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Arrival Time and Bohmian Mechanics: It Is the Theory Which Decides What We Can Measure

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Abstract: In this work, we analyze recent proposals by Das and Dürr (DD) to measure the arrival time distributions of quantum particles within the framework of de Broglie Bohm theory (or Bohmian mechanics). We also analyze the criticisms made by Goldstein Tumulka and Zanghì (GTZ) of these same proposals, and show that each protagonist is both right and wrong. In detail, we show that DD's predictions are indeed measurable in principle, but that they will not lead to violations of the no-signalling theorem used in Bell's theorem, in contradiction with some of Das and Maudlin's hopes.

Keywords: arrival time; quantum detection theory; Bohmian mechanics

1. Introduction

The concept of arrival time for a quantum particle at a spatial point has been a subject of considerable controversy since the theory was founded in the 1920s–1930s [1] (Similar difficulties arise in defining travel and dwell times in quantum mechanics [2,3]). At the more technical level, the main difficulty stems from the lack of consensus on the definition of a self-adjoint operator or POVM (positive operator valued measure) for the arrival time τ of a particle, and on the probability distribution $\mathcal{P}^{\Psi}(\tau)$ associated with these arrival times. Numerous proposals have been made over the years, none of them unanimously accepted (for exhaustive reviews of the problem, see [4–7]).

Remarkably, within the framework of de Broglie Bohm (dBB) theory [8,9], also called Bohmian mechanics—which is an alternative deterministic interpretation of quantum mechanics that re-establishes the notion of trajectory for particles—it is possible to define unambiguously the arrival time of a quantum particle at any point in space based on the precise calculation of the trajectory passing through that point [4,10–13].

However, one of the problems associated with this dBB definition of arrival times concerns its link with the notion of quantum observable and POVM. Although the dBB definition is well suited to the far-field regime, where it allows us to recover and justify standard results used in particle collision physics, it generally leads to difficulties in the near-field regime. In particular, it has been shown that the dBB definition leads to fundamental problems when the flow of particles across a surface is associated with the phenomenon of 'back-flow' [4]. In this back-flow regime, which is a purely undulatory phenomenon in which several waves with a wave vector pointing in a common direction generate an effective wave vector pointing in the opposite direction (for reviews and general discussions, see [14–19]), the same Bohmian trajectory crosses a predefined detection spatial zone D several times (i.e., at different times) from different sides [10]. The arrival time is therefore not uniquely defined, and we must add a condition on the first passage, the second passage, etc., of the particle in the detection region D [20,21]. Furthermore, the probability distribution of (first) arrival times given by dBB theory depends on the probability current $\mathbf{J}^{\Psi}(\mathbf{x},t)$ (more precisely on the so-called 'truncated' probability current distribution [22] associated with this multiplicity of passages on the detector). However, it has been shown [23,24] that, in general, the probability current $\mathbf{J}^{\Psi}(\mathbf{x},t)$ is not associated with a POVM due to the presence of back-flow. Since the notion of POVM is generally

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accepted as the most accurate theoretical description of a quantum observable, it seems a priori impossible to consider dBB arrival times as generally measurable. However, since these back-flow phenomena are usually confined to interference zones or near-field regions of very limited spatio-temporal extension, it is generally accepted that observing this regime would be very difficult and has not yet been achieved.

More fundamentally, the notion of a Bohmian arrival or travel time is still very controversial. For example, it was claimed that (in the context of tunneling times) '*Bohm's theory can make a definite prediction when standard quantum mechanics can make none at all*' [25] (similar speculations were discussed in [9] p. 215 and [26] pp. 53–55). This is a very strong statement which, if justified, would break the empirical equivalence between dBB mechanics and orthodox quantum mechanics. This would give a strong advantage to dBB theory and of course it generates controversy (see the discussion in [21]). In the present work, we will nevertheless insist on the possibility of preserving the empirical equivalence between dBB theory and traditional quantum mechanics. We will seek to demonstrate that there is no reason to doubt the relevance of this empirical equivalence. The key point here is not that dBB theory predicts results that contradict orthodox quantum mechanics. What we actually find is that, thanks to dBB mechanics, we can analyze and interpret empirical data (i.e., predicted by standard formalism) in terms of probabilities for Bohmian arrival times.

One other controversy is perhaps that the probability current $\mathbf{J}^{\Psi}(\mathbf{x}, t)$ is not unambiguously defined [13] (we can always add a term $\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{x}, t)$ to $\mathbf{J}^{\Psi}(\mathbf{x}, t)$ without altering the local conservation law $\partial\rho^{\Psi} + \boldsymbol{\nabla} \cdot \mathbf{J}^{\Psi} = 0$). There is thus a form of underdetermination concerning the uniqueness of dBB dynamics [27]. As a result, the physical meaning to be attributed to these arrival times based on the probability current is questionable. Nevertheless, there is a strong consensus concerning the far-field regime where the dBB trajectories are reduced to straight lines and which corresponds to the scattering regime without back-flow [28,29] (in this regime, a nice agreement between time of flight measurements and dBB prediction has been recently analyzed for a double-slit experiment [30]). It should be noted, in this context, that some authors [31–33] oppose this indetermination of the Bohmian dynamics on the basis of the relativistic extension of the dBB theory using the Dirac equation(see also [34–36]), and Holland's work [37] showing that Lorentz invariance fixes the arbitrariness in the form of the current.

A fundamental problem, not unrelated to arrival times in quantum mechanics and dBB theory, concerns the observability of trajectories predicted by dBB mechanics. For many years, it was accepted that these trajectories were hidden and therefore unobservable. However, the development of experimental methodologies based on the notion of weak measurements and weak values [38] has changed all that. Experiments and theories [39–42] have shown how to observe such Bohmian trajectories. Since this reconstruction of dBB trajectories is carried out statistically, it in no way violates the uncertainty principle. Moreover, recent work has emphasized the operational character of weak measurements and the definition of the observed dBB velocity field [43,44]. This is clearly linked to the present debate on arrival times, as we can see that it is the measurement procedure (and as we will, see some methods of post-selection and data analysis) that is at the heart of the problem of the definition of relevant physical quantities involved in the dBB theory. At the end of this work, we will make a link with weak measurements, showing that these methods can also contribute to solving the problem of arrival times.

Recently, it has been proposed by Siddhant Das and Detlef Dürr (hereinafter DD) to use the dBB approach for arrival times within the framework of the Pauli equation (i.e., the non-relativistic limit of the Dirac equation) for particles with spin-1/2 [45,46]. Going far beyond previous works [31–36] based on the Dirac equation, the authors have defined a precise regime, in principle physically attainable, where the presence of back-flow is not confined to the near-field domain. Importantly, they found a spin-dependent distribution of arrival time $\mathcal{P}_{\text{dBB}}^{\Psi_{\hat{\mathbf{s}}}}(\Sigma, \tau)$ (where Σ is the detector surface, τ the dBB arrival time, and $\Psi_{\hat{\mathbf{s}}}$ the wave function of the particle with spin direction along the direction $\mathbf{s}$) with interesting consequences for back-flow. This, of course, reopens the debate on the observation of Bohmian arrival times [47].

Moreover, in a recent comment [48], Goldstein Tumulka and Zanghì (hereinafter GTZ) critically assessed the proposition and calculations made in [45,46] and showed that if DD results were exact, and if the dBB first arrival time spin-dependent probability distribution could be identified with a POVM, then a contradiction occurs. Therefore, GTZ conclude that something must be wrong in the predictions given in [45,46] concerning the observation of dBB arrival times.

Yet, as Das and Aristarhov stressed in a reply [49], DD never actually claimed that their proposition for a dBB first arrival time measurement was reducible to a POVM, and thus far from contradicting [45,46] the results of GTZ [48], only show that indeed the Bohmian first arrival time probability distribution cannot be associated with a standard POVM. Therefore, the real question asked by GTZ and DD is whether or not '*the statistics of the outcomes of any quantum experiment are governed by a POVM*' [48] and only by POVM.

Here, to answer this question, we assess the analyses conducted by GTZ and DD. We show that while mathematically correct, GTZ's conclusions are physically unjustifed. In particular, we emphasize that GTZ's too strong reliance on POVM (which can be summarized by the slogan 'POVM and only POVM') is mostly a prejudice of the orthodox theory of quantum measurements that must be generally abandoned in light of the dBB theory. As we show, although POVMs are necessary, they are not sufficient to describe a Bohmian measurement process. Moreover, we also stress that contrary to DD's claims, experimental observation of the first arrival time probability distribution requires explicit consideration of the detector physics during the measurement process. Three key messages emerge from our analysis. First (i), in agreement with POVM no-go theorems [23,24], there is no universal detector for arrival time. Second (ii), every arrival time detector built for working at time t is in general a very invasive device and could prohibit subsequent detections at later time $t' > t$ (even if the measurement at time t did not actually occur because the detector did not fire; this is an instance of negative-result quantum measurements). In other words, in the dBB framework, one must distinguish between probability of being here at time t and probability of being detected here at time t. Finally (iii), first arrival times are defined within the dBB framework and as such require a post-selection of the data. The whole procedure is thus theory-laden. In the end, we show that when all these features are taken into account, nothing prohibits the experimental observation of the first arrival time probability distribution predicted by DD (e.g., in connection with weak measurement procedures introduced above [38–40,42,43]).

Moreover, as a follow-up of the previous studies by DD, it must be mentioned that the philosopher Tim Maudlin has on various occasions on social media [50–52] discussed the possibility of using the results obtained by DD for the spin-dependent time arrival probability distribution in order to develop new dBB-based faster-than-light communications protocols involving pairs of entangled spin-1/2 particles. This 'Bell telephone' possibility clearly contradicts the no-signalling theorem deduced from quantum mechanics in the context of Bell's theorem. More precisely, we demonstrate in quantum field theory that local commutativity and microcausality impose this no-signalling constraint [53,54]. As stressed by Bell, 'It is as if there is some kind of conspiracy, that something is going on behind the scenes which is not allowed to appear on the scenes' [55]. In fact, dBB theory emphasizes the crucial role of Born's rule in this derivation, as shown by Valentini [56], and Born's rule is fundamentally linked to the existence and use of POVMs in quantum mechanics. Unless we relax Born's rule, i.e., abandon 'quantum equilibrium' [56], or modify quantum mechanics, it is thus impossible in the quantum framework to exploit the violation of Bell's inequalities (i.e., the nonlocality of dBB theory) to transmit a signal faster than light. Therefore, in a recent extension of their original comment, GTZ stress [57] that DD's results, assuming Born's rule, also strongly contradict the no-signalling theorem and therefore conflict with standard quantum mechanics.

However, we demonstrate in this article that a new analysis of the problem, in particular in relation to point (iii) above, can remove the paradoxes. In fact, according to our analysis, it becomes possible to measure the probability distribution predicted by DD without violating the no-signalling theorem, thereby ruling out the possibility of supraluminal transmission channels contradicting Bell's theorem.

The layout of our article is as follows. In Section 2, we briefly review dBB theory and show, with the help of one typical example, how it allows us to give physical answers to questions that look devious in the orthodox quantum theory. In Section 2, we also analyze the nature of measurements in dBB theory and stress the limitation of the notion of POVM. In Section 3, we review the arrival time problem in the dBB theory and discuss DD's and Maudlin's proposals as well as the counter-analysis by GTZ. In Sections 4–6, we discuss the theory of detectors and the impact this has on dBB theory. In particular, we define regimes of strong and weak coupling for detections. Finally, in the last Section 7, we resume the first arrival time problem in the dBB theory and DD's proposal involving the Pauli or Dirac equation for a particle with spin-1/2 and show how we can in principle measure the distribution predicted by DD.

In particular, we will show that the arrival time distribution can be reconstructed using low-efficiency detectors so as not to perturb the wave function too much. As we shall see, the key point of this procedure is based on a highly non-linear post-selection method presuming knowledge of the underlying dBB dynamics (these dynamics being themselves in principle analyzable by weak measurement methods).

2. Bohmian Inference

The concept of experimental measurement, and more precisely of so-called direct experimental measurement, has always been a source of debate in physics since its foundation. Take for example Rutherford's experiment where a beam of α particles passes through a thin film of gold. From the deviation of α particles at high angles (i.e., in backscattering), Rutherford deduced (or rather induced) the existence of atomic nuclei acting as very compact centers of diffusion. Seen in hindsight, however, it is impossible to deduce a theory from this Rutherford experiment. As Albert Einstein understood perfectly well, the best we can say or infer is that in having a theory, we can define what is measurable, or not, and then compare the predictions to the results. In other words, any measurement is necessarily indirect and presupposes a theoretical model. As he explained in 1926 to Heisenberg, who claimed to be able to build a quantum theory by limiting himself only to what is observable:

> '*it is the theory which alone decides what is measurable*'. [58]

This is the heart of the hypothetico-deductive method!

In quantum physics, it is the forgetting of this elementary truth concerning the scientific method which is responsible for numerous errors of interpretation. So, let us take Young's famous two-slit experiment. According to Bohrian quantum doxa, it is impossible to interpret the observation of interference fringes using the concept of a continuous trajectory followed by individual particles. For Bohr and Heisenberg, for example, this would indeed amount to saying that the trajectory of a particle passing through hole A is influenced by the existence of hole B through which it, however, did not pass! From a classical perspective, this is a priori nonsense. But all this shows is that certain classical 'Newtonian' prejudices oppose a simple interpretation of Young's slit experiment in terms of trajectory.

However, we know that the pilot-wave dBB theory developed by Louis de Broglie in 1927 [59–61] and rediscovered by Bohm in 1952 [62,63] makes it possible to precisely explain this interference experiment using trajectories [8,9]. In this dBB theory, the trajectories of the particles are strongly curved by the presence of potentials of a specifically quantum nature which free themselves from the overly Newtonian prejudices of Bohr and Heisenberg. We can notice that the Newtonian reading made by Heisenberg and Bohr is biased. Indeed, in his seminal work *Opticks* published in 1704 [64], Newton sought to explain the observation of diffraction and Newton's rings using a theory (called by him the 'access theory') involving forces acting on the particles of light, and which, in many aspects, anticipates the notion of Bohmian quantum potential. In turn, this quantum theory of dBB, giving meaning to the notion of trajectory, makes it possible to define and characterize what is a 'good' measurement, or not, in complete agreement with Einstein's hypothetico-deductivism.

To be more precise, we remind the reader that the dBB velocity of a particle is given in the simplest non-relativistic spinless theory by the de Broglie guidance formula [8,9]:

$$\frac{d}{dt}\mathbf{x}(t) := \mathbf{v}^{\Psi}(\mathbf{x}(t),t) = \frac{\mathbf{J}^{\Psi}(\mathbf{x}(t),t)}{|\Psi(\mathbf{x}(t),t)|^2} = \mathrm{Im}[\frac{\boldsymbol{\nabla}\Psi(\mathbf{x}(t),t)}{m\Psi(\mathbf{x}(t),t)}] = \frac{\boldsymbol{\nabla}S^{\Psi}(\mathbf{x}(t),t)}{m} \tag{1}$$

where $\mathbf{J}^{\Psi}$ is the probability current, m the particle mass, and S^{Ψ} the phase of Ψ (i.e., the quantum Hamilton–Jacobi action). This first-order differential equation $\frac{dx}{v_x^{\Psi}} = \frac{dy}{v_y^{\Psi}} = \frac{dz}{v_z^{\Psi}} = dt$ can be integrated (at least numerically) and defines the Bohmian trajectories of the particle. In particular, trajectories obtained from these first-order dynamics can in general not cross [9].

An important feature of the dBB theory concerns probability and statistics. Indeed, from the law of conservation and the definition of the probability current, the dBB theory shows that if an ensemble of similarly prepared particles are statistically $\rho_{t_0}^{\Psi} := |\Psi|^2(\mathbf{x}(t_0),t_0)$ distributed at an initial time t_0, this will be so at any other time t: $\rho_t^{\Psi} := |\Psi|^2(\mathbf{x}(t),t)$. In other words, from this property, called equivariance, Born's rule $\rho^{\Psi} := |\Psi|^2$ is naturally consistent with the dBB theory, and therefore, the statistical predictions of standard quantum mechanics can be recovered within this framework [8,9].

Moreover, in the double-slit experiment, all of this has huge consequences. Consider the case of a single electron wave function

$$\Psi(x,y,z,t) = \Psi_0(x-a/2,y,z,t) + \Psi_0(x+a/2,y,z,t) \tag{2}$$

where $\Psi_0(x,y,z,t)$ is a propagating wave packet initially (i.e., at time $t=0$) centered on the origin and subsequently propagating along the z direction while it also spreads. Assuming $\Psi_0(x,y,z,t) = \Psi_0(-x,y,z,t)$, we thus deduce from Equations (1) and (2) that dBB trajectories cannot cross the symmetry plane $x=0$. Furthermore, suppose that at time $t=0$ the wave function $\Psi_0(x,y,z,t=0)$ has a finite spatial support $\Delta_0(\mathbf{0})$ such that the two wave packets $\Psi_0(x-a/2,y,z,t=0)$ and $\Psi_0(x+a/2,y,z,t=0)$ are not overlapping (i.e., $\Delta_0(a/2,0,0) \cap \Delta_0(-a/2,0,0) = \emptyset$). Thus, dBB theory allows us to retrodict; if we record the particle at the plane z in the zone $x>0$, we can indeed retrodict that the particle was necessarily coming at $t=0$ from the wave packet located in the upper side of the screen $z=0$, i.e., centered on $x=+a/2$. The converse is true for a particle detected in the region $x<0$, allowing us to infer that it was coming from the lower wave packet centered on $x=-a/2$.

With the dBB theory, the probability for the particle detected at time t to come from the wave packet centered on $x=a$ is thus rigorously

$$\mathcal{P}_{+a}^{\Psi}(t) = \int_{x\geq 0} d^3\mathbf{x}|\Psi(\mathbf{x},t)|^2 = \int_{\mathbf{x}-\hat{\mathbf{x}}\frac{a}{2}\in\Delta_0} d^3\mathbf{x}|\Psi_0(x-a/2,y,z,t=0)|^2 = \frac{1}{2} \tag{3}$$

with a similar and symmetric expression for $\mathcal{P}_{-a}^{\Psi}(t)$. This is easily deduced from the two properties that (i) the dBB trajectories cannot cross and ii) the conservation of the probability fluid is preserved along trajectories, i.e., $\rho^{\Psi}(\mathbf{x}(t),t)\delta^3\mathbf{x}(t) = \rho^{\Psi}(\mathbf{x}(t=0),t=0)\delta^3\mathbf{x}(t=0)$. According to dBB theory, we also have $\rho^{\Psi}(\mathbf{x}(t),t) = |\Psi|^2(\mathbf{x}(t),t)$, a probabilistic rule that was assumed by de Broglie even before Max Born!

For the present discussion, we emphasize that we can write

$$\mathcal{P}_{\pm a}^{\Psi}(t) = \langle\Psi(t)|\hat{O}_{\pm a}|\Psi(t)\rangle \tag{4}$$

with the operators $\hat{O}_{+a} = \int_{x\geq 0} d^3\mathbf{x}|\mathbf{x}\rangle\langle\mathbf{x}|$, $\hat{O}_{-a} = \int_{x\leq 0} d^3\mathbf{x}|\mathbf{x}\rangle\langle\mathbf{x}|$ defined as sums of projectors, i.e., a special case of POVM (in dBB theory, any measurement procedure is necessarily reduced to a spatial localization measurement, and so the use of projectors $|\mathbf{x}\rangle\langle\mathbf{x}|$ is unavoidable [9,65]).

We briefly remind the reader that mathematically speaking, POVMs $\hat{O}_n$ are linear self-adjoint operators (i.e., $\hat{O}_n = \hat{O}_n^\dagger$) acting on a Hilbert space $\mathcal{H}$, such that $\sum_n \hat{O}_n = \hat{I}$. These operators obey the positivity condition $\hat{O}_n > 0$ which actually reads $\langle\Psi|\hat{O}_n|\Psi\rangle > 0$ whatever the state $|\Psi\rangle \in \mathcal{H}$. This last condition is naturally needed in order to interpret $\langle\Psi|\hat{O}_n|\Psi\rangle$ as a probability. We note that, rigorously speaking, a POVM denotes the set of all linear operators satisfying the previous conditions. By extension, it is common to call any member of the preceding family a POVM, and we will continue to use this convention hereafter. The theory of quantum measurement ultimately relies on the concept of POVM (for an introduction to POVM and its use in quantum information processing, see [66,67], and for a more general and precise discussion related to the measurement process and dBB theory, see [24,65,68,69]). These operators indeed constitute fundamental mathematical tools formalizing the generalized von Neumann measurements coupling a system S to a pointer M. Moreover, in the dBB theory relying on spatial measurements, the fundamental role is played by projectors $|q\rangle\langle q|$ (where q is a coordinate vector in the configuration space of the system). We include in the Appendix A a brief description of the POVM measurement formalism applied to dBB theory for particles. This approach is non-ambiguous at least in the non relativistic regime for a particle with or without spin and by extension for Dirac (fermionic) relativistic particles (the extension to bosonic quantum fields is also possible but relies on different beables or hidden variables than particle positions q and will not be considered here).

Going back to our previous example with the double-slit experiment, it is central to observe that while $\mathcal{P}^{\Psi}_{\pm a}(t)$ are obtained from standard POVMs, the Bohmian algorithm to interpret these experimentally observable quantities as physical properties associated with the system at time $t = 0$ does not work for an arbitrary wave function. Indeed, the previous example strongly relies on the symmetry of Ψ_S. For a different superposition (for example, by adding a phase: $\Psi'_S(x, y, z, t) = \Psi_0(x - a/2, y, z, t) + e^{i\chi}\Psi_0(x + a/2, y, z, t)$), the interpretation of $\langle\Psi'(t)|\hat{O}_{+a}|\Psi'(t)\rangle$ as a probability $\mathcal{P}^{\Psi'}_{+a}$ for the particle to be initially in the upper wave packet will not generally hold! It will, however, work for the cases $\chi = 0$ or π.

In other words, in general, Equation (4) does not define genuine Bohmian 'which-path' observables. Moreover, for an arbitrary wave function $\Psi(x, y, z, t) = \alpha\Psi_0(x - a/2, y, z, t) + \beta\Psi_0(x + a/2, y, z, t)$, it will be possible to define other POVMs

$$\hat{O}^{(\Psi)}_{\pm a} = \int_{\mathbf{x}\in\Delta^{(\Psi)}_{\pm a}} d^3\mathbf{x}|\mathbf{x}\rangle\langle\mathbf{x}| \tag{5}$$

where $\Delta^{(\Psi)}_{\pm a}$ are spatial domain images of $\Delta_0(\pm a, 0, 0)$ through the Bohmian dynamical map $\mathbf{x}(t) = \mathbf{F}^{(\Psi)}_t(\mathbf{x}_0, t = 0)$, i.e., $\Delta^{(\Psi)}_{\pm a} = F^{(\Psi)}_t(\Delta_0(\pm a, 0, 0), t = 0)$ is the volume of the fluid carried by the particles during the evolution $F^{(\Psi)}_t$. In the dBB formalism, this can written

$$\begin{aligned}\mathcal{P}^{\Psi}_{\pm a}(t) &= \int \mathbb{I}_{\Delta^{(\Psi)}_{\pm a}}(\mathbf{x}(t))\rho^{\Psi}(\mathbf{x}(t), t)d^3\mathbf{x}(t) \\ &= \int \mathbb{I}_{\Delta^{(\Psi)}_{\pm a}}(\mathbf{x}(t))\rho^{\Psi}(\mathbf{x}_0, t = 0)d^3\mathbf{x}_0\end{aligned} \tag{6}$$

where $\mathbb{I}_{\Delta^{(\Psi)}_{\pm a}}(\mathbf{x}(t)) = \int_{\mathbf{u}\in\Delta^{(\Psi)}_{\pm a}} d^3\mathbf{u}\delta^3(\mathbf{x}(t) - \mathbf{u})$ is an indicator function.

In fact, all of this can be interpreted in another way. An experimenter content with measuring the spatial distribution of particle arrival on the detection screen will generally not be able to trace the notion of the path followed and thus obtain 'which-path' type information without performing a post-analysis on the events detected. So, in our example, the experimenter will be able to post-select the events detected in the x-positive region in order to obtain physical information. It is the theory, in this case that of de Broglie Bohm, that makes it possible to interpret and give meaning to the raw data.

It is clearly an example of Einstein's credo 'the theory decides what is to be measured'!

Having explained this, we are now ready to discuss the relationship between the concept of arrival time in Bohmian mechanics and the notion of POVM.

3. Can We Observe Bohmian First Arrival Time? (First Round)

The notion of arrival time can be precisely defined in dBB theory. Consider a region of space, say a Σ surface, then for a given wave function Ψ_t, the dBB trajectories $\mathbf{x}(t)$ arbitrarily integrated from an initial time $t_0 = 0$ and passing through this surface define the successive arrival times of the particle on this surface. In general, these times are not unique, as the particle can zig-zag around Σ. In cases where we can define a dBB first instant of arrival τ_Σ^Ψ on Σ (which is generally true for time-dependent problems where the wave function $\Psi_t$is non-stationary), we formally write [22,28–30,45,46]:

$$\tau_\Sigma^\Psi = \inf\{t : \mathbf{x}(t) \in \Sigma\} \tag{7}$$

The distribution of arrival time is generally obtained from the probability current $\mathbf{J}^\Psi(\mathbf{x}, t)$ projected onto the detection surface element having the direction $\mathbf{n}(\mathbf{x})$. Considering an infinitesimal surface $d\Sigma_\mathbf{x}$, the number of particles crossing this surface during an infinitesimal interval of time δt around t is given by

$$\mathcal{P}_{\text{dBB}}^\Psi(\mathbf{x}, t)\delta t := |\mathbf{J}^\Psi(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})| d\Sigma_\mathbf{x} \delta t = \rho^\Psi(\mathbf{x}(t), t)\delta^3\mathbf{x}(t) = \rho^\Psi(\mathbf{x}(t_0), t_0)\delta^3\mathbf{x}(t_0) \tag{8}$$

where we used the conservation of probability in the second and third lines with the volume $\delta^3\mathbf{x}(t) = |\mathbf{v}^\Psi(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})| d\Sigma_\mathbf{x}\delta t$ and where $\mathbf{x}(t_0)$ are the initial coordinates for the dBB trajectory connected to $\mathbf{x}(t)$. Moreover, if we consider only what is occurring in this time window δt, then $\mathcal{P}_{\text{dBB}}^\Psi(\mathbf{x}, t) = \rho^\Psi(\mathbf{x}(t_0), t_0)\delta^3\mathbf{x}(t_0)\frac{1}{\delta t}$ can be interpreted as an arrival time distribution for the elementary surface $d\Sigma$. Integrating over the surface Σ and post-selecting only on those events corresponding to a first arrival, the probability distribution of first arrival reads

$$\mathcal{P}_{\text{dBB}}^\Psi(\Sigma, \tau) = \int_\Sigma |\mathbf{J}^\Psi(\mathbf{x}, \tau = \tau_\Sigma^\Psi) \cdot \mathbf{n}(\mathbf{x})| d\Sigma_\mathbf{x} = \int_V \delta(\tau - \tau_\Sigma^\Psi)\rho^\Psi(\mathbf{x}_0, t_0) d^3\mathbf{x}_0 \tag{9}$$

where $\mathbf{x}_0 = \mathbf{x}(t_0)$. These two very general expressions are equivalent and were used by DD [30,45,46] partly based on previous works by Leavens [4,10,70] and Dürr and collaborators [28,29] (for other related works, see [22,71,72]).

We stress that the absolute value is required since the particle can come from the wrong side in the presence of back-flow. In this regime, the probability current is backpropagating, i.e., $\mathbf{J}^\Psi(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) < 0$ even if the incident wave packet Ψ contains only propagative components Ψ_k (i.e. plane waves) which separately satisfy $\mathbf{J}^{\Psi_k}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}) > 0$. In other words, back-flow can be seen as an interference effect specific to wave mechanics. We also emphasize that in general, $\mathcal{P}_{\text{dBB}}^\Psi(\Sigma, \tau)$ is not normalized, i.e., $\int d\tau \mathcal{P}_{\text{dBB}}^\Psi(\Sigma, \tau) \leq 1$, because not every trajectory is necessarily crossing Σ.

In usual scattering experiments where the incident wave packet Ψ_{t_0} is well localized in space and where the detection the surface is located in the far-field (the far-field is the regime where $r \gg \lambda$, with r a typical distance between the source and the detector and λ a typical wavelength), we can completely neglect back-flow. In this regime, the distribution of first arrival times becomes the distribution of arrival times altogether [28,29]. There is no longer any need to involve post-selection in arrival times, and the probability distribution reduces to the standard formula used in collision physics (e.g., for evaluating scattering cross-sections), regardless of any knowledge of Bohmian theory. We emphasize that semiclassical and far-field regimes are often used in the orthodox quantum interpretation but these approximations appear only as limiting special cases of the dBB framework situations where trajectories are classical-like (i.e., because the quantum potential is negligible). In the dBB framework, all kinds of vagueness concerning classicality can be easily removed and the physical interpretation of $\mathcal{P}_{\text{dBB}}^\Psi(\Sigma, \tau)$ is non-ambiguous even in regimes where the far-field positivity condition $\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})>0$ does not hold anymore. The ontological clarity of classical physics is thus recovered even in the quantum regime!

Yet, the fact that the probability $\mathcal{P}^{\Psi}_{\mathrm{dBB}}(\Sigma,\tau)$ can be mathematically constructed from the notion of Bohmian trajectories does not explain how this probability can be measured. Indeed, quantum mechanics is highly contextual and one should clearly distinguish the probability of being from the probability of being detected. In fact, it is well accepted that in the far-field regime, i.e., in the absence of back-flow, the $\mathcal{P}^{\Psi}_{\mathrm{dBB}}(\Sigma,\tau)$ distribution is directly measurable, in line with results obtained in studies of scattering and collision processes. This is also what emerges from the observation of diffraction and interference phenomena, also observed in the far-field and in very good agreement with the predictions for $\mathcal{P}^{\Psi}_{\mathrm{dBB}}(\Sigma,\tau)$ given by Bohmian theory.

Nevertheless, nothing is less certain for the more general regime where the back-flow phenomenon is predicted by theory. The measurability of $\mathcal{P}^{\Psi}_{\mathrm{dBB}}(\Sigma,\tau)$ can then be questioned. This is exactly the regime considered by DD [45,46] in a situation involving a spin-1/2 particle exhibiting a back-flow-like phenomenon even quite far from the source. To justify the measurability of $\mathcal{P}^{\Psi}_{\mathrm{dBB}}(\Sigma,\tau)$ in this regime, DD points out that a precise theory of detection is by no means necessary to understand the far-field regime in very good agreement with the theory. Similar statements were given by Dürr and Teufel in a known textbook:

> '*We should follow the common practice of quantum physics and henceforth not worry about the presence of detectors, simply taking for granted that the detection is designed in such a way that it does not mess up the trajectories too much*' [29] p. 347.

In DD's view, the same could be expected in the new situation, even in the presence of back-flow. They wrote:

> '*We expect that in the experiment proposed in this paper the same will be true, i.e., the detection event should not be drastically disturbed by the presence of the detector*'. [45]

However, it is not difficult to show that this necessarily leads to difficulties and even paradoxes such as those highlighted by GTZ [48].

The Specific Problem Considered by Das and Dürr

To keep the description of the situation described by DD as simple as possible, we recall that it considers a spin-1/2 particle confined in a cylindrical guide with symmetry axis z. Initially, the particle is described by a strongly localized wave function

$$\Psi_{\hat{\mathbf{s}}}(\rho,z,t_0)=\chi_{\hat{\mathbf{s}}}\cdot\Phi(\rho,z,t_0) \tag{10}$$

where $\chi_{\hat{\mathbf{s}}}$ is a two-component spinor such that $\chi^{\dagger}_{\hat{\mathbf{s}}}\boldsymbol{\sigma}\chi_{\hat{\mathbf{s}}}=\hat{\mathbf{s}}$ ($\hat{\mathbf{s}}$ is a unit vector defining the spin direction and $\boldsymbol{\sigma}=[\sigma_x,\sigma_y,\sigma_z]$ are the Pauli matrices). Initial particle spatial confinement along the z direction is provided by a potential well. When this is removed on one side only, the wave packet moves towards $z>0$ while preserving the structure of the spinor χ, which remains unchanged. The wave function then becomes $\Psi_{\hat{\mathbf{s}}}(\rho,z,t)=\chi_{\hat{\mathbf{s}}}\cdot\Phi(\rho,z,t)$, the spatial part, preserving its rotation invariance over time. The dBB theory applied to the Pauli equation leads to a probability current along the z direction:

$$J_z^{\Psi_{\hat{\mathbf{s}}}}(\mathbf{x},t)=|\Phi(\rho,z,t)|^2\frac{\partial_z S(\rho,z,t)}{m}+\frac{\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\varphi}}}{2m}\partial_\rho|\Phi(\rho,z,t)|^2. \tag{11}$$

In this formula, the first term is a convective current reminiscent of the formula used for a spinless particle (S is the phase of the wave packet $\Phi(\rho,z,t)$). The second term is a spin current associated with the magnetic structure of the electron ($\hat{\boldsymbol{\varphi}}$ is a unit vector for the polar angle direction). We stress that Equation (11) is an application of the non-relativistic Gordon formula $\mathbf{J}^{\Psi}=\frac{1}{2mi}(\Psi^{\dagger}\overleftrightarrow{\boldsymbol{\nabla}}\Psi)+\frac{1}{2m}\boldsymbol{\nabla}\times[\Psi^{\dagger}\boldsymbol{\sigma}\Psi]$ for the probability current of an electron. In the relativistic regime, it is more convenient to use Dirac current $\mathbf{J}^{\Psi}=\Psi^{\dagger}\boldsymbol{\alpha}\Psi$ using the bispinor Ψ (for previous studies using the Dirac equation, see [31–35]).

Two regimes are clearly identifiable. Firstly, in the longitudinal case where the spin vector is aligned with the $\pm z$ direction, only the convective current survives. The dBB theory then gives the same trajectories as for a spinless particle, and in particular the absence of back-flow. The second regime is more interesting and corresponds to the case of a purely transverse spin in the $\pm x$ direction, for example. In this case, the spin current reads $\pm\frac{\cos\varphi}{2}\partial_\rho|\Phi(\rho,z,t)|^2$ and can clearly change sign. In the configuration considered by DD, the spin current can more than compensate for the positive convective current, and so in some cases we obtain a back-flow $J_z^{\Psi_{\hat{\mathbf{s}}}} < 0$.

Using these predictions for the probability current, we can construct probability distributions for the first arrival times on a given cross-section Σ of the wave guide at $z = const > 0$ in both longitudinal and transverse spin configurations. The distribution $\mathcal{P}_{\text{dBB}}^{\Psi_{\pm\hat{\mathbf{z}}}}(\Sigma,\tau)$ for the longitudinal case is similar for both $\pm z$ possibilities (i.e., $\mathcal{P}_{\text{dBB}}^{\Psi_{\hat{\mathbf{z}}}}(\Sigma,\tau) = \mathcal{P}^{\Psi_{-\hat{\mathbf{z}}}}(\Sigma,\tau)$). Qualitatively, the distribution starts from zero for $\tau = 0$, approaches a maximum, then slowly decreases to zero for τ tending towards infinity. This probability distribution gives the same result as for the spinless particle case. The transverse configuration is more surprising. We have first a rotational invariance $\mathcal{P}_{\text{dBB}}^{\Psi_{\hat{\mathbf{s}}}}(\Sigma,\tau) = \mathcal{P}_{\text{dBB}}^{\Psi_{\hat{\mathbf{s}}'}}(\Sigma,\tau)$ for any choice of the transverse spin vector (for example, $\hat{\mathbf{s}} = +\hat{\mathbf{x}}$ or $\hat{\mathbf{s}}' = -\hat{\mathbf{x}}$), which was expected based on the cylindrical symmetry of the problem. Qualitatively, the probability distribution for the transverse case resembles the longitudinal one. The probability starts from zero at $\tau = 0$, approaches a maximum, and decreases. Here, however, the distribution is more peaked and the decay more pronounced. Remarkably, after a characteristic time τ_{max}, the distribution rigorously cancels out and remains so for any time $\tau > \tau_{max}$.

It is at this point that GTZ deduce a contradiction. Assuming that the distribution of arrival times is given by a POVM and that we have $\mathcal{P}_{\text{dBB}}^{\Psi_{\hat{\mathbf{s}}}}(\Sigma,\tau) = \langle\Psi_{\hat{\mathbf{s}}}|\hat{O}(\Sigma,\tau)|\Psi_{\hat{\mathbf{s}}}\rangle$, GTZ show that it would imply

$$\langle\Psi_{\hat{\mathbf{z}}}|\hat{O}(\Sigma,\tau)|\Psi_{\hat{\mathbf{z}}}\rangle + \langle\Psi_{-\hat{\mathbf{z}}}|\hat{O}(\Sigma,\tau)|\Psi_{-\hat{\mathbf{z}}}\rangle = \langle\Psi_{\hat{\mathbf{x}}}|\hat{O}(\Sigma,\tau)|\Psi_{\hat{\mathbf{x}}}\rangle + \langle\Psi_{-\hat{\mathbf{x}}}|\hat{O}(\Sigma,\tau)|\Psi_{-\hat{\mathbf{x}}}\rangle. \quad (12)$$

However, from the above-mentioned symmetries of the arrival time distribution, that would imply $\mathcal{P}_{\text{dBB}}^{\Psi_{\hat{\mathbf{z}}}}(\Sigma,\tau) = \mathcal{P}^{\Psi_{\hat{\mathbf{x}}}}(\Sigma,\tau)$, which is in general not true (in particular for τ_{max}). Therefore, as shown by GTZ, the dBB first arrival time distribution cannot be identified with a POVM.

This result is unavoidable and since any experimental quantum statistics are assumed to be represented by POVM, this seems to imply that $\mathcal{P}_{\text{dBB}}^{\Psi_{\cdot}}(\Sigma,\tau)$ is not measurable.

There is another very good reason justifying this result. If $\mathcal{P}_{\text{dBB}}^{\Psi_{\hat{\mathbf{s}}}}(\Sigma,\tau)$ was a POVM, then we would have a way of beating the no-signalling theorem. We could in fact use a Bell-type scenario with an entangled pair of EPR spin-1/2 particles to send signals faster than light! The idea has been recently proposed by Maudlin in several interviews referring to DD work [50–52]. The point is that if we have an EPR–Bohm pair $|\Psi_{12}\rangle \propto |\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle$ and if we send one of the particles to Pluto where Alice can measure the spin value along one arbitrary direction, then Bob on earth could, by recording the first arrival time distribution of the second particle (send into the wave guide proposed by DD), know instantaneously the value of the spin measured by Alice just by observing an event at time $\tau > \tau_{max}$ (see Figure 1). Critically, this Maudlin–DD proposal is based on the assumption that the distribution of first arrival times is identifiable with a POVM, i.e., $\langle\Psi_{12}|\hat{O}_2(\Sigma,\tau)|\Psi_{12}\rangle$. Moreover, POVMs are central for deriving the no-signalling theorem in relativistic quantum mechanics and this result is central to guaranteeing a peaceful relationship between quantum mechanics and Einstein's relativity theory. It is therefore a priori highly desirable that the $\mathcal{P}_{\text{dBB}}^{\Psi}(\Sigma,\tau)$ distribution not be a POVM; otherwise, it would jeopardize all quantum field theory! Clearly, Equation (12) agrees with the no-signalling theorem since the sum is independent of the spin basis chosen and Bob not knowing the result of Alice must observe a random mixture. Taken all together, this does not leave much hope for the measurability of the Bohmian distribution $\mathcal{P}_{\text{dBB}}^{\Psi}(\Sigma,\tau)$.

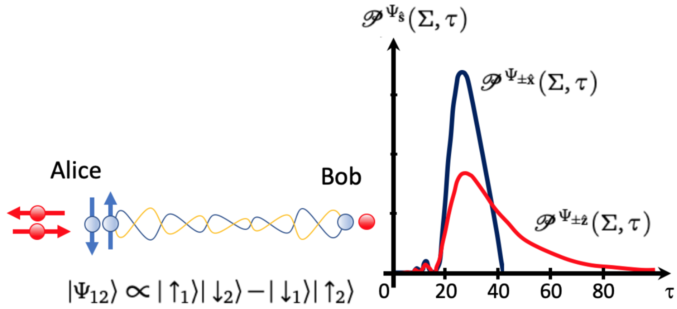

Figure 1. Principle of the experiment proposed by Das and Maudlin to build a faster-than-light Bell telephone. A pair of spin-1/2 particles in a perfectly entangled Einstein–Podolsky–Rosen (EPR) state is separated and analyzed by two agents, Alice and Bob. Alice can measure the spin of her particle along the $\pm\hat{\mathbf{s}}$ unit directions. Specifically, she considers the case $\pm\hat{\mathbf{z}}$ (longitudinal) and $\pm\hat{\mathbf{x}}$ (transverse). On his side, Bob measures the first arrival time distribution $\mathcal{P}^{\Psi_{\hat{s}}}_{\text{dBB}}(\Sigma,\tau)$ for his particle (still ignoring the spin of his particle). Let us say he is measuring at a time $\tau \gg \tau_{max}$ at which the distribution $\mathcal{P}^{\Psi_{\hat{x}}}_{\text{dBB}}(\Sigma,\tau)$ vanishes but $\mathcal{P}^{\Psi_{\hat{z}}}_{\text{dBB}}(\Sigma,\tau)$ does not (the distributions are here taken and freely adapted from [45]). If Bob, in his remote lab, detects an event at $\tau \gg \tau_{max}$, then can he deduce that Alice was measuring her spin along the longitudinal direction $\pm\hat{\mathbf{z}}$? Her measurement affects nonlocally the dBB dynamics of the particle detected by Bob. This is a form of faster-than-light communication contradicting Bell's no-signalling theorem [53].

Of course, there remains the possibility that DD's and Maudlin's results and analyses are correct, in which case it could be that performing the experiment could indeed defeat the POVM-based no-signalling theorem. This seems highly speculative, however, since this would imply a whole new physics beyond the standard theory of measurement based on POVMs and this would call into question the peaceful consensus between quantum mechanics and relativity theory. Therefore, it is likely that another, more nuanced answer is the right one. In what follows, we shall show both that GTZ's criticisms are too strong and that DD (and Maudin) are too confident about the measurability of the $\mathcal{P}^{\Psi}_{\text{dBB}}(\Sigma,\tau)$ distribution.

Let us start by looking at the problem in a more general way, and try to answer the two points (i) and (ii) mentioned in the introduction. Since the ideal dBB probability distribution $\mathcal{P}^{\Psi}_{\text{dBB}}(\Sigma,\tau)$ depends on the projected current $|\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$, the first question is whether we can associate a POVM with $|\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$ (i.e., $|\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})| \equiv \langle\Psi|\hat{O}_{\mathbf{x},t}|\Psi\rangle$). This is a natural hypothesis since the current depends bilinearly—more precisely, sesquilinearly [23,24]—on ψ and $\psi^{\dagger}$. The answer is no, and was given by Dürr and colleagues [23,24]. In fact, [23] was not interested in $|\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$ but in $\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})$, but the answer is the same. Let us summarize the reasoning: Assume two wave functions Ψ_1 and Ψ_2 such that $\mathbf{J}^{\Psi_1}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) > 0$ and $\mathbf{J}^{\Psi_2}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) > 0$ are true at point $\mathbf{x}$ and time t, but such that for the wave functions $\Psi_+ = \frac{\Psi_1+\Psi_2}{\sqrt{2}}$ and $\Psi_- = \frac{\Psi_1-\Psi_2}{\sqrt{2}}$, we have $\mathbf{J}^{\Psi_+}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) > 0$ and $\mathbf{J}^{\Psi_-}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) < 0$ (such situations can occur during interference experiments). If we assume that $|\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$ is associated with a POVM, then by definition, we must have:

$$|\mathbf{J}^{\Psi_1}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})| + |\mathbf{J}^{\Psi_2}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})| = |\mathbf{J}^{\Psi_+}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})| + |\mathbf{J}^{\Psi_-}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})| \quad (13)$$

Furthermore, we also have

$$\mathbf{J}^{\Psi_1}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) + \mathbf{J}^{\Psi_2}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) = \mathbf{J}^{\Psi_+}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) + \mathbf{J}^{\Psi_-}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x}) \quad (14)$$

Clearly, the two relations contradict each other, so $|\mathbf{J}^{\Psi}(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$ cannot generally be a POVM. This is a central result that rules out any possibility of $\mathcal{P}^{\Psi}_{\text{dBB}}(\Sigma,\tau)$ being a POVM! Note that this proof was obtain in 2013 [23], 10 years before the DD and GTZ results [45,48] and is not dependent on spin. This is therefore a very robust result.

But now comes the crux. What is the physical meaning of a POVM $\hat{O}$? Besides mathematics, this operator is just a tool, an algorithm, such that for any wave function Ψ, the quantity $\langle\Psi|\hat{O}|\Psi\rangle$ gives us a probability. Physically speaking, it means that we actually have a very precisely defined experimental context or setup (i.e., with external fields, mechanical frames and so on) such that we can record statistical data proportional to $\langle\Psi|\hat{O}|\Psi\rangle$. The fact that $|\mathbf{J}^\Psi(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$ is not a POVM implies that there is no experimental configuration such that the amount of statistical information recorded is directly proportional to $|\mathbf{J}^\Psi(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$ and this is the case whatever the Ψ wave function chosen.

However, we should be careful with this theorem. Indeed, this result in no way implies the non-existence of a POVM, or more precisely of an experimental context, which—in some situations and for some Ψ—could approximately imply a probability approaching $|\mathbf{J}^\Psi(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$. Far from that, we all know that detectors are not universal but instead have an optimum operating range outside of which reliable measurement is no longer possible. Therefore, even if it is not possible to build a universal (POVM) detector such that $|\mathbf{J}^\Psi(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|$ is a probability, one expects to have POVM detectors such that

$$\mathcal{P}^\Psi_{\text{detec.}}(\mathbf{x},t) = \eta^\Psi|\mathbf{J}^\Psi(\mathbf{x},t)\cdot\mathbf{n}(\mathbf{x})|d\Sigma_\mathbf{x} \tag{15}$$

where η^Ψ is an efficiency coefficient which is in general a complicated function of the quantum state Ψ. $\mathcal{P}^\Psi_{\text{detec.}}(\mathbf{x},t)$ depends on the detector used and therefore only reproduces approximately the dBB flux predictions given by Equation (8). In the next sections, we will consider the implications of this possibility.

4. The Fabry–Perot Ideal Absorbing Medium for a Plane Wave

We first consider the non-relativistic problem for spinless particles. We start by supposing a spinless nonrelativistic plane wave $\Psi^{(0)} = e^{ik_1 z}e^{ik_x x}e^{-i\frac{k^2}{2m}t}$ impinging on an absorbing Fabry–Perot slab located between the surfaces $z=0$ and $z=d$ (details concerning this model are given in Appendix B). The number of particles absorbed by the slab intuitively gives us the number of particles detected, and we can a priori define the arrival time distribution as

$$\mathcal{P}^\Psi_{\text{detec.}}(\Sigma,t) = \Sigma J_z^{\Psi^{(0)}}[1-|R|^2-|T|^2] \tag{16}$$

where $|R|^2$ and $|T|^2$ are the reflection and transmission coefficients, respectively (in the absence of absorption, we would have $1-|R|^2-|T|^2=0$), and where $J_z^{\Psi^{(0)}} = v\cos\theta > 0$ is the incident current ($v=k/m$ is the de Broglie velocity and θ the incidence angle with respect to the z axis). We can alternatively write

$$\mathcal{P}^\Psi_{\text{detec.}}(\Sigma,t) = -2\Sigma\int_{z=0}^{z=d} dz \text{Im}[V_{eff}]|\Psi|^2(x,y,z) \tag{17}$$

where V_{eff} is an effective dissipative potential such that $\text{Im}[V_{eff}] < 0$ (for a discussion of complex potentials in scattering theory, see [73]).

This complex potential implies a violation of unitarity and the local conservation law is modified as

$$\partial_t|\Psi|^2 = -\boldsymbol{\nabla}\cdot\mathbf{J}^\Psi + 2\text{Im}[V_{eff}]|\Psi|^2 \tag{18}$$

In this model, the violation of unitarity is reminiscent of a coupling with an external bath allowing inelastic scattering through the medium [74]. Absorbing effective potentials have often been used in quantum optics since the 1990s in order to model attenuators and losses [75]. In other words, these complex potentials are associated with transmission channels that we can neglect or that we can associate with trapped particles moving outside the domain of propagation considered. More physically, the medium constituting the slab is filled with absorbing atoms with individual extinction cross-section $\sigma_{ext} = \sigma_{scat} + \sigma_{abs}$ and we have $-2\text{Im}[V_{eff}] = Nv\sigma_{ext}$, where N is the absorbing atom density in the slab. A typical way of justifying such a model is to consider scattering of a particle by a potential well $V(x)$ having one bound energy state $E_0 < 0$ and a continuum of propagative modes $E_k > 0$ coupled to a bath with a continuum of energy levels (plus a vaccum state). The coupling allows us to derive an effective potential associated with absorption by the well, i.e., $V(x) \to V(x) + \Delta E - i\Gamma/2$ with $\Gamma > 0$ a decay constant associated with dissipation and absorption.

Moreover, even in this simple model, the physical interpretation must be conducted carefully and take into account the whole experimental configuration. This is so because dBB trajectories are in general highly contextual and nonclassical. The situation is non-ambiguous if the incident wave is actually a 'plane wave packet' with finite lateral extension (see Figure 2) as analyzed, for example, by Norsen in [76]. In the absence of the slab, the dBB trajectories would be straight lines (i.e., if we neglect the dispersion of the wave packet). An ideal projective detector, as discussed in standard textbooks, would simply be placed in the beam and the physical interaction would be neglected. Here, however, if we take into account the presence of the absorbing medium associated with the detector, the dBB trajectories are modified. If this disturbance does not change the trajectories upstream of the detector too much, the statistical predictions will not be affected too much and the detector can be considered efficient. More precisely, far away from the slab in the $z < 0$ domain, the incident and reflected contribution are non-overlapping and the dBB trajectories are straight lines with constant velocity v. In the vicinity of the slab, the incident and reflected part interfere and the dBB dynamics is much more complicated. As shown in detail in [76], the particle trajectories oscillate around main trajectories sketched in Figure 2 (see also [8,9,77]). The stream lines separate two regions of the initial plane wave packet. The trajectories drawn in blue are reflected and constitute a fraction $|R|^2$ of the incident flux [76]. The trajectories drawn in red are transmitted through the slab. Moreover, because of Equation (18), a fraction of the particles crossing the medium are continuously absorbed (i.e., detected) along the trajectories. After integrating Equation (18) over a dBB trajectory $\mathbf{x}(t)$, we obtain

$$|\Psi|^2(\mathbf{x}(\mathbf{t}), t) = |\Psi|^2(\mathbf{x}(\mathbf{t_0}), t_0)e^{-\int_{t_0}^{t} dt'[\boldsymbol{\nabla}\cdot\mathbf{v}^{\Psi}_{t'} + 2\text{Im}[V_{eff}]]} \quad (19)$$

which contains an exponentially decaying term involving $\text{Im}[V_{eff}] < 0$.

An ideal detector would be such that $|T|^2 \simeq 0$ and $|R|^2 \simeq 0$. In general, this is not so. Ultimately, as shown in Appendix B, in the weak-coupling regime where absorption is very low, the incident flow is weakly disturbed (the medium is nearly transparent, $|T|^2 \simeq 1$ and $|R|^2 \simeq 0$) and the arrival time probability reads $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, \tau) \simeq \Sigma d N \sigma_{ext} |\mathbf{J}^{\Psi^{(0)}}|$, which is independent of the incidence angle θ. In this regime, the detector does not record the ideal dBB probability Equation (8). Moreover, as shown in Appendix B, even if $|T|^2 \simeq 0$, it is in general not possible with this simple model to have $|R|^2 \simeq 0$. This is not only a question of numerical factor $\eta^{\Psi} = [1 - |R|^2 - |T|^2]$, but also an important experimental issue since the reflected and transmitted beams can subsequently disturb and even ultimately prohibit the current flow in other places where different detectors could be located.

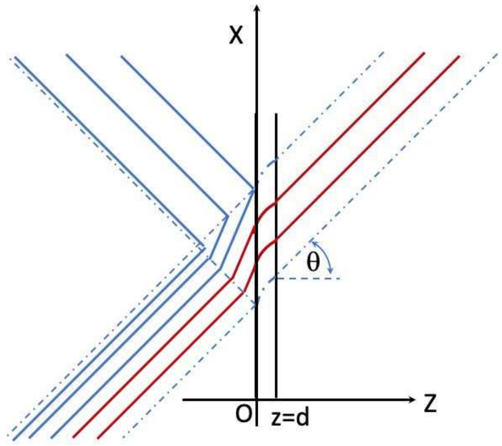

Figure 2. Visualization of the typical dBB trajectories scattered by a thin slab corresponding to a potential barrier. The dBB trajectories cannot cross and therefore the reflected (blue curves) trajectories and transmitted (red curves) trajectories are not overlapping. This is an idealized and schematic representation based on rectilinear rays inspired by the work of Norsen [76] (the dashed lines show the limits of these idealized beams). In his work, Norsen considers a 1D problem and looks at motion in the $t - x$ plane (t being time), whereas here, we are looking at a stationary problem in the $x - z$ plane.

5. Ideal Absorption of a Plane Wave by a Perfectly Matched Layer

The previous model of Section 4 was too simple, since we considered a homogeneous potential barrier. The idea to optimize the detector by using stratified media and complex potential has been considered in [17,78,79]. However, we consider here a more idealized approach. Indeed, in principle, an ideal detector can be obtained using a stratified medium known as a perfectly matched layer (PML), often used in numerical calculations [80] (our method differs from the Robin boundary condition approach advocated by Tumulka [81]). Consider here a one-dimensional problem and let $\Psi^{(0)} = e^{ikz}e^{-i\frac{k^2}{2m}t}$ be a plane wave solution of the Schrödinger equation with $k = \sqrt{(2mE)}$. We then suppose an ideal absorbing medium located in the region $z > 0$ such that the new wave function reads

$$\Psi^{(abs)}(z,t) = e^{ikz}e^{-\int_{-\infty}^{z} dz'\chi(z')}e^{-i\frac{k^2}{2m}t} \tag{20}$$

with $\chi(z) \geq 0$ an absorption function (ideally) vanishing for $z < 0$. As shown in Appendix C, we can immediately check that $\Psi^{(abs)}$ is a solution of the equation

$$\partial_z^2\Psi^{(abs)} + 2m(E - V_{eff})\Psi^{(abs)} = 0 \tag{21}$$

with the effective complex potential

$$V_{eff}(z) = \frac{\chi^2(z) - \chi'(z)}{2m} - i\chi(z)\frac{k}{m}. \tag{22}$$

corresponding to a dissipative (absorbing) medium or detector. Importantly, for this medium, there is no reflected wave (i.e., $R = 0$) and the transmitted wave is exponentially decaying as $|\Psi^{(abs)}(z,t)|^2 = e^{-2\int_{-\infty}^{z} dz'\chi(z')}$ for $z > 0$. If the exponential factor is very large, the transmission goes to zero very quickly as required for a good detector. Going back to the detecting slab considered previously, we can still apply Equation (17) for a plane wave at normal incidence if the function $\chi(z)$ ideally vanishes for $z < 0$ and $z > d$ (see Figure 3 for a more realistic situation where $\chi(z)$ is a continuous function). We have

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t) = 2\Sigma\frac{k}{m}\int_{z=0}^{z=d} dz'\chi(z')e^{-2\int_0^{z'} dz''\chi(z'')}. \tag{23}$$

In the particular case where $\chi(z) = \chi_0 > 0$ is constant for $0 < z < d$ (i.e., $\chi(z) = \chi_0\theta(z)\theta(d-z)$), we obtain the effective potential

$$V_{eff}(z) = \frac{\chi_0^2\theta(z)\theta(d-z) - \chi_0(\delta(z) - \delta(d-z))}{2m} - i\chi_0\theta(z)\theta(d-z)\frac{k}{m} \tag{24}$$

and Equation (23) reduces to

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t) = \Sigma\frac{k}{m}(1 - e^{-2\chi_0 d}) = \Sigma J_z^{\Psi^{(0)}}(1 - e^{-2\chi_0 d}) \tag{25}$$

with $J_z^{\Psi^{(0)}} = \frac{k}{m} = v > 0$ and Σ the detector cross-section. This detector has an efficiency $\eta^{\Psi} = 1 - e^{-2\chi_0 d}$. In the limit where $\chi_0 d \to +\infty$, we thus have $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t) \to \Sigma J_z^{\Psi^{(0)}}$, which recovers the dBB arrival time distribution Equation (8).

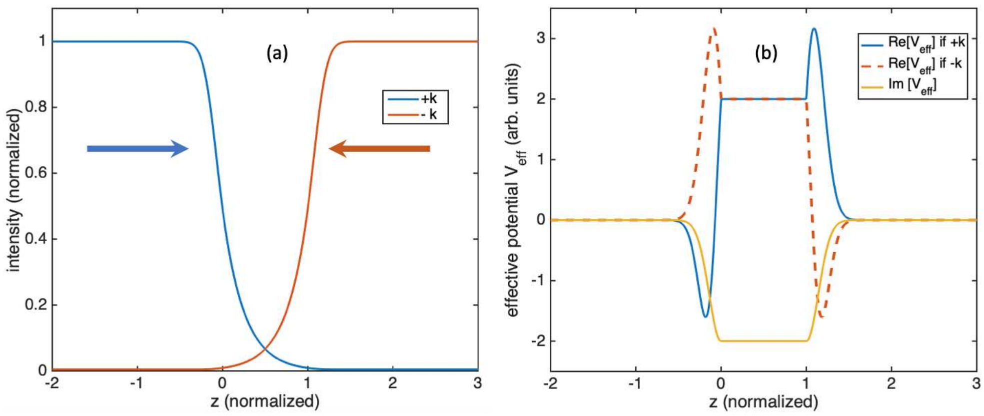

Figure 3. A typical complex potential associated with a PML for a particle detector. In (**a**), we show the transmitted intensity if the detector is optimized for either a plane wave propagating along the $+z$ direction (blue curve) or a wave moving along the $-z$ direction (red curve). (**b**) The real parts $\text{Re}[V_{eff}]$ of the potentials are shown (red dashed curve for the $-z$ incident direction and blue curve for the $+z$ direction) and compared with the imaginary part $\text{Im}[V_{eff}] < 0$ of the potential (orange curve) which takes the same form in both $\pm z$ cases.

It must be stressed that the detector is optimized here for a given wavevector k and that in general, for a different choice $k \to k + \delta k$, the potential will not act as a perfect absorber (i.e., in general, the reflectivity $R \neq 0$ for $\delta k \neq 0$). In a similar way, observe that k and χ do not necessarily have to be positive and that we can develop a detector adapted to a counterpropagating wave $\propto e^{-ikz}$ with $-k = -\sqrt{(2mE)} < 0$ as well. From Equation (22), we still have $\text{Im}[V_{eff}] = -\chi_0\theta(z)\theta(d-z)\frac{k}{m} < 0$ if $\chi(z) = -\chi_0\theta(z)\theta(d-z) < 0$, and this again corresponds to an absorbing medium (the choice $\chi(z) = +\chi_0\theta(z)\theta(d-z) > 0$ would have involved an anti-thermodynamical medium with gain, i.e, emitting particles instead of absorbing them). From Equation (25), we now have

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t) = \Sigma\frac{k}{m}(e^{2\chi_0 d} - 1) = \Sigma|J_z^{\Psi^{(0)}}|(e^{2\chi_0 d} - 1). \tag{26}$$

This detector is a priori associated with a different efficiency $\eta^{\Psi} = e^{2\chi_0 d} - 1$. However, this is mostly a problem of convention concerning the role of the input and exit sides. If we instead normalize the field by its value at $z = d$ and not at $z = 0$ (this is natural, since the wave is counter-propagating and decaying in the $-z$ direction), we recover $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t) = \Sigma|J_z^{\Psi'^{(0)}}|(1 - e^{2\chi_0 d})$ with the efficiency $\eta^{\Psi'} = 1 - e^{-2\chi_0 d}$ as in Equation (25) and now $J_z^{\Psi'^{(0)}} = -\frac{k}{m}e^{-2\chi_0 d}$ is associated with the incident plane wave $\Psi'^{(0)} = e^{-\chi_0 d}e^{-ikz}e^{-i\frac{k^2}{2m}t}$ and Equation (20) is replaced by $\Psi'^{(abs)}(z,t) = e^{-\chi_0 d}e^{ikz}e^{-\int_{-\infty}^{z} dz'\chi(z')}e^{-i\frac{k^2}{2m}t}$ with $\chi(z) = -\chi_0\theta(z)\theta(d-z) < 0$.

However, we emphasize that the new effective potential for the back-propagating wave is actually different from Equation (24) since we have

$$V_{eff}(z) = \frac{\chi_0^2\theta(z)\theta(d-z) + \chi_0(\delta(z) - \delta(d-z))}{2m} - i\chi_0\theta(z)\theta(d-z)\frac{k}{m} \tag{27}$$

the real part of which differs from that deduced from Equation (24). This demonstrates that it is not possible to use the same absorbing medium for the e^{ikz} and e^{-ikz} cases. If we had (wrongly) used Equation (24) for the e^{-ikz} case, we would have obtained an additional contribution in the form of a reflected plane wave proportional to e^{ikz} in the $z > d$ domain (i.e., the medium would not act as an idealized absorber for the counterpropagating wave).

Of course the discussion is based on an idealized medium, and the presence of Dirac distributions in the potential $V_{eff}(z)$ of Equations (24) and (27) shows that the $\chi = const.$ conditions are too strict. Moreover, these pathological features can be remodeled by considering smooth potentials removing the discontinuities. An example is developed in Appendix D. The conclusions we obtained before are, however, very general (the detector can even be optimized for any plane wave having the associated wavevector components k_x, k_z). As illustrated in Figure 3, the field transmitted through the medium is strongly attenuated without reflection. Also, the potential is characterized by $\text{Im}[V_{eff}] < 0$, and as before, we require two different potentials optimized either for the e^{ikz} (forward) or e^{-ikz} (backward) cases.

6. Generalization for Wave Packets and Time-Dependent Problems

The previous model, based on the interaction of a plane wave with an absorbing medium, can in principle be generalized to the case of a superposition of plane waves forming a wave packet. This is necessary in order to consider the problem associated with backflow. To do this, we will consider a particular case where the problem seems to be treatable with sufficient precision and rigor. Let us consider the case where the initial wave function $\Psi^{(0)}(\mathbf{x}, t)$, i.e., in the absence of a detector, is developable in Taylor series in the vicinity of a point $\mathbf{x}_0$. More specifically, we assume a constant energy E, i.e., $\Psi^{(0)}(\mathbf{x}, t) \propto e^{-iEt}$ (our wave packet or beam is thus physically a sum of plane waves having the same energy E but different wave vectors) and write $\Psi^{(0)}(\mathbf{x}, t) \simeq \Psi^{(0)}(\mathbf{x}_0)e^{-iEt}e^{i\boldsymbol{\nabla}S(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)+O((\mathbf{x}-\mathbf{x}_0)^2)}$. This is equivalent to assuming that the wave function is locally equivalent to a plane wave with an effective wavevector $\mathbf{k}_{eff}(\mathbf{x}_0) = \boldsymbol{\nabla}S(\mathbf{x}_0)$. As shown by Berry [18], in general, such wave packets can easily develop back-flow in the vicinity of a point $\mathbf{x}_0$.

A simple case is given by the superposition of two plane waves

$$\Psi^{(0)}(\mathbf{x}, t) = (e^{i\mathbf{k}_1\cdot\mathbf{x}} + \alpha e^{i\mathbf{k}_2\cdot\mathbf{x}})e^{-iEt} \tag{28}$$

with $k_{1z} > 0, k_{2z} > 0$ and $\alpha \in \mathbb{C}$, a constant. The probability current reads

$$\mathbf{J}^{\Psi}(\mathbf{x}) = \frac{\mathbf{k}_1}{m} + \frac{\mathbf{k}_1}{m}|\alpha|^2 + \frac{\mathbf{k}_1 + \mathbf{k}_2}{m}|\alpha|\cos\Phi \tag{29}$$

with $\Phi = (\mathbf{k}_2 - \mathbf{k}_1) \cdot \mathbf{x} + \text{Arg}[\alpha]$. We are looking for situations where $J_z^{\Psi^{(0)}} < 0$ (back-flow) and for a given $|\alpha|$, we naturally impose $\Phi = \pi$ for points where this back-flow is stronger. Writing $J_z^{\Psi} = \frac{k_{2z}}{m} f(|\alpha|)$ with

$$f(|\alpha|) = |\alpha|^2 - (1 + \frac{k_{1z}}{k_{2z}})|\alpha| + \frac{k_{1z}}{k_{2z}} \tag{30}$$

we easily find the minimum for $|\alpha|_{min} = \frac{1}{2}(1 + \frac{k_{1z}}{k_{2z}})$. In particular, as shown in Appendix E, we easily obtain the effective wavevector $\mathbf{k}_{eff}(\mathbf{x}_0)$ and we have $k_{eff,z}(\mathbf{x}_0 = 0) = -k_{2z}$ for $\alpha_{min} = -\frac{1}{2}(1 + \frac{k_{1z}}{k_{2z}})$.

It is thus possible to study numerically the trajectory back-flow effect near a point where $\Phi \simeq \pi$ (e.g., near $\mathbf{x}_0 = 0$). This is illustrated in Figure 4 for a typical example. In this example, we indeed have $k_{eff,z}(\mathbf{x}_0 = 0) = -k_{2z}$ and we also have $|\mathbf{k}_{eff}(\mathbf{x}_0 = 0)| \simeq 1.11|k_2|$, meaning that the effective wavelength $\lambda_{eff} \simeq 0.9\lambda_0$ is very close to the initial values of each plane wave.

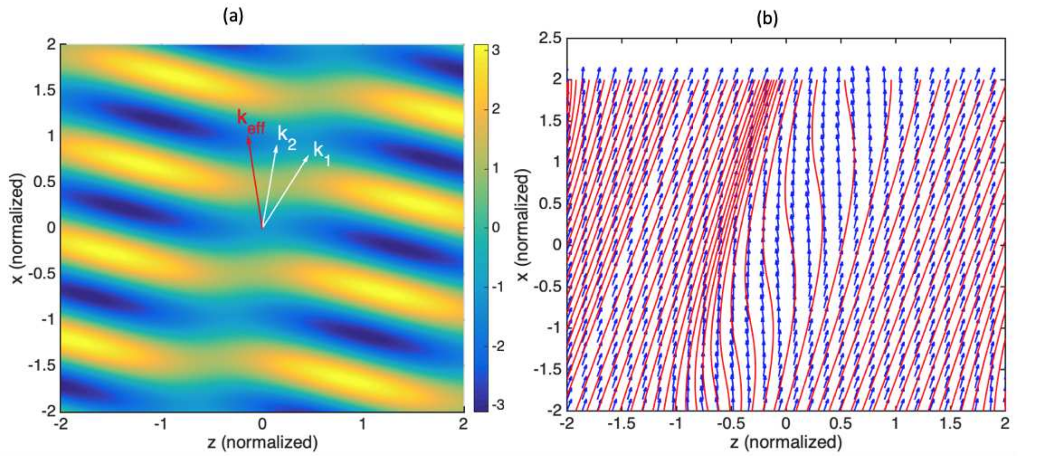

Figure 4. (**a**) Map of the real part of the wave function $\text{Re}[\Psi^{(0)}(z,x)]$ in presence of back-flow (see Equation (28)) for the case $\mathbf{k}_1 = [k_{1z} = k\cos(\pi/3), k\sin(\pi/3))]$ and $\mathbf{k}_2 = [k_{2z} = k\cos(9\pi/20), k\sin(9\pi/20))]$ and for $k = 2\pi$ (the white arrows show these wave vectors normalized to k). The effective wave vector at the origin $\mathbf{k}_{eff}(0,0)$(blue arrows) has a negative z component due to local back-flow. (**b**) Map of the dBB velocity vector (blue arrows) and trajectories(red lines) associated with map (**a**).

As shown in Figure 4, the interference zone where the back-flow is visible is sufficiently extended to imagine a detector localized in the region $\mathbf{x}_0 \simeq 0$ and able to observe the phenomenon. In order to be more precise, we can use the model or perfectly matched layer detectors and introduce a local potential barrier $V_{eff}(z)$ (see Equation (24) or Equation (27)) adapted to any regions of the interference field $\Psi^{(0)}(\mathbf{x})$, and this for both the normal and back-flow regimes.

The great specificity of these detectors is of course that they are highly optimized for well-defined regions of space (i.e., in the near environment of particular points $\mathbf{x}_0$, $\mathbf{x}_1$, ...), where the initial wave function $\Psi^{(0)}(\mathbf{x}, t)$ moves. Obviously, these detectors are highly invasive in the sense that they are optimized to cancel local reflectivity at the chosen point $\mathbf{x}_0$, $\mathbf{x}_1$, ... for a given local effective wave vector $\mathbf{k}_{eff}(\mathbf{x}_0), \mathbf{k}_{eff}(\mathbf{x}_1), \ldots$. Transmission is also zero, which implies high local absorption associated with high detection efficiency $\eta^{\Psi}(\mathbf{x}_0), \eta^{\Psi}(\mathbf{x}_1), \ldots$.

Moreover, faraway from the detector, the wave function is in general disturbed due to diffraction and scattering by the potential V_{eff}. Indeed, consider a single detector centered on point $\mathbf{x}_0 := [x_0, y_0, z_0]$. We assume that the detector has a finite volume δV located between the planes $z = z_0$ and $z_0 + d$, where d can be arbitrarily small if the detector is very efficient (due to a fast decay of the wave propagating in the absorbing medium). The transverse extension of the medium in the x and y direction is also limited around $x = x_0$ and $y = y_0$ (i.e., ideally over few wavelengths $\lambda_{eff}(\mathbf{x}_0)$). Therefore, the wave function in the vicinity of point $\mathbf{x}_0$ is at a first approximation given by the theory of Section 5.

However, due to the finite extension of the detector, important deviations must occur in the far-field. If we write $K_E^{(0)}(\mathbf{x}|\mathbf{x}_1)$, the time-independent Green's function for the Schrödinger equation in vacuum ($\mathbf{x}$ is the observation point and $\mathbf{x}_1$ the source point), we have

$$EK_E^{(0)}(\mathbf{x}|\mathbf{x}_1) = \frac{-\boldsymbol{\nabla}^2}{2m}K_E^{(0)}(\mathbf{x}|\mathbf{x}_1) + \delta^3(\mathbf{x} - \mathbf{x}_1) \tag{31}$$

and the usual retarded solution is

$$K_E^{(0)}(\mathbf{x}|\mathbf{x}_1) = -2m\frac{e^{i\sqrt{(2mE)}|\mathbf{x}-\mathbf{x}_1|}}{4\pi|\mathbf{x} - \mathbf{x}_1|} \tag{32}$$

From the Green theorem, the wave function scattered by the effective potential V_{eff} (with a negative imaginary part: $\text{Im}[V_{eff}] < 0$) is given by

$$\Psi(\mathbf{x}) = \Psi^{(0)}(\mathbf{x}) + \int_{\delta V} d^3\mathbf{x}_1 K_E^{(0)}(\mathbf{x}|\mathbf{x}_1)V_{eff}(\mathbf{x}_1)\Psi(\mathbf{x}_1) \tag{33}$$

where the integration is taken over the finite volume δV of the detector, i.e., for points $\mathbf{x}_1$ surrounding $\mathbf{x}_0$.

The scattered field $\Psi_{scat}(\mathbf{x}) = \Psi(\mathbf{x}) - \Psi^{(0)}(\mathbf{x})$ is equivalently writen using a surface integral over the closed boundary $\delta\Sigma$ surrounding δV. From the Huygens–Fresnel theory applied to Schrödinger's equation, we deduce:

$$\Psi_{scat}(\mathbf{x}) = -\oint_{\delta\Sigma} d\Sigma_1 \mathbf{n}_1 \cdot [\Psi(\mathbf{x}_1)\boldsymbol{\nabla}_{\mathbf{x}_1}\frac{K_E^{(0)}(\mathbf{x}|\mathbf{x}_1)}{2m} - \frac{K_E^{(0)}(\mathbf{x}|\mathbf{x}_1)}{2m}\boldsymbol{\nabla}_{\mathbf{x}_1}\Psi(\mathbf{x}_1)] \tag{34}$$

where $\mathbf{n}_1$ is the outwardly oriented unit vector normal to the surface element $d\Sigma_1$ at point $\mathbf{x}_1 \in \delta\Sigma$. In the case above of a sensing volume lying between planes $z = z_0$ and $z_0 + d$ and assuming a current $J_z^\Psi(\mathbf{x}_0) > 0$, the integration surface reduces approximately to the input face $\delta\Sigma_{in}$ at $z = z_0$ and we have

$$\Psi_{scat}(\mathbf{x}) \simeq -\int_{\delta\Sigma_{in}} dx_1 dy_1 [\Psi(\mathbf{x}_1)\partial_{z_1}\frac{K_E^{(0)}(\mathbf{x}|\mathbf{x}_1)}{2m} - \frac{K_E^{(0)}(\mathbf{x}|\mathbf{x}_1)}{2m}\partial_{z_1}\Psi(\mathbf{x}_1)] \tag{35}$$

with $R = |\mathbf{x} - \mathbf{x}_1|$. In general, this scattered field does not vanish, and in order to obtain a converging expression, we replace the surface integral at z_0 by a plane located at $z_0 - \epsilon$ such that $\chi(z_0 - \epsilon) \simeq 0$. Equation (35) thus reads

$$\Psi_{scat}(\mathbf{x}) \simeq \Psi^{(0)}(\mathbf{x}_0)\int_{\delta\Sigma_{in}} dx_1 dy_1 \frac{e^{ikR}}{4\pi R}(ik\hat{\mathbf{R}} \cdot \hat{\mathbf{z}} + ik_z)e^{i\mathbf{k}_{||}\cdot(\mathbf{x}_1 - \mathbf{x}_0)} \tag{36}$$

with $\mathbf{R} = \mathbf{x} - \mathbf{x}_1$, $\hat{\mathbf{R}} = \mathbf{R}/|\mathbf{R}|$ and the vector $k_z\hat{\mathbf{z}} + \mathbf{k}_{||} := \boldsymbol{\nabla}S(\mathbf{x}_0)$, as explained before (for $k_z > 0$). For points in the shadow region, near the detector, the scattered contribution $\Psi_{scat}(\mathbf{x})$ strongly compensates the incident term $\Psi^{(0)}(\mathbf{x})$ and the full wave function approximately vanishes. However, in general, this implies that the scattered wave interferes with the incident one and this will disturb the probability current $\mathbf{J}^\Psi$ as well as the dBB trajectories in the vicinity of the strongly absorbing detector located at $\mathbf{x}_0$. The detector is thus invasive and the disturbed trajectory flow will in general prohibit a subsequent measurement of the incident current $\mathbf{J}^\Psi$ at a different point $\mathbf{x}_0'$ located near $\mathbf{x}_0$. We stress that Equation (36) must be multiplied by the coefficient -1 if the current is counterprogating, i.e., if we have a local back-flow with $k_z < 0$ (this is because in this regime, the input face at $z = z_0$ contributing to the integral is replaced by the output face at $z = z_0 + d$). We thus have:

$$\Psi_{scat}(\mathbf{x}) \simeq -\Psi^{(0)}(\mathbf{x}_0) \int_{\delta\Sigma_{in}} dx_1 dy_1 \frac{e^{ikR}}{4\pi R} (ik\hat{\mathbf{R}} \cdot \hat{\mathbf{z}} + ik_z) e^{i\mathbf{k}_{||}\cdot(\mathbf{x}_1 - \mathbf{x}_0)} \tag{37}$$

and all the conclusions concerning the invasiveness of the detector are of course also valid in this back-flow regime.

The previous difficulties concerning ideal detectors are very general and will apply to current measurements for time-dependent problems. This, in principle, is central for time of flight and arrival time measurements. Qualitatively, the problem involves absorbing detectors modeled by time-dependent dissipative potentials $V_{eff}(\mathbf{x}, t)$. The potential is supposedly acting only in a small region of space δV surrounding a point $\mathbf{x}_0$ during a time interval δt surrounding a time t_0.

The central formula in the above (non-relativistic) analysis is Equation (18)

$$-\partial_t |\Psi|^2 = \boldsymbol{\nabla} \cdot \mathbf{J}^\Psi - 2\mathrm{Im}[V_{eff}]|\Psi|^2 \tag{38}$$

in which the sink term $-2\mathrm{Im}[V_{eff}]|\Psi|^2 \geq 0$ represents the local absorption of the medium characterized by an effective dissipative potential with $\mathrm{Im}[V_{eff}] < 0$. The probability of absorbing a particle is thus generally given by

$$\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) = -2 \int_{\delta\Omega} d^4x \mathrm{Im}[V_{eff}(\mathbf{x}, t)] |\Psi|^2(\mathbf{x}, t) \tag{39}$$

where $\delta\Omega$ is a 4-volume in space-time where the detector is active and the effective potential $V_{eff}(\mathbf{x}, t) \neq 0$. This effective potential is associated with relaxation and dissipation and can ultimately be justified by interactions with a thermal bath (see Section 4). We stress that $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega)$ is a POVM since $-\mathrm{Im}[V_{eff}(\mathbf{x}, t)] \geq 0$ and $|\Psi|^2(\mathbf{x}, t) = \langle \mathbf{x}|\Psi_t|\mathbf{x}\rangle$ (we have the additivity for two disjoint regions $\delta\Omega_1$ and $\delta\Omega_2$ $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega_1 \cup \delta\Omega_2) = \mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega_1) + \mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega_2)$). The measurement of the probability $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega)$ is thus physically unambiguous and must agree in both the orthodox and Bohmian quantum interpretations.

In practice, however, it is extremely difficult to build a detector with space-time resolution. The basic idea, for example, would be to introduce a time-shutter that opens and closes in a narrow time window δt. Within this time window, the incident particle is likely to pass through and interact with the absorbing medium of the detector. However, the wave theory of time-shutters and transient phenomena linked to diffraction in time is complex (see, for example, [82–84]) and we will not go into it here. Calculations not reproduced here show in particular that the presence of the shutter strongly disturbs the incident wave field (e.g., due to the presence of back-scattering), and this will of course have an impact on the $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega)$ probability. Another method could be to use a dynamical potential barrier [85], or alternatively a metal plate that rejects secondary electrons when subjected to a local excitation in space-time [86] (this approach has been used in an interferometry experiment involving He atoms [87] and analyzed using dBB dynamics [30]). For a review of detection methods relevant for arrival time measurement, see [4,5] as well as [88–90].

Limiting our description to an effective absorption potential and Equation (39), it will in general be difficult to reduce the probability to the simple dBB formula, Equation (8). Going back to Equation (38) and integrating over a four-volume $\delta\Omega = \delta V \times \delta t$ we obtain by applying Gauss theorem:

$$\int_{\delta V} d^3x |\Psi(\mathbf{x}, t)|^2 - \int_{\delta V} d^3x |\Psi(\mathbf{x}, t + \delta t)|^2 \\ + \int_t^{t+\delta t} dt \oint_\Sigma d^2\Sigma_{\mathbf{x}} \cdot \mathbf{n}_{\mathbf{x}} \mathbf{J}^\Psi(\mathbf{x}, t) = -2 \int_\Omega d^4x \mathrm{Im}[V_{eff}(\mathbf{x}, t)] |\Psi|^2(\mathbf{x}, t) \tag{40}$$

where $\mathbf{n_x}$ is the inward-oriented unit vector normal to the closed boundary Σ surrounding the volume δV of the detector and $d^2\Sigma_{\mathbf{x}}$ is a surface element of Σ. If the detector is efficient, we naturally expect $\int_{\delta V} d^3x|\Psi(\mathbf{x},t+\delta t)|^2 \simeq 0$. Similarly, for an efficient and compact detector reducing to a slab, we must have $\int_{\delta V} d^3x|\Psi(\mathbf{x},t)|^2 \ll \int_t^{t+\delta t} dt \oint_\Sigma d^2\Sigma_{\mathbf{x}}\mathbf{n_x} \cdot \mathbf{J}^\Psi(\mathbf{x},t) \simeq \delta t \int_{\Sigma_{in}} d^2\Sigma_{\mathbf{x}}\mathbf{n_x} \cdot \mathbf{J}^\Psi(\mathbf{x},t)$ where the only important contribution of the surface integral comes from the entrance surface of the detector Σ_{in} (the sign $\int_{\Sigma_{in}} d^2\Sigma_{\mathbf{x}}\mathbf{n_x} \cdot \mathbf{J}^\Psi \geq 0$ is thus naturally imposed but we can introduce a minus sign if we need to consider a back-flow process). Therefore, Equation (39) reduces to

$$\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) = -2\int_\Omega d^4x \text{Im}[V_{eff}(\mathbf{x},t)]|\Psi|^2(\mathbf{x},t) \simeq \delta t \int_{\Sigma_{in}} d^2\Sigma_{\mathbf{x}}\mathbf{n_x} \cdot \mathbf{J}^\Psi(\mathbf{x},t) \quad (41)$$

which is recovering the dBB flux result, Equation (8), with a detecting efficiency $\eta \simeq 1$. Of course, the present analysis is only an approximation. It cannot be general, since by definition, Equation (39) is a POVM, whereas from the theorem of Dürr et al. [23] (derived in Section 3, $\mathcal{P}^\Psi_{\text{dBB}}(\mathbf{x},\tau)$ is not a POVM! We insist on the fact that if we use the perfectly matched detector layer studied previously, then the surface integral $\int_{\Sigma_{in}} d^2\Sigma_{\mathbf{x}} \cdot \mathbf{n_x}\mathbf{J}^\Psi(\mathbf{x},t)$ depends on the initial wave function $\Psi^{(0)}$ existing in the absence of a detector. In this case, we were justifying the possibility of measuring the local dBB distribution. In other words, we have the local equivalence (and for this wave function): $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) \simeq \delta t\mathcal{P}^\Psi_{\text{dBB}}(\mathbf{x},\tau) \simeq \delta t\mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\mathbf{x},\tau)$.

Furthermore, like in the stationary regime, the far-field wave function will be in general strongly modified since we will have

$$\Psi(\mathbf{x},t) = \Psi^{(0)}(\mathbf{x},t) + \int_{\delta\Omega} d^4x_1 K^{(0)}(\mathbf{x},t|\mathbf{x}_1,t_1)V_{eff}(\mathbf{x}_1,t_1)\Psi(\mathbf{x}_1,t_1) \quad (42)$$

where $K^{(0)}(\mathbf{x},t|\mathbf{x}_1,t_1) = -i(\frac{m}{2\pi i(t-t_1)})^{3/2}e^{i\frac{m(\mathbf{x}-\mathbf{x}_1)^2}{2(t-t_1)}}\Theta(t-t_1)$ is the retarded Schrödinger Green function for the time-dependent problem. This implies that the mere presence of an efficient absorbing detector in the space-time region $\delta\Omega$ will in general disturb and influence the surrounding environement. In particular, it will generally affect other detectors, as we will now see.

7. General Discussions and Conclusions: Can We Observe Bohmian First Arrival Time? (Second Round)

7.1. Weak Coupling versus Strong Coupling: Advantages and Limitations

The previous results show that, in general, it should not be impossible to measure the dBB probability distribution $\mathcal{P}^\Psi_{dBB}(\mathbf{x},t) := |\mathbf{J}^\Psi(\mathbf{x},t) \cdot \mathbf{n}(\mathbf{x})|d\Sigma_{\mathbf{x}}$ of the first arrival times at a given space-time point. However, as we show now, the procedure is generally very invasive and may prevent this distribution from being measured in several space-time regions in the same experiment. This point is crucial and is in line with the result obtained on the no-signalling theorem concerning the impossibility of measurements that would contradict special relativity.

Indeed, the central problem we have seen in previous sections is that an effective detector modifies the Ψ wavefield in its immediate environment by scattering. This is in some ways reminiscent of Renninger's results on null or negative measurements [91]: A non-measurement or non-detection of a particle by a localised screen impacts the wave function of the particle outside that screen. The effect can never be neglected specifically if the wavelength of Ψ is comparable to the detector size.

In Bohmian mechanics, the consequences are unavoidable in order to understand how the presence of detectors can affect subsequent potential interactions or detections. The problem is fundamentally linked to relativistic causality. For example, consider a quantum particle following a Bohmian trajectory in space-time, as shown in Figure 5a. Along this trajectory, we can place absorbing detectors at the space-time points A, B, and C

(with time $t_C > t_B > tA$). From an intuitive point of view, if the detector observes a particle at A, this naturally prevents subsequent observations at B and C. In other words, intuitively, if, in an experiment, we position three detectors at A, B and C, only the first detector, which is assumed to be very efficient, will be able to potentially observe a particle (this probability of observation being given by Born's rule $|\Psi|^2$ for this point A) and we will never observe particles at B or C because the mere presence of the detector at A screens out the other detectors and precludes interactions. This intuitive description is, however, approximative and essentially classical. It presupposes that the presence of the detectors at A, B, C is not influencing the incident wave function Ψ. However, what we saw in previous sections of this article is precisely the opposite; the detectors generally disturb the wave function Ψ.

As we saw, we can basically distinguish two regimes. In the 'weak coupling regime', the detector is highly inefficient and $\eta \ll 1$. In this regime, the Bohmian trajectories can be considered as approximately non-modified: Most of the particles going through region A of the previous example will not be absorbed by the detector and only a small fraction of the incident particles will contribute to the recording signal at A. Moreover, in this weak coupling regime, nothing prohibits the detectors at position B or C from firing if the particle has not been detected at A (and B if we consider detection at C). Since we can ultimately suppose that the incident wave function is not disturbed (i.e., we can neglect scattering in Equation (36)), this implies that at the lowest order of approximation, the probabilities of detection $\mathcal{P}^{\Psi^{(0)}}_{\text{detec.}}(\mathbf{x}, t)$ at A, B or C are just calculated by ignoring the presence of the other detectors and using the incident wave function $\Psi^{(0)}$.

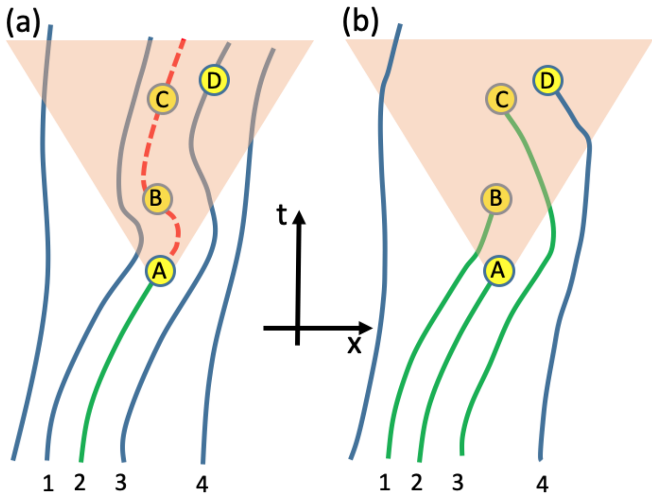

Figure 5. Bohmian trajectories $1, 2, 3, 4, \ldots$ in presence of position-time detectors. (**a**) In the idealized weak coupling regime, detectors at points A, B, C or D are not perturbed by spurious scattering and the system is sensitive to the incident wave function $\Psi^{(0)}$ existing in the absence of detectors. However, one must add a clock to distinguish precisely the first arrival, second arrival, etc., at a given position (the dashed line shows that A, B and C are belonging to the same dBB path). This requires a precise knowledge of the dBB trajectories. (**b**) In the strong regime, only one pass is needed; the detectors absorb with high efficiency the incoming particles. However, this strongly affects the wave function $\Psi \neq \Psi^{(0)}$ and disturbs the dBB motion on other detectors.

There are clearly advantages and disadvantages to considering the weak coupling regime. Starting with the advantages, we can see, by returning to Figure 5a, that in this regime, we can define detection experiments involving several points, A, B, C, etc., and associated with complex geometries. From this point of view, measuring the probability distribution $\mathcal{P}^{\Psi^{(0)}}_{\text{detec.}}(\Sigma, t)$ is not in principle a problem. We can imagine, for example, a set of weakly absorbing detectors distributed in a finite region of space-time in order to have access in the same experiment to the probability distribution of arrival times associated with the initial wave function $\Psi^{(0)}$.

However, this weak coupling regime leads to two important problems. Firstly, as we have seen in the previous sections, in the non-relativistic regime, if the efficiency of the detector decreases, the arrival time probability approaches $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, \tau) \simeq \Sigma d N \sigma_{ext} |\mathbf{J}^{\Psi^{(0)}}|$, which is independent of the incidence angle θ and which shows that in the non-relativistic regime in which the calculations are carried out, the detector is insensitive to the direction of the probability current. Going back to Equation (39), we have for an individual detector centered on A:

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \simeq -2\delta t \int_{\delta V} d^3x \text{Im}[V_{eff}(\mathbf{x}, t)] |\Psi^{(0)}|^2(\mathbf{x}, t) \sim \epsilon \delta t \delta V |\Psi^{(0)}|^2(\mathbf{x}_A, t_A) \quad (43)$$

with $\epsilon \sim -\frac{2}{\delta V} \int_{\delta V} d^3x \text{Im}[V_{eff}(\mathbf{x}, t)]$ a characteristic rate (in the model used previously—see Section 4—we have $\epsilon = -2\text{Im}[V_{eff}] = N v \sigma_{ext}$ and $\delta V = \Sigma d$, which allows us to recover the formula $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, \tau) \simeq \Sigma d N \sigma_{ext} |\mathbf{J}^{\Psi^{(0)}}|$). The detection probability is therefore no longer simply related to the ideal Bohmian probability Equation (8) $\mathcal{P}^{\Psi}_{dBB}(\mathbf{x}, t) := |\mathbf{J}^{\Psi}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})| d\Sigma_{\mathbf{x}}$ associated with the probability current $\mathbf{J}^{\Psi}(\mathbf{x}, t)$.

The second problem with weak coupling arises from the very fact that the dBB trajectories obtained are unperturbed with respect to the initial wave function $\Psi^{(0)}$ (see Figure 5a). It is therefore possible to imagine a detector that is sensitive to the first pass (at A), the second pass (at B), the third pass (at C), etc. However, if the detector can observe a particle corresponding to the second pass at B, for example (which may be associated with a backflow phenomenon), a clock is needed that can determine when the particles arrive and tell when it is a first, second or third pass. In practice, this requires knowledge of the dBB trajectories and is therefore dependent on the initial wave function. This problem seems to be even more fundamental than the first, as it concerns the very notion of measurement. In quantum measurement theory, which is based on the notion of POVM, it is presupposed that any good measurement requires detection equipment that will function independently of the chosen initial wave function $\Psi^{(0)}$. This is what is implied by the POVM formalism that reduces any observable probability to an expression of the type $\mathcal{P}^{\Psi}_a = \langle \Psi^{(0)} | \hat{O}_a | \Psi^{(0)} \rangle$, which contains an operator $\hat{O}_a$ (independent of $|\Psi^{(0)}\rangle$) and the wave function itself $|\Psi^{(0)}\rangle$, which averages the operator $\hat{O}_a$. Here, however, in order to define the first, second, etc., arrival times, the precise knowledge of the dBB trajectories is needed. This clearly seems to violate the natural formulation of quantum measurement based on POVM and therefore does not look appealing. For the reasons mentioned above, the weak coupling regime seems a priori unsuitable. This analysis confirms some of the worries of GTZ [48,57]. We will see below that, far from being the case, the alleged defects will in fact turn out to be advantageous. However, temporarily ignoring this point, it seems natural at this stage to focus on the other regime, i.e., the strong coupling regime with highly efficient detectors.

In the strong coupling regime, the detection efficiency is high, i.e., $\eta \sim 1$. As we have seen in this regime, the probability of detection approaches the ideal Bohmian formula Equation (8) $\mathcal{P}^{\Psi}_{dBB}(\mathbf{x}, t) := |\mathbf{J}^{\Psi}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})| d\Sigma_{\mathbf{x}}$. Moreover, as the probability of absorption is high, this seems de facto to prohibit the detection of second, third, etc., passes; in principle, only the first pass could be measured. This therefore seems intuitively desirable for a procedure for measuring the first arrival time distribution of a particle in a given zone of space (in agreement with the idealized classical picture of an absorption).

Alas, this image is, of course, an oversimplification because, as we have analyzed above, the simple registration of a strongly absorbing detector in A will disturb the detector's immediate environment by scattering. Thus, in general, the wave function is locally modified and the dBB trajectories are strongly perturbed. In the situation shown in Figure 5a, the Bohmian trajectories are causally disturbed by diffraction in the future light-cone emerging from point A. In fact, even if the particle passing through A is well absorbed, this in no way precludes particle detection at B and C, since other disturbed trajectories may reach regions B, C, etc., as shown in Figure 5b. Other regions may also be affected, such as the one where detector D is located, which was not on the initial dBB trajectory through ABC!

We can obtain a theorem concerning this issue. Indeed, from [23] and results discussed in Sections 3 and 6 (in particular, Equation (41)), we know that $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \simeq \delta t \mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\mathbf{x}, \tau)$ cannot generally be true for every wave function Ψ since the left-hand side of the relation is a POVM, whereas the right-hand side is not. Let us assume that $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \simeq \delta t \mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\mathbf{x}, \tau)$ is (approximately) true for a specific wave function Ψ. We can thus consider several efficient detectors in regions $\delta\Omega_1$, $\delta\Omega_2$, and suppose that for a given wave function Ψ, we have

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\cup_i \delta\Omega_i) = \sum_i \mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega_i) \simeq \sum_i \delta t_i \mathcal{P}^{\Psi_i}_{\text{dBB}}(\mathbf{x}_i, \tau_i) \tag{44}$$

Here, Ψ_i is the local wave function in the region $\delta\Omega_i$ that in principle includes the scattering contributions from other detectors that could causally interact with detector i (i.e., located in its past light-cone). Physically, we know some cases where Equation (44) is certainly true or a very good approximation (e.g., typically in the far-field domain [30,86]). But what we would ideally like to obtain is the stronger result

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\cup_i \delta\Omega_i) = \sum_i \mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega_i) \simeq \sum_i \delta t_i \mathcal{P}^{\Psi_i^{(0)}}_{\text{dBB}}(\mathbf{x}_i, \tau_i) \tag{45}$$

which depends on the initial wave function $\Psi_i^{(0)} := \Psi^{(0)}(\mathbf{x}_i, t_1)$ unperturbed by the presence of detectors and calculated at the various space-time points where the detectors are actually located. For a single isolated detector, we know a priori that this is possible, but for a set of detectors, the question remains open. However, we can easily show that this is actually impossible. Indeed, since the various detectors in regions $\delta\Omega_1$, $\delta\Omega_2$, etc., are strongly efficient, they actually record the probability of first arrival in these regions. The situation is thus the one sketched in Figure 5b. But points A, B, C in this example are located on the same initial unperturbed trajectory. Therefore, in order for Equation (45) to be true, we would have to have detectors located at B and C which are sensitive to the first passage of the particle at this point (in accordance with the definition and Figure 5b), and yet according to Equation (45), in reality, these detectors would measure the second passage (for B) and the third passage (for C). Another way of saying this is that, in accordance with the definition of the strong coupling regime, the detector in A should be able to observe a particle, while those in B and C should observe nothing in disagreement with Equation (45), which authorizes detection in B or C. So there is a contradiction and we must conclude that it is impossible to have such a configuration, and therefore, Equation (45) cannot be true in situations involving several points on the same trajectory!

This clearly undermines DD's position [45,46] that the presence of a detector should not be taken into account when analyzing arrival times (it also undermines some predictions made in [22] concerning back-flow). In fact, we are faced with two alternatives:

(1) We use detectors operating in the weak coupling regime, but then we have to amend the implicit assumption that any measurement is based solely on a POVM, and we have to add a post-selection and filtering condition (post-analysis) taking into account the dBB dynamics. Also, in this regime (at least if we neglect spin), $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, \tau) \simeq \Sigma dN \sigma_{ext} |\mathbf{J}^{\Psi^{(0)}}| \neq \mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\Sigma, \tau)$.

(2) We use a strongly coupled detection regime, but then we generally have to abandon the idea of being able to directly measure the dBB distribution of arrival times based on Equation (8). In case 2, we could of course eliminate the problem by limiting the analysis to the detection at only one single space-time point located in the small region $\delta\Omega$. But in turn, this would mean that a single experiment could not measure all the distributions $\mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\Sigma, \tau)$. We would need several experiments in order to reconstruct the distribution of arrival times. Additionally, in this situation, nothing would prohibit us from recording the distribution at say, point B of Figure 5b. Indeed, since there is no detector at A, there is no scattering from region A disturbing the local motion at B. Therefore, like in the weak coupling regime, we need to add a post-selection

depending on the full dBB dynamics in order to filter out such detection events. Like for the weak coupling regime, this clearly contradicts the assumption of a purely POVM-based quantum measurement procedure.

There are other issues that we have mentioned and which we must now consider, as they play a central role in the analysis of the work of DD and GTZ. Indeed, DD's predictions involve spin-1/2 particles, so we need to include a magnetic current term (see Equation (11)) in our analysis. As we shall see, this has a strong impact on both regime 1 (weak coupling) and regime 2 (strong coupling).

7.2. The Spin-Dependent Problem and the Measurement of the First Arrival Time Distribution of Das and Dürr

In the previous analysis, we did not include the spin-1/2 required in the work of DD [45,46] and GTZ [48,57]. For this purpose, we need to consider the dynamics of electron using either the Pauli or Dirac wave equation. Taking the relativistic Dirac wave equation, we have for the electron bi-spinor $\Psi(x) \in \mathbb{C}^4$:

$$i\gamma^\mu \partial_\mu \Psi(x) = m\Psi(x) + e\gamma^\mu A_\mu(x)\Psi(x) \tag{46}$$

where $\gamma^0 = \beta$ and $\boldsymbol{\gamma} = \beta\boldsymbol{\alpha}$ are Dirac matrices, $e = -|e|$ the electron charge and $A_\mu(x)$ the external electromagnetic field at space-time point $x := [t, \mathbf{x}]$. By an obvious generalization of the previous non-relativistic analysis, we can define an absorbing detector involving complex 4-vector potential $A^\mu := [A^0 = \Phi, \mathbf{A}] = \mathrm{Re}[A^\mu_{eff}] + i\mathrm{Im}[A^\mu_{eff}]$. The local conservation law for the 4-current $J^\mu = \overline{\Psi}\gamma^\mu\Psi := [J^0 = \rho^\Psi = \Psi^\dagger\Psi, \mathbf{J}^\Psi = \Psi^\dagger\boldsymbol{\alpha}\Psi]$ is deduced from Equation (46) and reads

$$-\partial_\mu J^\mu := -\partial_t \rho^\Psi - \boldsymbol{\nabla}\cdot\mathbf{J}^\Psi = -2e\mathrm{Im}[A^\mu_{eff}]J_\mu = -2e\mathrm{Im}[\Phi_{eff}]\rho^\Psi + 2e\mathrm{Im}[\mathbf{A}_{eff}]\cdot\mathbf{J}^\Psi \tag{47}$$

which generalizes Equation (18) obtained in the non-relativistic regime for spinless particles.

From this relation, we can (a priori) extend the analysis of Section 6 and define the probability of absorption by a detector located in the volume $\delta\Omega$

$$\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) = -2e\int_{\delta\Omega} d^4x \mathrm{Im}[A^\mu_{eff}]J_\mu(\mathbf{x}, t) \tag{48}$$

which generalizes Equation (39) obtained in the non-relativistic regime for spinless particles. However, unlike in the non-relativistic regime, there is an obvious difficulty; the scalar product $-e\mathrm{Im}[A^\mu_{eff}]J_\mu(\mathbf{x}, t)$ has no imposed sign. More precisely, in the non-relativistic regime, the quantity $-\mathrm{Im}[V_{eff}]|\Psi|^2(\mathbf{x}, t)$ could always be positive if the medium obeys a natural causal and entropic condition $-\mathrm{Im}[V_{eff}] \geq 0$ associated with inelastic scattering and dissipation in the medium (i.e., due to coupling with a thermal bath). Of course, media with gain (producing particles) such that $-\mathrm{Im}[V_{eff}] \leq 0$ are also potentially possible, but we could always imagine making a choice between the two alternatives. In the case of lossy media, interpreting Equation (39) as the probability of absorption associated with a POVM was therefore straightforward. In the relativistic regime, we cannot in general be sure that the intrinsic properties of the medium will always impose a value of $-e\mathrm{Im}[A^\mu_{eff}]J_\mu(\mathbf{x}, t)$ strictly positive or negative. It means that for a given field $\mathrm{Im}[A^\mu_{eff}]$, $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) \geq 0$ (loss) or $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) \leq 0$ (gain), depending on the wave function Ψ. Therefore, Equation (48) is not generally a POVM. Still, this quantity always has a physical meaning. If $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) \geq 0$, it represents a probability of absorption, and alternatively, if $\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) \leq 0$, then $-\mathcal{P}^\Psi_{\text{detec.}}(\delta\Omega) \geq 0$ represents a probability of emission (gain). The crux is that it depends explicitly on the wave function Ψ used. This implies that in the relativistic regime, the concept of POVM must be used cautiously. We stress that the present analysis is in line with works applying Bohmian mechanics to quantum field theory (QFT) where source/sink terms must be added in order to explain creation/annihilation of particles by fields [92].

Furthermore, supposing that for a given wave function Ψ, we indeed have $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \geq 0$ and that we are considering a strongly efficient detector, we must have, as in Section 6:

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) = -2e \int_{\delta\Omega} d^4x \text{Im}[A^{\mu}_{eff}] J_{\mu}(\mathbf{x}, t) \simeq \delta t \int_{\Sigma_{in}} d^2\Sigma_{\mathbf{x}} \mathbf{n}_{\mathbf{x}} \cdot \mathbf{J}^{\Psi}(\mathbf{x}, t) \tag{49}$$

where the surviving contribution comes from a surface integral on the entrance side of the detector (there is no ambiguity here, since the absorption condition $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \geq 0$ fixes the direction of the decay of the wave function inside the detector).

However, there is now a new difficulty. Indeed, the Dirac current J^{μ} can be separated into a convective current, a magnetic term and an absorbing term using the so-called Gordon formula:

$$J_{\mu} = \frac{i}{2m}(\overline{\Psi} \overleftrightarrow{\mathrm{D}}_{\mu} \Psi) - \frac{1}{2m}\partial^{\nu}(\overline{\Psi}\sigma_{\nu\mu}\Psi) - \frac{e}{m}\overline{\Psi}\sigma_{\mu\nu}\Psi \text{Im}[A^{\nu}_{eff}] \tag{50}$$

with $\sigma_{\nu\mu} = \frac{i}{2}[\gamma_{\nu}, \gamma_{\mu}]$, $D^{\mu} = \partial^{\mu} + ie\text{Re}[A^{\mu}_{eff}]$. The absorbing term is usually not present since the electromagnetic field is supposed to be real-valued. Here, this is not the case and we must in general include this contribution. In the non-relativistic limit, this reduces to $\rho^{\Psi} \simeq \Psi^{\dagger}\Psi$ and

$$\mathbf{J}^{\Psi} \simeq \frac{1}{2m}(\Psi^{\dagger} \overleftrightarrow{\boldsymbol{\pi}} \Psi) + \frac{1}{2m}\boldsymbol{\nabla} \times [\Psi^{\dagger}\boldsymbol{\sigma}\Psi] + \frac{e}{m}\text{Im}[\mathbf{A}_{eff}] \times \Psi^{\dagger}\boldsymbol{\sigma}\Psi \tag{51}$$

where $\boldsymbol{\pi} = \frac{\boldsymbol{\nabla}}{i} - e\text{Re}[\mathbf{A}_{eff}]$ and $\Psi \in \mathbb{C}^2$ is now a bispinor. Now the problem is that the divergence of the magnetic term in Equation (50) or (51) cancels out, and consequently, the associated surface integral calculated using Gauss's theorem over a closed surface surrounding the detector region (in the 3D or 4D formalism) vanishes in Equation (52). Therefore, in the non-relativistic regime, Equation (52) actually reduces to

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \simeq \delta t \int_{\Sigma_{in}} d^2\Sigma_{\mathbf{x}} \mathbf{n}_{\mathbf{x}} \cdot [\frac{1}{2m}(\Psi^{\dagger} \overleftrightarrow{\boldsymbol{\pi}} \Psi) + \frac{e}{m}\text{Im}[\mathbf{A}_{eff}] \times \Psi^{\dagger}\boldsymbol{\sigma}\Psi]. \tag{52}$$

The most important consequence is that an efficient detector (i.e., $\eta \sim 1$) cannot register a signal proportional to the total current flow; the detector is not sensitive to the magnetic term $\frac{1}{2m}\boldsymbol{\nabla} \times [\Psi^{\dagger}\boldsymbol{\sigma}\Psi]$. But it is precisely this term that plays a crucial role in the analysis of DD and GTZ, with disastrous consequences for the analysis of DD in this regime. More precisely, in the case of a simple detector where the coupling is via the scalar field $V_{eff} := e\Phi_{eff}$ ($A^{\mu}_{eff} := [\Phi_{eff}, 0]$), we recover the analysis conducted in the previous sections for spinless particles (indeed, we can always impose $-2e\text{Im}[\Phi_{eff}]\rho^{\Psi} \geq 0$, imposing $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \geq 0$), but now we see that the detector will only be sensitive to the convective term, which in the example of DD, reads

$$\mathbf{J}^{\Psi_{\hat{\mathbf{s}}}}_{conv.}(\mathbf{x}, t) = |\Phi(\rho, z, t)|^2 \frac{\boldsymbol{\nabla} S(\rho, z, t)}{m} \tag{53}$$

without the magnetic term $\frac{-1}{m}\hat{\mathbf{s}} \times \boldsymbol{\nabla}|\Phi(\rho, z, t)|^2$ of Equation (11). In the work of DD [46], an explicit formula is given for the current, which in their system of normalized units (see Equation (18) in [46]), reads:

$$\mathbf{J}^{\Psi_{\hat{\mathbf{s}}}}_{conv.}(\mathbf{x}, t) = |\Phi(\rho, z, t)|^2 \frac{tz}{1 + tz^2}\hat{\mathbf{z}} \tag{54}$$

which is directed parallel to the axis of the waveguide. It is interesting to note that a dBB dynamics without any magnetic term is often considered as a good alternative (see discussions in [8,9]). We can debate endlessly the motivations for the different dynamics, but in the end, we see that the detectors are in the present regime ignoring the spin term. The dBB trajectories deduced from this truncated current are thus straight lines parallel to the z axis and the velocity of the particle is $\frac{tz}{1+tz^2}\hat{\mathbf{z}}$. In this regime, there is no observed back-flow. Of course, for a Bohmian, the question of which velocity is the good one is fundamental, but from the point of view of detection theory, only the convective term plays a role and the trunctated dynamics without the spin term is the only relevant one.

Most importantly, this convective term generates a probability distribution of first arrival times that is independent of the orientation of the incident spin $\hat{\mathbf{s}}$. The analytical formula is again given in [46] for the convective current of Equation (54) (see Equations (51) and (52) in [46]), and we have:

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\tau, L) = \frac{4L^3}{\lambda_0\sqrt{\pi}} \frac{\tau e^{-\frac{L^2}{1+\tau^2}}}{(1+\tau^2)^{5/2}} \tag{55}$$

where $z = L$ is the position of the detector localized along a cross-section of the wave guide and λ_0 is a constant [46].

We stress that for DD, the distribution of first arrival times given by Equation (55) is only valid in the longitudinal cases where $\hat{\mathbf{s}} = \pm\hat{\mathbf{z}}$, whereas here, in the presence of strongly efficient detectors, this distribution is actually valid for every spin orientation $\hat{\mathbf{s}}$! In particular, this distribution is never vanishing, i.e., there is no critical time τ_{max} for which $\mathcal{P}^{\Psi}_{\text{detec.}}(\tau, L) = 0$ for $\tau > \tau_{max}$. This is very different from the predictions obtained by DD [45,46] with transverse spins. Clearly, this implies that in assuming the strong coupling regime, all paradoxes of DD and GTZ disappear.

More precisely, in the strong coupling regime, where detectors are inherently sensitive only to the first arrival time, we see that GTZ's analysis [48] is clearly validated to the detriment of DD's conclusions [45]. Indeed, in this regime, Equation (12) is trivially true, as there is no longer any spin dependence ($\mathcal{P}^{\Psi}_{\text{detec.}}(\tau, L)$ is, for all practical purposes, a POVM). Bell's theorem is also safe; tt is not possible to use this type of experiment to send faster-than-light signals, as the distribution is invariant to spin basis shifting. Again, this agrees with GTZ [48].

At this stage, you could say that the die was cast; GTZ were right and DD were wrong. However, we should not jump to conclusions. We have not yet analyzed the problem in terms of weak coupling detection.

As we said, this weak coupling regime has two inherent shortcomings. The first concerns the notion of POVM, which the approach seems to cast doubt on, since the notion of dBB trajectory must be taken into account in order to make arrival time predictions. The second problem stems from the fact that the $\mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\Sigma, \tau)$ distribution measured in the non-relativistic regime depends only on the norm of the probability current $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, \tau) \simeq \Sigma d N \sigma_{ext} |\mathbf{J}^{\Psi^{(0)}}|$. However, the second problem was obtained in the context of a non-relativistic theory for spinless particles. In the context of Dirac or Pauli theory, this problem can in fact be corrected. Indeed, starting from Equation (48), we deduce that in general, we have

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) = -2e \int_{\delta\Omega} d^4x \Big(\text{Im}[\Phi_{eff}]\Psi^\dagger\Psi(\mathbf{x}, t) - \text{Im}[\mathbf{A}_{eff}] \cdot \Psi^\dagger\boldsymbol{\alpha}\Psi(\mathbf{x}, t)\Big) \tag{56}$$

where Ψ is a Dirac bispinor. In the spinless case, only the first term appears, and in the weak coupling regime, we have indeed an absorption probability proportional to $\Psi^\dagger\Psi(\mathbf{x}, t)$, i.e., to the density of probability. In the Dirac–Pauli theory, this is possible if the detector is scalar, i.e., if $\text{Im}[\mathbf{A}_{eff}] = 0$. But for a spin-1/2 particle, this is not the only option. We can in principle develop an experimental configuration with an absorbing field such that $\text{Im}[\Phi_{eff}] = 0$ but $\text{Im}[\mathbf{A}_{eff}] \neq 0$. In this alternative, the 'probability' of detection/gain reads

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) = +2e \int_{\delta\Omega} d^4x \text{Im}[\mathbf{A}_{eff}] \cdot \Psi^\dagger\boldsymbol{\alpha}\Psi(\mathbf{x}, t) \sim \epsilon \delta t \delta V \mathbf{n} \cdot \mathbf{J}^{\Psi}_{total}(\mathbf{x}_{\text{detec.}}, t_{\text{detec.}}) \tag{57}$$

with $\epsilon\mathbf{n} \sim +\frac{2}{\delta V}\int_{\delta V} d^3x \mathrm{Im}[\mathbf{A}_{eff}(\mathbf{x},t)]$, a coupling efficiency (compare with Equation (43)). This 'probability' of local absorption/gain depends on the full Dirac current at the detector position, and thus, in principle, it is possible to build such a detector which in the weak coupling regime would give a signal directly related to the dBB probability predicted by DD [45,46]. Moreover, in order to have absorption and not gain, we must locally define the field $\mathrm{Im}[\mathbf{A}_{eff}(\mathbf{x},t)]$ in order to have $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) \geq 0$ in the small 4-volume $\delta\Omega$. This procedure is wave function-dependent, since for a given field $\mathrm{Im}[\mathbf{A}_{eff}(\mathbf{x},t)]$, we cannot impose the sign of Equation (57) for every wave function! Again, $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega)$ given by Equation (56) is not generally a POVM, but still, its physical interpretation in terms or loss or gain is obvious. The fact that it strongly depends on Ψ shows once more that in the relativistic regime, local interactions do not simply lead to POVM. Yet, by specifically and locally engineering a field $\mathrm{Im}[\mathbf{A}_{eff}(\mathbf{x},t)]$, we can imagine developing detectors adapted to a given wave function Ψ.

Note, however, that at the very end of a measurement process, POVMs are still used. A localized detector with $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) > 0$ or $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega) <$ properties will behave either as an absorber or as an emitter, depending on the case and the Ψ wave function chosen. If we know a priori by calculation (i.e., dBB trajectories) how it will behave, then in the end, the experimenter will either have to count absorbed particles (e.g., trapped in the detectors or channelled to particle-counting outputs in the far-field) or emitted particles, once again sent and redirected to more conventional counters in the far-field regime. In these final counting regimes, we ultimately end up using $\int_\Delta dq|q\rangle\langle q|$ projectors associated with POVMs in given regions of space Δ (i.e., in the far-field). The fact that dBB theory is ultimately based on such spatial location and counting experiments was already pointed out by de Broglie and Bohm and justifies the use of POVMs. Clearly, however, there is no universal detection procedure, and we need to add elements foreign to POVMs in order to perform good dBB physics.

Of course, here we are just building a proof of principle, and we can see that there is nothing in the laws of physics to prevent the construction of such detectors. However, more work clearly needs to be conducted to define a precise and efficient design for such a Dirac or Pauli current detector that would allow us to trace the Bohmian distribution of arrival times predicted by DD. This goes far beyond the scope of the present study. Now we have seen that POVMs are not the end of the story, but before concluding, however, it remains to return to the second objection against the weak coupling regime, namely the weakening of the exclusive use of POVMs, which is assumed as a postulate by GTZ, among others, and is associated with a post-analysis of the data.

7.3. Beyond the Standard-only POVM-Based Quantum Measurement Procedure

The fact that the notion of POVM appears in any probabilistic analysis in quantum mechanics, and more specifically, in Bohmian mechanics, is not surprising, as we pointed out at the beginning of this article. However, the belief that one can limit oneself to using POVMs to interpret quantum experiments within the framework of dBB theory is based on prejudiced beliefs and demonstrates that a better understanding of the importance of the Bohmian approach can be obtained. The problem is actually much more general than the one we analyzed in the previous subsection and which concerned only the Dirac–Pauli equation and the relativistic regime (or the regime of the Pauli equation with spin).

Let us return to the example of the two-slit experiment discussed at the beginning of this article in Section 2. As we showed then, it is possible, thanks to the dBB theory, to retrodict the passage of the particle through one or the other of the apertures while detecting interference fringes. To accomplish this, we need to know precisely the shape of the wave function $\Psi(\mathbf{x},t)$ used in the experiment (and in particular, its phase) in order to calculate the Bohmian velocity field and thus obtain the 'which path' information. Of course, the whole method relies on the validity of Born's rule on quantum probabilities, and in the end, this necessarily implies the use of POVM in the analysis. However, knowledge of the theoretical Bohmian trajectory allows us to find information that we would say is 'hidden' if we did not know this dBB theory. In other words, we start with the raw measurements using Born's rule, but we have to carry out post-analysis or filtering to process the data and highlight the correct Bohmian information. This new methodology has been strongly advocated in recent years by Detlef Dürr, and I consider this point to be a major contribution to our understanding of Bohmian theory. Clearly, this is far from being accepted by the whole 'Bohmian' community, but the lack of consensus shows, in my opinion, even more the importance of the classical physics prejudices that have survived among the dBB community.

Dürr often illustrated his argument by using Einstein's reply quoted in Section 2:

'*it is the theory which alone decides what is measurable*'. [58]

which Einstein gave to Heisenberg in 1926 when the latter claimed that he could build a theory using only the notion of the observable. What Einstein reminded or taught Heisenberg [58] was that every scientific theory begins with a quasi-metaphysical act; a theory has to be postulated, and this act, although motivated by previous observations, is free. Then comes the prediction, and empirical data can only be interpreted within a precise theoretical framework. This is the heart of the hypothetico-deductive method advocated by Boltzmann and Einstein. Here, we are interested in the dBB theory, and therefore, following Einstein's hypothetico-deductive method, the analysis of data must include the Bohmian dynamics in order to be predictive.

This is clearly the case for the retrodiction obtained in the two-slit experiment, which enabled us to trace back to the 'which path' information thanks to precise knowledge of the dBB trajectories (interestingly, it was only by forgetting the fundamentally quantum character of these dBB trajectories that Heisenberg and many others after him thoughtlessly deduced that Bohmian dynamics was surreal and that trajectory interpretation could not explain wave–particle duality). This is also clearly the case here (and in agreement with the conclusions of Dürr and Das [45,46]) for the analysis of the first arrival times of particles on a detector. Going back to the above analysis of strong and weak regimes, what we have deduced is indeed the need to explicitly take into account the Bohmian dynamics in order to be predictive and to reconstruct the first arrival time probability $\mathcal{P}^{\Psi}_{dBB}(\mathbf{x}, t)$ from the raw data.

This suggests the following experimental scenario for measuring the arrival time distribution predicted by DD. Use a set of detectors operating in the weak coupling regime in agreement with Equation (57) and sensitive to the probability current including the spin term. This set of detectors is distributed in space-time in such a way as to map the probability density of Bohmian arrival times $\mathcal{P}^{\Psi^{(0)}}_{\text{detec}}(\Sigma, \tau) \geq 0$. This requires a specific engineering of the local fields $\text{Im}[\mathbf{A}_{eff}(x_i)]$ in each spatio-temporal region $\delta\Omega_i$ where the detectors are located in order to impose $\mathcal{P}^{\Psi^{(0)}}_{\text{detec}}(\Sigma, \tau) \geq 0$. As we are working in the weak coupling regime, the initial wavefunction is very weakly perturbed, enabling us to carry out a single experiment without changing the protocol from point to point in space-time. However, in this weak coupling regime, we also need to perform a post-analysis to filter out signals that may or may not be associated with first arrival times, second arrival times, etc. This is clearly wave function-dependent and shows that elements foreign to POVM must be considered. Although the method is based on good absorbing detectors, it can only be interpreted physically if this post-analysis is carried out. This is very much in line with the principle described above in the two-slit example.

It could be argued that if knowledge of Bohmian trajectories is necessary in order to carry out this post-analysis or post-selection, this would be a purely theoretical element based on unobservable trajectories, i.e., 'hidden variables'. In reality, dBB trajectories are not unobservable in principle. As mentioned in the introduction, weak measurement protocols [38] can be used to map the velocity field [39,40,42–44] and then trace the trajectories followed by the particles. In principle, then, we could imagine a pre-experiment that would first map dBB trajectories for the $\Psi^{(0)}_t$ initial wave function we are interested in (in principle, this could also involve the regions of detectors but this would be very difficult, since it would require near-field measurements). Only once we know these dBB trajectories can we carry out the post-selection required in our arrival time experiment to reconstruct the Bohmian probability distribution.

It is worth noting that Dürr has long insisted on the limitation imposed by orthodox measurement theory of restricting itself to so-called linear procedures [24,29,65], which is at the heart of von Neumann's approach. However, he noted that weak measurements are part of non-linear measurement procedures that go beyond the simple use of POVMs [24,65]. This is also clearly the case with arrival time measurement, and our analysis strongly confirms his own intuitions and work with Das [45,46].

It is also important to note that approaches based on weak measurements have theoretically focused on non-relativistic problems (i.e., Schrodinger equation) [43] or the relativistic Klein Gordon equation for spinless particles [44]. However, experimental measurements [39,40,42] are based on electromagnetic trajectories which, in all rigor, take into account the polarization aspects associated with light propagation [93,94]. In the problem we are interested in here, it would be necessary to be able to define weak measurement protocols adapted to the relativistic Dirac equation, probably by analogy with what we have already accomplished in the optical and electromagnetic domain. This opens interesting research perspectives.

Clearly, with this protocol, we can bypass the objections raised by GTZ [48,57]. First of all, Equation (12) presupposed that arrival time measurements were entirely based on the notion of POVM. However, although our detectors here are fundamentally absorbing, since we have $\mathcal{P}^{\Psi^{(0)}}_{\text{detec}}(\Sigma, \tau) \geq 0$, detector engineering together with post-selection are specifically Bohmian and non-linear (they vary strongly with the wave functions used). Remarkably, the present protocol, while not based only on POVM, is physically acceptable and does not contradict any fundamental law.

Secondly, the objections related to Bell's theorem and no-signalling theorem are also answered. Indeed, in the 'Bell–Maudlin–Das experiment' described in Section 2, it is assumed that Bob can measure the distribution of dBB arrival times independently of knowledge of the spin of his particles. This makes sense in a procedure based solely on POVMs. But here, the design of the detectors and the method of post-selecting the data require knowledge of the wave function and therefore of the particle spin. Bob cannot establish the $\mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\Sigma, \tau)$ probability distribution without prior knowledge of the wave function and spin of the objects measured by Alice. Moreover, Bob could still decide to use a fixed setup such that the potentials $\text{Im}[\mathbf{A}_{eff}(x_i)]$ at the various points of the detector are uniquely defined for all the wave functions. If he is fixing the setup in such a way, then the various 'probabilities' $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega)$ for each elementary volume $\delta\Omega$ of the detector are not necessarily positive (this is reminiscent of the presence or loss and gain in the general dynamics). This is the case in particular if we have back-flow, as in DD's setup [45]. The detector is thus not always working correctly, and sometimes, part of the full detector emits particles instead of absorbing them. Moreover, if we consider the full Dirac current with convective and magnetic contributions in the configuration developed by DD [46] (see Equation (11)), we can easily prove (see Appendix F) that the full integrated signal $\mathcal{P}^{\Psi}_{\text{detec.,full}}(\Sigma, t) \simeq \eta \int d^2\Sigma_{\mathbf{x}} J_z^{\Psi_{\hat{\mathbf{s}}}}(\mathbf{x}, t)$ recorded by the detector without post-selection is given by

$$\mathcal{P}^{\Psi}_{\text{detec.,full signal}}(\Sigma, \tau) = \eta \frac{4L^3}{\lambda_0\sqrt{\pi}} \frac{\tau e^{-\frac{L^2}{1+\tau^2}}}{(1+\tau^2)^{5/2}}. \tag{58}$$

with $\eta \ll 1$ This is precisely (up to the η coefficient) the spin-independent distribution considered in [46] (see Equation (55)). In other words, because the signal is spin-independent, it cannot be used to violate non-signalling and sent a signal. Bob can, of course, decide to post-select data in order to reconstruct the first arrival dBB distribution (which is spin-dependent). But in order to do so, he must already know the spin measured by Alice in order to correlate the information!

To sum up and conclude, in this work, we have analyzed in detail DD's proposal [45,46] to measure particle arrival times using dBB theory. We have compared their work with the criticism made by GTZ [48,57]. To this end, we have studied in detail the notion of particle detection in quantum mechanics in the context of DBB theory. We concluded that both DD [45,46] and GTZ [48,57] were both right and wrong. More specifically, DD were right in believing that their specific Bohmian predictions involving the back-flow phenomenon could be observed. However, they were wrong to believe that the impact of detector physics could be neglected in their analyses. To be sure, the dBB $\mathcal{P}^{\Psi^{(0)}}_{\text{dBB}}(\Sigma, \tau)$ distribution is merely an ideal, theoretical formulation of particle flow in space-time. However, only the $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega)$ probability associated with absorption or, more generally, interaction phenomena makes sense in the context of a complete physical theory, and dBB theory as such is no exception to this fundamental fact.

In this work, we have clearly demonstrated the existence of two regimes: weak and strong coupling, corresponding to low and high detection or absorption efficiency, respectively. The strong coupling regime is the most natural, as it corresponds to the experimenter's natural expectation and it will lead to first arrival time distributions. In this regime, the detection probability $\mathcal{P}^{\Psi}_{\text{detec.}}(\delta\Omega)$ (which is a POVM in the non-relativistic regime) reduces approximately to the dBB probability, Equation (8) $\mathcal{P}^{\Psi}_{dBB}(\mathbf{x}, t) := |\mathbf{J}^{\Psi}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x})| d\Sigma_{\mathbf{x}}$, which is not a POVM. Since this is true only for some wave functions Ψ, there is no paradox. However, the method is highly invasive and strongly disturbs the wave function and dBB trajectories, which in general can lead to major technical difficulties. Therefore, it would be impossible to engineer complex time arrival detectors adapted to several wave functions $\Psi_1, \Psi_2, \ldots$, some presenting back-flow, others not. What is more, in the relativistic domain (requiring the Dirac equation) or in the Pauli equation regime for spin $1/2$ electrons, the notion of POVM is even more difficult to apply and we have seen that it is very hard to make a measurement approaching the dBB prediction because the spin magnetic current is generally undetected.

We deduced that the weak coupling regime was ultimately more appropriate for measuring the probability distribution predicted by DD. However, there is a price to pay. First of all, we must learn to give up the tenacious belief that only physics based on the notion of POVM has the right to be quoted. In fact, in the dBB framework, it is necessary to abandon this prejudice as soon as we seek to analyze trajectories (as we have shown with several examples). In keeping with Einstein's credo that 'only theory decides what is to be measured', we have shown that, in order to measure the $\mathcal{P}^{\Psi_{\hat{\mathbf{s}}}}_{\text{dBB}}(\Sigma, \tau)$ probability distribution predicted by DD [45,46] (and which depends on spin orientation $\hat{\mathbf{s}}$), in the weak coupling regime, we must necessarily carry out a post-analysis or post-selection to filter and classify the events detected, corresponding to first detection, second detection, etc. This point is fundamental and strongly contradicts GTZ's conclusions, which rely solely on the notion of POVM in their critical analysis.

In the end, however, we agree with GTZ [48,57] on two points. Firstly, the physics of the detector cannot be neglected in the analysis, as pointed out above (although this actually constitutes a weaker agreement with GTZ, who hastily concluded that no dBB arrival time measurement was possible based on their POVM analysis, whereas we demonstrate the opposite here); secondly, it is impossible within the framework of 'standard' non-modified quantum mechanics, in which the dBB theory is embedded (i.e., without questioning the foundations and without adding new physics at a 'sub-quantum' level), to contradict the results of Bell's theorem and violate the no-signalling condition[95]. In fact, an analysis obtained within the framework of the weak coupling regime shows that DD and Maudlin's proposal could not lead to such violations and to a hypothetical transmission of detectable supraluminal information. In our view, this is fortunate, as it means that the dBB theory is still completely empirically equivalent to the orthodox approach (in areas where these approaches are comparable). Of course, new physics is always possible [49], but it is by no means necessary here to agree with the results of DD and GTZ [45,46,48,57].

At a more fundamental level, our work should not be seen as an attempt to prove the correctness or truth of the dBB interpretation (contrary to hypotheses that have been discussed in the past [25]). As we have shown, the dBB theory fits in very well with the theoretical framework of quantum mechanics, and allows us to recover all its empirical content. In this field of measurement theory, Bohmian and orthodox quantum mechanics are empirically equivalent. Of course, the ontological clarity and absence of a measurement problem (i.e., the absence of a wavefunction collapse) is a great advantage for dBB theory. However, other ontological approaches could undoubtedly predict other trajectories and at the same time account for arrival time experiments. Also, as mentioned in the introduction, we could just add an arbitrary $\boldsymbol{\nabla} \times \mathbf{F}(\mathbf{x}, t)$ term to the local current in order to obtain a new Bohmian ontology. The general methodology here would be to develop detectors adapted to these new probability currents and dBB dynamics. This would clearly define a new distribution of probability for the arrival times and we see no reason or physical law which could prevent us from imagining detectors for such alternative theories. From a philosophical point of view, this leads us to be more modest about our preferred theories, while at the same time encouraging more comparative analysis of different approaches.

At a practical level, the greatest impact of our research will concern the possibility of constructing detectors for Bohmian arrival times in any situation, such as those involving back-flow. From a practical point of view, this will give a positive answer to a problem that has been debated since the 1990s, while generalizing it to problems involving spin particles.

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Appendix A. POVM and dBB Theory

In order to describe a quantum measurement, we start with a subsystem S wave function $|\psi_0^S\rangle \in \mathcal{H}^S = \sum_n c_n |n^S\rangle$ expanded in a complete basis $|n^S\rangle$ and initially uncoupled to a pointer M wave function $|\Phi_0^M\rangle \in \mathcal{H}^M$. During a generalized von Neumann measurement, the interaction between S and M is characterized by an unitary evolution operator $\hat{U}^{SM}$ acting on the full Hilbert space $\mathcal{H}^S \otimes \mathcal{H}^M$, and it leads to entanglement:

$$|\Psi_0^{SM}\rangle = |\psi_0^S\rangle|\Phi_0^M\rangle = (\sum_n c_n |n^S\rangle)|\Phi_0^M\rangle \xrightarrow{\hat{U}^{SM}} |\Psi_t^{SM}\rangle = \sum_n c_n |\Psi_n^{SM}\rangle \quad \text{(A1)}$$

where we have $\langle n^S|m^S\rangle = \delta_{nm} \to \langle \Psi_n^{SM}|\Psi_m^{SM}\rangle = \delta_{nm}$. We stress that in standard projective von Neumann measurements, $|\Psi_t^{SM}\rangle = |n^S\rangle|\Phi_t^M\rangle$, but here we consider a more general case. In the dBB framework, the physical probabilities are defined in the configuration space, and therefore, if q is the spatial coordinates for the S sub-system and ξ the spatial coordinates for the M sub-system, we initially have the wave function $\Psi_0^{SM}(q, \xi)$, which evolves as $\Psi^{SM}(q, \xi, t)$ at time t. We can thus rewrite

$$\Psi_0^{SM}(q, \xi) = \psi_0^S(q)\Phi_0^M(\xi) = (\sum_n c_n \psi_n^S(q)\Phi_0^M(\xi) \xrightarrow{\hat{U}^{SM}} \Psi^{SM}(q, \xi) = \sum_n c_n \Psi_n^{SM}(q, \xi, t) \quad \text{(A2)}$$

In order for an observer to legitimately speak of a quantum measurement, the physical variables of the pointer, which in the dBB framework are necessarily the coordinates ξ, M must move by an observable quantity in such a way that we can experimentally distinguish the different eigenvalues n, m, etc., associated with the different states $|n^S\rangle$, $|m^S\rangle$, etc. To do so in a non-ambiguous way, we must be sure that the different waves functions $\Psi_n^{SM}(q,\xi,t)$, $\Psi_m^{SM}(q,\xi,t)$, etc., are non-overlapping in the $\xi-$configuration space. In other words, these wave functions $\Psi_n^{SM}(q,\xi,t)$, $\Psi_m^{SM}(q,\xi,t)$, etc., must have finite disjoint supports Δ_n, Δ_m, etc., in the $\xi-$configuration space such that

$$|\Psi_n^{SM}(q,\xi,t)|^2 \cdot |\Psi_m^{SM}(q,\xi,t)|^2 = 0 \quad \text{if } n \neq m. \tag{A3}$$

The probability $\mathcal{P}_n$ of finding the pointer in the zone Δ_n of the $\xi-$configuration space is therefore given by

$$\begin{aligned}\mathcal{P}_n &= \int dq \int_{\Delta_n} d\xi |\Psi^{SM}(q,\xi,t)|^2 = |c_n|^2 \int dq \int_{\Delta_n} d\xi |\Psi_n^{SM}(q,\xi,t)|^2 \\ &= |c_n|^2 \int dq \int d\xi |\Psi_n^{SM}(q,\xi,t)|^2 = |c_n|^2 \langle \Psi_n^{SM}|\Psi_n^{SM}\rangle = |c_n|^2\end{aligned} \tag{A4}$$

It can be rewritten as

$$\mathcal{P}_n = \langle \Psi_t^{SM}|\hat{\Pi}_n^S|\Psi_t^{SM}\rangle = \langle \Psi_0^{SM}|(\hat{U}^{SM})^{-1}\hat{\Pi}_n^S\hat{U}^{SM}|\Psi_0^{SM}\rangle \tag{A5}$$

where $\hat{\Pi}_n^S = \int_{\Delta_n} d\xi|\xi\rangle\langle\xi|$ is the sum of projectors in the cell Δ_n. It is equivalent to

$$\mathcal{P}_n = \langle \psi_0^S|\hat{O}_n^S|\psi_0^S\rangle \tag{A6}$$

where $\hat{O}_n^S$ is a POVM defined by

$$\hat{O}_n^S = \langle \Phi_0^M|(\hat{U}^{SM})^{-1}\hat{\Pi}_n^S\hat{U}^{SM}|\Phi_0^M\rangle \tag{A7}$$

that explicitly reads

$$\hat{O}_n^S = \iint dq_f dq_0 |q_f\rangle\langle q_0|\mathcal{M}^{SM}(q_f,q_0) \tag{A8}$$

with

$$\mathcal{M}^{SM}(q_f,q_0) = \textstyle\int_{\Delta_n} d\xi \iiint dq d\xi_f d\xi_0 \Phi_0^{M^*}(\xi_f)\Phi_0^M(\xi_0)K^{SM}(q,\xi;q_0,\xi_0)K^{SM^*}(q,\xi;q_f,\xi_f) \tag{A9}$$

and the propagator $K^{SM}(q,\xi;q_0,\xi_0) = \langle q,\xi|\hat{U}^{SM}|q_0,\xi_0\rangle$. We stress that we have the condition $(\mathcal{M}^{SM}(q_0,q_f))^* = \mathcal{M}^{SM}(q_f,q_0)$ that implies the self-adjointedness $\hat{O}_n^S = (\hat{O}_n^S)^\dagger$ required in the definition of a POVM. To complete our definition of a dBB POVM, we observe that we have $\sum_n \hat{O}_n^S = \hat{I}$, and $\langle \psi_0^S|\hat{O}_n^S|\psi_0^S\rangle \geq 0$ whatever $|\psi_0^S\rangle$.

The previous analysis was limited to spinless systems. If we consider systems of particles with spins, we replace the wave function $\Psi^{SM}(q,\xi,t)$ by $\Psi_{i^S,j^M}^{SM}(q,\xi,t)$, where i^S and j^M are discrete spin indices for the S and M subsystems. From Equation (A1), we still have $\mathcal{P}_n = \langle \psi_0^S|\hat{O}_n^S|\psi_0^S\rangle$, where the POVM reads

$$\hat{O}_n^S = \sum_{i_f^S,i_0^S} \iint dq_f dq_0 |q_f\rangle\langle q_0|\mathcal{M}_{i_f^S,i_0^S}^{SM}(q_f,q_0) \tag{A10}$$

with $\mathcal{M}_{i_f^S,i_0^S}^{SM}(q_f,q_0) = (\mathcal{M}_{i_0^S,i_f^S}^{SM}(q_0,q_f))^*$ such that

$$\begin{aligned}\mathcal{M}_{i_f^S,i_0^S}^{SM}(q_f,q_0) = \sum_{i^S,j^M,j_f^M,j_0^M} \int_{\Delta_n} d\xi \iiint dq d\xi_f d\xi_0 {\Phi_{0,j_f^M}^M}^*(\xi_f)\Phi_{0,j_0^M}^M(\xi_0) \\ K_{i^S,j^M|i_0^S,j_0^M}^{SM}(q,\xi;q_0,\xi_0){K^{SM}}^*_{i^S,j^M|i_f^S,j_f^M}(q,\xi;q_f,\xi_f)\end{aligned} \tag{A11}$$

and $K_{i^S,j^M|i_0^S,j_0^M}^{SM}(q,\xi;q_0,\xi_0) = \langle q,\xi,i^S,j^M|\hat{U}^{SM}|q_0,\xi_0,i_0^S,j_0^M\rangle$.

Appendix B. Scattering by an Absorbing Fabry–Perot Detector

We model the absorbing medium by a set of atoms with individual extinction cross-section $\sigma_{ext} = \frac{4\pi}{k}\text{Im}[f_0]$, where $k = mv$ is the momentum of the incident particle and f_0 the (complex-valued) inelastic scattering amplitude associated with a spherically symmetric wave $\Psi_s \simeq \frac{f_0 e^{ikr}}{r}$. In the regime where the density N of the absorbing atom is not too high, the wave function propagating in the medium obeys the equation

$$\boldsymbol{\nabla}^2\Psi + (k^2 + 4\pi f_0 N)\Psi = \boldsymbol{\nabla}^2\Psi + 2m(E - V_{eff})\Psi = 0 \quad \text{(A12)}$$

corresponding to a medium having an effective propagation index $n_{eff} = \sqrt{[1 + \frac{4\pi f_0 N}{k^2}]}$, i.e., to a medium characterized by an effective (complex-valued) potential $V_{eff} = -\frac{4\pi f_0 N}{2m}$ with $\text{Im}[V_{eff}] = -N\frac{k}{2m}\sigma_{ext} < 0$. The time-dependent Schrödinger evolution in this potential $i\partial_t\Psi_t = [\frac{-\boldsymbol{\nabla}^2}{2m} + V_{eff}]\Psi_t$ leads to the conservation law Equation (18) containing a dissipation term $2\text{Im}[V_{eff}]|\Psi|^2$ due to the violation of unitarity in this effective model.

We now consider a plane wave incident on such a medium supposed to be confined in a (Fabry–Perot) slab between the parallel surfaces $z = 0$ and $z = d$. The incident plane wave reads $\Psi^{(0)} = e^{ik_1 z}e^{ik_x x}e^{-i\frac{k^2}{2m}t}$, where $k_x = k\sin\theta$, $k_1 = k\cos\theta = \sqrt{k^2 - k_x^2}$ are, respectively, the x and z wavevector components, θ is the incidence angle, and k the wavevector associated with the kinetic energy $\frac{k^2}{2m}$. In the presence of the Fabry–Perot slab, the wave functions in the regions $z < 0$ and $z > d$ read, respectively:

$$\begin{aligned} \Psi_< &= (e^{ik_1 z} + Re^{-ik_1 z})e^{ik_x x}e^{-i\frac{k^2}{2m}t} \\ \Psi_> &= Te^{ik_1 z}e^{ik_x x}e^{-i\frac{k^2}{2m}t} \end{aligned} \quad \text{(A13)}$$

Fresnel's reflection and transmission coefficients R, T are given by standard formulas:

$$\begin{aligned} R &= \frac{r}{1 - r^2 e^{i\delta}}(1 - e^{i\delta}) \\ T &= \frac{k_2}{k_1}\frac{te^{i\delta/2}}{1 - r^2 e^{i\delta}} \end{aligned} \quad \text{(A14)}$$

where $r = \frac{k_1 - k_2}{k_1 + k_2}$, $t = 2\frac{k_1}{k_1 + k_2}$ are the single interface Fresnel's coefficients (with $k_2 = \sqrt{k_1^2 + 4\pi f_0 N}$ the $z-$wavevector component in the absorbing medium), and $\delta = 2k_2 d$ is a complex phase shift.

In the medium, for $0 < z < d$, the wave function reads $\Psi_{\text{inside}} = (Ce^{ik_2 z} + De^{-ik_2 z})e^{-ik_x x}e^{-i\frac{k^2}{2m}t}$ with

$$\begin{aligned} C &= \frac{1}{2}[1 + R + \frac{k_1}{k_2}(1 - R)] \\ D &= \frac{1}{2}[1 + R - \frac{k_1}{k_2}(1 - R)] \end{aligned} \quad \text{(A15)}$$

The dBB trajectories can be computed in the different regions using the probability current $\mathbf{J}^\Psi = \text{Im}[\Psi^\dagger\boldsymbol{\nabla}\Psi]/m$. We have for $z < 0$

$$\begin{aligned} J_z^\Psi &= \frac{k_1}{m}(1 - |R|^2), \\ J_x^\Psi &= \frac{k_x}{m}(1 + |R|^2 + 2|R|\cos(2k_1 z - arg(R))) \end{aligned} \quad \text{(A16)}$$

leading to the trajectory equation $\frac{dz}{dx} = \frac{J_z^\Psi}{J_x^\Psi}$ in the interfering region:

$$\frac{dz}{dx} = \cot\theta \frac{1 - |R|^2}{1 + |R|^2 + 2|R|\cos(2k_1 z - arg(R))} \tag{A17}$$

The mean trajectory, around which the particle oscillates, obeys the equation $\frac{dz}{dx} = \cot\theta \frac{1-|R|^2}{1+|R|^2}$, which has a geometrical interpretation, as shown in [76] and Figure 2. In the slab for $0 < z < d$, we similarly obtain

$$\frac{dz}{dx} = \frac{\frac{k'_2}{kx}(|C|^2 e^{-2k''_2 z} - |D|^2 e^{2k''_2 z}) + 2\frac{k''_2}{kx}|DC|\sin(\xi)}{|C|^2 e^{-2k''_2 z} - |D|^2 e^{2k''_2 z} + 2|DC|\cos(\xi)} \tag{A18}$$

with $\xi = 2k'_2 z + arg(D) - arg(C)$, $k'_2 = \text{Re}[k_2]$, $k''_2 = \text{Im}[k_2]$. This defines a very complicated motion [61,76]. In the transmitted region $z > d$, we have $\frac{dz}{dx} = \cot\theta$, as it should be.

From Equation (18), we can calculate the difference between the probability current flows through the surfaces $z = 0$ and $z = d$:

$$\mathcal{I}_{z=0} - \mathcal{I}_{z=d} = \Sigma \cdot v\cos\theta[1 - |R|^2 - |T|^2] = \Sigma \cdot N\sigma_{ext} v \int_{z=0}^{z=d} dz|\Psi|^2(x,y,z) \tag{A19}$$

with $\mathcal{I}_{z=0} = \int_\Sigma dxdy J_z^\Psi(x,y,z=0)$, $\mathcal{I}_{z=d} = \int_\Sigma dxdy J_z^\Psi(x,y,z=d)$ and Σ is the whole lateral surface of the slab. Importantly, $\Sigma \cdot N\sigma_{ext} v \int_{z=0}^{z=d} dz|\Psi|^2(x,y,z)$ represents the probability of absorption by the slab per unit time, i.e., it defines the fraction of incident particles trapped by the detector per unit time or the arrival time probability density $\mathcal{P}^\Psi(\Sigma,\tau)$ (here, the situation is time-independent).

Two extreme regimes are relevant for the present discussion. First, in the weak coupling regime with a low density N and small cross-section σ_{ext}, we have a semi-transparent medium $r \simeq 0$ implying $R \simeq 0$ and $T \simeq e^{ik_2 d}$, i.e., $|T|^2 \simeq e^{-\frac{N\sigma_{ext} d}{\cos\theta}}$. Thus, we obtain

$$\mathcal{P}^\Psi(\Sigma,\tau) \simeq \Sigma \cdot N d\sigma_{ext} v = \Sigma \cdot N d\sigma_{ext}|\mathbf{J}^{\Psi^{(0)}}| \tag{A20}$$

which is proportional to the norm of the initial probability current and does not depend on the incidence angle θ. In the second 'strong absorption' regime, we assume $\text{Im}[\delta] \gg 1$ and thus $|T|^2 \simeq |t|^4 |\frac{k_2}{k_1}|^2 e^{-2\text{Im}[\delta]} \to 0$ and $|R|^2 \to |r|^2$. We thus obtain the arrival time density of probability:

$$\mathcal{P}^\Psi(\Sigma,\tau) \simeq \Sigma \cdot v\cos\theta[1 - |r|^2] = \Sigma \cdot J_z^{\Psi^{(0)}}[1 - |r|^2] \tag{A21}$$

Note that in the limit where the medium is strongly absorbing, we have $r \to -1$, and therefore, the probability of absorbing a particle tends to vanish as well.

Appendix C. Perfectly Matched Layer Detectors: General Derivations

We write

$$\Psi^{(abs)} = e^{ik_z f(z)} e^{i\mathbf{k}_{||}\cdot\mathbf{x}_{||}} e^{-i\frac{k^2}{2m}t}. \tag{A22}$$

We immediately check that we have

$$\frac{1}{f'(z)}\partial_z\left(\frac{1}{f'(z)}\partial_z e^{ik_z f(z)}\right) = -k_z^2 e^{ik_z f(z)} \tag{A23}$$

or equivalently

$$\partial_z^2\Psi^{(abs)} - \frac{f''(z)}{f'(z)}\partial_z\Psi^{(abs)} + k_z^2(f'(z))^2\Psi^{(abs)} = 0. \tag{A24}$$

The first-order derivative can be eliminated by using the condition $\frac{1}{f'(z)}\partial_z e^{ik_z f(z)} = ik_z e^{ik_z f(z)}$, and therefore, we have

$$\partial_z^2\Psi^{(abs)} - ik_z f''(z)\Psi^{(abs)} + k_z^2(f'(z))^2\Psi^{(abs)} = 0. \tag{A25}$$

Writing $k_z f(z) = k_z z + i\int_{-\infty}^{z} dz'\chi(z')$ ($\chi(z)$ defining the absorption of the system), $f'(z) = 1 + i\frac{\chi(z)}{k_z}$, $f''(z) = i\frac{\chi'(z)}{k_z}$, $(\partial_x^2 + \partial_y^2)\Psi^{(abs)} = -\mathbf{k}_{||}^2\Psi^{(abs)}$ we thus deduce

$$\boldsymbol{\nabla}^2\Psi^{(abs)}(z,\mathbf{x}_{||},t) + 2m(E - V_{eff}(z))\Psi^{(abs)}(z,\mathbf{x}_{||},t) = 0 \tag{A26}$$

with the effective complex potential

$$V_{eff}(z) = \frac{\chi^2(z) - \chi'(z)}{2m} - i\chi(z)\frac{k_z}{m}. \tag{A27}$$

Appendix D. Perfectly Matched Layer Detectors: A Particular Model

In relation with Appendix C, we now impose

$$\chi(z) = \chi_0 \cdot [\theta(-z)e^{-az^2} + \theta(z)\theta(d-z) + \theta(z-d)e^{-a(z-d)^2}] \tag{A28}$$

where $1/\sqrt{a}$ defines a characteristic length over which the potential V_{eff} rises continuously around the two zones $z \simeq 0$ and $z \simeq d$. With this choice, the function $f(z) = z + \frac{i}{k_z}\int_{-\infty}^{z} dz'\chi(z')$ in Equation (A22) reads

$$\begin{aligned} f(z) &= z + \frac{i}{2k_z}\chi_0\sqrt{(\frac{\pi}{a})}[1 + erf(\sqrt{a}z)] \quad &\text{if } z \leq 0 \\ f(z) &= z + \frac{i}{k_z}\chi_0[z + \frac{1}{2}\sqrt{(\frac{\pi}{a})}] \quad &\text{if } 0 \leq z \leq d \\ f(z) &= z + \frac{i}{k_z}\chi_0[d + \frac{1}{2}\sqrt{(\frac{\pi}{a})}(1 + erf(\sqrt{a}(z-d)))] \quad &\text{if } d \leq z. \end{aligned} \tag{A29}$$

where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x dz e^{-z^2}$. From Equation (A27), we deduce V_{eff} with $\chi'(z) = -2\chi_0 a \cdot [z\theta(-z)e^{-az^2} + (z-d)\theta(z-d)e^{-a(z-d)^2}]$. As shown in Figure 3, the potential is a continuous function of z (with slope discontinuities at $z = 0$ and $z = d$ arising from the second-order derivative $\chi''(z)$).

In analogy with Equation (23), we define the probability $\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t)$

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t) = 2\Sigma\frac{k}{m}\int_{-\infty}^{+\infty} dz\chi(z)e^{-2\int_{-\infty}^{z} dz\chi(z)} \tag{A30}$$

which reads

$$\mathcal{P}^{\Psi}_{\text{detec.}}(\Sigma, t) = \Sigma\frac{k}{m}[e^{-\xi}(1 - e^{-2\chi_0 d}) + \xi F(\xi) + e^{-2\chi_0 d}\xi G(\xi)] \tag{A31}$$

with $\xi = \chi_0\sqrt{(\frac{\pi}{a})}$, $F(\xi) = \frac{2}{\sqrt{\pi}}\int_{-\infty}^{0} dz e^{-z^2}e^{\xi(1+erf(z))}$, and $G(\xi) = \frac{2}{\sqrt{\pi}}\int_0^{+\infty} dz e^{-z^2}e^{\xi(1+erf(z))}$. In the limit $\xi \to 0$ (i.e. $a \to +\infty$), we have $F(0) = G(0) = 1$ and we recover the result, Equation (25).

The present analysis for a detector adapted to a plane wave $\propto e^{+ik_z z}$ with $k_z > 0$

can be used to define the absorbing medium corresponding to an incident plane wave propagating in the opposite direction, i.e., $\propto e^{-ik_z z}$. For this, we write the previous solution, Equation (A22), as $\Psi^{(abs)} = e^{ik_z z}e^{-F(z)}e^{i\mathbf{k}_{||}\cdot\mathbf{x}_{||}}e^{-i\frac{k^2}{2m}t}$ and we define the counterpropagating wave as the one obtained under the transformation $z \to d - z$. We write the new wave:

$$\tilde{\Psi}^{(abs)} = e^{-ik_z z}e^{-F(d-z)}e^{i\mathbf{k}_{||}\cdot\mathbf{x}_{||}}e^{-i\frac{k^2}{2m}t} = e^{-ik_z\tilde{f}(z)}e^{i\mathbf{k}_{||}\cdot\mathbf{x}_{||}}e^{-i\frac{k^2}{2m}t} \quad \text{(A32)}$$

where we omitted a phase constant. We have the transformation $F(Z) = \int_{-\infty}^{Z} dz\chi(z) \to F(d-Z) = \int_{Z}^{+\infty} dz\chi(d-z)$, and therefore, $\tilde{f}(Z) = Z - i\frac{1}{k_z}\int_{Z}^{+\infty} dz\chi(d-z)$, $\tilde{f}'(Z) = 1 + i\frac{1}{k_z}\chi(d-Z)$, $\tilde{f}''(Z) = -i\frac{1}{k_z}\frac{d}{dz}\chi(z)|_{z=d-Z}$. Finally, we deduce

$$\partial_z^2\tilde{\Psi}^{(abs)} + ik_z\tilde{f}''(z)\tilde{\Psi}^{(abs)} + k_z^2(\tilde{f}'(z))^2\tilde{\Psi}^{(abs)} = 0. \quad \text{(A33)}$$

and therefore,

$$\boldsymbol{\nabla}^2\tilde{\Psi}^{(abs)}(z,\mathbf{x}_{||},t) + 2m(E - \tilde{V}_{eff}(z))\tilde{\Psi}^{(abs)}(z,\mathbf{x}_{||},t) = 0 \quad \text{(A34)}$$

with the new effective complex potential adapted to the counterpropagative wave:

$$\tilde{V}_{eff}(Z) = \frac{\chi^2(d-Z) - \frac{d}{dz}\chi(z)|_{z=d-Z}}{2m} - i\chi(d-Z)\frac{k_z}{m}. \quad \text{(A35)}$$

With the example of Equation (A28), we have $\chi(d-z) = \chi(z)$, and $\frac{d}{dz}\chi(z)|_{z=d-Z} = -\chi'(Z)$, and therefore,

$$\tilde{V}_{eff}(Z) = \frac{\chi^2(Z) + \chi'(Z)}{2m} - i\chi(Z)\frac{k_z}{m}. \quad \text{(A36)}$$

This can be compared with Equation (A27) for the choice Equation (A28). The two effective potentials differ by the sign in front of $\chi'(z)$.

Appendix E. Back-Flow with Two Plane Waves

From

$$\Psi^{(0)}(\mathbf{x},t) = (e^{i\mathbf{k}_1\cdot\mathbf{x}} - \frac{1}{2}(1 + \frac{k_{1z}}{k_{2z}})e^{i\mathbf{k}_2\cdot\mathbf{x}})e^{-iEt} \quad \text{(A37)}$$

obtained with $\alpha = \alpha_{min} = -\frac{1}{2}(1 + \frac{k_{1z}}{k_{2z}})$ in Equation (28), we deduce the probability current $z-$component at the point $\mathbf{x}_0 = 0$:

$$J_z^{\Psi^{(0)}}(\mathbf{x}_0 = 0) = \frac{k_{2z}}{m}[|\alpha|^2 - (1 + \frac{k_{1z}}{k_{2z}})|\alpha| + \frac{k_{1z}}{k_{2z}}] = -\frac{k_{2z}}{4m}(1 - \frac{k_{1z}}{k_{2z}})^2 < 0. \quad \text{(A38)}$$

Similarly, we have $|\Psi^{(0)}(\mathbf{x}_0 = 0)|^2 = (1 - |\alpha|)^2 = \frac{1}{4}(1 - \frac{k_{1z}}{k_{2z}})^2$. This allows us to define an effective ($z-$component) wavevector:

$$k_{eff,z}(\mathbf{x}_0 = 0) = m\frac{J_z^{\Psi^{(0)}}(\mathbf{x}_0 = 0)}{|\Psi^{(0)}(\mathbf{x}_0 = 0)|^2} = -k_{2z}. \quad \text{(A39)}$$

We can easily deduce the other components of $\mathbf{k}_{eff}(\mathbf{x}_0 = 0)$. In particular, the quantum potential reads

$$Q^{\Psi^{(0)}}(\mathbf{x}_0 = 0) = \frac{-\boldsymbol{\nabla}^2|\Psi^{(0)}|}{2m|\Psi^{(0)}|}|_{\mathbf{x}_0=0} = \frac{(\mathbf{k}_1 - \mathbf{k}_2)^2}{m}\frac{(1 + \frac{k_{1z}}{k_{2z}})}{(1 - \frac{k_{1z}}{k_{2z}})^2} \quad \text{(A40)}$$

Appendix F. The Full Arrival Time Distribution with Non-Efficient Detectors

We start with Equation (11) and consider the full signal $\mathcal{P}^{\Psi}_{\text{detec., full signal}}(\Sigma,t) \simeq \eta \int d^{\Sigma}_{\mathbf{x}} J^{\Psi_{\hat{\mathbf{s}}}}_{z}(\mathbf{x},t)$, which reads

$$\mathcal{P}^{\Psi}_{\text{detec.,full signal}}(\Sigma,t) = \eta \int_0 R d\rho\rho \oint d\varphi \Big(|\Phi(\rho,z=L,t)|^2 \frac{\partial_z S(\rho,z=L,t)}{m} + \frac{\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\varphi}}}{2m}\partial_\rho |\Phi(\rho,z=L,t)|^2\Big). \quad \text{(A41)}$$

Since we are working in the weak coupling regime, the current $J^{\Psi_{\hat{\mathbf{s}}}}_{z}(\mathbf{x},t)$ can be negative, and this is associated with back-flow. The contributions of back-flow are negative in Equation (A43). However, it is not difficult to see that the second term of the integral associated with the spin-magnetic current vanishes. This is trivially so for the longitudinal case, where $\hat{\mathbf{s}} = \pm\hat{\mathbf{z}}$. For the transverse cases, it is sufficient to consider the case $\hat{\mathbf{s}} = +\hat{\mathbf{x}}$, the other cases being equivalent due to rotational invariance of the problem. If $\hat{\mathbf{s}} = +\hat{\mathbf{x}}$, we have

$$\oint d\varphi \frac{\hat{\mathbf{s}}\cdot\hat{\boldsymbol{\varphi}}}{2m}\partial_\rho |\Phi(\rho,z=L,t)|^2 = \oint d\varphi \frac{\cos(\varphi)}{2m}\partial_\rho |\Phi(\rho,z=L,t)|^2 = 0 \quad \text{(A42)}$$

as required. Therefore, we have

$$\mathcal{P}^{\Psi}_{\text{detec.,full signal}}(\Sigma,t) = \eta \int_0 R d\rho\rho \oint d\varphi |\Phi(\rho,z=L,t)|^2 \frac{\partial_z S(\rho,z=L,t)}{m}. \quad \text{(A43)}$$

which is spin-independent and considers only the convective current. We have recovered DD's result [46] (see Equation (55)):

$$\mathcal{P}^{\Psi}_{\text{detec.,full signal}}(\Sigma,\tau) = \eta \frac{4L^3}{\lambda_0\sqrt{\pi}} \frac{\tau e^{-\frac{L^2}{1+\tau^2}}}{(1+\tau^2)^{5/2}}. \quad \text{(A44)}$$

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