

$$\phi_k(p, f_k) = \phi_i(p, t_{ik} f_k) \,, \qquad f_i = t_{ik} f_k \,. \tag{B.14}$$

To do this along the entire total space, it is required the following consistency conditions on the transition functions t_{ij} following the same procedure above

$$\begin{aligned} t_{ii} &= e \,, & p &\in U_i \,, \\ t_{ij} &= t_{ji}^{-1} \,, & p &\in U_i \cup U_j \,, \\ t_{ij}\, t_{jk}\, t_{ki} &= 1 \,, & p &\in U_i \cup U_j \cup U_k \,. \end{aligned} \tag{B.15}$$

The third condition is known as the cocycle condition. These conditions are sufficient and enough to define a bundle $\pi : E \to M$. One example is the tangent bundle $\pi : TM \to M$, where T_pM is the fiber at p, V^μ are the fibre coordinates such that the local trivialization

$$\phi_i(p, \{V^\mu\}) = \phi_j(p, \{\widetilde{V}^\nu = e^\nu_\mu(p)V^\mu\}) \,, \tag{B.16}$$

with transitions functions taking values in the group $GL(d, \mathbb{R})$. The same analysis can be done for the frame bundle with base $\{e_a\}$ where $\phi_i(p, \{V^\alpha\}) = \phi_j(p, \{\widetilde{V}^\beta = e^\beta_\alpha(p)V^\alpha\})$, with transition function taking values in the structure group. These two bundles are known as vector bundles because the fibres are vector spaces. Another example is a spin tangential structure which means that the transition $\tilde{t}_{ij}$ now takes values in $\mathrm{Spin}(d)$, the double cover of $SO(d)$. We need to be very careful because the $SO(d)$-valued transition functions t_{ij} can lift to $\pm\tilde{t}_{ij}$, which means that $\tilde{t}_{ij}\,\tilde{t}_{jk}\,\tilde{t}_{ki} = \pm 1$. A spin structure is only possible if we can choose the set of transition functions such that $\tilde{t}_{ij}\,\tilde{t}_{jk}\,\tilde{t}_{ki} = 1$. This is important because to have fermions in a curve manifold the spin structure is essential.

We need the notion of a principal bundle. These are bundles where the fibre is the group G viewed as a manifold, i.e. there is a smooth map $\sigma : G \times G \to G$, and the transition function takes values on G. The consistency conditions are basically the same as before. There is also a base manifold M such that $\pi : P \to M$ denotes the principal G-bundle over M, for short sometimes we denote it as P_G only. By looking at a local trivialization $\phi_i^{-1}(u) = (p, g_i)$, for $u \in \pi^{-1}(U_i)$. Here, it is important the notion of left and right action, although they are equivalent we need to choose one to work with; the right action, for instance

$$ua = \phi_i(p, g_i a) \,, \tag{B.17}$$

from which follows that

$$ua = \phi_i(p, g_i a) = \phi_j(p, t_{ji} g_i a) = \phi_j(p, g_j a), \tag{B.18}$$

where by using the compatibility conditions, the local trivialization is independent of the chosen right action. The same can be shown for the left action. One can also show that the action of G on the fiber at p is transitive and free.

Another important concept in fibre bundles is a map known as a (local) section $s : M \to E$ defined by the property that $\pi \circ s = \mathrm{id}_M$. For instance, in a trivial bundle, this means $M \ni p \longmapsto (p, \sigma(p) = f) \in M \times F$, with $\sigma : M \to F$. In a nontrivial bundle, this is not always true globally. In a principal bundle, there is a way to define a local section by the canonical local trivialization $s_i = \phi_i(p, e)$ such that

$$s_i(p) = \phi_j(p, t_{ji}(p)e) = s_j(p)t_{ji}(p) \,. \tag{B.19}$$

This is an object that transforms non-trivially under the structure group. For principal bundles, charged fields can be thought of as sections of the *associated* bundle P_ρ. This is a vector bundle constructed from P_G and a σ-representation of G over a vector space V defined by

$$P_\sigma = P_G \times V/((p \cdot g, v) \sim (p, \sigma(g) \cdot v)) \tag{B.20}$$

where $v \in V$, and $\sim$ means identification of the points $(p \cdot g)$ and $(p, \sigma(g) \cdot v)$ of $P_G \times V$.

For the purpose of anomalies, there is a more convenient way to describe principal bundles, see [235,236] for introductory material on algebraic topology. This is convenient for various reasons, for instance for the classification of isomorphism classes of bundles. Let us introduce this convenient description known as universal bundle construction. We consider the following fibration $G \to EG \to BG$ where BG is the classifying space of G given by the quotient EG/G by a free G-action, EG is a contractible space. The important point is that this defines G-bundle over EG. For any space M we can define isomorphism classes of bundles $P_G \to M$ over M by homotopy classes of maps, for a specific representative in a homotopy class $f : M \to BG$ such that $P_G = f^*(EG)$, where f^* is the pullback map. We say that isomorphism classes of bundles are equivalent to homotopy classes of maps $[M, BG]$. This construction allows us to determine characteristic classes of bundles as the pullback of generators of the cohomology ring[1] of the classifying space $H^*(BG, \mathbb{Z})$ to the cohomology of the base manifold $H^*(M, \mathbb{Z})$ due to naturality of characteristic classes under pullbacks. In few words, characteristic classes essentially measure the nontriviality of a bundle and are constructed to classify bundles up to isomorphism which is usually more convenient than the homotopy classes of maps.

An important observation is to ask how we transport objects (vectors, tensors, fermions, and so on) along a curve. This led to the introduction of connections. For example, for tensors, there exists an extremely important connection known as the Levi-Civita connection ∇ which is compatible with the tensor metric $\nabla g = 0$. With this, it is possible to construct a covariant derivative in terms of Christoffel indices compatible with the $GL(d, \mathbb{R})$ tangent structure. For studying fermions on a curve space this is not that useful. We need to move to the frame bundle basically because there are no fermion representations in $GL(d, \mathbb{R})$. We need to lift the tangent structure to a spin structure as we already explained. This introduces a connection to parallel transport fermions, known as the spin connection, which allows coupled fermions to gravity. The same can be done for principal bundles, now the connection is a matrix-valued 1-form potential A taking values in the Lie algebra of G. This is basically the gauge vector boson A^a_μ, where μ is a vector index, a a Lie algebra index. This permits us to define the usual covariant derivative of Dirac action.

It is important to keep in mind that any mathematical object we introduce is always locally, in a particular chart. Then, we need to try to extend the local information through different patches as we show in the figure. By doing that task with the connection we find that a connection on one chart is gauge equivalent to another connection provided that

$$A^g = g^{-1} A\, g + g^{-1}\, , \mathrm{d}\, g \tag{B.21}$$

$$g : U_i \to G\,. \tag{B.22}$$

Roughly speaking, if the chart (U_i, A) and the chart (U_j, A^g), with 1-form connections A and A^g different only by the transformation (B.21), then we consider it a gauge equivalent and gluing process can be carried out. This is behind the principle of gauge *redundancy* (also described as gauge symmetry but, strictly speaking, this is not quite right) of gauge theories, like $U(1)$ Maxwell theory or the successful Standard Model of Particle Physics[2] but it is important to remark that the gauge vector boson in this physics language must be thought of properly as connections on principal bundles.

An important feature of connections is the fact that permit us to define gauge invariant objects in the sense that they are not subject to any gauge equivalence and are globally well defined. We will mainly consider the spin connection rather than the Levi-Civita connection because this is more natural to see gravity as a gauge theory where the gauge

[1]Here we have considered cohomology of spaces with coefficients in the integers. Coefficients like e.g. $\mathbb{R}$, $\mathbb{Z}_2$, reals, mod 2 integers are also possible

[2]For this case particle physics theorists for many decades looked only at some specific patch without much worry about the global topology of the fiber bundle P_G with connection A.

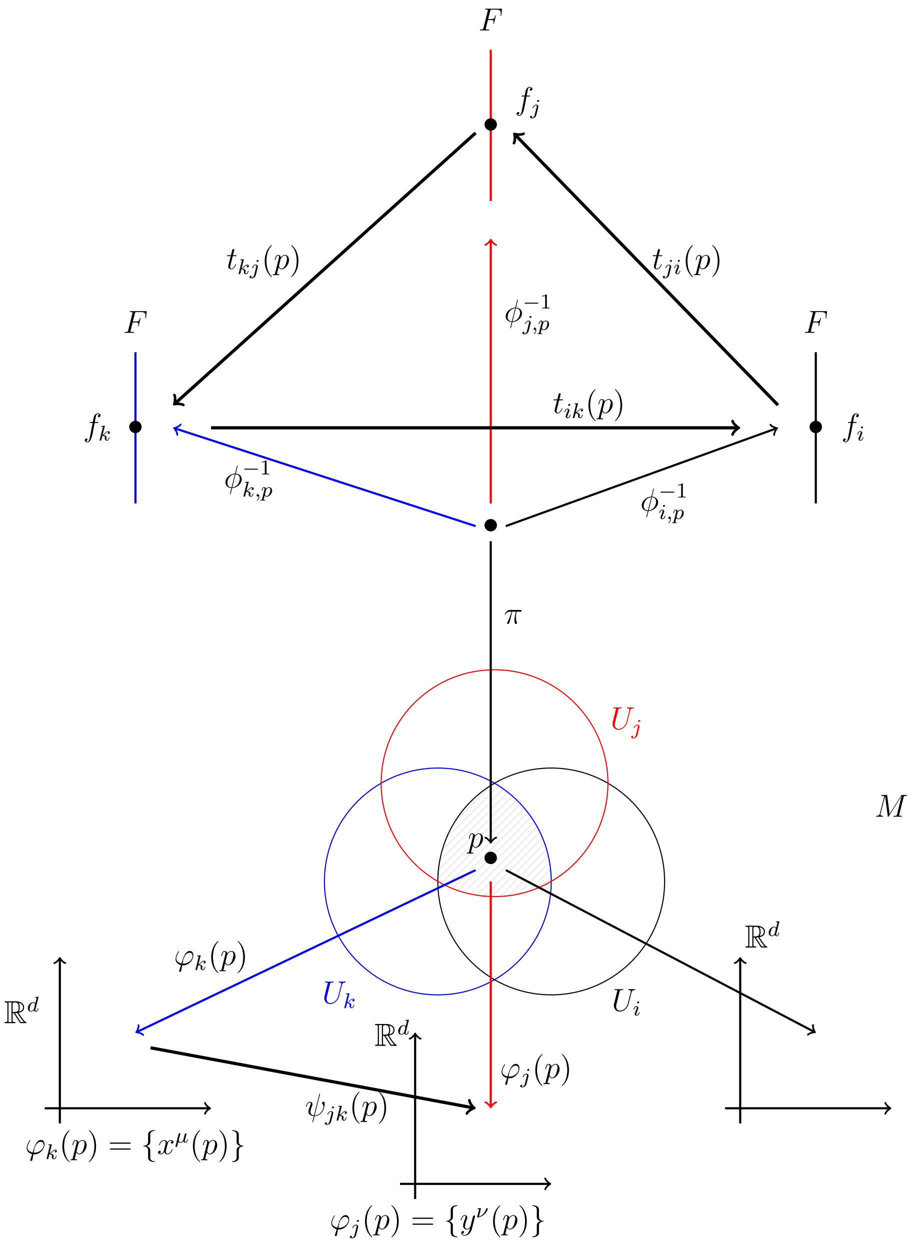


Figure B.1: In the bottom of this picture we show a set of three nonempty intersecting charts such that any structure must be subject to certain consistency conditions to carry out this covering process through an entire space. The top part shows the attachment of the bundle structure to the base space on the bottom. This structure must as well be subject to gluing consistency conditions on three nonempty charts. It turns out that this is a necessary and sufficient condition to properly define any structure over a topological space.

group is $SO(d)$, and also to have access to spinor representations, as we said. As can be seen from (B.3) by the transformation $e^\alpha \to L^\alpha{}_\beta(p)\, e^\beta$, the $L(p)$'s transformations are local (depends on the coordinates) elements of $SO(d)$ since $L^\alpha{}_\beta L^\gamma{}_\delta \delta_{\alpha\beta} = \delta_{\gamma\delta}$. The spin connection is a 1-form connection $\omega = \omega^\alpha{}_{\beta,\mu}\mathrm{d}x^\mu$ satisfying an anti-symmetry property $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$. The tensor valued Riemann curvature 2-form R is defined in terms of the spin connection by

$$R^\alpha{}_\beta = \mathrm{d}\omega^\alpha{}_\beta + \omega^\alpha{}_\beta\, \omega^\beta{}_\gamma\,, \tag{B.23}$$

this is an anti-symmetric matrix, so this can be brought to the following form by an appropriate orthogonal transformation

$$R \to \begin{pmatrix} 0 & -w_1 \\ w_1 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & -w_n \\ w_n & 0 \end{pmatrix} \oplus \ldots\,. \tag{B.24}$$

From this information, we can define a characteristic polynomial through $\det(I + R)$ which we denote by $p(R)$ such that (for a mathematical treatment of characteristic classes, see [199])

$$p(R) = \Pi_i(1 + w_i^2) = 1 + p_1(R) + p_2(R) + \cdots p_n(R)\,, \tag{B.25}$$

where each $p_s := p_s(R)$ is known as an s-th Pontrjagin class, a $4s$-form, defined as

$$p_s(R) = \sum_{i_1<\cdots<i_s} w_{i_1}^2 \cdots w_{i_s}^2\,. \tag{B.26}$$

We write down a few of them explicitly in terms of traces of the 2-form R because, for the purpose of anomalies, it is usual to express the anomaly polynomial in these terms. Nonetheless, we will stick to the notation in terms of the characteristic polynomials. The 4, 8 and 12-form are given by

$$\begin{aligned} p_1 &= -\frac{1}{2}\mathrm{tr}\, R^2\,, \\ p_2 &= \frac{1}{8}\left[\left(\mathrm{tr}\, R^2\right)^2 - 2\,\mathrm{tr}\, R^4\right]\,, \\ p_3 &= -\frac{1}{48}\left[\left(\mathrm{tr}\, R^2\right)^3 - 6\left(\mathrm{tr}\, R^2\right)\left(\mathrm{tr}\, R^4\right) + 8\,\mathrm{tr}\, R^6\right]\,, \end{aligned} \tag{B.27}$$

where we have introduced a normalization such that $R \to R/2\pi$. With this Pontrjagin classes, we can define the so-called A-roof or Dirac genus as

$$\hat{\mathrm{A}}(R) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(7\,p_1^2 - 4\,p_2) - \frac{1}{967680}(31\,p_1^3 - 44\,p_1\,p_2 + 16\,p_3) + \cdots \tag{B.28}$$

This polynomial plays an important role in the determination of the gravitational contribution to anomalies from chiral fields coupled to gravity.

From the 1-form connection $A = A_\mu \mathrm{d}x^\mu$, where we define $A_\mu = A_\mu^a T^a$ with the T^a the generators of the Lie algebra of G represented in specific representations. With this, we can define the curvature 2-form or field strength F, also a matrix-valued form by

$$F = \mathrm{d}A + A \wedge A\,, \tag{B.29}$$

containing information on the gauge bundle. This can be used to define the Chern character, another polynomial in terms of F by

$$\mathrm{ch}_{\mathsf{r}}(\mathrm{i}F) = \mathrm{tr}_{\mathsf{r}}\exp(\mathrm{i}F)\,, \tag{B.30}$$

$$= \mathrm{ch}_0(\mathrm{i}F) + \mathrm{ch}_1(\mathrm{i}F) + \cdots + \mathrm{ch}_n(\mathrm{i}F)\,, \tag{B.31}$$

with the $2k$-form

$$\mathrm{ch}_k(\mathrm{i}F) = \frac{1}{k!}\mathrm{tr}_{\mathsf{r}}\,(\mathrm{i}F)^k \ , \tag{B.32}$$

where r denotes the specific representation we are working with, in order to evaluate the trace tr over the corresponding generators, and we have normalized $\mathrm{i}F \to \mathrm{i}F/2\pi$ (usually we will omit to write the imaginary number but we must keep it in mind). This defines the k-th Chern character. These are the essential objects in the study of anomalies.

B.2 Anomaly polynomials

There is a very precise way to determine the anomaly polynomial of a theory involving all the anomalous chiral degrees of freedom through the Atiyah-Singer index theorem [76]. For a spin-$1/2$ Weyl fermion (namely, for spin-$1/2$ section of the associated bundle P_{r}) coupled to a matrix-valued G-connection A transforming in the representation r, the index density is given by (see [52, 83] for a physicist-friendly treatment of this)

$$\mathcal{I}_{\frac{1}{2}}(\mathcal{D}^{S^+\otimes P_{\mathsf{r}}}) = \hat{A}(TM)\,\mathrm{ch}_{\mathsf{r}}(F)\Big|_{2n} \tag{B.33}$$

For a spin-$3/2$ fermion that couples to the tangent bundle via a vector index and couples to the frame bundle via the spin index, it has been determined its anomaly also via an index density given

$$\mathcal{I}_{\frac{3}{2}}(\mathcal{D}^{S^+\otimes TM}) = \hat{A}(TM)\,(\mathrm{ch}(TM)-1)\Big|_{2n} \tag{B.34}$$

We find it very useful to define the following general formula

$$\begin{aligned}\hat{A}(TM)\,(\mathrm{ch}_{\mathsf{r}}(TM)-k) =& 2n-k+\frac{24+k-2n}{24}\,p_1 \\ &+\frac{1}{5760}\left[(240-7k+14n)\,p_1^2-4(240-k+2n)\,p_2\right]\end{aligned} \tag{B.35}$$

$$+\frac{1}{967680}\left[(504+31k-62n)\,p_1^3-4(504+11k-22n)\,p_1\,p_2+16(504+k-2n)\,p_3\right]+\cdots$$

where $k=1$ describes the anomaly of the gravitino in (B.34). For general k, we observe the following. The anomaly of neutral spin-$1/2$ fermions is given by

$$\mathcal{I}_{\frac{1}{2}}(\mathcal{D}^{S^\pm}) = \pm\hat{A}(TM) \tag{B.36}$$

as follows from (B.33). Therefore, in combination, the index density given the anomaly of a gravitino and $\pm k$ neutral fermions can be combined into (B.35).

Another important polynomial is the Hirzebruch polynomial $L(R)$ which describes the anomaly polynomial of a self-dual tensor field as

$$\mathcal{I}_{\mathrm{sd}} = -\frac{1}{8}L(R) \tag{B.37}$$

$$= -\frac{1}{8}-\frac{1}{24}\,p_1+\frac{1}{360}(p_1^2-7\,p_2)-\frac{1}{967680}(256\,p_1^3-1664\,p_1\,p_2+7936\,p_3)+\cdots \tag{B.38}$$

Appendix C

Bordism theory

C.1 Anomalies and bordisms

In order to detect anomalies of a fermion or any theory $\mathcal{T}$ in the main text we assumed that all the structure, tangential plus principal or any other structure denoted here as ζ, on a d-dimensional manifold, extends to a $d+1$-manifold X. Afterward, we focus on a manifold such that $X = \partial Y$ where Y is a $d+2$ manifold with the structure ζ also extended to Y. Each of these steps may or may not be possible, namely, there may exist manifolds X which cannot be realized as boundaries of other manifolds Y. It is important to have a way to know when this is possible. The mathematical theory accounted for this is known as bordism theory [237] for the classical reference and [238] for a set of recent lecture notes.

Consider the set of all closed $d+1$-dimensional manifolds $\mathcal{C}_{d+1}$. Consider X and X' as two elements in $\mathcal{C}_{d+1}$. Then, one can introduce the following equivalence relation. We say that X and X' are equivalent if there exists a $d+2$ manifold Y such that $\partial Y = X \sqcup \overline{X}'$, where $\overline{X}$ refers to the manifold X with reversed structure. For instance, if X is orientable then $\overline{X}$ has a reversed orientation. This equivalence relation is reflexive, transitive, and symmetric, it defines an equivalence class. Therefore, we can refine our set into $\mathcal{C}_{d+1}/\sim$, the set of equivalence classes of closed manifolds connected by a $d+2$-manifold. We can furthermore endow $\mathcal{C}_{d+1}/\sim$ with an abelian structure by means of the disjoint union where the identity is denoted by $[\varnothing]$, the inverse by $[\overline{X}]$ such that $[X]+[X'] = [X \sqcup X']$ satisfies the axioms of an abelian structure. We say that X and X' are bordant if they belong to the same class $[X]$. The abelian group of bordism classes will be denoted as $\Omega^{\zeta}_{d+1}(\mathrm{pt})$. The class label by $\varnothing$ means that there exists X such that $X \sim \varnothing$, which consequently means that $\partial W = X$. This is almost exactly what we assumed to evaluate perturbative anomalies in the main part of this work. From this discussion we can appreciate that does not make much sense to look for cancellation of nonperturbative anomalies if we are already unable to cancel the perturbative ones. It is also clear from this discussion that perturbative anomalies are evaluated on closed $(d+2)$-manifolds. This suggests that we need to look at the bordism group $\Omega^{\zeta}_{d+2}(\mathrm{pt})$. The group equivalence classes of closed $(d+2)$-manifolds with ζ structure.

Nevertheless, up to this point, we have only looked at manifolds that are bordant to $\varnothing$, it might happen that $[X] \in \Omega^{\zeta}_{d+1}(\mathrm{pt})$ corresponds to a nontrivial class in bordism, hence there is no way to detect this classes with (3.40). Assume that we have already cancelled the anomalies discussed in the previous paragraph. Hence, the next step is to think in an object that detects nontrivial classes in $\Omega^{\zeta}_{d+1}(\mathrm{pt})$. For the case of fermions is already known that the Atiyah-Patodi-Singer (APS) η-invariant is the appropriate topological invariant to evaluate nonperturbative anomalies [59, 60, 134], namely

$$Z(X_{\text{closed}}) = \exp(2\pi \mathsf{i}\eta(X_{\text{closed}})) \,, \tag{C.1}$$

where X_{closed} must be thought of as a representative of the class $[X]$ alluded before. Behind this, we are assuming that η is actually a bordism invariant, which is indeed true provided that perturbative anomalies cancel. To see this, suppose that X_{closed}, $X \in [X]$, are in the same class, then

$$\exp\big(2\pi \mathsf{i}\eta(X \sqcup \overline{X}_{\text{closed}})\big) = \exp(2\pi \mathsf{i}\eta(\partial Y)) = \exp\left(-2\pi \mathsf{i} \int_Y \mathcal{I}_{d+2}\right) , \tag{C.2}$$

where we have used the APS index theorem and the fact that Index is always an integer by definition. Therefore,

$$\exp\big(2\pi \mathsf{i}\eta(X \sqcup \overline{X}_{\text{closed}})\big) = 1 \,. \tag{C.3}$$

It turns out that the η-invariant under disjoint or gluing operation behaves as the sum of the individual pieces and $\eta(\overline{X} = -\eta(X)$, hence

$$\exp(2\pi \mathsf{i}\eta(X)) = \exp(2\pi \mathsf{i}\eta(X_{\text{closed}})) \,. \tag{C.4}$$

This is the reason why we are allowed to evaluate the η-invariant in a convenient generator of the class $[X]$.

Notice that, the η-invariant is a group homomorphism

$$\eta : \Omega^{\zeta}_{d+1}(\text{pt}) \to \mathbb{R}/\mathbb{Z} \tag{C.5}$$

where we can think of $\mathbb{R}/\mathbb{Z} \simeq U(1)$, and $\Omega^{\zeta}_{d+1}(\text{pt})$ is the set of bordant manifolds with ζ-structure. Thus, the η-invariant could be thought of as belonging to the group of homomorphisms

$$\eta \in \mathrm{Hom}(\Omega^{\zeta}_{d+1}(\text{pt}), U(1)) \,. \tag{C.6}$$

Classification of this problem for invertible field theories has been conjectured [95] to be equivalent to the classification of anomalies of a d-dimensional theory that appears as the boundary of such invertible phases, see [98, 185] treatment of this conjecture, while for a friendly treatment, see [110, 188]. As we said in the Introduction 1, this approach has been pushed forward in many places, go there for a sample of reference. Also, the canonical reference to compute bordism groups is [239].

This seems to offer a rigorous and complete classification of anomalies in chiral fermion theories, though it requires the computation of abstract mathematical objects.

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