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Do some virtual bound states carry torsion trace?

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This paper presents theoretical arguments that certain virtual bound states carry the trace component of affine torsion. The motivation for this work is that Einstein–Cartan theory, which extends general relativity by including torsion to model intrinsic angular momentum, is becoming more credible. We are not aware of any situation for which there is evidence or substantial argument for the presence of torsion trace, except in the continuum theory of edge dislocations in crystals. The main evidence for the hypothesis consists of analogies between the structure of virtual bound states and (a) geometry of dislocations in crystal lattices, which are modeled with torsion; and (b) modeling of intrinsic angular momentum by torsion in Einstein–Cartan theory and the theory of micro-elasticity. The work focuses on conjectured presence of torsion in para-positronium, which intermediates annihilation of an electron and a positron with opposite z-spins. If the virtual bound state carries torsion, then the local law of conservation of angular momentum can hold over the spacelike separation during annihilation.

Keywords: Torsion; virtual bound state; positronium; spin; angular momentum; dislocations; Einstein–Cartan theory.

1. Introduction

This paper presents theoretical arguments to support the hypothesis that certain virtual bound states carry the trace component of affine torsion. The motivation for this work is that Einstein–Cartan (EC) theory, which extends general relativity (GR) by including torsion to model intrinsic angular momentum (AM), is becoming more credible [8, 20, 23]. We are not aware of any situation for which there is evidence or substantial argument for the presence of torsion trace, except in the continuum theory of edge dislocations in crystals. The evidence presented here is of two types that match the structure of virtual bound states: (a) angular momentum

and conservation thereof are represented as torsion in EC theory and in continuum theories of exchange of intrinsic and orbital AMs; and (b) screw and edge dislocations in crystal lattices, which are represented as torsion, illustrate the geometry of torsion.

We define "intrinsic" angular momentum to be the AM on too small a scale to be modeled as orbital AM in a particular classical model; that is, intrinsic AM cannot be modeled by displacement fields (in solids) or velocity fields (in fluids). Intrinsic AM includes but is not limited to quantum spin.

1.1. *Torsion and angular momentum*

This paper contains enough information about the role of affine torsion in physics to support the main hypothesis of this work. This work is not a comprehensive review of torsion in physics. It discusses affine torsion as a classical tensor field defined in Élie Cartan's general theory of affine connections [2, 4–7, 13, 14, 24] and as identified with intrinsic AM in EC theory.

Below are main events in the use of torsion to represent intrinsic AM.

(a) Early in the 20th century, Eugene Cosserat pioneered the field of micro-elasticity in continuum mechanics, which allows points to have torque and classical intrinsic AM. Cosserat showed that exchange of intrinsic and orbital AMs requires an intrinsic AM tensor and a nonsymmetric momentum tensor during the exchange. This is the earliest example of modeling intrinsic AM with affine torsion of which we are aware [9].
(b) In the 1920s, Élie Cartan developed the theory of affine connections, in which Riemannian curvature (rotational curvature) and affine torsion (translational curvature) are natural companions in Riemann–Cartan affine geometry. Several times in the 1920s, Cartan urged Einstein to extend GR to include torsion. In 1929, Einstein replied that he did not understand Cartan's description of torsion, and he did not know what physical process torsion would model [10].
(c) In the early 1960s, EC theory was essentially completed with the interpretation of it as a gauge theory of the Poincaré group, and by identifying torsion as intrinsic AM [12, 22]. EC theory is a modest extension of GR that enables conservation of AM to include exchange of intrinsic and orbital AMs, which GR cannot do because of its symmetric momentum tensor [8].

Torsion has applications in EC theory that are important in gravitational theory but are not relevant to our current purpose. (a) Repulsive torsion force interacts with ultra-high squared density of intrinsic AM to eliminate gravitational singularities. (b) Current cosmological models ignore intrinsic AM that comes from two sources: fluid turbulence [22–24], and exchange of intrinsic and orbital AMs across scales from galactic clusters ($\sim 10^{23}$ m) to orbits of moons ($\sim 10^9$ m).

Capozziello *et al.* discuss several methods of decomposition of torsion into irreducible representations, and alternative applications of torsion, including theories in which torsion is defined as the gradient of a scalar field [3].

Jiminez *et al.* assert that three classes of gravitational gauge theories are acceptable: (a) GR, (b) flat space theories based on torsion, and (c) flat space theories based on nonmetricity [11].

According to [8], only three known theories satisfy all the empirical tests of gravitational theories: GR, EC theory, and teleparallel gravity, which uses torsion in a flat space.

We prefer EC theory, for four reasons:

(a) EC theory makes the least invasive changes to GR that enable exchange of intrinsic and orbital AMs. In the absence of intrinsic AM, EC theory is identical to GR.
(b) Torsion appears in EC theory exactly where it appears in Cartan's master structure equations for affine differential geometry; EC theory has no ad-hoc torsion terms.
(c) EC theory can be derived from GR with no additional assumptions [20].
(d) EC theory has been more thoroughly studied than the alternatives.

1.2. *Torsion and geometry of lattice defects*

Physical theories that use affine torsion to model continuum distributions of defects in affine lattices provide insight into the geometry of torsion. Since the 1950s, the continuum theory of dislocations models distributions of dislocations and disclinations with torsion and Riemannian curvature, respectively [1, 15]. The discrete theory of dislocations provides the best intuitive geometrical model of discrete torsion.

1.3. *Results*

This paper uses insights from EC theory and the theory of dislocations to support the conjecture that certain virtual bound states carry torsion trace that enables micro-scale conservation of AM along spacelike separations. This work focuses on annihilation of two spin-1/2 particles with opposite z-spins that are connected over a spacelike separation by a virtual bound state, for example, annihilation of an electron and a positron with opposite z-spins, mediated by para-positronium.

The annihilation of two Dirac particles joined at the ends of their worldlines by a virtual bound state has these analogs in the fields discussed here:

• In the discrete theory of dislocations, the configuration that is analogous to para-positronium is two screw dislocations whose dislocation lines end, and the ends are joined by an edge dislocation.
• In the continuum theory of dislocations, the analogous configuration is two densities of antisymmetric torsion (screw dislocations) of opposite helicities, such

that, wherever the two densities decline together, torsion trace appears so that the conservation law of dislocation lines is satisfied.

- In the Einstein–Cartan theory, the analogous configuration is two densities of antisymmetric torsion (Dirac particles) of opposite helicities with spacelike separation, such that, wherever the two densities decline together, torsion trace appears so that conservation of AM is satisfied. In the Riemann–Cartan geometry, terms quadratic in torsion appear, so that zero net torsion does not imply zero torsion.

- The law of conservation of dislocation lines and the law of conservation of total AM in EC theory are both applications of the contracted first Bianchi identity of Riemann–Cartan geometry.

Models of dislocations in terms of torsion are presented in Sec. 2. The model of intrinsic AM in terms of torsion in EC theory is described in Sec. 3. The model of virtual bound states in terms of holonomy and torsion is described in Sec. 4. Speculations about future developments related to this work are discussed in Sec. 5. The paper is concluded in Sec. 6. The interpretations of torsion in Riemann–Cartan geometry, the theory of dislocations, and EC theory are compared in Table A.1 in Appendix A.

2. Dislocations in Lattices Illustrate Geometry of Torsion

The theory of dislocations in affine lattices provides a well-accepted physical application of torsion [16, 21, 25]. We begin with an affine defect that is not a dislocation and is more familiar to physicists than torsion.

2.1. *Disclinations*

A disclination is a discrete unit of Riemannian curvature. Figure 1 is a graphical representation of a disclination with less than 2π rad around its vertical centerline.
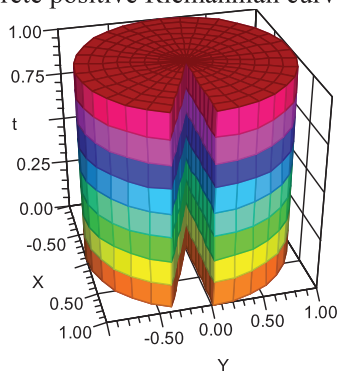

Fig. 1. Graphical representation of disclination in three dimensions.

In a real disclination, the two faces of the angular deficit are joined together. This dispiration is a discrete version of Riemannian curvature $R_{\mu\nu\mu\nu}$. To construct a two-dimensional version of a disclination, excise a wedge of material of angle θ from a flat disk.

This yields a horizontal slice of the 3D graphic in Fig. 1. If you join the edges of the cut, the disk becomes a conical surface. The angular deficit at the apex of the cone is θ, and the area deficit of a disk centered at the apex of radius r is $1/2\theta r^2$. The tip of the cone has a Dirac delta of Riemannian curvature of magnitude θ. The remainder of the cone has no intrinsic curvature.

To construct a surface with positive Riemannian curvature, remove many small angular wedges whose centers are distributed around the disk. The limit of many small wedges yields a surface with positive curvature.

A disclination that has more than 2π rad at the center represents discrete negative curvature. Negative Ricci curvature can occur GR in the presence of a positive cosmological constant, which introduces repulsive acceleration. A distribution of negative disclinations provides a discrete geometric model for dark energy via the cosmological constant.

2.2. *Discrete dislocations in lattices*

In crystalline materials, a dislocation exists in a neighborhood where traversing a loop that would be closed in a perfect lattice is not closed. For example, in a perfect cubic lattice, a loop with 10 lattice steps in direction $+x$, 10 in direction $+y$, 10 in direction $-x$, and 10 in direction $-y$, is closed. If the loop path surrounds a dislocation line, then the path does not close. We borrow from materials science the term "Burghers vector" to denote the "failure-to-close vector" when we traverse a loop that would otherwise close in a perfect lattice, but surrounds a dislocation in our defected lattice.

Two types of dislocations are most common in materials: screw dislocations and edge dislocations.

2.2.1. *Screw dislocations*

A screw dislocation has a Burghers vector that is orthogonal to the plane of the loop around the dislocation. The graphical representation looks like a screw, or a parking garage ramp, as in Fig. 2.

2.2.2. *Edge dislocations*

Figure 3 illustrates edge dislocations, which have Burghers vector in the plane of the loop used to measure the dislocation.

In Fig. 3(a), if you travel around a loop in the zy-plane that surrounds the center, you are translated one lattice spacing in the $+z$-direction. (Travel 4 units in the
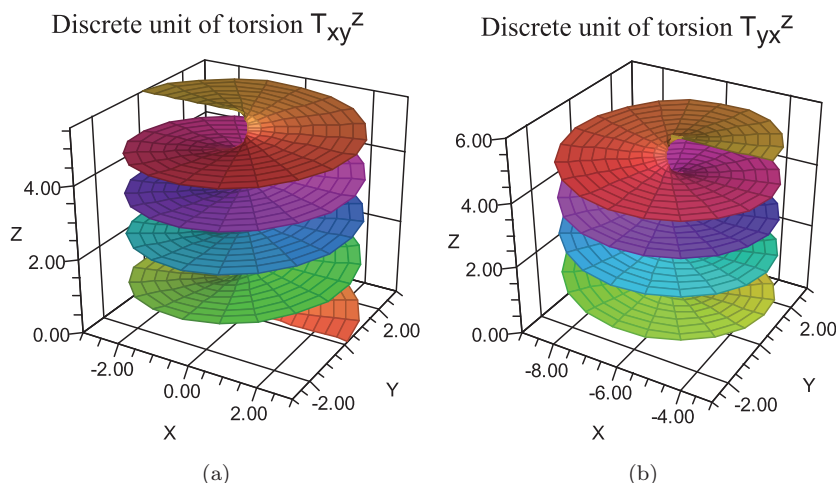
Discrete unit of torsion $T_{xy}{}^z$

Discrete unit of torsion $T_{yx}{}^z$

Fig. 2. Graphical representations of screw dislocations. (a) Right-hand screw dislocation. Discrete unit of torsion $T_{xy}{}^z$ and (b) Left-handed screw dislocation. Discrete unit of torsion $T_{yx}{}^z$.
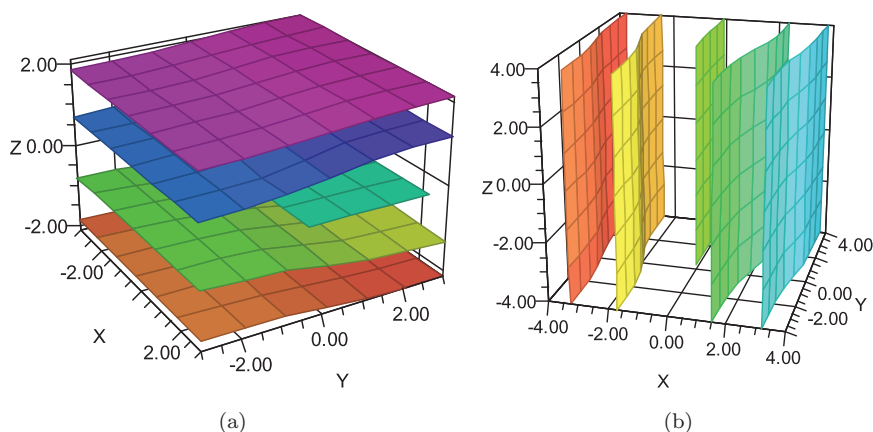
Fig. 3. Graphical representations of edge dislocations.

direction $+z$, 4 units in the direction $+y$, 4 units in the direction $-z$, and 4 units in the direction $-y$. You finish 1 unit above the point where you started.) The graph looks like a parking garage with many levels, where one level covers only half the area of the others; that half-level has an edge where you could fall off if there were no guard rails. This dislocation is a discrete unit of torsion $T_{zy}{}^z$.

Figure 3(b) is a graphical representation of the same edge dislocation with different coordinate orientation. If you travel around a loop in the yx-plane that surrounds the center, you are translated one lattice spacing in the $+x$-direction. This dislocation is a discrete unit of torsion $T_{yx}{}^x$.

2.2.3. *Other defects in affine lattices*

Defects in affine lattices can be classified by their dimension:

- Zero-dimensional defects (point defects): Vacancies (missing atom or ion), interstitials (atoms or ions not located at a regular lattice position), Frenkel defects (displaced atoms or ions that create a void and a nearby inclusion), and substitutions (atoms or ions at regular lattice locations but not the usual kind of atoms or ions).
- One-dimensional defects: Dislocations and disclinations. These are called "line defects" because the Burghers vector continues along a line that cannot end inside a crystal; this is the conservation law for dislocation lines. The lines can terminate at two-dimensional defects where the regular lattice structure breaks down.
- Two-dimensional defects: Grain boundaries, phase boundaries, domain walls, stacking faults, and free surfaces.
- Three dimensional defects: Precipitates and inhomogeneities.

2.3. *Continuum distribution of dislocations*

In Riemannian geometry, rotational holonomy is a discrete version of Riemannian (rotational) curvature:

$$\text{rotational curvature} = \text{limit}_{\text{area}\to 0}(\log(\text{rotational holonomy}))/$$

$$(\text{area enclosed by the loop}). \tag{1}$$

(Fine point: log is the inverse of the exponential map; log maps the structure group to its Lie algebra.)

In Riemann–Cartan geometry, translational holonomy is a discrete version of affine torsion:

$$\text{torsion} = \text{limit}_{\text{area}\to 0}\log(\text{translational holonomy})/$$

$$(\text{area enclosed by the loop}). \tag{2}$$

In the continuum theory of dislocations, torsion represents the density of dislocations. The two common types of dislocations have characteristic components of torsion:

- Screw dislocations have Burghers vectors (in the z-direction) orthogonal to the plane of the loop (the xy-plane); they are represented by the torsion $T_{xy}{}^z$, or any other permutation of the orthogonal directions $\{x, y, z\}$. The antisymmetrized indices x and y indicate the plane of a loop.
- Edge dislocations have Burghers vectors (in the z-direction) lying in the plane of the loop (the zy−plane); they are represented by the torsion trace $T_{zy}{}^z$, or any other permutation of $\{x, y, z\}$. The antisymmetrized indices z and y indicate the plane of a loop.

3. Einstein–Cartan Theory Describes Intrinsic Angular Momentum with Torsion

3.1. *Background*

EC theory is the first theory of fundamental physics that uses affine torsion. EC theory is a gauge theory of the Poincaré group. Torsion is identified with intrinsic AM:

- A stationary spinning particle with AM polarized in the xy-plane (conventionally described as AM in the z-direction) is represented as a discrete unit of torsion T_{xy}^t. Its translational holonomy (Burghers vector) is a timelike vector; parallel translation around a spacelike equatorial loop translates the observer into the future or the past.
- A stationary spin-1/2 Dirac field with z-spin polarized in the $+z$-direction has a spin tensor equal to antisymmetrized torsion $T_{[x,y,t]}$. Dirac fields couple only to totally antisymmetric torsion.

Although EC theory is a classical theory of physics, the quantum spin fits well in EC theory.

3.2. *Conservation of angular momentum*

In the continuum theory of dislocations, the conservation law for dislocations is

$$D_k(T_{ij}^k + (\text{torsion trace terms})_{ij}^k) = \text{sum of rotational curvature terms}, \qquad (3)$$

where D is the covariant derivative operator and $i, j, k \in \{1, 2, 3\}$ [2, p. 101; 13, p. 121].

The law of conservation of AM in EC theory describes the exchange of intrinsic and orbital angular momentums,

$$D_k(J_{ij}^k) + P_{[ij]} = 0, \qquad (4)$$

where

- D is the covariant derivative operator;
- $J_{ij}^k = $ Intrinsic AM $= S_{ij}^k/\kappa$, $\kappa = (8\pi G/c^4)$, $G = $ Universal gravitational constant;
- $S_{ij}^k = T_{ij}^k + (\delta_i^k T_{jm}^m - \delta_j^k T_{im}^m)$ is the modified torsion, which includes torsion trace;
- $P_{[i,j]}$ is the antisymmetric part of the linear 4-momentum. It is the local rate of loss of orbital AM;
- $i, j, \lambda\{0, 1, 2, 3\}$. The equations of EC theory are simplified by distinguishing the spacetime indices μ, ν, λ, etc. and the current indices (or fiber indices) i, j, k, etc. We ignore the distinction in this paper.

Equation (4) has a simple intuitive interpretation:

- The first term, the divergence of the modified torsion, is the rate at which intrinsic AM increases (or decreases if negative) at a spacetime location.
- The second term, the antisymmetric part of the momentum tensor, is the rate at which orbital AM increases (or decreases if negative) at a spacetime location.
- Equation (4) states that, since total AM is conserved, the sum of the changes in intrinsic and orbital AMs must equal zero.

The law of conservation of dislocations and the law of conservation of AM in EC theory are the same identity in Riemann–Cartan geometry: the contracted first Bianchi identity. This identity in different fields has the same geometric meaning, though it has different physical interpretations.

3.3. *Assessment of Einstein–Cartan theory*

Although no empirical evidence supports EC theory at this time, the arguments for the theory are substantial.

(a) EC theory, teleparallel gravity (which alters the geometric basis of GR), and GR are the only theories of gravitation known to satisfy all empirical tests of GR [8, 23].
(b) EC theory can model exchange of intrinsic and orbital AMs. GR cannot do this because its momentum tensor is symmetric:
  — The essence of turbulence is transfer of orbital AM to smaller scales [17].
  — Current theories of turbulence do not track intrinsic AM. Present research suggests that transfer of AM in turbulence ends with classical intrinsic AM [18, 19].
  — Classical turbulence is common in cosmology. Therefore, a master classical theory of spacetime should be able to model these phenomena, at least in theory.
(c) EC theory, plus quantum field-theoretic models of matter, removes many of the singularities in GR, e.g. in black holes and in some Big Bang cosmological models.
(d) Torsion appears in EC theory only in the places where it occurs in Cartan's master structure equations for affine geometry. EC theory has no ad-hoc torsion terms.
(e) Classical EC theory can be derived from GR without additional assumptions [20].
(f) Mathematically, EC theory is that it includes translational symmetries of spacetime in the structure group. So, EC theory can properly derive linear momentum as the Noether current of spacetime translations.

These advantages arguably make EC theory a more credible master theory of classical spacetime physics than GR.

4. Annihilation of Two Spin-1/2 Particles

This work focuses on a configuration of two spin-1/2 particles with opposite z-spins that are separated by a spatial distance, and that annihilate via a virtual bound state that connects the ends of the worldlines of the annihilating particles. For example, an electron and a positron with opposite z-spins can annihilate at a short distance by forming a virtual bound state, para-positronium, which has $S = 0$. Ortho-positronium has $S = 1$ and $M = -1, 0, 1$, so the internal AM content is more complicated.

Just as the torsion trace of an edge dislocation enables screw dislocations to annihilate at a distance while preserving dislocation lines, the torsion carried by para-positronium enables spin-1/2 particles to annihilate at a distance while locally conserving AM.

4.1. *Two screw dislocations whose ends are joined by edge dislocation*

Two screw dislocations whose ends are joined by an edge dislocation form a geometric analog for two spin-1/2 particles that annihilate at a spacelike separation mediated by a virtual bound state.

Figure 4 represents the configuration of two screw dislocations of opposite helicities whose ends are joined by an edge dislocation:

The thought experiment below helps to visualize the overall structure of this configuration of dislocations:

(a) Rotate in the xy-plane around the screw dislocation on the right. This moves you up the screw, which means the Burghers vector is in the direction $+z$.

(b) When you reach the top of the screw, turn the dislocation line (and the plane of rotation) smoothly 90° to the left (in the $-x$-direction), so that the core of the dislocation is horizontal.

- The left-handed screw dislocation on the left is depicted in Figure 2-b. Its torsion is $T_{yx}{}^z$.
- The right-handed screw on the right is depicted in Figure 2-a. Its torsion is $T_{xy}{}^z$.
- The edge dislocation has the orientation depicted in Figure 3-a. Its torsion is $T_{zy}{}^z$.
- Above the screw and edge dislocations is a flat layer of affine lattice without defects, with part of the layer removed to expose the edge dislocation.
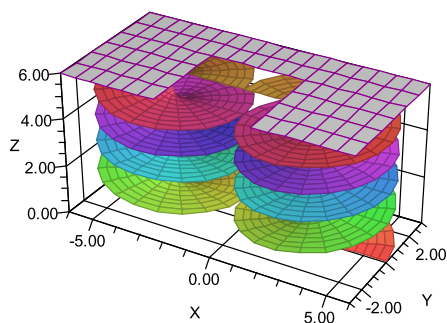

Fig. 4. Two screws dislocations of opposite helicities joined by an edge dislocation.

(c) Along the horizontal dislocation line, rotate one time around a loop in the zy-plane, using the same number of lattice spacings in the directions $+z$, $+y$, $-z$, and $-y$. The path does not close, and reveals a Burghers vector in the $+z$-direction.

(d) When you reach the left end of the edge dislocation, turn the dislocation line (and the plane of rotation) smoothly $90°$ downward (in the $-z$-direction), so that the core of the dislocation is vertical pointing downward on the left screw.

(e) Rotate around the screw dislocation on the left in the yx-plane. This moves you down the screw, which means the Burghers vector is in the direction $-z$.

All three dislocations have Burghers vector (translational holonomy) in the $+z$-direction. The configuration satisfies the conservation law for dislocations: dislocation lines do not end, and the Burghers vector is constant. The edge dislocation permits the screw dislocations to terminate without violating the conservation of dislocation lines.

4.2. *Two spin-1/2 particles of opposite z-spins joined by a virtual bound state*

The configuration of dislocations depicted in Fig. 4 is a 3D nonrelativistic model for two spin-1/2 particles of opposite z-spins joined by a virtual bound state in relativistic quantum field theory. The dislocation model can be adapted to a relativistic model of spin-1/2 particles by replacing the z-direction in the dislocation model with a timelike direction in relativistic field theory. The translational holonomy (Burghers vector) in spacetime points to a timelike direction instead of the z-direction. For a Dirac particle, parallel translation once around the spiral translates to a future or past time. In Fig. 4, we have the following:

- The left-handed screw on the left is depicted in Fig. 2(b). Its torsion is $T_{yx}{}^{t}$.
- The right-handed screw on the right is depicted in Fig. 2(a). Its torsion is $T_{xy}{}^{t}$.
- The edge dislocation has the orientation depicted in Fig. 3(a). Its torsion is $T_{ty}{}^{t}$.

The Burghers vector (angular momentum vector) lies in the time direction, as in Dirac particles. However, it is a strange form of intrinsic AM because its Burghers vector (AM vector) lies in the space time plane of the loop used to measure it.

We conjecture that this argument holds for any virtual bound state that mediates the annihilation of two particles with canceling nonzero spins.

5. Further Developments

5.1. *Torsion in quantum gravity and other quantum theories*

Intrinsic AM is generally more important in quantum theories than in their classical counterparts. Therefore, EC theory is likely a better classical starting point for a theory of quantum gravity than is GR.

Spin is omnipresent in quantum field theory. EC theory already assumes that all spin-1/2 particles have torsion. We expect that, in a future quantum field theory, torsion will be present wherever spin is present. This paper is a narrow instance of how and where torsion will appear.

5.2. *Quantum entanglement?*

Quantum entanglement seems to require instant transport of information across a macroscopic spacelike separation between two entangled particles. The model proposed here suggests that certain virtual bound states use torsion trace to transport AM over a spacelike separation, thereby enabling local conservation of AM in spacetime. We might speculate that some gauge fields, analogous to torsion for spin currents, enable entangled particles to share information over macroscopic distances without violating the local structure of spacetime.

6. Conclusion

This work provides theoretical arguments to support the conjecture that para-positronium and possibly other virtual bound states carry components of torsion, including torsion trace. The geometry and intrinsic AM in para-positronium are analogous to those of a configuration of two screw dislocations connected by an edge dislocation, i.e. two pieces of antisymmetric torsion of opposite helicities connected by a spacelike line that carries torsion trace. The key properties of torsion that are used in this argument are based on analogies with two applications of torsion in classical physical theories:

- The correspondence of torsion with intrinsic AM is the central feature of EC theory.
- The local geometry of antisymmetric torsion and torsion trace are illustrated by screw and edge dislocations in affine lattices in crystallography.

The conservation law of total AM in EC theory and the conservation of dislocation lines in lattices have the same mathematical foundation as the contracted first Bianchi identity of Riemann–Cartan geometry.

Using torsion trace to represent a spacelike flow of intrinsic AM allows the local law of conservation of AM to hold across a spacelike separation when para-positronium mediates annihilation of two Dirac particles.

We are unaware of any experimental tests that might confirm that these virtual bound states carry torsion. The best supports for the conjecture are: (a) EC theory claims that all intrinsic AM is modeled by torsion; and (b) torsion enables conservation of total AM across the micro-scale spacelike separation between the fermions.

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Appendix A. Riemann–Cartan Geometry

Riemann–Cartan geometry is the common foundation of the theory of dislocations and Einstein–Cartan theory.

Table A.1. Concepts of Riemann–Cartan geometry in dislocation theory and Einstein–Cartan theory.

Riemann–Cartan geometry	Theory of dislocations	Einstein–Cartan theory	Quantum theory
Rotational holonomy (discrete rotational curvature)	Disclination	Continuum limit is Riemannian curvature	Need quantum theory of gravity
Rotational curvature (Riemannian curvature)	Locked-in strain	Riemannian curvature (gravitation)	Need quantum theory of gravity
Translational holonomy (discrete translational curvature)	Burghers vector of discrete dislocation, or continuum density of dislocations	Intrinsic AM of a discrete particle	Conjecture: spin of discrete particle
Torsion (translational curvature)	Density of dislocations	Density of intrinsic AM	Conjecture: density of intrinsic AM
Helical torsion (Burghers vector is orthogonal to the plane of the loop)	$T_{xy}{}^z =$ Screw dislocations loop in the xy-plane, Burghers vector in z-direction	For spin-1/2 fields: $T_{xy}{}^t =$ spin with loop in xy-plane and translation in t-direction	Conjecture: spin density of spin-1/2 Dirac particle with spin polarization in z-direction
Torsion trace	$T_{zy}{}^z =$ Density of edge dislocations with loop in zy-plane, Burghers vector in z-direction	$T_{ty}{}^t =$ Density of spin with loop in the ty-plane and translation in t-direction	Conjecture: spin density of virtual bound state
Contracted first affine Bianchi identity	Conservation of dislocation lines. Dislocation line cannot end inside a crystal grain	Conservation of AM: Δspin + Δorbital AM $= 0$	Conjecture: conservation of AM

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