Holographic Complexity and Spacetime Locality

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Abstract

This thesis is devoted to the formulation of a new result in [1–3] that establishes a connection between holographic complexity in the form of the so-called Complexity = Volume proposal and the gravitational clumping of matter, within the AdS/CFT correspondence. The main result of the thesis, the 'Momentum/Volume Complexity (PVC) correspondence', formalizes the recurrent idea that the gravitational clumping of matter increases the complexity of the quantum state. The PVC correspondence works for perturbations of finite entropy thermal states beyond the scrambling time, where the linear growth of complexity is associated to the frozen momentum of the excitation in the black hole interior. It generalizes previous 'Momentum/Size' correspondences in the literature.

The exact PVC correspondence of this thesis works for any normalizable spherically symmetric state in arbitrary dimensions and for any normalizable state in 2+1 dimensions. Its proof is based on the kinematics of the Momentum Constraint of General Relativity.

There are two physical obstructions for an exact PVC correspondence in more general situations. The first one is intrinsic to the topology of space, and it arises in the presence of spatial wormholes connecting different asymptotic boundaries. In this case, the spatial wormhole can stretch without any matter whatsoever. The second obstruction comes from the impossibility to define a local notion of gravitational momentum, and in particular it arises for pure gravity solutions consisting of gravitational waves. A Generalized PVC correspondence is formulated to include this latter case, derived from the Codazzi equation, which assigns a contraction of the Weyl tensor to the purely gravitational contribution of the momentum.

The central notion of 'infall momentum' has a Newtonian version which explicitly captures the intuitive idea that matter clumping increases complexity. A relativistic generalization of this version also exists. Finally, the value of VC for states with small backreaction is given in terms of a radial 'moment of inertia' that quantifies the degree of clumping of matter.

Other work developed during the thesis and related to the topics covered in this manuscript is [4–6].

Resumen

Esta tesis se compone de los artículos [1–3] en los que se establece una conexión entre la noción de complejidad holográfica dada por la denominada prescripción de Complejidad = Volumen, y el grado de compresión de la materia debido a la atracción gravitatoria, en el contexto de la correspondencia AdS/CFT. El resultado prinicpal de esta tesis formaliza la idea recurrente de que la atracción gravitatioria de la materia aumenta la complejidad cuántica del estado del sistema. Cuando el colpaso gravitatorio genera un agujero negro en AdS, la complejidad crece linealmente después de la termalización global, y este crecimiento lineal está capturado por el momento propio de la materia en el interior del agujero negro, que se encuentra congelado debido a la acumulación de las hipersuperficies extremales. Este resultado generaliza correspondencias previas entre el momento y el tamaño del operador que han aparecido recientemente en la literatura.

La correspondencia PVC de esta tesis captura de forma exacta cualquier configuración esféricamente simétrica de materia normalizable en cualquier dimensión, además de cualquier estado en 2+1 dimensiones. Su demostración está basada en la cinemática de la restricción inicial de momento de la Relatividad General.

Existen dos obstrucciones principales para extender la correspondencia Complejidad / Momento exacta a situaciones más generales. La primera es intrínseca a la topología del espacio, y aparece cada vez que este incluya agujeros de gusano que conecten distintas regiones asintóticas. La segunda restricción se manifiesta por la imposibilidad de definir una noción local de momento gravitatorio, y en particular aparece cuando el espaciotiempo es una solución dinámica de gravedad pura sin materia, formada por ondas gravitacionales. Esta última situación se puede incluir en una generalización de la correspondencia PVC, derivada de las ecuaciones de Codazzi, en la que se le asigna una contribución dada por una contracción del tensor de Weyl al momento puramente gravitatorio del sistema.

La noción central de 'momento de caída' tienen una versión Newtonianan que explícitamente captura la idea intuitiva de que la compresión de la materia aumenta la complejidad. Existe también una generalización relativista de esta versión. Finalmente, el valor de VC para estados con *backreaction* pequeña estña dada en términos de un 'momento de inercia radial' que cuantifica el grado de compresión de la materia.

Otro trabajo de investigación realizado durante el transcurso de la tesis doctoral, y relacionado con los temas que cubre este manuscrito es [4–6].

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Chapter I

Foundations

1 Introduction

The long chain of scientific breakthroughs that took place in Physics over the last century completely revolutionized our way to conceive reality. The world at subatomic scales appears to be governed by a handful of physical principles: the laws of quantum mechanics, the symmetries of special relativity, and the principle of spacetime locality of matter and its interactions. Together they logically spawn the sophisticated framework of Quantum Field Theory, upon which our most precise description of the constituents of the Universe, the Standard Model of particle physics, is formulated.

Attaining such an incredibly precise and satisfactory description of the electroweak and strong interactions, it was a matter of time for an attempt to include gravity into this paradigm to materialize [7]. The solid foundations of QFT remarkably require that the gravitational interaction is transmitted by means of a massless particle of spin 2, the graviton, which necessarily couples in a universal way to all forms of energy [8], providing a microscopic *raison d'être* of Einstein's theory of General Relativity.

In the theory of quantum gravity, the gravitational vertex is weighted by the coupling $g \sim \sqrt{G}\, E$, where G is Newton's constant and E is the center-of-mass energy of the process. Any reasonable gravitational system thus becomes strongly coupled at the Planck scale, $M_P^{-1} = \sqrt{G} \sim 10^{-33}\,\text{cm}$, or even before, depending on other high-dimension operators present in the effective theory.

As in many other physical systems, the breakdown of the perturbative expansion typically points towards the existence of unidentified degrees of freedom underlying close to the cutoff scale, which are ultimately responsible of bringing the theory back into a controlled regime. The question then arises: what are these degrees of freedom in the case of gravity?

Perhaps a solid indication of what these degrees of freedom are *not* is provided already by a classic result of Weinberg and Witten [9]. In essence, any attempt to describe the graviton as a composite state of more elementary local constituents can be disregarded, since such a bound state will never be able to reproduce the Lorentz transformation properties of the graviton. This result arguably precludes the possibility of having an ultraviolet-complete description of gravity without abandoning the route of QFT at some point below the Planck scale.

String Theory is, for multiple reasons, our most prominent candidate for an ultraviolet completion of quantum gravity. In the perturbative regime, the graviton is naturally accommodated in the spectrum of a closed fundamental string of size $\ell_s \gtrsim M_P^{-1}$, together with a zoo of different particles arising from different vibration modes of the string. The theory contains a long list of features which captivate the aesthetically minded, including: supersymmetry, extra dimensions, complete unification of matter and the interactions, UV-finiteness of the perturbative S-matrix, dualities between all the seemingly different weakly coupled corners of the theory, etc. Even if it is yet to be seen whether the conditions in our Universe can be accommodated into the landscape of string vacua, string theory definitely exceeded the expectations of many providing ultraviolet-complete models of quantum gravity even at the non-perturbative level.

At any rate, due to the large hierarchy between the TeV scale and the Planck (or string) scale, it seems implausible that these ideas can be put under direct experimental scrutiny in the near future. Fortunately for us, gravity happens to be radically different at the fundamental level, and it is hides its marvelous features on objects of the macroscopic world. These objects are black holes.

1.1 The Holographic Principle

Black holes stand out as one of a kind among the plethora of exotic gravitational phenomena. They are perfectly suited for a classical description within the theory of General Relativity, and yet they provide a window to the fundamental structure of the quantum theory.

In the realm of astrophysics, black holes of a few kilometers in size are known to generically form from the gravitational collapse of large stars in the final stages of their lives. Almost a century after their theoretical prediction, stellar mass black holes have been finally observed in binary systems via their gravitational wave signal [10]. In addition, supermassive black holes of $10^6 - 10^9$ solar masses which inhabit the galactic centers have more recently been observed, including Sagittarius A* in the center of our own galaxy [11].

Figure 1: *Supermassive black hole in the galactic center of M87.*

Classically, a black hole of mass M is an object of infinite entropy. Its near-horizon region is able to accommodate excitations with arbitrarily large redshift which, without modifying the mass of the black hole, effectively dissipate for an exterior observer once they cross the horizon. Black holes are literally characterized as spacetime holes out of which nothing can scape, and therefore possess a vanishing temperature. The existence of such zero-temperature cosmic reservoirs becomes problematic at the level of the second law of thermodynamics. Basically, throwing anything at them from the outside world decreases the entropy of the Universe.

The tension between black hole entropy and the second law of thermodynamics is only resolved in the quantum theory. It was Bekenstein who first considered the thought-experiment of finding the most entropic way to form a black hole of mass M starting from matter which satisfies the laws of quantum mechanics [12, 13]. Such an upper bound in the entropy now exists, and it arises from the fact that the minimum energy of an excitation is inversely proportional to its wavelength, $\delta E = \lambda^{-1}$. To count on something as falling into the black hole, one must now be sure that its wavelength fits inside the Schwarzschild radius. A back-of-the envelope calculation then reveals that black holes have finite entropy proportional to their area in Planck units.

The area of the black hole is a promising candidate for its coarse-grained entropy since, within General Relativity, it is a quantity which cannot decrease with time for any process involving matter with positive null energy. However, it was really not until the much more sophisticated analysis by Hawking [14] that the area could be taken seriously as an entropy in the standard thermodynamic sense. Hawking made the brilliant observation that black holes do emit thermal radiation, for the reason that the local inertial vacuum of the quantum fields across the horizon looks thermally populated for an observer who sits outside.

In light of this observation, black holes do behave as ordinary quantum mechanical systems from an outside perspective, and in particular they follow the standard laws of thermodynamics, with an entropy given by the acclaimed Bekenstein-Hawking formula

$$S = \frac{\text{Area}}{4G} \,. \tag{1.1}$$

Unlike in any local physical system, the entropy of the black hole scales with its area, which hints that the fundamental degrees of freedom of the black hole rearrange in such a way that they live on the (stretched) horizon, with an approximate density of one degree of freedom per Planck area.

A formula like (1.1) has dramatic consequences for the interior locality. It seems to point out that the interior space is not fundamental, and that it really emerges from the properties of the fundamental degrees of freedom placed at the horizon. A caricature of this phenomenon exists in optics, known as a *hologram*. A hologram consists of a two dimensional surface that is able to encode the image of a three dimensional object. The object seems to emerge 'out of nowhere' when the observer lights up the system. In the case of a black hole, however, the hologram encodes all the interior space itself, and the encoding is physically much more subtle, since there is no simple analog of 'lighting up' the black hole from the outside to see what lies inside.

Furthermore, the black hole entropy accounts for an overwhelming majority of the total microstates of any gravitational system at sufficient large energy in a finite volume. This can also involve situations in which the volume is microscopic. Indeed, the scattering amplitude of a few particles at a center-of-mass energy of $E \gg M_P$ will be entropically dominated, at the level of accessible intermediate states, by the production a 'large' black hole resonance of radius $R \sim GE \gg M_P^{-1}$ and entropy $S \sim GE^2$ which will evaporate in a time $t \sim G^2 E^3$, spitting gravitons and other particles throughout the process of evaporation. For the S-matrix experimentalist who collects the outcomes of the scattering, it will be impossible to resolve distances shorter than GE in this experiment. In this way, the existence of black holes poses an end to the Wilsonian paradigm at the Planck scale. Trying to resolve shorter distances is fundamentally impossible since the UV and IR degrees of freedom start to mix at the Planck scale.

These considerations regarding black holes as holograms, together with the entropic dominance of black hole microstates in any system lead, after quite a while, to the radical proposal by t' Hooft [15] and later by Susskind [16] to promote the black hole entropy to a physical principle, dubbed the *holographic principle*, which views space itself as an emergent structure arising from the intricate configuration of some putative holographic degrees of freedom that live very far away.

For many reasons, the exact nature of the holographic degrees of freedom that could describe our Universe remains elusive. In other types of 'Universes', however, namely those with negative cosmological constant, the structure of space is such that gravity and matter live inside an infinite 'box', called AdS space, which allows to naturally place the holographic degrees of freedom on the walls of the box.

1.2 AdS/CFT

String theory, as a consistent theory of quantum gravity with black holes, passes the test and is able to successfully implement the holographic principle in its full glory. In fact, for many, the (second) largest triumph of the theory is that it allows to reproduce the Bekenstein-Hawking entropy (1.1) microscopically for certain BPS black holes with Ramond-Ramond fluxes by directly counting states of strings ending on D-branes at weak coupling in the type IIB string theory [17, 18]. More impressively, string theory accommodates the strong version of the holographic principle in a series of ultraviolet-complete models of gravitational holograms in AdS space.

The original model proposed by Maldacena [19] consists of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions with gauge group $SU(N)$ and gauge coupling g. In the type IIB string, this theory arises naturally as the low-energy limit of the worldvolume theory of a stack of N coincident D3-branes.[1] At strong t' Hooft coupling $\lambda = g^2 N$, the system admits a dual supergravity description in terms of a black D3-brane solution with N units of Ramond-Ramond five-form flux, which develops an $\text{AdS}_5 \times \mathbf{S}^5$ near horizon geometry. In certain low energy limit, the massless closed strings in the asymptotic region decouple in both pictures, which allows for the definition of the full type IIB string theory on the throat in terms of the strong coupling regime of the SYM theory. The curvature radius of the AdS_5 and of the $\mathbf{S}^5$ are given by $\ell \sim \lambda^{1/4}\, \ell_s$ in units of the string length. The ten-dimensional Planck scale is $\ell_P \sim N^{-1/4}\, \ell$ and therefore a very large value of N is required so that the classical solution can be trusted.

In addition to the original example, there have been quite a variety of different models worked out in different numbers of AdS dimensions by adding more sophisticated ingredients and following a similar

[1] Modulo a $U(1)$ collective mode.

route, or by starting from different corners of string theory (see eg. [19–21]). The common thread in all of these examples is that the string/M theory living inside the AdS_{d+1} 'box' emerges as a collective phenomenon of a strongly interacting quantum system living on the d-dimensional walls of the box.

The Dictionary

The boundary description displays conformal symmetry, realizing the asymptotic symmetries of the dual gravitational theory in AdS. These theories of quantum gravity are then, like any other CFT, fully specified by means of the conformal data $\{J_i, \Delta_i, C_{ij}^k\}$ of spins J_i, conformal dimensions Δ_i and OPE coefficients C_{ij}^k. The conformal data, already at this level, has to satisfy the highly non-trivial constraints imposed by the associativity of the OPE.

Sensible holographic CFTs moreover must have a very particular spectrum of states in radial quantization. In the low-lying part of the spectrum, with $\Delta \ll N^2$, they contain local primaries $\mathcal{O}$ which behave as free fields in the large-N limit, even if the theory is strongly coupled. Indeed, these fields generically incorporate large amounts of anomalous dimension which can scale with some power of the coupling λ. In SYM, these *generalized free fields* are constructed from properly normalized single traces of products of gauge-invariant operators, like $\mathcal{O}(x) = \mathrm{Tr}\, F^n(x)$, where F is the gauge field strength, and $n \ll N$.

In the large-N limit, the correlation functions of these fields factorize

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle \sim \langle \mathcal{O}(x_1)\mathcal{O}(x_2)\rangle \langle \mathcal{O}(x_3)\mathcal{O}(x_4)\rangle + (\text{permutations}) \,. \tag{1.2}$$

In fact, the consistency of the conformal block decomposition of this 4-point function (1.2) in the direct channel requires that there exists, apart from the generalized free field algebra, a tower of conformal primaries corresponding to 'multi-particle operators', like $\mathcal{O}^2(x)$, $\mathcal{O}^3(x)$, as well as more complicated fields of higher spin [22, 23]. The generalized free fields therefore generate a low-lying Fock space inside the full CFT.

The presence of such an integrable substructure in any holographic CFT becomes obvious once the bulk dual of a generalized free field $\mathcal{O}$ is identified. The dual corresponds simply to a supergravity/string excitation ϕ of small mass, $m\ell \sim O(N^0)$, in AdS. Both operators are related via the so-called *extrapolate dictionary* [24, 25]

$$\mathcal{O}(x) = \lim_{r\to\infty} r^{\Delta}\, \phi(r, x) \,, \tag{1.3}$$

that is, the generalized free field provides the normalizable boundary value to the bulk field, where r is the Fefferman-Graham radial coordinate. The mass m of the bulk field ϕ is related to the conformal dimension Δ and spin of the operator $\mathcal{O}$. [2] For example, the precise relation for scalar fields is

$$\Delta \;=\; \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2\ell^2}\,. \tag{1.4}$$

A compelling example of this relation is the duality between the bulk AdS graviton $h_{\mu\nu}$ and a generalized free field of spin $J = 2$ and conformal dimension $\Delta = d$, which must necessarily be conserved because it saturates the unitarity bound. This field is no other than the energy-momentum tensor of the CFT, $T_{\mu\nu}$. The holographic construction is therefore able to evade the no-go theorem of Weinberg and Witten by modifying the number of dimensions of the space where the fundamental description lives.

At large but finite N, the non-planar contributions render the above subspace an approximate Fock space, where conformal dimensions receive $1/N$ corrections and OPE coefficients between generalized free fields no longer vanish. The latter correspond to bulk three-point interactions of the supergravity fields, which are of order $1/N$ in AdS units. It is then tempting to 'bootstrap' the exact bulk theory in a $1/N$ expansion, inferring the necessary Witten diagrams that match a consistent conformal block decomposition of the boundary correlation functions (see [22, 23] and references therein). It is plausible, howver, that only few solutions exist to this problem, all supersymmetric, namely those holographic CFTs that are handed in known examples of AdS/CFT.

Black holes

At high energies, an overwhelming amount of black hole states is required for the proper functioning of the holographic duality. The boundary description is a theory with spacetime local degrees of freedom, in which the high-energy spectrum must obey the 'Cardy formula'

$$\rho(\Delta) \;\sim\; \exp\!\Big(c^{\frac{1}{d}}\,\Delta^{\frac{d-1}{d}}\Big)\,, \tag{1.5}$$

where $c \sim N^2$ is the central charge, a measure of the local number of degrees of freedom of the theory at high enough energies $\Delta \gtrsim N^2$.

Practically all of the states accounted for in (1.5) come from completely new primaries at high-energies, which do not originate as multiparticle states of the low-lying Fock space. The latter produces a density of states of the form

$$\rho(\Delta) \;\sim\; \exp\!\Big(\alpha\,\Delta^{\frac{d}{d+1}}\Big)\,, \tag{1.6}$$

where $\alpha \sim O(N^0)$, and $1 \ll \Delta \ll \lambda^{1/4}$. The N-dependence of the free energy shows that there exists a first-order phase transition, between a 'confined' phase at low temperatures and a 'deconfined' phase

[2] The supergravity description requires of the existence of a gap in the spectrum of the CFT, controlled by $\lambda^{1/4}$ for SYM, for the lightest state of spin greater than 2.

at high temperatures, in the large-N limit of any holographic CFT on the sphere [26, 27].

In the bulk, the density of states (1.6) accounts for the supergravity in AdS in $d+1$ dimensions, which is effectively a local theory below the string scale, of $O(N^0)$ species. The high-energy phase, on the other hand, is represented by the dominance of large black hole states in AdS above the Hawking-Page temperature $T \gtrsim \ell^{-1}$, which was found long before the microscopic interpretation [28]. [3] The bulk Bekenstein-Hawking entropy (1.1) of large AdS black holes provides the right scaling (1.5) of a local system in one less dimension.

Bulk locality

As described above, the gravitational hologram isometrically embeds the supergravity and stringy excitations on top of AdS into the Hilbert space of the CFT on the sphere, $\mathcal{H}_{\text{CFT}}$. This approximate Fock space of the CFT is commonly called the *code subspace* of empty AdS, $\mathcal{H}_{\text{code}}$, for reasons that will become apparent in the next item.

The identification (1.3) between the asymptotic boundary conditions for the bulk fields and the generalized free field algebra opens up the possibility to derive the approximate local structure of the bulk in terms of the CFT data [30–33]. Consider a local supergravity field $\Phi(X)$, where X is some spacetime point in AdS. [4] In the large-N limit, the field $\Phi(X)$ is a free field in the bulk, and its boundary conditions are given by (1.3). Finding a CFT representation of $\Phi(X)$ then translates to finding a 'spacelike' Green's function of the Klein-Gordon equation which propagates the bulk point X into a boundary point y. Such a Green's function $K(X, y)$ exists [31–33] and, albeit non-unique, it provides a boundary representation of the bulk field

$$\Phi(X) = \int \mathrm{d}^d y \, K(X, y) \, \mathcal{O}(y) \, . \tag{1.7}$$

The HKLL representation (1.7) corresponds to a smeared local operator over the CFT spacetime. When propagated into a Cauchy slice of the CFT, the operator (1.7) becomes explicitly non-local. It consistently reproduces the two main features of an emergent local free bulk: (i) microcausality, $[\Phi(X), \Phi(X')] = 0$ for X and X' spacelike-separated points, and (ii) short-distance vacuum correlations, $\Phi(X)\Phi(X') \to |X - X'|^{1-d}$ as $X \to X'$. These expressions do not hold as operator equations in the CFT, they can only hold within the code subspace $\mathcal{H}_{\text{code}}$.

At finite but large N, the field $\Phi(X)$ is no longer free, and the representation (1.7) needs to be improved in a $1/N$ expansion by adding suitable interaction vertices, weighted by powers of $1/N$, which

[3] In realistic models, an intermediate stringy regime, which develops a Hagedorn density of sates at high temperatures, will control the details of the phase transition (see [29] and references therein).

[4] Gravitational dressing can be added by attaching a spacelike geodesic Wilson line from X to the boundary. For the sake of simplicity, we omit such a technical discussion here.

now allow to propagate multiple boundary points $(y_1, ..., y_n)$ into X. [5]

Entanglement builds space

Plenty of new ideas have flourished in these last twenty years concerning the precise way in which the boundary system encodes the quantum information of the bulk in AdS/CFT systems. Arguably the most intriguing interpretation is that quantum entanglement is at the root of the emergence of a connected space, an idea which is often referred to with the slogan 'ER = EPR' [34, 35]. The avatar of ER=EPR is the eternal black hole in AdS, which possesses a spatial wormhole connecting two asymptotic regions (see Fig. 2). The fine-grained description is the *thermofield-double state* living in two completely independent copies of the CFT [36],

$$|\text{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta \frac{E_n}{2}} \, |n\rangle_L \otimes |n^*\rangle_R \ . \tag{1.8}$$

The thermofield-double contains a large amount of bipartite entanglement, in fact given by (1.1), between the CFTs. On the contrary, product states of the two CFTs correspond to disconnected bulk geometries, since both systems appear uncorrelated. Nevertheless, bipartite entanglement is not enough to ensure geometric connectivity. The state (1.8) moreover has a very unique entanglement spectrum, which captures the particular correlations of a short ER bridge. A small perturbation a scrambling time away in the past is able to completely destroy these correlations [37].

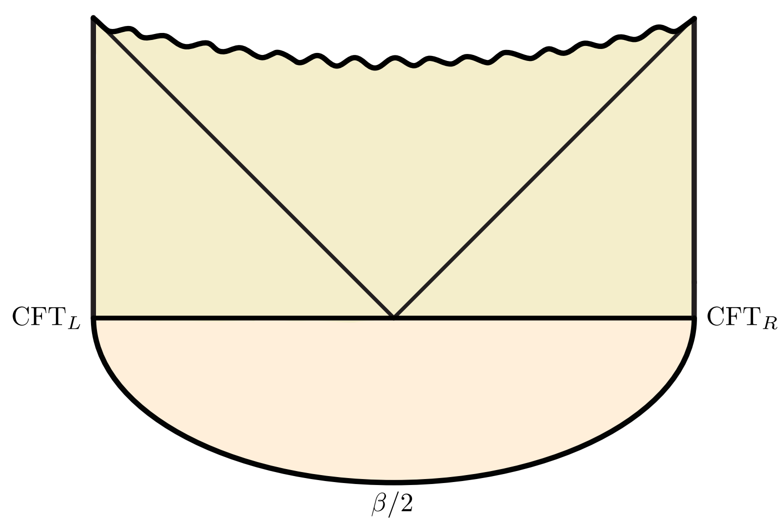

Figure 2: *The thermofield double state $|TFD\rangle$ is dual to an eternal black hole. In the euclidean section, the semi-circle represents the path integral that prepares the state on $\mathcal{H}_L \otimes \mathcal{H}_R$, and the filling of this circle corresponds to the euclidean AdS-Schwarzschild geometry. The bulk state of the quantum fields is the Hartle-Hawking state.*

[5] For example, at order $1/N$, only a three-point vertex contributes, and Φ has an improved HKLL representation in terms of smeared local and bilocal fields in the CFT.

Ryu and Takayanagi proposed a formula which strengthens this connection [38]. Consider the simplest case of the ground state of the CFT on the sphere, $|0\rangle$, and some bipartition of the Hilbert space $\mathcal{H}_{\text{CFT}} = \mathcal{H}_A \otimes \mathcal{H}_{\bar{A}}$, in terms of the degrees of freedom in some spatial region A and on its complement $\bar{A}$ (see Fig. 3). [6] The properties of the state on $\mathcal{H}_A$ are described by the mixed state $\rho_A = \text{Tr}_{\bar{A}}\, |0\rangle\langle 0|$. In particular, the overall amount of entanglement between A and $\bar{A}$ on this state is measured by the von Neumann entropy of the reduced density matrix, $S(\rho_A) = -\text{Tr}_A\,(\rho_A \log \rho_A)$. The proposal is that, in the large-N limit, the entanglement entropy is given by

$$S(\rho_A) = \frac{\text{Area}(\chi_A)}{4G}\,, \tag{1.9}$$

where χ_A is a bulk codimension-2 minimal surface, called the RT surface, anchored to ∂A and homologous to A. Even if the RT formula (1.9) looks structurally like the Bekenstein-Hawking entropy formula (1.1), and in fact it does reduces to the latter in certain cases, the notion of entropy involved in (1.9) is a much more fine-grained property of the actual quantum state $|0\rangle$.

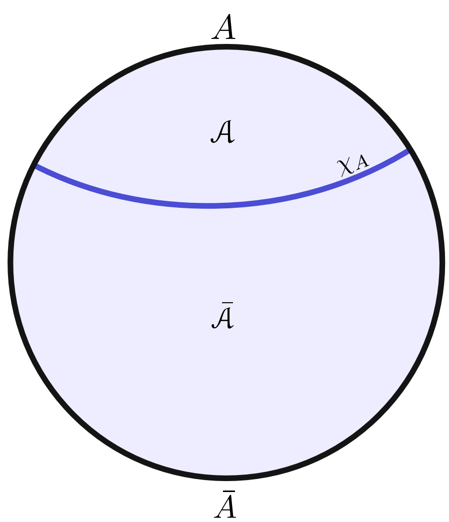

Figure 3: *Cauchy slice of empty AdS. The RT surface χ_A divides the bulk in two regions $\mathcal{A}$ and $\bar{\mathcal{A}}$.*

The RT formula was re-derived from 'first principles' in the work of Lewkowycz and Maldacena [40] (see also [41]). In this work, the replica trick is complemented in holographic systems *á la* Gibbons-Hawking by the rule of filling the bulk geometry with any allowable on-shell configuration which respects the asymptotic boundary conditions, and evaluating the on-shell gravitational action on each of these saddles. After the analytic continuation of the replica-symmetric geometry in the number of replicas, and taking this number close to one, the only contribution of the Einstein-Hilbert action which happens

[6] Strictly speaking, in the continuum, the Hilbert space of the CFT does not factorize, and only UV finite quantities, like the relative entropy, make sense. These have to be defined in terms of local operator subalgebras and the Tomita-Takesaki theory (see [39] and references therein). A similar situation occurs for the case of lattice gauge theories in finite volume, except that in this case entanglement entropy is finite. We skip such a level of rigor here, and think naively about the CFT on a lattice and forget about the gauge constraints.

to survive in this limit is localized on the RT surface χ_A and it reproduces (1.9) exactly.

Generalizations of the RT formula have also been extensively studied. On the one hand, the covariant version of the RT formula, valid for any time-dependent geometry, is the so-called HRT formula [42], in which χ_A appears as an extremal surface, rather than a minimal one. The second kind of generalization involves $1/N$ corrections, which in bulk terms correspond to the quantum corrections of the bulk supergravity fields. Using the same replica trick, but now keeping track of one-loop determinants and of backreaction, leads to the FLM formula [43]

$$S(\rho_A) \;=\; S_W(\chi_A) \;+\; S(\rho_{\mathcal{A}})\;, \tag{1.10}$$

where $S_W(\chi_A)$ is a local 'Wald-entropy' associated to χ_A, which at leading order in G reduces to the area term. On the other hand, the RT surface divides the bulk Hilbert space into two regions, $\mathcal{H}_{\text{code}} = \mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\bar{\mathcal{A}}}$, where $\mathcal{A}$ is a homology hypersurface between the RT surface and A, and $\bar{\mathcal{A}}$ is its complement (see Fig. 3). The quantity $S(\rho_{\mathcal{A}})$ represents the entanglement entropy of the bulk state between $\mathcal{A}$ and its complement. [7] The bulk domain of dependence of $\mathcal{A}$, denoted $D(\mathcal{A})$, is called the *entanglement wedge* of A.

Under linearized variations of the state, the 'first law of entanglement', together with (1.10), outputs the equality of modular Hamiltonians, or equivalently of relative entropies, of the subregions A and $\mathcal{A}$, within the code subspace [47]. A theorem follows from this assumption, ensuring the recovery of all the information in the entanglement wedge of A solely from the degrees of freedom of A [48]. Explicit ways to implement this reconstruction have also been provided in [49, 50].

The bulk-to-boundary map is redundant, since a bulk point X can belong to many different entanglement wedges, and hence the local operator $\Phi(X)$ has many representations within different subregions of the boundary system. This redundancy is crucial to guarantee that, no matter the part of the boundary $\bar{A}$ that one loses access to, the information of $\Phi(X)$ can be recovered from any A, as long as the bulk point lies within the entanglement wedge $X \in D(A)$. In this way, the gravitational hologram behaves very much like a *quantum error correcting code* [51] which maps the 'logical' bulk information into the 'physical' boundary in a way that it protects this information against the 'noise' which destroys arbitrary parts of the boundary [52].

Such a direct link between the emergence of space, the RT formula and error correction has lead to the appearance of discrete toy models which qualitatively recreate these basic features of AdS/CFT. The logic is that entanglement serves as a highly efficient way to parametrize the Hilbert space of a many-body quantum system, in particular, when trying to look for ground states of local Hamiltonians. A possible ansatz for the state is given in terms of a *tensor network*, a graph that inputs some inner structure for the wavefunction $\Psi_{i_1,\dots,i_N}$ by giving a geometric meaning to its entanglement. Precisely,

[7] Alternatively, the QES prescription [44] serves as a further generalization of the FLM formula (1.10), valid in principle to all orders in $1/N$, which has been used in recent approaches to the black hole information paradox [45, 46].

the entanglement entropy of some subregion is upper bounded by the minimal cut through the network which bipartites the system accordingly.

Holographic tensor networks are rooted on discrete realizations of the RT formula, in which the graph is a representation of the emergent hyperbolic geometry itself (see Fig. 4). This geometry emerges non-linearly from the entanglement structure of the ground state of a critical many body quantum system, which possess polynomially decaying correlations, as opposed to the exponentially decaying correlations in a gapped system. Some particular tensor network representations of holographic states specialize on capturing the scale invariant details of the entanglement structure, like MERA [53, 54], while others favor the exact realization of the RT formula and, at the same time, straightforwardly generalize to holographic quantum error correcting codes [55, 56]. [8]

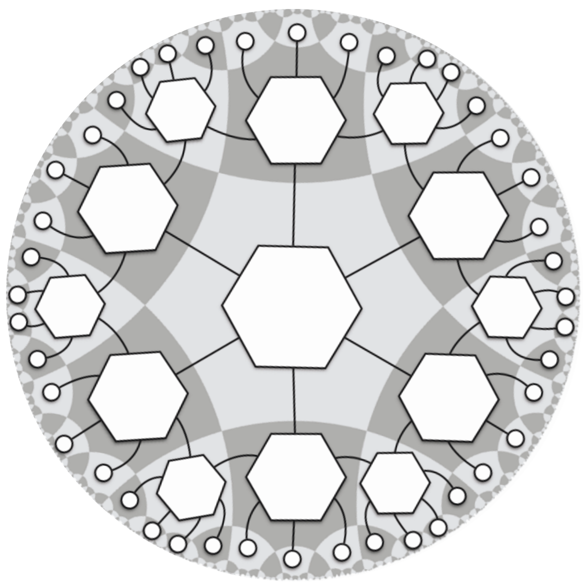

Figure 4: *HaPPY tensor network representing the ground state of the holographic system on the spatial* $\mathbf{S}^1$*. Each hexagon represents a so-called absolutely maximally entangled state on six parties. The tensor network is designed to saturate the RT formula. The complexity of the state, measured by the number of tensors in the network, is proportional to the volume of the slice.*

Holographic Volume Complexity

Quantum complexity has been identified as a key notion in the development of the holographic dictionary for its promise to offer a peek into the interior of black holes [58]. In the Complexity = Volume (VC) prescription, the complexity of a state is given by

$$\mathcal{C}(|\Psi\rangle) = \frac{\text{Vol}(\Sigma)}{G\ell} , \tag{1.11}$$

[8] A generic tensor network with hyperbolic geometry will also saturate the RT formula with very high probability, provided that the bond dimension of the network is large (see [57]).

where Σ is a extremal spacelike hypersurface, anchored to the boundary slice in which the state $|\Psi\rangle$ lives [59–61]. The heuristic motivation for this proposal is that it measures the overall amount of emergent space, a quantity which has a direct interpretation in tensor network jargon, namely as the computational complexity of the 'circuit' that efficiently represents the entanglement structure of the holographic state $|\Psi\rangle$ (see Fig. 4).

For high-temperature thermofield double states (1.8), the tensor network is a discrete representation of the ER bridge [35, 62]. Under the action of the Hamiltonian $H_L + H_R$, the interior cylindrical circuit grows a layer of S tensors per thermal time, yielding a total rate of

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} \sim TS\,, \tag{1.12}$$

in analogy with the interior volume growth of the black hole (see Fig. 5).

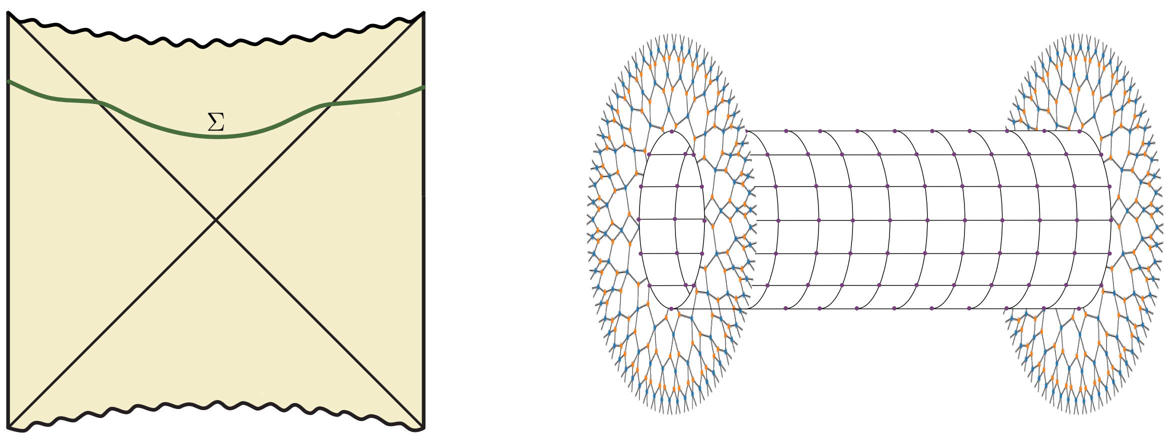

Figure 5: *On the left, the extremal volume slice Σ is a geometric ER bridge that grows linearly in time. On the right, the tensor network representation of the time-evolved thermofield double state. Under time evolution with $H_L + H_R$, the circuit grows a layer of tensors per thermal time, in analogy with the volume of Σ in the black hole interior. The right figure has been taken from* [63].

As opposed to entanglement entropy, the definition of VC purely in terms of the holographic variables remains elusive for the time being (see however [64–66] for some work in this direction), and we shall not try to address this issue here. At any rate, extremal spatial volumes parametrized by codimension-one boundary data are interesting quantities in any putative holographic description. Whether they are literally related to some sort of computational complexity of the boundary system is an open question, but it is certain that there exists a notion of 'volume complexity' induced from the bulk description.

Chapter II

Momentum/Size Correspondence

2 Momentum and Size

A recurrent idea since the early days of holography is that there is a somewhat implicit relation between notions of 'complexity' of the encoded quantum information in the boundary system and the degree of gravitational clumping of matter in the bulk [16]. Recent studies have substantiated this claim, providing quantitative evidence, within the framework of AdS/CFT and of its two-dimensional counterpart, in favor of the relation between a radial component of the momentum of an infalling particle and the rate of growth in 'size' of the dual operator [67–73]. In this section, we review the so-called momentum/size (PS) correspondence, which originally motivated [1–3].

2.1 Rindler momentum and fast scrambling

The proposal is that the 'size' of an operator can be characterized by a mechanical momentum of an effective particle in the bulk [67, 68]. The bulk particle is injected by the 'small' operator $\mathcal{O}$ on the boundary, acting for simplicity on a thermal reference state $\rho_\beta = e^{-\beta H}/Z(\beta)$ at, say $t = 0$. If the resulting state is evolved in time

$$e^{-itH}\,\mathcal{O}\,\rho_\beta\,e^{iHt} \;=\; \mathcal{O}_{-t}\,\rho_\beta\;, \tag{2.1}$$

any increase of complexity is attributed to the increase in 'size' of the operator when evolved to the past, in what we usually refer to a 'precursor': $\mathcal{O}_{-t} = e^{-itH}\mathcal{O}\,e^{itH}$. The state (2.1) can be interpreted as a heavy particle state falling through the bulk.

Consider a high-temperature reference state ρ_β describing a large black hole in AdS. The effective particle will fall towards the horizon and eventually will reach the near horizon region. In the vicinity of the regular horizon, we can pick polar Rindler coordinates (t, ρ) which approximate the metric as

$$\mathrm{d}s^2 \approx -\kappa^2\rho^2\mathrm{d}t^2 + \mathrm{d}\rho^2 + \mathrm{d}s_\perp^2\;, \tag{2.2}$$

where $\mathrm{d}s_\perp^2$ is a metric along the horizon which formally sits at $\rho = 0$, and $\kappa = \frac{2\pi}{\beta}$ is the surface gravity.

The motion of the radially infalling matter particle can be taken as $X = \rho_0$ in terms of the adapted reference frame $\tau = \rho\sinh(\kappa t)$ and $X = \rho\cosh(\kappa t)$, where τ is the proper time of the co-moving frame to the particle. In terms of the Rindler coordinates, the trajectory follows $\rho \approx \rho_0\,\exp(-\kappa t)$ at late Rindler times. The proper Rindler-radial momentum satisfies

$$P_\rho \;=\; -\frac{\mathrm{d}\rho}{\mathrm{d}\tau} \;\approx\; \kappa\,\rho_0\,e^{\kappa t}\;. \tag{2.3}$$

Since the surface gravity coincides with the fast-scrambling Lyapunov exponent, $\kappa = \lambda_L$, the idea is to relate P_ρ and operator size $\mathcal{S}(\mathcal{O})$, measured by the decay of a suitable out-of-time-order correlator [74]. In this case, both terms grow exponentially in time, so that the qualitative behavior only establishes $P_\rho \;\sim\; \mathcal{S}(\mathcal{O})$ as proportional to the size, or any of its higher time derivatives.

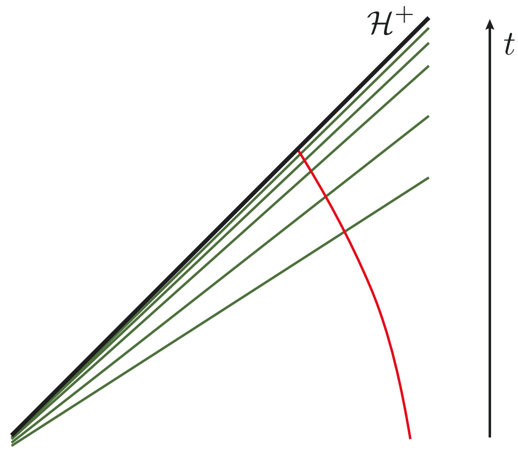

Figure 6: *Standard notions of momentum in the PS correspondence are defined in terms of near-horizon dynamics, using radial and time coordinates which remain outside the horizon.*

2.2 Free fall in a near extremal throat

The precise relation between momentum and size can be elucidated by considering the free fall of the effective particle in the presence of a near-extremal magnetic Reissner–Nordström black hole in four dimensions [69–71].

In the near horizon region, the geometry now contains an $AdS_2 \times \mathbf{S}^2$ throat (see Fig. 7), which in the s-wave sector is effectively described by Jackiw-Teitelboim (JT) gravity, with action

$$I_{\rm JT}[\mathcal{M}] \;=\; S_0\, \chi(\mathcal{M}) \;+\; \int_{\mathcal{M}} \phi \left(R + \frac{2}{\ell^2} \right) \;+\; \int_{\partial\mathcal{M}} 2\phi \left(K - \frac{1}{\ell} \right) \;. \tag{2.4}$$

Here $\chi(\mathcal{M})$ is the Euler characteristic of $\mathcal{M}$, and S_0 is the extremal entropy. The dilaton acts as a Lagrange multiplier which fixes the geometry to be locally AdS_2, $R = -2/\ell$, where $\ell = r_+$ is the radius of the higher dimensional black hole. The only gravitational degree of freedom of this system is located at the cutoff AdS boundary $\partial\mathcal{M}$, roughly speaking, at the local maximum of the higher dimensional s-wave potential barrier. Its dynamics is described by the Schwarzian action, which can be reinterpreted as a non-relativistic 'boundary particle' on a constant electric field [75]. The Schwarzian breaks the boundary reparametrization invariance of the Einstein-Hilbert action into an $SL(2, \mathbf{R})$ subgroup corresponding to the isometries of AdS_2 (see [76]).

An analogous pattern of symmetry breaking arises in the infrared sector of the large-N Sachdeev-Ye-Kitaev (SYK) model. This quantum mechanical model consists of N interacting Majorana fermions $\{\psi_i\}$ with quenched disorder given by the Hamiltonian

$$H \;=\; \sum_{i,j,k,l} J_{ijkl}\; \psi_i \psi_j \psi_k \psi_l \tag{2.5}$$

where the couplings J_{ijkl} are independent gaussian random variables with zero mean and variance

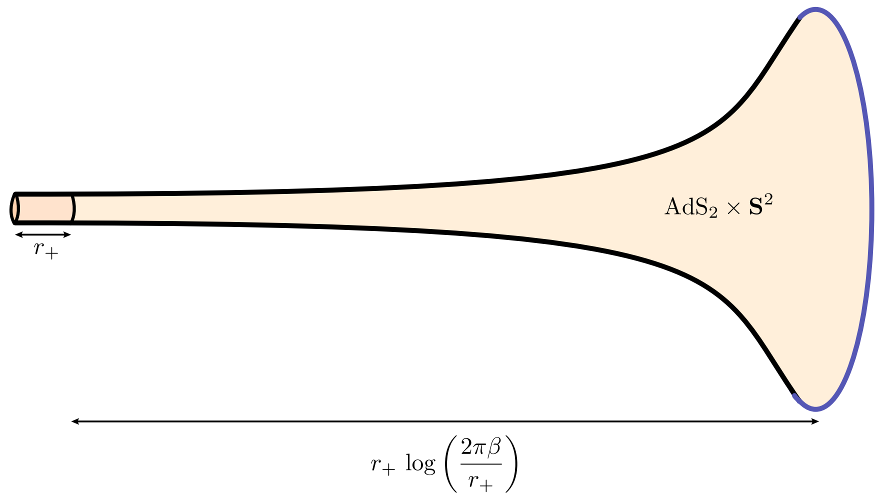

Figure 7: *Near horizon geometry of a near extremal RN black hole with horizon r_+ and inverse temperature $\beta \gg r_+$. The Rindler region sits at the end of the AdS$_2$ throat. The proper distance of the throat is enhanced by a factor of $\log(\beta/r_+)$. The blue boundary particle sits at the maximum of the potential barrier.*

$\overline{J^2} \sim \mathcal{J}^2/N^3$. In the large-$N$ limit, the theory becomes 'classical' in terms of the bilocal master field $G(\tau,\tau') = \frac{1}{N}\sum_i \langle \psi_i(\tau)\psi_i(\tau')\rangle$, and develops an emergent reparametrization invariance of this field in the deep infrared, $E \ll \mathcal{J}$. This symmetry gets spontaneously broken by the conformal solution $G \sim (\tau-\tau')^{-2\Delta}$, as well as explicitly broken by a small term, of order $E/\mathcal{J}$, into a $SL(2,\mathbf{R})$ subgroup. The effective theory of this soft mode is given, at leading order in a derivative expansion, by the same Schwarzian that governs the dynamical cutoff boundary of JT gravity (see [76, 77]). [9]

The parameter matching in the JT/SYK correspondence can be performed comparing the thermodynamic entropy of both systems at low temperatures. In the large-N SYK model with $\beta\mathcal{J} \gg 1$, the entropy scales as $S_\beta \approx S_0 + \frac{\alpha N}{\beta \mathcal{J}}$, where α is an $O(1)$ coefficient that determines the specific heat [77]. Remarkably, the model has a huge ground state degeneracy in the large-N limit, given by $S_0 = \gamma N$, where γ is another $O(1)$ coefficient. On the gravitational side, $S_\beta \approx S_0 + \frac{4\pi^2 r_+^3}{G\beta}$ is the near-extremal entropy of the four-dimensional black hole, in terms of the extremal radius r_+ and the four-dimensional Planck scale $\sqrt{G}$, and $S_0 = \pi\frac{r_+^2}{G}$ is the extremal entropy. The duality then requires

$$\mathcal{J} \sim \frac{1}{r_+}\,, \tag{2.6}$$

$$N \sim \frac{r_+^2}{G}\,. \tag{2.7}$$

modulo some numerical constants that can be found for instance in [70]. [10]

[9] No Goldstone modes arise from this 'symmetry breaking', since the $SL(2,\mathbf{R})$ has to be thought of as a redundancy in $G(\tau,\tau')$, rather than a normal symmetry which acts on the Hilbert space.

[10] In the putative holographic dual of the full SYK model, there is no sub-AdS locality, since the wavelength of all the excitations is parametrically controlled by the AdS size $\mathcal{J}^{-1} \sim r_+ = \ell$.

In this controlled framework, it is possible create a particle on the top of the near-extremal throat, basically by applying some single-fermion operator, like ψ_1, on top of the low-temperature thermal state of the SYK model, $\rho_\beta = e^{-\beta H}/Z(\beta)$. The typical energy of the excitation will be controlled by $\mathcal{J}$, which according to (2.6) agrees with the energy at the maximum of the potential barrier of the near extremal black hole. The strategy is to follow the motion of the effective particle along the throat, and try to extend the notion of Rindler-momentum (2.3) to the full AdS_2 region in order to match the dual behavior of the quantum mechanical 'size' of the precursor $\psi_1(t)$ in the SYK model.

Operator growth

To define a notion of operator size, it is necessary to first select a basis of operators and attribute a natural size to each of the elements of the basis. For a system with a random Hamiltonian, any choice of basis is completely ad-hoc, and the notion of size will therefore lack of physical meaning. If, however, the Hamiltonian possesses some degree of locality, it then makes sense to distinguish between small and large operators. In the SYK model, the Hamiltonian (2.5) is 4-local in the fermion basis, and hence it is natural to assign size n to the operator $\psi_{i_1}\psi_{i_2}...\psi_{i_n}$ with $i_1 < i_2 < ... < i_n$, which basically characterizes the number of different fermions that the operator contains.

Consider a time-evolved operator $\mathcal{O}(t) = e^{-iHt}\,\mathcal{O}\,e^{iHt}$ acting on the $2^{\frac{N}{2}}$-dimensional Hilbert space of the SYK model. [11] The operator admits an expansion in the fermion basis of the form

$$\mathcal{O}(t) \;=\; \sum_{n=1}^{N}\;\sum_{i_1<...<i_n} c_{i_1...i_n}(t)\, 2^{\frac{n}{2}}\;\psi_{i_1}\psi_{i_2}...\psi_{i_n}\;, \tag{2.8}$$

where $c_{i_1...i_p}(t)$ is interpreted as its 'wavefunction'. The *size* of the operator is simply

$$\mathcal{S}(\mathcal{O}(t)) \;\equiv\; \sum_{n=1}^{N}\;\sum_{i_1<...<i_n} |c_{i_1...i_n}(t)|^2\, n\;. \tag{2.9}$$

The size measured in this way can be recast as the square of the (anti)commutator of $\mathcal{O}(t)$ with one-site fermion operators, averaged over all sites,

$$\mathcal{S}(\mathcal{O}(t)) \;=\; \frac{1}{2}\sum_{i=1}^{N} 2^{-\frac{N}{2}}\,\mathrm{Tr}\left(\{O(t),\psi_i\}^\dagger\,\{O(t),\psi_i\}\right)\;. \tag{2.10}$$

In this latter form, it is manifest that the evolution of (2.10) towards saturation is dominated by the behavior of the out-of-time-order correlator (OTOC) at infinite temperature, and in fact it is possible to show that

$$N - \mathcal{S}(\mathcal{O}(t)) = (-1)^{|\mathcal{O}|}\sum_{i=1}^{N} 2^{-\frac{N}{2}}\,\mathrm{Tr}\left(O(t)^\dagger\psi_i\,O(t)\psi_i\right)\;, \tag{2.11}$$

[11] The operator is for simplicity assumed to be normalized, that is, $2^{-\frac{N}{2}}\,\mathrm{Tr}(\mathcal{O}^\dagger\mathcal{O}) = 1$.

where $(-1)^{|\mathcal{O}|} = 0, 1$ depending on whether the operator $\mathcal{O}$ is bosonic or fermionic, respectively. The OTOC serves to diagnose quantum chaos in a very precise way, namely by reading off the 'Lyapunov exponent' [74].

At finite temperature, the notion of size (2.9) can be generalized. Basically, it is convenient to smear the operator $\mathcal{O}$ in order to wash out the effects of very high energetic modes and isolate the physics associated with the thermal scale β (see e.g. the definition of the OTOC in [74]). A way to do this in the formulations (2.10) or (2.11) is to perform an euclidean time evolution by $\beta/2$ of each operator, that is, consider $\mathcal{O}\,\rho_\beta^{1/2}$ instead. The *finite-temperature size* of the operator is defined as

$$\mathcal{S}_\beta(\mathcal{O}) \equiv \frac{\mathcal{S}(\mathcal{O}\,\rho_\beta^{1/2}) - \mathcal{S}(\rho_\beta^{1/2})}{\delta_\beta} , \tag{2.12}$$

where δ_β is a normalization such that the size of a single fermion is one, $\sum_i \mathcal{S}_\beta(\psi_i) = 1$.

Qi and Streicher found a really simple formula for the growth of the thermal size of a fermion operator, in the low-temperature limit of the large-N SYK model [78]

$$\mathcal{S}_\beta(\psi_1(t)) \approx 2\frac{\beta^2\mathcal{J}^2}{\pi^2} \sinh^2\left(\frac{\pi t}{\beta}\right) \tag{2.13}$$

At early times, $t \ll \beta$, the operator grows quadratically as $\mathcal{J}^2\, t^2$, while at late times $t \gtrsim \beta$, the growth becomes exponential with fast scrambling Lyapunov exponent $\lambda_L = \frac{2\pi}{\beta}$, again saturating the bound in [74, 79].

Phenomenological PS correspondence

With the Qi-Streicher formula (2.13) at hand, it is now possible to find a generalization of the notion of Rindler-momentum P_ρ that captures the initial growth of the fermion operator in the SYK model.

The AdS_2 metric is

$$\mathrm{d}s^2 \approx -e^{-2\rho/r_+}\,\mathrm{d}t^2 + \mathrm{d}\rho^2 . \tag{2.14}$$

This near extremal throat extends from the cutoff boundary at $\rho = 0$, up to $\rho = r_+ \log\left(\frac{\beta}{2\pi r_+}\right)$, where the Rindler region starts, and the metric becomes (2.2) for a proper length $\Delta\rho = r_+$ (see Fig. 7). [12]

Consider the motion of the effective particle in free fall through the AdS_2 throat. From time-translation symmetry, the proper energy of the particle is conserved, which gives the condition $\mathrm{d}t/\mathrm{d}\tau = e^{2\rho/r_+}$, where τ is the proper time of the particle. This condition allows to solve explictly for the

[12] In the exact near-extremal metric, there is a small transition region of length $\sim r_+$ between the AdS throat and the Rindler region. The details of this region are unimportant for the purposes of matching the two parametric behaviors of (2.13) up to $O(1)$ numerical constants.

trajectory of the particle

$$\rho(t) = \frac{r_+}{2} \log\left(1 + \frac{4t^2}{r_+^2}\right) . \tag{2.15}$$

The natural candidate for an extension of the Rindler momentum (2.3) on the throat is the proper AdS-radial momentum, which behaves as

$$P_\rho = -\frac{1}{r_+}\frac{\mathrm{d}\rho}{\mathrm{d}\tau} = \frac{2t}{r_+^2} \approx 2\mathcal{J}^2\, t , \tag{2.16}$$

where we used (2.6) to relate r_+^{-1} to the SYK energy scale $\mathcal{J}$. [13] This behavior is prolonged until $t \approx \beta/\pi$, the moment at which the particle reaches the Rindler region. In the Rindler region, the proper radial momentum again follows an exponential growth

$$P_\rho \approx \frac{2\mathcal{J}^2\,\beta}{\pi}\, e^{\frac{2\pi}{\beta}t}. \tag{2.17}$$

Comparing these two regimes with (2.13), we conclude that the proper radial momentum phenomenologically agrees with the time derivative of the size of the dual operator in the SYK model, which establishes the precise PS correspondence

$$\boxed{P_\rho \approx \frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{S}_\beta(\psi_1(t))}\, , \tag{2.18}$$

or $P = \dot{\mathcal{S}}$ for short.

Additionally, the PS correspondence gives an intuition on why the scrambling time is smaller for a near extremal black hole, given by

$$t_s \sim \beta \log \frac{S - S_0}{\delta S} , \tag{2.19}$$

where δS is the entropy of the effective matter particle. The reason is that the operator is already 'big', of size $\mathcal{J}^2\beta^2$, when it enters the Rindler region of exponential growth. Therefore, it takes less time for the operator to fully scramble. The extremal degrees of freedom of the black hole do not decouple from the dynamics of the operator, as the naive interpretation of (2.19) suggests. What really happens is that the apparent decoupling is an artifact of the emergent AdS throat at low temperatures [69].

Proof of the PS correspondence

A formal proof of the PS correspondence $P = \dot{\mathcal{S}}$ has also been provided in [71]. The proof follows from the analysis of the representation of the $SL(2,\mathbf{R})$ symmetry generators in the semiclassical Hilbert space of JT gravity plus matter, $I_{\mathrm{JT}} + I_{\mathrm{matter}}$, where the latter only couples to the metric. In the bulk, the group of symmetries is generated by the 'boost' generator B, the global Hamiltonian E, and the

[13] We added the prefactor r_+^{-1} to define a momentum with units of energy. The 'proper mass' of the particle is of order $r_+^{-1} \sim \mathcal{J}$.

radial momenum operator P (see Fig. 8). Together they form the algebra

$$[B, E] = -i\mathcal{J}^{-1} P \tag{2.20}$$

$$[B, P] = -i\mathcal{J}\, E \tag{2.21}$$

$$[P, E] = -i\mathcal{J}\, B\ , \tag{2.22}$$

where, for convenience, we have added dimension $\mathcal{J}$ to the momentum P, while B and E remain dimensionless. These symmetries are, in the strict sense, gauge redundancies of the Schwarzian description, and therefore all of the physical states of the Hilbert space must be uncharged under them.

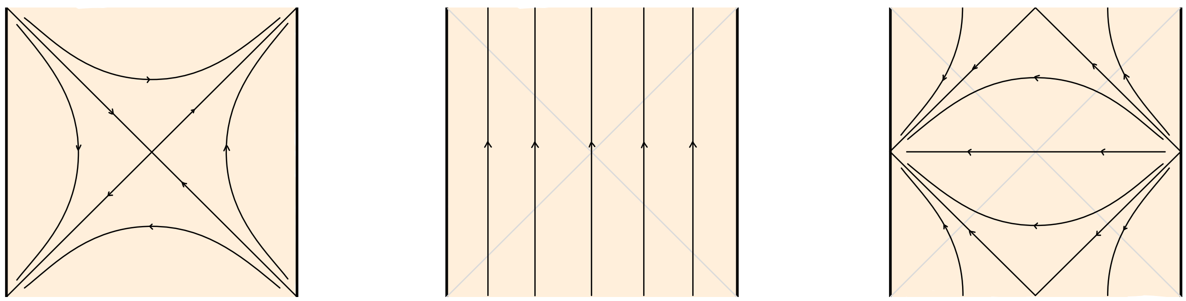

Figure 8: *Orbits of the $SL(2, \mathbf{R})$ generators. From left to right, the modular Hamiltonian B, the global energy E and the momentum P.*

In the SYK, the $SL(2, \mathbf{R})$ reparametrization subgroup only emerges in the deep infrared, and hence, it will act non-trivially on high-energy states. To construct the generators, it is therefore convenient to take the formal limit of large N and zero temperature $\beta\mathcal{J} \to \infty$, while keeping the ratio $\frac{N}{\beta J}$ fixed. In this regime, it is possible to translate the symmetry generators from the exact Schwarzian theory (see [71]), which on the doubled SYK Hilbert space $\mathcal{H}_L \otimes \mathcal{H}_R$ read

$$B \;=\; \frac{\beta}{2\pi}\,(H_R - H_L)\ , \tag{2.23}$$

$$E \;=\; \frac{\beta}{2\pi}\,(H_R + H_L + H_{\text{int}}) \;-\; E_0\ , \tag{2.24}$$

and $P = i\mathcal{J}[B, E]$. The global energy E has an extra bilocal interaction between both boundaries, which has the form $H_{\text{int}} = i\mu \sum_i \psi_i^L \psi_i^R$, and $\frac{\mu}{\mathcal{J}}$ is a parameter which depends on $\beta\mathcal{J}$. The form of this term was originally elucidated in [80] to couple the two boundaries, making the wormhole traversable. [14] The constant E_0 ensures that the value of E for the thermofield-double vanishes, that is, it satisfies the $SL(2, \mathbf{R})$ gauge constraints.

The remarkable feature of (2.24) is that the operator $\mathcal{S} = i \sum_i \psi_i^L \psi_i^R + \frac{N}{2}$ connects to the previous definition of size (2.12). Indeed, it is possible to rewrite the thermal size in terms of the expectation

[14] H_{int} has an analogous version for a general QFT in flat space. The global Hamiltonian has a interaction term in the boundary that separates both Rindler wedges. This is consistent with the fact that the global time-evolution connects both Rindler regions, while the modular Hamiltonian does not.

value of $\mathcal{S}$ in the doubled Hilbert space [78]

$$\mathcal{S}_\beta(\mathcal{O}) \;=\; \frac{\langle \text{TFD}|\, \mathcal{O}_R^\dagger \mathcal{S} \mathcal{O}_R \,|\text{TFD}\rangle \,-\, \langle \text{TFD}|\, \mathcal{S} \,|\text{TFD}\rangle}{\delta_\beta}\;. \tag{2.25}$$

Consider a state of the form $|\Psi(t)\rangle \;=\; \mathcal{O}_R(t)\,|\text{TFD}\rangle$, where t is the right boundary time generated by H_R. At the level of expectation values on $|\Psi(t)\rangle$, we can replace $B = -i\frac{\beta}{2\pi}\frac{\mathrm{d}}{\mathrm{d}t}$, which leads to

$$P = i\mathcal{J}[B,E] \;=\; \frac{\beta\mathcal{J}}{2\pi}\frac{\mathrm{d}E}{\mathrm{d}t} \;=\; \frac{\beta^2\mathcal{J}\mu}{4\pi^2}\frac{\mathrm{d}\mathcal{S}}{\mathrm{d}t}\;. \tag{2.26}$$

More explicitly, writing down the expectation values, and using (2.25) yields the familiar version of the PS correspondence

$$\boxed{\langle\Psi(t)|\, P \,|\Psi(t)\rangle \;=\; \frac{\beta^2\mu\delta_\beta\mathcal{J}}{4\pi^2}\,\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{S}_\beta(\mathcal{O}(t))}\,, \tag{2.27}$$

where the prefactor can be shown to be an $O(1)$ number, as in (2.18). The momentum P generates spatial translations along the $-\partial_\rho$ vector field in AdS_2, and therefore it coincides with the kinematical quantity P_ρ when evaluated for heavy particle states.

2.3 Remarks

The PS correspondence opens up a fascinating way to re-interpret the origin of the gravitational attraction, namely as the tendency for an operator to scramble between the holographic degrees of freedom. [15] It relates a linear kinematical quantity, P, whose evolution is governed by the laws of General Relativity, to a fine-grained measure of the quantum state $\dot{\mathcal{S}}$ in the internal large-N space, whose evolution is completely governed by quantum chaotic dynamics.

The derivation of $P = \dot{\mathcal{S}}$ presented above, however, gives the impression to rely on the presence of a large black hole originating the background gravitational field in which the effective particle propagates. Gravitational attraction, on the other hand, is a universal feature of all forms of energy. A more satisfactory correspondence should be applicable to a much broader class of reference states, as well as to higher dimensional standard AdS/CFT setups.

Additionally, the putative generalization of ‘size’ must have all sorts of phenomenology, depending on the reference state [72, 73]. For instance, in the low-lying part of the spectrum of the CFT, an oscillatory behavior of this ‘size’ is generically expected, since the spectrum is nearly integrable. In the bulk, these oscillations occur from the trajectory of the heavy particle in the empty AdS potential.

[15] In some sense, the PS correspondence resonates with some of the ideas in [81], even if the notion of entropy is replaced by ‘complexity’ here.

Chapter III

Momentum / Complexity Correspondence

3 Operator Complexity

Measures of operator complexity have received considerable recent attention in studies of information scrambling in many-body quantum systems [54, 78, 82–90]. One motivation is the characterization of operator complexity in holographic systems. A recurring theme in this context is the notion that gravitational 'clumping' increases complexity of the dual quantum state. If a black hole is formed, this is realized in the most extreme way, as the complexity keeps growing linearly well after the black hole has equilibrated its exterior geometry. However, the growth of complexity occurs for any gravitational infall of matter, however dilute, as indicated by explicit calculations for collapsing thin shells [91, 92]. A time-reversal transformation to a situation with matter outflow should instead decrease the complexity, suggesting that there is a relation between some average 'infall momentum' and the rate of complexity change.

The bulk particle is 'injected' by the 'small' operator $\mathcal{O}$ on the boundary, acting on some reference state $\mathcal{O}\,|\Psi\rangle$ at, say $t = 0$. If the resulting state is evolved in time

$$e^{-itH}\,\mathcal{O}\,|\Psi\rangle = e^{-itH}\,\mathcal{O}\,e^{itH}\,e^{-itH}\,|\Psi\rangle = \mathcal{O}_{-t}\,|\Psi\rangle_t\ , \tag{3.1}$$

any increase of complexity can be attributed partly to the increase in complexity of the time-evolved reference state $|\Psi\rangle_t$, and partly to the increase in complexity of the operator when evolved to the past, in what we usually refer to a 'precursor': $\mathcal{O}_{-t} = e^{-itH}\mathcal{O}\,e^{itH}$. If the increase in complexity of the reference state can be neglected or somehow subtracted, we can define the complexity of the operator $\mathcal{O}_{-t}$ in terms of the complexity of the evolved state. The state (3.1) can be interpreted as a heavy particle state falling through the bulk. More precisely, we may define the operator complexity in terms of the state complexity by the subtraction

$$\mathcal{C}_{\mathcal{O}}(t) = \mathcal{C}\left[\mathcal{O}_{-t}|\Psi\rangle_t\right] - \mathcal{C}\left[|\Psi\rangle_t\right]\ , \tag{3.2}$$

with some appropriate normalization. In practice, this definition must be supplemented by some definite prescription for the state complexity such as, for example, the size (2.9), the VC (1.11) or other complexity proposals [93–95].

Let us suppose that the state (3.1) can be interpreted as a heavy particle falling through the bulk. Then, the proposal of a momentum/complexity correspondence (PC correspondence for short) amounts to a relation of the form

$$\frac{\mathrm{d}\mathcal{C}_{\mathcal{O}}}{\mathrm{d}t} = P_{\mathcal{C}}\ , \tag{3.3}$$

where $\mathcal{C}_{\mathcal{O}}$ is the complexity of the operator, and $P_{\mathcal{C}}$ is a suitable component of the mechanical momentum of the associated particle. On general grounds, the right-hand side of (3.3) has an inherent ambiguity, since we must specify which particular momentum component is the relevant one, and this selects a particular coordinate system.

A particular case of (3.3) is the momentum/size correspondence reviewed in Chapter II, which involves the particle fall towards the horizon, as indicated in Fig. 6, and an interpretation of $\mathcal{C}_{\mathcal{O}}(t)$ in terms of operator size $\mathcal{S}_\beta(\mathcal{O}(t))$ defined by (2.12) in the quantum mechanical dual system. In that case, the momentum $P_\rho = -\frac{\mathrm{d}\rho}{\mathrm{d}\tau}$ is defined with respect to the proper radial coordinate ρ in the near horizon region, which interpolates between the AdS proper momentum and the Rindler proper momentum for near extremal throats.

A limitation of the PS correspondence is that, for it to hold for all times, it needs to be strictly interpreted in the large N limit, where the operator never fully scrambles. In systems with finite size, operator growth as such should stop at the scrambling time, of order $t_s \sim \lambda_L^{-1} \log N_{\text{eff}}$, where N_{eff} is the effective number of degrees of freedom. In the picture of bulk infall, the scrambling time corresponds to the particle reaching the stretched horizon, a timelike layer situated about one Planck length away from the horizon (see Fig. 9).

An interesting question is whether it is possible to establish a different type of PC correspondence for operator complexity that would operate at times much larger than the scrambling time. In this regime, complexity and size are not expected to be proportional: while operator size should saturate, an operator complexity defined as in (3.2) should grow linearly at long times, with a slope proportional to the average energy injected in the system by the action of the operator. This is expected in tensor-network or quantum circuit definitions of complexity, but it also seems to hold in different definitions of operator complexity, such as K-complexity [85], which was recently shown to exhibit the characteristic linear growth at late times [86, 87].

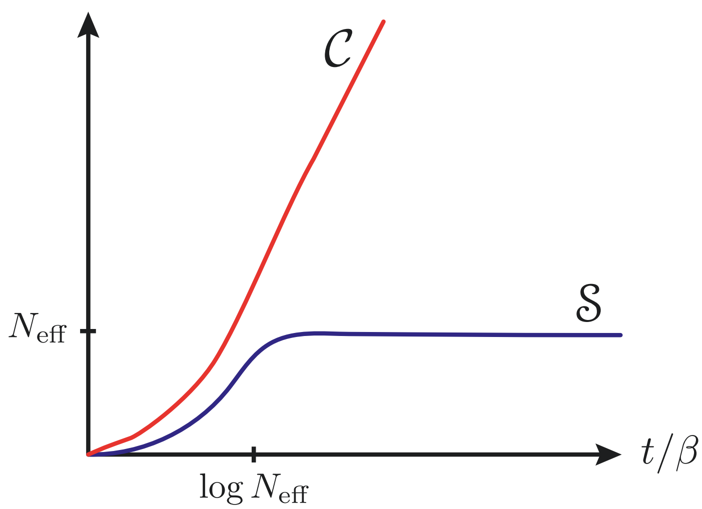

Figure 9: *For a fast scrambler, complexity and size are proportional before scrambling. Size saturates at the scrambling time, while complexity keeps growing linearly for a much longer timescale.*

4 Momentum and Complexity of Thin Shells

In this section we show that, adopting VC prescription (1.11) as the definition of (3.2), a momentum/volume complexity (PVC) correspondence of the form (3.3) exists at all times, for operators that are dual to spherical shells falling on timelike trajectories. The momentum $P_{\mathcal{C}}$ is that of the shells, measured with respect to a particular radial coordinate which we specify. More precisely, we find

$$\frac{\mathrm{d}\mathcal{C}_{\mathcal{O}}}{\mathrm{d}t} = P_{\mathcal{C}}(t) = -\int_{\Sigma_t} N^{\mu}_{\Sigma}\, T_{\mu\nu}\, \mathcal{C}^{\nu}_{\Sigma} \,, \tag{4.1}$$

where Σ_t is a maximal-volume surface anchored at boundary time t, the basic ingredient of the VC definition, N_{Σ} is the unit normal to Σ_t and $\mathcal{C}_{\Sigma}$ is a suitable radial vector field defined on Σ_t. In this form of the PVC correspondence, the shells only contribute through their energy momentum tensor, and the 'suitable coordinate system' to measure the momentum is obtained by foliating the bulk spacetime with the extremal-volume surfaces themselves. Therefore, we expect (4.1) to have a much wider generality than the thin-shell dynamics which was used for its derivation, and we will indeed give a more general proof of this correspondence in section 5. The compatibility of a constant late-time complexity rate and a constant bulk matter momentum results form the late-time accumulation of maximal surfaces in the black hole interior, a well-known property of the VC prescription.

4.1 Thin-shell operators and states

For a holographic CFT defined on a spherical spatial manifold $\mathbf{S}^{d-1}$ of radius ℓ, we consider its gravity dual on AdS_{d+1}, also taken to have curvature radius ℓ. A thin shell of dust injected from the AdS boundary can be represented in the CFT by the action of a formal product operator

$$\mathcal{O}_{\text{shell}} \sim \prod_{D_\Lambda \in \mathcal{P}_\Lambda} \phi_{\Lambda, D_\Lambda} \,, \tag{4.2}$$

where $\mathcal{P}_\Lambda$ is a partition of the sphere in domains D_Λ of size Λ^{-1}, the regularization cutoff. The operators ϕ_{Λ,D_Λ} can be seen as bulk operators, applied at radius of order $r_\Lambda \sim \Lambda\,\ell^2$, and smeared over the domain D_Λ. The idea is to use ϕ_{Λ,D_Λ} to inject a heavy bulk particle at radius r_Λ. Although we imagine specifying the operators in bulk effective field theory, we can always regard it as a CFT operator by a bulk-boundary reconstruction map, say using the HKKL formulation [31, 32].

These operators are 'big' in the sense of the spatial structure, but are 'simple' in holographic terms, since they are constructed from operators near the boundary of AdS. By appropriately choosing ϕ_{Λ,D_Λ}, we can generate a semiclassical state whose subsequent evolution is parametrized as the collapse of the shell of particles in the bulk geometry. In the case that the local factors ϕ_{Λ,D_Λ} are engineered with very massive bulk fields, or equivalently CFT operators with very large conformal weight, we can regard the shell as composed of classical massive particles forming a dust cloud with density σ and four-velocity field u^μ.

For the purposes of this section, we define the operator complexity in terms of the general prescription (3.2), where the state complexity is regarded as computed with the VC prescription. For technical convenience, we shall take the high-temperature thermofield double state as the reference state on the Hilbert space of two copies of the CFT, and the shell state is injected on the Right CFT as indicated in Figure 10, at times much larger than the thermalization time T^{-1}, where T is the Hawking temperature of the black hole.

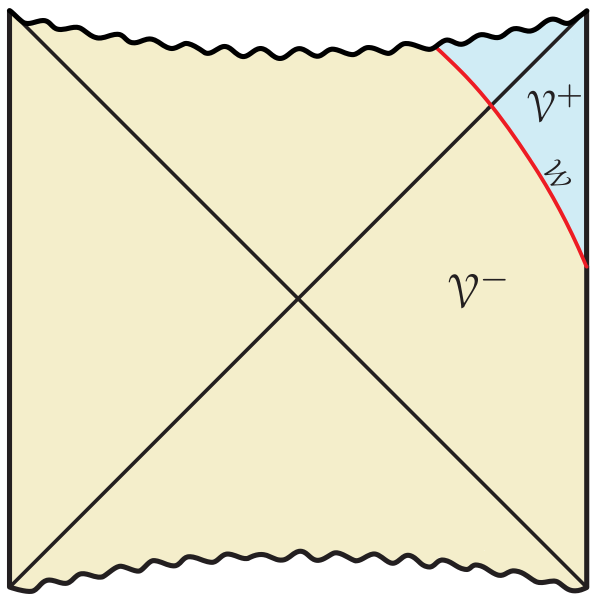

Figure 10: *Penrose diagram of the collapsing shell geometry. The shell is injected in the bulk at late times compared with T^{-1}, causing the initial black hole of mass M_- to grow up to the bigger mass M_+. The worldvolume of the matter shell is labelled $\mathcal{W}$ and sets the boundary between the two black hole spacetimes $\mathcal{V}^\pm$.*

The complexity of the shell operator is defined in terms of bulk quantities as

$$\mathcal{C}\left[\mathcal{O}_{\text{shell}}\right] = \frac{d-1}{8\pi\, G\, \ell}\left[\text{Vol}(\Sigma_{\text{bh+shell}}) - \text{Vol}(\Sigma_{\text{bh}})\right] \,, \tag{4.3}$$

where Σ denotes the extremal codimension-one hypersurface with given asymptotic boundary conditions, defined in the eternal black hole spacetime with and without the shell. The concrete prefactor in (4.3) is chosen for convenience of normalization. From now on shall measure bulk lengths in units of curvature radius, so that we set $\ell = 1$.

The worldvolume of the thin shell is a codimension-one timelike manifold $\mathcal{W}$ which divides the spacetime manifold in two regions: $\mathcal{V}^+$ is a Schwarzschild-AdS solution of mass M_+ which we identify as 'exterior' or 'right' region, and $\mathcal{V}^-$, a similar solution of mass M_- referred to as the 'interior' or 'left' region. The ADM energy of the shell is given by $M_+ - M_-$ and is assumed to be positive. Spherical symmetry holds globally in the full spacetime, whereas stationarity is broken at $\mathcal{W}$. Both $\mathcal{V}^\pm$ have smooth Killing vectors which are timelike in the asymptotic regions and spacelike inside event horizons. Denoting these vectors as $\xi_\pm = \partial/\partial t_\pm$, where $t_\pm$ are adapted coordinates, we can write a standard form of the metric on both sides of $\mathcal{W}$:

$$\mathrm{d}s_\pm^2 = -f_\pm \mathrm{d}t_\pm^2 + f_\pm^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{d-1}^2, \tag{4.4}$$

where

$$f_\pm = 1 + r^2 - \frac{16\pi G M_\pm}{(d-1)V_\Omega r^{d-2}} , \tag{4.5}$$

and $V_\Omega = \mathrm{Vol}(\mathbf{S}^{d-1})$. The shell dynamics follows from Einstein's equations, which take the form of junction conditions (cf. [96, 97]). Denoting the induced metric on $\mathcal{W}$ as

$$\mathrm{d}s_\mathcal{W}^2 = -\mathrm{d}\tau^2 + R(\tau)^2\mathrm{d}\Omega_{d-1}^2 , \tag{4.6}$$

in terms of the shell's proper time τ and its radius $R(\tau)$, continuity of the spacetime metric across $\mathcal{W}$ implies the first junction condition,

$$f_\pm(R)\left(\frac{\mathrm{d}t_\pm}{\mathrm{d}\tau}\right)^2 - \frac{1}{f_\pm(R)}\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 = 1 . \tag{4.7}$$

The second junction condition establishes the jump of the extrinsic curvature across $\mathcal{W}$ as proportional to the stress-energy on the shell's world-volume. For a thin shell of dust we have

$$T_{\mu\nu} = \sigma\, u_\mu\, u_\nu\, \delta(\ell) , \tag{4.8}$$

where u^μ is the four-velocity field of the shell and σ is the surface density. The coordinate ℓ measures proper distance away from $\mathcal{W}$ in the orthogonal spacelike direction, increasing towards the exterior region; in other words, the normal unit vector $N_\mathcal{W} = \partial/\partial\ell$ satisfies $N_\mathcal{W}^2 = 1$ and $u_\mu\, N_\mathcal{W}^\mu = 0$. For spherically infalling dust the density $\sigma(R)$ must be inversely proportional to the shell's volume, that is to say, the total rest mass

$$m = \sigma\, V_\Omega\, R^{d-1} \tag{4.9}$$

remains constant.

The second junction condition specifies the jump in extrinsic curvature across $\mathcal{W}$,

$$\sqrt{\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 + f_-(R)} - \sqrt{\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 + f_+(R)} = \frac{8\pi G}{d-1}\,\sigma\, R . \tag{4.10}$$

The particular conditions of spherical symmetry and stationarity along $\mathcal{V}^\pm$ allow us to write the junction conditions in terms of the Killing vectors $\xi_\pm$, an expression that will be useful later. Using that $\xi_\mu = g_{t\mu}$ and the explicit form of the metric (4.4) we find

$$(u \cdot \xi)_\pm = -f_\pm \frac{\mathrm{d}t_\pm}{\mathrm{d}\tau} \,. \tag{4.11}$$

Furthermore, since $\xi_\pm$ are orthogonal to the angular spheres, the normalization implies

$$g_{\mu\nu}\, \xi^\mu_\pm\, \xi^\nu_\pm = (\xi_\pm)^2 = -(u \cdot \xi_\pm)^2 + (N_\mathcal{W} \cdot \xi_\pm)^2 = -f_\pm \,, \tag{4.12}$$

an expression which determines $N_\mathcal{W} \cdot \xi_\pm$ once we know $u \cdot \xi_\pm$. Using (4.11) and (4.12) we may recast the two junction conditions as jumping rules for the Killing vectors, namely the component normal to $\mathcal{W}$ is continuous

$$N_\mathcal{W} \cdot \xi_+ \Big|_\mathcal{W} = N_\mathcal{W} \cdot \xi_- \Big|_\mathcal{W} \,, \tag{4.13}$$

whereas the component tangential to $\mathcal{W}$ jumps like the extrinsic curvature,

$$(u \cdot \xi_+ - u \cdot \xi_-)\Big|_\mathcal{W} = \sqrt{\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 + f_-(R)} - \sqrt{\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 + f_+(R)} = \frac{8\pi G}{d-1}\, \sigma\, R \,. \tag{4.14}$$

Equivalently, we can say that both junction conditions boil down to the jump rule:

$$(\Delta \xi^\mu)_\mathcal{W} \equiv \left(\xi^\mu_+ - \xi^\mu_-\right)\Big|_\mathcal{W} = -\frac{8\pi G}{d-1}\, \sigma\, R\, u^\mu \,. \tag{4.15}$$

One more presentation of the shell dynamics is obtained by extracting from (4.10) the ADM mass of the shell as a constant of motion:

$$M_\text{shell} = M_+ - M_- = m\sqrt{\left(\frac{\mathrm{d}R}{\mathrm{d}\tau}\right)^2 + f_-(R)} - \frac{4\pi G}{(d-1)V_\Omega}\frac{m^2}{R^{d-2}} \,. \tag{4.16}$$

This can be interpreted as a kinetic contribution proportional to the shell's rest mass m, corrected by a gravitational self-energy term. In fact, the constancy of m suggests a natural $(1+1)$-dimensional picture in terms of an effective particle of mass m, moving in the two-dimensional section of the metric obtained by simply deleting the angular directions:

$$\mathrm{d}s^2_{1+1} = \bar{g}_{\alpha\beta}\, \mathrm{d}x^\alpha\, \mathrm{d}x^\beta = -f_-(r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f_-(r)} \,. \tag{4.17}$$

In particular, the shell energy (4.16) can be obtained as the canonical energy from the effective action of a free particle

$$S_\text{eff} = \int \mathrm{d}\lambda\, L_\text{eff} = -m \int \mathrm{d}\lambda \sqrt{\bar{g}_{\alpha\beta}\frac{\mathrm{d}x^\alpha}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\beta}{\mathrm{d}\lambda}} \,, \tag{4.18}$$

provided we can neglect the gravitational self-energy effects.

4.2 Proof of the PVC correspondence for thin shells

Our goal is to derive a PVC correspondence relation by direct evaluation of the left hand side of (3.3), with $\mathcal{C}_{\text{shell}}$ defined as in (4.3). This will allow us to identify the correct component of 'radial momentum'. The complexity being defined through the VC prescription, we start with a preliminary discussion of extremal-volume surfaces in the relevant geometries.

Extremal volumes

Let a codimension-one spacelike surface Σ be defined by the embedding functions $X^\mu(y^a)$, with y^a coordinates along the hypersurface. The volume functional reads

$$V\,[\Sigma] \;=\; \int_\Sigma \mathrm{d}^d y\,\sqrt{h}\;, \tag{4.19}$$

where $h_{ab} \;=\; \partial_a X^\mu\,\partial_b X^\nu\, g_{\mu\nu}(X)$ is the induced metric on Σ. [16] Under a generic variation δX^μ the volume varies as

$$\delta V = \int_\Sigma (\text{e.o.m.})_\mu\, \delta X^\mu + \int_{\partial\Sigma} \mathrm{d}S^a\, \partial_a X_\mu\, \delta X^\mu \;. \tag{4.20}$$

where

$$(\text{e.o.m.})_\mu \;=\; -\frac{1}{\sqrt{h}}\,\partial_a\left(\sqrt{h}\,h^{ab}\,g_{\mu\nu}\,\partial_b X^\nu\right) \;+\; \frac{1}{2}\,h^{ab}\,\partial_a X^\rho\,\partial_b X^\sigma\,\partial_\mu g_{\rho\sigma} \tag{4.21}$$

vanishes precisely when the hypersurface Σ is extremal. In this case, the variation reduces to a boundary term,

$$\delta V\Big|_{\text{extremal}} = \int_{\partial\Sigma} \mathrm{d}S^a\, e^\mu_a\, \delta X_\mu\;, \tag{4.22}$$

where we have defined the vector fields $e^\mu_a = \partial_a X^\mu$ tangent to Σ.

For the geometry of interest here, Σ is a cylindrical manifold of topology $\mathbf{R}\times\mathbf{S}^{d-1}$, the boundary having two disconnected components consisting of spheres at the left and right spatial infinities. We shall use the same future-directed time variables on both boundaries and take a left-right symmetric time configuration $t_L = t_R = t$, so that we can write the following boundary conditions at the regularization surfaces $r = r_\Lambda$:

$$\delta X^\mu_\pm\Big|_{r=r_\Lambda} = \pm\delta t\;\xi^\mu_\pm\Big|_{r=r_\Lambda}\;, \tag{4.23}$$

where the $\pm$ signs account for the fact that the left-side Killing vector ξ_- is past-directed at large radii. Spherical symmetry allows us to parametrize the induced metric on extremal surfaces in the form

$$\mathrm{d}s^2_\Sigma = h_{ab}\,\mathrm{d}y^a\,\mathrm{d}y^b = \mathrm{d}y^2 + g(y)\,\mathrm{d}\Omega^2_{d-1}\;, \tag{4.24}$$

[16] We use latin indices for coordinates on the hypersurface Σ and greek indices for general coordinates in the full spacetime.

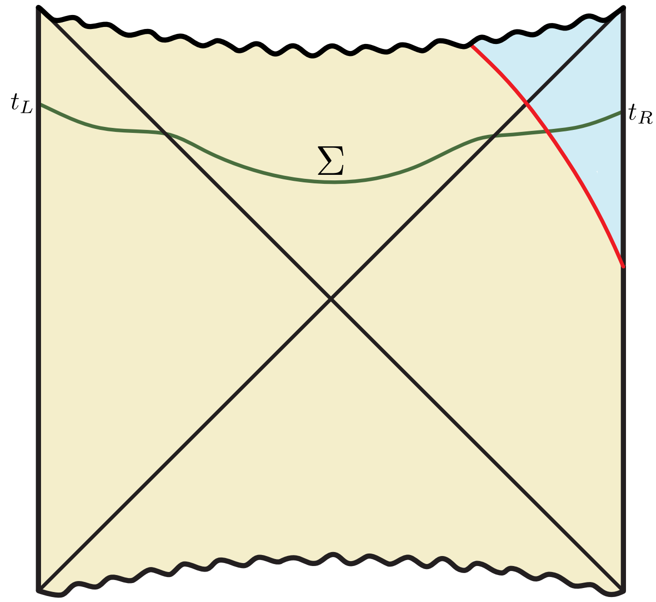

Figure 11: *Extremal codimension-one surface Σ of interest. Its boundary $\partial\Sigma$ consists of two spheres at infinity, located at times $t_L = t_R = t$.*

where y is a radial coordinate running over the real line, with $y = \pm\infty$ corresponding respectively to the left and right boundaries of Σ. In these coordinates, we can picture $e^\mu_y = \partial_y X^\mu$ as a unit-normalized, radial, spherically symmetric, right-pointing vector field. Denoting the spheres at infinity by $\mathcal{S}_{\pm\infty}$ we can rewrite the volume variation of extremal surfaces (4.22) as

$$\delta V\Big|_{\text{extremal}} = \delta t \left[\int_{\mathcal{S}_{+\infty}} e^\mu_y (\xi_+)_\mu + \int_{\mathcal{S}_{-\infty}} e^\mu_y \, (\xi_-)_\mu \right] , \tag{4.25}$$

where we have absorbed the sign assignments in (4.23) into a reversal of orientation for the left-boundary integral. Namely, both integrals in (4.25) are now written as scalar integrals over the boundary spheres.

This expression for the volume dependence with asymptotic time is useful because the featured integrals turn out to be Noether charges. If we view the volume functional (4.19) as an action on a collection of fields X^μ defined over Σ, the isometries of the $\mathcal{V}^\pm$ portions are interpreted as 'internal symmetries' of the this field theory, with their corresponding Noether currents. The time-translation symmetries associated to $\xi_\pm$ induce Noether currents of the form [17]

$$J_a = e^\mu_a \, \xi_\mu \, , \qquad \nabla_a J^a = 0 \, . \tag{4.26}$$

In particular, the integral of the radial component J_y over any fixed-y section $\mathcal{S}_y$ is a Noether charge

[17] In order to prove conservation, we just use $\xi_\mu = g_{t\mu}$ and evaluate the equation of motion from (4.21).

which is conserved under transport in the y direction:

$$\Pi[\mathcal{S}_y] = \int_{\mathcal{S}_y} e^\mu_y \, \xi_\mu \, , \qquad \partial_y \Pi[\mathcal{S}_y] = 0 \, . \tag{4.27}$$

Identification of the PC component

We have now the machinery in place to evaluate (4.3). The formula (4.25) implies

$$\frac{\mathrm{d}V}{\mathrm{d}t} = \Pi_+ + \Pi_- \, , \tag{4.28}$$

in terms of the Noether charges $\Pi_\pm \equiv \Pi[\mathcal{S}_{\pm\infty}]$ on right and left boundaries (a similar result was derived in [91, 92] for null shells). The normalization of the operator complexity requires the subtraction of the same expression, evaluated on the Noether charges $\Pi^{(0)}_\pm$ of the eternal black hole geometry without infalling shell, namely

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{C}[\mathcal{O}_{\text{shell}}] = \frac{d-1}{8\pi G} \left[\Pi_+ - \Pi^{(0)}_+ + \Pi_- - \Pi^{(0)}_- \right] \, . \tag{4.29}$$

Left-right symmetry of the eternal black hole geometry implies $\Pi^{(0)}_+ = \Pi^{(0)}_-$, whereas we can also set $\Pi_- \approx \Pi^{(0)}_-$ at the left regularization boundary because, for shells that enter the geometry at very late times, their worldvolume $\mathcal{W}$ remains very far from the left boundary. Hence, near the left regularized boundary, the extremal surface Σ is very well approximated by that of the eternal black hole. As we remove the regularization, in the limit $r_\Lambda \to \infty$, we must actually obtain $\Pi_- = \Pi^{(0)}_-$. This allows us to remove all explicit reference to the eternal black hole geometry and write

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{C}[\mathcal{O}_{\text{shell}}] = \frac{d-1}{8\pi G} \left[\Pi_+ - \Pi_- \right] \, . \tag{4.30}$$

Furthermore, the conservation of Noether charges in either $\mathcal{V}^+$ or $\mathcal{V}^-$ allows us to bring the Noether charges to both sides of the shell's worldvolume:

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{C}[\mathcal{O}_{\text{shell}}] = \frac{d-1}{8\pi G} (\Delta\Pi)_{\mathcal{W}} = \frac{d-1}{8\pi G} \int_{\mathcal{S}_{\mathcal{W}}} e^\mu_y \, (\Delta\xi_\mu)_{\mathcal{W}} \, , \tag{4.31}$$

where $(\Delta\xi^\mu)_{\mathcal{W}} = (\xi^\mu_+ - \xi^\mu_-)\big|_{\mathcal{W}}$ is the jump of the Killing vectors across $\mathcal{W}$ and $\mathcal{S}_{\mathcal{W}}$ is the sphere at the intersection $\Sigma \cap \mathcal{W}$. Using now the junction conditions in the form (4.15), we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\, \mathcal{C}[\mathcal{O}_{\text{shell}}] = -\int_{\mathcal{S}_{\mathcal{W}}} \sigma \, R \, e^\mu_y \, u_\mu \, . \tag{4.32}$$

We can now elaborate (4.32) in various ways in order to flesh out the PC-duality interpretation. First, we define a 'complexity field' over Σ as a rescaling of the e^μ_y field:

$$\mathcal{C}^\mu_\Sigma \equiv -r \, e^\mu_y \, . \tag{4.33}$$

Second, we define a density of proper momentum along the shell's worldvolume

$$\mathcal{P}^\mu \equiv \sigma\, u^\mu \ . \tag{4.34}$$

With these definitions we can rewrite (4.31) as

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{C}[\mathcal{O}_{\text{shell}}] = P_{\mathcal{C}} = \int_{\mathcal{S}_{\mathcal{W}}} \mathcal{P}_\mu\, \mathcal{C}^\mu_\Sigma \ , \tag{4.35}$$

a relation which identifies the precise component of momentum which is dual to complexity growth, namely the projection of the proper momentum along the direction of the complexity vector field $\mathcal{C}^\mu_\Sigma$. It is a particular radial component with inward orientation and appropriate normalization.

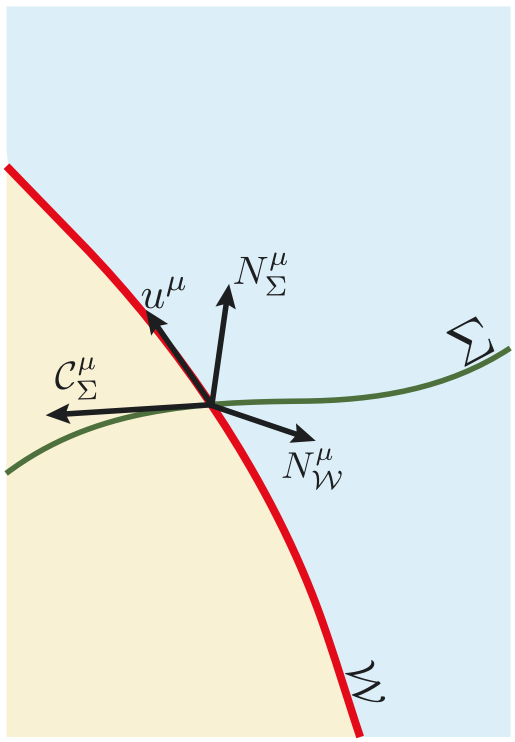

Figure 12: *Configuration of relevant vectors at the intersection sphere $\mathcal{S}_\mathcal{W} = \Sigma \cap \mathcal{W}$.*

A second presentation of this result has the virtue of hiding some of the peculiarities of the concrete system we have considered so far. In fact, no explicit geometrical information about the shell's worldvolume $\mathcal{W}$ is needed in order to express the PC duality relation. To see this, let us consider the expression

$$-\int_\Sigma N^\mu_\Sigma\, T_{\mu\nu}\, \mathcal{C}^\nu_\Sigma \ , \tag{4.36}$$

where N_Σ is the unit timelike normal to Σ. It measures the flux through Σ of a suitably normalized momentum component along Σ. Upon explicit evaluation for the spherical shell, using (4.8), we find

$$-\int \mathrm{d}y \int_{\mathcal{S}_y} \sigma\,(N_\Sigma \cdot u)\,(\mathcal{C}_\Sigma \cdot u)\,\delta(\ell)\;. \tag{4.37}$$

Furthermore, $\delta(\ell) = \delta(y - y_{\mathcal{W}})\,|d\ell/dy|^{-1}$, where $y_{\mathcal{W}}$ is the value of the y coordinate at the shell's intersection. From the definition of the $\mathcal{W}$-normal we have $d\ell/dy = \partial_y X^\mu\,\partial_\mu \ell\; = e_y \cdot N_{\mathcal{W}}$, which allows us to collapse the integral to the intersection sphere $\mathcal{S}_{\mathcal{W}}$:

$$\int_{\mathcal{S}_{\mathcal{W}}} \sigma\, R\, \frac{(N_\Sigma \cdot u)\,(e_y \cdot u)}{(e_y \cdot N_{\mathcal{W}})}\;, \tag{4.38}$$

where we have used (4.33). To further reduce this integral we notice that N_Σ and e_y are orthogonal and unit normalized, as well as the pair u and $N_{\mathcal{W}}$, so that we have $N_\Sigma \cdot u\; = -N_{\mathcal{W}} \cdot e_y$, where the minus sign accounts for the timelike character of both N^μ_Σ and u^μ. This simplifies (4.38) and recovers (4.32). Hence, we have established the more intrinsic form of the PC relation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{C}[\mathcal{O}_{\text{shell}}] = P_{\mathcal{C}} = -\int_\Sigma N^\mu_\Sigma\, T_{\mu\nu}\, \mathcal{C}^\nu_\Sigma\;. \tag{4.39}$$

In this version, all explicit reference to the details of the bulk state gets reduced to its stress-energy tensor. The vector fields N_Σ and $\mathcal{C}_\Sigma$ are defined in terms of the extremal surface, whose detailed geometry is also determined by $T_{\mu\nu}$ through the back reaction on the geometry. Indeed, the form of (4.39) should remain valid for spherical shells with any internal equation of state, including those corresponding to branes which change the AdS radius of curvature across $\mathcal{W}$. Furthermore, the role of the Noether charges in the derivation of (4.35) and (4.39) makes it clear that it applies as well to spherical thin shells collapsing in vacuum AdS and forming a one-sided black hole.

More generally, we expect that any spherical matter distribution can be approximated by a limit of many concentric thin shells, so that (4.39) should remain valid for *any* matter bulk distribution with spherical symmetry. The generalization to one-sided collapse of thin shells with arbitrary equations of state, but still maintaining spherical symmetry, is explained in Appendix B. A first step towards lifting the spherical symmetry requirement is presented in Appendix C, which considers a formal collapse of a rotating shell in AdS_3.

4.3 Late time limit and the black hole interior

One chief motivation behind this work is the elucidation of the very late time regime of operator complexity growth in the light of the PC duality. Any definition of operator complexity with the structure of equation (3.2) will assign a linear growth at late times. In particular, given that state complexities are expected to grow proportionally to $E_\Psi\, t$, where E_Ψ is a characteristic energy of the state, the subtracted definition for operator complexity gives a slope proportional to $E_{\mathcal{O}}\, t$, where $E_{\mathcal{O}}$

is the additional energy injected by the operator $\mathcal{O}$. Translated to our gravitational set up, we expect a late time behavior

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}[\mathcal{O}_{\text{shell}}]\Big|_{\text{late}} \approx M_+ - M_- = M_{\text{shell}} \,. \tag{4.40}$$

We would like to check that our PVC relation satisfies this expected asymptotic behavior. A simple check can be performed in the limit of very large AdS black holes. This coincides with the situation where the infalling shells have small gravitational self-energy at all times that are relevant for the calculation.

The key point is to notice that, at late times, the extremal surfaces Σ_t accumulate in the interior of the black hole, exponentially converging [18] to a limiting surface Σ_∞ (cf. [58, 98]). For a shell that enters the black hole very late, this surface interpolates between the limiting surfaces $(\Sigma_\infty)_\pm$ associated to the early and late black holes of mass $M_\pm$ (cf. Figure 13). In terms of the interior Schwarzschild radial coordinates, let $\widetilde{r}_\pm$ denote the saturation radii, defined by the local extremization of the 'volume Lagrangian' $r^{d-1}\sqrt{|f(r)|}$. By explicit calculation we find, in the limit of very large AdS black holes

$$\widetilde{r}^{\,d} \approx \frac{8\pi G M}{(d-1)V_\Omega} \,. \tag{4.41}$$

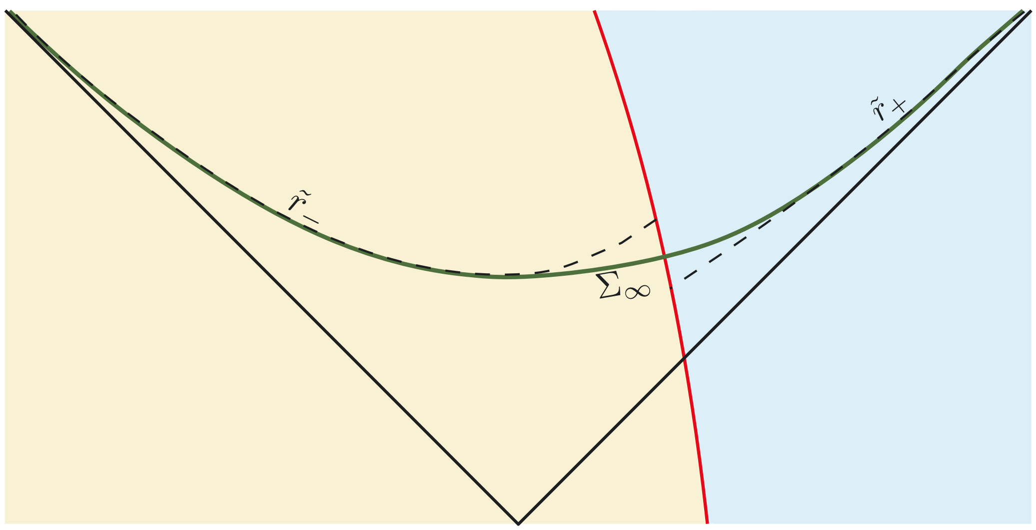

Figure 13: *The saturation slice Σ_∞ interpolates between the extremal surface barrier inside $\widetilde{r}_-$ and outside $\widetilde{r}_+$.*

We can now make use of the 'movability' of the Noether charges $\Pi_\pm$ to evaluate then away from $\mathcal{W}$, but still inside the black hole interior, in a region where Σ_t is well-approximated by a constant-r surface. Let us denote the angular spheres at such points by $\widetilde{\mathcal{S}}_\pm$. Then, equation (4.30) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}[\mathcal{O}_{\text{shell}}]\Big|_{\text{late}} \approx \frac{d-1}{8\pi G}\left(\Pi\left[\widetilde{\mathcal{S}}_+\right] - \Pi\left[\widetilde{\mathcal{S}}_-\right]\right) \,. \tag{4.42}$$

[18] See Appendix A for an quantitative discussion of this phenomenon.

In computing the Noether charges, we notice that $\xi_\pm = \partial/\partial t_\pm$ are approximately tangent to Σ_t in the saturation region. Hence, we can write $e^\mu_y \approx \xi^\mu/\sqrt{\xi^2}$ and the Noether integrals are simply

$$\Pi[\widetilde{\mathcal{S}}_\pm] \approx \int_{\widetilde{\mathcal{S}}_\pm} \sqrt{\xi^2} = V_\Omega\, \widetilde{r}_\pm^{\,d-1} \sqrt{|f(\widetilde{r})_\pm|} \approx V_\Omega\, \widetilde{r}_\pm^{\,d} \approx \frac{8\pi G M_\pm}{d-1} \,. \tag{4.43}$$

In the last equality we have made use of (4.41) and the approximation of a large AdS black hole. Therefore, upon subtraction we conclude the proof of (4.40).

An important observation regarding this result is the fact that the vector fields $\mathcal{C}^\mu$ and e^μ_y do differ significantly in the interior saturation region, because the rescaling factor $\widetilde{r}$ is non trivial there, and yet this rescaling is crucial to obtain the expected long-time asymptotics. Therefore, the peculiar normalization (4.33) of the momentum component along Σ is necessary for the consistency of the results.

We can obtain further insight into the rationale behind the linear complexity growth by passing to the effective particle description. Again neglecting self-energy corrections, we can envision the dynamics of the shell as that of a probe particle of mass m falling through the $(1+1)$-dimensional metric (4.17). The PC duality relation admits the two-dimensional representation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{C}[\mathcal{O}_{\text{shell}}] = P_\mathcal{C} = P_\alpha\, \mathcal{C}^\alpha \,, \tag{4.44}$$

where $P^\alpha = m\, u^\alpha$, with α a two-dimensional index. Picking for example the standard (r,t) coordinates, we have

$$P_\mathcal{C} = -r\left(\frac{\partial t}{\partial y}\, P_t + \frac{\partial r}{\partial y} P_r\right) \,. \tag{4.45}$$

Let us introduce an adapted coordinate for the radial 'complexity field' $\mathcal{C}^\alpha = -r e^\alpha_y$, namely we define a rescaled radial coordinate χ such that

$$\mathcal{C}^\alpha = \left(\frac{\partial}{\partial \chi}\right)^\alpha = -r\, e^\alpha_y = -r\, \left(\frac{\partial}{\partial y}\right)^\alpha \,, \tag{4.46}$$

or, equivalently

$$\frac{\partial}{\partial \chi} = -r\,\frac{\partial}{\partial y} \,. \tag{4.47}$$

Using the so-defined χ coordinate, we can simplify (4.45) so that

$$P_\mathcal{C} = P_t\,\frac{\partial t}{\partial \chi} + P_r\,\frac{\partial r}{\partial \chi} = P_\chi \,. \tag{4.48}$$

To the extent that we are only interested in describing the particle motion to the past of the saturation surface Σ_∞, we may use a time slicing given by the extremal surfaces Σ_t themselves, and coordinate the spacetime in terms of (t,χ,Ω). In this frame, the complexity momentum coincides with the χ-canonical momentum, provided we stay within the probe approximation:

$$P_{\mathcal{C}} = P_{\chi} = \frac{\partial L_{\text{eff}}}{\partial \dot{\chi}} \ . \tag{4.49}$$

This brings our general formalism into contact with the discussion of canonical Rindler momentum in the introduction. However, the present treatment is capable of describing the late-time behavior of the complexity. In particular, the use of a time slicing adapted to the extremal surfaces leads to the phenomenon of *saturation* in the black-hole interior. This freezes the value of the momentum at a constant value for asymptotically large values of t, thereby explaining why a linear growth of complexity can be compatible with a PC-type formula (3.3).

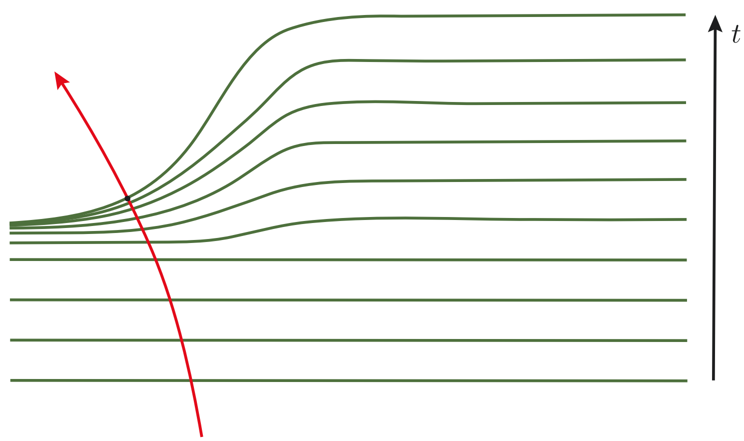

Figure 14: *The late-time saturation of the time slicing in the interior of the black hole results in a frozen momentum component, as required for any PC formula which should apply in a regime of linear complexity growth.*

5 Momentum and Complexity: A Proof

A key property of the PVC correspondence (4.39) is that all dynamical assumptions about the shells are concealed inside the energy-momentum tensor $T_{\mu\nu}$. Therefore, it is natural to suspect that a PVC relation of this form could have a much wider degree of generality. In this section we confirm this expectation, showing that the content of (4.39) is essentially the Momentum Constraint of General Relativity (GR).

5.1 PVC From The Momentum Constraint

We shall work with spacetimes X asymptotic to global AdS$_{d+1}$ with $d \geqslant 2$. The bulk state is described as a solution of Einstein equations with energy momentum tensor $T_{\mu\nu}$, and the asymptotic behavior of a normalizable state. We shall adopt units such that the asymptotic radius of curvature of AdS is $\ell = 1$, although most of our results still hold in the flat spacetime limit $\ell \to \infty$. The VC formula is taken to be

$$\mathcal{C}\,[\Sigma_t] = \frac{d-1}{8\pi G}\,\text{Vol}(\Sigma_t)\;, \tag{5.1}$$

a regularized volume of an extremal codimension-one hypersurface Σ_t, anchored at boundary time t, which labels the real line in $\mathbf{R} \times \mathbf{S}^{d-1}$, the conformal boundary of X. For notational simplicity we will often suppress the time label in Σ_t, with the implicit understanding that a choice of Σ is equivalent to a choice of boundary time.

To fix notation, $g_{\mu\nu}$ denotes the metric on X and h_{ab} the induced metric on Σ, with world-volume coordinates y^a. Latin indices are raised and lowered with h_{ab}, whereas greek indices are operated with $g_{\mu\nu}$. The embedding of Σ into X is described by the functions $X^\mu(y^a)$, with tangent frame vector fields $e^\mu_a = \partial_a X^\mu$. The extrinsic curvature of Σ is denoted K_{ab}, and its trace $K = h^{ab}K_{ab}$ will vanish throughout our discussion, since we are focusing on extremal-volume surfaces. Finally, the future-pointing, unit timelike normal to Σ is denoted N^μ_Σ.

We begin by deriving a useful equation for the rate of VC. Since Σ is extremal, its first-order variation with respect to a variation of the anchoring surface is a boundary term of the form

$$\delta\mathcal{C}\,[\Sigma] = \frac{d-1}{8\pi G}\int_{\partial\Sigma} \delta X_\Sigma\;, \tag{5.2}$$

where $(\delta X_\Sigma)_a = e^\mu_a\,\delta X_\mu$ is the embedding variation, pulled back to Σ. For a rigid time translation δt at the boundary, we have $\delta X_\Sigma = \delta t\,(\partial_t)_\Sigma$, where ∂_t denotes the time-translation vector in X, which is asymptotically a Killing vector. Dividing by δt we obtain an ADM-like expression for the rate of VC:

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \frac{d-1}{8\pi G}\int_{\partial\Sigma}(\partial_t)_n = \frac{d-1}{8\pi G}\int_{\partial\Sigma} dS^a\, e^\mu_a\,(\partial_t)_\mu\;. \tag{5.3}$$

This equation represents the complexity rate as the integral of $(\partial_t)_n = e_n \cdot \partial_t$ over the boundary of the extremal surface, where $e_n^\mu = e_a^\mu \, n_{\partial\Sigma}^a$, with $n_{\partial\Sigma}$ the outward pointing normal to $\partial\Sigma$. Since e_n is tangent to Σ, the integrand is sensitive to the asymptotic bending of Σ by the presence of non-trivial geometry in the bulk. More precisely, we pick the term of order $1/r^{d-1}$, for r the radius of an angular sphere which regularizes $\partial\Sigma$.

Given any 'current' J^a defined on Σ, which has the same boundary integral as $(\partial_t)_n$,

$$\int_{\partial\Sigma} J_n = \int_{\partial\Sigma} (\partial_t)_n \;, \tag{5.4}$$

we can use Stokes theorem to write the VC rate as a bulk integral of its 'source' over the extremal surface:

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \frac{d-1}{8\pi G} \int_\Sigma \nabla_a J^a \;. \tag{5.5}$$

A strategy to obtain a PVC equation is to make a clever choice of J, in such way that it is sourced by a momentum density. A simple example is provided by the well-known case of spherical thin shells, whose PVC relation (4.39) can be derived in this language by choosing $J_a = (\partial_t)_\mu \, e_a^\mu$. In this approximation scheme ∂_t is a Killing vector except for jumps at the worldvolume of the shells, so that the integral (5.5) localizes to delta-function contributions, with coefficients controlled by the junction conditions (cf. [97] for a review). This derivation shows that the PVC relation is independent of any choice of equation of state on the world-volume of the shells.

Exact PVC

In order to pursue this strategy in more general terms, we can work backwards by seeking a natural GR equation that uses the momentum density over a spacelike surface. The obvious candidate is the so-called Momentum Constraint (MC): given any Cauchy surface Σ, initial data h_{ab} and K_{ab} are constrained by the equation (cf. [97])

$$\nabla^a K_{ab} - \nabla_b K = -8\pi G \, \mathcal{P}_b \;, \tag{5.6}$$

where $\mathcal{P}_b = -N_\Sigma^\mu \, T_{\mu\nu} \, e_b^\nu$ is the pulled-back momentum flux through Σ. For the purposes of this work, we can simplify this equation by setting $K = 0$, since Σ is taken to be extremal.

In order to integrate the MC we must introduce a tangent vector field on Σ. Anticipating its role in what follows, we shall refer to this field, C_Σ, as the 'infall' vector field, despite the fact that at this point it is completely arbitrary. Multiplying (5.6) by C^b and integrating by parts we obtain the equivalent expression

$$\int_\Sigma \mathcal{P}_C = -\frac{1}{8\pi G} \int_{\partial\Sigma} dS^a \, K_{ab} \, C^b + \frac{1}{8\pi G} \int_\Sigma K^{ab} \, \nabla_a C_b \;, \tag{5.7}$$

where $\mathcal{P}_C = \mathcal{P}_a \, C^a$ is the momentum component that is being selected by the C-field. The left hand side has the form of the momentum integral we are seeking, whereas we have a boundary term in the

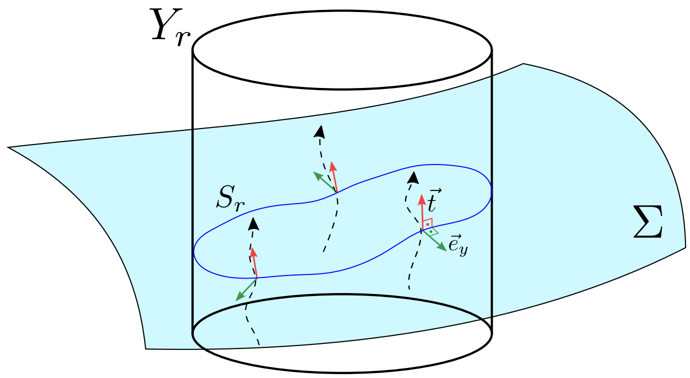

Figure 15: *For each point in S_r, the time coordinate is chosen to properly parametrize a geodesic (dashed lines) on Y_r. $\vec{e}_y$ is then picked to be orthogonal to S_r.*

right-hand side that we could try to interpret as $\mathrm{d}\mathcal{C}/\mathrm{d}t$. In other words, we would like to set

$$J_a = -\frac{1}{d-1}\, K_{ab}\, C^b \;, \tag{5.8}$$

and fix the behavior of C^b at the boundary so that we satisfy (5.4). This can be analyzed by means of a local computation as follows. In the vicinity of Σ, we may choose coordinates such that the metric reads

$$ds^2_X \to \frac{dr^2}{r^2} + r^2\left(-dt^2 + \gamma_{ij}(r,t,\theta)\, d\theta^i\, d\theta^j\right) \;\text{ as }\; r\to\infty \;. \tag{5.9}$$

Here, r is a Fefferman–Graham coordinate which foliates X by timelike codimension-one submanifolds Y_r. The angles θ^j parametrize the intersection $S_r = Y_r \cap \Sigma$, of spherical topology and induced metric proportional to γ_{ij}, which is itself asymptotic to a unit round $(d-1)$-sphere, up to normalizable corrections of order $1/r^d$. The crucial simplifying property of (5.9) is the choice of time coordinate, which is geodesic and orthogonal to S_r (cf. Figure 15).

The induced metric on Σ can be written near the boundary as

$$ds^2_\Sigma \to dy^2 + r^2(y)\,\gamma_{ij}(y,\theta)\, d\theta^i\, d\theta^j \;, \tag{5.10}$$

for some function $r(y)$ asymptotic to $\sinh(y)$ as $y\to\infty$. This allows us to write the normal one-form as $N_\Sigma = e^t_y\, dr - e^r_y\, dt$, and compute the extrinsic curvature $K_{ab} = e^\mu_a\, e^\nu_b\, \nabla_\mu\, N_\nu$. The relevant component turns out to be K_{yy} which, using the traceless character, $K=0$, may be evaluated as $K_{yy} = -r^{-2}\,\gamma^{ij}\, K_{ij}$. Explicitly

$$K_{yy} = -(d-1)\, r\, e^t_y - \frac{1}{2r^2}\, e^r_y\, \gamma^{ij}\partial_t\gamma_{ij} - \frac{r^2}{2}\, e^t_y\, \gamma^{ij}\partial_r\gamma_{ij} \;. \tag{5.11}$$

An asymptotic analysis reveals the large-r scalings $e^r_y \sim r$, $e^t_y \sim 1/r^{d+1}$, $\partial_t\gamma_{ij}\sim 1/r^d$ and $\partial_r\gamma_{ij}\sim 1/r^{d+1}$, so that the right hand side of (5.11) is dominated by the first term: $K_{yy}\approx -(d-1)\, r\, e^t_y$. Since $e_y\cdot\partial_t = -r^2\, e^t_y$, we learn that (5.4) can be satisfied provided the C-field is chosen with the boundary conditions

$$C_\Sigma \to -r(y)\,\partial_y \quad \text{as} \quad y \to \infty \,. \tag{5.12}$$

This is exactly the result that was found 'empirically' for the case of thin shells in [1], justifying the name 'infall field' which, from this point of view, is nothing but the condition for the integrated Momentum Constraint to compute the complexity rate.

We are now ready to assemble all the pieces and write down a 'generalized PVC' relation. Defining a total 'C-momentum' through Σ and a 'remainder' by the expressions

$$P_C\,[\Sigma] = \int_\Sigma \mathcal{P}_C \,, \qquad R_C\,[\Sigma] = \frac{1}{8\pi G}\int_\Sigma K^{ab}\,\nabla_a C_b \,, \tag{5.13}$$

we have established

$$\boxed{\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} \;=\; P_C\,[\Sigma] + R_C\,[\Sigma]}\,. \tag{5.14}$$

This shows that part of the complexity rate at time t can always be attributed to momentum flow through Σ_t. In fact, a sufficient condition can be placed on the 'infall field' which ensures the vanishing of the remainder. The extrinsic curvature K_{ab} being symmetric and traceless, we can write the remainder in the form

$$R_C\,[\Sigma] = \frac{1}{8\pi G}\int_\Sigma K^{ab}\,\left(\nabla_{(a}C_{b)} - \frac{1}{d}\,h_{ab}\,\nabla\cdot C\right)\,. \tag{5.15}$$

The term in parenthesis is proportional to the conformal Lie derivative, which vanishes if the C-field is a conformal Killing vector (CKV). This happens for any spherically symmetric state, for which the infall field has exactly the form (E.5) throughout Σ. The same is true of any solution of Einstein's equations in $2+1$ dimensions, because Σ is then two-dimensional. In both these cases, the induced metric on Σ is conformal to the Poincaré ball $ds^2_{\text{ball}} = dz^2 + \sinh^2(z)\,d\Omega^2_{d-1}$, with a rescaling factor which approaches unity at $\partial\Sigma$. The Poincaré ball provides a 'canonical' infall field $C_\Sigma = -\sinh(z)\,\partial_z$ which is a radial CKV on Σ with the appropriate boundary conditions (E.5).

Therefore, we conclude that any spacetime in $2+1$ dimensions [19] and any spherically symmetric state in arbitrary dimensions satisfies an exact PVC relation

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} \;=\; P_C\,[\Sigma]\,. \tag{5.16}$$

It is notable that we obtained all these results with no extra hypothesis on the nature of the matter, i.e. no positivity conditions on $T_{\mu\nu}$ were required. This suggests that the nature of the PVC relation is essentially kinematical once we take into account the constraints of GR.

[19] Modulo boundary gravitons in AdS_3, which do not satisfy the boundary conditions (5.9) globally on the asymptotic boundary $\mathbf{R}\times\mathbf{S}^1$.

5.2 Obstructions

The most important exception to an exact PVC relation is provided by gravitational waves. In this case, the Weyl tensor of X does not vanish, and embedded hypersurfaces will in general fail to be conformally trivial. In the absence of a canonical choice of C_Σ in the bulk, a remainder correction will be present generically. This is natural from the physical point of view, since a black hole could be formed by colliding gravitational waves, and the linear growth of complexity must eventually build up at long times even if $T_{\mu\nu} = 0$ all along.

Approximate PVC relations should exist in the context of the linearized gravity approximation. If X contains gravitational waves perturbing a spherically symmetric background X_0, it should be possible to establish an approximate PVC relation of the form

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} \approx -\int_{\Sigma_0} N_0^\mu \left(T_{\mu\nu} + t_{\mu\nu}\right) C_0^\nu \,, \tag{5.17}$$

where $t_{\mu\nu}$ is a pseudotensor of Landau–Lifshitz type and the normal, N_0, and infall, C_0, vectors are referred to the surface Σ_0, extremal with respect to the background geometry X_0. If the gravitational waves can be fully related to matter sources, the $t_{\mu\nu}$ contribution will be hierarchically smaller than the matter contribution.

A different type of obstruction to an exact PVC correspondence occurs when we have wormholes. The simplest example which captures the relevant issues is the Einstein–Rosen bridge of an eternal black hole. In vacuum, the extremal surfaces are spherically symmetric cylinders of topology $\mathbf{R}\times\mathbf{S}^{d-1}$, with a $\mathbf{Z}_2$ reflection symmetry between left and right sides, acting on $\mathbf{R}$ in the standard fashion. Radial CKVs exist, but the asymptotic boundary conditions are necessarily incompatible with the 'infall' interpretation in both boundaries: if the C_Σ field is 'infall' on the right side, it must be 'outfall' on the left side. Revisiting the asymptotic boundary conditions for the C-field (5.4) and (5.8) we see that an inversion of C is correlated with an inversion of the time-translation vector ∂_t, namely the equation $\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = P_C\,[\Sigma]$ holds when we interpret the complexity rate as measured with respect to the Killing Hamiltonian $H_K = H_R - H_L$. In this case one obtains $\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t_K} = 0$ for the vacuum solution, where the K label stands for the choice of time variable dual to H_K. The same is true for any $\mathbf{Z}_2$-symmetric momentum configuration, such as identical collapsing matter distributions on both sides. In order to get $\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t_K} > 0$ we need a sufficient amount of 'outfall' in the left side. [20]

For the case of a vacuum Einstein–Rosen bridge, it is certainly possible to define CKVs with appropriate infall conditions in the vicinity of each boundary, but these choices are necessarily incompatible with each other in the bulk; at some point the conformal Lie derivative must be non-zero, and a contribution from the remainder is turned on. For instance, if we want to compute the standard complexity rate with respect to the TFD Hamiltonian $H_{\mathrm{TFD}} = H_R + H_L$, we must introduce a defect in the interior

[20] Note that in this case, (5.14) can be viewed as the generalization of the PVC correspondence provided in [71] for the case of AdS_2, in particular, equation (6.114) of that paper.

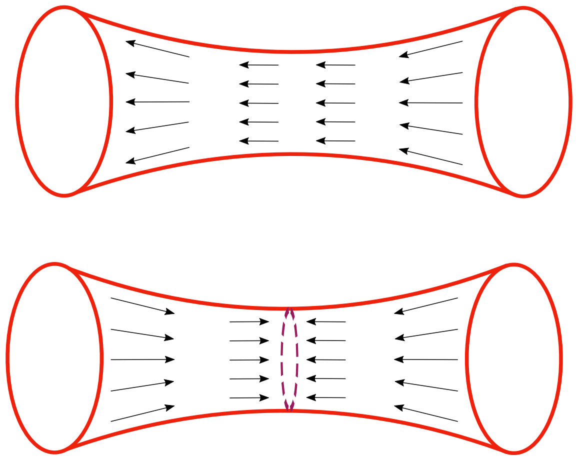

Figure 16: *On the top panel, a CKV field on the Einstein–Rosen bridge is in-falling on one side and 'out-falling' on the other. On the bottom panel, insisting on being in-falling on both sides forces a discontinuous jump through a defect in the interior.*

along which the C-field switches its orientation from 'right-infall' to 'left-infall' (cf. Figure 16). If we do this at the $\mathbf{S}^{d-1}$ sitting at the fixed point of the $\mathbf{Z}_2$ reflection, we obtain a delta function contribution to the integrand of the remainder. A simple calculation reveals then the standard result $\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = 2M$.

6 Generalized Momentum and Complexity

Notwithstanding the important cases that show an exact or approximate PVC correspondence, the remainder term in (5.14) does not vanish in general. The simplest and more important instance occurs when considering pure gravity solutions. Gravitational wave scattering with black hole formation is the crucial example of a process with a non-vanishing rate of complexity growth, which must come entirely from the remainder term (5.15). The purpose of this section is to propose a generalization of (5.14) in which the notion of 'infall momentum' is suitably generalized, in a way that can handle pure gravity solutions. The very existence of such a generalization is quite remarkable, given that no strictly local definition of energy-momentum exists for the purely gravitational degrees of freedom.

6.1 Generalized PVC from the Codazzi Equation

The key idea in obtaining a generalized PVC relation is to replace the momentum constraint by a more general starting point. The natural candidate is the Codazzi equation (cf. [97])

$$\nabla_c K_{ab} - \nabla_b K_{ac} = N^\mu R_{\mu\nu\sigma\rho}\, e^\nu_a\, e^\sigma_b\, e^\rho_c \,, \tag{6.1}$$

since the momentum constraint (5.6) is contained in its trace. Notice however that (6.1) involves a projection of the full Riemann tensor instead of simply the Ricci tensor. Therefore, the difference between (6.1) and (5.6) is proportional to the Weyl tensor $W_{\mu\nu\rho\sigma}$. Since gravitational waves are precisely characterized by a non-vanishing Weyl tensor, the Codazzi equation has the right ingredients for the kind of generalization that we are seeking.

Following the same steps of the previous section, we want to integrate (6.1) over the extremal surface Σ. In doing so, we need to contract the three free indices with an 'infall' rank-3 tensor field, M^{abc}, enjoying the same symmetry properties as the Codazzi equation, namely antisymmetry in the last two indices, $M^{abc} = -M^{acb}$, and the cyclic identity $M^{abc} + M^{bca} + M^{cab} = 0$. Using these symmetry properties, the Ricci decomposition of the Riemann tensor and Einstein's equation, we can rewrite (6.1) as

$$-\, M^{abc}\, \nabla_c K_{ab} = \frac{8\pi G}{d-1}\, \mathcal{P}_C + \frac{1}{2}\, \mathcal{W}_M \,, \tag{6.2}$$

where $\mathcal{P}_C = \mathcal{P}_a C^a$, and the C-field

$$C^b = h_{ac}\, M^{abc} \tag{6.3}$$

is an infall vector field induced by the infall tensor field. The density

$$\mathcal{W}_M = -N^\mu W_{\mu\nu\rho\sigma}\, e^\nu_a\, e^\rho_b\, e^\sigma_c\; M^{abc} \tag{6.4}$$

is the contraction of the infall tensor field with the pulled-back, projected Weyl tensor. Integrating now by parts we find

$$-\int_{\partial\Sigma} \mathrm{d}S_a \, K_{bc} \, M^{bca} \;=\; \frac{8\pi G}{d-1} \int_\Sigma \mathcal{P}_C \;+\; \frac{1}{2}\int_\Sigma \mathcal{W}_M \;-\; \int_\Sigma K_{ab} \, \nabla_c \, M^{abc} \; . \tag{6.5}$$

An interpretation of the left-hand side of (6.5) as computing $\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t}$ requires that we set

$$J^a \;=\; -\, K_{bc} \, M^{bca} \tag{6.6}$$

where $J^a = \partial_t^\mu e_\mu^a$ satisfies the condition (5.4).

Finally, defining the remainder term

$$R_M[\Sigma] = -\frac{d-1}{8\pi G} \int_\Sigma K_{ab} \, \nabla_c \, M^{abc} \; , \tag{6.7}$$

and the integrated 'Weyl momentum'

$$W_M[\Sigma] = \frac{d-1}{16\pi G} \int_\Sigma \mathcal{W}_M \; , \tag{6.8}$$

we readily obtain a tensor generalization of (5.14)

$$\boxed{\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = P_C[\Sigma] + W_M[\Sigma] + R_M[\Sigma]} \; . \tag{6.9}$$

This equation is completely general, under the assumption that the boundary condition (6.6) is satisfied. It generalizes (5.14) in two ways. First, it contains information about bulk dynamics that goes beyond the mere initial-value constraints of GR, since it stems from the Codazzi equation. Second, it requires a generalization of the notion of 'infall' vector field into a tensor infall field with many more components. An immediate consequence of this increase in degrees of freedom materializes when we consider sufficient conditions for the remainder to vanish.

If the infall tensor M^{abc} can be further restricted so that the remainder (6.7) vanishes, that is to say, if M^{abc} can be chosen satisfying the conditions:

$$-\; K_{ab} \, M^{abc}\Big|_{\partial\Sigma} = (\partial_t)^c\Big|_{\partial\Sigma} \; , \qquad K_{ab} \, \nabla_c M^{abc} = 0 \; , \tag{6.10}$$

then we obtain the remainder-free, generalized PVC correspondence

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = P_C[\Sigma] + W_M[\Sigma] \; . \tag{6.11}$$

Its main novelty compared to the restricted PVC (5.16) is the presence of a purely gravitational contribution to the infall momentum, $W_M[\Sigma]$, formally depending on the Weyl curvature. In particular, states with no 'clumping' matter, having $P_C = 0$, still pick the Weyl contribution to the complexity rate. This we will see explicitly in the next section, in a particular example.

We now collect a few observations regarding our proposed generalization of the PVC correspondence.

- The generalized relations (6.9) and (6.11) reduce to the 'restricted' ones (5.14) and (5.16) when the infall tensor field admits the factorized ansatz

$$M^{abc} = \frac{1}{d-1}\left(h^{ac}C^{b} - h^{ab}C^{c}\right) , \tag{6.12}$$

in terms of some 'infall' vector field C^a. For sufficiently localized bulk states, this ansatz can be used to solve the box boundary condition (6.6) when we place the box at infinity in an asymptotically flat or AdS spacetime. In the AdS case, the solution involves a C-field with asymptotic behavior (5.12). In Appendix E we extend this result to the asymptotically flat case. Conceptually, the combination of (6.12) and (5.12) shows that, asymptotically, the 'infall' interpretations of M^{abc} and C^a reduce to one another.

- We can look for sufficient conditions for the vanishing of the remainder (6.7), which would generalize the conformal Killing condition $\nabla_{(a}C_{b)} = \frac{\nabla\cdot C}{d}\, h_{ab}$. Given that Σ_t is extremal, with $K = 0$, the remainder (6.7) vanishes if the symmetrized divergence of M^{abc} is a conformal rescaling, $\nabla_c\, M^{(ab)c} = \Phi\, h^{ab}$ for some scalar function Φ on Σ. Taking traces, we can compute Φ and obtain the equivalent trace-free transversality condition

$$\nabla_c \left(M^{(ab)c} - \frac{1}{d}\, h^{ab}\, h_{ef}\, M^{efc}\right) = 0 \, . \tag{6.13}$$

We can expect that finding transverse tensors satisfying (6.13) on Σ should be easier than finding conformal Killing vectors on Σ, simply as a consequence of the existence of many more degrees of freedom in $M^{(ab)c}$. Within the factorized ansatz (6.12), the M-transversality condition (6.13) reduces to the conformal Killing condition for the C-field, giving the most general solution of (6.13) in spherically symmetric spacetimes or any $(2+1)$-dimensional solution, precisely those cases in which the restricted PVC is exact (see Appendix D).

- The generalized PVC relation presented here is bound to suffer from similar topological obstructions as the restricted PVC. For an eternal black hole state with spherical symmetry, extremal hypersurfaces Σ_t have two disconnected boundaries, the infall tensor field satisfies the factorized ansatz (6.12) and the generalized 'Weyl momentum' still vanishes when evaluated on the extremal surfaces. This means that the result of section 5.2 still applies, i.e. the rate of complexity growth with forward time variables on both sides satisfies (6.9) with a non-zero remainder term $R_M[\Sigma_t]$, which in this example happens to coincide with (5.15). Therefore, the example of the eternal black hole implies that an exact PVC relation with vanishing remainder will always require some topological assumptions about the extremal surfaces.

6.2 An Explicit Check of the Generalized PVC

In this subsection we present a detailed verification of (6.10) and (6.11) on a non-trivial exact solution of Einstein's equations: a gravitational pp-wave. Gravitational waves are usually discussed in the context of a perturbative expansion around a flat background. In such cases, one usually makes the ansatz

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \,, \tag{6.14}$$

where $h_{\mu\nu}$ is treated as a small perturbation and its dynamics is determined by the linearized GR theory. The standard analysis shows that gravitational waves are transversely polarized, that is to say, in a suitable coordinate frame, a wave travelling in the z direction will only distort the metric in the transverse directions $y^i = \{x, y\}$, yielding a metric of the form

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + (\delta_{ij} + h_{ij}(u))\mathrm{d}y^i\mathrm{d}y^j, \tag{6.15}$$

where we have defined the light-like coordinates $u = t - z$ and $v = t + z$.

Exact non-perturbative gravitational wave solutions to the Einstein equations are highly idealized objects, but nevertheless exist and might be useful for pedagogical purposes. One way of defining them is through a direct generalization of (6.15), dropping the requirement that $h_{\mu\nu}$ is small and considering solutions of the form

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + g_{ij}(u)\mathrm{d}y^i\mathrm{d}y^j \,, \tag{6.16}$$

known as the Rosen form of a gravitational plane wave with parallel propagation (PP) solution. In order to be able to perform some calculations, we will sacrifice some generality by sticking to the following particular ansatz (cf. [99–101])

$$\mathrm{d}s^2 = -\mathrm{d}u\mathrm{d}v + L^2(u)\left(e^{2\beta(u)}\mathrm{d}x^2 + e^{-2\beta(u)}\mathrm{d}y^2\right) \,, \tag{6.17}$$

where the functions $L(u)$ and $\beta(u)$ are to be determined by the Einstein equations. As the solution represents a null wave by construction, the only component of the Ricci tensor that is excited is

$$R_{uu} = -2L^{-1}\left(L'' + (\beta')^2 L\right) \,, \tag{6.18}$$

where the primes stand for $\mathrm{d}/\mathrm{d}u$. Demanding a purely gravitational solution therefore requires

$$L'' + (\beta')^2 L = 0 \tag{6.19}$$

to hold. Once this condition is satisfied, the manifold still possess a non-trivial Riemann curvature, with the non-zero elements given by

$$\begin{aligned} R_{uxux} &= -e^{2\beta} L\,(2L'\beta' + L\beta'')\,, \\ R_{uyuy} &= -e^{-2\beta} L\,(2L'\beta' + L\beta'')\,. \end{aligned} \tag{6.20}$$

We shall check the generalized PVC correspondence for a region X defined by the slice delimited by $-\ell \leqslant z \leqslant \ell$. The boundary ∂X has left and right disconnected components $z = \pm\ell$. The extremal surface has the form $\Sigma = \gamma \times \mathbf{R}^2$, where the curve γ is anchored at ∂X on times t_L and t_R (see Fig. 17). We will fix the conventional normalization of the complexity by choosing the 'AdS size' to equal the coordinate edge of X in the z direction, namely we set $\ell_{\text{AdS}} = \ell$.

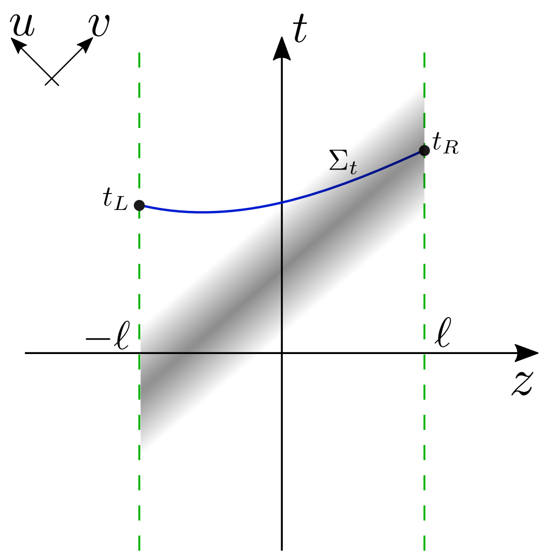

Figure 17: *Schematic representation of the extremal surface for a pp-wave pulse traversing the box of size 2ℓ from left to right. The transverse $\mathbf{R}^2$ plane is not shown.*

In checking (6.11), we shall factor the formally infinite transverse volume of the $\mathbf{R}^2$ component, namely we aim to compute both sides of

$$\frac{\mathrm{d}V_\gamma}{\mathrm{d}t} = \frac{\ell}{2V[\mathbf{R}^2]} \int_\Sigma \mathcal{W}_M\;, \tag{6.21}$$

where $V_\gamma = V[\Sigma]/V[\mathbf{R}^2]$ is the longitudinal volume of a symmetrically anchored curve, with $t_L = t_R = t$.

Computation of the Extremal Volume

Picking an arbitrary parametrization of γ, the longitudinal volume is given by

$$V_\gamma = \int_{\lambda_L}^{\lambda_R} \mathrm{d}\lambda \sqrt{-L(u)^4 \dot{v}\dot{u}}\,, \tag{6.22}$$

where the dot stands for d/dλ and, as we see, the dependence on the function $\beta(u)$ drops off the determinant of the induced metric on the slice, significantly simplifying the extremalization problem. Furthermore, we may choose the coordinate u itself as our λ parameter, yielding the effective lagrangian

$$\mathcal{L} = \sqrt{-L(u)^4 \dot{v}}\,. \tag{6.23}$$

As the action does not depend explicitly on v we can obtain a first integral for the Euler-Lagrange equations from the canonical momentum associated to v

$$p = \frac{\partial \mathcal{L}}{\partial \dot{v}} = \frac{L(u)^2}{2\sqrt{-\dot{v}}}\,. \tag{6.24}$$

Feeding this back into the action, we get that the on-shell volume will be

$$V_\gamma = \frac{1}{2p}\int_{u_R}^{u_L} \mathrm{d}u\; L(u)^4 = 2p\,[v(u)]_{u_R}^{u_L}\,, \tag{6.25}$$

meaning that with this choice of parameters $v(u)$ itself measures (up to a multiplicative constant) the volume along the slice. Integrating (6.24) we get

$$v(u) = -\frac{1}{4p^2}\int \mathrm{d}u\; L(u)^4 + c\,, \tag{6.26}$$

and we can fix the value of the constants p and c by imposing the boundary conditions $v(u_{L,R}) = v_{L,R}$. Solving for p we can write the volume purely in terms of the unknown function $L(u)$ and the boundary values

$$V_\gamma = \sqrt{(v_R - v_L)\int_{u_R}^{u_L} \mathrm{d}u\; L(u)^4}\,, \tag{6.27}$$

which is a completely general expression for arbitrary boundaries. For the simpler setup $z_R = -z_L = \ell$ and considering also a symmetric evolution $t_R = t_L = t$, we may calculate the rate of growth, which gives

$$\frac{\mathrm{d}V_\gamma}{\mathrm{d}t} = \frac{1}{4p}\left[L(u_L)^4 - L(u_R)^4\right]. \tag{6.28}$$

As a check, we see that the trivial flat solution ($L(u) = 1$ and $\beta(u) = 0$) gives the expected behaviour, i.e. an extremal surface given by the straight line $v(u) = -u + 2t$, corresponding to the fixed-t surface. The volume of this surface is simply $V = 2\ell$, and of course its growth rate vanishes.

Notice that any other non-trivial profile for $L(u)$ enjoying the symmetry $L(u_L)^4 = L(u_R)^4$ will have vanishing rate as well. This is for example the case of pp-waves that have a finite extension in the u direction (e.g. a compactly supported pulse). The volume may change as the wave enters or leaves the region but will stay constant if the metric at the boundaries remains flat.

Computation of the Gravitational Infall Momentum

After solving explicitly the variational problem for the pp-wave spacetime (6.17), we will now evaluate the gravitational infall momentum induced by the Weyl tensor. To begin with, recall that the embedding functions $X^\mu = \{u, v(u), x, y\}$ defining the surface are given by

$$v(u) = -\frac{1}{4p^2}\int \mathrm{d}u\; L(u)^4 + c\,, \tag{6.29}$$

so, choosing the coordinates on Σ to be $y^a = \{u, x, y\}$ we can readily calculate the tangent vectors

$$e^\mu_a = \frac{\partial X^\mu}{\partial y^a} = \begin{pmatrix} 1 & -\frac{L(u)^4}{4p^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{6.30}$$

and the induced metric on the slice

$$h_{ab} = g_{\mu\nu} e^\mu_a e^\nu_b = \begin{pmatrix} \frac{L(u)^4}{4p^2} & 0 & 0 \\ 0 & L(u)^2 e^{2\beta(u)} & 0 \\ 0 & 0 & L(u)^2 e^{-2\beta(u)} \end{pmatrix}. \tag{6.31}$$

The timelike normal vector is

$$N^\mu = \left(\frac{2p}{L(u)^2}, \frac{L(u)^2}{2p}, 0, 0\right)\,, \tag{6.32}$$

which allows us to compute the extrinsic curvature of the surface

$$K_{ab} = \nabla_\mu N_\nu e^\mu_a e^\nu_b = \begin{pmatrix} -\frac{LL'}{p} & 0 & 0 \\ 0 & \frac{2p\, e^{2\beta}(L'+L\beta')}{L} & 0 \\ 0 & 0 & \frac{2p\, e^{-2\beta}(L'-L\beta')}{L} \end{pmatrix}, \tag{6.33}$$

which is of course traceless ($K = 0$) due to the extremal character of Σ.

We now have all the ingredients at hand to determine the infall tensor field satisfying the differential equation and boundary conditions

$$K_{ab}\nabla_c M^{abc} = 0\,, \tag{6.34}$$

$$-\ell\, K_{ab} M^{abc}\Big|_{\phi\Sigma} = (\partial_t)^\mu\, e^c_\mu\Big|_{\phi\Sigma}\,. \tag{6.35}$$

The dynamical equation gives

$$e^{2\beta}\left(L' + L\beta'\right)\left(M^{xxu}\left(3L' + L\beta'\right) + LM'^{xxu}\right) \tag{6.36}$$

$$+ \quad e^{-2\beta}\left(L' - L\beta'\right)\left(M^{yyu}\left(3L' - L\beta'\right) + LM'^{yyu}\right) = 0\,,$$

from which it is easy to find a particular solution given by

$$M^{yyu} \;=\; A\,e^{\beta}L^{-3}\,, \tag{6.37}$$

$$M^{xxu} \;=\; B\,e^{-\beta}L^{-3}\,. \tag{6.38}$$

A and B are integration constants to be fixed by the boundary conditions (6.35) which become

$$\left.\frac{e^{-\beta}\left(2pL\beta'\left(A - Be^{2\beta}\right) - 2pL'\left(A + Be^{2\beta}\right)\right)}{L^4}\right|_{u=u_L,u_R} = \frac{1}{2\ell} - \left.\frac{2p^2}{\ell L^4}\right|_{u=u_L,u_R}. \tag{6.39}$$

Actually, since (6.35) is a vectorial equation, two more scalar equations impose certain algebraic conditions on other tensor components of M^{abc}. These components will nevertheless be annihilated upon contraction with the Weyl tensor (6.20) in the generalized PVC formula, so we will not calculate them here. Solving for the constants A and B we get

$$\ell\,A \;=\; \frac{e^{\beta_L+\beta_R}\left(e^{\beta_R}\left(4p^2 - L_L^4\right)\left(L_R' + L_R\beta_R'\right) - e^{\beta_L}\left(4p^2 - L_R^4\right)\left(L_L' + L_L\beta_L'\right)\right)}{4pe^{2\beta_R}\left(L_L' - L_L\beta_L'\right)\left(L_R' + L_R\beta_R'\right) - 4pe^{2\beta_L}\left(L_L' + L_L\beta_L'\right)\left(L_R' - L_R\beta_R'\right)}\,, \tag{6.40}$$

$$\ell\,B \;=\; \frac{e^{\beta_R}\left(4p^2 - L_R^4\right)\left(L_L' - L_L\beta_L'\right) - e^{\beta_L}\left(4p^2 - L_L^4\right)\left(L_R' - L_R\beta_R'\right)}{4pe^{2\beta_R}\left(L_L' - L_L\beta_L'\right)\left(L_R' + L_R\beta_R'\right) - 4pe^{2\beta_L}\left(L_L' + L_L\beta_L'\right)\left(L_R' - L_R\beta_R'\right)}\,, \tag{6.41}$$

where we used the notation $L_{L,R} = L(u_{L,R})$. Embedding the tensor into the four-dimensional spacetime $M^{\nu\rho\sigma} = M^{abc}e_a^{\nu}e_b^{\rho}e_a^{\sigma}$, and contracting it with the Weyl and normal vectors we can finally obtain the longitudinal portion of the gravitational infall momentum

$$\begin{aligned}
\frac{\ell}{2V[\mathbf{R}^2]}\int_\Sigma \mathcal{W}_M \;&=\; -\frac{\ell}{2V[\mathbf{R}^2]}\int_\Sigma N^\mu\,W_{\mu\nu\rho\sigma}\,M^{\nu\rho\sigma} \\
&=\; \frac{\ell}{2}\int_{u_R}^{u_L}\mathrm{d}u\left(Ae^{-\beta(u)} - Be^{\beta(u)}\right)\left(2L'(u)\beta'(u) + L(u)\beta''(u)\right) \\
&=\; \frac{1}{4p}\left[L(u_L)^4 - L(u_R)^4\right]\,,
\end{aligned} \tag{6.42}$$

which of course is exactly (6.28), the same result that we obtained with the direct extremalization procedure. The vanishing of the total gravitational infall momentum for a perfectly contained pulse, having $L(u_L) = L(u_R)$, is consistent with the idea that the pp-wave is 'infalling' from the point of view of the left boundary, but is equally 'outfalling' from the point of view of the right boundary.

Observations

We conclude this section with a few observations regarding these explicit computations.

- The pp-wave example illustrates that a form of the PVC correspondence holds for pure gravitational waves in asymptotically flat spacetime but, more emphatically, the 'box' can have a finite size, defined by some conventional coordinate condition. If the M-tensor can be defined as satisfying (6.10), the generalized PVC holds with no need to make special physical arrangements to define the walls of the box. In this sense we can say that the generalized PVC correspondence is a quasilocal property of the bulk dynamics.

- Even though we chose a purely gravitational solution in order to maximize the differences with our previous analysis for collapsing matter solutions in section 4.1, notice that most of the details go through even after dropping the Ricci-flatness condition (6.19). In such case, the ansatz (6.17) describes in general a mixture of a gravitational and a (null) matter pp-wave with an energy-momentum tensor given by

$$T_{\mu\nu} = -\frac{2}{8\pi G} L^{-1} \left(L'' + (\beta')^2 L \right) \delta^u_\mu \delta^u_\nu \,. \tag{6.43}$$

  Of course, for this model to describe real matter one should ask $T_{\mu\nu}$ to satisfy certain null energy conditions, which in turn will impose some restrictions on the functions $L(u)$ and $\beta(u)$. At any rate, our observation here is that this non-trivial Ricci curvature does not change any of our analysis from (6.29) to (6.41) since all quantities on the slice depend only on first order derivatives of the metric, yielding formally identical results for the $M^{\mu\nu\sigma}$ tensor field. The total volume variation does however pick an additional term from the matter momentum

$$\frac{\mathrm{d}V_\gamma}{\mathrm{d}t} = \frac{\ell}{V[\mathbf{R}^2]} \left[\frac{1}{2} \int_\Sigma \mathcal{W}_M - \int_\Sigma N^\mu T_{\mu\nu} C^\nu \right] \,, \tag{6.44}$$

  where the 'infall vector field' C^μ defined in (6.3) can be easily obtained from our solution for M^{abc} yielding

$$C^a = \ell^{-1} L^{-1}(u) \left(A e^{-\beta(u)} + B e^{\beta(u)} \right) \delta^a_u \,. \tag{6.45}$$

  As it can be readily checked, the sum of the two contributions in (6.44) nicely recovers the correct result (6.28) again without any additional contribution. The technical reason behind this relies on the existence of the total derivative

$$\frac{\mathrm{d}F(u)}{\mathrm{d}u} = \sqrt{h} \left[\frac{8\pi G}{d-1} \mathcal{P}_C + \frac{1}{2} \mathcal{W}_M \right] \,, \tag{6.46}$$

$$F(u) = -\left(A e^{-\beta} + B e^{\beta} \right) L' + \left(A e^{-\beta} - B e^{\beta} \right) L \beta' \,, \tag{6.47}$$

  which allows us to perform the integral over Σ for arbitrary functions $L(u)$ and $\beta(u)$, including the Ricci and conformally flat solutions as well as any generic mixture of gravitational and matter waves.

- As it is well known, the very concept of local energy and momentum becomes ill-defined when we try to adapt it to gravity itself, where only perturbative notions of an approximate energy-momentum pseudotensor $t_{\mu\nu}$ have been proposed (cf. [102–104]). For that reason we do not expect to find a clear general relation between complexity growth and energy inflow for pure gravity solutions. We find however amusing that our example admits an interpretation along these lines. In particular, it is possible to find a 'gravitational inflow vector' $\widetilde{C}^{\mu}$ which allows us to re-write the integrand in (6.42) in a similar fashion as the matter piece, i.e.

$$W_{\mu\nu\rho\sigma}M^{\nu\rho\sigma} = \widetilde{t}_{\mu\nu}\widetilde{C}^{\nu} \tag{6.48}$$

with

$$\widetilde{C}^{a} = \ell^{-1}L^{-1}(u)\left(Ae^{-\beta(u)} - Be^{\beta(u)}\right)\delta^{a}_{u}\,, \tag{6.49}$$

$$\widetilde{t}_{\mu\nu} = 2\left(\frac{2L'\beta' + L\beta''}{L}\right)\delta^{u}_{\mu}\delta^{u}_{\nu}\,, \tag{6.50}$$

where we can identify $\widetilde{t}_{\mu\nu}$ as the 'square root' [21] of the Bel-Robinson tensor (cf. [106]), an object that is constructed purely from the Weyl tensor and satisfies the dominant property $\widetilde{t}_{\mu\nu}u^{\mu}u^{\nu} \geqslant 0$ for any future-pointing vector u^{μ} (cf. [107]). It would be interesting to investigate whether this formal analogy still holds beyond the particular example at hand.

[21] See [105] for a proper definition of this object.

7 Matter Infall and Complexity

To gain intuition about the infall momentum, we invoke the PVC correspondence (5.14) in this section to study the evolution of complexity for states with dilute matter which backreacts slightly on the geometry. We will just work to leading order in the backreaction, given by some parameter $\varepsilon \ll 1$ that controls the deviation of the metric $g_{\mu\nu} = \mathbf{g}_{\mu\nu} + \varepsilon\, \delta g_{\mu\nu}$ from the background metric $\mathbf{g}_{\mu\nu}$. We will assume that this reference state is 'trivial', in the sense that background metric $\mathbf{g}_{\mu\nu}$ is a spherically symmetric vacuum solution. Under these assumptions about the reference state, all the time-dependence of the complexity of the state can be traced back to the operator that creates the matter, as in the definition (3.2).

The background C-field is just $\mathbf{C}^\mu = -r\, \mathbf{e}_y^\mu$ in terms of the radial tangent field to the background slice $\mathbf{\Sigma}$, with unit normal $\mathbf{N}^\mu$. Since $T_{\mu\nu} \sim \varepsilon$, at leading order in the backreaction ε, the infall momentum can be evaluated on the background slice,

$$P[\Sigma] \;=\; -\int_{\mathbf{\Sigma}} \mathbf{N}^\mu \, T_{\mu\nu}\, \mathbf{C}^\nu \;. \tag{7.1}$$

The remainder term (5.15) must vanish in the reference state for this choice of background C-field. Moreover, the extinsic curvature of the background slice vanishes identically, $\mathbf{K}_{ab} = 0$, by virtue of the staticity of spacetime. The only term that can contribute at leading order in ε then comes from

$$R_C[\Sigma] \;=\; \frac{1}{8\pi G}\int_{\mathbf{\Sigma}} \delta K^{ab}\, \left(\nabla_{(a}\mathbf{C}_{b)} - \frac{1}{d}\,\mathbf{h}_{ab}\, \nabla\cdot\mathbf{C}\right)\;, \tag{7.2}$$

where δK^{ab} represents the linear variation of the extrinsic curvature of the slice Σ due to backreaction, and $\mathbf{h}_{ab}$ is the induced metric of the background slice $\mathbf{\Sigma}$. Since the $\mathbf{C}$-field is a CKV of the background slice, then the term inside the parenthesis exactly vanishes, making $R_C[\Sigma] = 0$. Therefore the PVC correspondence becomes exact at leading order in the backreaction

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} \;=\; P_{\text{infall}} \;=\; -\int_{\mathbf{\Sigma}} \mathbf{N}^\mu \, T_{\mu\nu}\, \mathbf{C}^\nu \;, \tag{7.3}$$

where, from now on in this section, we neglect all contributions of order $O(\varepsilon^2)$.

7.1 Newtonian limit

For any state satisfying an exact PVC relation (5.16), the radial C-field is conformal to the canonical C-field of the Poincaré ball, which vanishes at the 'center'. This vanishing point may be moved by the action of the asymptotic isometries, such as translations in Minkowski spacetime, but a given globally defined infall field will always have a 'center'. This suggests that the infall momentum behaves like angular momentum does: an arbitrary center must be specified, although any center is a valid reference point.

The important notion of 'infall momentum' can be further elucidated by taking the Newtonian limit of (7.3) for a collection of point particles. We can have these particles moving deep inside AdS, in a region of size $\ell \ll \ell_{\rm AdS}$ or work directly in asymptotically flat spacetime. Fixing the reference system at the point where $\mathbf{C} = 0$, the complexity rate in the Newtonian approximation is the total infall momentum for the particle system:

$$\frac{\rm d}{\rm dt}\,\mathcal{C}_{\rm Newtonian} \;=\; P_{\rm infall} \;=\; -\frac{1}{\ell_{\rm AdS}}\sum_i \mathbf{x}_i \cdot \mathbf{p}_i \;, \tag{7.4}$$

where we have momentarily restored the dependence on the 'box' length scale $\ell_{\rm AdS} = 1$, an arbitrary choice in this Newtonian discussion. We see that it is indeed a sort of 'radial-inward' version of the angular momentum, constructed with scalar products rather than vector products. Just like angular momentum, the so-defined 'infall momentum' is not invariant under translations or boosts, and a special role is played by the center of mass $\mathbf{X} = \sum_i m_i \mathbf{x}_i / \sum_i m_i$.

Suppose our system has a number of distant clusters, so that each of them can be regarded as approximately isolated. The total infall momentum can be decomposed in 'intrinsic' and 'orbital' parts:

$$P_{\rm infall} = \sum_\alpha P_{\rm infall}[\mathbf{X}_\alpha] - \sum_\alpha \mathbf{P}_\alpha \cdot \mathbf{X}_\alpha \;, \tag{7.5}$$

where $\mathbf{X}_\alpha$ is the center of mass of the α-cluster and $\mathbf{P}_\alpha$ its total momentum. In this expression, $P_{\rm infall}[\mathbf{X}_\alpha]$ accounts for the 'intrinsic' infall momenta within each cluster, measured with respect to its center of mass. Hence, 'compositeness' of effective particles is incorporated through an additive term for each particle, something analogous to 'spin'.

Infall momentum has the crucial property of being a total derivative, $P_{\rm infall} = \frac{\rm d}{\rm dt}\,\mathcal{I}_{\rm clump}$, where

$$\mathcal{I}_{\rm clump} = -\frac{1}{2}\sum_i m_i \mathbf{x}_i^2 \tag{7.6}$$

is a sort of 'spherical' moment of inertia which measures the degree of 'clumping' of the matter. Hence we find that, within the Newtonian approximation, the complexity is completely determined, up to an additive constant, by the degree of matter 'clumping'

$$\mathcal{C}_{\rm Newtonian} = \mathcal{C}_0 + \mathcal{I}_{\rm clump} \;. \tag{7.7}$$

Complexity and Newtonian Force

Unlike angular momentum, infall momentum will not be conserved in general. Its time derivative, proportional to the second derivative of the complexity, is

$$\frac{\mathrm{d}^2\mathcal{C}}{\mathrm{d}t^2} \;=\; \frac{\mathrm{d}}{\mathrm{dt}}\, P_{\mathrm{infall}} \;=\; -2T - \sum_i \mathbf{x}_i \cdot \mathbf{F}_i \;, \tag{7.8}$$

where $\mathbf{F}_i$ is the Newtonian force acting on the i-th particle and T is the total kinetic energy.

If the internal dynamics of the system is described by a potential $V(\mathbf{x}_i)$ which is a homogeneous function of degree k, Euler's theorem implies

$$\frac{\mathrm{d}}{\mathrm{dt}}\, P_{\mathrm{infall}} \;=\; -2T + kV = -(k+2)T + kE \;, \tag{7.9}$$

with E the conserved total energy. For a gravitational system, $k = -1$, which is either unbound or marginally bound, $E \geqslant 0$, the time derivative of the infall momentum, or equivalently, the second derivative of the complexity, is guaranteed to be negative. For stably bounded systems, on the other hand, the time-average of the infall momentum must vanish, $\frac{1}{2t_0}\int_{-t_0}^{t_0} \mathrm{d}t\, P_{\mathrm{infall}}(t) \;=\; 0$, where t_0 corresponds to a suitable time-window, and complexity remains constant on average.

7.2 Relativistic Matter

Let us now consider relativistic matter in asymptotically flat spacetime $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\, \mathrm{d}\Omega_{d-1}^2$, where we again fix the reference system $r = 0$ at the center of infall. At leading order in the backreaction, the PVC correspondence (7.3) is given in terms of the constant global-time background slice $\mathbf{\Sigma}$, with unit normal $\mathbf{N} = \partial_t$ and C-field $\mathbf{C} = -r\,\partial_r$. The infall momentum in this case is

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{dt}} \;=\; P_{\mathrm{infall}} \;=\; \int \mathrm{d}r\, \mathrm{d}\Omega_{d-1}\, r^d\, T_{tr} \;. \tag{7.10}$$

This relativistic infall momentum also happens to be a total derivative. The relativistic matter is conserved at leading order in the backreaction, with respect to the background metric, $\nabla_\mu T^{\mu\nu} = 0$. From this condition, it is straightforward to see that indeed

$$r^d\, T_{tr} \;=\; \partial_t \left(r \int_0^r \mathrm{d}r'\, r'^{d-1}\, T_{tt} \right) \;, \tag{7.11}$$

up to total derivatives that vanish under the integration over the $\mathbf{S}^{d-1}$. Plugging this expression in (7.10), and interchanging the order of the radial integrals gives, $P_{\mathrm{infall}} = \frac{\mathrm{d}}{\mathrm{dt}}\, \mathcal{I}_{\mathrm{clump}}$, for the relativistic 'clumping' moment of inertia

$$\mathcal{I}_{\mathrm{clump}} \;=\; -\frac{1}{2} \int \mathrm{d}r\, r^2\, \mathcal{E}(r) \;, \tag{7.12}$$

where $\mathcal{E}(r) = \int \mathrm{d}\Omega_{d-1}\, r^{d-1}\, T_{tt}$ is the energy density per radial unit. The power of this expression relies in the fact that the 'spherical' moment of intertia now includes the effect in the complexity of any classical field with small backreaction. Moreover, we emphasize that our derivation applies for any dilute distribution of energy T_{tt}, which can have an arbitrary profile along the $\mathbf{S}^{d-1}$.

The complexity of the relativistic system is given again by the degree of matter 'clumping'

$$\mathcal{C} = \mathcal{C}_0 + \mathcal{I}_{\text{clump}}\,. \tag{7.13}$$

Complexity and Relativistic Dynamics

There is also a generalization of (7.8) to the relativistic case. To derive it, let us start by writing the difference $P_{\text{infall}}(t_2) - P_{\text{infall}}(t_1)$ as a flux over the boundaries of the spatial region $\mathcal{W}_{(t_1,t_2)}$ that extends for $t \in [t_1, t_2]$. The difference of infall momenta corresponds to the flux of the vector field $v^\mu = -T^\mu{}_\nu\, \mathbf{C}^\nu$ evaluated at $\partial\mathcal{W}_{(t_1,t_2)}$. That is, [22]

$$P_{\text{infall}}(t_2) \;-\; P_{\text{infall}}(t_1) \;=\; \int_{\partial W_{(t_1,t_2)}} \mathbf{N}_\mu\, v^\mu\,. \tag{7.14}$$

Applying Stoke's theorem, the difference in infall momenta has the following expression as a spacetime integral over $\mathcal{W}_{(t_1,t_2)}$,

$$P_{\text{infall}}(t_2) \;-\; P_{\text{infall}}(t_1) \;=\; \int_{W_{(t_1,t_2)}} T_{\mu\nu}\, \nabla^\mu\, \mathbf{C}^\nu\,. \tag{7.15}$$

The covariant derivative of the background C-field can be written as $\nabla^\mu\, \mathbf{C}^\nu \;=\; -\,\mathbf{e}^\mu_a\, \mathbf{e}^\nu_b\, \mathbf{h}^{ab}$. Decomposing the spacetime integral using the $\mathbf{\Sigma}$ slices, and taking $t_2 \to t_1$ we get

$$\frac{\mathrm{d}^2\mathcal{C}}{\mathrm{d}t^2} \;=\; \frac{\mathrm{d}P_{\text{infall}}}{\mathrm{d}t} \;=\; -\int_{\mathbf{\Sigma}} p_{\mathbf{\Sigma}} \tag{7.16}$$

where $p_{\mathbf{\Sigma}} = T_{ab}\, \mathbf{h}^{ab}$ is the 'total pressure' along the slice $\mathbf{\Sigma}$. The average pressure can also be expressed as $p = \epsilon + T$, in terms of the energy density $\epsilon = \mathbf{N}^\mu\, T_{\mu\nu}\, \mathbf{N}^\nu$ and the trace $T = T^\mu_\mu$. [23]

As an example, consider a perfect fluid with $T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p\, \mathbf{g}_{\mu\nu}$, the total pressure is positive provided that the pressure p and the energy density ρ of the fluid are positive, $p_{\mathbf{\Sigma}} \geqslant dp$. The second derivative of the complexity for such an unbound system is negative.

[22] Here we assume that the fields decay sufficiently fast as $r \to \infty$.

[23] Note that, for the Newtonian case, (7.8) vanishes for a rotating rigid body along the center of infall, since in that case $\mathbf{x}_i \cdot \mathbf{F}_i = m_i \dot{\mathbf{x}}_i^2$. For the relativistic case, the rotating body has $p = 0$ in (7.16). The 'centripetal force field' generates negative radial pressure $T_{rr} < 0$ which cancels the angular component.

8 Conclusions

Recent progress in gravitational holography has lead to some speculation regarding the fundamental origin of the gravitational attraction. The most radical ideas point towards gravity and quantum mechanics being two sides of the same coin, after all. Quantitatively, this connection is encapsulated in the momentum/size correspondence reviewed in section 2 of this thesis, which characterizes the gravitational attraction exerted by a black hole as the tendency of a 'small' probe operator to scramble under the influence of a chaotic Hamiltonian.

However, any correspondence which relies on the notion of 'operator size' will have its own limitations to describe finite-entropy systems after the scrambling time. In particular, the PS correspondence is unable to describe the experience of the infalling particle in the black hole interior. Furthermore, black holes and fast scrambling seems crucial on its derivation, while on the other hand, gravitational attraction is a universal feature of all forms of energy.

In this thesis, we have presented a momentum/complexity correspondence, in the context of the Complexity = Volume prescription, which: (i) extends to arbitrary late times after scrambling, keeping track of the experience of the infalling matter in the black hole interior, and (ii) is valid for *any* spherically symmetric state of matter, however dilute, in higher dimensions, as well as for any state in 2+1 dimensions. The PVC correspondence formalizes the idea that the gravitational clumping of matter increases the complexity of the quantum state.

In section 4 we have presented the original 'phenomenological' derivation of the correspondence, following the evolution of thin shell operators impinging on double-sided AdS black holes. The key to the construction is to measure the momentum with respect to a bulk time foliation by the same maximal surfaces that one uses to compute the VC.

Next, in section 5, we have shown that the momentum/complexity correspondence is implicit in the Complexity=Volume prescription, as a result of the Momentum Constraint in General Relativity. The PVC correspondence is based on two ingredients that were advanced in the thin-shell analysis of section 4: the use of maximal-volume hypersurfaces as the time foliation to measure the momentum, and a particular choice of momentum component along the extremal surfaces, determined by an appropriate 'infall field' C_Σ. In formulas

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \int_\Sigma \mathcal{P}_C + R_C\,[\Sigma]\;, \tag{8.1}$$

where $\mathcal{P}_C = -N_\Sigma^\mu\, T_{\mu\nu}\, C_\Sigma^\nu$ is the infall momentum. The infall field is required to have fixed boundary conditions at infinity, but otherwise the freedom implicit in its specification is reflected in the existence of a 'remainder' correction $R_C[\Sigma]$ to the PVC relation. The remainder vanishes if C_Σ extends to the bulk as a conformal Killing vector, something that is guaranteed for any spacetime in $2+1$ dimensions and any spherically symmetric spacetime in arbitrary dimensions. From the physical point of view, the most important exception is provided by gravitational waves. This is natural in some sense, since we know that there is simply no candidate for a local measure of purely gravitational momentum to be integrated over Σ.

In section 6 we have presented a further generalization of the idea that certain holographic complexity/momentum correspondences are largely implicit in the dynamics of Einstein gravity. The main realization is that the PVC correspondence of section 5 admits a nontrivial generalization into a fully gravitational PVC stemming from the Codazzi equation:

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \int_\Sigma \mathcal{P}_C + \int_\Sigma \mathcal{W}_M \ . \tag{8.2}$$

The main novelty of this generalized PVC relation is the occurrence of a new contribution to the 'rate of gravitational clumping', measured by an appropriate flux of the Weyl tensor across the extremal surface, $\mathcal{W}_M = -\frac{d-1}{16\pi G}\, N^\mu_\Sigma\, W_{\mu\nu\rho\sigma}\, M^{\nu\rho\sigma}_\Sigma$. A crucial technical ingredient is the generalization of the notion of 'infall vector field' C_Σ into a rank-3 'infall tensor field' M_Σ with the same symmetry structure as the Codazzi equation itself. In order for (8.2) to be true, M^{abc}_Σ must be chosen to satisfy the equation $K_{ab}\, \nabla_c M^{abc}_\Sigma = 0$ throughout Σ, with boundary conditions (6.6). We have explicitly checked that these requirements can be met in an exact pure-gravity pp-wave solution of Einstein's equations.

In section 7, we have shown that the central concept of 'infall momentum' has a Newtonian version which explicitly captures the intuitive idea that matter clumping increases complexity. A relativistic generalization of this version also exists. Finally, the value of VC for states with small backreaction is given in terms of a radial 'moment of inertia' that quantifies the degree of clumping of matter.

The PVC correspondence presented in this thesis opens many avenues for future research. For instance, it would be interesting to check the complexity slope (4.40) for the thin shells by direct evaluation of the infall momentum of the shell. This requires detailed control of the precise location of the intersection sphere $\mathcal{S}_\mathcal{W}$ in the black hole interior. It is also interesting to check whether a transient exists for early times which shows a measurable Lyapunov exponent. This is a nontrivial fact, given that our time foliation is quite different from a near-horizon Rindler system. In particular, such chaotic transients were numerically identified in [92, 108] in VC computations relevant to situations which are similar, although not identical, to the set up studied in section 4.

Another interesting open problem is to find a generalization of the PVC correspondence to include non-trivial boundary dynamics in AdS/CFT examples. This includes the VC of 'cosmological' constructions driven by time-dependent states in the CFT, as in [109], and 'boundary gravitons' in $2+1$ dimensions [64, 110–112]. It would be interesting to study the detailed solutions of (5.12) that arise in these situations, where the PVC relation is expected to contain additional 'boundary' contributions beyond the bulk infall momenta.

The generalized PVC correspondence also poses a number of interesting questions. While (8.2) is certainly more general than (8.1), we are still lacking a more precise physical interpretation of the Weyl-momentum $\mathcal{W}_M$. It would be interesting to explore possible connections to pseudo-local energy notions based on the Bel–Robinson tensor, as suggested at the end of the section 6.2. Further elucidation along these lines will follow form a careful analysis of weak-field expansions around the asymptotic factorized ansatz (6.12).

At a purely mathematical level, it would be interesting to delimitate the reach of sufficient conditions such as the symmetrized transversality condition on the infall tensor (6.13). The answer is guaranteed to be nontrivial, for at least two reasons. First, the explicit solution we have found for M_Σ^{abc} in the pp-wave example does *not* satisfy (6.13). Therefore, we know that in cases that are sufficiently far from the factorized ansatz (6.12), the transversality condition is too strong. Second, even when the factorized ansatz works and (6.13) reduces to the conformal Killing condition, topological obstructions can prevent the remainder from vanishing.

We end with a digression on the more general significance of PVC relations like (8.1) and (8.2). First of all, our proposed PVC correspondences are tailor-made for the VC prescription. By now, a plethora of different complexity proposals exist [64,65,93–95,111–115] and it would be interesting to see if analogous momentum/complexity correspondences can be formulated. When addressing this question, one should keep in mind that subtly different notions of complexity may exist in the boundary description. As a simple example of this fact, we can consider operator K-complexity [85–88, 116], which is conceptually different from circuit complexity, yet it shows analogous 'phenomenology' in certain situations.

At any rate, we know that extremal spatial volumes parametrized by codimension-one boundary data are interesting quantities in any putative holographic description. Whether they are literally related to some sort of computational complexity is an open question, but it is certain that there exists a notion of 'volume complexity' induced from the bulk description. In this context, one can imagine using the PVC formula as a basis for its elucidation. Since the right hand side of (8.2) is a local bulk integral, we can expect that a sufficiently powerful prescription of bulk operator reconstruction can be used to give an operational definition of $\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t}$ in the dual holographic picture (CFT or otherwise). A further integration determines the 'volume complexity' up to a constant, mimicking the strategy followed before to determine the Newtonian limit of the complexity in equations (7.6) and (7.7). In this context, it becomes interesting to investigate the relation between the PVC correspondence and other structural properties of holographic complexity, such as the first and second laws of complexity [117–119].

Conclusiones

El progreso reciente en holografía gravitacional ha generado cierta especulación a la hora de reinterpretar el origen fundamental de la atracción gravitatoria. Las ideas más radicales apuntan a que la gravedad y la mecánica son dos caras de la misma moneda, después de todo. Cuantitativamente, esta conexión está capturada en la correspondencia momento/tamaño presentada en la sección 2 de esta tesis, que caracteriza la atracción gravitatoria ejercida por un agujero negro como la tendencia de un operador de prueba a mezclarse entre todos los grados de libertad del agujero negro bajo la acción de un Hamiltoniano caótico.

Sin embargo, cualquier correspondencia que se base en la noción de 'tamaño del operador' tiene sus propias limitaciones cuando se trata de describir sistemas de entropía finita después del *scrambling time*. En particular, la correspondencia PS es incapaz de describir la experiencia de la partícula en caída libre en el interior del agujero negro. Además, los agujeros negros y el caos maximal parecen ingredientes cruciales en su derivación, mientras que, por el contrario, la atracción gravitatoria es una característica universal de todas las formas de energía.

En esta tesis, hemos presentado una correspondencia momento/complejidad, en el contexto de la prescripción Complejidad = Volumen, que satisface las siguientes propiedades: (i) es válida para tiempos arbitrariamente largos después del *scrambling time*, y contiene la información de la particula en caída libre en el interior del agujero negro (ii) su rango de aplicabilidad incluye *cualquier* estado esféricamente simétrico de materia, sin importar su densidad, en dimensiones superiores, junto con cualquier estado en 2+1 dimensiones. La correspondencia PVC formaliza la idea de que la compresión gravitatoria de la materia incrementa la complejidad cuántica del estado.

En la sección 4 hemos descrito la derivación 'fenomenológica' original de la correspondencia, siguiendo la evolución de los operadores de tipo corteza que caen a un agujero negro eterno en AdS. El punto clave en la construcción es el hecho de medir el momento con respecto a la foliación temporal dada por las superficies maximales que se utilizan para calcular la VC.

Después, en la sección 5, hemos demostrado que la correspondencia momento/complejidad queda implícita en la prescripción Complejidad = Volumen, como resultado de la restricción inicial de momento de la Relatividad General. La correspondencia PVC está basada en dos ingredientes que fueron avanzados en el análisis de las cortezas de la sección 4: el uso de hipersuperficies de volumen extremal como la foliación temporal para medir el momento, y una elección particular de la componente del momento a lo largo de las superficies extremales, determinada por el correspondiente 'campo de caída' C_Σ. En fórmulas

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \int_\Sigma \mathcal{P}_C + R_C\,[\Sigma]\;, \tag{8.3}$$

donde $\mathcal{P}_C = -N^\mu_\Sigma T_{\mu\nu} C^\nu_\Sigma$ es el momento de caída. El campo de caída satisface ciertas condiciones de contorno en el infinito, pero a pesar de eso, la libertad en definirlo está implícita en la especificación del resto $R_C[\Sigma]$ en la relación PVC. El resto se anula si C_Σ se extiende a lo largo de la superficie como un vector de Killing conforme, una propiedad que está garantizada para cualquier espaciotiempo en $2+1$ dimensiones y para cualquier espaciotiempo esféricamente simétrico en dimensiones arbitrarias. Desde el punto de vista físico, la excepción más importante a que el resto se anule se debe a las ondas gravitacionales. Esto es natural en cierto sentido, ya que es bien sabido que no existe simplemente una medida local del momento puramente gravitacional para ser integrado a través de Σ.

En la sección 6 hemos presentado una generalización más de la idea de que ciertas correspondencias momento/complejidad se encuentran implícitas en la dinámica de la gravedad de Einstein. La realización principal es que la correspondencia PVC de la sección 5 admite una generlalización no trivial a una correspondencia PVC totalmente gravitacional que emerge de la ecuación de Codazzi:

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \int_\Sigma \mathcal{P}_C + \int_\Sigma \mathcal{W}_M \ . \tag{8.4}$$

La mayor novedad de esta relación PVC generalizada es la existencia de una nueva contribución al ritmo de compresión gravitatoria medida por un flujo apropiado del tensor de Weyl a lo largo de la superficie extremal, $\mathcal{W}_M = -\frac{d-1}{16\pi G}\, N^\mu_\Sigma\, W_{\mu\nu\rho\sigma}\, M^{\nu\rho\sigma}_\Sigma$. Un ingrediente técnico crucial es la generalización de la noción del 'campo vectorial de caída' C_Σ al 'campo tensorial de caída' de rango 3 M_Σ con las mismas simetrías que la ecuación de Codazzi. Para que (8.4) sea válida, el campo M^{abc}_Σ tiene que elegirse de tal forma que satisfaga la ecuación $K_{ab}\,\nabla_c M^{abc}_\Sigma = 0$ a lo largo de Σ, con las condiciones de contorno dadas por (6.6). Hemos corroborado explícitamente que estos requerimientos se pueden satisfacer para una solución exacta a las ecuaciones de Einstein sin materia del tipo *pp-wave*.

En la sección 7, hemos demostrado que el concepto central de 'momento de caída' tiene una versión Newtonianan que captura explícitamente la idea de que la compresión de la materia aumenta la complejidad. Existe también una noción relativista del 'momento de caída'. Finalmente, el valor de VC para estados con *backreaction* pequeña viene dada en términos de un 'momento radial de inercia' que cuantifica el grado de compresión de la materia.

La correspondencia PVC presentada en esta tesis abre muchas puertas para investigar en el futuro. Por ejemplo, sería interesante corroborar la pendiente de la complejidad (4.40) para las cortezas mediante la evaluación directa del momento de caída de la corteza. Esto requeriría un control detallado del lugar de la esfera de intersección $\mathcal{S}_\mathcal{W}$ en el interior del agujero negro. Sería también interesante investigar si existe una región transitoria a tiempos cortos que muestre un exponente de Lyapunov realmente medible. Esto es un hecho no trivial, dado que nuestra foliación temporal es bastante distinta de las coordenadas de Rindler en la región cercana al horizonte. Estas transiciones caóticas fueron determinadas en particular en [92, 108] en cálculos de VC correspondientes a situaciones que similares, pero no del todo idénticas, a las consideradas en la sección 4.

Otro problema interesante es encontrar una generalización de la correspondencia PVC que incluya una dinámica no trivial de la frontera en ejemplos de AdS/CFT. Esto incluye la VC de las construcciones 'cosmológicas' regidas por estados dependientes del tiempo en la CFT, como en [109], y 'gravitones de frontera' en $2+1$ dimensiones [64,110–112]. Sería interesante estudiar soluciones detalladas a (5.12) que surgiŕıan en estas situaciones en las que se espera que la correspondencia PVC contenga contribuciones adicionales de frontera además del momento de caída.

La correspondencia PVC generalizada también genera un gran numero de preguntas interesantes. Mientras que es ciertamente más general que, todavía nos falta una interpretación física más precisa del 'momento de Weyl' $\mathcal{W}_M$. Sería interesante explorar las posibles conexiones con las nociones pseudo-locales de energía basadas en el tensor de Bel–Robinson, como sugerimos en el final de la sección 6.2. La elucidación de estas nociones requreriría un análisis detallado de las expansiones de campo débil en torno al ansatz asintóticamente factorizado (6.12).

A nivel matemático, sería interesante delimitar el alcance de las condiciones suficientes como la condición de la transversalidad simétrica del tensor de caída (6.13). La respuesta está garantizada a ser no trivial, al menos por dos razones. La primera es que la solución explícita que hemos encontrado para M_Σ^{abc} en el ejemplo de la *pp-wave* no satisface (6.13). Por ello, sabemos que casos suficientemente alejados de la solución factorizada (6.12), la condición de transversalidad es demasiado restrictiva. Segundo, aunque la solución factorizada funcione y (6.13) se reduzca a la condición de vector de Killing conforme, las restricciones topológicas pueden prevenir de que el resto se anule.

Finalizamos con una disgresión en el significado más general de relaciones PVC como (8.1) y (8.2). Primero, las correspondencias PVC que hemos propuesto están específicamente diseádas para la prescripción VC. Actualmente, existe una plétora de propuestas distinta para la complejidad holgráfica [64,65,93–95,111–115] y sería interesante saber si existen correspondencias momento/complejidad análogas para estas otras propuestas. Cuando se trata esta cuestión, uno debe tener en mente que pueden existir nociones de complejidad que difieran sutilmente en la CFT. Como simple ejemplo de este hecho, podemos considerar la K-complejidad del operador [85–88,116], que es conceptualmente distinta a la complejidad computacional medida por un circuito cuántico, y aún así muestra una 'fenomenología' similar en ciertas situaciones.

De cualquier modo, sabemos que los volumenes espaciales extremales parametrizados por datos de la frontera de codimensión uno son cantidades interesantes en cualquier descripción holográfica hipotética. Si estas cantidades se corresponden literalmente con algún tipo de complejidad computacional o no es un problema abierto, pero lo que está claro es que existe una noción de 'complejidad de volumen' inducida desde la descripción dual. En este contexto, uno se podría imaginar tomar la correspondencia PVC como base para su elucidación. Por el hecho de que la parte derecha de (8.4) es una cantidad local en el lado gravitatorio, podemos esperar que una prescripción lo suficientemente poderosa del operador en el *bulk* se pueda reconstruir para dar una definición operacional a $\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t}$ en el dual holográfico (CFT u otro). Una integración más en el tiempo determina la 'complejidad de volumen', salvo una constante, adoptando la estrategia que hemos seguido para determinar el límite Newtoniano de la complejidad en las ecuaciones (7.6) and (7.7). En este contexto, se vuelve interesante investigar la relación entre la correspondencia PVC y otras propiedades estructurales de la complejidad cuántica, como la primera y segunda ley de la complejidad [117–119].

Appendices

A Late time accumulation of maximal slices

In this appendix, we show proof of the exponentially fast accumulation of maximal slices in the black hole interior. For that matter, we will work within the benchmark case of an eternal black hole, whose metric is given in Eddington-Finkelstein coordinates by

$$\mathrm{d}s^2 \;=\; -f(r)\,\mathrm{d}u^2 \;+\; 2\,\mathrm{d}u\,\mathrm{d}r + r^2\mathrm{d}\Omega_{d-1}^2\,. \tag{A.1}$$

By spherical symmetry, the maximal surface can be written as a direct product $\Sigma = \gamma \times \mathbf{S}^{d-1}$, with γ a curve in the $u - r$ plane. Exploiting this symmetry we can reduce thus the problem of volume extremalization to that of a spacelike geodesic in the effective two-dimensional spacetime

$$\mathrm{d}s_\gamma^2 \;=\; r^{2(d-1)}(-f(r)\,\mathrm{d}u^2 \;+\; 2\,\mathrm{d}u\,\mathrm{d}r)\;, \tag{A.2}$$

so that the effective volume functional is given by

$$V[\Sigma]V_\Omega^{-1} \;=V[\gamma] = \int \mathrm{d}\lambda\, r^{d-1}\sqrt{-f(r)\,\dot{u}^2 + 2\,\dot{u}\,\dot{r}}\;, \tag{A.3}$$

where λ is an arbitrary spacelike parameter and the dot stands for $\mathrm{d}/\mathrm{d}\lambda$. The Lagrangian in (A.3) enjoys a conserved charge associated to the static Killing

$$\Pi \;=\frac{\partial \mathcal{L}_\gamma}{\partial \dot{u}} = r^{d-1}\,\frac{-f(r)+\dot{r}}{\sqrt{-f(r)+2\,\dot{r}}}\;, \tag{A.4}$$

where Π is guaranteed to be positive by the spacelike character of the geodesic $\mathrm{d}s_\gamma^2 > 0$ and we have taken the convinient gauge choice $\lambda = u$. Feeding the conserved charge into the equations of motion for $r(u)$ we get

$$\dot{r} \;=\; f(r) \;+\; \frac{\Pi^2}{r^{2(d-1)}} \;+\; \frac{\Pi}{r^{d-1}}\sqrt{\frac{\Pi^2}{r^{2(d-1)}} + f(r)}\;. \tag{A.5}$$

Upon the imposition of reflection symmetry in our setup ($t_L = t_R = t$), the boundary conditions can be recasted to be $\dot{r}(u_i) = 0$ and $r(u_\infty) = r_\infty$ for $u_i = r_*(r_i)$, $u_\infty = t$ the values of the parameter at the symmetric turning point and boundary respectively. In terms of the turning point radius r_i we can get a simple expression for Π

$$\Pi \;=\; r_i^{(d-1)}\,\sqrt{-f(r_i)}\;. \tag{A.6}$$

An implicit relation between t and r_i can be obtained integrating (A.5)

$$\int_{u_i}^{u_\infty} du \;\;=\;\; \int_{r_i}^{r_\infty} dr\,\frac{r^{2(d-1)}}{g^{1/2}(r)\,\left(\Pi + g^{1/2}(r)\right)}\;. \tag{A.7}$$

where we have defined the function

$$g(r) = r^{2(d-1)} f(r) - r_i^{2(d-1)} f(r_i) , \tag{A.8}$$

which vanishes at the minimal radius r_i. Breaking up the radial integral into an inner an outer piece and substituting the boundary conditions, we can obtain an expression for the boundary time

$$t = \int_{r_i}^{r_h} dr \frac{r^{2(d-1)}}{g^{1/2}(r) \left(\Pi + g^{1/2}(r)\right)} + h(r_h, r_i, r_\infty) . \tag{A.9}$$

where $h(r_h, r_i, r_\infty)$ is a finite function for all values of its parameters. As we see from the structure of the zeros of $g(r)$, the integral above contains a pole at $r = r_i$. In order to approximate the integral (A.9) we may expand $g(r)$ to second order around r_i

$$g(r) = \alpha(\widetilde{r}_i - r_i)(r - r_i) + \frac{\alpha}{2}(r - r_i) + \ldots . \tag{A.10}$$

where α is a positive constant depending on the parameters of the black hole and $\widetilde{r}_i$ is the asymptotic limiting surface. The necessity to go up to second order in the expansion is revealed by the vanishing of the linear term in the late time limit corresponding to $r_i \to \widetilde{r}_i$. Feeding (A.10) into (A.9) and expanding the rest of the integral to zero order we get

$$t \approx \frac{r_i^{2(d-1)}}{\Pi} \int_{r_i}^{r_h} dr \left[\alpha(\widetilde{r}_i - r_i)(r - r_i) + \frac{\alpha}{2}(r - r_i)\right]^{-1/2} + \text{ finite} . \tag{A.11}$$

which can be solved exactly

$$t \approx -\frac{r_i^{2(d-1)}}{\Pi(\alpha/2)^{1/2}} \log(r_i - \widetilde{r}_i) + \text{ finite} . \tag{A.12}$$

Inverting this expression we get the desired result, i.e. the exponentially fast saturation of maximal slices in the black hole interior

$$r_i - \widetilde{r}_i \approx b\, e^{-t/a} , \tag{A.13}$$

where a and b approach constant values in the late time limit.

B One-sided PVC correspondence

In this appendix, we extend the regime of validity of the PVC correspondence (4.39) to situations in which there is a spherically symmetric thin shell living in the AdS vacuum. We introduce a slightly more general formalism to manifestly show that the same PVC formula holds for any spherical thin shell irrespectively of its internal equation of state.

We start from a single holographic CFT on $\mathbf{S}^{d-1}$ and take the CFT vacuum as the reference state to define the operator complexity (3.2). Using the VC prescription, the bulk definition is

$$\mathcal{C}\left[\mathcal{O}_{\text{shell}}\right] = \frac{d-1}{8\pi G}\left[\text{Vol}(\Sigma_{\text{AdS+shell}}) - \text{Vol}(\Sigma_{\text{AdS}})\right] \,, \tag{B.1}$$

where Σ is the extremal hypersurface of interest, defined in empty AdS with and without the shell respectively. A peculiarity of this choice of reference state is that its complexity is constant in time, and this makes the rate of (B.1) to depend only on the extremal hypersurface on the spacetime with the shell. This extremal volume hypersurface Σ will be topologically a ball anchored to the asymptotic sphere $\mathcal{S}_\infty$ at boundary time t. A generic infinitesimal deformation of its embedding function $\delta X^\mu = \delta\varepsilon\, N^\mu_\Sigma + \delta\kappa^a\, e^\mu_a$ will produce the volume variation

$$\delta V[\Sigma]\big|_{\text{extremal}} \;=\; \int_\Sigma \nabla_a \delta\kappa^a \;=\; \int_{\mathcal{S}_\infty} dS_a\, \delta\kappa^a \,, \tag{B.2}$$

as in (4.22), which in this case follows from the tracelessness of the extrinsic curvature of Σ. In particular, for time translations of the boundary sphere, we need to take the tangent deformation to asymptotically become $(\delta\kappa_a)|_{\mathcal{S}_\infty} = (\partial_t \cdot e_a)\,\delta t$. From (B.2), the rate of extremal volume then reads

$$\frac{\text{d}V}{\text{d}t} \;=\; \int_\Sigma \nabla_a\, \rho^a \;\equiv\; \Pi \,, \tag{B.3}$$

for ρ^a any tangent vector that asymptotically approaches $\partial_t \cdot e^a$.

For spherically symmetric thin shell configurations, there will be two timelike Killing vectors $\xi^\mu_\pm$ individually defined on each of the regions of spacetime $\mathcal{V}^\pm$ glued by the worldvolume $\mathcal{W}$. Taking ℓ as a normal coordinate to $\mathcal{W}$, we can define the Killing vector field globally as $\xi^\mu = \xi^\mu_-\,\Theta(-\ell) \,+\, \xi^\mu_+\,\Theta(\ell)$, where Θ is the step function. The Killing condition is then broken due to a possible discontinuity across $\mathcal{W}$

$$\nabla_{(\mu}\,\xi_{\nu)} \;=\; (N_{\mathcal{W}})_{(\mu}\, (\Delta\xi)_{\nu)}\; \delta(\ell) \,, \tag{B.4}$$

where we used that $\partial_\mu \Theta(\ell) \,=\, \delta(\ell)\,(N_{\mathcal{W}})_\mu$, for $N^\mu_{\mathcal{W}}$ the $\mathcal{W}$-normal. The global piecewise Killing ξ^μ asymptotically becomes the time translation generator ∂^μ_t, and therefore it is possible to choose its projection to the extremal hypersurface $\xi^a = \xi \cdot e^a$ to play the role of the tangent vector in (B.3). The projection of (B.4) into Σ reads

$$\nabla_{(a}\,\xi_{b)} \;=\; \delta(\ell)\,\big(N_{\mathcal{W}}\cdot e_{(a}\big)\,\big(\Delta\xi\cdot e_{b)}\big) \,+\, (\xi\cdot N_{\Sigma})\,K_{ab}\;, \tag{B.5}$$

where K_{ab} the extrinsic curvature of Σ. This second term breaks the Killing condition as a consequence of the original Killing ξ^{μ} failing to be tangent to Σ. Nevertheless, for the extremal hypersurface Σ the trace of this term vanishes, which makes this tangent vector to be conserved away from $\mathcal{W}$

$$\nabla_a\,\rho^a \;=\; \delta(\ell)\;(N_{\mathcal{W}}\cdot e_a)\;h^{ab}\;(\Delta\xi\cdot e_b)\;. \tag{B.6}$$

This tangent vector precisely agrees with the Noether current (4.26) arising from the internal time-translation symmetry of the volume functional.

In this framework, we thus find that the rate of the operator complexity is proportional to a localized quantity on $\mathcal{W}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{C}[\mathcal{O}_{\text{shell}}] \;=\; \frac{d-1}{8\pi G}\int_{\Sigma}\delta(\ell)\;(N_{\mathcal{W}}\cdot e_a)\;h^{ab}\;(\Delta\xi\cdot e_b)\;, \tag{B.7}$$

namely the discontinuity of the stationary Killing vector field.

To evaluate the discontinuity of the Killing vector across $\mathcal{W}$, let us focus on the codimension two sphere of intersection $\mathcal{S}_{\mathcal{W}} = \Sigma\cap\mathcal{W}$. We define the spacelike tangent to Σ which is orthogonal to $\mathcal{S}_{\mathcal{W}}$ and unit norm, denoted by e_y^{μ}. Similarly, we define the timelike tangent to $\mathcal{W}$, denoted u^{μ}, as the one orthogonal to $\Sigma\cap\mathcal{W}$ and unit norm. From spherical symmetry $\xi^{\mu}_{\pm}|_{\mathcal{W}}$ will be orthogonal to the spheres, and an identical argument to the one provided in section 4.1 determines that the only discontinuity will be tangent to $\mathcal{W}$ and with value

$$(\Delta\xi^{\mu})_{\mathcal{W}} \;=\; -\frac{8\pi G}{d-1}\,(S_{\rho\sigma}\,u^{\rho}\,u^{\sigma}\,R)\;u^{\mu}\;, \tag{B.8}$$

where $S_{\mu\nu}$ is the induced energy-momentum on $\mathcal{W}$, and R is the radius of $\mathcal{S}_{\mathcal{W}}$. Substituting in (B.7) and noting that $N_{\mathcal{W}}\cdot e_y$ can be written as $-N_{\Sigma}\cdot u$ from the argument given in 4.2, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{C}[\mathcal{O}_{\text{shell}}] \;=\; \int_{\Sigma}(T_{\mu\nu}u^{\mu}u^{\nu})\;r\;(N_{\Sigma}\cdot u)\,(u\cdot e_y)\;. \tag{B.9}$$

The one-sided version of the PVC correspondence then follows from the decomposition $u^{\mu}u^{\nu} = -g^{\mu\nu} + N^{\mu}_{\mathcal{W}}N^{\nu}_{\mathcal{W}} + g^{\mu\nu}_{\mathcal{S}_{\mathcal{W}}}$, where the last term is the induced metric on $\mathcal{S}_{\mathcal{W}}$, and from the thin-shell condition $T_{\mu\nu}\,N^{\nu}_{\mathcal{W}} = 0$. Upon the definition of the 'complexity field' $\mathcal{C}^{\mu}_{\Sigma} = -r\,e^{\mu}_{y}$, we arrive at the desired formula

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{C}[\mathcal{O}_{\text{shell}}] \;=\; -\int_{\Sigma}N^{\mu}_{\Sigma}\,T_{\mu\nu}\,\mathcal{C}^{\nu}_{\Sigma}\;. \tag{B.10}$$

This derivation of the PVC correspondence certainly clarifies that the PVC formula applies to any spherically symmetric thin shell in AdS, including branes that separate AdS patches of different curvature radius.

C Rotating thin shell in AdS_3

In this Appendix we use the language developed in Appendix B to begin exploring less symmetric configurations. We consider the particular example of a rotating thin shell that collapses in AdS_3, corresponding to a stationary but not static exterior spacetime. This solution will be treated formally in the sense that we do not insist in the physical consistency of the shell's energy momentum tensor. The main interest of this simple exercise is to show that formula (B.10) continues to apply with the same complexity field $\mathcal{C}^{\mu}_{\Sigma}$, despite the existence of 'shear' components in the jumping conditions for the Killing vectors.

The outside spacetime $\mathcal{V}^+$ consists of a rotating BTZ solution (cf. [120])

$$\mathrm{d}s_+^2 \;=\; -f_+(r)\,\mathrm{d}t_+^2 \;+\; \frac{\mathrm{d}r^2}{f_+(r)} + r^2\left(\mathrm{d}\phi_+ \;-\; \frac{a}{r^2}\,\mathrm{d}t_+\right)^2 \;, \tag{C.1}$$

with blackening factor

$$f_+(r) \;=\; r^2 - \mu^2 \;+\; \frac{a^2}{r^2}\;, \tag{C.2}$$

for $a \;=\; 4\,GJ$ and $\mu^2 \;=\; 8\,GM$ the ADM angular momentum and mass, respectively. We choose the inner spacetime $\mathcal{V}^-$ to be pure AdS_3

$$\mathrm{d}s_-^2 \;=\; -(1+r^2)\,\mathrm{d}t_-^2 \;+\; \frac{\mathrm{d}r^2}{1+r^2} + r^2\,d\phi_-^2\;. \tag{C.3}$$

The worldvolume of the shell $\mathcal{W}$ will have metric

$$\mathrm{d}s^2_{\mathcal{W}} = -\mathrm{d}\tau^2 \;+\; R(\tau)^2\,\mathrm{d}\psi^2\;, \tag{C.4}$$

where ψ is a co-rotating angle. Demanding for the continuity of the metric across $\mathcal{W}$ translates then to the set of conditions

$$\psi \;=\; \phi_- \;=\; \phi_+ - \,\omega(R)\,t_+ \;+\theta(\tau)\;, \tag{C.5}$$

$$-1 \;=\; -f_-(R)\,(\dot{t}_-)^2 \;+\; \frac{(\dot{R})^2}{f_-(R)} \;=\; -f_+(R)\,(\dot{t}_+)^2 \;+\; \frac{(\dot{R})^2}{f_+(R)}\;, \tag{C.6}$$

where the angular frequency of the shell is basically $\omega(R) \;=\; a/R^2$, and the function $\theta(\tau)$ accounts for the variation in the angular frequency of the shell due to its shrinking

$$\dot{\theta}(\tau) \;=\; \dot{\omega}(R)\,t_+\;. \tag{C.7}$$

The discontinuity in the extrinsic curvature on $\mathcal{W}$ as seen from $\mathcal{V}^{\pm}$ will be sourced by the induced energy-momentum tensor of the shell $S_{\mu\nu}$. Since the interior frame is co-rotating with the shell, the situation is the same as for the spherically symmetric collapse in section 4.1, for which we already know the components of the extrinsic curvature. The calculation from the exterior frame is a little more involved, but it can be done by using the precise form of the outward pointing $\mathcal{W}$-normal $(N_{\mathcal{W}})_\mu = \dot{t}_+ (\mathrm{d}r)_\mu - \dot{R}\,(\mathrm{d}t_+)_\mu$ and velocity field $u^\mu = \dot{R}\,\partial_r^\mu + \dot{t}_+\,\partial_{t_+}^\mu + \omega\,\dot{t}_+\,\partial_{\phi_+}^\mu$. The second junction conditions can then be expressed as

$$S^\tau{}_\tau = \frac{1}{8\pi G}\,\frac{\beta_+ - \beta_-}{R} \tag{C.8}$$

$$S^\psi{}_\psi = \frac{1}{8\pi G}\,\frac{\dot{\beta}_+ - \dot{\beta}_-}{\dot{R}} \tag{C.9}$$

$$S^\tau{}_\psi = -\frac{1}{8\pi G}\,\omega R \tag{C.10}$$

where $\beta_\pm = \sqrt{\dot{R}^2 + f_\pm(R)}$.

Let us proceed to calculate the discontinuity in the stationary Killing vector

$$\Delta\xi^\mu \;=\; -(\Delta\xi\cdot u)\,u^\mu \;+\; \frac{(\Delta\xi\cdot\partial_\psi X)}{R^2}\,\partial_\psi X^\mu \;+\; (\Delta\xi\cdot N_{\mathcal{W}})\,N_{\mathcal{W}}^\mu\,. \tag{C.11}$$

It is straightforward to evaluate all these projections, an using (C.8) and (C.10) we can write them as

$$\Delta\xi^\mu \;=\; -\left(8\pi G S_{\tau\tau} R\right)\,u^\mu \;-\; \left(8\pi G\, S_{\tau\psi}\,\frac{1}{R}\right)\partial_\psi X^\mu\,. \tag{C.12}$$

Plugging this result in (B.7), and noting that the extremal hypersurface Σ will in this case intersect $\mathcal{W}$ on a constant τ circle, we have that the angular discontinuity of the Killing does not contribute to the rate of the complexity since $(N_{\mathcal{W}}\cdot\partial_\psi X)$ vanishes. Moreover, the contribution from the Killing discontinuity in the u^μ direction has the same form as in the spherically symmetric case, and hence we obtain the same PC duality

$$\frac{\mathrm{d}}{\mathrm{d}t}\,\mathcal{C}[\mathcal{O}_{\text{shell}}] \;=\; -\int_\Sigma N_\Sigma^\mu\,T_{\mu\nu}\,\mathcal{C}_\Sigma^\nu\,, \tag{C.13}$$

where the 'complexity field' $\mathcal{C}_\Sigma^\mu = -r\,e_y^\mu$. It is tempting to conjecture that the 'complexity field' $\mathcal{C}_\Sigma^\mu$ persists to be inward pointing tangent to Σ and orthogonal to $\Sigma\cap\mathcal{W}$ for more general situations of thin shells gluing two stationary spacetimes $\mathcal{V}^\pm$ together.

D Recovering the Exact PVC for Special Cases

In this appendix, we show that the factorized *ansatz* (6.12) for the infal tensor M^{abc} is the most general solution of the trace-free transversality condition (6.13) for generic 2+1 dimensional spacetimes as well as for spherically symmetric solutions in higher dimensions. The boundary condition (6.6) reduces to (4.23) for the C-field, which is now restricted to be a conformal Killing vector. With previous knowledge of the required asymptotic behavior for the C-field in AdS, we also comment on how the generalized PVC reduces to the exact PVC for extremal volume slices anchored to the asymptotic boundary of AdS.

Let us first consider a generic spacetime in $2+1$ dimensions. The number of algebraically independent components of the infall tensor M^{abc} for $d=2$ is 2, which precisely coincides with the number of trace-free transversality conditions (6.13). In order to solve them explicitly, we will choose coordinates locally on Σ such that

$$\mathrm{d}s_\Sigma^2 \;=\; e^{2\omega(z,\bar{z})}\,\mathrm{d}z\,\mathrm{d}\bar{z}\;, \tag{D.1}$$

where $z \;=\; y+i\phi$, and $\omega(z,\bar{z})$ some real function. For the case of asymptotically AdS spacetimes, the metric (D.1) asymptotes the Poincaré disk metric $\omega \;\sim\; y$ as $y\to\infty$. We will suitably choose the two independent components of M^{abc} to be the real and imaginary parts of $M^{zz\bar{z}}$ in these complex coordinates. The set of conditions (6.13) becomes particularly simple in these coordinates

$$\partial_{\bar{z}}\,\left(e^{2\omega}\,M^{zz\bar{z}}\right) \;=\; 0\;, \tag{D.2}$$

$$\partial_{z}\,\left(e^{2\omega}\,M^{\bar{z}\bar{z}z}\right) \;=\; 0\;. \tag{D.3}$$

It is straightforward to see that the most general solution of these equations is

$$M^{zz\bar{z}}(z,\bar{z}) \;=\; 2\,g(z)\,e^{-2\omega(z,\bar{z})}\;, \tag{D.4}$$

for some holomorphic function $g(z)$. The C-field obtained by taking the trace of this infall tensor is precisely $C^z \;=\; g(z)$. In two dimensions, every vector field of this form is locally a conformal Killing vector. The key observation is that this infall tensor field factorizes as $M^{abc} \;=\; h^{ac}\,C^b - h^{ab}\,C^c$. It then becomes clear the reason why the general solution of (6.13) can be constructed from an infall C-field which is a conformal Killing vector. In AdS, the required asymptotic boundary condition is $C^y \sim -1$ for the case of the unnormalized y coordinate. The unique holomorphic extension of this condition is to set $g(z) \;=\; -1$ throughout Σ. This way, we obtain the canonical C-field (cf. Figure **??**) which is orthogonal to the constant y lines, inward pointing, and has a norm that depends on the point in question, $C^2 \;=\; e^{2\omega}$. This infall field certainly coincides with the inward radial conformal Killing vector of the Poincaré disk. In fact, the Weyl-momentum vanishes in 2+1 dimensions as the Weyl tensor is exactly zero, which, together with the above definition of the C-field, shows how the generalized PVC reduces the exact PVC correspondence for any geometric state in 2+1 dimensions.

Let us now consider the case of a spherically symmetric spacetime in higher dimensions. Assuming that Σ inherits spherical symmetry, the induced metric can be written as

$$\mathrm{d}s_\Sigma^2 \;=\; \mathrm{d}y^2\;+r^2(y)\,\mathrm{d}\Omega_{d-1}^2\;, \tag{D.5}$$

where y is an outward directed coordinate normal to the spheres. Moreover, it is natural to assume that the most generic infall tensor M^{abc} is isotropic under $SO(d)$ (cf. [121]), up to possible terms that do not contribute to the boundary condition, and hence can be considered as pure gauge redundancies (tangent diffeomorphisms that die off asymptotically). The only irreducible isotropic rank-3 tensor is ϵ^{abc} for the case of $SO(3)$, but still this tensor does not lie in the same irreducible representation of $GL(d)$ as the infall tensor. Therefore, any isotropic M^{abc} will necessarily be reducible into products of lower-rank tensors. The most general irreducible isotropic rank-2 tensor is of the form $f(y)\, h_{ab}$, where h_{ab} is the spherically symmetric metric (D.5). The most general isotropic vector is orthogonal to the spheres with an angle-independent norm, $C = C(y)\, \partial_y$. With these building blocks in hand, there are two ways to construct an isotropic M^{abc}, i.e. from a rank-2 tensor and a vector $C^a\, h^{bc}$, or alternatively from three vectors $C_1^a\, C_2^b\, C_3^c$. The latter belongs to the totally symmetric representation of $GL(d)$, and hence it vanishes when projected into the representation of M^{abc}. Projecting the former provides then with the most general isotropic infall tensor

$$M^{abc} \;=\; \frac{1}{d-1}\, \left(h^{ac}\, C^b \,-\, h^{ab} C^c \right) \,, \tag{D.6}$$

which is again of the factorized form (6.12). In asymptotically AdS spacetimes, the required asymptotic boundary condition (6.35) will be satisfied by the canonical inward radial C-field on the Poincaré ball $C \,=\, -\, r(y)\, \partial_y$. For any such factorized M^{abc}, the Weyl-momentum density will vanish due to tracelessness and antisymmetry of the Weyl tensor

$$\mathcal{W}_M \;=\; -\frac{1}{d-1}\, N^\mu\, W_{\mu\nu\rho\sigma}\, \left(h^{\nu\sigma}\, C^\rho \,-\, h^{\nu\rho}\, C^\sigma \right) \;=\; \frac{2}{d-1}\, N^\mu\, W_{\mu\nu\rho\sigma}\, C^\sigma \, \left(g^{\nu\rho} + N^\nu N^\rho \right) \;=\; 0\,, \tag{D.7}$$

which, together with the characterization of the C-field, leads to the exact PVC correspondence for any spherically symmetric normalizable state in $d+1$ dimensional asymptotically AdS spacetimes.

E Asymptotic Boundary Conditions

In this appendix, we extend the analysis of section 5 of asymptotically AdS boundary conditions to include the asymptotically flat case. We elaborate on the asymptotic boundary conditions for the C-field and M-field that solve (5.12) and (6.6) in both cases.

To start, we might adopt asymptotic coordinates in the vicinity of Σ such that the metric reads

$$\mathrm{d}s_X^2 \;\to\; \frac{\mathrm{d}r^2}{r^a} \,-\, r^a\, \mathrm{d}t^2 \,+\, r^2\, \gamma_{ij}(r,t,\theta)\, \mathrm{d}\theta^i \mathrm{d}\theta^j \quad \text{as} \quad r \,\to\, \infty\,. \tag{E.1}$$

where $a = 2$ is the AdS case and $a = 0$ is the flat case.

Here, r is an ‘asymptotically radial’ coordinate which foliates X by timelike codimension-one submanifolds Y_r. In the case of AdS, it corresponds to a Fefferman-Graham coordinate for a particular conformal frame. The angles θ^j parametrize the intersection $S_r \,=\, Y_r \cap \Sigma$, of spherical topology and induced metric proportional to γ_{ij}, which is itself asymptotic to a unit round sphere, up to normalizable corrections of order $1/r^d$. The time coordinate is chosen to be geodesic on Y_r and orthogonal to S_r.

The induced metric on Σ can be written near the boundary as

$$ds^2_\Sigma \to dy^2 + r^2(y)\,\gamma_{ij}(y,\theta)\,d\theta^i\,d\theta^j\ , \tag{E.2}$$

for some function $r(y)$ which asymptotically $r \sim a/2\,\sinh y\,+\,(1-a/2)\,y$ as $y \to \infty$. This allows us to write the normal one-form as $N_\Sigma = e^t_y\,dr - e^r_y\,dt$, and compute the extrinsic curvature $K_{ab} = e^\mu_a\,e^\nu_b\,\nabla_\mu\,N_\nu$. The relevant component turns out to be K_{yy} which, using the traceless character, $K = 0$, may be evaluated as $K_{yy} = -r^{-2}\,\gamma^{ij}\,K_{ij}$. Explicitly

$$K_{yy} = -\frac{d-1}{r^{1-a}}\,e^t_y - \frac{1}{2r^a}\,e^r_y\,\gamma^{ij}\,\partial_t\gamma_{ij} - \frac{r^a}{2}\,e^t_y\,\gamma^{ij}\,\partial_r\gamma_{ij}\ . \tag{E.3}$$

Asymptotically, $\partial_t\gamma_{ij} \sim 1/r^{d+1-a/2}$ and $\partial_r\gamma_{ij} \sim 1/r^{d+1}$. For $a = 2$, this is nothing but the requirement that the solution is asymptotically AdS, with the round metric on the conformal boundary. For asymptotically flat spacetimes, one of the defining properties is that all the derivatives of the metric perturbation decay with the same inverse power law of the radius. An asymptotic analysis of the $K = 0$ condition reveals the large-r scalings $e^r_y \sim r^{a/2}$, $e^t_y \sim 1/r^{d-1+a}$, so that the right hand side of (E.3) is dominated by the first term:

$$K_{yy} \approx -\frac{d-1}{r^{1-a}}\,e^t_y\ . \tag{E.4}$$

Since $e_y \cdot \partial_t = -\,r^a\,e^t_y$, we learn that (4.23) can be satisfied provided the C-field is chosen with the boundary conditions

$$C \to -\frac{1}{b}\,r(y)\,\partial_y \quad \text{as} \quad y \to \infty\ . \tag{E.5}$$

This is exactly the same result that was found for asymptotically AdS boundary conditions in section 5, justifying the name 'infall field' for the C-field. Similarly, the M-field satisfying (6.6) will asymptotically factorize as in (6.12) for the C-field given by (E.5).

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