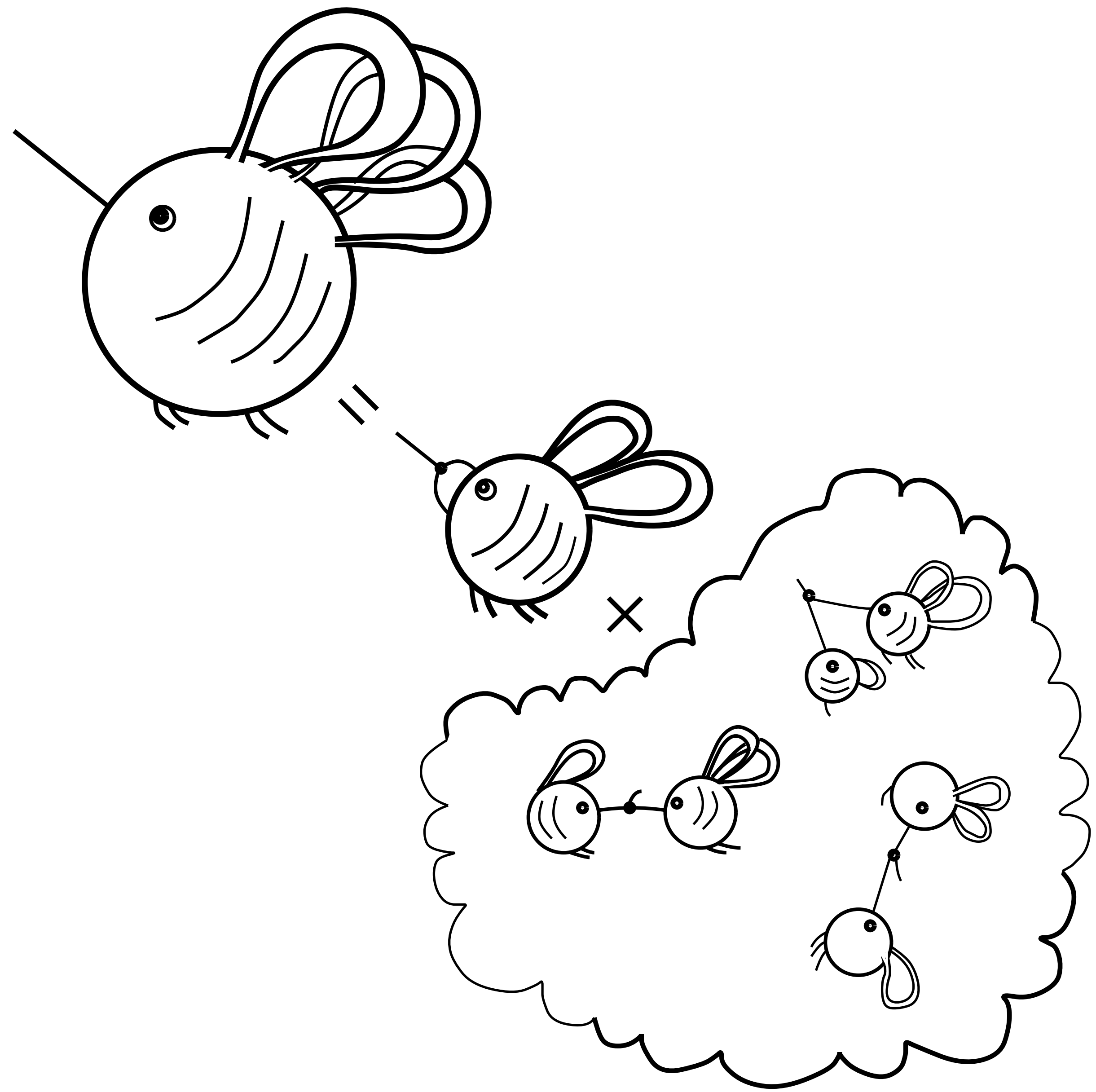
Petr Dunin-Barkovskiy

Gromov-Witten theory and spectral curve topological recursion



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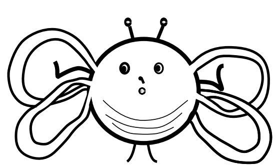

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Gromov-Witten theory
and
spectral curve topological recursion

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Petr Dunin-Barkovskiy

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Faculteit: Faculteit der Natuurwetenschappen, Wiskunde en Informatica

Contents

Chapter 1

Introduction

This thesis deals with problems in the fields of algebraic geometry and mathematical physics related to Gromov–Witten theory, spectral curve topological recursion and Hurwitz numbers.

The next section, Section 1.1, gives an introduction for non-experts. Then in Sections 1.2–1.4 we introduce and define the objects studied in the thesis. This allows us to formulate the results of the thesis in Section 1.5. Chapters 2–6 constitute the main body of the thesis; they are based on papers [104, 103, 101, 100, 102] by the author, in collaboration with S.Shadrin, L.Spitz; N.Orantin, S.Shadrin, L.Spitz; M.Mulase, P.Norbury, A.Popolitov, S.Shadrin; M.Kazarian, N.Orantin, S.Shadrin, L.Spitz; and N.Orantin, A.Popolitov, S.Shadrin respectively[1]. Finally, Chapter 7 provides a popular summary in Dutch.

1.1 Introduction for non-experts

One of the central concepts in algebraic geometry is a concept of an *algebraic curve*. Usually (and it is indeed the case for this thesis) people work with *complex* algebraic curves. In this case these *curves* are actually two-dimensional surfaces. We call them *curves* since such a curve can be represented as a solution to an algebraic equation in two-dimensional (complex) space. When we say "algebraic equations" we mean equations polynomial in the independent variables.

Gromov-Witten theory deals with maps (i.e., essentially, embeddings) of algebraic curves into a given complex manifold (i.e. some multidimensional space). This theory originated from physics, more precisely from string theory. *Gromov-Witten invariants* are numbers which, essentially, count the number of ways one can embed curves of a given type into a given complex manifold. Gromov-Witten invariants are very interesting since they are both interesting entities from the string theory point of view, and because they turned out to be very useful in algebraic geometry. Moreover, it was found that they are also connected to a completely different area of mathematical physics, namely, to the theory of integrable systems.

Spectral curve topological recursion is a quite general technique which has applications

[1] Formal remark on co-authorship, required by the Promotieregelement of the University of Amsterdam: all authors of all these papers have equally contributed to all obtained results.

in many different branches of mathematics and physics. This technique, given certain initial data, namely, a *spectral curve* (which is an algebraic curve equipped with some additional structure), produces so-called *n-point functions* on this spectral curve. It turns out that for a vast array of problems in such areas of science as algebraic geometry, mathematical physics, topology and combinatorics, these n-point functions appear to be *generating functions* for numbers arising in these problems, if one takes appropriate initial data. When we say that these n-point functions are the generating functions for these numbers we mean that they *encode* these numbers in such a way that the numbers appear as coefficients in series expansions of these functions. Spectral curve topological recursion is very interesting since it is one and the same procedure that for appropriate choices of initial data produces answers to many seemingly unrelated problems.

Hurwitz numbers count *covers* of sphere by two-dimensional compact surfaces (compact here, essentially, means that they are not infinite and have no boundary). Cover is a sufficiently nice map from the surface to the sphere, i.e. to each point of the surface one puts into correspondence a point of the sphere. To almost every point of the sphere (except for a finite number of the so-called *ramification points*) correspond the same number of points of the surface, which is called the *degree* of the cover. Then, if one specifies the behavior in these ramification points, it turns out that there is only a finite number of possible covers, up to certain equivalence. This number is called the Hurwitz number. Hurwitz numbers are interesting since they have interpretations in terms of combinatorics and topology, and also play a role in algebraic geometry and mathematical physics.

The thesis mostly studies connections between the above concepts. One of the main results of the thesis is a way to apply (*local* version of) the spectral curve topological recursion to any Gromov-Witten theory. Namely, we propose a way (and prove that it works) to choose initial data in the (local) spectral curve topological recursion such that the resulting n-point functions will generate Gromov-Witten invariants for any given target complex manifold.

Another result is related to the so-called *quantum* spectral curve equation. It turns out that in some cases it is possible to show that certain generating functions (called wave functions) satisfy quantized versions of the spectral curve equation. Quantization here means that the variables in the equation are replaced with differential operators in the same way as it happens in quantum physics. In the thesis it is shown that the wave function for the Gromov–Witten invariants of sphere (i.e. for the target manifold being the sphere) satisfies the corresponding quantum spectral curve equation.

Another main result is the new, combinatorial, proof of the so-called ELSV formula. This formula relates *simple* Hurwitz numbers (particular type of Hurwitz numbers with ramification profiles specified in a certain way) to the so-called Hodge integrals, which are entities similar to Gromov–Witten invariants. Its original proof (and all the other subsequent ones) involved complicated geometric considerations. Here we prove it in a simple combinatorial way by, first, proving that these Hurwitz numbers are polynomial expressions in numbers which describe the ramification profile, then using this to prove the spectral curve topological recursion for these simple Hurwitz numbers, and finally using the above result on spectral curve topological recursion for Gromov–Witten theories to build the connection to Hodge integrals.

Finally, we prove (in a combinatorial way) the spectral curve topological recursion for the problem of counting *bi-colored maps*. Bi-colored maps are, essentially, ways to subdivide a given two-dimensional surface into polygons and paint them in black and white such that white polygons only border black ones and vice versa. It turns out that these numbers can be generated by the spectral curve topological recursion with the appropriate initial data.

1.2 Frobenius manifolds and Givental theory

In this section we introduce Frobenius manifolds, moduli spaces of algebraic curves, Gromov–Witten theory, cohomological field theory and Givental theory.

1.2.1 Frobenius manifolds

A Frobenius manifold is a differential-geometric structure that was introduced by Dubrovin in the early 1990's as a mathematical framework for the study of two-dimensional topological field theory in genus zero [25, 26]. It has appeared to be a quite universal structure that has many naturally arising examples. In particular, Frobenius manifolds can serve as a classification tool for (dispersionless) bi-Hamiltonian hierarchies of hydrodynamic type [28, 27]. Nowadays there is a number of standard textbooks on Frobenius manifolds, see [26, 78, 62].

Here we introduce Frobenius manifolds following [26].

Definition 1.2.1. An algebra A over $\mathbb{C}$ is called (commutative) *Frobenius algebra* if:

1. It is a commutative associative $\mathbb{C}$-algebra with a unity e.
2. It is supplied with a $\mathbb{C}$-bilinear symmetric nondegenerate inner product

$$A \times A \to \mathbb{C}, \quad a,\, b \mapsto (a, b)$$

being invariant in the following sense:

$$(ab, c) = (a, bc)$$

Definition 1.2.2. M is *Frobenius manifold* if a structure of Frobenius algebra is specified on any tangent plane T_tM at any point $t \in M$ smoothly depending on the point such that

1. The invariant inner product $(\ ,\ )$ is a flat metric on M.
2. The unity vector field e is covariantly constant w.r.t. the Levi-Civitá connection ∇ for the metric $(\ ,\ )$

$$\nabla e = 0$$

3. Let

$$c(u, v, w) := (u \cdot v, w)$$

(a symmetric 3-tensor). We require the 4-tensor

$$(\nabla_z c)(u, v, w)$$

to be symmetric in the four vector fields u, v, w, z.

4. A vector field E must be determined on M such that
$$\nabla(\nabla E) = 0$$
and that the correspondent one-parameter group of diffeomorphisms acts by conformal transformations of the metric $(\ ,\ )$ and by rescalings on the Frobenius algebras T_tM.

Locally Frobenius manifold can be defined in terms of a *Frobenius prepotential* $F(t)$ as follows.

Definition 1.2.3. Function $F = F(t)$, $t = (t^1, \dots, t^n)$ is a *Frobenius prepotential* if its third derivatives
$$c_{\alpha\beta\gamma}(t) := \frac{\partial^3 F(t)}{\partial t^\alpha \partial t^\beta \partial t^\gamma}$$
obey the following equations

1. Normalization:
$$\eta_{\alpha\beta} := c_{1\alpha\beta}(t)$$
is a constant nondegenerate matrix. Let
$$(\eta^{\alpha\beta}) := (\eta_{\alpha\beta})^{-1}.$$
We will use the matrices $(\eta_{\alpha\beta})$ and $(\eta^{\alpha\beta})$ for raising and lowering indices.

2. Associativity: the functions
$$c^\gamma_{\alpha\beta}(t) := \eta^{\gamma\epsilon} c_{\epsilon\alpha\beta}(t)$$
(summation over repeated indices is assumed here) for any t must define in the n-dimensional space with a basis e_1, ..., e_n a structure of an associative algebra A_t
$$e_\alpha \cdot e_\beta = c^\gamma_{\alpha\beta}(t) e_\gamma.$$
Note that the vector e_1 will be the unity for all the algebras A_t:
$$c^\beta_{1\alpha}(t) = \delta^\beta_\alpha.$$

3. $F(t)$ must be quasihomogeneous function of its variables:
$$F(c^{d_1} t^1, \dots, c^{d_n} t^n) = c^{d_F} F(t^1, \dots, t^n) \tag{1.1}$$
for any nonzero c and for some numbers d_1, ..., d_n, d_F.

Coordinates t such as above on the Frobenius manifold are called the *flat coordinates.*

It will be convenient to rewrite the quasihomogeneity condition in the infinitesimal form introducing the *Euler vector field*
$$E = E^\alpha(t) \partial_\alpha$$
as
$$\mathcal{L}_E F(t) := E^\alpha(t) \partial_\alpha F(t) = d_F \cdot F(t)$$
$E(t)$ is a linear vector field
$$E = \sum_\alpha d_\alpha t^\alpha \partial_\alpha$$
generating the scaling transformations (1.1).

Definition 1.2.4. Point of a Frobenius manifold is called *semisimple* if at this point the Frobenius algebra structure is nondegenerate, i.e. there is no tangent vector that squares to zero.

Near a semisimple point of a Frobenius manifold one can introduce the so-called *canonical coordinates*, which are defined as coordinates u^i that have the property

$$\frac{\partial}{\partial u^i} \cdot \frac{\partial}{\partial u^j} = \delta_{ij} \frac{\partial}{\partial u^j}, \tag{1.2}$$

where δ_{ij} is the Kronecker delta.

Define $\Delta_i := 1/(\partial_i, \partial_i)$ to be the inverse of the square of the length of the i^{th} canonical basis element, and call $\{\partial/\partial v^i := \Delta_i^{1/2} \partial/\partial u^i\}$ the normalized canonical basis in the tangent space.

Let U be the matrix of canonical coordinates $U = \operatorname{diag}(u^1, \ldots, u^N)$ and denote by Ψ the transition matrix from the flat to the normalized canonical bases. That is, denoting $\mathrm{d}t = (\mathrm{d}t^1, \ldots, \mathrm{d}t^N)^{\mathrm{T}}$ and $\mathrm{d}u = (\mathrm{d}u^1, \ldots, \mathrm{d}u^N)^{\mathrm{T}}$, one has

$$\Delta^{-1/2} \mathrm{d}u = \Psi \mathrm{d}t, \tag{1.3}$$

where $\Delta = \operatorname{diag}(\Delta_1, \ldots, \Delta_N)$.

1.2.2 Gromov-Witten theory

Let us introduce Gromov–Witten theory, following, e.g. [87].

First, let us introduce moduli spaces of algebraic curves. The moduli space of curves $\mathcal{M}_{g,n}$, $g \geq 0$, $n \geq 0$, $2g - 2 + n > 0$, parametrizes smooth complex curves of genus g with n ordered marked points. It is a smooth complex orbifold of dimension $3g - 3 + n$.

The space $\overline{\mathcal{M}}_{g,n}$ is a compactification of $\mathcal{M}_{g,n}$. It parametrizes stable curves of genus g with n ordered marked points. A stable curve is a possibly reducible curve with possible nodes, such that the order of its automorphism group is finite. Genus of a stable curve is the arithmetic genus, namely, the genus of the smooth curve that we get if the replace each node (given locally by the equation $xy = 0$) with a cylinder (given locally by the equation $xy = \epsilon$). The space $\overline{\mathcal{M}}_{g,n}$ is a smooth compact complex orbifold.

There is a number of natural mappings between the moduli spaces of curves.

First, there are projections $\pi \colon \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ that forget the last marked point. Note that there is a subtlety related to the fact that when we forget a marked point a stable curve can become unstable.

Second, there is a 2-to-1 mapping $\sigma \colon \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}$ whose image is the boundary divisor of irreducible curves with one node.

Third, there are mappings $\rho \colon \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g,n}$, $g_1 + g_2 = g$, $n_1 + n_2 = n$, whose images are the other irreducible boundary divisors of the compactification of $\overline{\mathcal{M}}_{g,n}$.

Let L_i the line bundle over $\overline{\mathcal{M}}_{g,n}$, whose fiber over a point $x \in \overline{\mathcal{M}}_{g,n}$ represented by a curve C_g with marked points $x_1, \ldots, x_n$ is equal to $T^*_{x_i} C_g$. Denote by $\psi_i \in H^2(\overline{\mathcal{M}}_{g,n})$ the first Chern class of L_i. ψ_i are referred to as *psi-classes*.

Given a complex manifold X, one can consider $[X_{g,k,\deg}]$, the moduli space of degree deg stable maps to X of genus-g curves with k marked points. There is a natural projection from $[X_{g,k,\deg}]$ to $\overline{\mathcal{M}}_{g,k}$ which consists of forgetting the map to X. Note that there is a subtlety related to the fact that an unstable curve can be mapped to X in a stable way. One can consider pullback of psi-classes to $[X_{g,k,\mathrm{deg}}]$ with respect to this projection. We will denote them as ψ_i too.

Gromov–Witten theory studies the so-called Gromov–Witten invariants (also called Gromov–Witten correlators), defined as follows (left hand side is just a notation):

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g := \sum_{\mathrm{deg}} \int_{[X_{g,k,\mathrm{deg}}]^{\mathrm{vir}}} ev_1^*(e_{i_1})\psi_1^{d_1}\, ev_2^*(e_{i_2})\psi_2^{d_2}\cdots ev_k^*(e_{i_k})\psi_k^{d_k}, \quad (1.4)$$

where , ev_i is the evaluation map at the i^{th}.

Gromov–Witten *potential* $\mathcal{F}_g$ of genus g is the following generating function for the correlators ($v^{d,i}$ are formal variables):

$$\mathcal{F}_g = \sum \frac{\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g}{|\operatorname{Aut}((i_m,d_m)_{m=1}^k)|} v^{d_1,i_1}\cdots v^{d_k,i_k}, \quad (1.5)$$

where $|\operatorname{Aut}((i_m,d_m)_{m=1}^k)|$ denotes the number of automorphisms of the collection of multi-indices (i_m,d_m) and where the sum is such that it includes each monomial $v^{d_1,i_1}\cdots v^{d_k,i_k}$ exactly once.

It turns out [78] that *the genus zero potential without descendants*, i.e.

$$F = \mathcal{F}_0\big|_{v^{d,i}=0,\ d>0} \quad (1.6)$$

is a Frobenius prepotential.

Partition function of the Gromov–Witten theory is defined as follows:

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g\right), \quad (1.7)$$

where $\hbar$ is a formal parameter.

Celebrated result of Kontsevich–Witten states that the partition function for the Gromov-Witten theory of a point (i.e. for $X = \{\mathrm{pt}\}$) is the tau-function of the KdV integrable hierarchy. We will denote this partition function as Z_{KdV}.

1.2.3 Cohomological field theory

Here we introduce cohomological field theory, again following [87].

Cohomological field theory is a generalization of Gromov-Witten theory where one drops the target complex manifold and takes just some vector space V in place of its cohomology. Roughly speaking, a CohFT is a system of cohomology classes on the moduli spaces of curves with the values in the tensor powers of V, compatible with all natural mappings between the moduli spaces.

The formal definition is the following. We fix a vector space $V = \langle e_1,\ldots,e_s\rangle$ (e_1 will play a special role) with a non-degenerate scalar product η. A cohomological field theory is a system of cohomology classes $\alpha_{g,n} \in H^*(\overline{\mathcal{M}}_{g,n}, V^{\otimes n})$ satisfying the properties:

1. $\alpha_{g,n}$ is equivariant with respect to the action of S_n on the labels of marked points and components of $V^{\otimes n}$.

2. $\sigma^*\alpha_{g,n} = (\alpha_{g-1,n+2}, \eta^{-1})$; $\rho^*\alpha_{g,n} = (\alpha_{g_1,n_1+1} \cdot \alpha_{g_2,n_2+1}, \eta^{-1})$ (in both cases we contract with the scalar product the two components of V corresponding to the two points in the preimage of the node under normalization).

3. $(\alpha_{0,3}, e_1 \otimes e_i \otimes e_j) = \eta_{ij}$, $\pi^*\alpha_{g,n} = (\alpha_{g,n+1}, e_1)$ (again, we contract the component of V corresponding to the last marked point with e_1).

Then *correlators* in cohomological field theory are defined as follows:

$$\langle \tau_{d_1}(e_{i_1}) \cdots \tau_{d_n}(e_{i_n}) \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} \prod_{j=1}^{n} \psi_j^{d_j} \cdot \left(\alpha_{g,n}, \otimes_{j=1}^{n} e_{i_j}\right) \tag{1.8}$$

Genus-g potentials and partition function are defined for cohomological field theories in exactly the same way as for Gromov–Witten theories above.

The statement about the genus zero potential without the descendants being a Frobenius prepotential is also true for any cohomological field theory.

1.2.4 Givental theory

Givental theory [52, 53, 54] is one of the most important tools in the study of Gromov-Witten invariants of target varieties and general cohomological field theories. It allows, in particular, to obtain explicit relations between the partition functions of different theories, reconstruct higher genera correlators from the genus 0 data, and establish general properties of semi-simple theories.

The core of the theory is Givental's formula that gives a formal Gromov-Witten potential associated to a semi-simple Frobenius structure. Teleman proves [93] that the formal Gromov-Witten potential associated to the Frobenius structure of a target variety with semi-simple quantum cohomology coincides with the actual Gromov-Witten potential in all genera.

In order to write Givental formula, let us construct an operator series $R(z) = \sum_{k \geq 0} R_k z^k$ in the following way.

Recursively define the off-diagonal entries of R_k in normalized canonical coordinates by solving the equation

$$\Psi^{-1}\mathrm{d}(\Psi R_{k-1}) = [\mathrm{d}U, R_k]. \tag{1.9}$$

using $R_0 = \mathbf{I}$ as a base case. Construct the diagonal entries of R_k by integrating the next equation

$$\Psi^{-1}\mathrm{d}(\Psi R_k) = [\mathrm{d}U, R_{k+1}] \tag{1.10}$$

using the fact that the diagonal entries of $[\mathrm{d}U, R_{k+1}]$ are equal to zero. To fix the integration constant, use the Euler equation

$$R_k = -(i_E \mathrm{d}R_k)/k, \tag{1.11}$$

where $E = \sum u^i \partial_i$ is the Euler field.

This procedure recursively defines R_k for all k.

Let us consider the following reexpansion:

$$R(z) = \sum_{l=0}^{\infty} R_l z^l = \exp\left(\sum_{l=1}^{\infty} r_l z^l\right). \tag{1.12}$$

Then we denote by $(r_l z^l)\hat{}$ the following differential operator:

$$\begin{aligned}(r_l z^l)\hat{} := & -(r_l)^i_{\mathbf{1}} \frac{\partial}{\partial v^{l+1,i}} + \sum_{d=0}^{\infty} v^{d,i} (r_l)^j_i \frac{\partial}{\partial v^{d+l,j}} \\ & + \frac{\hbar}{2} \sum_{m=0}^{l-1} (-1)^{m+1} (r_l)^{i,j} \frac{\partial^2}{\partial v^{m,i} \partial v^{l-1-m,j}}.\end{aligned} \tag{1.13}$$

Here the indices $i, j \in \{1, \dots, N\}$ on r_l correspond to the basis $\{e_1, \dots, e_N\}$ of V, and the index $\mathbf{1}$ corresponds to the unit vector $e_{\mathbf{1}}$. When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η.

The quantization $\hat{R}$ of series $R(z)$ is then defined by

$$\hat{R} = \exp\left(\sum_{l=1}^{\infty} \left((-1)^l r_l z^l\right)\hat{}\right). \tag{1.14}$$

Givental formula then allows to reproduce all ancestor correlators of the given (homogenous) cohomological field theory in the following way:

$$Z = \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}, \tag{1.15}$$

where

$$\mathcal{T} := Z_{\mathrm{KdV}}(\{u^{d,1}\}) \cdots Z_{\mathrm{KdV}}(\{u^{d,N}\});$$

$\hat{\Delta}$ replaces the variables of i^{th} KdV τ-function according to $u^{d,i} = \Delta_i^{1/2} v^{d,i}$ and replaces $\hbar$ with $\Delta_i \hbar$, while $\hat{\Psi}$ is the change of variables $v^{d,i} = \Psi^i_\nu t^{d,\nu}$.

1.3 Spectral curve topological recursion

The theory developed by Eynard and Orantin (see [48, 44, 18]), is a procedure, called spectral curve topological recursion, that takes the following objects as input. First, a particular Riemann surface, which is called the *spectral curve*. Second, two functions x and y on this surface, and, third, a choice of a bi-differential on this surface, which we will call the *two-point function* (which also is often called Bergman kernel). And, occasionally, a particular extra choice of a coordinate on an open part of the Riemann surface. The output of the topological recursion is a set of n-forms $\omega_{g,n}$, whose expansion in this additional coordinate generates interesting numbers.

In some cases these numbers are correlators of a matrix model (that was the original motivation for introducing the topological recursion; it is a natural generalization of the reconstruction procedure for the correlators of a certain class of matrix models, see, e.g. [4]), in some other cases they appear to be related to Gromov-Witten theory and to various intersection numbers on the moduli space of curves.

Note that this spectral curve topological recursion is unrelated to the topological recursion occurring in the theory of moduli spaces of curves.

One of the ways to think about the input data of the topological recursion theory is to say that the $(g,n)=(0,1)$ part of a partition function in some geometrically motivated theory determines the spectral curve; the $(g,n)=(0,2)$ part of a partition function determines the two-point function, and the rest of the correlators can be reconstructed from these two via topological recursion, in terms of a proper expansion of $\omega_{g,n}$ (see [31]).

The topological recursion theory is often used to reproduce known partition functions, extracts from them some higher genus correlators which were up to now unreachable and gives new non-trivial relations for the correlators, see e. g. [45].

Local version of the spectral curve topological recursion is defined as follows.

For $N\in\mathbb{N}^*$, we call *times* a set of N families of complex numbers $\{h^i_k\}_{k\in\mathbb{N}}$ for $i=1,\dots,N$ and *jumps* another set of $N\times N$ infinite families of complex numbers $\left\{B^{i,j}_{k,l}\right\}_{(k,l)\in\mathbb{N}^2}$ for $i,j=1,\dots,N$. We finally define a set of canonical coordinates $\{a_i\}_{i=1}^N\in\mathbb{C}^N$ subject to $a_i\neq a_j$ for $i\neq j$.

Definition 1.3.1. For all $i,j\in\{1,\dots,N\}$, we define the following set of analytic functions and differential forms in a neighborhood of $0\in\mathbb{C}$:

$$x^i(z):=z^2+a_i\,,\quad y^i(z):=\sum_{k=0}^{\infty}h^i_k z^k \tag{1.16}$$

and

$$B^{i,j}(z,z')=\delta_{i,j}\frac{dz\otimes dz'}{(z-z')^2}+\sum_{k,l=0}^{\infty}B^{i,j}_{k,l}z^k z'^l dz\otimes dz'. \tag{1.17}$$

For $2g-2+n>0$, we define the genus g, n-point correlation functions $\omega^{i_1,\dots,i_n}_{g,n}(z_1,\dots z_n)$ recursively by

$$\omega^{i_0,i_1,\dots,i_n}_{g,n+1}(z_0,z_1,\dots,z_n):=\sum_{j=1}^{N}\operatorname*{Res}_{z\to0}\frac{\int_{-z}^{z}B^{i_0,j}(z_0,\cdot)}{2\left(y^j(z)-y^j(-z)\right)\mathrm{d}x^j(z)}\times$$
$$\left(\omega^{j,j,i_1,\dots,i_n}_{g-1,n+2}(z,-z,z_1,\dots,z_n)+\sum_{A\cup B=\{1,\dots,n\}}\sum_{h=0}^{g}\omega^{j,\mathbf{i}_A}_{h,|A|+1}(z,\mathbf{z}_A)\omega^{j,\mathbf{i}_B}_{g-h,|B|+1}(-z,\mathbf{z}_B)\right), \tag{1.18}$$

where for any set A, we denote by $\mathbf{z}_A$ (resp., $\mathbf{i}_A$) the set $\{z_k\}_{k\in A}$ (resp., $\{i_k\}_{k\in A}$), and where the base of the recursion is given by

$$\omega^i_{0,1}(z):=0;\qquad\omega^{i,j}_{0,2}(z,z'):=B^{i,j}(z,z'). \tag{1.19}$$

1.4 Hurwitz numbers and covers of sphere

1.4.1 Simple Hurwitz numbers

Simple Hurwitz numbers $h^\circ_{g,\mu}=h^\circ_{g;\mu_1,\dots,\mu_n}$ enumerate ramified coverings of the 2-sphere by a connected genus g surface, where the ramification profile over infinity is given by the partition $\mu = (\mu_1, \ldots, \mu_n)$, there are simple ramifications over $\mathfrak{b}(g, \mu) = 2g - 2 + n + |\mu|$ fixed points, and there are no further ramifications.

Hurwitz numbers play an important role in the interaction of combinatorics, representation theory of symmetric groups, integrable systems, tropical geometry, matrix models, and intersection theory of the moduli spaces of curves.

1.4.2 Bi-colored maps

Here we discuss the problem of enumeration of *bi-colored maps*. They are decompositions of closed two-dimensional surfaces into polygons of black and white color glued along their sides, considered as combinatorial objects. We count such decompositions of two-dimensional surfaces into a fixed set of polygons with some appropriate weights. This problem is then equivalent to enumeration of Belyi functions with fixed type of local monodromy data over its critical values (following [23], we call such functions *hypermaps*), which is a special case of a more general Hurwitz problem.

Belyi functions are objects of principle importance in algebraic geometry; they allow to detect the algebraic curves defined over the field of algebraic numbers. There is a way to study them in terms of "dessins d'enfants", that is, some embedded graphs in two-dimensional surfaces, see [69] for a survey or [1] for some recent developments.

The local monodromy data of a Belyi function can be controlled by the choice of three partitions of the degree of the function. We consider a special generating function for enumeration of Belyi functions. Namely, we fix the length of the first partition to be n and we introduce some formal variables $x_1, \ldots, x_n$ to control the first partition as an n-point function; we introduce auxiliary parameters t_i, $i \geq 1$, in order to control the number of parts of length i in the second partition as a generating function; and we take the sum of all possible choices of the third partition so that the genus of the surface is equal to $g \geq 0$. This way we get some functions $W_n^{(g)}(x_1, \ldots, x_n)$ that also depend on formal parameters t_i, $i \geq 0$.

More precisely, we have the following.

We are interested in the enumeration of covers of $\mathbb{P}^1$ branched over three points. These covers are defined as follows.

Definition 1.4.1. Consider m positive integers $a_1, \ldots, a_m$ and n positive integers $b_1, \ldots, b_n$. We denote by $\mathcal{M}_{g,m,n}(a_1, \ldots, a_m | b_1, \ldots, b_n)$ the weighted count of branched covers of $\mathbb{P}^1$ by a genus g surface with $m + n$ marked points $f\colon (\mathcal{S}; q_1, \ldots, q_m; p_1, \ldots, p_n) \to \mathbb{P}^1$ such that

- f is unramified over $\mathbb{P}^1 \backslash \{0, 1, \infty\}$;
- the preimage divisor $f^{-1}(\infty)$ is $a_1 q_1 + \ldots a_m q_m$;
- the preimage divisor $f^{-1}(1)$ is $b_1 p_1 + \ldots b_n p_n$;

Of course, a cover f can exist only if $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. In this case $d = b_1 + \cdots + b_n$ is called the degree of a cover.

These covers are counted up to isomorphisms preserving the marked points $p_1, \ldots, p_n$ pointwise and covering the identity on $\mathbb{P}^1$. The weight of a cover is equal to the inverse order of its automorphism group.

The n-point correlation function is defined by

$$\Omega^{(a)}_{g,n}(x_1,\ldots,x_n) := \sum_{m=0}^{\infty} \sum_{\substack{1\le a_1,\ldots,a_m\le a\\ 0\le b_1,\ldots,b_n}} \mathcal{M}_{g,m,n}(a_1,\ldots,a_m|b_1,\ldots,b_n) \prod_{i=1}^{m} t_{a_i} \prod_{j=1}^{n} b_j x_i^{-b_j-1}. \quad (1.20)$$

1.5 Results

1.5.1 Inversion symmetry through Givental group action

Here we present the main result of Chapter 2. This result expresses *inversion symmetry* (a nontrivial symmetry of Frobenius manifolds) through Givental group action.

In [26], Dubrovin derived some symmetries of Frobenius manifolds coming from the elementary Schlesinger transformations of the associated special ODE. One type of transformations, the so-called Legendre-type transformations, refers to the possible choices of flat coordinates for the associated pencil of flat connections that let it be integrated to a solution of the WDVV equation (we are not sure that it is presented in that way anywhere, but implicitly it is explained in [76, 77]). Another transformation is called the *inversion symmetry* and it really looks completely unexpected in terms of the solution of the WDVV equation and its flat coordinates.

Recently, Liu, Xu, and Zhang studied the action of the inversion symmetry on the integrable hierarchies associated to Frobenius manifolds [75]. They described the action of the inversion symmetry on the principal (dispersionless) hierarchies completely; it turns out to be a particular reciprocal transformation. They made some interesting conjectures on the topological deformations of those hierarchies and the genus expansion of the corresponding tau-function.

Given a Frobenius manifold M with flat coordinates $(t^1,\ldots,t^n)$ and potential F, inversion transformation consists of the following change of coordinates:

$$\begin{aligned}
\hat{t}^1 &= \frac{1}{2}\frac{t_\sigma t^\sigma}{t^n},\\
\hat{t}^\alpha &= \frac{t^\alpha}{t^n} \text{ for } \alpha \neq 1,\, n,\\
\hat{t}^n &= -\frac{1}{t^n},
\end{aligned}$$

together with the following change of the potential and the metric:

$$\begin{aligned}
&\hat{F}(\hat{t}) = (t^n)^{-2}\left[F(t) - \frac{1}{2}t^1 t_\sigma t^\sigma\right] = (\hat{t}^n)^2 F + \frac{1}{2}\hat{t}^1\hat{t}_\sigma\hat{t}^\sigma,\\
&\hat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}.
\end{aligned}$$

Main theorem of Chapter 2 is as follows:

Theorem 1.5.1. *The inversion transformation is given by the Givental transformation* $\hat{R} = \exp\left(\sum_{k\geq 1}\left(r_k z^k\right)\hat{}\right)$ *with*

$$r_1 = \begin{pmatrix} 0 & \dots & 0 & 1 \\ 0 & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & 0 \end{pmatrix},$$
$$r_k = 0, \quad k > 1.$$

More precisely, if $\hat{F}(\hat{t})$ *is the inversion transformation of* $F(t)$*, then the local expansion of* $\hat{F}(\hat{t})$ *at* $(0, \dots, 0, -1)$ *is the same as the genus zero part without descendants of the* $\hat{R}$*-transformed potential of the cohomological field theory corresponding to the local expansion of* $F(t)$ *at* $(0, \dots, 0, 1)$*.*

1.5.2 Identification of the Givental formula with the spectral curve topological recursion procedure

Here we present the main result of Chapter 3.

Suppose some local spectral curve is given. For any $i \in \{1, \dots, N\}$ and $k \in \mathbb{Z}_{\geq 0}$ define

$$W_k^i(z) := \sum_{j=1}^{N} d\left(\left(-\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \xi_0^i(z, j)\right).$$

Theorem 1.5.2. *Let* R *be some series of operators on an* N*-dimensional vector space* V *as in Section 3.1. Let* $Z = \hat{R}\hat{\Delta}\mathcal{T}$*, where* $\mathcal{T}$ *is a product of* N *KdV* τ*-functions, be the partition function of the corresponding semi-simple cohomological field theory.*

Define a local spectral curve by the following data

$$\check{B}_{p,q}^{i,j} := [z^p w^q]\frac{\delta^{ij} - \sum_{s=1}^{N} R_s^i(-z)R(-w)_s^j}{z+w} \tag{1.21}$$

and

$$\check{h}_k^i := [z^{k-1}]\left(-R(-z)\right)_{\mathbf{1}}^i \tag{1.22}$$
$$h_1^i := -\frac{1}{2\sqrt{\Delta^i}}. \tag{1.23}$$

Let $\omega_{g,n}$ *be the genus* g*,* n*-pointed topological recursion invariant of this spectral curve and denote by*

$$\Omega(\{v^{d,i}\}) = \left(\sum_{g,d} \omega_{g,d}(z_1, \dots, z_d)\Big|_{W_d^i(z_m) = v^{d,i}} \hbar^{g-1}\right)$$

their sum after a change of variables $W_k^i(z_m) \leftrightarrow v^{d,i}$ *for all* m*. Then the partition function of the cohomological field theory and the topological recursion invariants agree in the following sense:*

$$Z(\{v^{d,i}\}) = \exp\left(\Omega(\{v^{d,i}\})\right). \tag{1.24}$$

1.5.3 Quantum spectral curve for the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$

Here we present the main result of Chapter 4.

Let us recall the Gromov–Witten theory for the case of $X = \mathbb{P}^1$.

The descendant Gromov-Witten invariants of $\mathbb{P}^1$ are defined by

$$\left\langle \prod_{i=1}^{n} \tau_{b_i}(\alpha_i) \right\rangle_{g.n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \psi_i^{b_i} ev_i^*(\alpha_i), \tag{1.25}$$

where $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}$ is the virtual fundamental class of the moduli space,

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d) \longrightarrow \mathbb{P}^1$$

is a natural morphism defined by evaluating a stable map at the i-th marked point of the source curve, $\alpha_i \in H^*(\mathbb{P}^1,\mathbb{Q})$ is a cohomology class of the target $\mathbb{P}^1$, and ψ_i is the tautological cotangent class in $H^2(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d),\mathbb{Q})$. We denote by 1 the generator of $H^0(\mathbb{P}^1,\mathbb{Q})$, and by $\omega \in H^2(\mathbb{P}^1,\mathbb{Q})$ the Poincaré dual to the point class. We assemble the Gromov-Witten invariants into particular generating functions as follows. For every (g,n) in the stable sector $2g-2+n>0$, we define the *free energy* of type (g,n) by

$$F_{g,n}(x_1,\ldots,x_n) := \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n}. \tag{1.26}$$

Here the degree d is determined by the dimension condition of the cohomology classes to be integrated over the virtual fundamental class. We note that (1.26) contains the class $\tau_0(1)$. For unstable geometries, we introduce two functions

$$S_0(x) := x - x\log x + \sum_{d=1}^{\infty} \left\langle -\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}} \right\rangle_{0,1}^{d}, \tag{1.27}$$

$$S_1(x) := -\frac{1}{2}\log x + \frac{1}{2}\sum_{d=0}^{\infty} \left\langle \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x^{b+1}} \right)^2 \right\rangle_{0,2}^{d}. \tag{1.28}$$

Main result of Chapter 4 is as follows:

Theorem 1.5.3. *The wave function*

$$\Psi(x,\hbar) := \exp\left(\frac{1}{\hbar} S_0(x) + S_1(x) + \sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x,\ldots,x) \right) \tag{1.29}$$

satisfies the quantum curve equation of an infinite order

$$\left[\exp\left(\hbar\frac{d}{dx}\right) + \exp\left(-\hbar\frac{d}{dx}\right) - x \right] \Psi(x,\hbar) = 0. \tag{1.30}$$

Moreover, the free energies $F_{g,n}(x_1,\ldots,x_n)$ as functions in n-variables, and hence all the Gromov-Witten invariants (1.25)*, can be recovered from the equation* (1.30) *alone, using the mechanism of the spectral curve topological recursion.*

1.5.4 Polynomiality of Hurwitz numbers and a new proof of the ELSV formula

Here wi present the main results of Chapter 5.

The ELSV formula [33] gives an expression for connected Hurwitz numbers in terms of intersection numbers on the moduli space of curves:

$$h^{\circ}_{g,\mu} = \mathfrak{b}(g,\mu)! \prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g^{\vee}(1)}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}. \tag{1.31}$$

The Bouchard-Mariño conjecture [12] (proved by now in several different papers) is also a relation of Hurwitz numbers to matrix models. Consider the spectral curve

$$x = ye^{-y} \tag{1.32}$$

equipped with the two-point function

$$\frac{dydy'}{(y-y')^2}. \tag{1.33}$$

Then the n-point functions $w_{g,n}$ produced from this data via the spectral curve topological recursion are equal to

$$\sum_{\mu_1,\dots,\mu_n} \frac{h^{\circ}_{g;\mu_1,\dots,\mu_n}}{\mathfrak{b}(g,\mu)!} \, \mu_1 \dots \mu_n \, x_1^{\mu_1-1} \dots x_n^{\mu_n-1} dx_1 \dots dx_n. \tag{1.34}$$

These two statements are known to be equivalent [36], see also [89]. We revisit this equivalence and present this argument in a new way.

Let us describe the existing proofs of both statements. All proofs of the ELSV formula [33, 59, 82, 74] are based, either directly or, as the original one, indirectly, on the computation of the Euler class of the fixed locus of the $\mathbb{C}^*$-action on the space of (relative stable) maps to $\mathbb{C}\mathrm{P}^1$. All mathematically rigorous proofs of the Bouchard-Mariño conjecture [43, 79] use the ELSV formula and the Laplace transform of the so-called cut-and-join equation for Hurwitz numbers, the basic equation that also allows to reconstruct them recursively. There is one more proof of the Bouchard-Mariño conjecture in [11] that goes through the construction of a matrix model for Hurwitz numbers and a direct derivation of the topological recursion, but it will require plenty of subtle analytic work to make it really mathematically rigorous. Of course, since the ELSV formula is proved independently, the fact [36, 89] that the two statements are equivalent implies the Bouchard-Mariño conjecture as well.

There is still a number of interesting questions on both statements. The first question is whether it is possible to prove the Bouchard-Mariño conjecture independently of the ELSV formula. The second question is whether there exists any way to derive the ELSV formula combinatorially, rather than via the computation of the Euler class mentioned above. For example, all Hurwitz numbers can be computed combinatorially, either using the character formula, or, equivalently, using the semi-infinite wedge formalism, or recursively via the cut-and-join equation. On the other hand, the intersection number in the ELSV formula can also be computed combinatorially. Indeed, we can use the

Mumford formula [80] for the Chern characters of the Hodge bundle in order to reduce the intersection number in the ELSV formula to intersection numbers of ψ-classes, and any intersection number of ψ-classes can be computed using the Witten-Kontsevich theorem [98, 67]. The third question, posed e.g. in [96, 57], is the following. The structure of the ELSV formula implies some polynomiality property of Hurwitz numbers, that is

$$h^{\circ}_{g;\mu_1,\dots,\mu_n} = \mathfrak{b}(g,\mu)! \left(\prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!}\right) P_{g,n}(\mu_1,\dots,\mu_n),$$

where $P_{g,n}(\mu_1,\dots,\mu_n)$ are some polynomials in $\mu_1,\dots,\mu_n$. Though this fact is completely combinatorial, the only way to prove it known up to now is to use the ELSV formula. So, the third question we consider here is whether it is possible to prove this polynomiality in some direct way, without any usage of the ELSV formula.

We provide full answer to all three questions. It is organized in the following way. First, we prove in Section 5.1 the polynomiality of Hurwitz numbers directly from the definition in terms of the semi-infinite wedge formalism. Our argument is a refinement of an argument by Okounkov and Pandharipande in [83]. Then, using the polynomiality property of Hurwitz numbers we are able to derive in Section 5.2 the Bouchard-Mariño conjecture directly from the cut-and-join equation. Then, since we have an equivalence of the Bouchard-Mariño conjecture and the ELSV formula, we immediately derive the ELSV formula in a new way. In Section 5.3 we review the correspondence between the topological recursion and the Givental theory, with a special focus on the 1-dimensional case, and in Section 5.4 we provide a (slightly refined) proof of the equivalence of the ELSV formula and the Bouchard-Mariño conjecture.

Theorem 1.5.4. *The Hurwitz numbers $h^{\circ}_{g;\mu_1,\dots,\mu_n}$ for $(g,n) \notin \{(0,1),(0,2)\}$ can be expressed as follows:*

$$h^{\circ}_{g;\mu_1,\dots,\mu_n} = (2g+|\mu|+n-2)! \left(\prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!}\right) P_{g,n}(\mu_1,\dots,\mu_n), \tag{1.35}$$

where $P_{g,n}(\mu_1,\dots,\mu_n)$ is some polynomial in $\mu_1,\dots,\mu_n$.

Basically this theorem gives the form of the ELSV formula without specifying the precise formulas for the coefficients. This property (in a bit stronger form) was conjectured in [56] and then proved in [57], with the help of the ELSV formula. Still, the question whether this property can be derived without using the ELSV formula remained open [96]. This is precisely what we do in Chapter 5: we prove this statement without using the ELSV formula.

Define the generating function for the connected Hurwitz numbers $h^{\circ}_{g;\mu}$ in the following way:

$$H^{\circ}_{g,n} := \sum_{\mu_1,\dots,\mu_n \in \{1,2,\dots\}} \frac{h^{\circ}_{g;\mu_1,\dots,\mu_n}}{\mathfrak{b}(g,\mu)!} x_1^{\mu_1}\dots x_n^{\mu_n}. \tag{1.36}$$

Theorem 1.5.4 implies that, for $(g,n) \notin \{(0,1),(0,2)\}$,

$$H^{\circ}_{g,n} = \sum_{\substack{k_1,\dots,k_n \in \\ \{0,1,\dots,K_{g,n}\}}} c_{k_1\dots k_n} \prod_{i=1}^{n} \sum_{\mu_i=1}^{\infty} \frac{\mu_i^{\mu_i+k_i}}{\mu_i!} x_i^{\mu_i}, \tag{1.37}$$

where $c_{k_1\dots k_n}$ are the coefficients of the polynomials $P_{g,n}$ from Theorem 5.1.1, and $K_{g,n}$ is the highest power appearing in $P_{g,n}$.

Define

$$\rho_k(x) := \sum_{m=1}^{\infty} \frac{m^{m+k}}{m!} x^m. \tag{1.38}$$

for $(g,n) \notin \{(0,1),(0,2)\}$, and

$$W_{g,n}(t_1,\dots,t_n) = \sum_{\substack{k_1,\dots,k_n \in \\ \{0,1,\dots,K_{g,n}\}}} c_{k_1\dots k_n} \prod_{i=1}^{n} \rho_{k_i+1}(t_i). \tag{1.39}$$

In the unstable cases we define the functions $W_{g,n}$ by setting explicitly

$$W_{0,1}(t_1) = 0, \tag{1.40}$$

$$W_{0,2}(t_1,t_2) = \frac{t_1^2(t_1+1)t_2^2(t_2+1)}{(t_2-t_1)^2}. \tag{1.41}$$

In Chapter 5 we give a new proof, using the above polynomiality result, the following

Theorem 1.5.5 (Bouchard-Mariño conjecture)**.** *The polynomials $W_{g,n}$ can be determined by the either of the following recursive formulas*

$$\begin{aligned} W_{g,n}(t_1,t_{L'}) &= -\operatorname*{res}_{z=0}\Big(K(z,t_1)\,\widetilde{W}_{g,n}\big(\frac{1}{z},\frac{1}{z};t_{L'}\big)\Big) \\ &= \operatorname*{res}_{z=0}\Big(K(z,t_1)\,\widetilde{W}_{g,n}\big(\frac{1}{z},\frac{1}{\sigma(z)};t_{L'}\big)\Big) \\ &= -\operatorname*{res}_{z=0}\Big(K(z,t_1)\,\widetilde{W}_{g,n}\big(\frac{1}{\sigma(z)},\frac{1}{\sigma(z)};t_{L'}\big)\Big) \end{aligned}$$

where

$$K(z,t_1) = \frac{t_1^2(1+t_1)}{2(1-z\,t_1)(1-\sigma(z)\,t_1)}\,\frac{z\,dz}{z+1}$$

and $\sigma(z)$ is defined by

$$(1+z)\,e^{-z} = (1+\sigma(z))\,e^{-\sigma(z)}. \tag{1.42}$$

Using this and the Gromov–Witten/spectral curve topological recursion correspondence discussed in the previous section allows us to give a new proof of the ELSV formula:

$$h^{\circ}_{g;\mu_1,\dots,\mu_n} = \mathfrak{b}(g,\mu)!\left(\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g}{\prod_{i=1}^{n}(1-\mu_i\psi_i)}\right)\prod_{i=1}^{n}\frac{\mu_i^{\mu_i}}{\mu_i!}. \tag{1.43}$$

1.5.5 Spectral curve topological recursion for counting of bi-colored maps

Here we present the main result of Chapter 6.

Since the combinatorial problem of counting bi-colored maps allows us to arrange te answers into generating functions $W_n^{(g)}(x_1,\dots,x_n)$, it makes sense to check whether these functions $W_n^{(g)}(x_1,\dots,x_n)$ can be reproduced via the *topological recursion* [48].

The theory of spectral curve topological recursion, actually, has initially occurred as a way to solve a set of loop equations satisfied by the correlation functions of a particular class of matrix model [40, 16, 47, 18].

In fact, bi-colored maps are a standard representation of correlation functions of a two matrix model, see a survey in [35] or more recent paper [5], and the topological recursion in this case is derived in [18]. However, the general question that one can pose there is whether there is any way to relate the topological recursion to the intrinsic combinatorics of bi-colored maps. There are two steps of derivation of the topological recursion in [18]. First, using skillfully chosen changes of variables in the matrix integral, one can define the *loop equations* for the correlation functions [38]. Then, via a sequence of formal computations, one can determine the spectral curve and prove the topological recursion.

The loop equations of a formal matrix model are equivalent to some combinatorial properties of bi-colored maps [95]. In this thesis we exhibit these combinatorial relations deriving the loop equations directly from the intrinsic combinatorics of the bi-colored maps. This procedure can be generalized for deriving combinatorially the loop equations of an arbitrary formal matrix model. This allows us to prove the topological recursion for the functions $W_n^{(g)}(x_1, \ldots, x_n)$ in a purely combinatorial way.

As a motivating example, we use a recent conjecture posed by Do and Manescu in [23]. They considered the enumeration problem for a special case of our bi-colored maps, where all polygons of the white color have the same length a. In this case, they conjectured that this enumeration problem satisfies the topological recursion and proposed a particular spectral curve. So, as a special case of our result, we prove their conjecture, and it appears to be a purely combinatorial proof. Though similar problems were considered a lot recently [66, 7, 8], the conjecture of Do and Manescu was not covered there.

There is a general principle that associates to a given spectral curve its quantization, which is a differential operator called *quantum spectral curve* [60]. Conjecturally, this operator should annihilate the wave function, which is, roughly speaking, the exponent of the generating series of functions $\int^x \cdots \int^x W_n^{(g)}(x_1, \ldots, x_n) dx_1 \cdots dx_n$. We show that this general principle works in this case, namely, we derive the quantum spectral curve directly from the same combinatorics of loop equations. This generalizes the main result in [23] .

The combinatorics that we use in the analysis of bi-colored maps is in fact of a more general nature. The same idea of derivation of the loop equations can be used in more general settings. In particular, we show how it would work for the enumeration of 4-colored maps, where the topological recursion was derived from the loop equations by Eynard in [39].

All this allows us to give a combinatorial proof of the following:

Corollary 1.5.6. *The generating series $\Omega_{g,k}^{(a)}(x_1, \ldots, x_k)$ can be computed by topological recursion with a genus 0 spectral curve*

$$E^{(a)}(x, y) = y \left(\sum_{i=1}^{a} t_i y^{i-1} - x \right) + 1 = 0 \tag{1.44}$$

and the genus 0 2-point function defined by the corresponding Bergmann kernel, i. e.

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} \tag{1.45}$$

for a global coordinate z on the genus 0 spectral curve.

We would like to mention that most of the results of paper [102], on which Chapter 6 is based, were derived independently by Borot [10] while this paper was being written, and the combinatorial approach to loop equations in Section 6.3 was also independently derived by Eynard [34, Chapter 8].

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Chapter 2

Givental graphs and inversion symmetry

This chapter is based on paper [104], joint work with S. Shadrin and L. Spitz. In this chapter we express Dubrovin's inversion transformation through the Givental group action.

In Section 2.1 we recall Y.-P. Lee's formulas for the operators of the infinitesimal deformation and explain them in terms of graphs. In Section 2.2 we use the graphical representation of the Givental group action in order to find a particular group element that performs the inversion symmetry. In Section 2.3 we reproduce the elementary Schlesinger transformation that was the origin of the inversion symmetry (for that we heavily use the results obtained in [15] in multi-KP approach to Frobenius manifold structures). Finally, in Section 2.4 we reproduce the formulas of Liu, Xu, and Zhang for the transformation of the Hamiltonians of the principle hierarchy under the inversion symmetry (this comes as a very special case of the general deformation formulas for the Hamiltonians obtained in [14]).

2.1 Givental group action as a sum over graphs

In this section we explain an interpretation of the Givental group action [52, 54] on cohomological field theories as a sum over graphs.

2.1.1 Cohomological field theories and Frobenius manifolds

Consider the space of partition functions for n-dimensional cohomological field theories

$$Z = \exp(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g) \tag{2.1}$$

in variables $t^{d,\mu}$, $d \geq 0$, $\mu = 1,\dots,n$. Such a partition function is always tame; the weighted degree of any monomial $\hbar^g t^{d_1,\mu_1}\cdots t^{d_k,\mu_k}$ occurring with non-zero coefficient is not more than $3g-3+k$, where the weight of $\hbar$ is 0, and the weight of $t^{d,\mu}$ is d. There is a fixed scalar product η on the vector space $V := \langle e_1,\dots,e_n\rangle$ of primary fields corresponding to the indices $\mu = 1,\dots,n$. Furthermore, we will denote by $e_{\mathbf{1}}$ the vector in V that plays the role of the unit in the underlying family of Frobenius algebras.

In this chapter, we will always work in flat coordinates, that is, $\eta_{\alpha\beta} = \delta_{\alpha,n-\beta}$ and $e_{\mathbf{1}} = e_1$.

The information of the genus zero part of a cohomological field theory is equivalent to the information of a Frobenius manifold. That is, given a cohomological field theory with genus zero partition function $\mathcal{F}_0$, we obtain the potential F of a Frobenius manifold by

$$F(t^1, \ldots, t^n) = \mathcal{F}_0(t^{d,\mu})|_{t^{d,\mu}=0 \text{ for } d>0}$$

where we identify $t^\mu := t^{0,\mu}$.

On the other hand, given a Frobenius manifold we can uniquely reconstruct the genus zero descendant part using topological recursion ([78]). Although the construction we describe below is for the full genus expansion of a cohomological field theory, it can be restricted to the genus zero part (with or without descendants), and thus interpreted as an action on the space of Frobenius manifolds. This is what we will do in Example 2.1.3 and the subsequent sections.

Notation 2.1.1. Define the so-called correlators

$$\langle \tau_{d_1}(\alpha_1)\tau_{d_2}(\alpha_2)\cdots\tau_{d_k}(\alpha_k)\rangle_g$$

by

$$\mathcal{F}_g = \sum \frac{\langle \tau_{d_1}(\alpha_1)\tau_{d_2}(\alpha_2)\cdots\tau_{d_k}(\alpha_k)\rangle_g}{|\operatorname{Aut}((\alpha_i, d_i)_{i=1}^k)|} t^{d_1,\alpha_1}\cdots t^{d_k,\alpha_k}, \tag{2.2}$$

where $|\operatorname{Aut}((\alpha_i, d_i)_{i=1}^k)|$ denotes the number of automorphisms of the collection of multi-indices (α_i, d_i) and where the sum is such that it includes each monomial $t^{d_1,\alpha_1}\cdots t^{d_n,\alpha_n}$ exactly once.

2.1.2 Differential operators

Let us remind the reader of the original formulation, due to Y.-P. Lee, of the infinitesimal Givental group action in terms of differential operators [70, 71, 72].

Consider a sequence of operators $r_l \in \operatorname{Hom}(V, V)$, $l \geq 1$, such that the operators with odd (resp., even) indices are symmetric (resp., skew-symmetric). Then we denote by $(r_l z^l)\hat{}$ the following differential operator:

$$\begin{aligned}(r_l z^l)\hat{} := &-(r_l)^\mu_{\mathbf{1}} \frac{\partial}{\partial t^{l+1,\mu}} + \sum_{d=0}^\infty t^{d,\nu}(r_l)^\mu_\nu \frac{\partial}{\partial t^{d+l,\mu}} \\ &+ \frac{\hbar}{2}\sum_{i=0}^{l-1}(-1)^{i+1}(r_l)^{\mu,\nu}\frac{\partial^2}{\partial t^{i,\mu}\partial t^{l-1-i,\nu}}.\end{aligned} \tag{2.3}$$

Givental observed that the action of the operators

$$\hat{R} := \exp(\sum_{l=1}^\infty (r_l z^l)\hat{})$$

on formal power series preserves tameness. The main theorem of [51] states that this action preserves the property that Z is the generating function of the correlators of a cohomological field theory with the target space (V, η) (see also [65, 93]).

Remark 2.1.2. The action of the operators described above is usually referred to as the action of the *upper triangular group*. There is also a *lower triangular group* action, but we do not consider it in the present chapter.

2.1.3 Expressions in terms of graphs

We now describe the Givental action in terms of graphs. Consider a connected graph γ of arbitrary genus, and with leaves. To such a graph we assign some additional structure. First, we choose an orientation on each edge of the graph, in an arbitrary way (the contribution of a graph will not depend on these choices). Second, to each element of the graph (a leaf, an edge, a vertex) we associate some tensor over the vector space $V[[z]]$ (where z is a formal variable) that also depends on $\hbar$ and $t^{d,\mu}$ for $d \geq 0$ and $1 \leq \mu \leq n$. This graph equipped with an additional structure of such a type we denote by $\check{\gamma}$.

Notation 2.1.3. By a *half-edge*, we mean either an edge together with a choice of one of the two adjacent vertices, or a leaf. If we want to talk only about the first of these two, we will use *half of an internal edge*.

Leaves

Leaves are decorated by one of two types of vectors. The first type corresponds to the second term of the operator (2.3) and is given by

$$\mathcal{L} := \exp\left(\sum_{l=1}^{\infty} r_l z^l\right)\left(\sum_{d=0}^{\infty}\sum_{\mu=1}^{n} e_\mu t^{d,\mu} z^d\right). \tag{2.4}$$

The second type of decoration is given by the vector

$$\mathcal{L}_0 := -z \cdot \left(\exp(\sum_{l=1}^{\infty} r_l z^l) - \mathbf{I}\right)(e_{\mathbf{1}}) \tag{2.5}$$

and corresponds to the dilaton shift (the first term of the operator (2.3)).

Edges

An edge is already oriented. We expect to decorate it with a bivector. Using the scalar product we can turn any (skew-)symmetric operator into a bivector. However, this requires a choice of sign. Some choice of sign was already made in the differential operator (2.3) when we used the symbol $(r_l)^{\mu\nu}$. Let us fix this choice. In the case of a skew-symmetric operator, the bivector is also skew-symmetric, so we have to use the orientation of the underlying edge in order to fix the ambiguity. It will be obvious later on that nothing depends on the choice of orientations on edges.

So, we are going to assign a bivector $\mathcal{E} \in (V[[z]])^{\otimes 2}$ to an oriented edge. The first copy of $V[[z]]$ is associated to the input, the second to the output of the oriented edge. For clarity, we will denote the formal variable corresponding to the first copy by z, and the one corresponding to the second copy by w. We put

$$\mathcal{E} = \tilde{\mathcal{E}}\eta,$$

where $\tilde{\mathcal{E}} \in \mathrm{Hom}(V,V)[[z,w]]$ is given by

$$\tilde{\mathcal{E}} := -\hbar \cdot \frac{\exp\left(\sum_{l=1}^{\infty}(-1)^{l-1} r_l z^l\right)\exp\left(\sum_{l=1}^{\infty} r_l w^l\right) - \mathbf{I}}{z+w}.$$

Let us rewrite this formula in a more convenient way. Denote by $r(z)$ the series $\sum_{l=1}^{\infty} r_l z^l$. Then $\tilde{\mathcal{E}}$ is equal to

$$\begin{aligned}\tilde{\mathcal{E}} &= -\hbar \cdot \frac{\exp(r(z)^*)\exp(r(w)) - \mathbf{I}}{z+w} \\ &= -\hbar \cdot \frac{\exp\left(-r(-z)\right)\exp(r(w)) - \mathbf{I}}{z+w}.\end{aligned} \tag{2.6}$$

(cf. the same formula in [93]).

The change of the orientation of an edge corresponds to the replacement of an operator with its adjoint and the simultaneous interchange of z and w. From Equation (2.6) it is obvious that $\tilde{\mathcal{E}}^*|_{z\leftrightarrow w} = \tilde{\mathcal{E}}$. Using the symmetry of the metric, we see that nothing depends on the choice of orientations on edges.

Vertices

The collection of correlators of order n corresponding to a formal power series $\mathcal{F}_g$ in variables $t^{d,\mu}$ can be considered as a tensor $\mathcal{V}_g[n] \in (V^*[[z]])^{\otimes n}$. Namely, the tensor $\mathcal{V}_g[n]$ sends $e_{\mu_1} z_1^{d_1} \otimes \cdots \otimes e_{\mu_n} z_n^{d_n}$ to
the correlator $\langle \tau_{d_1}(e_{\mu_1}) \cdots \tau_{d_n}(e_{\mu_n})_g \rangle$ (which is just a number), and we extend this definition linearly.

We want to apply an element of the Givental group to the series Z; this means that we decorate the vertices of index n exactly by the tensor

$$\mathcal{V}[n] := \sum_{g\geq 0} \hbar^{g-1} \mathcal{V}_g[n]. \tag{2.7}$$

Contraction of tensors

Consider a decorated graph $\check{\gamma}$. We have associated vectors in $V[[z]]$ to leaves and bivectors in $(V[[z]])^{\otimes 2}$ to edges (the former depending on $\hbar$ and $t^{d,\mu}$, the later depending on $\hbar$). Furthermore, for each edge we have associated one copy of $V[[z]]$ with the input of the edge and the other with the output. At each vertex, we now contract the tensor $\mathcal{V}[n]$ with the tensor product of the decorations of the half edges corresponding to the vertex, where n is the index of the vertex. The result is a number depending on $\hbar$ and $t^{d,\mu}$ which we denote by $\mathcal{C}(\check{\gamma})$.

The final formula

Finally, we sum over all possible decorated graphs like this, weighted by the inverse order of their automorphisms to obtain a formal power series in $t^{d,\mu}$ that also depends on $\hbar$. In a formula:

$$\log(\hat{R}(Z)) = \sum_{\check{\gamma}\in\check{\Gamma}} \frac{1}{\#\,\mathrm{Aut}(\check{\gamma})} \mathcal{C}(\check{\gamma}) \tag{2.8}$$

where $\check{\Gamma}$ denotes the set of all decorated graphs as above, and $\mathrm{Aut}(\check{\gamma})$ is the set of automorphisms of $\check{\gamma}$. From now on we will use a decorated graph and the function of $\hbar$ and $t^{d,\mu}$ assigned to it by the graphical formalism interchangeably.

It follows directly from the combinatorics of graphs that the result is represented as a formal power series of the same form as in Equation (2.1).

Remark 2.1.4. Note that for any graph the only choice in the decoration is that for each leaf, it can either be decorated by $\mathcal{L}$ or $\mathcal{L}_0$.

Furthermore, the decorations on the edges and leaves are defined as sums. Using the linearity of the functions with which we contract at the vertices, we can replace a graph with a leaf or edge decorated with a sum by a sum of graphs which only differ from the original one by replacing this sum with its individual terms. We will use this freedom in computations; thus, we will work graphs that are not elements of $\check{\Gamma}$ as well.

Remark 2.1.5. *The formal variable z.* The contraction of tensors couples the power of the formal variable z to the first index of the variable $t^{d,\mu}$. Thus, in the context of cohomological field theory, the power of z should be interpreted as keeping track of the power of the ψ-class appearing at the corresponding half-edge.

The trivial example

We discuss the trivial example of the Givental action, that is, we assume that $r_l = 0$, $l = 1, 2, \dots$. In this case $\mathcal{E} = 0$, so the only connected graphs that give a non-trivial contribution are the graphs with one vertex and no edges. Furthermore, $\mathcal{L}_0$ is also zero, so we only need to compute

$$\frac{1}{n!}\mathcal{V}[n](\underbrace{\mathcal{L}\otimes\cdots\otimes\mathcal{L}}_{n\ times})$$

which is the nth homogeneous component of $\sum_{g\geq 0}\hbar^{g-1}\mathcal{F}_g$, as we can see directly from the definition of $\mathcal{V}[n]$. Therefore, the sum over all graphs just gives us the initial series Z.

Dilaton equation and topological recursion relation

We remind the reader of the well-known topological recursion relation and dilaton equation [98] which hold for any cohomological field theory.

In terms of correlators, the *dilaton equation* is given by

$$\langle \tau_1(1)\tau_{b_1}(\alpha_1)\dots\tau_{b_k}(\alpha_k)\rangle_g = (2g-2+k)\,\langle \tau_{b_1}(\alpha_1)\dots\tau_{b_k}(\alpha_k)\rangle_g \tag{2.9}$$

for any g.

In terms of graphical formalism, the dilaton equation has the following interpretation: whenever we are given a graph with a leaf that is marked by $e_1 z$, the dilaton equation allows us to remove that leaf entirely, at the same time multiplying the resulting graph by $(2g-2+k)$, where k is the number of leaves/edges going from the corresponding vertex (the removed leaf is not counted).

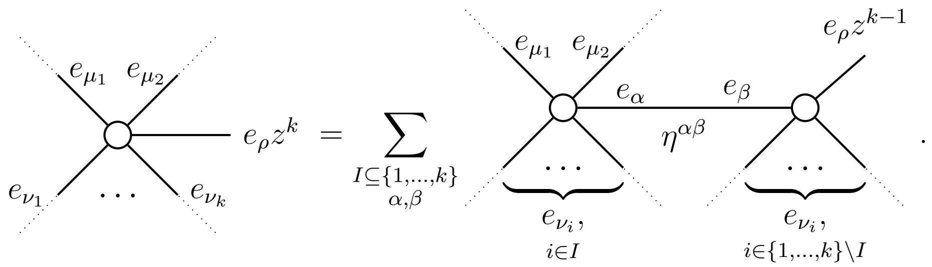
Consider the generating function for descendant classes

$$D = \exp\left(\sum_{d,\alpha} t^{d,\alpha}\tau_d(\alpha)\right)$$

Then the genus zero *topological recursion relation* takes the following form (for $d_1 > 0$):

$$\langle \tau_{d_1}(\alpha_1)\tau_{d_2}(\alpha_2)\tau_{d_3}(\alpha_3)D\rangle_0 = \sum_{\lambda,\sigma} \langle \tau_{d_1-1}(\alpha_1)\tau_0(\lambda)D\rangle_0 \, \eta^{\lambda\sigma} \, \langle \tau_0(\sigma)\tau_{d_2}(\alpha_2)\tau_{d_3}(\alpha_3)D\rangle_0 \quad (2.10)$$

The topological recursion relation has the following graphical interpretation. Whenever we are given a graph with a leaf marked by $e_i z^k$ for some i and $k > 0$, we can remove a ψ-class (lower the power of z) in the following way. Pick any two other half-edges on the same vertex (vertices in graphs that have a non-zero contribution are always at least trivalent) and split the vertex into two vertices connected by an edge marked by $\sum_{\alpha,\beta} \eta^{\alpha\beta} e_\alpha \otimes e_\beta$. Put the two chosen half-edges on one vertex and the original leaf on the other, now marked by $e_i z^{k-1}$. Take the sum over all possible distributions of the other half edges of the original vertex over the two new vertices. It is easy to see that this procedure does not depend on the choice of two half-edges, and represents the topological recursion relation. In an equation (dotted lines represent either edges which connect the vertices to some other parts of the graph or just leaves):



Example; inversion symmetry in two dimensions

To illustrate the graphical formalism in practice, we explicitly compute one of the terms of the two-dimensional case of *the inversion transformation* defined and studied in general in Section 2.2. Let F_0 be the potential of a two-dimensional Frobenius manifold given by

$$F_0(t^1,t^2) = \frac{(t^1)^2t^2}{2} + \sum_{k\geq 3} \frac{\sigma_k}{k!}(t^2)^k \quad (2.11)$$

for some set of numbers $\{\sigma_k | k \geq 3\}$, and let $r(z) = \sum_k r_k z^k$ be the matrix series given by

$$r := r_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad r_k = 0 \text{ for all } k > 1. \quad (2.12)$$

As above, using the topological recursion relation in genus zero we can consider $F_0(t^1,t^2)$ as a restriction to the small phase space of some full descendant genus zero potential $\mathcal{F}_0\left(\{t^{1,d},t^{2,d}\}_{d=0}^{\infty}\right)$, identifying t^1, t^2 with $t^{1,0}, t^{2,0}$ and setting all other variables equal to zero. Define $\tilde{\mathcal{F}}_0$ to be the genus zero part of $\log(\exp(\widehat{r(z)})\exp(\hbar^{-1}\mathcal{F}_0))$. We compute the coefficient $\tilde{\sigma}_5$ of $(t^{2,0})^5$ in $\tilde{\mathcal{F}}_0$ using the graphical formalism (as usual, we regard $\tilde{\mathcal{F}}_0$ as the exponential generating series for its coefficients).

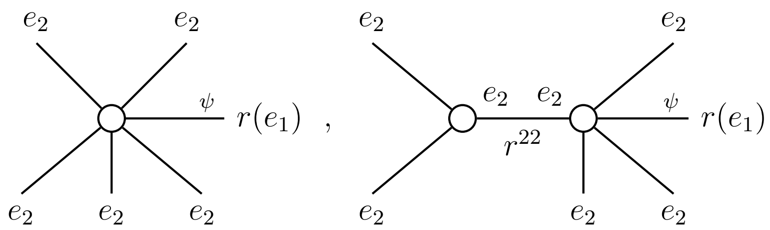


Figure 2.1: Two of the graphs contributing to $\tilde{\sigma}_5$ where one of the leaves is decorated using $\mathcal{L}_0$. Note that in this case, both of their contributions are zero, because $\mathcal{L}_0 = 0$.

Since the variables $t^{d,\mu}$ only appear in the formalism when we have leaves decorated by $\mathcal{L}$, graphs contributing to $\tilde{\sigma}_5$ must have precisely five leaves decorated by $\mathcal{L}$. Furthermore, in these decorations, only the terms which depend solely on $t^{0,2}$ out of all $t^{d,\mu}$ contribute to $\tilde{\sigma}_5$. By equation (2.12) we have

$$\exp\left(\sum_{l=1}^{\infty} r_l z^l\right) = 1 + rz, \tag{2.13}$$

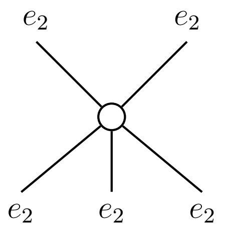
therefore, after total expansion using the linearity of Remark 2.1.4, leaves that were originally decorated by $\mathcal{L}$ have at most one ψ-class.

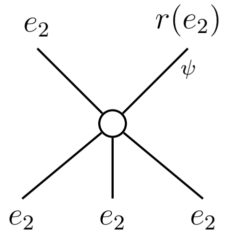
In principle, there could be extra leaves which are decorated by $\mathcal{L}_0$ (note that the variables $t^{d,\mu}$ do not appear in $\mathcal{L}_0$). We have drawn two graphs with such leaves in Figure 2.1. However, it follows immediately from equation (2.12) that $\mathcal{L}_0 = 0$, so the dilaton term plays no role in this computation.

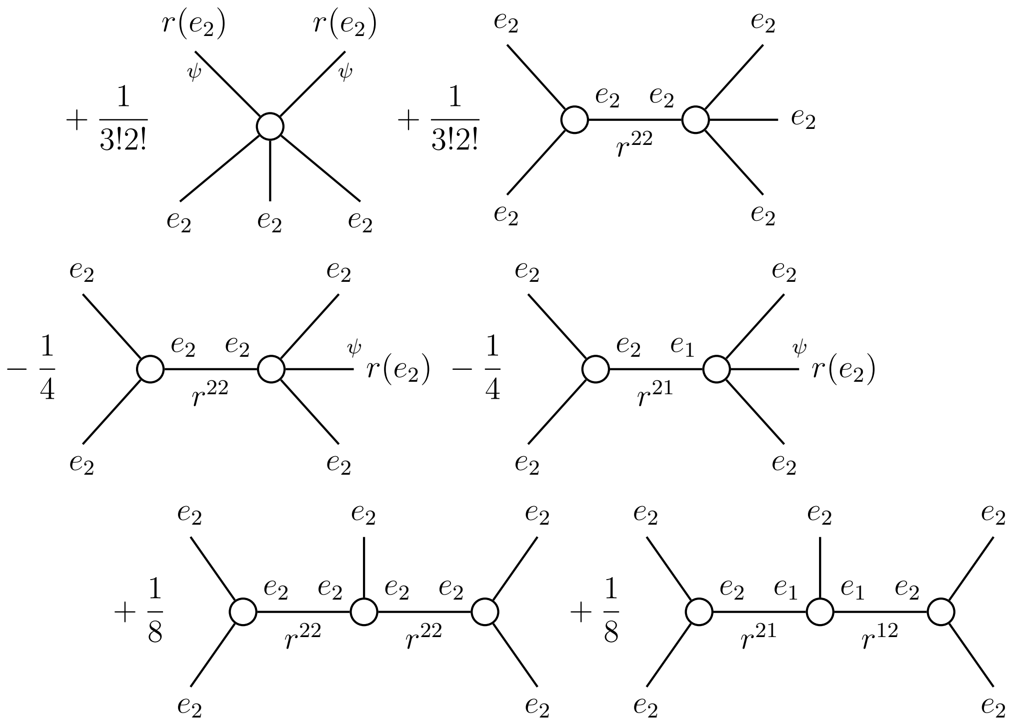
Once again using equation (2.12), we see that in this case the edge decoration simplifies to

$$\mathcal{E} = -\sum_{\mu,\nu} r^{\mu,\nu} e_\mu \otimes e_\nu. \tag{2.14}$$

By the tameness property, any vertex for which the total number of ψ-classes (that is, the total power of z) at half-edges connected to it is equal to some d, must have valence at least $d+3$ for the graph to have a non-zero contribution. Taking into account that vertices at which no ψ-class appears must have either precisely three leaves, two of which are decorated with e_1 and one with e_2, or only leaves decorated with e_2, it is easy to see that $\tilde{\sigma}_5$ is given by the following sum:

$\frac{1}{5!}\tilde{\sigma}_5 = \frac{1}{5!}$ 

$+\frac{1}{4!}$ 



Let us explain the notation. The coefficients in front of the graphs are just the inverse orders of the corresponding automorphism groups. The labels at the leaves are the ones coming from $\mathcal{L}$, where we have left out the variables $t^{d,\mu}$, and where we have replaced z by ψ to remind the reader that it keeps track of the power of ψ-class at that leaf. The decorations at the edges are split between the input, output and middle of the edge. For instance, an edge decorated by $r^{22}e_2 \otimes e_2$ is shown with a label e_2 near the input and output of the edge, and a label r^{22} in the middle. The minus signs in the third line come from the minus sign in equation (2.14).

Note that in the original description of the algorithm, the first three graphs would have appeared as one graph with the sums of different decortions on the leaves, as would the second three graphs and also the last two graphs. We have used the linearity described in Remark 2.1.4 to write them as the sums of graphs that appear above.

To get the result of this computation we first note that $r^{\mu\nu} = r^{\mu}_{\rho}\eta^{\rho\nu}$. In our case this means that $r^{11} = 1$, and all other entries are 0. Thus, only the first three terms survive. Using either the dilaton equation or topological recursion, and using that $r(e_2) = e_1$ in this case, we see immediately that

$$\tilde{\sigma}_5 = \sigma_5 + 10\sigma_4 + 20\sigma_3.$$

This agrees with formula (2.22) for the inversion transformed potential, as it should.

2.1.4 Equivalence of descriptions

It follows directly from the standard correspondence between differential operators of the type (2.3) and Feynman-type formulas in terms of graphs ([53], cf. [86]) that the descriptions of the Givental group action given in Sections 2.1.2 and 2.1.3 are equivalent.

For simplicity, we will first assume that $r_l(e_{\mathbf{1}}) = 0$ for all l, allowing us to ignore the dilaton term. In that case, the only thing we have to show is that

$$\begin{aligned}\hat{R} &:= \exp\left(\sum_{d\geq 0, l\geq 1} t^{d,\nu}(r_l)^\mu_\nu \partial_{d+l,\mu} + \frac{\hbar}{2}\sum_{i,j\geq 0}(-1)^{i+1}\partial_{i,\mu}(r_{i+j+1})^{\mu\nu}\partial_{j,\nu}\right) \\ &= \exp\left(\sum_{d\geq 0, l\geq 1} t^{d,\nu}(r_l)^\mu_\nu \partial_{d+l,\mu}\right)\exp\left(\sum_{k,l\geq 0}(V_{k,l})^{\mu\nu}\partial_{k,\mu}\partial_{l,\nu}\right)\end{aligned} \tag{2.15}$$

where $\partial_{d,\mu} = \dfrac{\partial}{\partial t^{d,\mu}}$ and $V_{k,l}$ is defined by

$$-\frac{\hbar}{2}\frac{\exp(-r(-z))\exp(r(w)) - \mathbf{I}}{z+w} = \sum V_{k,l} z^k w^l$$

and we assume summation over repeated Greek indices (we will do so for the rest of this section). Equation (2.15) follows from the Campbell-Baker-Hausdorff formula in the following way. Write

$$X = \sum_{l\geq 1, d\geq 0}(r_l)^\mu_\nu t^{d,\nu}\partial_{d+l,\mu}, \quad Y = \frac{\hbar}{2}\sum_{i,j\geq 0}(-1)^{i+1}(r_{i+j+1})^{\mu\nu}\partial_{i,\mu}\partial_{j,\nu}$$

for the linear and quadratic parts in the exponent in $\hat{R}$ respectively. Then Y commutes with any (iterated) commutator of X and Y containing at least one copy of Y. Therefore, it follows from Campbell-Baker-Hausdorff that $e^{X+Y} = e^X e^Z$, where

$$\begin{aligned}Z &:= \frac{-e^{-\operatorname{ad}_X}+1}{\operatorname{ad}_X}Y = \sum_{p\geq 0}\frac{(-1)^p(\operatorname{ad}_X)^p}{(p+1)!}Y \\ &= \frac{\hbar}{2}\sum_{p\geq 0}\sum_{s+t=p}\sum_{f_1,\dots,f_s\geq 0}\sum_{g_1,\dots,g_t\geq 0}\sum_{i,j\geq 0}\frac{\binom{p}{s}}{(p+1)!}(-1)^{i+1+f_1+\dots+f_s+s}\cdot \\ &\qquad \cdot (r_{f_s}\cdots r_{f_1}r_{i+j+1}r_{g_1}\cdots r_{g_t})^{\mu\nu}\partial_{i+f_1+\dots+f_s,\mu}\partial_{j+g_1+\dots+g_t,\nu}.\end{aligned} \tag{2.16}$$

In the last equality we use the fact that r_l is symmetric when l is odd, and skew-symmetric when l is even. Writing $Z = \sum_{k,l} Z_{kl}\partial_k\partial_l$, it is easy to see that

$$(z+w)\sum_{k,l} Z_{kl}z^k w^l = -\frac{\hbar}{2}(\exp(-r(-z))\exp(r(w)) - \mathbf{I}) \tag{2.17}$$

by expanding the right hand side and using the equality

$$\frac{1}{k!(n-k-1)!n} + \frac{1}{(k-1)!(n-k)!n} = \frac{1}{k!(n-k)!}.$$

This completes the proof of the equivalence of descriptions for the case where $r_l e_{\mathbf{1}} = 0$.

For the general case, it is clear that replacing X by

$$\tilde{X} = X_1 + X_2 = -\sum_{l\geq 1}(r_l)^\mu_{\mathbf{1}}\partial_{l+1,\mu} + \sum_{l\geq 1, d\geq 0}(r_l)^\mu_\nu t^{d,\nu}\partial_{d+l,\mu},$$

will not affect any of the arguments made above. That is, since X_1 commutes with Y, the same argument proves that $e^{\tilde{X}+Y} = e^{\tilde{X}} e^{Z}$. Therefore, it remains to show that

$$\exp\left(-\sum_{l\geq 1}(r_l)^\mu_{\mathbf{1}}\partial_{l+1,\mu} + \sum_{d\geq 0, l\geq 1}(r_l)^\mu_\nu t^{d,\nu}\partial_{d+l,\mu}\right) = \exp\left(\sum_{d\geq 0, l\geq 1}(r_l)^\mu_\nu t^{d,\nu}\partial_{d+l,\mu}\right)\left(\sum_{l\geq 1}(W_l)^\mu_{\mathbf{1}}\partial_{l,\mu}\right) \tag{2.18}$$

where W_l is defined by

$$\sum_{l\geq 1} W_l z^l = (-z)\left(\exp\left(\sum_{l\geq 1} r_l z^l\right) - \mathbf{I}\right).$$

Since X_1 commutes with any iterated commutator of X_1 and X_2 including X_1 at least once, we have $e^{X_1+X_2} = e^{X_2}e^T$, where

$$\begin{aligned} T &:= \frac{-e^{-\operatorname{ad}_{X_2}}+1}{\operatorname{ad}_{X_2}} X_1 \\ &= -\sum_{p,l}\sum_{f_1,\dots,f_p}\frac{1}{(p+1)!}(r_{f_p}\cdots r_{f_1} r_l)^\mu_{\mathbf{1}}\partial_{f_1+\dots+f_p+l+1,\mu} = \sum_{l\geq 1}(W_l)^\mu_{\mathbf{1}}\partial_{l,\mu}. \end{aligned} \tag{2.19}$$

This completes the proof of the equivalence of the graphical and operator representation of Givental's theory.

2.2 Inversion transformation

The so-called *inversion transformation* is an important example of a transformation that gives a discrete symmetry of Frobenius structures. Namely, if one applies this transformation to any given Frobenius manifold, the resulting object is again a Frobenius manifold.

It turns out that in terms of the Givental group action one can express this transformation in a particularly nice way.

Let us recall the definition of the inversion transformation ([26]). Given a Frobenius manifold M with flat coordinates $(t^1,\dots,t^n)$ and potential F, this transformation consists of the following change of coordinates:

$$\begin{aligned} \hat{t}^1 &= \frac{1}{2}\frac{t_\sigma t^\sigma}{t^n}, \\ \hat{t}^\alpha &= \frac{t^\alpha}{t^n} \text{ for } \alpha \neq 1,\, n, \\ \hat{t}^n &= -\frac{1}{t^n}, \end{aligned}$$

together with the following change of the potential and the metric:

$$\begin{aligned} \hat{F}(\hat{t}) &= (t^n)^{-2}\left[F(t) - \frac{1}{2}t^1 t_\sigma t^\sigma\right] = (\hat{t}^n)^2 F + \frac{1}{2}\hat{t}^1\hat{t}_\sigma\hat{t}^\sigma, \\ \hat{\eta}_{\alpha\beta} &= \eta_{\alpha\beta}. \end{aligned}$$

We will also need the inverse of the inversion transformation:

$$
\begin{aligned}
t^1 &= \frac{1}{2}\frac{\hat{t}_\sigma \hat{t}^\sigma}{\hat{t}^n},\\
t^\alpha &= -\frac{\hat{t}^\alpha}{\hat{t}^n} \text{ for } \alpha \neq 1,\, n,\\
t^n &= -\frac{1}{\hat{t}^n}.
\end{aligned}
$$

We prove the following

Theorem 2.2.1. *The inversion transformation is given by the Givental transformation* $\hat{R} = \exp\left(\sum_{k\geq 1}\left(r_k z^k\right)\hat{}\,\right)$ *with*

$$
\begin{aligned}
r_1 &= \begin{pmatrix} 0 & \dots & 0 & 1\\ 0 & \dots & 0 & 0\\ \vdots & \ddots & \vdots & \vdots\\ 0 & \dots & 0 & 0 \end{pmatrix},\\
r_k &= 0, \quad k > 1.
\end{aligned}
$$

More precisely, if $\hat{F}(\hat{t})$ *is the inversion transformation of* $F(t)$*, then the local expansion of* $\hat{F}(\hat{t})$ *at* $(0,\dots,0,-1)$ *is the same as the genus zero part without descendants of the* $\hat{R}$*-transformed potential of the cohomological field theory corresponding to the local expansion of* $F(t)$ *at* $(0,\dots,0,1)$*.*

Proof of Theorem 2.2.1. We are going to check that the coefficients of the local expansion of $\hat{F}(\hat{t})$ at $(0,\dots,0,-1)$ and the coefficients of the genus zero part without the descendants of the $\hat{R}$-transformed cohomological field theory potential corresponding to the local expansion of $F(t)$ at $(0,\dots,0,1)$ agree.

Let us determine the coefficients of $\hat{F}$. Recall that in flat coordinates, the metric is given by $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$. Thus, the potential has the form

$$
F(t) = \frac{1}{2}t^1\left(t^1 t^n + \dots + t^n t^1\right) - \frac{1}{2}t^1 t^1 t^n + H\left(t^2,\dots,t^n\right), \tag{2.20}
$$

for some function H.

Note that we consider cohomological field theories as well as Frobenius potentials to be defined up to addition of any terms of order 2 or lower in t's, so we disregard such terms here and below.

Computing the inversion-transformed potential, we have

$$
\begin{aligned}
\hat{F}(\hat{t}) = \frac{1}{2}\hat{t}^1\left(\hat{t}^1\hat{t}^n + \dots + \hat{t}^n\hat{t}^1\right) - \frac{1}{2}\hat{t}^1\hat{t}^1\hat{t}^n +\\
+ \frac{1}{8\hat{t}^n}\left(\hat{t}^2\hat{t}^{n-1} + \dots + \hat{t}^{n-1}\hat{t}^2\right)^2 + \hat{t}^n\hat{t}^n H\left(-\frac{\hat{t}^2}{\hat{t}^n},\dots,-\frac{\hat{t}^{n-1}}{\hat{t}^n},-\frac{1}{\hat{t}^n}\right).
\end{aligned} \tag{2.21}
$$

Recall the correlator notation for the coefficients of the potential and denote by

$$
\frac{1}{|\operatorname{Aut}((\alpha))|}\left\langle \hat{\tau}_0\left(\alpha_1\right)\dots\hat{\tau}_0\left(\alpha_N\right)\right\rangle_H^{\mathrm{I}}
$$

the coefficient of $\hat{t}^{\alpha_1}\dots\hat{t}^{\alpha_N}$ in the inversion-transformed potential coming from the last term of (2.21), and by

$$\frac{1}{|\operatorname{Aut}((\alpha))|}\left\langle\hat{\tau}_0\left(\alpha_1\right)\dots\hat{\tau}_0\left(\alpha_N\right)\right\rangle_Q^{\mathrm{I}}$$

the coefficient of $\hat{t}^{\alpha_1}\dots\hat{t}^{\alpha_N}$ in the inversion-transformed potential coming from the second-to-last term of (2.21), where $|\operatorname{Aut}((\alpha))|$ denotes the number of automorphisms of the collection of indices α_i.

We are interested in the local expansion near $(0,\dots,0,-1)$, so we put $\hat{t}^n=-1+\epsilon$. Then, for the last term we have

$$\begin{aligned}
&(1-\epsilon)^2 H\left(\frac{\hat{t}^2}{1-\epsilon},\dots,\frac{\hat{t}^{n-1}}{1-\epsilon},\frac{1}{1-\epsilon}\right)=\\
&\quad=\sum_{N+p\geq 3}\ \sum_{2\leq\alpha_1\leq\dots\leq\alpha_N\leq n-1}\frac{\hat{t}^{\alpha_1}\dots\hat{t}^{\alpha_N}\epsilon^p\left(1-\epsilon\right)^{2-p-N}}{|\operatorname{Aut}((\alpha))|\,p!}H_{\alpha_1\dots\alpha_N\underbrace{n\dots n}_{p}}\\
&\quad=\sum_{\substack{N+p\geq 3\\ k\geq 0}}\ \sum_{2\leq\alpha_1\leq\dots\leq\alpha_N\leq n-1}\frac{\binom{N+k+p-3}{k}}{|\operatorname{Aut}((\alpha))|\,p!}H_{\alpha_1\dots\alpha_N\underbrace{n\dots n}_{p}}\hat{t}^{\alpha_1}\dots\hat{t}^{\alpha_N}\epsilon^{p+k},
\end{aligned}\tag{2.22}$$

where H with subscripts stands for the value of the respective multiple partial derivative of H taken at $(0,\dots,0,1)$. In terms of correlators this means that

$$\begin{aligned}
&\langle\hat{\tau}_0(\alpha_1)\dots\hat{\tau}_0(\alpha_N)\,(\hat{\tau}_0(n))^q\rangle_H^{\mathrm{I}}\\
&\qquad=\sum_{p+k=q}\frac{q!}{p!}\binom{N+k+p-3}{k}H_{\alpha_1\dots\alpha_N\underbrace{n\dots n}_{p}}
\end{aligned}\tag{2.23}$$

for $2\leq\alpha_1,\dots,\alpha_N\leq n-1$.

For the second-to-last term we have

$$\begin{aligned}
&\frac{1}{8\hat{t}^n}\left(\hat{t}^2\hat{t}^{n-1}+\dots+\hat{t}^{n-1}\hat{t}^2\right)^2=\\
&\qquad-\sum_{\substack{2\leq\alpha\leq\beta\leq n+1-\beta\leq n+1-\alpha\leq n-1\\ k}}\frac{1}{|\operatorname{Aut}_2(\alpha,\beta)|}\epsilon^k\hat{t}^\alpha\hat{t}^{n+1-\alpha}\hat{t}^\beta\hat{t}^{n+1-\beta},
\end{aligned}\tag{2.24}$$

where $|\operatorname{Aut}_2(\alpha,\beta)|$ is defined in the following way. Define $\bar{\alpha}=n+1-\alpha$ for any $2\leq\alpha\leq n-1$. Then $|\operatorname{Aut}_2(\alpha,\beta)|=1$ if all four numbers $\alpha,\beta,\bar{\alpha}$ and $\bar{\beta}$ are pair-wise different, $|\operatorname{Aut}_2(\alpha,\beta)|=2$ if two of them are equal, but not equal to the other two, and $|\operatorname{Aut}_2(\alpha,\beta)|=8$ if all of them coincide. In terms of correlators, this means that for $2\leq\alpha,\beta,\bar{\alpha},\bar{\beta}\leq n-1$

$$\left\langle\hat{\tau}_0\left(\alpha\right)\hat{\tau}_0\left(\bar{\alpha}\right)\hat{\tau}_0\left(\beta\right)\hat{\tau}_0\left(\bar{\beta}\right)\left(\hat{\tau}_0\left(n\right)\right)^k\right\rangle_Q^{\mathrm{I}}=-\frac{k!\,|\operatorname{Aut}((\alpha,\beta,\bar{\alpha},\bar{\beta}))|}{|\operatorname{Aut}_2(\alpha,\beta)|},\tag{2.25}$$

while Q-correlators of any other form vanish.

Since the Givental transformation acts trivially on cubic terms, the part containing $\hat{t}^1$ obviously coincides with what is coming from the Givental transformation.

Let us describe the situation on the Givental side. The main point we are going to use is the very simple form of the matrices r_l, where the only non-zero entry of r_1 is $(r_1)^1_n = 1$, and r_l is identically zero for all other l. Furthermore, we are interested only in decorated graphs where $t^{d,\mu}$ with $d > 0$ do not enter the decorations, since we aim at recovering the Frobenius potential, which is the genus zero part without descendants. Thus we will write $t^\mu := t^{0,\mu}$ to simplify expressions.

Taking all this into account, by equation (2.4) we have $\mathcal{L} = ze_1t^n + \sum_{\mu=1}^n e_\mu t^\mu$ for the decoration of ordinary leaves, and no dilaton leaves since the expression (2.5) vanishes entirely in our case. Furthermore, for the internal edges we have $\mathcal{E} = -e_1 \otimes e_1$ by equation (2.6).

Let us find which decorated graphs will give a nonzero contribution. We see that z always comes coupled to e_1, which allows us to use the dilaton equation (2.9) to express all graphs with z entering their decorations in terms of graphs without z entering their decorations.

Since the contraction with the tensor associated with a vertex is a linear operation, we can represent a given graph as a sum of 2^k graphs, where k is the number of the leaves, such that instead of the sum $ze_1t^n + \sum_{\mu=1}^n e_\mu t^\mu$ on each leaf we will have either just ze_1t^n or just $\sum_{\mu=1}^n e_\mu t^\mu$. Then the dilaton equation implies that the contribution of each of these 2^k graphs is a multiple of the contribution of a graph resulting from removal of all ze_1t^n leaves from the given graph. All these resulting graphs then obviously do not have any z entering into their decorations.

Let us then find which graphs with no z in the decorations after using the dilaton equation can give a non-zero contribution. The claim is that they are either single-vertex ones with any number of leaves, or trivalent ones with no more than two internal edges going out of each vertex. Recall that the tensors $\mathcal{V}[n]$ appearing on n-valent vertices are built from the coefficients of the nth homogeneous part of the original potential. Then the claim follows from the form of the decorations we have on the internal edges, namely $-e_1 \otimes e_1$, and the fact that t^1 enters the the original potential only in cubic terms. More precisely, if a vertex has an internal edge going from it, then, since the corresponding tensor $\mathcal{V}[n]$ gets contracted with $-e_1 \otimes e_1$, n should be equal to 3 because only $\mathcal{V}[3]$ has non-zero components with one of the indices equal to 1.

Furthermore, if there is only one internal edge going from a given vertex, then there are two leaves attached to this vertex decorated by $\sum_{\mu=1}^n e_\mu t^\mu$. Taking the linearity into account, we can represent the given decorated graph as a sum of n^2 graphs for which these two leaves are decorated by $e_\mu t^\mu$ and $e_\nu t^\nu$ for $\mu, \nu \in \{1, \dots, n\}$ respectively. From the form of the original potential (2.20) it follows that out of these graphs only the ones with $2 \leq \mu \leq n-1$ and $\nu = n + 1 - \mu$ give a non-zero contribution.

By similar considerations a vertex with more than two internal edges attached to it will contribute zero, and on a vertex with precisely two internal edges attached to it only e_nt^n survives as the decoration of the single leaf attached to it.

With help of the linearity property we now totally expand all the decorated graphs we have after applying the Givental transformation. By the consideration above, we are left with the following sum, where each graph is of course multiplied by the inverse order of its automorphism group. First, there are all possible graphs with one vertex and any number of leaves decorated by $e_\mu t^\mu$ for any μ and any number of leaves decorated by ze_1t^n. Second, there are all possible trivalent graphs with at least one internal edge in total and no more than two internal edges going from each vertex, with $-e_1 \otimes e_1$ decorating internal edges, $e_n t^n$ decorating the single leaf attached to a vertex with two internal edges, and $e_\mu t^\mu$ and $e_{n-\mu+1} t^{n-\mu+1}$ for $2 \leq \mu \leq n-1$ decorating two leaves attached to a vertex with only one internal edge going from it. Furthermore, all graphs obtained from the above trivalent ones by adding any number of leaves decorated by $z e_1 t^n$ to any number of vertices are also included in the sum. One can find the graphical representation of all of these graphs below.

Thus we have described all relevant graphs giving the Givental-transformed potential. Now let us show that the Frobenius potential recovered from them precisely coincides with the inversion-transformed potential.

The contribution coming from the one-vertex graphs turns out to coincide with the last term of (2.21). More precisely, if we denote the contribution of these one-vertex graphs to the coefficient of $\hat{t}^{\alpha_1} \dots \hat{t}^{\alpha_N} \epsilon^q$ in the Givental-transformed potential by

$$\frac{1}{|\operatorname{Aut}((\alpha))|\, q!} \langle \hat{\tau}_0(\alpha_1) \dots \hat{\tau}_0(\alpha_N)\, (\hat{\tau}_0(n))^q \rangle_H^{\mathrm{G}}$$

we have

$$\frac{1}{|\operatorname{Aut}((\alpha))|\, q!} \langle \hat{\tau}_0(\alpha_1) \cdots \hat{\tau}_0(\alpha_N)\, (\hat{\tau}_0(n))^q \rangle_H^{\mathrm{G}} = \\ = \frac{1}{|\operatorname{Aut}((\alpha))|} \sum_{p+k=q} \frac{1}{k!\, p!} \left\langle \tau_0(\alpha_1) \cdots \tau_0(\alpha_N)\, (\tau_0(n))^p\, (\tau_1(1))^k \right\rangle \tag{2.26}$$

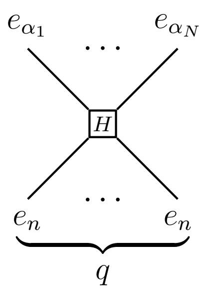
(here on the right hand side the correlator corresponds to original, non-transformed potential). Using the dilaton equation (2.9) we get

$$\begin{aligned} &\langle \hat{\tau}_0(\alpha_1) \cdots \hat{\tau}_0(\alpha_N)\, (\hat{\tau}_0(n))^q \rangle_H^{\mathrm{G}} \\ &\quad = q! \sum_{p+k=q} \frac{(N+k+p-3) \cdots (N+p-2)}{p!\, k!} \cdot \\ &\qquad \cdot \langle \tau_0(\alpha_1) \cdots \tau_0(\alpha_N)\, (\tau_0(n))^p \rangle \\ &\quad = \sum_{p+k=q} \frac{q!}{p!} \binom{N+k+p-3}{k} \langle \tau_0(\alpha_1) \dots \tau_0(\alpha_N)\, (\tau_0(n))^p \rangle \\ &\quad = \sum_{p+k=q} \frac{q!}{p!} \binom{N+k+p-3}{k} H_{\alpha_1 \dots \alpha_N \underbrace{n \dots n}_{p}}, \end{aligned} \tag{2.27}$$

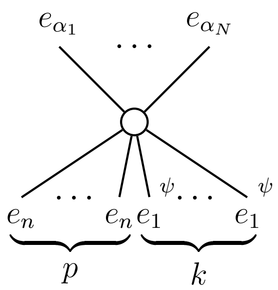
which exactly coincides with $\langle \hat{\tau}_0(\alpha_1) \dots \hat{\tau}_0(\alpha_N)\, (\hat{\tau}_0(n))^q \rangle_H^{\mathrm{I}}$ on the inversion-transformed side (2.23).

In terms of graphs, equations (2.26) and (2.27) can be expressed in the following way:

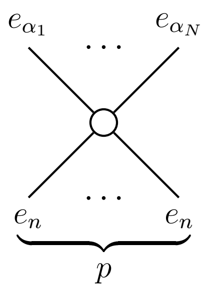
$$\langle \hat{\tau}_0(\alpha_1) \dots \hat{\tau}_0(\alpha_N)\, (\hat{\tau}_0(n))^q \rangle_H^{\mathrm{G}} =: \tag{2.28}$$



$$= \sum_{p+k=q} \binom{q}{k}$$



$$= \sum_{p+k=q} \frac{q!}{p!} \binom{N+k+p-3}{k}$$



Now we look at the graphs with a non-zero number of internal edges, which turn out to correspond precisely to the second-to-last term of (2.21). By the discussion above about graphs, they contribute only to correlators of the form

$$\left\langle \hat{\tau}_0(\alpha)\, \hat{\tau}_0(\bar{\alpha})\, \hat{\tau}_0(\beta)\, \hat{\tau}_0\left(\bar{\beta}\right) \left(\hat{\tau}_0(n)\right)^k \right\rangle_Q^{\mathrm{G}}$$

(where the index G attached to correlator means that it corresponds to the Givental-transformed potential, and the index Q means that we take just the part coming from graphs with at least one internal edge), and this contribution is the following (for $2 \leq \alpha, \beta, \bar{\alpha}, \bar{\beta} \leq n-1$):

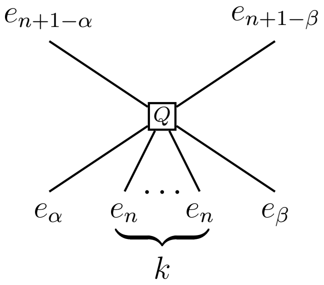
$$\begin{aligned}
&\frac{\left\langle \hat{\tau}_0(\alpha)\, \hat{\tau}_0(\bar{\alpha})\, \hat{\tau}_0(\beta)\, \hat{\tau}_0\left(\bar{\beta}\right) \left(\hat{\tau}_0(n)\right)^k \right\rangle_Q^{\mathrm{G}}}{k!\, |\operatorname{Aut}((\alpha,\beta,\bar{\alpha},\bar{\beta}))|} \\
&\quad = \frac{1}{|\operatorname{Aut}_2(\alpha,\beta)|} \left(\sum_{m_1+m_2=k} \frac{1}{m_1!\, m_2!} \left\langle \tau_0(\alpha)\, \tau_0(\bar{\alpha}) \left(\tau_1(1)\right)^{m_1} \tau_0(\mu) \right\rangle \right. \\
&\qquad\qquad (r_1)^{\mu\nu} \left\langle \tau_0(\nu)\, \tau_0(\beta)\, \tau\left(\bar{\beta}\right) \left(\tau_1(1)\right)^{m_2} \right\rangle \\
&\quad + \sum_{\substack{m_1+m_2+m_3\\=k-1}} \frac{1}{m_1!\, m_2!\, m_3!} \left\langle \tau_0(\alpha)\, \tau(\bar{\alpha}) \left(\tau_1(1)\right)^{m_1} \tau_0(\mu_1) \right\rangle (r_1)^{\mu_1\nu_1} \\
&\quad \left\langle \tau_0(\nu_1)\, \tau_0(n) \left(\tau_1(1)\right)^{m_2} \tau_0(\mu_2) \right\rangle (r_1)^{\mu_2\nu_2} \left\langle \tau_0(\nu_2)\, \tau_0(\beta)\, \tau\left(\bar{\beta}\right) \left(\tau_1(1)\right)^{m_2} \right\rangle \\
&\qquad\qquad + \cdots \\
&\quad + \left\langle \tau_0(\alpha)\, \tau_0(\bar{\alpha})\, \tau_0(\mu_1) \right\rangle (r_1)^{\mu_1\nu_1} \left\langle \tau_0(\nu_1)\, \tau_0(n)\, \tau_0(\mu_2) \right\rangle (r_1)^{\mu_2\nu_2} \cdots \\
&\qquad \left. \cdots \left\langle \tau_0(\nu_k)\, \tau_0(n)\, \tau_0(\mu_{k+1}) \right\rangle (r_1)^{\mu_{k+1}\nu_{k+1}} \left\langle \tau_0(\nu_{k+1})\, \tau_0(\beta)\, \tau\left(\bar{\beta}\right) \right\rangle \right). \qquad (2.29)
\end{aligned}$$

The contribution of each product of correlators on the right hand side is given by $(-1)^p\, m_1! \cdot \cdots \cdot m_{p+1}!$, where $p+1$ is the number of correlators in the product. Thus, the result is equal to

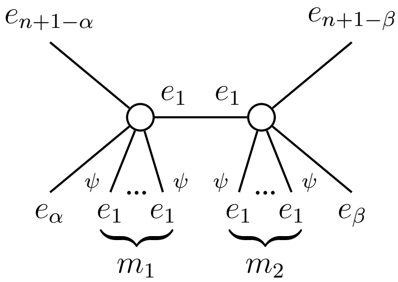
$$\left\langle \hat{\tau}_0(\alpha)\,\hat{\tau}_0(\bar{\alpha})\,\hat{\tau}_0(\beta)\,\hat{\tau}_0\left(\bar{\beta}\right)(\hat{\tau}_0(n))^k \right\rangle_Q^{\mathrm{G}} = \frac{k!\,|\operatorname{Aut}((\alpha,\beta,\bar{\alpha},\bar{\beta}))|}{|\operatorname{Aut}_2(\alpha,\beta)|}\sum_{p=1}^{k+1}(-1)^p\binom{k+1}{p} = -\frac{k!\,|\operatorname{Aut}((\alpha,\beta,\bar{\alpha},\bar{\beta}))|}{|\operatorname{Aut}_2(\alpha,\beta)|}, \quad (2.30)$$

which coincides with what we have on the inversion-transformed side (2.25). This concludes the proof. In terms of graphs formula (2.29) takes the following form:

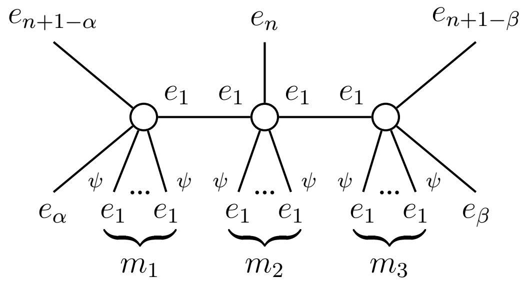
$$\left\langle \hat{\tau}_0(\alpha)\,\hat{\tau}_0(\bar{\alpha})\,\hat{\tau}_0(\beta)\,\hat{\tau}_0\left(\bar{\beta}\right)(\hat{\tau}_0(n))^k \right\rangle_Q^{\mathrm{G}} =:$$



$$= \frac{k!\,|\operatorname{Aut}((\alpha,\beta,\bar{\alpha},\bar{\beta}))|}{|\operatorname{Aut}_2(\alpha,\beta)|}\Bigg(-\sum_{m_1+m_2=k}\frac{1}{m_1!\,m_2!}$$

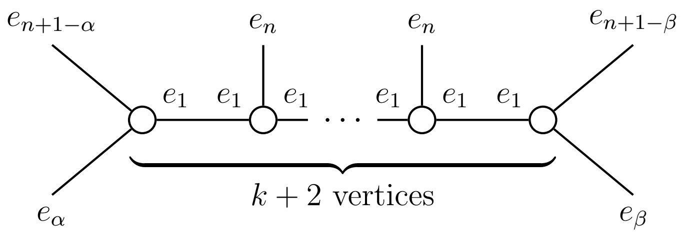


$$+\sum_{m_1+m_2+m_3=k-1}\frac{1}{m_1!\,m_2!\,m_3!}$$



$$+ \quad \cdots$$

$$+(-1)^{k+1}$$



$$\Bigg) = -\frac{k!\ \operatorname{Aut}(\alpha,\beta,\bar{\alpha},\bar{\beta})}{\#\operatorname{Aut}_2(\alpha,\beta)}. \quad (2.31)$$

□

Remark 2.2.2. Note that in this Section we used neither semi-simplicity of the Frobenius structure nor the Euler vector field. The Givental group element that we obtained acts perfectly without any extra assumptions, except for the analyticity of $F(t)$ at point $(0,\dots,0,1)$. Even this last assumption is not necessary due to the following reasons. Since inversion transformation is singular at the origin, in Dubrovin's original formulation analyticity at some point other than the origin is implicitly assumed. This domain of analyticity can very well not include $(0,\dots,0,1)$. However, one can deal with this in Givental approach by considering not only the action of $\hat{R}$-operator, but also the action of $\hat{\Psi}$-operator [52], which has a simple form. This will make formulas a bit less nice, so for this reason we consider here only the case when $F(t)$ is analytical at $(0,\dots,0,1)$. Going to the more general case with the help of $\hat{\Psi}$-operator is rather straightforward.

2.3 Relation to Schlesinger transformations

In the semi-simple case, the inversion transformation of Frobenius structures originates from a Schlesinger transformation of a special differential operator [26]:

$$\Lambda = \partial_z - U - \frac{1}{z}[\Gamma(u), U], \tag{2.32}$$

where U is the diagonal matrix of canonical coordinates

$$U = \begin{pmatrix} u^1 & & \\ & \ddots & \\ & & u^n \end{pmatrix} \tag{2.33}$$

and Γ is the Darboux-Egoroff matrix.

With the help of the results of [15] and our known form of $\hat{R}$-matrix we are now able to reproduce the formula for the Schlesinger transformation for the rotation coefficients γ_{ij} from [26]:

$$\begin{aligned} \hat{\gamma}_{ij} &= \gamma_{ij} - A_{ij}, \\ A_{ij} &= \frac{\sqrt{\partial_i t_1 \partial_j t_1}}{t_1} \end{aligned} \tag{2.34}$$

in Givental approach. We prove the following

Proposition 2.3.1. *$\hat{R}$-transformation of rotation coefficients gives*

$$\hat{\gamma}_{ij} = \gamma_{ij} - \frac{\sqrt{\partial_i t_1 \partial_j t_1}}{1+t_1}. \tag{2.35}$$

Proof of Proposition 2.3.1. Following [15], for the infinitesimal deformation of γ we have (we write all of the indices explicitly in order to get all of the instances of the metric correctly):

$$\begin{aligned} (r_1 z)\hat{}\ \gamma^{ij} &= -\left(\Psi_0\right)^i_\alpha \left(r_1\right)^\alpha_\beta \eta^{\beta\gamma} \left(\Psi_0\right)^j_\gamma, \\ (r_1 z)\hat{}\ \left(\Psi_0\right)^i_\alpha &= \left(\Psi_1\right)^i_\beta \left(r_1\right)^\beta_\alpha - \left(\Psi_0\right)^i_\beta \left(r_1\right)^\beta_\gamma \eta^{\gamma\delta} \left(\Psi_0\right)^j_\delta \delta_{jk} \left(\Psi_1\right)^k_\alpha, \\ (r_1 z)\hat{}\ \left(\Psi_1\right)^i_\alpha &= \left(\Psi_2\right)^i_\beta \left(r_1\right)^\beta_\alpha - \left(\Psi_0\right)^i_\beta \left(r_1\right)^\beta_\gamma \eta^{\gamma\delta} \left(\Psi_0\right)^j_\delta \delta_{jk} \left(\Psi_2\right)^k_\alpha, \\ &\ \vdots \end{aligned} \tag{2.36}$$

Here by Ψ_i, $i = 0, 1, 2, \dots$, we denote the twisted wave functions of the multi-KP hierarchy as in [15].

Taking into account that r_1 has only one nonzero element, we see that this chain actually terminates in the sense that Ψ_2 never enters the expression for the total deformation of γ, and also taking into account that [26, 97]

$$\sum_{k=1}^{n} (\Psi_0)_1^k (\Psi_1)_1^k = t_1, \tag{2.37}$$

we arrive at the following formula for transformed rotation coefficients:

$$\begin{aligned} \hat{\gamma}^{ij} &= \gamma^{ij} - (\Psi_0)_1^i (\Psi_0)_1^j \left(1 - t_1 + (t_1)^2 - (t_1)^3 + \dots\right) \\ &= \gamma^{ij} - \frac{\sqrt{\partial_i t_1 \partial_j t_1}}{1 + t_1}. \end{aligned} \tag{2.38}$$

Here we should recall that in order to get our $\hat{R}$-matrix, we made a shift to the point $(0, \dots, 0, 1)$. Due to flat metric being anti-diagonal with unit components, we also have $t_1 = t^n$. This means that the right hand side of (2.38) actually coincides with that of (2.34), which proves the claim. □

2.4 Implications for integrable hierarchies

The result of Section 2.2 allows us to explicitly obtain inversion-transformed Hamiltonians of the principal hierarchy. We prove the following

Proposition 2.4.1. *Linear span of $\hat{R}$-transformed Hamiltonians of the principal hierarchy coincides with the linear span of inversion-transformed Hamiltonians obtained in [75].*

Proof of Proposition 2.4.1. In order to prove this proposition, we use the results of [14] for the deformation of $\Omega_{\alpha,p;\beta,q}$ under Givental transformation, where

$$\Omega_{\alpha,p;\beta,q} = \frac{\partial^2 F_0}{\partial t^{\alpha,p} \partial t^{\beta,q}}, \tag{2.39}$$

where F_0 is the total genus zero potential with descendants.

In the case of genus zero and for our $\hat{R}$-operator, for the infinitesimal deformation of Hamiltonians

$$\theta_{\alpha,p} = \Omega_{\alpha,p;1,0}, \tag{2.40}$$

we have (following [14]):

$$(r_1 z)\hat{}\ \theta_{\alpha,p} = U\ \theta_{\alpha,p} + \delta_\alpha^n \theta_{1,p+1}, \tag{2.41}$$

where operator U is given by

$$U = -v^n - \frac{1}{2} \sum_{\gamma=1}^{n} v^\gamma v^{n+1-\gamma} \frac{\partial}{\partial v^1} + v^n \sum_{\gamma=1}^{n} v^\gamma \frac{\partial}{\partial v^\gamma}. \tag{2.42}$$

This infinitesimal deformation can be exponentiated to give the inversion-transformed Hamiltonians:

$$\hat{\theta}_{\alpha,p}\left(\hat{v}\right) = \left(\exp\left(U\left(v\right)\right)\theta_{\alpha,p}\left(v\right) + \delta^n_\alpha \exp\left(U\left(v\right)\right)\theta_{1,p+1}\left(v\right)\right)\Big|_{v=\hat{v}}. \tag{2.43}$$

Now we are able to compare our results with the ones of [75], where the inversion-transformed Hamiltonians are given in a bit less explicit form:

$$\begin{aligned}
&\hat{\theta}^{LXZ}_{1,0}(\hat{v}) = -\frac{1}{v^n}, \quad \hat{\theta}^{LXZ}_{1,p}(\hat{v}) = -\frac{\theta_{n,p-1}(v)}{v^n}, \quad p \geq 1, \\
&\hat{\theta}^{LXZ}_{\alpha,p}(\hat{v}) = \frac{\theta_{\alpha,p}(v)}{v^n}, \quad 2 \leq \alpha \leq n-1, \ p \geq 0, \\
&\hat{\theta}^{LXZ}_{n,p}(\hat{v}) = \frac{\theta_{1,p+1}(v)}{v^n}, \quad p \geq 0,
\end{aligned} \tag{2.44}$$

Applying the inverse inversion transformation, we get (for $2 \leq \alpha \leq n-1$)

$$\begin{aligned}
\hat{\theta}^{LXZ}_{\alpha,p}(\hat{v}) &= -\hat{v}^n \theta_{\alpha,p}\left(\frac{\sum_{i=1}^n \hat{v}^i \hat{v}^{n+1-i}}{2\hat{v}^n}, -\frac{\hat{v}^2}{\hat{v}^n}, \dots, -\frac{\hat{v}^{n-1}}{\hat{v}^n}, -\frac{1}{\hat{v}^n}\right) \\
&= (1-\epsilon)\theta_{\alpha,p}\left(\hat{v}^1 - \frac{1}{2}\frac{\sum_{i=2}^{n-1} \hat{v}^i \hat{v}^{n+1-i}}{1-\epsilon}, \frac{\hat{v}^2}{1-\epsilon}, \dots, \frac{\hat{v}^{n-1}}{1-\epsilon}, \frac{1}{1-\epsilon}\right)
\end{aligned} \tag{2.45}$$

Now it's easy to see that the operator from (2.43) makes exactly this change of variables in the function $\theta_{\alpha,p}$, which proves the coincidence of Hamiltonians $\theta_{\alpha,p}$ for $2 \leq \alpha \leq n-1$. In an analogous way, for $\alpha = 1$ and $\alpha = n$ we see that our Hamiltonians do not coincide with the ones of [75] but are instead certain linear combinations of them, which is perfectly valid due to the fact that only the linear span of the collection of Hamiltonians is unambiguously defined. □

Remark 2.4.2. In principle, the result of [14] gives also a deformation formula for the Hamiltonians of the full Dubrovin-Zhang hierarchy that is reduced to Equation (2.41) in genus 0. An advantage of Equation (2.41) is that it is an ODE whose right hand side is *linear* in Hamiltonians, and therefore we can immediately write a nice closed formula for its solution. In the general case the right hand side appears to be quadratic. This still allows to integrate the corresponding ODE formally, but the resulting formulas don't say much about the inverse-transformed Hamiltonians. The same is true for the tau-functions.

Chapter 3

Identification of the Givental formula with the spectral curve topological recursion procedure

This chapter is based on paper [103], joint work with N. Orantin, S. Shadrin and L. Spitz. In this chapter we establish the identification of Givental formula and the local spectral curve topological recursion procedure with an appropriate choice of initial data. Then, as a corollary, we prove the Norbury–Scott conjecture on the spectral curve topological recursion for the Gromov–Witten theory of $\mathbb{C}\mathrm{P}^1$.

This chapter deals with the Givental theory and the spectral curve topological recursion theory; for more information on both theories we refer to [44, 73, 87] as possible sources. In Section 3.1 we recall the Givental theory, and present the Givental formula as a sum over graphs. In Section 3.2 we do the same for the topological recursion. In Section 3.3 we prove the theorem on identification of the two theories and provide a corresponding dictionary. In Section 3.4 we provide the computations showing that this identification works for the spectral curve proposed by Norbury and Scott for the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$.

3.1 Givental group action as a sum over graphs

In this section we review the Givental group action and we remind the reader how it can be used to write the partition function of an N-dimensional semi-simple cohomological field theory as an operator acting on the product of N KdV τ-functions. Using this, we write the partition function for such a cohomological field theory as a sum over decorated graphs. This is essentially the same as what was done in Chapter 2; in the present chapter the contributions are distributed in a slightly different way over the components of the graph to make the comparison with the topological recursion.

3.1.1 Givental group action

We remind the reader of the original formulation, due to Y.-P. Lee, of the infinitesimal Givental group action in terms of differential operators [70, 71, 72].

Consider the space of partition functions for N-dimensional cohomological field theories

$$Z = \exp\left(\sum_{g\geq 0} \hbar^{g-1}\mathcal{F}_g\right) \tag{3.1}$$

in variables $v^{d,i}$, $d \geq 0$, $i = 1, \ldots, N$. There is a fixed scalar product $\eta_{ij} = \delta_{ij}$ on the vector space $V := \langle e_1, \ldots, e_N\rangle$ of primary fields corresponding to the indices $i = 1, \ldots, N$. Furthermore, we will denote by $e_{\mathbf{1}}$ the vector in V that plays the role of the unit.

Later on we will also use the so-called *correlators*

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g$$

which correspond to the coefficients of formal power series $\mathcal{F}_g$ in the following way:

$$\mathcal{F}_g = \sum \frac{\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g}{|\operatorname{Aut}((i_m, d_m)_{m=1}^k)|} v^{d_1,i_1}\cdots v^{d_k,i_k}, \tag{3.2}$$

where $|\operatorname{Aut}((i_m, d_m)_{m=1}^k)|$ denotes the number of automorphisms of the collection of multi-indices (i_m, d_m) and where the sum is such that it includes each monomial $v^{d_1,i_1}\cdots v^{d_k,i_k}$ exactly once. Note that in the special case of a Gromov-Witten theory for some manifold X, these correlators carry the following meaning:

$$\langle \tau_{d_1}(e_{i_1})\tau_{d_2}(e_{i_2})\cdots\tau_{d_k}(e_{i_k})\rangle_g = \sum_{\deg}\int_{[X_{g,k,\deg}]} ev_1^*(e_{i_1})\psi_1^{d_1}\, ev_2^*(e_{i_2})\psi_2^{d_2}\cdots ev_k^*(e_{i_k})\psi_k^{d_k}, \tag{3.3}$$

where $[X_{g,k,\deg}]$ is the moduli space of degree deg stable maps to X of genus-g curves with k marked points, ev_i is the evaluation map at the i^{th} point and ψ correspond to ψ-classes.

Consider a sequence of operators $r_l \in \operatorname{Hom}(V, V)$ for $l \geq 1$, such that the operators with odd (resp., even) indices are symmetric (resp., skew-symmetric). Then we denote by $(r_l z^l)\hat{}$ the following differential operator:

$$(r_l z^l)\hat{} := -(r_l)^i_{\mathbf{1}}\frac{\partial}{\partial v^{l+1,i}} + \sum_{d=0}^{\infty} v^{d,i}(r_l)^j_i\frac{\partial}{\partial v^{d+l,j}} + \frac{\hbar}{2}\sum_{m=0}^{l-1}(-1)^{m+1}(r_l)^{i,j}\frac{\partial^2}{\partial v^{m,i}\partial v^{l-1-m,j}}.$$

Here the indices $i, j \in \{1, \ldots, N\}$ on r_l correspond to the basis $\{e_1, \ldots, e_N\}$ of V, and the index $\mathbf{1}$ corresponds to the unit vector $e_{\mathbf{1}}$. When we write r_l with two upper-indices we mean as usual that we raise one of the indices using the scalar product η.

Given such a sequence of operators r_l, we define an operator series $R(z)$ in the following way

$$R(z) = \sum_{l=0}^{\infty} R_l z^l := \exp\left(\sum_{l=1}^{\infty} r_l z^l\right). \tag{3.4}$$

The quantization $\hat{R}$ of this series is defined by

$$\hat{R} = \exp\left(\sum_{l=1}^{\infty}\left((-1)^l r_l z^l\right)\hat{}\,\right). \tag{3.5}$$

Givental observed that the action of such operators $\hat{R}$ on formal power series Z for which the number of ψ-classes (given by the first index of $v^{d,\mu}$) at any monomial of degree n is no more than $3g-3+n$, is well-defined. The main theorem of [51] states that this action preserves the property that Z is a generating function of the correlators of a cohomological field theory with target space (V,η) (see also [65, 93]).

Remark 3.1.1. Note that this definition of $\hat{R}$ differs from the one in Chapter 2 by the sign $(-1)^l$. It is needed here to agree with Givental's notation in Proposition 3.1.3, cf. [52, Proposition 7.3]. For the same reason, in order to agree with the conventions of Givental, we label in a matrix by the upper index the column and by the lower index the row.

3.1.2 Givental operator for a Frobenius manifold

Let $Z(\{t^{d,\mu}\})$ be the partition function of some N-dimensional semi-simple conformal cohomological field theory. We recall the construction (due to Givental [52, 53, 54], see also Dubrovin [24]) of an operator series $R(z)$ as in the previous section whose quantization takes the product of N KdV τ-functions to Z.

Let F be the restriction of $\log(Z)$ to the genus zero part without descendants. Denote $t^\mu := t^{0,\mu}$. Then F can be interpreted as a formal Frobenius manifold with metric

$$\eta_{\alpha\beta} = \frac{\partial^3 F}{\partial t^{\mathbf{1}}\partial t^\alpha \partial t^\beta} \tag{3.6}$$

and Frobenius algebra structure $c_{\alpha\beta}^\gamma$

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma}. \tag{3.7}$$

We can assume that $\eta_{\alpha\beta} = \delta_{\alpha+\beta,n+1}$ and $e_{\mathbf{1}} = e_1$. According to [26] it is always possible by an appropriate choice of these flat coordinates t^μ.

Canonical coordinates

Another set of coordinates is given by the *canonical coordinates* $\{u^i\}$ which can be found as solutions to Equation (3.54) from [26], and have the property that $\{\partial_i := \partial/\partial u^i\}$ forms a basis of canonical idempotents of the Frobenius algebra product. In these coordinates the metric is diagonal and the unit vector field is given by $e_{\mathbf{1}} = \partial_1 + \cdots + \partial_N$.

Define $\Delta_i := 1/(\partial_i,\partial_i)$ to be the inverse of the square of the length of the i^{th} canonical basis element, and call $\{\partial/\partial v^i := \Delta_i^{1/2}\partial/\partial u^i\}$ the normalized canonical basis in the tangent space. We denote the coordinates corresponding to this basis by v^i, and the formal variables corresponding to these coordinates by $v^{d,i}$. They are precisely the formal variables $v^{d,i}$ appearing in the previous section.

Let U be the matrix of canonical coordinates $U = \mathrm{diag}(u^1,\dots,u^N)$ and denote by Ψ the transition matrix from the flat to the normalized canonical bases. That is, denoting $\mathrm{d}t = (\mathrm{d}t^1,\dots,\mathrm{d}t^N)^{\mathrm{T}}$ and $\mathrm{d}u = (\mathrm{d}u^1,\dots,\mathrm{d}u^N)^{\mathrm{T}}$, one has

$$\Delta^{-1/2}\mathrm{d}u = \Psi \mathrm{d}t, \tag{3.8}$$

where $\Delta = \mathrm{diag}(\Delta_1,\dots,\Delta_N)$.

Remark 3.1.2. Note that Ψ obtained with the help of the definition above depends on the point p of the Frobenius manifold.

Recursion

Construct an operator series $R(z) = \sum_{k\geq 0} R_k z^k$ as in the previous section in the following way.

Recursively define the off-diagonal entries of R_k in normalized canonical coordinates by solving the equation

$$\Psi^{-1}\mathrm{d}(\Psi R_{k-1}) = [\mathrm{d}U, R_k]. \tag{3.9}$$

using $R_0 = \mathbf{I}$ as a base case. Construct the diagonal entries of R_k by integrating the next equation

$$\Psi^{-1}\mathrm{d}(\Psi R_k) = [\mathrm{d}U, R_{k+1}] \tag{3.10}$$

using the fact that the diagonal entries of $[\mathrm{d}U, R_{k+1}]$ are equal to zero. To fix the integration constant, use the Euler equation

$$R_k = -(i_E \mathrm{d}R_k)/k, \tag{3.11}$$

where $E = \sum u^i \partial_i$ is the Euler field (here we use the fact that we started with a conformal cohomological field theory).

This procedure recursively defines R_k for all k. The following proposition is essentially proved in Givental's papers [52, 53].

Proposition 3.1.3. *Let F be a local N-dimensional Frobenius manifold structure, semisimple at the origin, and let (R_k) be the series of operators constructed from this F by the recursive procedure described above, at the origin. Let Ψ and Δ be as above, taken at the origin as well. Then we have the following formula:*

$$\mathcal{F}_0 = \operatorname*{Res}_{\hbar=0} \mathrm{d}\hbar \cdot \log \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}. \tag{3.12}$$

Here $\mathcal{F}_0 = \mathcal{F}_0(\{t^{d,\mu}\})$ is the genus 0 descendant potential of cohomological field theory associated to F; $\mathcal{T}$ is the product of N KdV tau-functions,

$$\mathcal{T} := Z_{\mathrm{KdV}}(\{u^{d,1}\}) \cdots Z_{\mathrm{KdV}}(\{u^{d,N}\});$$

$\hat{\Delta}$ replaces the variables of i^{th} KdV τ-function according to $u^{d,i} = \Delta_i^{1/2} v^{d,i}$ and replaces $\hbar$ with $\Delta_i \hbar$, while $\hat{\Psi}$ is the change of variables $v^{d,i} = \Psi^i_\nu t^{d,\nu}$. The unit for the R-action is given by $(\Psi^1_1, \dots, \Psi^N_1)$.

Remark 3.1.4. In fact, using Teleman's result in [93], one has a refined version of Equation (3.12):

$$Z = \hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}. \tag{3.13}$$

Note that it holds for cohomological field theories. In the Gromov-Witten case, when quadratic terms in the potential cannot be neglected, there appears an additional complication, see the next remark below.

Remark 3.1.5. Givental's formula [52] for a Gromov-Witten total descendant potential (without the $(g = 1, n = 0)$-term),

$$Z = \hat{S}^{-1}\hat{\Psi}\hat{R}\hat{\Delta}\mathcal{T}, \tag{3.14}$$

also includes the operator $\hat{S}$, given by

$$\hat{S} = \exp\left(\sum_{l=1}^{\infty}(s_l z^{-l})^{\hat{}}\right), \tag{3.15}$$

where the operators $(s_l z^{-l})^{\hat{}}$ are defined in the following way (see, e. g., [51, Section 4.2]):

$$\sum_{l=1}^{\infty}(s_l z^{-l})^{\hat{}} = -\,(s_1)_1^{\mu}\frac{\partial}{\partial t^{0,\mu}} + \frac{1}{\hbar}\sum_{d=0}^{\infty}(s_{d+2})_{1,\mu}\,t^{d,\mu} \tag{3.16}$$

$$+\sum_{\substack{d=0\\ l=1}}^{\infty}(s_l)^{\mu}_{\nu}\,t^{d+l,\nu}\frac{\partial}{\partial t^{d,\mu}} + \frac{1}{2\hbar}\sum_{\substack{d_1,d_2\\ \mu_2,\mu_2}}(-1)^{d_1}(s_{d_1+d_2+1})_{\mu_1,\mu_2}\,t^{d_1,\mu_1}t^{d_2,\mu_2}.$$

Note that formula (3.15) for the quantization of S differs from the analogous formula (3.5) for R by a factor of $(-1)^l$ in the exponent, which agrees with the definition in Givental's papers [53, 52].

The matrices s_k are defined through the following relation:

$$S(z) = \sum_{k=0}^{\infty} S_k z^{-k} = \exp\left(\sum_{l=0}^{\infty} s_l z^{-l}\right), \tag{3.17}$$

where for $S(z)$, taken at a point p of the Frobenius manifold, we have (see [52]), for any points a and b of the Frobenius manifold,

$$(a, b\, S_p) := (a,b) + \sum_{k=0}^{\infty}\langle \tau_0(a)\exp\left(\tau_0(p)\right)\tau_k(b)\rangle_0\, z^{-1-k}. \tag{3.18}$$

Here on the left hand side the brackets stand for the scalar product on the tangent space to the Frobenius manifold at p, and we used an identification of the tangent space with the whole Frobenius manifold, since in this case the Frobenius manifold is itself a vector space. If p is the origin, we have just

$$(a, b\, S) := (a,b) + \sum_{k=0}^{\infty}\langle \tau_0(a)\tau_k(b)\rangle\;\; z^{-1-k}. \tag{3.19}$$

Note that this S action is defined in the general case when the total descendant genus 0 potential is known. For the case when only a Frobenius potential is specified, the choice of S is then called a *calibration* of the Frobenius manifold, see [53, 24] for related details. In the case of cohomological field theory when we disregard quadratic terms, the S action is trivial if p is taken to be the origin.

It turns out that in most of the relevant cases, e.g. for the Gromov-Witten theory of $\mathbb{CP}^1$ (see section 3.4.1 below), the only relevant term in equation (3.16) is

$$\sum_{\substack{d=0\\ l=1}}^{\infty}(s_l)^{\mu}_{\nu}\,t^{d+l,\nu}\frac{\partial}{\partial t^{d,\mu}},$$

since $(s_1)_1^\mu$ vanishes and all other terms just change the unstable terms in the potential.

This means, that in these cases $\hat{S}^{-1}$ just performs a linear change of formal variables $t^{d,\mu}$ in the following way:

$$t^{d,\mu} \mapsto \sum_{m=d}^{\infty} (S_{m-d})_\nu^\mu t^{m,\nu}. \tag{3.20}$$

3.1.3 Expressions in terms of graphs

In Chapter 2 the action of an operator series as in equation (3.5) is written as a sum over graphs. By Remark 3.1.4, this allows us to construct the potential of any semi-simple conformal cohomological field theory as a sum over graphs. Here we repeat the construction of Chapter 2 in a slightly different way that will be more convenient for the comparison with the topological recursion formalism. Furthermore, we also include the action of $\hat{\Delta}$. It is easy to see that the construction is equivalent to that of Chapter 2.

Notation 3.1.6. Let γ be any graph. By a half-edge we mean either a leaf or an edge together with a choice of one of the two vertices it is attached to. By $V(\gamma)$, $E(\gamma)$, $H(\gamma)$ and $L(\gamma)$ we denote the sets of vertices, edges, half-edges and leaves of γ. For any vertex v of γ, denote by $H(v)$ the set of half-edges connected to v.

Let $\widetilde{\Gamma}$ be the set of all connected graphs γ together with a choice of disjoint splitting $L(\gamma) = L^*(\gamma) \coprod L^\bullet(\gamma)$, a labelling of the vertices by pairs $(g,i) \in \mathbb{Z}_{\geq 0} \times \{1,\dots,N\}$ and a labelling of the elements of $H(\gamma)$ by non-negative integers, such that the label of a leaf in $L^\bullet$ is always greater than one. The elements of $L^*(\gamma)$ are called *ordinary leaves*, the elements of $L^\bullet$ are called *dilaton leaves*. We denote by Γ the subset of all graphs in $\widetilde{\Gamma}$ that are *stable*; that is, any vertex labelled $(0,i)$ for some i is of valence at least three.

For any graph γ denote by $\mathfrak{g}\colon V(\gamma) \to \mathbb{Z}_{\geq 0}$ and $\mathfrak{i}\colon V(\gamma) \to \{1,\dots,N\}$ the maps that associate to any vertex its first and second label respectively, and by $\mathfrak{k}\colon H(\gamma) \to \mathbb{Z}_{\geq 0}$ the map that associates to any half-edge its label. Denote by $\mathfrak{v}\colon L(\gamma) \to V(\gamma)$ the map that associates to each leaf the corresponding vertex, and by $\mathfrak{v}_1, \mathfrak{v}_2\colon E(\gamma) \to V(\gamma)$ and by $\mathfrak{h}_1, \mathfrak{h}_2\colon E(\gamma) \to H(\gamma)$ the maps that associate to an edge the first and second vertex, and the corresponding half-edges respectively.

Remark 3.1.7. The labels introduced above are used to keep track of different data for the trivial cohomological field theory; g is for the genus, i for the primary field in canonical coordinates and the labelling of the marked half-edges is for the power of ψ-class.

Remark 3.1.8. As in Chapter 2, edges of a graph in Γ are considered to be oriented (this allows to define the maps $\mathfrak{v}_1$ and $\mathfrak{v}_2$ unambiguously); the final result does not depend on the orientation.

Let $R(z)_j^i$ be the components of the operator series $R(z)$ in normalized canonical basis as computed in Section 3.1.2. To each part of a graph $\gamma \in \Gamma$ we assign some polynomial in formal variables $\hbar$ and $v^{d,i}$. Here $\hbar$ is used to keep track of the genus, while the first index of $v^{d,i}$ keeps track of the number of ψ-classes and the second index keeps track of the normalized canonical coordinate.

Leaves

To each ordinary leaf $l \in L^*$ marked by k attached to a vertex marked by the pair (g, i), we assign

$$(\mathcal{L}^*)^i_k(l) := [z^k] \left(\sum_{d \geq 0} \left((R(-z))^i_j v^{d,j} z^d \right) \right), \tag{3.21}$$

which corresponds to the second term in (3.4).

To a dilaton leaf $\lambda \in L^\bullet(\gamma)$ marked by k attached to a vertex marked by (g, i) we assign

$$(\mathcal{L}^\bullet)^i_k(\lambda) := [z^{k-1}] \left(-(R(-z))^i_{\mathbf{1}} \right), \tag{3.22}$$

which corresponds to the first term in (3.4), which is called the *dilaton shift*.

Edges

To an edge e connecting a vertex v_1 marked by (g_1, i_1) to a vertex v_2 marked by (g_2, i_2) and with markings k_1 and k_2 at the corresponding half-edges, we assign

$$\mathcal{E}^{i_1,i_2}_{k_1,k_2}(e) := [z^{k_1} w^{k_2}] \left(\hbar \cdot \frac{\delta^{i_1 i_2} - \sum_s (R(-z))^{i_1}_s (R(-w))^{i_2}_s}{z + w} \right). \tag{3.23}$$

Note that this does not depend on the choice of ordering of the vertices and that it follows from the fact that $R(z)$ can be written as $R(z) = \exp(\sum r_l z^l)$ that the numerator on the right-hand side is equal to the product of $(z + w)$ with some power series in z and w, so this definition makes sense.

Vertices

Let v be a vertex marked by (g, i) with n half-edges attached to it (this includes all ordinary and dilaton leaves and also half-edges that are parts of internal edges) labelled by $k_1, \ldots, k_n$. Then we assign to v the following expression:

$$\mathcal{V}^{(g,i)}_{\{k_1,\ldots,k_n\}}(v) := \hbar^{g-1} (\Delta_i)^{\frac{1}{2}(2g-2+n)} \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n}. \tag{3.24}$$

Z as a sum over graphs

It is easy to see that the sum over all graphs in Γ of the product of the contributions described above, weighted by the inverse order of the automorphism group of the graph, is equal to the graph-sum described in Chapter 2 (the only difference is that now we have specialized to the action on the trivial cohomological field theory, leading to ψ-class integrals (3.24) as vertex contributions). Thus, we recover the partition function Z of the cohomological field theory we started with as a sum over Γ:

$$(\hat{R}\hat{\Delta}\mathcal{T})(\{v^{d,j}\}) = \sum_{\gamma\in\Gamma} \frac{1}{|\operatorname{Aut}(\gamma)|}$$
$$\prod_{v\in V(\Gamma)} \hbar^{\mathfrak{g}(v)-1} (\Delta_{\mathfrak{i}(v)})^{\frac{1}{2}(2\mathfrak{g}(v)-2+\mathrm{val}(v))} \left\langle \prod_{h\in H(v)} \tau_{\mathfrak{k}(h)} \right\rangle_{\mathfrak{g}}$$
$$\prod_{e\in E(\gamma)} \mathcal{E}^{\mathfrak{i}(\mathfrak{v}_1(e)),\mathfrak{i}(\mathfrak{v}_2(e))}_{\mathfrak{k}(\mathfrak{h}_1(e)),\mathfrak{k}(\mathfrak{h}_2(e))}(e) \prod_{l\in L^*(\gamma)} (\mathcal{L}^*)^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(l) \prod_{\lambda\in L^\bullet(\gamma)} (\mathcal{L}^\bullet)^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(\lambda). \quad (3.25)$$

3.2 Topological recursion

In this section, we define a local version of the topological recursion and write the corresponding invariants as a sum over graphs, which allows us to compare it to the Givental action in the next section.

3.2.1 Local topological recursion

We define a local version of the topological recursion in the following way. The term local refers to the fact that the data are all defined locally around the canonical coordinates without any reference to the possible existence of a global manifold where these functions can be defined.

Definition 3.2.1. For $N \in \mathbb{N}^*$, we call times a set of N families of complex numbers $\{h^i_k\}_{k\in\mathbb{N}}$ for $i = 1, \ldots, N$ and jumps another set of $N \times N$ infinite families of complex numbers $\{B^{i,j}_{k,l}\}_{(k,l)\in\mathbb{N}^2}$ for $i, j = 1, \ldots, N$. We finally define a set of canonical coordinates $\{a_i\}_{i=1}^N \in \mathbb{C}^N$ subject to $a_i \neq a_j$ for $i \neq j$.

For all $i, j \in \{1, \ldots, N\}$, we define the following set of analytic functions and differential forms in a neighborhood of $0 \in \mathbb{C}$:

$$x^i(z) := z^2 + a_i\,, \quad y^i(z) := \sum_{k=0}^{\infty} h^i_k z^k \quad (3.26)$$

and

$$B^{i,j}(z, z') = \delta_{i,j} \frac{dz \otimes dz'}{(z - z')^2} + \sum_{k,l=0}^{\infty} B^{i,j}_{k,l} z^k z'^l dz \otimes dz'. \quad (3.27)$$

For $2g-2+n > 0$, we define the genus g, n-point correlation functions $\omega^{i_1,\ldots,i_n}_{g,n}(z_1, \ldots z_n)$ recursively by

$$\omega^{i_0,i_1,\ldots,i_n}_{g,n+1}(z_0, z_1, \ldots, z_n) := \sum_{j=1}^{N} \operatorname*{Res}_{z\to 0} \frac{\int_{-z}^{z} B^{i_0,j}(z_0, \cdot)}{2\,(y^j(z) - y^j(-z))\, \mathrm{d}x^j(z)} \times$$
$$\left(\omega^{j,j,i_1,\ldots,i_n}_{g-1,n+2}(z, -z, z_1, \ldots, z_n) + \sum_{A\cup B=\{1,\ldots,n\}} \sum_{h=0}^{g} \omega^{j,\mathbf{i}_A}_{h,|A|+1}(z, \mathbf{z}_A) \omega^{j,\mathbf{i}_B}_{g-h,|B|+1}(-z, \mathbf{z}_B) \right), \quad (3.28)$$

where for any set A, we denote by $\mathbf{z}_A$ (resp., $\mathbf{i}_A$) the set $\{z_k\}_{k\in A}$ (resp., $\{i_k\}_{k\in A}$), and where the base of the recursion is given by

$$\omega_{0,1}^{i}(z) := 0; \qquad \omega_{0,2}^{i,j}(z,z') := B^{i,j}(z,z'). \tag{3.29}$$

For convenience, in the sequel we denote

$$K^{i,j}(z,z') = \frac{\int_{-z}^{z} B^{i,j}(z',\cdot)}{2(y^j(z) - y^j(-z))dx^j(z)} \tag{3.30}$$

and

$$\omega_{g,n}(\vec{z}) = \sum_{\vec{i}} \omega_{g,n}^{\vec{i}}(\vec{z}) \ , \tag{3.31}$$

where the length of $\vec{z}$ and $\vec{i}$ is n.

3.2.2 Correlation functions and intersection numbers

The correlation functions built by this topological recursion can actually be written in terms of intersection of ψ classes on the moduli space of Riemann surfaces. This result is a slight generalization of [36, 37] to the local topological recursion.

3.2.3 One-branch point case

The link between the topological recursion formalism and intersection numbers on the moduli space of Riemann surfaces comes from the application of this formalism to the Airy curve. This case corresponds to $N = 1$ and:

$$x(z) = z^2 + a\,, \quad y(z) = z \quad \text{and} \quad B(z,z') = \frac{dz \otimes dz'}{(z-z')^2}. \tag{3.32}$$

Remark 3.2.2. Since there is only one branch point in this case, i.e. $N = 1$, we omit the superscript indicating which branch point we consider in the notations of this section.

For further convenience, we introduce two additional parameters by considering the curve

$$x(z) = z^2 + a\,, \quad y(z) = \alpha z \quad \text{and} \quad B(z,z') = \beta\frac{dz \otimes dz'}{(z-z')^2}, \tag{3.33}$$

the usual Airy curve being $\alpha = \beta = 1$. In this case, the topological recursion reads

$$\begin{aligned}\omega_{g,n+1}(z_0,z_1,\dots,z_n) :=& \operatorname*{Res}_{z\to 0} \frac{\beta}{2\alpha}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2-z^2)}\times \\ &\left(\omega_{g-1,n+2}(z,-z,z_1,\dots,z_n) + \sum_{A\cup B=\{1,\dots,n\}}\sum_{h=0}^{g}\omega_{h,|A|+1}(z,\mathbf{z}_A)\omega_{g-h,|B|+1}(-z,\mathbf{z}_B)\right)\end{aligned} \tag{3.34}$$

and one has

Lemma 3.2.3. *The correlation functions of the Airy curve can be expressed in terms of intersection numbers:*

$$\omega_{g,n}(z_1,\dots,z_n) = \left(-\frac{\beta}{2\alpha}\right)^{2g+n-2} \beta^{g+n-1} \sum_{\alpha_1,\dots,\alpha_n\geq 0} \langle \tau_{a_1}\dots\tau_{a_n}\rangle_{g,n} \prod_{i=1}^{n} \frac{(2\alpha_i+1)!!\,dz_i}{z_i^{2\alpha_i+2}}. \quad (3.35)$$

This lemma was proved many times by direct computation [41, 36, 44, 99], matching the topological recursion with the recursive definition of the intersection numbers.

As a side note, the first few correlation functions are

$$\omega_{0.3}(z_1,z_2,z_3) = -\frac{\beta^3}{2\alpha}\prod_{i=1}^{3}\frac{dz_i}{z_i^2}, \quad (3.36)$$

$$\omega_{0,4}(z_1,z_2,z_3,z_4) = \frac{\beta^5}{4\alpha^2}\prod_{i=1}^{4}\frac{dz_i}{z_i^2}\sum_{i=1}^{4}\frac{3}{z_i^2}, \quad (3.37)$$

$$\omega_{1,1}(z) = \frac{-\beta^2}{2\alpha}\frac{dz}{8z^4} \quad (3.38)$$

and

$$\omega_{1,2}(z_1,z_2) = \frac{\beta^4}{4\alpha^2}\frac{dz_1\,dz_2}{8z_1^2z_2^2}\left(\frac{5}{z_1^4}+\frac{5}{z_2^4}+\frac{3}{z_1^2z_2^2}\right). \quad (3.39)$$

Remark 3.2.4. It is important to remark that there exist different conventions in the literature for defining the topological recursion, mainly differing by a change of sign of the recursion kernel. The latter can be recovered by a change of sign $\alpha \to -\alpha$.

Let us now consider a deformation of the Airy curve which we will refer to as the KdV curve in the following. It has only one branch point, $N=1$, and reads

$$\begin{cases} x(z) = z^2 + a_i \\ y(z) = \alpha\displaystyle\sum_{k=1}^{\infty} h_k z^k \\ B(z,z') = \beta B_{\mathrm{KdV}}(z,z') = \beta dz\otimes dz'(z-z')^2 \end{cases}. \quad (3.40)$$

The corresponding correlation functions can also be expressed in terms intersection numbers as follows:

Lemma 3.2.5. *The correlation functions of the KdV curve read:*

$$\omega_{g,n}(z_1,\dots,z_n) = \left(-\frac{\beta}{2\alpha h_1}\right)^{2g+n-2} \beta^{g+n-1}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!} \sum_{\vec{\alpha}\in\mathbb{N}^{*m}}\prod_{k=1}^{m}(2\alpha_k+1)!!\frac{h_{2\alpha_k+1}}{h_1}\prod_{i=1}^{n}\frac{(2d_i+1)!!\,dz_i}{z_i^{2d_i+2}}\left\langle\prod_{j=1}^{n}\tau_{d_j}\prod_{k=1}^{m}\tau_{\alpha_k+1}\right\rangle_{g,n+m}. \quad (3.41)$$

Proof. Once again the proof can be found in the literature [41, 46, 36]. However, let us study a graphical interpretation of this result when considering an arbitrary convention for the topological recursion. For $f(z)$ an analytic function around $z \to 0$ and $\{T_k\}_{k\in\mathbb{Z}}$ a set of parameters, one can compute

$$\operatorname*{Res}_{Z_1\to 0}\operatorname*{Res}_{Z_2\to 0} K(Z_1,z)\Bigg\{\left(\sum_{k\geq 1} T_k Z_1^k\right) dZ_1\, K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 - \left(\sum_{k\geq 1} T_k (-Z_1)^k\right) dZ_1\, K(Z_2,Z_1) f(Z_2)\,[dZ_2]^2\Bigg\} \tag{3.42}$$

where the recursion kernel is the one of the Airy curve, i.e. the one for which $h_k = 0$ for $k \geq 2$:

$$K(z,z_0) = \frac{\beta}{2\alpha h_1}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2 - z^2)}. \tag{3.43}$$

One can move the integration contours to get

$$\operatorname*{Res}_{Z_1\to 0}\operatorname*{Res}_{Z_2\to 0} = \operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to 0} + \operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to Z_2} + \operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to -Z_2}. \tag{3.44}$$

The first term of the right hand side vanishes since the integrand does not have any pole at $Z_1 \to 0$. Let us now compute one of the other two terms:

$$\begin{aligned}\operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to Z_2} K(Z_1,z)\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 = \\ -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2}{2Z_2} f(Z_2)\operatorname*{Res}_{Z_1\to Z_2}\frac{dz}{2Z_1}\frac{1}{(z^2 - Z_1^2)}\frac{\beta}{2\alpha h_1}\frac{1}{(Z_1^2 - Z_2^2)}\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) = \\ = -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2 - Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_k}{4} Z_2^{k-2}.\end{aligned} \tag{3.45}$$

In the same way,

$$\begin{aligned}\operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to -Z_2} K(Z_1,z)\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 = \\ = -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2 - Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_k}{4}(-Z_2)^{k-2}.\end{aligned} \tag{3.46}$$

The sum of these two terms reads

$$\begin{aligned}\operatorname*{Res}_{Z_2\to 0}\operatorname*{Res}_{Z_1\to \pm Z_2} K(Z_1,z)\left(\sum_{k\geq 1} T_k Z_1^k dZ_1\right) K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2 = \\ = -\operatorname*{Res}_{Z_2\to 0}\frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2 - Z_2^2)}\frac{\beta}{2\alpha h_1}\sum_{k\geq 1}\frac{T_{2k}}{2}(Z_2)^{2k-2}\end{aligned} \tag{3.47}$$

and finally:

$$\begin{aligned}
&\operatorname*{Res}_{Z_1\to 0}\operatorname*{Res}_{Z_2\to 0} K(Z_1,z)\left\{\left(\sum_{k\geq 1} T_k Z_1^k\right) dZ_1\, K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2\right.\\
&\qquad\left.-\left(\sum_{k\geq 1} T_k(-Z_1)^k\right) dZ_1\, K(Z_2,Z_1) f(Z_2)\,[dZ_2]^2\right\} =\\
&\qquad= \operatorname*{Res}_{Z_2\to 0} \frac{\beta}{2\alpha h_1}\frac{dZ_2 dz}{2Z_2} f(Z_2)\frac{1}{(z^2-Z_2^2)}\left(-\frac{\beta}{2\alpha h_1}\right)\sum_{k\geq 1} T_{2k}(Z_2)^{2k-2}. \qquad (3.48)
\end{aligned}$$

On the other hand, plugging in the times h_k amounts to computing similar quantities:

$$\begin{aligned}
&\operatorname*{Res}_{z\to 0} \frac{\beta}{2\alpha h_1}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2-z^2)}\frac{1}{\left(1+\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}\right)} f(z)\,[dz]^2 =\\
&= \operatorname*{Res}_{z\to 0} \frac{\beta}{2\alpha h_1}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2-z^2)} f(z)\,[dz]^2\,(1-\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}+\left[\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k}\right]^2+\ldots) \qquad (3.49)
\end{aligned}$$

The first term of this sum is the Airy recursion kernel. The second one is of the shape of the preceding one with $T_{2k+2}=\frac{2\alpha h_{2k+1}}{h_1}$ for $k\geq 1$ so that:

$$\begin{aligned}
&-\operatorname*{Res}_{z\to 0} \frac{\beta}{2\alpha}\frac{dz_0}{2z\,dz}\frac{1}{(z_0^2-z^2)} f(z)\,[dz]^2\sum_{k=1}^{\infty}\frac{h_{2k+1}}{h_1}z^{2k} =\\
&\qquad= \operatorname*{Res}_{Z_1\to 0}\operatorname*{Res}_{Z_2\to 0} K(Z_1,z_0)\left\{g(Z_1)dZ_1\, K(Z_2,-Z_1) f(Z_2)\,[dZ_2]^2\right.\\
&\qquad\qquad\left.-g(-Z_1)dZ_1\, K(Z_2,Z_1) f(Z_2)\,[dZ_2]^2\right\} \qquad (3.50)
\end{aligned}$$

where

$$g(z) := \sum_{k\geq 1}\frac{2\alpha h_{2k+1}}{\beta h_1}z^{2k+2}. \qquad (3.51)$$

This same procedure can be applied to the other terms of the sum. The k^{th} order term can be written as a sequence of $k+1$ residues computed with the Airy recursion kernel with $g(z)dz$ on one of the outgoing legs. This computation shows that introducing non-vanishing times amounts to introducing a non-vanishing $\omega_{0,1}(z) := g(z)dz$ in the topological recursion.

It is often useful to represent the topological recursion in a graphical form by representing the interaction kernel $K(z,z_0)$ by an edge oriented from z_0 towards a trivalent vertex labelled by z and the function $\omega_{0,2}(z_1,z_2)$ by a non-oriented edge (see [48] for more details about this set of graphs). In this form, $\omega_{g,n}(z_1,\ldots,z_n)$ is a sum over trivalent graphs of genus g with n leaves labelled by the arguments $z_1,\ldots,z_n$. The preceding computation shows that the correlation functions of the KdV curve can be obtained from the correlation functions of the Airy curve by introducing a set of new leaves, called dilation leaves, in the definition of the graphs used. A dilation leave decorated by a label k is weighted by

$$(2d-1)!!\ \operatorname*{Res}_{z\to 0} g(z)\frac{dz}{z^{2d+1}} = (2d-1)!!\ \frac{2\alpha h_{2d-1}}{\beta}. \qquad (3.52)$$

Plugging this expression into the formula for the Airy correlation functions proves the result. $\square$

General case

In this section we give a formula for the correlation function of the local topological recursion.

Definition 3.2.6. Let $\Gamma_{g,n}$ be the subset of Γ (see Notation 3.1.6) consisting of graphs of genus g' such that $g'+\sum_{v\in V(\Gamma)}\mathfrak{g}(v)=g$ and with n ordinary leaves. Let us also introduce orderings on the ordinary leaves and denote by $\check{\Gamma}_{g,n}$ the set of all graphs from $\Gamma_{g,n}$ with all possible orderings on the ordinary leaves. For a given graph with a fixed ordering $\check{\gamma}\in\check{\Gamma}_{g,n}$ for an ordinary leaf of that graph $l\in L^*(\check{\gamma})$ we denote by $\mathfrak{m}(l)$ the index of this particular leaf (then $\mathfrak{m}(l)$ is an integer from 1 to n such that different leaves have different values $\mathfrak{m}(l)$ assigned to them).

Theorem 3.2.7. *The correlation functions can be written as a sum over decorated graphs whose vertices are weighted by intersection of ψ-classes on $\overline{\mathcal{M}}_{g,n}$, edges by the jumps, ordinary leaves by primitives of B and dilaton leaves by the times.*

For $2-2g-n<0$, one has

$$\omega_{g,n}(\vec{z})=\frac{1}{n!}\sum_{\check{\gamma}\in\check{\Gamma}_{g,n}}\prod_{v\in V(\check{\gamma})}\left(-2h_1^{\mathfrak{i}(v)}\right)^{2-2\mathfrak{g}(v)-\mathrm{val}(v)}\left\langle\prod_{h\in H(v)}\tau_{\mathfrak{k}(h)}\right\rangle_{\mathfrak{g}(v),\mathrm{val}(v)}$$

$$\prod_{e\in E(\check{\gamma})}\check{B}^{\mathfrak{i}(\mathfrak{v}_1(e)),\mathfrak{i}(\mathfrak{v}_2(e))}_{\mathfrak{k}(\mathfrak{h}_1(e)),\mathfrak{k}(\mathfrak{h}_2(e))}\prod_{l\in L^*(\check{\gamma})}\sum_{j=1}^{N}\mathrm{d}\xi^{\mathfrak{i}(\mathfrak{v}(l))}_{\mathfrak{k}(l)}(z_{\mathfrak{m}(l)},j)\prod_{\lambda\in L^\bullet(\check{\gamma})}\check{h}^{\mathfrak{i}(\mathfrak{v}(\lambda))}_{\mathfrak{k}(\lambda)}\tag{3.53}$$

with

$$\check{h}^i_k:=2(2k-1)!!h^i_{2k-1},\tag{3.54}$$

$$d\xi^i_d(z_\alpha,j):=\operatorname*{Res}_{z\to 0}\frac{(2d+1)!!dz}{z^{2d+2}}\int^z B^{i,j}(z,z_\alpha),\tag{3.55}$$

$$\check{B}^{i,j}_{d_1,d_2}:=B^{i,j}_{2d_1,2d_2}\,(2d_1-1)!!\,(2d_2-1)!!\tag{3.56}$$

and

$$\left\langle\prod_{i=1}^{n}\tau_{k_i}\right\rangle_{g,n}:=\int_{\overline{\mathcal{M}}_{g,n}}\psi_1^{k_1}\psi_2^{k_2}\dots\psi_n^{k_n}.\tag{3.57}$$

Proof. The proof is very similar to the one presented in [37, 68]. However, we prefer to present a completely graphical proof so that the link with the next sections becomes clear.

We follow the proof of [37]. From the definition, one can write the correlation functions as a sum over graphs with oriented and non-oriented arrows linking trivalent vertices resulting in the following expression:

$$\omega^{\vec{i}}_{g,n}(\vec{z})=\sum_{G\in\widehat{G}_{g,n}}\omega(G)\tag{3.58}$$

with $\widehat{G}_{g,n}$ the set of genus g trivalent graphs with one root and $n-1$ leaves labelled by the arguments z_i and a skeleton tree of oriented edges pointing from the root towards the leaves weighted by

$$\begin{aligned} \omega(G) \quad = \quad & \vec{\prod_{v\in V(G)}} \operatorname*{Res}_{Z_v\to 0} \prod_{e\in E_{\text{oriented}}(G)} K^{i(v_1(e)),i(v_2(e))}(Z_{v_1(e)},Z_{v_2(e)}) \\ & \prod_{e\in E_{\text{unoriented}}(G)} B^{i(v_1(e)),i(v_2(e))}(Z_{v_1(e)},Z_{v_2(e)}) \end{aligned} \tag{3.59}$$

where each leaf is considered as a one-valent vertex v and one denotes Z_v the variable z_i associated to this leaf in the correlation function, $E_{\text{oriented}}(G)$ is the set of oriented leaves of G and $E_{\text{unoriented}}(G)$ is the set of unoriented leaves of G (see [48] for further details). The product of residues $\vec{\prod\limits_{v\in V(G)}} \operatorname{Res}_{Z_v\to 0}$ is oriented following the arrows, i.e. one first computes the residue corresponding to the end of an arrow before the one associated to its root.

It is useful to remark, that, for any edge, oriented or not, one has two types of contributions. Indeed, the functions $B^{i,j}(z,z')$ have a singular part

$$B^{i,j}_{\text{KdV}}(z,z') := \delta_{i,j}\frac{dz\otimes dz'}{(z-z')^2} \tag{3.60}$$

and a regular part

$$B^{i,j}_{\text{reg}}(z,z') := \sum_{k,l=0}^{\infty} B^{i,j}_{k,l} z^k z'^l dz\otimes dz' \tag{3.61}$$

when $z\to z'$:

$$B^{i,j}(z,z') = B^{i,j}_{\text{KdV}}(z,z') + B^{i,j}_{\text{reg}}(z,z'). \tag{3.62}$$

In the same way, one has

$$K^{i,j}(z,z') = K^{i,j}_{\text{KdV}}(z,z') + K^{i,j}_{\text{reg}}(z,z'). \tag{3.63}$$

One can translate this by representing $B^{i,j}_{\text{KdV}}(z,z')$ (resp. $B^{i,j}_{\text{reg}}(z,z')$) by dashed (resp. dotted) unoriented edges and $K^{i,j}_{\text{KdV}}(z,z')$ (resp. $K^{i,j}_{\text{reg}}(z,z')$) by dashed (resp. dotted) oriented edges form z to z'. The preceding sum is thus transformed into a sum over graphs where the edges are dotted or dashed and weighted accordingly.

The dashed edges can be expressed in a slightly different way. Indeed, one has

$$B^{i,j}_{\text{reg}}(z,z') = \operatorname*{Res}_{z_1\to z}\operatorname*{Res}_{z_2\to z'} B^{i,i}_{\text{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\text{reg}}(z_1,z_2)\right] B^{j,j}_{\text{KdV}}(z_2,z') \tag{3.64}$$

and

$$K^{i,j}_{\text{reg}}(z,z') = \operatorname*{Res}_{z_1\to z}\operatorname*{Res}_{z_2\to \pm z'} B^{i,i}_{\text{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\text{reg}}(z_1,z_2)\right] K^{j,j}_{\text{KdV}}(z_2,z') \tag{3.65}$$

by a simple application of the Cauchy formula.

Remember that such an edge comes with integration of its boundary variables, thus, one typically has to compute

$$\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z'\to 0} g(z)K^{i,j}_{\mathrm{reg}}(z,z')f(z') \tag{3.66}$$

which reads

$$\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_1\to z}\operatorname*{Res}_{z'\to 0}\operatorname*{Res}_{z_2\to \pm z'} g(z)B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] K^{j,j}_{\mathrm{KdV}}(z_2,z')f(z'). \tag{3.67}$$

One can move the integration contours around 0 thanks to:

$$\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_1\to z} = \operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0} - \operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_1\to 0} \tag{3.68}$$

and

$$\operatorname*{Res}_{z'\to 0}\operatorname*{Res}_{z_2\to \pm z'} = \operatorname*{Res}_{z_2\to 0}\operatorname*{Res}_{z'\to 0} - \operatorname*{Res}_{z'\to 0}\operatorname*{Res}_{z_2\to 0}. \tag{3.69}$$

Since, the integrand does not have any pole as $z_1 \to 0$ nor $z_2 \to 0$, this shows that 3.66 is equal to

$$\operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_2\to 0} g(z)\, B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] \operatorname*{Res}_{z'\to 0} K^{j,j}_{\mathrm{KdV}}(z_2,z')\, f(z'). \tag{3.70}$$

In the same way, one gets that

$$\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z'\to 0} g(z)B^{i,j}_{\mathrm{reg}}(z,z')f(z') \tag{3.71}$$

is equal to

$$\operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0}\operatorname*{Res}_{z_2\to 0} g(z)\, B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\right] \operatorname*{Res}_{z'\to 0} B^{j,j}_{\mathrm{KdV}}(z_2,z')\, f(z'). \tag{3.72}$$

One can finally proceed in a similar way for re-expressing the weights of the root and the leaves by writing[1]

$$\operatorname*{Res}_{z'\to 0} K^{i,j}(z,z')f(z') = \operatorname*{Res}_{z_2\to 0}\left[\int^{z_2} B^{i,j}(z,z_2)\right] \operatorname*{Res}_{z'\to 0} K^{j,j}_{\mathrm{KdV}}(z_2,z')\, f(z') \tag{3.73}$$

and

$$\operatorname*{Res}_{z\to 0} g(z)B^{i,j}(z,z') = \operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z\to 0} g(z)\, B^{i,i}_{\mathrm{KdV}}(z,z_1)\left[\int^{z_1} B^{i,j}(z_1,z')\right]. \tag{3.74}$$

As a result, by applying this transformation to each dotted line, any graph is composed of a set of dotted subgraphs whose vertices have the same label separated by dashed lines. Since each subgraph with label i also includes a root and leaves, it is a contribution to the correlation functions obtained for the case $N = 1$, times h^i_k and vanishing jumps $B^{i,i}_{k,l} = 0$.

[1] Remark that, for the roots and leaves, in opposition to the inner edges, the functions are the full ones, not just the regular part.

In the sum over graphs, one can thus replace every sum over such sub-graphs by vertices of corresponding genus weighted by the correlation function for $N=1$, which reads

$$\omega_{g,n}^{\vec{i}}(\vec{z}) = \sum_{\gamma\in\Gamma_{g,n}} \Omega(\gamma) \tag{3.75}$$

where

$$\begin{aligned}\Omega(\gamma) \quad = \quad & \prod_{v\in V(\gamma)} \prod_{h\in H(v)} \mathop{\mathrm{Res}}_{Z_h\to 0} \omega^{\mathrm{KdV},i(v)}_{g(v),\mathrm{val}(v)} \Big(\{Z_h\}_{h\in H(v)} \Big) \\ & \prod_{e\in E(\gamma)} \int^{Z_{h_1(e)}} \int^{Z_{h_2(e)}} B_{\mathrm{reg}}^{i(v_1(h)),i(v_2(h))}\big(Z_{h_1(e)}, Z_{h_2(e)}\big) \\ & \prod_{h\in L^*(\gamma)} \int^{Z_h} B^{i,j}(Z_h, z_h)\end{aligned} \tag{3.76}$$

where $\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n)$ is the genus g, n-pointed correlation function obtained from the topological recursion in the case $N=1$ and the initial data:

$$\begin{cases} x(z) = z^2 + a_i \\ y(z) = \sum_{k=1}^{\infty} h_k^i z^k \\ B(z,z') = B_{\mathrm{KdV}}(z,z') = \frac{dz\otimes dz'}{(z-z')^2} \end{cases} . \tag{3.77}$$

As explained in the preceding section, it can be expressed in terms of intersection numbers:

$$\begin{aligned}\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n) = & \big(-2h_1^i\big)^{2-2g-n} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \\ & \sum_{\vec{\alpha}\in\mathbb{N}^{*m}} \prod_{k=1}^{m} (2\alpha_k+1)!! \frac{h^i_{2\alpha_k+1}}{h_1^i} \prod_{i=1}^{n} \frac{(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}} \\ & \left\langle \prod_{j=1}^{n} \tau_{d_j} \prod_{k=1}^{m} \tau_{\alpha_k+1} \right\rangle_{g,n+m}\end{aligned} \tag{3.78}$$

which can be made more symmetric under the exchange of the ordinary and dilation leaves by writing

$$\begin{aligned}\omega_{g,n}^{\mathrm{KdV},i}(z_1,\dots,z_n) = & \sum_{m=0}^{\infty} (-2h_1)^{2-2g-n-m} \frac{1}{m!} \\ & \sum_{\vec{\alpha}\in\mathbb{N}^{*m}} \prod_{k=1}^{m} (2\alpha_k-1)!! 2h^i_{2\alpha_k-1} \prod_{i=1}^{n} \frac{(2d_i+1)!!\, dz_i}{z_i^{2d_i+2}} \\ & \left\langle \prod_{j=1}^{n} \tau_{d_j} \prod_{k=1}^{m} \tau_{\alpha_k} \right\rangle_{g,n+m}\end{aligned} \tag{3.79}$$

Absorbing the factors of the form

$$\frac{(2d+1)!!\mathrm{d}z}{z^{2d+2}} \tag{3.80}$$

into the corresponding half-edge contribution, the weight of an inner edge becomes

$$\operatorname*{Res}_{z_1\to 0}\operatorname*{Res}_{z_2\to 0}\int^{z_1}\int^{z_2} B^{i,j}_{\mathrm{reg}}(z_1,z_2)\,\frac{(2d_1+1)!!\mathrm{d}z_1}{z_1^{2d_1+2}}\,\frac{(2d_2+1)!!\mathrm{d}z_1}{z_2^{2d_2+2}} \tag{3.81}$$

which is equal to

$$\check{B}^{i,j}_{d_1,d_2} := B^{i,j}_{2d_1,2d_2}\,(2d_1-1)!!\,(2d_2-1)!! \tag{3.82}$$

while the weight of the ordinary leaves becomes

$$\mathrm{d}\xi^i_d(z_\alpha,j) := \operatorname*{Res}_{z\to 0}\frac{(2d+1)!!\mathrm{d}z}{z^{2d+2}}\int^z B^{i,j}(z,z_\alpha), \tag{3.83}$$

where one considers both the singular and non-singular part of $B^{i,j}(z,z_\alpha)$. Collecting these contributions together proves the theorem. □

3.2.4 Change of scales

An important property of the correlation functions built in this way is their homogeneity property which reads

$$\forall\lambda\in\mathbb{C}\,,\ \omega_{g,n}(\vec{z}_N|x,\lambda y,B)=\lambda^{2-2g-n}\omega_{g,n}(\vec{z}_N|x,y,B) \tag{3.84}$$

One can thus get an additional factor λ^i by replacing $h^i_k\to\lambda^i h^i_k$ resulting in a rescaling of the weight of the vertices by $\left(\lambda^{\mathrm{i}(v)}\right)^{2-2g(v)-\mathrm{val}(v)}$.

3.2.5 Weights, Laplace transform and recursive definition

It is interesting to note that the weights of the edges are the coefficient of the Laplace transform of B:

$$\check{B}^{i,j}(u,v) := \sum_{(k,l)\in\mathbb{N}^2}\check{B}^{i,j}_{k,l}u^{-k}v^{-l} \tag{3.85}$$

is equal to

$$\check{B}^{i,j}(u,v)=\delta_{i,j}\frac{uv}{u+v}+ \\ \frac{\sqrt{uv}e^{ua_i+va_j}}{2\pi}\int_{x(z)-a_i\in\mathbb{R}^+}\int_{x(z')-a_j\in\mathbb{R}^+}B^{i,j}(z,z')e^{-ux(z)-vx(z')}. \tag{3.86}$$

In [37], it was proved that, if dx is a meromorphic form defined on a Riemann surface, $\check{B}^{i,j}(u,v)$ can be factorized and expressed in terms of some basic functions. Here, we will consider the converse and build $B^{i,j}_{k,l}$ by induction in such a way that there exist a set of functions $\{f_{i,j}(u)\}_{i,j=1}^N$ such that

$$\check{B}^{i,j}(u,v)=\frac{uv}{u+v}\left(\delta_{i,j}-\sum_{k=1}^N f_{i,k}(u)f_{k,j}(v)\right). \tag{3.87}$$

Let us define the coefficients $B^{i,j}_{k,l}$ recursively in terms of the initial data $B^{i,j}_{k,0}$ by imposing that

$$\xi^i_{d+1}(z,j) := -2\frac{d\xi^i_d(z,j)}{dx^{[j]}(z)} - \sum_{k=1}^{\mathfrak{b}} \check{B}^{i,k}_{d,0}\,\xi^k_0(z,j), \tag{3.88}$$

or, in terms of the Laplace transform

$$\begin{aligned} f^i_d(u,j) &:= \frac{\sqrt{u}}{2\sqrt{\pi}} \int_{x(z)-a_j\in\mathbb{R}^+} e^{-u(x(z)-a_j)} dx^j(z)\,\xi^i_d(z) \\ &= \delta_{i,j}(-1)^d u^d - \sum_{d'} \check{B}^{i,j}_{d,d'} u^{-d'-1}, \end{aligned} \tag{3.89}$$

it reads

$$f^i_{d+1}(u,j) := -2uf^i_d(u,j) - \sum_{k=1}^{N} \check{B}^{i,k}_{d,0}\,f^k_0(z,j). \tag{3.90}$$

With this definition, one has

$$\check{B}^{i,j}(u,v) = \frac{uv}{u+v}\left(\delta_{i,j} - \sum_{k=1}^{\mathfrak{b}} f^k_0(u,i) f^k_0(v,j)\right). \tag{3.91}$$

3.3 Identification of the two theories

In this section we show how to find a local spectral curve corresponding to any semi-simple conformal Frobenius manifold.

Suppose some local spectral curve is given. For any $i \in \{1,\ldots,N\}$ and $k \in \mathbb{Z}_{\geq 0}$ define

$$W^i_k(z) := \sum_{j=1}^{N} d\left(\left(-\frac{1}{z}\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \xi^i_0(z,j)\right).$$

Theorem 3.3.1. *Let R be some series of operators on an N-dimensional vector space V as in Section 3.1. Let $Z = \hat{R}\hat{\Delta}\mathcal{T}$, where $\mathcal{T}$ is a product of N KdV τ-functions, be the partition function of the corresponding semi-simple cohomological field theory.*

Define a local spectral curve by the following data

$$\check{B}^{i,j}_{p,q} := [z^p w^q] \frac{\delta^{ij} - \sum_{s=1}^N R^i_s(-z) R(-w)^j_s}{z+w} \tag{3.92}$$

and

$$\check{h}^i_k := [z^{k-1}]\left(-R(-z)\right)^i_{\mathbf{1}} \tag{3.93}$$

$$h^i_1 := -\frac{1}{2\sqrt{\Delta^i}}. \tag{3.94}$$

Let $\omega_{g,n}$ be the genus g, n-pointed topological recursion invariant of this spectral curve and denote by

$$\Omega(\{v^{d,i}\}) = \left(\sum_{g,d} \omega_{g,d}(z_1,\ldots,z_d)\Big|_{W^i_d(z_m)=v^{d,i}} \hbar^{g-1}\right)$$

their sum after a change of variables $W^i_k(z_m) \leftrightarrow v^{d,i}$ for all m. Then the partition function of the cohomological field theory and the topological recursion invariants agree in the following sense:

$$Z(\{v^{d,i}\}) = \exp\left(\Omega(\{v^{d,i}\})\right). \tag{3.95}$$

Proof. In Sections 3.1 and 3.2 we have given representations of Z and $\omega_{g,n}$ as sums over the set Γ (in fact, in the case of $\omega_{g,n}$ this set is $\check{\Gamma}$ rather than Γ, but after changing the variables $W^i_k(z_m) \leftrightarrow v^{d,i}$ we can take the sum over orderings and arrive at the sum over Γ acquiring an additional factor of $n!$, which cancels with the corresponding factor in (3.53)). We prove the theorem by showing that the contribution of each individual graph to Z is equal to the contribution to Ω.

Let $\gamma \in \Gamma$ be some graph. Note that on both sides we assign the same weight to the vertices of γ, namely to a vertex labelled (g,i) with n half-edges attached to it labelled $d_1,\ldots,d_n$ we associate

$$(-2h^i_1)^{2-2g-n}\left\langle \tau_{d_1}\cdots\tau_{d_n}\right\rangle_{g,n}. \tag{3.96}$$

Furthermore, by equation (3.92), any edge in γ contributes the same to Z and Ω.

Let l be an ordinary leaf of γ labelled by k attached to a vertex labelled by (g,i). We use induction on k to show that the contribution to Z is the same as the contribution to Ω.

The contribution of l to Z is given by

$$\mathcal{L}^i_k(l) = [z^k]\left(\sum_d (R(-z))^i_j v^{d,j} z^d\right) = \sum_{d=0}^{k} (-1)^{k-d}(R_{k-d})^i_j v^{d,j}. \tag{3.97}$$

When $k=0$, the contribution of l to Ω is given by

$$\sum_j d\xi^i_0(z_j,j) = W^i_0. \tag{3.98}$$

Since $(R_0)^i_j = \delta^i_j$, the contributions to Z and Ω agree when $k=0$.

Now suppose that they agree for some $k \in \mathbb{Z}_{\geq 0}$. That is, suppose that

$$\sum_j d\xi^i_k(z^j,j) = \sum_{l=0}^{k}(-1)^{k-l}(R_{k-l})^i_s W^s_l. \tag{3.99}$$

Then, using Equation (3.88), the contribution of the leaf to Ω for the index $k+1$ is given by

$$\begin{aligned}\sum_j d\xi^i_{k+1}(z^j,j) &= \sum_j d\left(-2\frac{\partial \xi^i_k(z^j,j)}{\partial x^j} - \sum_{t=1}^{N}\check{B}^{i,t}_{k,0}\xi^t_0(z^j,j)\right)\\ &= \sum_j d\left(-\frac{1}{z^j}\frac{\partial}{\partial z^j}\xi^i_k(z^j,j) - \sum_{t=1}^{n} -(-1)^{k+1}(R_{k+1})^i_t\xi^t_0(z^j,j)\right)\\ &= \sum_{l=0}^{k}(-1)^l(R_l)^i_t W^t_{k+1-l} + (-1)^{k+1}(R_{k+1})^i_t W^t_0 = \sum_{l=0}^{k+1}(-1)^l(R_l)^i_t W^t_{k+1-l},\end{aligned} \tag{3.100}$$

where we used equation (3.92) to write

$$\check{B}^{i,t}_{k,0} = -(-1)^{k+1}(R_{k+1})^i_t. \tag{3.101}$$

This completes the induction, and since it is clear that the dilaton leaves contribute the same in both cases, it also completes the proof of the theorem. □

Remark 3.3.2. The theorem above deals with the potential of a cohomological field theory written in terms of formal variables $v^{d,i}$ corresponding to normalized canonical basis. In order to pass to flat coordinates one can change the variables in the following way:

$$v^{d,i} = \Psi^i_\mu t^{d,\mu}. \tag{3.102}$$

On the spectral curve side it will correspond to changing the variables W^i_k in the following way:

$$W^i_k = \Psi^i_\mu V^\mu_k. \tag{3.103}$$

Thus, the theorem holds in the same form for the potential of cohomological field theory written in terms of formal variables $t^{d,\mu}$, only one should identify $t^{d,\mu}$ with V^μ_d.

Remark 3.3.3. Above we established the correspondence between cohomological field theories and symplectic invariants of spectral curves. However, as noted in Remark 3.1.5, in the case of Gromov-Witten theories we cannot disregard quadratic terms. So, in the formula for the total descendent potential an additional operator $\hat{S}$ appears. In some cases, again see Remark 3.1.5, it performs only a linear change of formal variables $t^{d,\mu}$ on which the potential depends. Thus, to establish the correspondence in this case, one has to change the variables W^i_k in precisely the same way, and then identify the resulting variables with $t^{d,\mu}$, similar to the case of previous remark. Occasionally, the changes of variables preformed by $\hat{\Psi}$ and $\hat{S}^{-1}$ can be a re-expansion of $\omega_{g,n}$ in a new coordinate on the spectral curve. We explain this procedure in detail for the case of $\mathbb{C}\mathrm{P}^1$ below in section 3.4.

Remark 3.3.4. The system of equations obtained via a Laplace transform from the equations of Givental for the R-matrix (that is, the so-called equations of deformed flat connection) is studied in detail in [24, Section 5]. This gives, in particular, a recipe to reconstruct the two-point function directly from the Frobenius structure bypassing the reconstruction of the R-matrix. This also explains why we call the critical values $a_1, \ldots, a_N$ of x the canonical coordinates.

3.4 The Norbury-Scott conjecture

In this section we recall and prove the Norbury-Scott conjecture on the stationary sector of the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$.

3.4.1 Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$

The Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$ is discussed from the geometric point of view in many sources, see e. g. [82]. Givental proved in [52] that his formula for the formal Gromov-Witten potential coincides with the geometric Gromov-Witten potential of $\mathbb{C}\mathrm{P}^1$, so we discuss it here only from the Givental point of view, ignoring the geometric background. The same computations one can find in [92, 91].

The underlying structure of Frobenius manifold is determined by the following solution of the WDVV equation

$$\frac{1}{2}(t^1)^2t^2 + e^{t^2}, \tag{3.104}$$

and the scalar product given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.105}$$

All ingredients of the Givental formula depend on a particular choice of the point on the Frobenius manifold, and in this case we choose the point $(0,0)$ in the coordinates (t_1, t_2).

We perform a direct computation following the recipe of Givental in [53], see also Section 3.1.2. As a possible choice of the canonical coordinates, we use

$$u^1 = t^1 + 2\exp(t^2/2); \tag{3.106}$$
$$u^2 = t^1 - 2\exp(t^2/2). \tag{3.107}$$

In particular, for $t^1 = t^2 = 0$ we have $u^1 = -u^2 = 2$. Then,

$$\Delta_1^{-1} = \left\langle \frac{\partial}{\partial u^1}, \frac{\partial}{\partial u^1} \right\rangle = \frac{\exp(-t^2/2)}{2}; \tag{3.108}$$
$$\Delta_2^{-1} = \left\langle \frac{\partial}{\partial u^2}, \frac{\partial}{\partial u^2} \right\rangle = \frac{-\exp(-t^2/2)}{2}, \tag{3.109}$$

so we can choose the square roots as

$$\Delta_1^{-1/2} = \frac{\exp(-t^2/4)}{\sqrt{2}}; \tag{3.110}$$
$$\Delta_2^{-1/2} = \frac{-i\exp(-t^2/4)}{\sqrt{2}}, \tag{3.111}$$

and for this choice we have the following matrix of transition from the basis given by $(\partial/\partial t^1, \partial/\partial t^2)$ to the normalized canonical basis:

$$\Psi = \begin{pmatrix} \frac{\exp(-t^2/4)}{\sqrt{2}} & \frac{-i\exp(-t^2/4)}{\sqrt{2}} \\ \frac{\exp(t^2/4)}{\sqrt{2}} & \frac{i\exp(t^2/4)}{\sqrt{2}} \end{pmatrix}. \tag{3.112}$$

It is the matrix $\Psi = \Psi^i_\alpha$, where α labels the rows and corresponds to the flat basis, while i labels the columns and corresponds to the normalized canonical basis.

The recipe of reconstruction of the matrix R from [53] gives at the origin the matrix $R(\zeta) = \sum_{k=0}^\infty R_k\zeta^k$, where

$$R_k = \frac{(2k-1)!!(2k-3)!!}{2^{4k}k!} \cdot \begin{pmatrix} -1 & (-1)^{k+1}2ki \\ 2ki & (-1)^{k+1} \end{pmatrix} \tag{3.113}$$

The S matrix is given by the derivatives of the deformed flat coordinates, computed in [27, Example 3.7.9] At the origin we have:

$$\begin{aligned} S(\zeta^{-1}) =& \mathbf{I} + \zeta^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k}}{(k!)^2} \begin{pmatrix} 1 - 2k\left(\frac{1}{1} + \cdots \frac{1}{k}\right) & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k-1}}{(k!)^2} \begin{pmatrix} 0 & -2\left(\frac{1}{1} + \cdots \frac{1}{k}\right) \\ \frac{1}{k+1} & 0 \end{pmatrix}. \end{aligned} \tag{3.114}$$

(Note once again that we are using the convention that the matrices are acting on vector rows, opposite to the standard one).

The unit vector at the origin in the normalized canonical basis is equal to

$$e = (1,0) \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \end{pmatrix} = \left(\frac{1}{\sqrt{2}}, \frac{-i}{\sqrt{2}} \right). \tag{3.115}$$

Therefore, the dilaton leaves (cf. Equation (3.22)) in the Givental formula for $\mathbb{C}\mathrm{P}^1$ at the origin are

$$(\mathcal{L}^\bullet)^1_{k+1} = \frac{1}{\sqrt{2}} \cdot \frac{(-1)^{k+1}\left((2k-1)!!\right)^2}{k! 2^{4k}}; \tag{3.116}$$

$$(\mathcal{L}^\bullet)^2_{k+1} = \frac{i}{\sqrt{2}} \cdot \frac{\left((2k-1)!!\right)^2}{k! 2^{4k}} \tag{3.117}$$

for $k \geq 0$.

Proposition 3.4.1. *The Gromov-Witten potential of* $\mathbb{C}\mathrm{P}^1$*,*

$$Z_{\mathbb{C}\mathrm{P}^1}(\hbar, \{t^{\ell,1}, t^{\ell,2}\}_{\ell=0}^{\infty}), \tag{3.118}$$

is obtained from $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$ *(understood as a sum over graphs in the sense of Section 3.1.3 and written down in the normalized canonical basis, that is, in the variables* $v^{d,i}$*,* $d \geq 0$*,* $i = 1,2$*) via a linear change of variables given by*

$$\sum_{m \geq k}^{\infty} \left(t^{m,1}, t^{m,2}\right) S_k \zeta^{m-k} = \sum_{\ell=0}^{\infty} \left(v^{\ell,1}, v^{\ell,2}\right) \zeta^\ell \cdot \Psi^{-1}, \tag{3.119}$$

and a correction of the unstable terms (that is, (g,n)*-correlators with* $2g - 2 + n \leq 0$*).*

Proof. In order to get the Gromov-Witten potential of $\mathbb{C}\mathrm{P}^1$ as given by the Givental formula, we have to apply the $\hat{\Psi}$- and $\hat{S}^{-1}$-action to the expression in terms of graphs discussed in Section 3.1.3 that corresponds to $\hat{R}\hat{\Delta}Z_{\mathrm{KdV}}^{\otimes 2}$. The $\hat{\Psi}$-action is just a linear change of variable by definition. The general S-action is discussed in [51, Section 4.2]. It is a combination of a shift of variables that vanishes in our case (indeed, $(1,0)S_1 = (0,0)$), the linear change of variables that we have in the statement of Proposition, and a correction of unstable terms that is not essential for us. □

3.4.2 The Norbury-Scott conjecture

Norbury and Scott [81] propose the following construction. They consider a spectral curve given by

$$\begin{cases} x &= z + \frac{1}{z}; \\ y &= \log z, \end{cases} \tag{3.120}$$

and the standard two-point function

$$B(z, z') = \frac{dz \otimes dz'}{(z - z')^2}. \tag{3.121}$$

Via topological recursion they obtain the n-forms $\omega_{g,n}$ that they consider in the global variable x, and they conjecture the following theorem (they prove it for $g = 0, 1$):

Theorem 3.4.2. *For $2g - 2 + n > 0$, we have:*

$$\prod_{j=1}^{n} \left(- \operatorname*{Res}_{x_j = \infty} \frac{1}{(a_j + 1)!} x_1^{a_j+1} \right) \omega_{g,n}(x_1, \dots, x_n) = \langle \prod_{j=1}^{n} \tau_{2,a_j} \rangle_g, \tag{3.122}$$

where $\langle \prod_{j=1}^{n} \tau_{2,a_j} \rangle_g$ is the corresponding correlator in $Z_{\mathbb{CP}^1}$, that is, the coefficient of $\hbar^{g-1} \prod_{j=1}^{n} t_{2,a_j} / |Aut(a_1, \dots, a_n)|$ in $\log Z_{\mathbb{CP}^1}$.

In the rest of this section we prove this theorem, identifying all ingredients of the topological recursion with the corresponding parts of the Givental formula.

3.4.3 Proof of the Norbury-Scott conjecture

Local coordinates near the branch points

We denote the local coordinates by $z_1 = \sqrt{x - 2}$ and $z_2 = \sqrt{x + 2}$. Then we have:

$$x = z_1^2 + 2 \text{ near } x = 2,\ z = 1,\ z_1 = 0; \tag{3.123}$$

$$x = z_2^2 - 2 \text{ near } x = -2,\ z = -1,\ z_2 = 0. \tag{3.124}$$

Therefore,

$$z = 1 + \frac{z_1^2}{2} \pm z_1 \sqrt{1 + \frac{z_1^2}{4}}; \tag{3.125}$$

$$z = -1 + \frac{z_2^2}{2} \pm i z_2 \sqrt{1 - \frac{z_2^2}{4}}. \tag{3.126}$$

In both cases we choose $+$ for $\pm$.

Expansion of y

Recall that $y = \log z$. The direct computation shows:

$$y = \int \frac{dz_1}{\sqrt{1 + \frac{z_1^2}{4}}}; \tag{3.127}$$

$$y = \int \frac{-i\, dz_2}{\sqrt{1 - \frac{z_2^2}{4}}}; \tag{3.128}$$

Note that

$$\frac{1}{\sqrt{1+\frac{z_1^2}{4}}} = 1 + \sum_{k=1}^{\infty} z_1^{2k} \cdot \frac{(-1)^k (2k-1)!!}{k! 2^{3k}}; \tag{3.129}$$

$$\frac{-i}{\sqrt{1-\frac{z_2^2}{4}}} = -i + \sum_{k=1}^{\infty} z_2^{2k} \cdot \frac{(-i)\cdot(2k-1)!!}{k! 2^{3k}}. \tag{3.130}$$

Therefore

$$y = z_1 + \sum_{k=1}^{\infty} z_1^{2k+1} \cdot \frac{(-1)^k (2k-1)!!}{k! 2^{3k}(2k+1)}; \tag{3.131}$$

$$y = -iz_2 + \sum_{k=1}^{\infty} z_2^{2k+1} \cdot \frac{(-i)\cdot(2k-1)!!}{k! 2^{3k}(2k+1)}. \tag{3.132}$$

Thus the coefficients $\check{h}^i_{k+1}$, $k \geq 0$, are given by the following formulas:

$$\check{h}^1_{k+1} = 2 \cdot \frac{(-1)^k \left((2k-1)!!\right)^2}{k! 2^{3k}}; \tag{3.133}$$

$$\check{h}^2_{k+1} = 2 \cdot \frac{(-i)\cdot\left((2k-1)!!\right)^2}{k! 2^{3k}}. \tag{3.134}$$

Matrix $f_{i,j}(w)$

We use the following definition of the matrix $f_{ij}(w)$ (cf. Equation (3.87)):

$$f_{ij}(w) = \delta_{ij} - w\check{B}^{[ij]}(0, w^{-1}), \tag{3.135}$$

where $w = v^{-1}$. We use $\tilde{B}^{ij}_{0,l} = (B^{ij}_{\text{reg}})_{0,2l}(2l-1)!!$, and the following expressions:

$$B^{11}_{\text{reg}}(0, z_1) = \left[\frac{dz(z_1') \otimes dz(z_1)}{(z(z_1') - z(z_1))^2} - \frac{dz_1' \otimes dz_1}{(z_1' - z_1)^2}\right]_{z_1'=0} \tag{3.136}$$

$$B^{12}_{\text{reg}}(0, z_2) = \left[\frac{dz(z_1') \otimes dz(z_2)}{(z(z_1') - z(z_2))^2}\right]_{z_1'=0} \tag{3.137}$$

$$B^{21}_{\text{reg}}(0, z_1) = \left[\frac{dz(z_2') \otimes dz(z_1)}{(z(z_2') - z(z_1))^2}\right]_{z_2'=0} \tag{3.138}$$

$$B^{22}_{\text{reg}}(0, z_2) = \left[\frac{dz(z_2') \otimes dz(z_2)}{(z(z_2') - z(z_2))^2} - \frac{dz_2' \otimes dz_2}{(z_2' - z_2)^2}\right]_{z_2'=0} \tag{3.139}$$

Therefore,

$$B^{11}_{\mathrm{reg}}(0,z_1) = \frac{1}{z_1^2}\left(\frac{1}{\sqrt{1+\frac{z_1^2}{4}}}-1\right) \tag{3.140}$$

$$B^{12}_{\mathrm{reg}}(0,z_2) = \frac{i}{4(1-\frac{z_2^2}{4})^{3/2}} \tag{3.141}$$

$$B^{21}_{\mathrm{reg}}(0,z_1) = \frac{i}{4(1+\frac{z_1^2}{4})^{3/2}} \tag{3.142}$$

$$B^{22}_{\mathrm{reg}}(0,z_2) = \frac{1}{z_2^2}\left(\frac{1}{\sqrt{1-\frac{z_2^2}{4}}}-1\right) \tag{3.143}$$

So, we have the following expansions:

$$B^{11}_{\mathrm{reg}}(0,z_1) = \sum_{k=0}^{\infty} z_1^{2k}\cdot\frac{(-1)^{k+1}(2k+1)!!}{(k+1)!2^{3(k+1)}} \tag{3.144}$$

$$B^{12}_{\mathrm{reg}}(0,z_2) = \sum_{k=0}^{\infty} z_2^{2k}\cdot\frac{i(2k+1)!!}{(k)!2^{3k+2}} \tag{3.145}$$

$$B^{21}_{\mathrm{reg}}(0,z_1) = \sum_{k=0}^{\infty} z_2^{2k}\cdot\frac{i(-1)^k(2k+1)!!}{(k)!2^{3k+2}} \tag{3.146}$$

$$B^{22}_{\mathrm{reg}}(0,z_2) = \sum_{k=0}^{\infty} z_2^{2k}\cdot\frac{(2k+1)!!}{(k+1)!2^{3(k+1)}}. \tag{3.147}$$

The formulas for $f_{ij}(w)$ are then

$$f_{11}(w) = 1+\sum_{k=1}^{\infty} w^k\cdot\frac{(-1)^{k+1}(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{3.148}$$

$$f_{12}(w) = \sum_{k=1}^{\infty} w^k\cdot\frac{-i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{3.149}$$

$$f_{21}(w) = \sum_{k=1}^{\infty} w^k\cdot\frac{(-1)^k i(2k-1)!!(2k-3)!!}{(k-1)!2^{3k-1}} \tag{3.150}$$

$$f_{22}(w) = 1+\sum_{k=1}^{\infty} w^k\cdot\frac{-(2k-1)!!(2k-3)!!}{k!2^{3k}} \tag{3.151}$$

This coincides with the formula for the $\sum_{k=0}^{\infty} R_k 2^k(-w)^k$ at the point $(0,0)$.

Comparison of the coefficient of (g,n,m)-vertex

In this section we consider a vertex of genus g with n attached half-edges or ordinary leaves, and m dilaton leaves, with an associated intersection number $\langle\prod_{i=1}^n \tau_{d_i}\prod_{i=1}^m \tau_{a_i+1}\rangle_{g,n+m}$. There are vertices of type 1 and type 2, depending on the canonical coordinate that we associate to the vertex. We compare the coefficients that we associate to these vertices in the Givental case, using the data from Section 3.4.1 in Formula (3.25), and in the case of local topological recursion, using the data from Sections 3.4.3-3.4.3 in Formula (3.53).

The coefficients that we have in Formula (3.53) (at the vertex of the type 1 and 2 resp.):

$$(-2)^{2-2g-n-m} \qquad \text{and} \qquad (2i)^{2-2g-n-m}. \tag{3.152}$$

Let us compute how these coefficients change if we take into account all the differences between R-matrix and the dilaton leaves. First, observe that the extra factor of 2^k in R_k and, in addition, an extra factor of $\sqrt{2}$ that we have to put by hand on each ordinary leave give us together the extra factors of

$$2^{\sum_{i=1}^n d_i} 2^{n/2} \qquad \text{and} \qquad 2^{\sum_{i=1}^n d_i} 2^{n/2}. \tag{3.153}$$

Then the quotient of the contributions of the dilaton leaves gives us the extra factors of

$$2^{\sum_{i=1}^m (a_i+1)} 2^{m/2} (-1)^m \qquad \text{and} \qquad 2^{\sum_{i=1}^m (a_i+1)} 2^{m/2} (-1)^m. \tag{3.154}$$

Let us assign by hand an extra factor of $(-1)^{2g-2+n}$ to each (g,n,m)-vertex. This way we get the following coefficients:

$$2^{g-1+n/2+m/2} \qquad \text{and} \qquad 2^{g-1+n/2+m/2} i^{2g-2+n+m}. \tag{3.155}$$

These coefficients are precisely

$$\left((\Delta_1)^{1/2}\right)^{2g-2+n+m} \qquad \text{and} \qquad \left((\Delta_2)^{1/2}\right)^{2g-2+n+m}. \tag{3.156}$$

Therefore, the coefficient of $\prod_{k=1}^{\hat{n}} W_{d_k}^{i_k}$ in a graph of global genus $\hat{g}$ with $\hat{n}$ marked leaves in the Formula (3.53) for the set up of Norbury-Scott, multiplied by

$$2^{\hat{n}/2}(-1)^{2\hat{g}-2+\hat{n}} = (-\sqrt{2})^{\hat{n}}, \tag{3.157}$$

is equal to the coefficient of $\prod_{k=1}^{\hat{n}} t^{d_k,i_k}$ in the same graph in Formula (3.25). This extra factor will be taken into account via rescaling of the variables by $-\sqrt{2}$.

The Ψ-action

Let us apply the Ψ-operator to the leaves. After comparing the R-action with graph expansion given formulas (3.25) and (3.53), and taking into account the extra factor of $-\sqrt{2}$, we have the following identification of the marking on the leaves:

$$\sum_{a-b=c} (t^{a,1}, t^{a,2}) S_b = \left(W_c^1, W_c^2\right) \Psi^{-1}/(-\sqrt{2}). \tag{3.158}$$

Here

$$W_0^1 = \left.\frac{dz}{(1-z)^2}\right|_{z=z(z_1)} + \left.\frac{dz}{(1-z)^2}\right|_{z=z(z_2)} \tag{3.159}$$

$$W_0^2 = \left.\frac{i\,dz}{(1+z)^2}\right|_{z=z(z_1)} + \left.\frac{i\,dz}{(1+z)^2}\right|_{z=z(z_2)}, \tag{3.160}$$

and

$$W_c^i = d\left(\left(-2\frac{d}{dx}\right)^c \int W_0^i\right), \tag{3.161}$$

so we can work in the global coordinate z rather than in the local coordinates z_1, z_2.

Since

$$\Psi^{-1}/(-\sqrt{2}) = \begin{pmatrix} \frac{-1}{2} & \frac{-1}{2} \\ \frac{-i}{2} & \frac{i}{2} \end{pmatrix}, \tag{3.162}$$

we have:

$$\sum_{a-b=c} (t^{a,1}, t^{a,2}) S_b = \left(U_c^1, U_c^2\right), \tag{3.163}$$

where

$$U_0^1 = \frac{1}{2}\left(-\frac{dz}{(1-z)^2} + \frac{dz}{(1+z)^2}\right) \tag{3.164}$$

$$U_0^2 = \frac{-1}{2}\left(\frac{dz}{(1-z)^2} + \frac{dz}{(1+z)^2}\right) \tag{3.165}$$

and

$$U_c^i = d\left(\left(-2\frac{d}{dx}\right)^c \int U_0^i\right), \qquad i = 1, 2; c = 0, 1, 2, \dots. \tag{3.166}$$

The S-action

The S-action is just a linear change of variables prescribed by Equation (3.163). This means that we replace each U_c^i with a linear combination of times $t^{a,j}$, $a \geq c$, where the coefficient of $t^{a,2}$ (this is the series of variables corresponding to the stationary sector) is equal to

$$\begin{cases} 0, & \text{if } a-c \text{ is even;} \\ \frac{1}{(k+1)\cdot(k!)^2}, & \text{if } a-c = 2k+1. \end{cases} \tag{3.167}$$

for $i = 1$, and

$$\begin{cases} \frac{1}{(k!)^2}, & \text{if } a-c = 2k; \\ 0, & \text{if } a-c \text{ is odd.} \end{cases} \tag{3.168}$$

for $i = 2$.

Norbury and Scott make the same kind of a linear change of variables, with the coefficient of $t^{a,2}$ in U_c^j, $j = 1, 2$, given by

$$-\operatorname*{Res}_{x=\infty} \frac{1}{(a+1)!} x^{a+1} U_c^j = \frac{1}{(a+1)!} \operatorname*{Res}_{z=0} \left(z + \frac{1}{z}\right)^{a+1} U_c^j. \tag{3.169}$$

In order to complete the proof of Theorem 3.4.2, we have to check two things: (1) that the Norbury-Scott formula for the contribution depends only on the difference $a-c$; (2) that for $c = 0$ Equation (3.169) gives exactly the same coefficients as we have in Equations (3.167) and (3.168).

The first thing follows directly from the formula. Indeed,

$$-\oint \frac{x^{a+1}}{(a+1)!} d\left(\left(-2\frac{d}{dx}\right)^c \int U_0^j\right) = \oint \frac{x^a}{(a)!}\left(\left(-2\frac{d}{dx}\right)^c \int U_0^j\right) dx \tag{3.170}$$
$$= 2^c \oint \frac{x^{a-c}}{(a-c)!}\left(\int U_0^j\right) dx$$
$$= -2^c \oint \frac{x^{a+1-c}}{(a+1-c)!} d\left(\int U_0^j\right).$$

In particular, we see that the coefficient is equal to 0 if $a < c$.

Then, a direct computation shows that

$$\begin{aligned}
&\frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} U_0^1 \\
&= \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{1}{2}\left(-\frac{dz}{(1-z)^2}+\frac{dz}{(1+z)^2}\right) \\
&= \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{-2z\,dz}{(1-z^2)^2} \\
&= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-2}{(2k+2)!}\left(\binom{2k+2}{0}\cdot(k+1)+\binom{2k+2}{1}\cdot k+\cdots\binom{2k+2}{k}\cdot 1\right) & \text{if } a = 2k+1. \end{cases} \\
&= \begin{cases} 0, & \text{if } a \text{ is even;} \\ \frac{-1}{(k+1)(k!)^2} & \text{if } a = 2k+1. \end{cases}
\end{aligned} \tag{3.171}$$

and

$$\begin{aligned}
&\frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} U_0^2 \\
&= \frac{1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+1} \frac{-1}{2}\left(\frac{dz}{(1-z)^2}+\frac{dz}{(1+z)^2}\right) \\
&= \frac{-1}{(a+1)!}\operatorname*{Res}_{z=0}\left(z+\frac{1}{z}\right)^{a+2} \frac{z\,dz}{(1-z^2)^2} \\
&= \begin{cases} \frac{-1}{(2k+1)!}\left(\binom{2k+2}{0}\cdot(k+1)+\binom{2k+2}{1}\cdot k+\cdots\binom{2k+2}{k}\cdot 1\right) & \text{if } a = 2k; \\ 0, & \text{if } a \text{ is odd} \end{cases} \\
&= \begin{cases} \frac{-1}{(k!)^2} & \text{if } a = 2k; \\ 0, & \text{if } a \text{ is odd.} \end{cases}
\end{aligned} \tag{3.172}$$

We see that there is an extra factor of (-1) in all coefficients. This means that the (g,n)-correlation functions of Norbury-Scott differ from the stationary Gromov-Witten invariants of $\mathbb{C}\mathrm{P}^1$ by the factor of $(-1)^n$. But this factor is exactly the difference we must have because Norbury and Scott using a different convention on the sign in the topological recursion, cf. Remark 3.2.4. This completes the proof of Theorem 3.4.2.

Chapter 4

Quantum spectral curve for the Gromov-Witten theory of the complex projective line

This chapter is based on paper [101], joint work with M. Mulase, P. Norbury, A. Popolitov and S. Shadrin. In this chapter we prove the quantum spectral curve equation for the Gromov-Witten theory of $\mathbb{P}^1$.

The chapter is organized as follows. In Section 4.2 we start with a solution $\mathcal{W}_{g,n}$ to the topological recursion equation with respect to the spectral curve Σ of (4.13). It is a symmetric differential form of degree n on Σ^n. We then propose a unique mechanism to integrate $\mathcal{W}_{g,n}$ into a rational function. The goal of this section is to show that this primitive function is identical to (4.5). Then in Section 4.3, we re-write $\Psi(x,\hbar)$ in a different manner, only involving stationary Gromov-Witten invariants of $\mathbb{P}^1$. This formula allows us to express it in terms of a *semi-infinite wedge product* in Section 4.4. Using this formalism, we reduce the quantum curve equation (4.9) to a combinatorial equation (4.14) in Section 4.5. Equation (4.14) is then proved in Section 4.6 using representation theory of S_d, which in turn establishes (4.9). For completeness, we give an expression of (4.15) in terms of special values of the Laguerre polynomials in Section 4.7. Section 4.8 is devoted to the comparison of (4.9) and the Toda lattice equations of [84], in terms of the functions X_d of (4.15).

4.1 Quantum curves

The purpose of this Chapter is to construct the *quantum curve* for the Gromov-Witten invariants of the complex projective line $\mathbb{P}^1$. Quantum curves are conceived in the physics literature, including [2, 21, 22, 42, 60, 63]. They quantize the *spectral curves* of the theory, and are conjectured to capture the information of many topological invariants, such as certain Gromov-Witten invariants, quantum knot invariants, and cohomology of instanton moduli spaces for 4-dimensional gauge theory. In this chapter we show that the conjecture is indeed true for the Gromov-Witten theory of $\mathbb{P}^1$.

4.1.1 Spectral curves and quantum curves

When spectral curves appear in mathematics, they take various different forms, and even look as totally different objects. For example, they can be the mirror curve of a toric Calabi-Yau 3-fold, the SL_2-character variety of the fundamental group of a knot complement, or a Seiberg-Witten curve. In the context of the Gromov-Witten theory of $\mathbb{P}^1$, it is the *Landau-Ginzburg model*

$$x = z + \frac{1}{z}, \tag{4.1}$$

which is the homological mirror dual of $\mathbb{P}^1$ with respect to the standard Kähler structure. Our main theorem (Theorem 4.1.1 below) states that the quantization of (4.1), which we call a quantum curve, characterizes the exponential generating function of Gromov-Witten invariants of $\mathbb{P}^1$.

In a purely algebro-geometric setting, a quantum curve can be understood in the following way [30]. Let C be a non-singular complex projective algebraic curve, and η the tautological 1-form on the cotangent bundle T^*C. A spectral curve Σ is a complex 1-dimensional subvariety

$$\begin{tikzcd} \iota : \Sigma \arrow[r] & T^*C \arrow[d, "\pi"] \\ & C \end{tikzcd} \tag{4.2}$$

in the cotangent bundle, which is automatically a Lagrangian subvariety with respect to the standard symplectic form $-d\eta$. A quantum curve is an $\hbar$-deformed D-module on the 1-parameter formal family $C[[\hbar]]$ of the curve C, whose *semi-classical limit* coincides with the spectral curve Σ. On an affine piece of the base curve C with coordinate x, we can choose a generator P of the D-module and consider a *Schrödinger-like* equation

$$P(x,\hbar)\Psi(x,\hbar) = 0. \tag{4.3}$$

The construction of the quantum curve in this setting is established in [30] for $SL(2,\mathbb{C})$ Hitchin fibrations.

The geometric situation we consider here is slightly different. Instead of the cotangent bundle T^*C in (4.2), we have a surface X equipped with a $\mathbb{C}^*$-invariant holomorphic symplectic form and a spectral curve Σ is mapped into it. The base curve C is replaced by the quotient $X/\mathbb{C}^*$. For example, if a curve C admits a $\mathbb{C}^*$-action then the natural holomorphic symplectic form on $X = T^*C$ is $\mathbb{C}^*$-invariant. In local coordinates for T^*C, the $\mathbb{C}^*$-action is given by $c \cdot (w,z) = (cw, c^{-1}z)$ and the symplectic form is given by $dx \wedge (dz/z)$ where $x = wz$ is the quotient map $(w,z) \mapsto wz$ by the $\mathbb{C}^*$-action. Reflecting the $\mathbb{C}^*$-action, the quantum curve (4.3) becomes a differential equation of *infinite* order, or a *difference* equation.

We present here the first rigorous example of a direct connection between Gromov-Witten theory and quantum curves. Our construction requires the fermionic Fock space representation of the Gromov-Witten invariants [84], and a subtle combinatorial analysis based on representation theory of symmetric groups.

4.1.2 Main theorem

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ denote the moduli space of stable maps of degree d from an n-pointed genus g curve to $\mathbb{P}^1$. This is an algebraic stack of dimension $2g-2+n+2d$. The dimension reflects the fact that a generic map from an algebraic curve to $\mathbb{P}^1$ has only simple ramifications, which we can see from the Riemann-Hurwitz formula. The descendant Gromov-Witten invariants of $\mathbb{P}^1$ are defined by

$$\left\langle \prod_{i=1}^{n} \tau_{b_i}(\alpha_i) \right\rangle_{g.n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \psi_i^{b_i} ev_i^*(\alpha_i), \tag{4.4}$$

where $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)]^{vir}$ is the virtual fundamental class of the moduli space,

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \longrightarrow \mathbb{P}^1$$

is a natural morphism defined by evaluating a stable map at the i-th marked point of the source curve, $\alpha_i \in H^*(\mathbb{P}^1, \mathbb{Q})$ is a cohomology class of the target $\mathbb{P}^1$, and ψ_i is the tautological cotangent class in $H^2(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbb{Q})$. We denote by 1 the generator of $H^0(\mathbb{P}^1, \mathbb{Q})$, and by $\omega \in H^2(\mathbb{P}^1, \mathbb{Q})$ the Poincaré dual to the point class. We assemble the Gromov-Witten invariants into particular generating functions as follows. For every (g, n) in the stable sector $2g-2+n>0$, we define the *free energy* of type (g, n) by

$$F_{g,n}(x_1, \ldots, x_n) := \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n}. \tag{4.5}$$

Here the degree d is determined by the dimension condition of the cohomology classes to be integrated over the virtual fundamental class. We note that (4.5) contains the class $\tau_0(1)$. For unstable geometries, we introduce two functions

$$S_0(x) := x - x\log x + \sum_{d=1}^{\infty} \left\langle -\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}} \right\rangle_{0,1}^{d}, \tag{4.6}$$

$$S_1(x) := -\frac{1}{2}\log x + \frac{1}{2}\sum_{d=0}^{\infty} \left\langle \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x^{b+1}} \right)^2 \right\rangle_{0,2}^{d}. \tag{4.7}$$

The appearance of the extra terms, in particular the $\log x$ terms, will be explained in Section 4.3. We shall prove the following.

Theorem 4.1.1 (Main Theorem). *The wave function*

$$\Psi(x, \hbar) := \exp\left(\frac{1}{\hbar} S_0(x) + S_1(x) + \sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x, \ldots, x) \right) \tag{4.8}$$

satisfies the quantum curve equation of an infinite order

$$\left[\exp\left(\hbar \frac{d}{dx}\right) + \exp\left(-\hbar \frac{d}{dx}\right) - x \right] \Psi(x, \hbar) = 0. \tag{4.9}$$

Moreover, the free energies $F_{g,n}(x_1, \ldots, x_n)$ as functions in n-variables, and hence all the Gromov-Witten invariants (4.4), can be recovered from the equation (4.9) alone, using the mechanism of the ***topological recursion*** *of [18, 48].*

Remark 4.1.2. Put

$$S_m(x) := \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(x, \dots, x). \tag{4.10}$$

Then our wave function is of the form

$$\Psi(x, \hbar) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x)\right), \tag{4.11}$$

which provides the WKB approximation of the quantum curve equation (4.9). Thus the significance of (4.5) is that the exponential generating function (4.8) of the descendant Gromov-Witten invariants of $\mathbb{P}^1$ gives the solution to the *exact* WKB analysis for the difference equation (4.9).

Remark 4.1.3. For the case of Hitchin fibrations [30], the Schrödinger-like equation (4.3) is a direct consequence of the generalized topological recursion. In our current context, the topological recursion does not play any role in establishing (4.9).

Remark 4.1.4. Although the shape of the operator in (4.9) has a similarity with the Lax operator of the Toda lattice equations that control the Gromov-Witten invariants of $\mathbb{P}^1$ [84], we are unable to find any direct relations between these two apparently different equations. We present a detailed comparison of these equations in Section 4.8.

4.1.3 WKB approximation, topological recursion, and representation theory

The WKB analysis provides a perturbative quantization method of a classical mechanical problem. We can recover the classical problem corresponding to (4.9) by taking its semi-classical limit, which is the singular perturbation limit

$$\begin{aligned}\lim_{\hbar \to 0}\left(e^{-\frac{1}{\hbar}S_0(x)}\left[\exp\left(\hbar\frac{d}{dx}\right) + \exp\left(-\hbar\frac{d}{dx}\right) - x\right] e^{\frac{1}{\hbar}S_0(x)} e^{\sum_{m=1}^{\infty}\hbar^{m-1}S_m(x)}\right) \\ = \left(e^{S_0'(x)} + e^{-S_0'(x)} - x\right) e^{S_1(x)} = 0. \end{aligned}\tag{4.12}$$

In terms of new variables $y(x) = S_0'(x)$ and $z(x) = e^{y(x)}$, the semi-classical limit gives us an equation for the spectral curve

$$z \in \Sigma = \mathbb{C}^* \subset \mathbb{C} \times \mathbb{C}^* \xleftarrow{\exp} T^*\mathbb{C} = \mathbb{C}^2 \ni (x, y)$$

by

$$\begin{cases} x = z + \frac{1}{z} \\ z = e^y \end{cases}. \tag{4.13}$$

This is the reason we consider (4.9) as the quantization of the Landau-Ginzburg model (4.1).

It was conjectured in [81] that the stationary Gromov-Witten theory of $\mathbb{P}^1$ should satisfy the topological recursion of [18, 48] with respect to the spectral curve (4.13). We refer to [30, 31, 81] for a mathematical formulation of the topological recursion. The conjecture is solved in Chapter 3 as a corollary to its main theorem, which establishes the correspondence between the topological recursion and the Givental formalism.

The quantum curve equation (4.9) determines only the function Ψ, and by the $\hbar$-expansion, each coefficient $S_m(x)$. But then how do we possibly recover $F_{g,n}$ for each (g,n) as a function in n variables? Here comes the significance of the topological recursion of [18, 48], which was established in Chapter 3 for the case of the Gromov-Witten theory of $\mathbb{P}^1$. The scenario goes as follows. First we note that the semi-classical limit of (4.9) identifies the spectral curve (4.13). We then launch the topological recursion formalism of [18, 48] for this particular spectral curve, and obtain symmetric differential n-forms $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ on Σ^n. In this chapter we will present a canonical way to *integrate* these n-forms, which yields the free energy $F_{g,n}(x_1,\dots,x_n)$ for every (g,n) subject to $2g-2+n>0$. In this sense the single equation (4.9) knows the information of all Gromov-Witten invariants (4.4). This shows the power of quantum curves.

The key discovery of the present chapter is that the quantum curve equation (4.9) is equivalent to a recursion equation

$$\frac{x}{\hbar}\left(e^{-\hbar\frac{d}{dx}}-1\right)X_d(x,\hbar)+\frac{1}{1+\frac{x}{\hbar}}e^{\hbar\frac{d}{dx}}X_{d-1}(x,\hbar)=0 \tag{4.14}$$

for a rational function

$$X_d(x,\hbar)=\sum_{\lambda\vdash d}\left(\frac{\dim\lambda}{d!}\right)^2\prod_{i=1}^{\ell(\lambda)}\frac{x+(i-\lambda_i)\hbar}{x+i\hbar}. \tag{4.15}$$

Here λ is a partition of $d\geq 0$ with parts λ_i and $\dim\lambda$ denotes the dimension of the irreducible representation of the symmetric group S_d characterized by λ.

4.2 The functions $F_{g,n}$ in terms of Gromov-Witten invariants

The significance of the idea of quantum curves is that the single equation (4.3) captures all information of the topological invariants of the theory. The key process from this single equation to the topological invariants is the *integral form* of the mechanism known as the topological recursion of [18, 48]. We refer to [30, 31, 81] for mathematical formulation of the topological recursion. This section is devoted to providing the unique mechanism to integrate the topological recursion, for the context of the Gromov-Witten theory of $\mathbb{P}^1$.

Let us begin with a solution $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ to the topological recursion of [18, 48] associated with the spectral curve $\Sigma=\mathbb{C}^*$ defined by

$$\begin{cases} x(z)=z+\frac{1}{z} \\ y(z)=\log z \end{cases}. \tag{4.16}$$

This means that symmetric differential forms $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ of degree n on Σ^n for (g,n) in the stable range $2g-2+n>0$ are inductively defined by the following recursion formula:

$$
\begin{aligned}
&\mathcal{W}_{g,n}(z_1,\dots,z_n)\\
&\qquad = \frac{1}{2\pi i}\oint_{z=\pm 1}\frac{\int_z^{1/z}\mathcal{W}_{0,2}(\,\cdot\,,z_1)}{\mathcal{W}_{0,1}(1/z)-\mathcal{W}_{0,1}(z)}\Bigg[\mathcal{W}_{g-1,n+1}(z,1/z,z_2,\dots,z_n)\\
&\qquad\qquad + \sum_{\substack{g_1+g_1=g\\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}}\mathcal{W}_{g_1,|I|+1}(z,z_I)\mathcal{W}_{g_2,|J|+1}(1/z,z_J)\Bigg],
\end{aligned}
\tag{4.17}
$$

where the residue integral is taken with respect to the variable $z\in\Sigma$ on two small, positively oriented, closed loops around $z=1$ and $z=-1$, and for the index set $I\subset\{2,\dots,n\}$, we denote by $|I|$ its cardinality, and $z_I=(z_i)_{i\in I}$. For (g,n) in the unstable range, we define

$$
\mathcal{W}_{0,1}(z) := y(z)dx(z), \tag{4.18}
$$

$$
\mathcal{W}_{0,2}(z_1,z_2) := \frac{dz_1dz_2}{(z_1-z_2)^2}-\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}. \tag{4.19}
$$

The goal of this section is to derive the integral $F_{g,n}(z_1,\dots,z_n)$ of $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ in a consistent and unique way that has the x-variable expansion (4.5).

Remark 4.2.1. The second term of the right-hand side of (4.19) does not play any role in the topological recursion (4.17). It is included here for the consistency of the primitive $F_{0,2}(z_1,z_2)$ to be discussed in Section 4.3.

Definition 4.2.2. For $2g-2+n>0$, we define the *primitive* $F_{g,n}(z_1,\dots,z_n)$ of the n-form $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ to be a rational function on Σ^n that satisfies the following conditions:

$$
d_1\cdots d_nF_{g,n}(z_1,\dots,z_n)=\mathcal{W}_{g,n}(z_1,\dots,z_n); \tag{4.20}
$$

$$
F_{g,n}(z_1,\dots,z_{i-1},1/z_i,z_{i+1},\dots,z_n)=-F_{g,n}(z_1,\dots,z_n),\quad i=1,\dots,n; \tag{4.21}
$$

$$
F_{g,n}(z_1,\dots,z_n)\big|_{z_1=\cdots=z_n=0}=0. \tag{4.22}
$$

If it exists, then it is unique.

From now on, we need to relate functions or differential forms defined on the spectral curve $\Sigma=\mathbb{C}^*$ of (4.16) and on the base curve $\mathbb{C}$. We recall [31] that the inverse function of (4.1) for the branch near $z=0$ and $x=\infty$ is given by the generating function of the Catalan numbers

$$
z=z(x)=\sum_{m=0}^{\infty}\frac{1}{m+1}\binom{2m}{m}\frac{1}{x^{2m+1}}. \tag{4.23}
$$

By abuse of notation, for a function or a differential form $f(z)$ on Σ, we denote the pull-back via (4.23) simply by $f(x):=f(z(x))$.

It is established in Chapter 3 that the solution $\mathcal{W}_{g,n}$ of the topological recursion has the following x-variable expansion in terms of the stationary Gromov-Witten invariants of $\mathbb{P}^1$:

$$
\mathcal{W}_{g,n}(x_1,\dots,x_n)=\left\langle\prod_{i=1}^{n}\left(\sum_{b=0}^{\infty}(b+1)!\,\tau_b(\omega)\,\frac{dx_i}{x_i^{b+2}}\right)\right\rangle_{g,n}. \tag{4.24}
$$

There is no systematic mechanism to integrate this expression to obtain (4.5). Instead, we establish the following theorem in this section.

Theorem 4.2.3. *For every (g, n) in the stable sector $2g-2+n > 0$, there exists a primitive $F_{g,n}(z_1, \ldots, z_n)$ in the sense of Definition 4.2.2, such that its x-variable expansion is given by*

$$F_{g,n}(x_1, \ldots, x_n) = \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n}. \tag{4.25}$$

Remark 4.2.4. We need a different treatment for the unstable primitives $F_{0,1}(z)$ and $F_{0,2}(z_1, z_2)$. They are calculated in Section 4.3.

The rest of this section is devoted to proving this theorem. We start with recalling (in a bit reformulated way) some results of Chapter 3. The most important one is the formula for $\mathcal{W}_{g,n}(z_1, \ldots, z_n)$ in terms of the auxiliary functions $W_d^i(z)$ (defined below) with the *ancestor* Gromov-Witten invariants as its coefficients. We will then prove the existence of the anti-symmetric primitives of the functions W_d^i, and their x-expansions. This will then lead us to the proof of the above theorem, where we will also utilize the known relations between the ancestor and the descendant Gromov-Witten invariants.

4.2.1 Some results from Chapter 3

The ancestor Gromov-Witten invariants of $\mathbb{P}^1$ we need are

$$\left\langle \prod_{i=1}^{n} \bar{\tau}_{b_i}(\alpha_i) \right\rangle_{g,n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \bar{\psi}_i^{b_i} ev_i^*(\alpha_i), \tag{4.26}$$

where $\bar{\psi}_i$ denotes the pull back of the cotangent class on $\overline{\mathcal{M}}_{g,n}$ by the natural forgetful morphism

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Since we adopt a quantum field theoretic point of view in calculating Gromov-Witten invariants, we often call them *correlators*. The ancestor and descendant correlators do not agree. We will give a formula to determine one from the other in (4.31).

Let us define

$$W_0^1(z) := \frac{dz}{(1-z)^2}, \tag{4.27}$$

$$W_0^2(z) := \frac{idz}{(1+z)^2}, \tag{4.28}$$

$$W_k^i(z) := d\left(\left(-2\frac{d}{dx(z)} \right)^k \int W_0^i(z) \right), \qquad i = 1, 2; \quad k \geq 0. \tag{4.29}$$

Then for $g \geq 0$ and $n \geq 1$ with $2g-2+n > 0$, from Theorem 3.3.1 (as shown in the proof of Theorem 3.4.2), we have

$$\mathcal{W}_{g,n}(z_1, \ldots, z_n) = \sum_{\vec{d},\, \vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1}) \ldots \bar{\tau}_{d_n}(\tilde{e}_{i_n}) \rangle_g \frac{W_{d_1}^{i_1}(z_1)}{2^{d_1}\sqrt{2}} \cdots \frac{W_{d_n}^{i_n}(z_n)}{2^{d_n}\sqrt{2}}. \tag{4.30}$$

Here the sum over $\vec{d}$ and $\vec{i}$ are taken over all integer values $0 \leq d_k$ and $i_k = 1, 2$. Note that the coefficients of this expansion are the *ancestor* Gromov-Witten invariants. The cohomology basis for $H^1(\mathbb{P}^1, \mathbb{Q})$ is normalized as follows. First we denote by $e_1 = 1$ and $e_2 = \omega$. Using the normalization matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

we define

$$\tilde{e}_i = \left(A^{-1}\right)_i^{\mu} e_{\mu}.$$

In this section we use the Einstein convention and take summation over repeated indices.

With the help of the Givental formula, Proposition 3.4.1 relates the ancestor and the descendant correlators for $\mathbb{P}^1$ by

$$\begin{aligned} &\sum_{\vec{d},\,\vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1}) \ldots \bar{\tau}_{d_n}(\tilde{e}_{i_n}) \rangle_g \, v^{d_1,i_1} \ldots v^{d_n,i_n} \\ =&\sum_{\vec{d},\,\vec{\mu}} \langle \tau_{d_1}(e_{\mu_1}) \ldots \tau_{d_n}(e_{\mu_n}) \rangle_g \, t^{d_1,\mu_1} \ldots t^{d_n,\mu_n}, \end{aligned} \tag{4.31}$$

where $v^{d,i}$ and $t^{d,\mu}$ are formal variables related by the following formula:

$$v^{d,i} = A^i_{\mu} \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^{\mu}_{\nu} t^{m,\nu}. \tag{4.32}$$

Here $(\mathcal{S}_k)^{\mu}_{\nu}$ are the matrix elements of the Givental S-matrix and defined by

$$\begin{aligned} \mathcal{S}(\zeta^{-1}) = \sum_{k=0}^{\infty} \mathcal{S}_k \zeta^{-k} &= \mathbf{I} + \zeta^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k}}{(k!)^2} \begin{pmatrix} 1 - 2k\left(\frac{1}{1} + \cdots + \frac{1}{k}\right) & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k-1}}{(k!)^2} \begin{pmatrix} 0 & -2\left(\frac{1}{1} + \cdots + \frac{1}{k}\right) \\ \frac{1}{k+1} & 0 \end{pmatrix}. \end{aligned} \tag{4.33}$$

In the proof of Theorem 3.4.2 it was shown that the x^{-1}-expansion of $W^i_d(z)$ near $z = 0$ was given by the following formula:

$$W^i_d(z) = 2^d \sqrt{2}\, A^i_{\mu} \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^{\mu}_{\nu} \; \delta^{\nu}_2 \,(m+1)!\, \frac{dx}{x^{m+2}}, \tag{4.34}$$

where δ^i_j is the Kronecker delta symbol. The above formula, together with formulas (4.30)-(4.33), implies (4.24).

The first step of integrating $\mathcal{W}_{g,n}$ is to identify a suitable primitive of the differential 1-forms $W^i_d(z)$.

Proposition 4.2.5. *For given $i = 1, 2$ and $d \geq 0$, there exists a uniquely defined rational function $\theta^i_d(z)$ on Σ such that*

$$d\theta^i_d(z) = W^i_d(z), \tag{4.35}$$

$$\theta^i_d(1/z) = -\,\theta^i_d(z). \tag{4.36}$$

Moreover, the x^{-1}-expansion of $\theta^i_d(z)$ near $z = 0$ is given by

$$\theta^i_d(z(x)) = 2^d\sqrt{2}\, A^i_\mu \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \, \left(-\delta^\nu_1 \delta^m_0 \, \frac{1}{2} - \delta^\nu_2 \, m! \, \frac{1}{x^{m+1}}\right). \tag{4.37}$$

4.2.2 Proof of Proposition 4.2.5

It is easy to see by direct computation that the rational functions

$$\begin{aligned} \theta^1_0 &:= \frac{1}{1-z} - \frac{1}{2} \\ \theta^2_0 &:= -\frac{i}{1+z} + \frac{i}{2} \end{aligned} \tag{4.38}$$

are the unique solutions of (4.35) and (4.36) for $d = 0$.

Equation (4.29), together with condition (4.35), implies that if $\theta^i_d(z)$ exists, then it has to satisfy

$$\theta^i_d(z) = \left(-2\frac{d}{dx(z)}\right)^d \theta^i_0(z). \tag{4.39}$$

Since x is symmetric under the coordinate change $z \longmapsto 1/z$, we see that the right-hand side of equation (4.39) satisfies (4.36). This means that $\theta^i_d(z)$ *defined* by (4.39) is, for given i and d, indeed the unique solution of (4.35) and (4.36).

We denote by $\tilde{\theta}^i_d$ the right-hand side of (4.37). We wish to prove that the x^{-1}-expansion of $\theta^i_d(z)$ near $z = 0$ is given by $\tilde{\theta}^i_d$. Let us introduce the following notation:

$$\eta^\mu_d := \frac{1}{2^d\sqrt{2}} \left(A^{-1}\right)^\mu_i \, \theta^i_d. \tag{4.40}$$

Then we have

$$\eta_0 = \left(\frac{1}{1-z^2} - \frac{1}{2}, \frac{z}{1-z^2}\right), \tag{4.41}$$

$$\eta^\mu_k(z) = \left(-\frac{d}{dx(z)}\right)^k \eta^\mu_0, \tag{4.42}$$

and condition (4.37) becomes equivalent to the condition that the x^{-1}-expansion of η^μ_d near $z = 0$ is equal to $\tilde{\eta}^\mu_d$, where

$$\tilde{\eta}^\mu_d := \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \, \left(-\delta^\nu_1 \delta^m_0 \, \frac{1}{2} - \delta^\nu_2 \, m! \, \frac{1}{x^{m+1}}\right). \tag{4.43}$$

Let us prove formula (4.43) for $d = 0$. Note that $\mathcal{S}_0 = \mathbf{I}$, so for the constant term of $\tilde{\eta}_0$ we have

$$\left[\frac{1}{x^0}\right] \tilde{\eta}_0^\mu = -\delta_1^\mu \frac{1}{2}. \tag{4.44}$$

It is easy to see from (4.41) that η_0^i has the same constant term at $z = 0$.

For $k \geq 1$ we have

$$\begin{aligned}
\left[\frac{1}{x^{2k-1}}\right] \tilde{\eta}_0^1 &= -(2k-2)!\,(\mathcal{S}_{2k-2})_2^1 = 0, \\
\left[\frac{1}{x^{2k-1}}\right] \tilde{\eta}_0^2 &= -(2k-2)!\,(\mathcal{S}_{2k-2})_2^2 = -\frac{(2k-2)!}{((k-1)!)^2}, \\
\left[\frac{1}{x^{2k}}\right] \tilde{\eta}_0^1 &= -(2k-1)!\,(\mathcal{S}_{2k-1})_2^1 = -\frac{(2k-1)!}{k!\,(k-1)!}, \\
\left[\frac{1}{x^{2k}}\right] \tilde{\eta}_0^2 &= -(2k-1)!\,(\mathcal{S}_{2k-1})_2^2 = 0.
\end{aligned} \tag{4.45}$$

For the corresponding coefficients in the x^{-1}-expansion of η_0^μ near $z = 0$ we have ($k \geq 1$):

$$\begin{aligned}
\operatorname*{Res}_{z=0} x^{2k-2}(z)\, \eta_0^1\, dx(z) &= -\operatorname*{Res}_{z=0} z^{-2k} \left(1+z^2\right)^{2k-2} dz = 0, \\
\operatorname*{Res}_{z=0} x^{2k-2}(z)\, \eta_0^2\, dx(z) &= -\operatorname*{Res}_{z=0} z^{-2k+1} \left(1+z^2\right)^{2k-2} dz = -\frac{(2k-2)!}{((k-1)!)^2}, \\
\operatorname*{Res}_{z=0} x^{2k-1}(z)\, \eta_0^1\, dx(z) &= -\operatorname*{Res}_{z=0} z^{-2k-1} \left(1+z^2\right)^{2k-1} dz = -\frac{(2k-1)!}{k!\,(k-1)!}, \\
\operatorname*{Res}_{z=0} x^{2k-1}(z)\, \eta_0^2\, dx(z) &= -\operatorname*{Res}_{z=0} z^{-2k} \left(1+z^2\right)^{2k-1} dz = 0.
\end{aligned} \tag{4.46}$$

We see that the coefficients in (4.45) precisely coincide with the ones in (4.46). This implies that the x^{-1}-expansion of η_0^μ is indeed given by $\tilde{\eta}_0^\mu$.

By virtue of (4.42), we see that the x^{-1}-expansion of η_k^μ near $z = 0$ is given by the following formula (for $k \geq 1$):

$$\left(-\frac{d}{dx}\right)^k \eta_0^\mu = \sum_{m=0}^{\infty} (\mathcal{S}_m)_\nu^\mu \left(-\delta_2^\nu\,(m+k)!\,\frac{1}{x^{m+k}}\right) = \sum_{m=d}^{\infty} (\mathcal{S}_{m-k})_\nu^\mu \left(-\delta_2^\nu\, m!\,\frac{1}{x^{m+1}}\right).$$

This coincides with the formula for $\tilde{\eta}_k^\mu$ for $k \geq 1$. Thus, we have proved that the x^{-1}-expansion of η_k^μ is given by $\tilde{\eta}_k^\mu$, which, in turn, implies that Equation (4.37) holds. This concludes the proof of the proposition.

4.2.3 Proof of Theorem 4.2.3

Recall Equation (4.30) for $\mathcal{W}_{g,n}$:

$$\mathcal{W}_{g,n}(z_1,\ldots,z_n) = \sum_{\vec{d},\,\vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1}) \ldots \bar{\tau}_{d_n}(\tilde{e}_{i_n}) \rangle_g \frac{W_{d_1}^{i_1}(z_1)}{2^{d_1}\sqrt{2}} \cdots \frac{W_{d_n}^{i_n}(z_n)}{2^{d_n}\sqrt{2}}.$$

Since we know how to integrate every $W_d^i(z)$, we simply define

$$F_{g,n}(z_1,\dots,z_n) := \sum_{\vec{d},\,\vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1})\dots\bar{\tau}_{d_n}(\tilde{e}_{i_n})\rangle_g \, \frac{\theta^{i_1}_{d_1}(z_1)}{2^{d_n}\sqrt{2}}\cdots\frac{\theta^{i_n}_{d_n}(z_n)}{2^{d_n}\sqrt{2}}. \tag{4.47}$$

Then from Proposition 4.2.5, we see that (4.20) and (4.21) are automatically satisfied. We also know from Proposition 4.2.5 that the x^{-1}-expansion of $F_{g,n}$ near $z_1=\dots=z_n=0$ is given by

$$\begin{aligned} &F_{g,n}(x_1,\dots,x_n)\\ &\quad= \sum_{\vec{d},\,\vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1})\dots\bar{\tau}_{d_n}(\tilde{e}_{i_n})\rangle_g \prod_{k=1}^{n} A^{i_k}_{\mu_k} \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^{\mu_k}_{\nu_k} \left(-\delta_1^{\nu_k}\delta_0^m\,\frac{1}{2} - \delta_2^{\nu_k}\, m!\,\frac{1}{x_k^{m+1}}\right). \end{aligned}$$

Using (4.31) and (4.32), we find

$$\begin{aligned} &F_{g,n}(x_1,\dots,x_n)\\ &= \sum_{\vec{d},\vec{i}} \langle \tau_{d_1}(e_{\mu_1})\dots\tau_{d_n}(e_{\mu_n})\rangle_g \prod_{i=1}^{n}\left(-\delta_1^{\mu_i}\delta_0^{d_i}\,\frac{1}{2} - \delta_2^{\mu_i}\, d_i!\,\frac{1}{x^{d_i+1}}\right)\\ &= \left\langle \prod_{i=1}^{n}\left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x_i^{b+1}}\right)\right\rangle_{g,n}. \end{aligned} \tag{4.48}$$

The final condition (4.22) follows from the fact that $\langle(\tau_0(1))^n\rangle_{g,n} = 0$ for all g and n in the stable range. This concludes the proof of the theorem.

4.3 The shift of variable simplification

Let us now turn our attention toward proving (4.9) of Theorem 4.1.1. In this section, as the first step, we establish a formula for the wave function $\Psi(x,\hbar)$ of (4.8) involving only the stationary Gromov-Witten invariants.

Our starting point is

$$\begin{aligned} \log\Psi(x,\hbar) &= \frac{1}{\hbar}S_0(x) + S_1(x)\\ &\quad + \sum_{g,d=0}^{\infty}\sum_{\substack{n=1\\2g-2+n>0}}^{\infty} \frac{\hbar^{2g-2+n}}{n!}\left\langle\left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)^n\right\rangle^d_{g,n}. \end{aligned} \tag{4.49}$$

Using the string equation and some earlier results in [31], we shall give an expression for $\log\Psi(x,\hbar)$ purely in terms of the stationary sector. More precisely, we prove the following lemma.

Lemma 4.3.1. *The function* $\log \Psi(x,\hbar)$ *is a solution to the following difference equation:*

$$\exp\left(-\frac{\hbar}{2}\frac{d}{dx}\right)\log \Psi(x,\hbar) = \frac{1}{\hbar}\left(x - x\log x\right) + \sum_{g,d=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\left\langle\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)^n\right\rangle_{g,n}^{d}. \tag{4.50}$$

4.3.1 Expansion of S_0 and S_1

The functions $S_0(x)$ and $S_1(x)$ of (4.6) and (4.7) are derived from the first steps of the WKB method, that is, they are just imposed by the quantum spectral curve equation. In this subsection, we represent them in terms of the unstable $(0,1)$- and $(0,2)$-Gromov-Witten invariants.

First let us calculate these functions from the WKB approximation (4.11). After taking the semi-classical limit (4.12), we can calculate $S_1'(x)$ as follows:

$$\begin{aligned}
&e^{-\frac{1}{\hbar}S_0(x)-S_1(x)}\left(e^{\hbar\frac{d}{dx}} + e^{-\hbar\frac{d}{dx}} - x\right)e^{\frac{1}{\hbar}S_0(x)+S_1(x)}\\
&= e^{S_0'(x)+\hbar\left(\frac{1}{2}S_0''(x)+S_1'(x)\right)}e^{\hbar\frac{d}{dx}} + e^{-S_0'(x)+\hbar\left(\frac{1}{2}S_0''(x)-S_1'(x)\right)}e^{-\hbar\frac{d}{dx}} - x + O(\hbar^2)\\
&= e^{S_0'(x)}\left(1+\hbar\left(\frac{1}{2}S_0''(x)+S_1'(x)\right)\right) + e^{-S_0'(x)}\left(1+\hbar\left(\frac{1}{2}S_0''(x)-S_1'(x)\right)\right)\\
&\quad - x + O(\hbar^2)\\
&= \hbar\left(\frac{S_0''(x)}{2}\left(e^{S_0'(x)}+e^{-S_0'(x)}\right) + S_1'(x)\left(e^{S_0'(x)}-e^{-S_0'(x)}\right)\right) + O(\hbar^2).
\end{aligned}$$

The coefficient of $\hbar$ must vanish, hence we can solve for $S_1'(x)$. Since

$$S_0''(x) = \frac{d}{dx}S_0'(x) = \frac{d}{dx}\log z = \frac{\frac{d}{dz}\log z}{x'(z)} = \frac{\frac{1}{z}}{1-\frac{1}{z^2}} = \frac{1}{z-\frac{1}{z}},$$

we find

$$S_1'(x) = -\frac{1}{2}\frac{1}{z-\frac{1}{z}}\frac{z+\frac{1}{z}}{z-\frac{1}{z}} = -\frac{1}{2}\frac{z(z^2+1)}{(z^2-1)^2}. \tag{4.51}$$

It is proved in [31, Equation (7.9) and Theorem 7.7] that

$$\begin{aligned}
\sum_{d=0}^{\infty}\left\langle\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)\right\rangle_{0,1}^{d} &= \sum_{d=1}^{\infty}\left\langle\left(-\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}}\right)\right\rangle_{0,1}^{d}\\
&= -2z + \left(z+\frac{1}{z}\right)\log\left(1+z^2\right),
\end{aligned} \tag{4.52}$$

and

$$\sum_{d=0}^{\infty}\left\langle\prod_{i=1}^{2}\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x_i^{b+1}}\right)\right\rangle_{0,2}^{d} = -\log\left(1-z_1z_2\right). \tag{4.53}$$

One of the implications of the string equation is

$$\langle \tau_0(1)\tau_{b+1}(\omega)\rangle_{0,2}^d = \langle \tau_b(\omega)\rangle_{0,1}^d .$$

Using this form of the string equation and Equation (4.52), we calculate that

$$\begin{aligned}
&\sum_{d=1}^{\infty} \left\langle \left(-\frac{1}{2}\tau_0(1)\right)\left(-\frac{(2d-1)!\tau_{2d-1}(\omega)}{x_i^{2d}}\right)\right\rangle_{0,2}^{d} \\
&= \frac{1}{2}\frac{d}{dx}\left(-2z + \left(z+\frac{1}{z}\right)\log\left(1+z^2\right)\right) \\
&= \frac{1}{2}\log x + \frac{1}{2}\log z.
\end{aligned} \tag{4.54}$$

Note that the only condition we have for $S_0(x)$ is that $S_0'(x) = \log z$. Therefore, if we define

$$S_0(z) := F_{0,1}(z) = \int \mathcal{W}_{0,1}(z) = \int y(z)dx(z)$$

by formally applying (4.10) for $m = 0$, and impose the skew-symmetry condition (4.21) to the primitive $F_{0,1}(z)$, then from (4.52) we obtain

$$\begin{aligned}
S_0(x) &= \frac{1}{z} - z + \left(z+\frac{1}{z}\right)\log z \\
&= (x - x\log x) + \sum_{d=1}^{\infty}\left\langle\left(-\frac{(2d-2)!\tau_{2d-2}(\omega)}{x_i^{2d-1}}\right)\right\rangle_{0,1}^{d} .
\end{aligned} \tag{4.55}$$

The determination of $S_1(x)$ is trickier. Morally speaking, if we formally apply (4.10) for $m = 1$, then we obtain

$$S_1(x) = -\frac{1}{2}F_{0,2}(z,z) \tag{4.56}$$

for the primitive

$$\begin{aligned}
F_{0,2}(z_1,z_2) = \int^{z_1}\int^{z_2} &\mathcal{W}_{0,2}(z_1,z_2) \\
&= \int^{z_1}\int^{z_2}\left(\frac{dz_1dz_2}{(z_1-z_2)^2} - \frac{dx_1dx_2}{(x_1-x_2)^2}\right) \\
&\qquad = -\log(1-z_1z_2) + f(z_1) + f(z_2) + c.
\end{aligned} \tag{4.57}$$

Here we are imposing the condition that $F_{0,2}(z_1,z_2)$ is a symmetric function. The fact that $F_{0,2}$ is a primitive of $\mathcal{W}_{0,2}$ does not determine the function $f(z)$. Therefore, we are free to choose $f(z)$ so that the differential equation (4.51) holds. Obviously, we need to choose $f(z) = \dfrac{1}{2}\log z$. In this way, using (4.53) and (4.54) as well, we obtain

$$\begin{aligned}
S_1(x) &= -\frac{1}{2}\log\left(1-z^2\right) + \frac{1}{2}\log z \\
&= -\frac{1}{2}\log x + \frac{1}{2}\sum_{d=0}^{\infty}\left\langle\left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)^2\right\rangle_{0,2}^{d} .
\end{aligned} \tag{4.58}$$

Remark 4.3.2. This adjustment of the choice of $S_1(x)$ also appears in the Hitchin fibration case of [30]. Still we have one degree of freedom for choosing a constant c of (4.57). It does not matter to the linear quantum curve equation (4.9), because the constant term c only affects on the overall constant factor of Ψ of (4.8).

4.3.2 A new formula for $\log\Psi$

We use Equations (4.55) and (4.58) to rewrite the formula (4.49) for $\log\Psi$ in the following way:

$$\log\Psi(x,\hbar) = \sum_{g,d=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}(-1)^n}{n!}\Theta^d_{g,n}, \tag{4.59}$$

where

$$\Theta^0_{0,1} := -x + x\log x + \frac{\hbar}{2}\log x + \sum_{k=2}^{\infty}\left\langle\tau_0(1)^k\tau_{k-2}(\omega)\right\rangle^0_{0,k+1}\frac{(-1)^k\hbar^k}{2^k k!}\frac{(k-2)!}{x^{k-1}} \tag{4.60}$$

and

$$\Theta^d_{g,n} := \sum_{k=0}^{\infty}\sum_{b_1,\ldots,b_n=0}^{\infty}\left\langle\tau_0(1)^k\prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n+k}\frac{(-1)^k\hbar^k}{2^k k!}\frac{\prod_{i=1}^n b_i!}{x^{n+\sum_{i=1}^n b_i}}. \tag{4.61}$$

It is obvious that for dimensional reasons, $\Theta^0_{0,n}=0$ for any $n\geq 2$. Lemma 4.3.1 is then a direct corollary to the following statement.

Lemma 4.3.3. *The quantities defined in* (4.60) *and* (4.61) *are given by*

$$\Theta^0_{0,1} = -\left(x+\frac{\hbar}{2}\right) + \left(x+\frac{\hbar}{2}\right)\log\left(x+\frac{\hbar}{2}\right); \tag{4.62}$$

$$\Theta^d_{g,n} = \sum_{b_1,\ldots,b_n}\left\langle\prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n}\frac{\prod_{i=1}^n b_i!}{\left(x+\frac{\hbar}{2}\right)^{n+\sum_{i=1}^n b_i}}, \tag{4.63}$$

where in the second equation the sum is taken over all $b_1,\ldots,b_n\geq 0$ such that $\sum_{i=1}^n b_i = 2g+2d-2$.

4.3.3 Proof of Lemma 4.3.3

Since the difference between the definitions (4.60)-(4.61) and the values (4.62)-(4.63) is simply the elimination of $\tau_0(1)$, we prove Lemma 4.3.3 by using the string equation for the Gromov-Witten invariants of $\mathbb{P}^1$:

$$\left\langle\tau_0(1)^k\prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n+k} = \sum_{\substack{j=1\\ b_j>0}}^{n}\left\langle\tau_0(1)^{k-1}\tau_{b_j-1}(\omega)\prod_{\substack{i=1\\ i\neq j}}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n+k-1}, \tag{4.64}$$

where we assume $2g-2+n>1$ and $k>0$.

First, let us directly compute $\Theta^0_{0,1}$. Equation (4.64) implies that

$$\left\langle\tau_0(1)^k\tau_{k-2}(\omega)\right\rangle^0_{0,k+1} = \left\langle\tau_0(1)^{k-1}\tau_{k-3}(\omega)\right\rangle^0_{0,k} = \left\langle\tau_0(1)^2\tau_0(\omega)\right\rangle_{0,3} = 1. \tag{4.65}$$

Therefore,

$$\sum_{k=2}^{\infty}\left\langle \tau_0(1)^k\tau_{k-2}(\omega)\right\rangle_{0,k+1}^{0}\frac{(-1)^k\hbar^k}{2^k k!}\frac{(k-2)!}{x^{k-1}}$$
$$=\sum_{k=2}^{\infty}\frac{(-1)^k\hbar^k}{2^k k!}\frac{(k-2)!}{x^{k-1}}=\left(x+\frac{\hbar}{2}\right)\log\left(\frac{x+\frac{\hbar}{2}}{x}\right)-\frac{\hbar}{2}.$$

This proves Equation (4.62).

The proof of Equation (4.63) goes as follows. Recall that $g+d>0$ and $n>0$. Equation (4.64) implies that any correlator $\left\langle \tau_0(1)^k\prod_{i=1}^n\tau_{b_i}(\omega)\right\rangle_{g,n+k}^{d}$ can be represented as a linear combination of the correlators $\langle\prod_{i=1}^n\tau_{b_i}(\omega)\rangle_{g,n}^d$ with $\sum_{i=1}^n b_i=2g+2d-2$. Moreover, for any $k\geq 0$ and $c_1,\dots,c_n\geq 0$ such that $\sum_{i=1}^n c_i=k$, the coefficient of a particular correlator $\langle\prod_{i=1}^n\tau_{b_i}(\omega)\rangle_{g,n}^d$ in $\left\langle \tau_0(1)^k\prod_{i=1}^n\tau_{b_i+c_i}(\omega)\right\rangle_{g,n+k}^{d}$ is equal to

$$\frac{k!}{c_1!\cdots c_n!}.$$

Therefore, the total coefficient of $\langle\prod_{i=1}^n\tau_{b_i}(\omega)\rangle_{g,n}^d$ in $\Theta_{g,n}^d$ is equal to

$$\begin{aligned}
&\sum_{k=0}^{\infty}\sum_{\substack{c_1,\dots,c_n=0\\ c_1+\cdots c_n=k}}^{\infty}\frac{(-1)^k\hbar^k}{2^k k!}\frac{\prod_{i=1}^n(b_i+c_i)!}{x^{n+\sum_{i=1}^n(b_i+c_i)}}\frac{k!}{c_1!\cdots c_n!}\\
&=\frac{\prod_{i=1}^n(b_i)!}{x^{n+\sum_{i=1}^n(b_i)}}\sum_{k=0}^{\infty}\left(\frac{-\hbar}{2x}\right)^k\sum_{\substack{c_1,\dots,c_n\geq 0\\ c_1+\cdots+c_n=k}}\prod_{i=1}^n\frac{(b_i+c_i)!}{b_i!c_i!}.
\end{aligned}\tag{4.66}$$

On the other hand, expansion of the coefficient of $\langle\prod_{i=1}^n\tau_{b_i}(\omega)\rangle_{g,n}^d$ in Equation (4.63) is equal to

$$\begin{aligned}
&\frac{\prod_{i=1}^n(b_i)!}{\left(x+\frac{\hbar}{2}\right)^{n+\sum_{i=1}^n(b_i)}}=\prod_{i=1}^n\frac{(b_i)!}{\left(x+\frac{\hbar}{2}\right)^{b_i+1}}\\
&=\prod_{i=1}^n\frac{(b_i)!}{(x)^{b_i+1}}\sum_{c_i=0}^{\infty}\left(\frac{-\hbar}{2x}\right)^k\frac{(b_i+c_i)!}{b_i!c_i!}.
\end{aligned}\tag{4.67}$$

Since (4.66) and (4.67) are identical, we have proved Equation (4.63). This completes the proof of Lemma 4.3.1.

4.4 Reduction to the semi-infinite wedge formalism

In this Section we represent the formula for $\Psi(x,\hbar)$ in terms of the semi-infinite wedge formalism. We use the formula of Okounkov-Pandharipande [84] that relates the stationary sector of the Gromov-Witten invariants of $\mathbb{P}^1$ to the expectation values of the so-called $\mathcal{E}$-operators. In order to include the extra combinatorial factors that we have in the expansion of $\log\Psi(x,\hbar)$, we consider the $\mathcal{E}$-operators with values in formal differential operators.

4.4.1 Semi-infinite wedge formalism

In this subsection we recall very briefly some basic facts about the semi-infinite wedge formalism. For more details we refer to [84, 88].

Let us consider a vector space $V := \bigoplus_{c=-\infty}^{\infty} V_c$, where V_c is spanned by the basis vectors $\underline{a_1} \wedge \underline{a_2} \wedge \underline{a_3} \wedge \cdots$ such that $a_i \in \mathbb{Z} + 1/2$, $i = 1, 2, \ldots$, $a_1 > a_2 > a_3 \ldots$, and for all but a finite number of terms we have $a_i = 1/2 - i + c$. We denote by ψ_k the operator $\underline{k}\wedge\colon V_c \to V_{c+1}$, and by ψ_k^* the operator $\partial/\partial\underline{k}\colon V_c \to V_{c-1}$. Both are odd operators, and they satisfy the graded commutation relation $[\psi_i, \psi_i^*] = 1$, with all other possible pairwise commutators equal to zero.

We denote by $:\psi_i\psi_j^*:$ the normally ordered product, that is, $:\psi_i\psi_j^* := \psi_i\psi_j^*$ for $j > 0$ and $:\psi_i\psi_j^* := -\psi_j^*\psi_i$ for $j < 0$. We introduce the operators $\mathcal{E}_n(z)$, $n \in \mathbb{Z}$ as

$$\mathcal{E}_n(z) := \sum_{k\in\mathbb{Z}+1/2} \exp\left(z\left(k - \frac{n}{2}\right)\right) :\psi_{k-r}\psi_k^*: + \frac{\delta_{n0}}{\zeta(z)}, \tag{4.68}$$

where $\zeta(z) = \exp(z/2) - \exp(-z/2)$. These operators satisfy the commutation relation $[\mathcal{E}_n(z), \mathcal{E}_m(w)] = \zeta(nw - mz)\mathcal{E}_{n+m}(z + w)$.

For any operator $\mathcal{A} = \mathcal{E}_{n_1}(z_1)\cdots\mathcal{E}_{n_m}(z_m)$ we denote by $\langle|\mathcal{A}|\rangle$ the coefficient of the vector $v_\emptyset := \underline{-1/2} \wedge \underline{-3/2} \wedge \underline{-5/2} \wedge \cdots$ in the basis expansion of $\mathcal{A}v_\emptyset$. If we want to compute a particular correlator $\langle|\mathcal{E}_{n_1}(z_1)\cdots\mathcal{E}_{n_m}(z_m)|\rangle$, first we use the commutation relation for the $\mathcal{E}$-operators, and then appeal to the simple fact that $\mathcal{E}_n(z)|\rangle = 0$ for $n > 0$, $\langle|\mathcal{E}_n(z) = 0$ for $n < 0$, and $\langle|\mathcal{E}_0(z_1)\cdots\mathcal{E}_0(z_n)|\rangle = 1/\left(\zeta(z_1)\cdots\zeta(z_n)\right)$. In this section we are mostly interested in correlators for the form

$$\langle|\mathcal{A}|\rangle = \left\langle\left|\mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(z_i)\mathcal{E}_{-1}(0)^d\right|\right\rangle. \tag{4.69}$$

For the purpose of establishing the results in [84], Okounkov and Pandharipande considered the *disconnected* version of Gromov-Witten invariants and Hurwitz numbers. The disconnectedness here means we allow disconnected domain curves mapped to $\mathbb{P}^1$. For example, they establish in [84, Proposition 3.1, Equation 3.4] a formula for disconnected stationary Gromov-Witten invariants of $\mathbb{P}^1$, which reads

$$\sum_{b_1,\ldots,b_n\geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega)\right\rangle^{\bullet\, d} \prod_{i=1}^n x_i^{b_i+1} = \frac{1}{(d!)^2}\left\langle\left|\mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(x_i)\mathcal{E}_{-1}(0)^d\right|\right\rangle, \tag{4.70}$$

where $\langle\ \ \rangle^\bullet$ denotes the disconnected Gromov-Witten invariant. Counting the number of disconnected domain curves and connected ones are related simply by talking the logarithm. Thus we have

$$\sum_{g=0}^{\infty}\sum_{b_1,\ldots,b_n\geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega)\right\rangle_{g,n}^{d} \prod_{i=1}^n x_i^{b_i+1} = \log\left(\sum_{b_1,\ldots,b_n\geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega)\right\rangle^{\bullet\, d} \prod_{i=1}^n x_i^{b_i+1}\right).$$

This prompts us to introduce the *connected* correlator notation, corresponding to (4.70), as follows:

$$\sum_{g=0}^{\infty}\sum_{b_1,\ldots,b_n\geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega)\right\rangle_{g,n}^{d} \prod_{i=1}^n x_i^{b_i+1} = \frac{1}{(d!)^2}\left\langle\left|\mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(x_i)\mathcal{E}_{-1}(0)^d\right|\right\rangle^{\circ}. \tag{4.71}$$

The connected correlator is also known as the *cumulant* in probability theory, which is calculate via the inclusion-exclusion formula. In general, for an operator $\mathcal{A}$ of (4.69), we denote by $\langle|\mathcal{A}|\rangle^\circ$ the contribution coming from the single operator of the form $\mathcal{E}_0(\sum_{i=1}^n z_i)$ in the end, after applying the commutation relation successively. Of course in terms of generating functions, this simply means we take the logarithm of the expression.

4.4.2 A new formula for Ψ

Noticing that $\exp\left(\frac{\hbar}{2}\frac{d}{dx}\right)$ is an automorphism, from (4.50) we find

$$\log \Psi(x,\hbar) = \exp\left(\frac{\hbar}{2}\frac{d}{dx}\right) T(x),$$

where

$$T(x) := \sum_{g,d=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{b_1,\dots,b_n=0}^{\infty}\left\langle\prod_{i=1}^n \tau_{b_i}(\omega)\right\rangle_{g,n}^d \prod_{i=1}^n\left(-\frac{b_i!}{x^{b_i+1}}\right) + \frac{1}{\hbar}\langle\tau_{-2}(\omega)\rangle_{0,1}^0(x - x\log x).$$

Here we have used the convention of [84] that $\langle\tau_{-2}(\omega)\rangle_{0,1}^0 = 1$ and $\tau_{-1}(\omega)=0$. We are now ready to re-write the right-hand side in terms of expectation values of $\mathcal{E}$-operators. Corollary 4.4.2 of the following lemma is the main result of this section.

Lemma 4.4.1. *For any $d\geq 0$, $n\geq 1$, $(d,n)\neq(0,1)$, we have*

$$\sum_{g=0}^{\infty}\hbar^{2g-2+n}\sum_{b_1,\dots,b_n=0}^{\infty}\left\langle\prod_{i=1}^n \tau_{b_i}(\omega)\right\rangle_{g,n}^d \prod_{i=1}^n\left(-\frac{b_i!}{x_i^{b_i+1}}\right) = \frac{1}{(d!)^2\hbar^{2d}}\left\langle\left|\mathcal{E}_1(0)^d\prod_{i=1}^n\mathcal{E}_0\left(-\hbar\frac{\partial}{\partial x_i}\right)(\log x_i)\,\mathcal{E}_{-1}(0)^{-d}\right|\right\rangle^\circ. \quad (4.72)$$

For $d=0$ and $n=1$, we have

$$\frac{1}{\hbar}\langle\tau_{-2}(\omega)\rangle_{0,1}^0(x-x\log x) + \sum_{g=1}^{\infty}\hbar^{2g-1}\left\langle\prod_{i=1}^n\tau_{2g-2}(\omega)\right\rangle_{g,1}^0\left(-\frac{(2g-2)!}{x^{2g-1}}\right) = \left\langle\left|\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right|\right\rangle^\circ. \quad (4.73)$$

Here we denote by $\langle\quad\rangle^\circ$ the connected expectation value. This means that after the successive application of the commutation relation, all differential operators appear in one correlator. Of course for $d=0$, $n=1$, we have $\langle\mathcal{E}_0\rangle^\circ = \langle\mathcal{E}_0\rangle$. The following corollary is a straightforward application of Lemma 4.4.1.

Corollary 4.4.2. *We have the following expression for* $\log\Psi$*:*

$$\log\Psi(x,\hbar) = \sum_{d=0}^{\infty}\frac{1}{\hbar^{2d}(d!)^2}\left\langle\left|\mathcal{E}_1(0)^d\sum_{n=1}^{\infty}\frac{\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)^n}{n!}\mathcal{E}_{-1}(0)^d\right|\right\rangle^\circ. \quad (4.74)$$

4.4.3 Proof of Lemma 4.4.1

The starting point of the proof is (4.71). Note that only negative b_i contribution comes from $\langle \tau_{-2}(\omega)\rangle_{0,1}^0 = 1$, which is the coefficient of x_i^{-1} in $\langle |\mathcal{E}_0(x_i)|\rangle^\circ$.

Let $A(x) = \sum_{i=-1}^\infty a_i x^i$ be an arbitrary Laurent series. Observe that

$$A\left(-\hbar\frac{d}{dx}\right)(\log x) = a_{-1}\left(\frac{x - x\log x}{\hbar}\right) + a_0 \log x - \sum_{i=1}^\infty a_i \frac{(i-1)!\hbar^i}{x^i}. \tag{4.75}$$

We can apply this observation to the correlator

$$\frac{1}{(d!)^2}\left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(x_i)\,\mathcal{E}_{-1}(0)^{-d} \right| \right\rangle^\circ \tag{4.76}$$

and change $\mathcal{E}_0(x_i)$ to

$$\mathcal{E}_0\left(-\hbar\frac{\partial}{\partial x_i}\right)\log x_i.$$

If $(n,d) \neq (1,0)$, then we have a formal Laurent series in $x_1,\ldots,x_n$, where the degree of each variable in each term is less than or equal to -1. Together with the computation of the degree of $\hbar$, which is $\sum_{i=1}^n (b_i+1) - 2d = 2g-2+n$, we establish Equation (4.72).

If $(n,d) = (1,0)$, then it is sufficient to observe that $\langle|\mathcal{E}_0(x)|\rangle^\circ = x^{-1} + O(x)$. Thus we have one additional term $(x - x\log x)/\hbar$ as in (4.75), which is exactly the first term in Equation (4.73).

This completes the proof of Lemma 4.4.1, and hence, Corollary 4.4.2.

4.5 Reduction to a combinatorial problem

The expression (4.74) of $\log\Psi$ in the form of the vacuum expectation value of the operator product allows us to convert the quantum curve equation (4.9) into a combinatorial formula.

Our starting point is the Ψ-function represented in the form

$$\Psi(x,\hbar) = 1 + \sum_{d=0}^\infty \frac{1}{\hbar^{2d}(d!)^2}\left\langle \left| \mathcal{E}_1(0)^d \sum_{n=1}^\infty \frac{1}{n!}\mathcal{A}(x)^n \mathcal{E}_{-1}(0)^d \right| \right\rangle^\star, \tag{4.77}$$

where

$$\begin{aligned}\mathcal{A}(x) &= \exp\left(\frac{\hbar}{2}\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\\ &= \sum_{k\in\mathbb{Z}+\frac12}\exp\left(\left(-k+\frac12\right)\hbar\frac{d}{dx}\right)(\log x) :\psi_k\psi_k^*: \\ &\quad + B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x - x\log x}{\hbar}\right).\end{aligned} \tag{4.78}$$

Here $B(t) := t/(e^t-1)$ in (4.78) is the generating series of the Bernoulli numbers, and the notation $\langle - \rangle^\star$ in (4.77) means that in the computation of this expectation value using the commutation relations, we never allow any $\mathcal{E}_1(0)$ and $\mathcal{E}_{-1}(0)$ to commute directly. We need this requirement since we exponentiate the series (4.74), which does not have terms without $\mathcal{E}_0$-operators. The goal of this section is to prove Corollary 4.5.2.

Lemma 4.5.1. *We have*

$$\exp\left(\frac{1}{\hbar^2}\right)\Psi(x,\hbar) = \exp\left(B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x - x\log x}{\hbar}\right)\right)X, \tag{4.79}$$

where $X := \sum_{d=0}^{\infty} X_d/\hbar^{2g}$*, and* X_d *is given by*

$$\begin{aligned} X_d &= \frac{1}{(d!)^2}\left\langle\left|\mathcal{E}_1(0)^d \exp\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \log\left(x - \left(k-\frac{1}{2}\right)\hbar\right) :\psi_k\psi_k^*:\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle \\ &= \sum_{\lambda\vdash d}\left(\frac{\dim\lambda}{d!}\right)^2 \prod_{i=1}^{\infty}\frac{x+(i-\lambda_i)\hbar}{x+i\hbar}. \end{aligned} \tag{4.80}$$

Corollary 4.5.2. *The quantum spectral curve equation*

$$\left[\exp\left(\hbar\frac{d}{dx}\right) + \exp\left(-\hbar\frac{d}{dx}\right) - x\right]\Psi(x,\hbar) = 0$$

is equivalent to the following equation for the function X*:*

$$\left[\frac{1}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right) + x\exp\left(-\hbar\frac{d}{dx}\right) - x\right]X = 0. \tag{4.81}$$

Proof of Lemma 4.5.1. Corollary 4.4.2 implies that

$$\begin{aligned} &\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d \exp\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle^\circ}{\hbar^{2d}(d!)^2} \\ &= \log\Psi(x,\hbar) + \frac{1}{\hbar^2} + 1. \end{aligned} \tag{4.82}$$

Indeed, we add terms with $n = 0$, and it is easy to see that

$$\left\langle\left|\mathcal{E}_1(0)^d\mathcal{E}_{-1}(0)^d\right|\right\rangle^\circ = 0, \qquad d \geq 2,$$

and $\langle|\mathcal{E}_1(0)\mathcal{E}_{-1}(0)|\rangle^\circ = \langle|\mathrm{Id}|\rangle^\circ = 1$. Therefore,

$$\begin{aligned} &\exp\left(\frac{1}{\hbar^2}\right)\Psi(x,\hbar) = \\ &\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d \exp\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle}{\hbar^{2d}(d!)^2}. \end{aligned} \tag{4.83}$$

From the definition of the operator $\mathcal{E}_0$, we have

$$\begin{aligned}
&\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\\
&=\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\left(\sum_{k\in\mathbb{Z}+1/2}\log\left(x-k\hbar\right):\psi_k\psi_k^*:\right)\\
&\quad+\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\frac{-\hbar\frac{d}{dx}}{\exp\left(-\frac{1}{2}\hbar\frac{d}{dx}\right)-\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)}\left(\frac{x-x\log x}{\hbar}\right)\\
&=\sum_{k\in\mathbb{Z}+1/2}\log\left(x-\left(k-\frac{1}{2}\right)\hbar\right):\psi_k\psi_k^*:+B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x-x\log x}{\hbar}\right).
\end{aligned}\tag{4.84}$$

Now define

$$A_1=\sum_{k\in\mathbb{Z}+1/2}\log\left(x-\left(k-\frac{1}{2}\right)\hbar\right):\psi_k\psi_k^*: \tag{4.85}$$

$$A_2=B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x-x\log x}{\hbar}\right). \tag{4.86}$$

Since A_1 and A_2 commute, we have $\exp(A_1+A_2)=\exp(A_2)\exp(A_1)$. Furthermore, since A_2 is a scalar operator, we have

$$\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d\exp(A_2)\exp(A_1)\mathcal{E}_{-1}(0)^d\right|\right\rangle}{\hbar^{2d}(d!)^2}=\exp(A_2)\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d\exp(A_1)\mathcal{E}_{-1}(0)^d\right|\right\rangle}{\hbar^{2d}(d!)^2}.$$

This is exactly the right-hand side of Equation (4.79). □

Proof of Corollary 4.5.2. We just have to show that

$$\begin{aligned}
\exp(-A_2)\exp\left(\hbar\frac{d}{dx}\right)\exp(A_2)&=\frac{1}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right);\\
\exp(-A_2)\exp\left(-\hbar\frac{d}{dx}\right)\exp(A_2)&=x\exp\left(-\hbar\frac{d}{dx}\right);\\
\exp(-A_2)x\exp(A_2)&=x.
\end{aligned}$$

The last equality is tautological, and the first two are obtained by a straightforward computation. □

For completeness, let us also explain Equation (4.80). It is based on several standard facts about the semi-infinite wedge formalism. For any partition $\lambda=(\lambda_1\geq\lambda_2\geq\lambda_3\geq\dots)$ we associate a basis vector $v_\lambda\in V_0$ given by

$$\underline{\left(\lambda_1-\frac{1}{2}\right)}\wedge\underline{\left(\lambda_2-\frac{3}{2}\right)}\wedge\underline{\left(\lambda_3-\frac{5}{2}\right)}\wedge\cdots. \tag{4.87}$$

Then, we have $\mathcal{E}_{-1}(0)^d v_\emptyset = \sum_{\lambda\vdash d} \dim\lambda \cdot v_\lambda$, $\left\langle \right| \mathcal{E}_1(0)^d v_\lambda = \dim\lambda$, and the fact that for any constants a_n, $n \in \mathbb{Z}+1/2$, v_λ is an eigenvector of the operator $\sum_{n\in\mathbb{Z}+1/2} a_n : \psi_n\psi_n^* :$ with the eigenvalue $\sum_{i=1}^{\infty}\left(a_{\lambda_i-i+1/2} - a_{-i+1/2}\right)$. Therefore, v_λ is an eigenvector of the operator

$$A_1 = \exp\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \log\left(x - \left(k-\frac{1}{2}\right)\hbar\right) : \psi_k\psi_k^* :\right) \tag{4.88}$$

with the eigenvalue

$$\exp\left(\sum_{i=1}^{\infty} \log\left(x+(i-\lambda_i)\hbar\right) - \log\left(x+i\hbar\right)\right) = \prod_{i=1}^{\infty} \frac{x+(i-\lambda_i)\hbar}{x+i\hbar}, \tag{4.89}$$

and the total weight of the vector v_λ in $\left\langle \left| \mathcal{E}_1(0)^d A_1 \mathcal{E}_{-1}(0)^d \right| \right\rangle$ is $(\dim\lambda)^2$. This implies Equation (4.80).

4.6 Key combinatorial argument

We have shown that the quantum curve equation (4.9) is equivalent to a combinatorial equation (4.81), which is indeed a first-order recursion equation for X_d of (4.80) with respect to the index d. In this section we prove (4.81).

Let $\lambda \vdash d$ be a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{\ell(\lambda)} > 0)$ of $d \geq 1$. We can always append it with $d-\ell(\lambda)$ zeros $\lambda_{\ell(\lambda)+1} := 0, \dots, \lambda_d := 0$ at the end so that we would have a partition of d of length d with non-negative parts. Throughout this section we use this convention that a partition of d has length d.

Consider the following sum over all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d)$ of $d \geq 1$:

$$X_d := \sum_{\lambda\vdash d} \frac{1}{H_\lambda^2} \prod_{i=1}^{d} \frac{x+(i-\lambda_i)\hbar}{x+i\hbar}. \tag{4.90}$$

Here $H_\lambda := \prod_{ij} h_{ij}$, where h_{ij} is the hook length at the vertex (ij) of the corresponding Young diagram, so that $d!/\prod h_{ij}$ is the dimension of the irreducible representation corresponding to λ. Or equivalently, it is the number of the standard Young tableaux of this shape. We use the convention that $X_0 := 1$.

In this Section we prove the following key combinatorial lemma.

Lemma 4.6.1. *The series $X := \sum_{d=0}^{\infty} X_d/\hbar^{2g}$ satisfies the following equation:*

$$\left[\frac{1}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right) + x\exp\left(-\hbar\frac{d}{dx}\right) - x\right] X = 0. \tag{4.91}$$

Proof. In fact, (4.91) is a direct consequence of the following more refined statement.

Lemma 4.6.2. *For any $d \geq 1$ we have*

$$\frac{1}{x/\hbar+1}\exp\left(\hbar\frac{d}{dx}\right) X_{d-1} + \left[\frac{x}{\hbar}\exp\left(-\hbar\frac{d}{dx}\right) - \frac{x}{\hbar}\right] X_d = 0. \tag{4.92}$$

Indeed, since $\left[x\exp\left(-h\frac{d}{dx}\right)-x\right]X_0=0$, the sum of Equation (4.92) for all $d\geq 1$ with coefficients $1/\hbar^{2d-1}$ yields Lemma 4.6.1. $\square$

To prove Lemma 4.6.2, we need to recall some standard facts on the hook length formula as well as a recent result of Han [61].

4.6.1 Hook lengths and shifted parts of partition

We use the following result from [61]. For a partition $\lambda\vdash d$, $d\geq 1$, we define the so-called g-function:

$$g_\lambda(y):=\prod_{i=1}^{d}(y+\lambda_i-i). \tag{4.93}$$

For any $\lambda\vdash d$, $d\geq 1$, we denote by $\lambda\setminus 1$ the set of all partitions of $d-1$ that can be obtained from λ (or rather the corresponding Young diagram) by removing one corner of λ.

Lemma 4.6.3 (Han [61]). *For every partition λ we have*

$$\frac{1}{H_\lambda}\left(g_\lambda(y+1)-g_\lambda(y)\right)=\sum_{\mu\in\lambda\setminus 1}\frac{1}{H_\mu}g_\mu(y). \tag{4.94}$$

Here y is a formal variable.

We need the following corollary of this lemma.

Corollary 4.6.4. *For an integer $d\geq 1$ we have*

$$\sum_{\lambda\vdash d+1}\frac{1}{H_\lambda^2}\left(g_\lambda(y+1)-g_\lambda(y)\right)=\sum_{\mu\vdash d}\frac{1}{H_\mu^2}g_\mu(y). \tag{4.95}$$

Proof. We recall that for any $\mu\vdash d$, $d\geq 1$, we have:

$$\sum_{\substack{\lambda\vdash d+1\\ \lambda\setminus 1\ni\mu}}\frac{1}{H_\lambda}=\frac{1}{H_\mu}. \tag{4.96}$$

Therefore,

$$\begin{aligned}\sum_{\mu\vdash d}\frac{1}{H_\mu^2}g_\mu(y)&=\sum_{\mu\vdash d}\frac{1}{H_\mu}\sum_{\substack{\lambda\vdash d+1\\ \lambda\setminus 1\ni\mu}}\frac{1}{H_\lambda}g_\mu(y)\\&=\sum_{\lambda\vdash d+1}\frac{1}{H_\lambda}\sum_{\substack{\mu\vdash d\\ \mu\in\lambda\setminus 1}}\frac{1}{H_\lambda}g_\mu(y)\\&=\sum_{\lambda\vdash d+1}\frac{1}{H_\lambda^2}\left(g_\lambda(y+1)-g_\lambda(y)\right).\end{aligned}$$

$\square$

4.6.2 Reformulation of Lemma 4.6.2 in terms of g-functions

We make the following substitution: $y := -x/\hbar$. Then we see that

$$X_d = \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y)}{\prod_{i=1}^d (y-i)}.$$

Moreover,

$$\begin{aligned} &\frac{1}{x/\hbar+1} \exp\left(\hbar \frac{d}{dx}\right) X_{d-1} + \left[\frac{x}{\hbar} \exp\left(-\hbar\frac{d}{dx}\right) - \frac{x}{\hbar}\right] X_d \\ &= \frac{-1}{y-1} \exp\left(-\frac{d}{dy}\right) X_{d-1} + \left[-y \exp\left(\frac{d}{dy}\right) + y\right] X_d. \end{aligned} \tag{4.97}$$

Observe that

$$\frac{-1}{y-1} \exp\left(-\frac{d}{dy}\right) X_{d-1} = -\sum_{\lambda \vdash d-1} \frac{1}{H_\lambda^2} \frac{g_\lambda(y-1)}{\prod_{i=1}^d (y-i)}; \tag{4.98}$$

$$-y \exp\left(\frac{d}{dy}\right) X_d = (d-y) \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y+1)}{\prod_{i=1}^d (y-i)};$$

$$y X_d = y \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y)}{\prod_{i=1}^d (y-i)}.$$

Using Corollary 4.6.4 we can rewrite the right hand side of Equation (4.98) as

$$\frac{-1}{y-1} \exp\left(-\frac{d}{dy}\right) X_{d-1} = \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y-1) - g_\lambda(y)}{\prod_{i=1}^d (y-i)}. \tag{4.99}$$

Therefore, the right hand side of Equation (4.97) is equal to

$$\frac{Y_d(y)}{\prod_{i=1}^d (y-i)}, \tag{4.100}$$

where

$$Y_d(y) := \sum_{\lambda \vdash d} \frac{(d-y) g_\lambda(y+1) + (y-1) g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2}. \tag{4.101}$$

Note that $Y_d(y)$ is a polynomial in y of degree $\leq d+1$, and Lemma 4.6.2 is equivalent to the following statement:

Lemma 4.6.5. *For any $d \geq 1$ we have $Y_d(y) \equiv 0$.*

4.6.3 Proof of Lemma 4.6.5

In this subsection we prove Lemma 4.6.5 and, therefore, Lemmas 4.6.2 and 4.6.1.

First of all, it is easy to check that for any $d \geq 1$ the polynomial $Y_d(y)$ has at least one root. Namely,

$$Y_d(d) = \sum_{\lambda \vdash d} \frac{(d-1) g_\lambda(d) + g_\lambda(d-1)}{H_\lambda^2} = 0. \tag{4.102}$$

Indeed, $g_\lambda(d)$ is not equal to zero only for $\lambda = (1, 1, \ldots, 1)$. In this case $g_\lambda(d) = d!$, $H_\lambda = d!$, and $(d-1)g_\lambda(d)/H_\lambda^2 = (d-1)/d!$. Notice that $g_\lambda(d-1)$ does not vanish only for $\lambda = (2, 1, 1, \ldots, 1, 0)$. In this case $g_\lambda(d-1) = -d \cdot (d-2)!$, $H_\lambda = d \cdot (d-2)!$, and $g_\lambda(d-1)/H_\lambda^2 = -(d-1)/d!$. Thus we see that always $Y_d(d) = 0$, establishing (4.102).

Now we proceed by induction. It is easy to check that $Y_1(y) \equiv 0$. Assume that we know that $Y_d(y) \equiv 0$. Corollary 4.6.4 then implies that

$$
\begin{aligned}
Y_d(y) &= \sum_{\lambda \vdash d} \frac{(d-y)g_\lambda(y+1) + (y-1)g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2} \\
&= \sum_{\lambda \vdash d+1} \frac{(d-y)g_\lambda(y+2) + (2y-d-1)g_\lambda(y+1) + (2-y)g_\lambda(y) - g_\lambda(y-1)}{H_\lambda^2} \\
&= \sum_{\lambda \vdash d+1} \frac{((d+1)-(y+1))g_\lambda(y+2) + ((y+1)-1)g_\lambda(y+1) + g_\lambda(y)}{H_\lambda^2} \\
&\quad - \sum_{\lambda \vdash d+1} \frac{((d+1)-y)g_\lambda(y+1) + (y-1)g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2} \\
&= Y_{d+1}(y+1) - Y_{d+1}(y).
\end{aligned}
$$

By assumption, we have $Y_d(y) \equiv 0$. Therefore, $Y_{d+1}(y+1) = Y_{d+1}(y)$ for any y. HenceY_{d+1} is constant. Since we have shown that $Y_{d+1}(d+1) = 0$, we conclude that $Y_{d+1} \equiv 0$.

This completes the proof of Lemmas 4.6.5, 4.6.2, and 4.6.1. Thus we have established the main theorem of this chapter.

4.7 Laguerre polynomials

In this section we prove a combinatorial expression for the functions

$$
X_d := \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \prod_{i=1}^{d} \frac{x + (i - \lambda_i)\hbar}{x + i\hbar}
$$

in terms of the *Laguerre polynomials* $L_n^{(\alpha)}(z)$.

The Laguerre polynomial is a solution of the differential equation

$$
z\frac{d^2}{dz^2}L_n^{(\alpha)}(z) + (\alpha + 1 - z)\frac{d}{dz}L_n^{(\alpha)}(z) + nL_n^{(\alpha)}(z) = 0,
$$

and has a closed expression

$$
L_n^{(\alpha)}(z) = \sum_{i=0}^{n} (-1)^i \binom{n+a}{n-i} \frac{z^i}{i!}.
$$

Here is a list of some properties of the Laguerre polynomials:

$$\frac{L_n^\alpha(z)}{\binom{n+\alpha}{n}} = 1 - \sum_{j=1}^{n} \frac{z^j}{a+j} \frac{L_{n-j}^{(j)}(z)}{(j-1)!}; \tag{4.103}$$

$$nL_n^\alpha(z) = (n+\alpha)L_{n-1}^\alpha(z) - zL_{n-1}^{\alpha+1}(z); \tag{4.104}$$

$$L_n^\alpha(z) = L_n^{\alpha+1}(z) - L_{n-1}^{\alpha+1}(z); \tag{4.105}$$

$$\binom{n+\alpha}{n} = \sum_{i=0}^{n} \frac{z^i}{i!} L_{n-i}^{(\alpha+i)}(z). \tag{4.106}$$

Proposition 4.7.1. *For any $d \geq 0$ we have:*

$$X_d = \frac{1}{d!}\left(1 - \sum_{m=1}^{d} \frac{1}{(m-1)!} L_{d-m}^{(m)}(1) \frac{\hbar}{x + m\hbar}\right).$$

Remark 4.7.2. This equation can be rewritten as

$$X_d = \frac{1}{d!} \frac{L_d^{x/\hbar}(1)}{L_d^{x/\hbar}(0)}.$$

Indeed, we just apply the identity (4.103) for $\alpha = x/\hbar$ and $z = 1$ and further observe that $\binom{n+x/\hbar}{n} = L_n^{(x/\hbar)}(0)$.

Proof of Proposition 4.7.1. It is obvious that both X_d and

$$\tilde{X}_d := \frac{1}{d!}\left(1 - \sum_{m=1}^{d} \frac{1}{(m-1)!} L_{d-m}^{(m)}(1) \frac{\hbar}{x + m\hbar}\right) \tag{4.107}$$

are rational functions with simple poles at $x/\hbar = -1, -2, \ldots, -d$. We have defined $X_0 := 1$, and it is easy to see that $Z_0 = 1$. Then we know (see Lemma 4.6.2) that all X_d are unambiguously determined by the equation

$$\frac{1}{\frac{x}{\hbar}+1} \exp\left(\hbar \frac{d}{dx}\right) X_d + \left[\frac{x}{\hbar} \exp\left(-\hbar \frac{d}{dx}\right) - \frac{x}{\hbar}\right] X_{d+1} = 0 \tag{4.108}$$

for all $d \geq 0$. In order to prove the proposition we check that $\{\tilde{X}_d\}_{d\geq 0}$ also satisfy this equation.

Indeed, observe that

$$\begin{aligned}
&\frac{1}{\frac{x}{\hbar}+1} \exp\left(\hbar \frac{d}{dx}\right) \tilde{X}_d \qquad (4.109)\\
&= \frac{1}{d!}\left(\frac{1}{\frac{x}{\hbar}+1} - \sum_{m=1}^{d} \frac{L_{d-m}^{(m)}(1)}{(m-1)!} \frac{1}{(\frac{x}{\hbar}+1)(\frac{x}{\hbar}+m+1)}\right)\\
&= \frac{1}{\frac{x}{\hbar}+1} \frac{1}{d!}\left(1 - \sum_{m=1}^{d} \frac{L_{d-m}^{(m)}(1)}{m!}\right) + \frac{1}{d!} \sum_{m=2}^{d+1} \frac{L_{d+1-m}^{(m-1)}(1)}{(m-1)!} \frac{1}{(\frac{x}{\hbar}+m)};
\end{aligned}$$

$$\begin{aligned}
&\frac{x}{\hbar}\exp\left(-\hbar\frac{d}{dx}\right)\tilde{X}_{d+1} \qquad (4.110)\\
&=\frac{1}{(d+1)!}\left(\frac{x}{\hbar}-\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}\frac{\frac{x}{\hbar}}{\frac{x}{\hbar}+m-1}\right)\\
&=\frac{\frac{x}{\hbar}}{(d+1)!}-\frac{1}{(d+1)!}\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}+\frac{1}{(d+1)!}\sum_{m=1}^{d}\frac{L^{(m+1)}_{d-m}(1)}{(m-1)!}\frac{1}{\frac{x}{\hbar}+m};
\end{aligned}$$

and

$$\begin{aligned}
&-\frac{x}{\hbar}\tilde{X}_{d+1}=\frac{1}{(d+1)!}\left(-\frac{x}{\hbar}+\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}\frac{\frac{x}{\hbar}}{\frac{x}{\hbar}+m}\right) \qquad (4.111)\\
&=\frac{-\frac{x}{\hbar}}{(d+1)!}+\frac{1}{(d+1)!}\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}-\frac{1}{(d+1)!}\sum_{m=1}^{d+1}\frac{L^{(m)}_{d+1-m}(1)}{(m-1)!}\frac{m}{\frac{x}{\hbar}+m}.
\end{aligned}$$

It is obvious that in the sum of the expressions (4.109), (4.110), and (4.111) the coefficient of $x/\hbar$ and the constant term vanish. So, we have to prove that the coefficient of each $1/(x/\hbar+m)$, $m=1,\dots,d+1$, vanishes.

The coefficient of $1/(x/\hbar+d+1)$ is equal to

$$\frac{1}{d!}\frac{L^{(d)}_0(1)}{d!}-\frac{1}{(d+1)!}\frac{(d+1)L^{(d+1)}_0(1)}{d!},$$

which is equal to zero since $L^{(d)}_0=L^{(d+1)}_0=1$.

The coefficient of $1/(x/\hbar+m)$, $2\le m\le d$, is equal to

$$\frac{1}{d!}\frac{L^{(m-1)}_{d+1-m}(1)}{(m-1)!}+\frac{1}{(d+1)!}\frac{L^{(m+1)}_{d-m}(1)}{(m-1)!}-\frac{1}{(d+1)!}\frac{mL^{(m)}_{d+1-m}(1)}{(m-1)!}. \qquad (4.112)$$

First we use Equation (4.104) for $z=1$, $\alpha=m$, and $n=d+m-1$:

$$(d+1-m)L^{(m)}_{d+1-m}(1)=(d+1)L^{(m)}_{d-m}(1)-L^{(m+1)}_{d-m}(1). \qquad (4.113)$$

We see then that expression (4.112) is equal to

$$\frac{1}{d!(m-1)!}\left(L^{(m-1)}_{d+1-m}(1)-L^{(m)}_{d+1-m}(1)+L^{(m)}_{d-m}(1)\right).$$

And this is equal to zero due to Equation (4.105) for $n=d+1-m$, $\alpha=m-1$, and $z=1$.

The coefficient of $1/(x/\hbar+1)$ is equal to

$$\frac{1}{d!}\left(1-\sum_{m=1}^{d}\frac{L^{(m)}_{d-m}(1)}{m!}\right)+\frac{1}{(d+1)!}L^{(2)}_{d-1}(1)-\frac{1}{(d+1)!}L^{(1)}_{d}(1).$$

Using that $L^{(2)}_{d-1}(1) = -dL^{(1)}_d(1) + (d+1)L^{(1)}_{d-1}(1)$ (which is Equation (4.104) for $z = 1$, $n = d$, and $\alpha = 1$) and $-L^{(1)}_d(1) + L^{(1)}_{d-1}(1) = -L^{(0)}_d(1)$ (which is Equation (4.105) for $z = 1$, $d = n$, and $\alpha = 0$), we see that this coefficient is equal to

$$\frac{1}{d!}\left(1 - \sum_{m=0}^{d} \frac{L^{(m)}_{d-m}(1)}{m!}\right).$$

This is equal to zero due to Equation (4.106) for $z = 1$, $\alpha = 0$, and $n = d$.

Thus we see that the sum of expressions (4.109), (4.110), and (4.111) is equal to zero. So, the functions $\tilde{X}_d$ satisfy Equation (4.108), and, therefore, $\tilde{X}_d = X_d$ for all $d \geq 0$. □

4.8 Toda lattice equation

In this section we recall, for completeness, the Toda lattice equation for the partition function of the Gromov-Witten invariants of $\mathbb{P}^1$. We show that the Toda lattice equation implies a quadratic relation for the functions X_d, $d \geq 0$, considered in the previous sections. It is an open question whether it is possible to relate the Toda lattice equation to the quantum spectral curve equation.

It is convenient to include a degree variable q in the free energy of the Gromov-Witten invariants of $\mathbb{P}^1$. Define

$$\mathcal{F}_g := \sum_d q^d \left\langle \exp\left\{\sum_{i\geq 0}^{\infty} \tau_i(\omega)t_i + \tau_0(1)t\right\}\right\rangle_g^d \tag{4.114}$$

(where we have switched off $\tau_k(1)$ for $k > 0$). Then $\mathcal{F} = \sum_{g=0}^{\infty} \mathcal{F}_g$ satisfies the Toda lattice equation:

$$\exp(\mathcal{F}(t+1) + \mathcal{F}(t-1) - 2\mathcal{F}(t)) = \frac{1}{q}\frac{\partial^2}{\partial t_0^2}\mathcal{F}(t), \tag{4.115}$$

which was conjectured by Eguchi-Yang [32] and proven by Dubrovin-Zhang [29] and Okounkov-Pandharipande [84, Equation (4.11)].

We specialize

$$\Phi(x,\hbar,q) := F\left(q = \frac{q}{\hbar^2},\ t = -\frac{1}{2},\ t_i = -i!\left(\frac{\hbar}{x}\right)^{i+1}\right) \tag{4.116}$$

and consider the function $\exp\{\Phi(x,\hbar,q) - \Phi(x,\hbar,0)\}$.

Lemma 4.8.1. *We have:*

$$\exp\{\Phi(x,\hbar,q) - \Phi(x,\hbar,0)\} = \sum_{d=0}^{\infty}\left(\frac{q}{\hbar^2}\right)^d X_d. \tag{4.117}$$

Proof. Indeed, tracing back the arguments of Sections 4.3 and 4.4 and taking q into account this time, it is easy to see that

$$\Phi(x,\hbar,q) = \sum_{d=0}^{\infty} \left(\frac{q}{\hbar^2}\right)^d \sum_{g=0}^{\infty} \left\langle \exp\left(-\frac{1}{2}\tau_0(1) - \sum_{i=0}^{\infty} \tau_i(\omega) i! \left(\frac{\hbar}{x}\right)^{i+1}\right)\right\rangle_g^d. \tag{4.118}$$

The proof of Lemma 4.5.1 implies that

$$\begin{aligned}\Phi(x,\hbar,0) &= \sum_{g=0}^{\infty} \left\langle \exp\left(-\frac{1}{2}\tau_0(1) - \sum_{i=0}^{\infty} \tau_i(\omega) i! \left(\frac{\hbar}{x}\right)^{i+1}\right)\right\rangle_g^0 \\ &= B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x - x\log x}{\hbar}\right)\end{aligned} \tag{4.119}$$

These two observations and Equation (4.79) imply Equation (4.117). □

By abuse of notation, we denote the function $\exp\{\Phi(x,\hbar,q) - \Phi(x,\hbar,0)\}$ also by X (so-called degree-weighted X). The quantum spectral curve equation for this degree-weighted X reads

$$\left[\frac{q}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right) + x\exp\left(-\hbar\frac{d}{dx}\right) - x\right] X = 0. \tag{4.120}$$

The Toda lattice equation combined with the string and the divisor equations implies the following equation for X:

Proposition 4.8.2. *We have:*

$$\frac{X(x+\hbar)X(x-\hbar)}{X(x)^2} = \frac{x+\hbar}{x}\frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\log X(x)\right). \tag{4.121}$$

Proof. We recall the divisor equation

$$\frac{\partial \mathcal{F}}{\partial t_0} = \frac{1}{2}t^2 + q\frac{\partial \mathcal{F}}{\partial q}. \tag{4.122}$$

Consider Equations (4.115). The result of Section 4.3 implies that the shifts in $\tau_0(1)$-coefficient t can be replaced by shifts of variable x with factor $\hbar$. Using this and Equation (4.122) we obtain equation for $\Phi(x,\hbar,q)$:

$$\exp\left(\Phi(x+\hbar,\hbar,q) + \Phi(x-\hbar,\hbar,q) - 2\Phi(x,\hbar,q)\right) = \frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\Phi(x,\hbar,q)\right). \tag{4.123}$$

From the definition of degree-weighted X it follows that

$$\frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\log X(x)\right) = \frac{\partial}{\partial q}\left(q\frac{\partial}{\partial q}\Phi_q(x)\right). \tag{4.124}$$

Therefore, in order to prove Equation (4.121), it is enough to show that

$$\frac{\exp\left(\Phi(x+\hbar,\hbar,0)\right)\exp\left(\Phi(x-\hbar,\hbar,0)\right)}{\exp\left(2\Phi(x,\hbar,0)\right)} = \frac{x}{x+\hbar}. \tag{4.125}$$

Indeed, using Equation (4.119) we can represent the left hand side of Equation (4.125) in the following form:

$$\begin{aligned}
&\exp\left(\left[\left(e^{t}+e^{-t}-2\right)B(t)\right]\Big|_{t=-\hbar\frac{d}{dx}}\left(\frac{x-x\log x}{\hbar}\right)\right) \qquad (4.126)\\
&=\exp\left(\left[\left(e^{t/2}-e^{-t/2}\right)^{2}\frac{t}{e^{t}-1}\right]\Bigg|_{t=-\hbar\frac{d}{dx}}\left(\frac{x-x\log x}{\hbar}\right)\right)\\
&=\exp\left(\log(x)-\log(x+\hbar)\right).
\end{aligned}$$

□

The homogeneous in q part of the Toda lattice equation (4.121) for the series expansion given by Equation (4.117) can be rewritten as

$$\frac{x}{x+\hbar}\sum_{a+b=d}X_a(x+\hbar)X_b(x-\hbar)=\sum_{a+b=d+1}X_a(x)X_b(x)\frac{(a-b)^2}{2} \qquad (4.127)$$

for any $d\geq 0$. We can use Equation (4.92) in order to rewrite this equation as

$$\begin{aligned}
&\sum_{a+b=d+1}\left(X_a(x)X_b(x-\hbar)-X_a(x-\hbar)X_b(x-\hbar)\right) \qquad (4.128)\\
&=\frac{\hbar^2}{x^2}\sum_{a+b=d+1}X_a(x)X_b(x)\frac{(a-b)^2}{2},
\end{aligned}$$

for any $d\geq 0$. It would be interesting to see whether Equation (4.127) or Equation (4.128) can be related to Equation (4.92).

Chapter 5

Polynomiality of Hurwitz numbers, Bouchard-Mariño conjecture, and a new proof of the ELSV formula

This chapter is based on paper [100], joint work with M. Kazarian, N. Orantin, S. Shadrin and L. Spitz. In this chapter we prove the polynomiality of simple Hurwitz numbers, which allows us to give a new proof of the Bouchard-Mariño conjecture, which, in turn, allows us to give a new proof of the ELSV formula.

This chapter is organized in the following way. First, we prove in Section 5.1 the polynomiality of Hurwitz numbers directly from the definition in terms of the semi-infinite wedge formalism. Our argument is a refinement of an argument by Okounkov and Pandharipande in [83]. Then, using the polynomiality property of Hurwitz numbers we are able to derive in Section 5.2 the Bouchard-Mariño conjecture directly from the cut-and-join equation. Then, since we have an equivalence of the Bouchard-Mariño conjecture and the ELSV formula, we immediately derive the ELSV formula in a new way. In Section 5.3 we review the correspondence between the topological recursion and the Givental theory, with a special focus on the 1-dimensional case, and in Section 5.4 we provide a (slightly refined) proof of the equivalence of the ELSV formula and the Bouchard-Mariño conjecture.

5.1 Polynomiality of the Hurwitz numbers

In this section we prove the following theorem:

Theorem 5.1.1. *The Hurwitz numbers $h^{\circ}_{g;\mu_1,\dots,\mu_n}$ for $(g,n) \notin \{(0,1),(0,2)\}$ can be expressed as follows:*

$$h^{\circ}_{g;\mu_1,\dots,\mu_n} = (2g+|\mu|+n-2)! \left(\prod_{i=1}^{n} \frac{\mu_i^{\mu_i}}{\mu_i!}\right) P_{g,n}(\mu_1,\dots,\mu_n), \tag{5.1}$$

where $P_{g,n}(\mu_1,\dots,\mu_n)$ is some polynomial in $\mu_1,\dots,\mu_n$.

Basically this theorem gives the form of the ELSV formula without specifying the precise formulas for the coefficients. This property (in a bit stronger form) was conjectured in [56] and then proved in [57], with the help of the ELSV formula. Still, the question whether this property can be derived without using the ELSV formula remained open [96]. This is precisely what we do here: we prove this statement without using the ELSV formula.

5.1.1 Infinite wedge

In this subsection we recall some basic facts from the theory of the semi-infinite wedge space following [83, 84, 64].

Let V be an infinite dimensional vector space with a basis labeled by the half integers. Denote the basis vector labeled by $m/2$ by $\underline{m/2}$, so $V = \bigoplus_{i \in \mathbb{Z}+\frac{1}{2}} \underline{i}$.

Definition 5.1.2. The semi-infinite wedge space $\mathcal{V}$ is the span of all wedge products of the form

$$\underline{i_1} \wedge \underline{i_2} \wedge \cdots \tag{5.2}$$

for any decreasing sequence of half integers (i_k) such that there is an integer c (called the charge) with $i_k + k - \frac{1}{2} = c$ for k sufficiently large. We denote the inner product associated with this basis by $(\cdot, \cdot)$.

Here we are mostly concerned with the zero charge subspace $\mathcal{V}_0 \subset \mathcal{V}$ of the semi-infinite wedge space, which is the space of all wedge products of the form (5.2) such that

$$i_k + k = \frac{1}{2} \tag{5.3}$$

for k sufficiently large.

Remark 5.1.3. An element of $\mathcal{V}_0$ is of the form

$$\underline{\lambda_1 - \frac{1}{2}} \wedge \underline{\lambda_2 - \frac{3}{2}} \wedge \cdots$$

for some integer partition λ. This follows immediately from condition (5.3). Thus, we canonically have a basis for $\mathcal{V}_0$ labeled by all integer partitions.

Notation 5.1.4. We denote by v_λ the vector labeled by a partition λ. The vector labeled by the empty partition is called the vacuum vector and denoted by $|0\rangle = v_\emptyset = \underline{-\frac{1}{2}} \wedge \underline{-\frac{3}{2}} \wedge \cdots$.

Definition 5.1.5. If $\mathcal{P}$ is an operator on $\mathcal{V}_0$, then we define the *vacuum expectation value* of $\mathcal{P}$ by $\langle \mathcal{P} \rangle := \langle 0|\mathcal{P}|0\rangle$, where $\langle 0|$ is the dual of the vacuum vector with respect to the inner product $(\cdot, \cdot)$, and called the covacuum vector. We will also refer to these vacuum expectation values as (disconnected) *correlators*.

We now define some operators on the infinite wedge space.

Definition 5.1.6. Let k be any half integer. Then the operator $\psi_k \colon \mathcal{V} \to \mathcal{V}$ is defined by $\psi_k \colon (\underline{i_1} \wedge \underline{i_2} \wedge \cdots) \mapsto (\underline{k} \wedge \underline{i_1} \wedge \underline{i_2} \wedge \cdots)$. It increases the charge by 1.

The operator ψ_k^* is defined to be the adjoint of the operator ψ_k with respect to the inner product $(\cdot, \cdot)$.

Definition 5.1.7. The normally ordered products of ψ-operators are defined in the following way

$$E_{ij} := :\psi_i\psi_j^*: := \begin{cases} \psi_i\psi_j^*, & \text{if } j > 0 \\ -\psi_j^*\psi_i & \text{if } j < 0 \ . \end{cases} \tag{5.4}$$

This operator does not change the charge and can be restricted to $\mathcal{V}_0$. Its action on the basis vectors v_λ can be described as follows: $:\psi_i\psi_j^*:$ checks if v_λ contains $\underline{j}$ as a wedge factor and if so replaces it by $\underline{i}$. Otherwise it yields 0. In the case $i = j > 0$, we have $:\psi_i\psi_j^*:(v_\lambda) = v_\lambda$ if v_λ contains $\underline{j}$ and 0 if it does not; in the case $i = j < 0$, we have $:\psi_i\psi_j^*:(v_\lambda) = -v_\lambda$ if v_λ does not contain $\underline{j}$ and 0 if it does. These are the only two cases where the normal ordering is important.

Notation 5.1.8. We denote by $\zeta(z)$ the function $e^{z/2} - e^{-z/2}$.

Definition 5.1.9. Let $n \in \mathbb{Z}$ be any integer. We define an operator $\mathcal{E}_n(z)$ depending on a formal variable z by

$$\mathcal{E}_n(z) = \sum_{k \in \mathbb{Z}+\frac{1}{2}} e^{z(k-\frac{n}{2})} E_{k-n,k} + \frac{\delta_{n,0}}{\zeta(z)}.$$

Note that

$$[\mathcal{E}_a(z), \mathcal{E}_b(w)] = \zeta\left(\det\left[\begin{smallmatrix} a & z \\ b & w \end{smallmatrix}\right]\right) \mathcal{E}_{a+b}(z+w) \tag{5.5}$$

and

$$\mathcal{E}_0(z)\big|0\big\rangle = \frac{1}{\zeta(z)}\big|0\big\rangle \tag{5.6}$$

and also that

$$\mathcal{E}_k(z)\big|0\big\rangle = 0, \quad k > 0. \tag{5.7}$$

Definition 5.1.10. In what follows we will use the following operator:

$$\mathcal{F}_2 := \sum_{k \in \mathbb{Z}+\frac{1}{2}} \frac{k^2}{2} E_{k,k}.$$

Definition 5.1.11. We will also need the following operators:

$$\alpha_k := \mathcal{E}_k(0), \quad k \neq 0.$$

5.1.2 Hurwitz numbers in the infinite wedge formalism

By $h_{g,\mu} = h_{g;\mu_1,\ldots,\mu_n}$ we denote the Hurwitz numbers for possibly disconnected covering surfaces. The character formula for the disconnected Hurwitz numbers $h_{g,\mu}$ implies that (see e.g. [83])

$$h_{g,\mu} = \left\langle e^{\alpha_1} \mathcal{F}_2^{\mathsf{b}(g,\mu)} \prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle. \tag{5.8}$$

Note the difference between our disconnected Hurwitz numbers $h_{g,\mu}$ and the ones in [83] (which are denoted by $\mathsf{C}_g(\mu)$ there). The difference is in a factor of $|\mathrm{Aut}(\mu)|$, the number of automorphisms of the partition.

Definition 5.1.12. Define the genus-generating functions for the disconnected Hurwitz numbers and for the connected ones as well:

$$h_\mu(u) := \sum_{g=1-n}^{\infty} \frac{u^{2g-2}}{\mathfrak{b}(g,\mu)!} h_{g,\mu}, \tag{5.9}$$

$$h^\circ_\mu(u) := \sum_{g=0}^{\infty} \frac{u^{2g-2}}{\mathfrak{b}(g,\mu)!} h^\circ_{g,\mu}. \tag{5.10}$$

Remark 5.1.13. Note that these two generating functions are related to each other through the inclusion-exclusion formula, namely, we have:

$$\begin{aligned}
&\text{for } n=1, && h_{\mu_1}(u) = h^\circ_{\mu_1}(u);\\
&\text{for } n=2, && h_{\mu_1,\mu_2}(u) = h^\circ_{\mu_1,\mu_2}(u) + h^\circ_{\mu_1}(u)h^\circ_{\mu_2}(u),\\
& && h^\circ_{\mu_1,\mu_2}(u) = h_{\mu_1,\mu_2}(u) - h_{\mu_1}(u)h_{\mu_2}(u);\\
&\text{for } n=3, && h_{\mu_1,\mu_2,\mu_3}(u) = h^\circ_{\mu_1,\mu_2,\mu_3}(u) + h^\circ_{\mu_1,\mu_2}(u)h^\circ_{\mu_3}(u) + h^\circ_{\mu_1,\mu_3}(u)h^\circ_{\mu_2}(u)\\
& && \qquad + h^\circ_{\mu_2,\mu_3}(u)h^\circ_{\mu_1}(u) + h^\circ_{\mu_1}(u)h^\circ_{\mu_2}(u)h^\circ_{\mu_3}(u),\\
& && h^\circ_{\mu_1,\mu_2,\mu_3}(u) = h_{\mu_1,\mu_2,\mu_3}(u) - h_{\mu_1,\mu_2}(u)h_{\mu_3}(u) - h_{\mu_1,\mu_3}(u)h_{\mu_2}(u)\\
& && \qquad - h_{\mu_2,\mu_3}(u)h_{\mu_1}(u) + 2h_{\mu_1}(u)h_{\mu_2}(u)h_{\mu_3}(u),
\end{aligned}$$

and so on.

We have

$$\begin{aligned}
h_\mu(u) &= u^{-|\mu|-n} \left\langle e^{\alpha_1} e^{u\mathcal{F}_2} \prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \right\rangle \\
&= u^{-|\mu|-n} \left\langle e^{\alpha_1} e^{u\mathcal{F}_2} \left(\prod_{i=1}^{n} \frac{\alpha_{-\mu_i}}{\mu_i} \right) e^{-u\mathcal{F}_2} e^{-\alpha_1} \right\rangle \\
&= u^{-|\mu|-n} \left\langle \prod_{i=1}^{n} \left(e^{\alpha_1} e^{u\mathcal{F}_2} \frac{\alpha_{-\mu_i}}{\mu_i} e^{-u\mathcal{F}_2} e^{-\alpha_1} \right) \right\rangle .
\end{aligned} \tag{5.11}$$

The second equality holds since $e^{-u\mathcal{F}_2}$ and $e^{-\alpha_1}$ fix the vacuum vector.

5.1.3 $\mathcal{A}$-operators

Now, following [83], we introduce certain operators that we use later on to rewrite the formula for Hurwitz numbers.

Definition 5.1.14. Define

$$\mathcal{A}(a,b) := \left(\frac{\zeta(b)}{b}\right)^a \sum_{k\in\mathbb{Z}} \frac{\zeta(b)^k}{(a+1)_k}\, \mathcal{E}_k(b), \tag{5.12}$$

where a and b are parameters and we use the standard notation:

$$(a+1)_k = \frac{(a+k)!}{a!} = \begin{cases} (a+1)(a+2)\cdots(a+k)\,, & k \geq 0\,,\\ (a(a-1)\cdots(a+k+1))^{-1}\,, & k \leq 0\,. \end{cases} \tag{5.13}$$

If $a \neq 0, 1, 2, \ldots$, the sum in (5.12) is infinite in both directions. If a is a non-negative integer, the summands with $k \leq -a-1$ in (5.12) vanish.

Note that Proposition 3 of [83] implies that the correlator

$$\langle \mathcal{A}(z_1, uz_1) \ldots \mathcal{A}(z_n, uz_n) \rangle \tag{5.14}$$

is well-defined for all $(z_1, \ldots, z_n) \in \Omega \subset \mathbb{C}^n$ and sufficiently small u, where

$$\Omega = \left((z_1, \ldots, z_n) \middle| |z_k| > \sum_{i=1}^{k-1} |z_i|, \ k = 1, \ldots, n \right). \tag{5.15}$$

Definition 5.1.15. Define the *connected correlator* of $\mathcal{A}$-operators

$$\langle \mathcal{A}(z_1, uz_1) \ldots \mathcal{A}(z_n, uz_n) \rangle^\circ$$

through the disconnected ones via the inclusion-exclusion formula (cf. Remark (5.1.13)).

Proposition 5.1.16.

$$h^\circ_{g;\mu_1\ldots\mu_n} = \mathfrak{b}(g,\mu)! \prod_{i=1}^n \left(\frac{\mu_i^{\mu_i - 1}}{\mu_i!} \right) [u^{2g-2+n}] \langle \mathcal{A}(\mu_1, u\mu_1) \ldots \mathcal{A}(\mu_n, u\mu_n) \rangle^\circ \tag{5.16}$$

where $[u^{2g-2+n}] \langle \mathcal{A}(\mu_1, u\mu_1) \ldots \mathcal{A}(\mu_n, u\mu_n) \rangle^\circ$ *stands for the coefficient of* u^{2g-2+n} *in* $\langle \mathcal{A}(\mu_1, u\mu_1) \ldots \mathcal{A}(\mu_n, u\mu_n) \rangle^\circ$.

Proof. The main part of the proof follows [83]. Note that

$$e^{u\mathcal{F}_2} \, \alpha_{-m} \, e^{-u\mathcal{F}_2} = \mathcal{E}_{-m}(um) \tag{5.17}$$

which is easy to see since $\mathcal{F}_2$ acts diagonally. From the commutation relations for $\mathcal{E}_i$ we see that

$$e^{\alpha_1} \, \mathcal{E}_{-m}(s) \, e^{-\alpha_1} = \frac{\zeta(s)^m}{m!} \sum_{k \in \mathbb{Z}} \frac{\zeta(s)^k}{(m+1)_k} \mathcal{E}_k(s). \tag{5.18}$$

The previous two formulas imply the following, for $m \in \{1, 2, 3, \ldots\}$ (Lemma 2 of [83]):

$$e^{\alpha_1} \, e^{u\mathcal{F}_2} \, \alpha_{-m} \, e^{-u\mathcal{F}_2} \, e^{-\alpha_1} = \frac{u^m \, m^m}{m!} \mathcal{A}(m, um). \tag{5.19}$$

Now we can rewrite formula (5.11) as

$$h_{\mu_1\ldots\mu_n}(u) = u^{-n} \prod_{i=1}^n \left(\frac{\mu_i^{\mu_i - 1}}{\mu_i!} \right) \langle \mathcal{A}(\mu_1, u\mu_1) \ldots \mathcal{A}(\mu_n, u\mu_n) \rangle. \tag{5.20}$$

Recall that the connected Hurwitz numbers can be expressed though the disconnected ones with the help of the inclusion-exclusion formula. Since the relation between connected and disconnected Hurwitz numbers is the same as the one between connected and disconnected correlators, we have:

$$h^\circ_{\mu_1\ldots\mu_n}(u) = u^{-n} \prod_{i=1}^n \left(\frac{\mu_i^{\mu_i - 1}}{\mu_i!} \right) \langle \mathcal{A}(\mu_1, u\mu_1) \ldots \mathcal{A}(\mu_n, u\mu_n) \rangle^\circ. \tag{5.21}$$

Comparing the coefficients in front of the same powers of u on the right hand side and on the left hand side we directly obtain the statement of the proposition. $\square$

Now we see that in order to prove Theorem 5.1.1 we only have to show that expressions $[u^{2g-2+n}] \langle \mathcal{A}(\mu_1, u\mu_1) \ldots \mathcal{A}(\mu_n, u\mu_n) \rangle^\circ$ are polynomial in $\mu_1, \ldots, \mu_n$.

5.1.4 Further properties of $\mathcal{A}$-operators

In this subsection we modify an expression for the connected correlators of $\mathcal{A}$-operators in order to exclude possible so-called *unstable terms*.

Definition 5.1.17. Let $\mathcal{A}_k$ be the coefficients of the expansion of the operator $\mathcal{A}(z, uz)$ in powers of z:

$$\mathcal{A}(z, uz) = \sum_{k \in \mathbb{Z}} \mathcal{A}_k \, z^k \, . \tag{5.22}$$

We will use the following theorem, due to Okounkov and Pandharipande:

Theorem 5.1.18 (Okounkov-Pandharipande, [83])**.**

$$[\mathcal{A}_k, \mathcal{A}_l] = (-1)^l \delta_{k+l-1} \, . \tag{5.23}$$

Definition 5.1.19. Define

$$\mathcal{A}_+(z, uz) := \sum_{k=1}^{\infty} \mathcal{A}_k \, z^k \, . \tag{5.24}$$

Notation 5.1.20. For any operator $\mathcal{P}(u)$ define

$$\langle \mathcal{P}(u) \rangle_k := [u^k] \, \langle \mathcal{P}(u) \rangle \qquad \text{(the coefficient of } u^k \text{ in } \langle \mathcal{P}(u) \rangle\text{)}.$$

If the operator $\mathcal{P}(u)$ is a product of $\mathcal{A}$-operators, then the connected correlator $\langle \mathcal{P}(u) \rangle^\circ$ makes sense, and we define

$$\langle \mathcal{P}(u) \rangle_k^\circ := [u^k] \, \langle \mathcal{P}(u) \rangle^\circ \qquad \text{(the coefficient of } u^k \text{ in } \langle \mathcal{P}(u) \rangle^\circ\text{)}.$$

Our next goal is to give a formula that would express a coefficient of u in a disconnected correlator of a product of $\mathcal{A}$-operators $\langle \mathcal{A}(z_1, uz_1) \dots \mathcal{A}(z_n, uz_n) \rangle_k$ in terms of connected ones (the "inverse" inclusion-exclusion formula, cf. formulas in Remark 5.1.13). We get a sum of the products of correlators, where the operators $\mathcal{A}(z_i, uz_i)$ are distributed in some way among the factors, and the degree of u is also specified for each factor so that the sum of all degrees in each product is equal to k. For further computations it is convenient to encode the summands of this expression in terms of Young tableaux with extra structure.

Definition 5.1.21. We denote by $\mathcal{Y}_{n,k}$ the set of $\{1, \dots, n\}$ Young tableaux (i. e. Young diagrams of size n with each box labeled by a number from 1 to n such that no two boxes are labeled by the same number) with certain conditions and additional row labels.

Namely, let y be such a tableau. Let $c_{i,j}(y)$ be the number in the i-th row and j-th column. Let $h(y)$ be the number of rows, and let $l_i(y)$ be the length of the i-th row. Now we are ready to describe the conditions.

First, the numbers in the rows should be ascending, i. e. for any i and for any $j_1 < j_2$ we have $c_{i,j_1}(y) < c_{i,j_2}(y)$. Second, the numbers in the first column that correspond to rows of the same length should be ascending, i. e., if $l_{i_1}(y) = l_{i_2}(y)$ and $i_1 < i_2$, then $c_{i_1,1}(y) < c_{i_2,1}(y)$.

By $\lambda_i(y) \in \{-1, 0, 1, \dots\}$ we denote additional labels that are assigned to all rows, and we require that $\sum_{i=1}^{h(y)} \lambda_i(y) = k$.

There is a one-to-one correspondence between the elements of $\mathcal{Y}_{n,k}$ and the terms in the expression for a disconnected correlator through the connected ones. Rows in y correspond to individual connected correlators in the product, while labels λ correspond to the Euler characteristics of these connected correlators. This can be expressed through the following formula:

$$\begin{aligned}&\langle \mathcal{A}(z_1, uz_1)\dots\mathcal{A}(z_n, uz_n)\rangle_k \qquad (5.25)\\ &= \sum_{y\in\mathcal{Y}_{n,k}}\prod_{i=1}^{h(y)}\left\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\dots\mathcal{A}(z_{c_{i,l_i(y)}(y)}, uz_{c_{i,l_i(y)}(y)})\right\rangle^\circ_{\lambda_i(y)}.\end{aligned}$$

The terms in this sum that contain either $\langle\mathcal{A}(z_i, uz_i)\rangle^\circ_{-1}$ or $\langle\mathcal{A}(z_i, uz_i)\mathcal{A}(z_j, uz_j)\rangle^\circ_0$ are called *unstable terms*. We would like to exclude all unstable terms. This way we get a summation over a subset $\widetilde{\mathcal{Y}}_{n,k}$ of $\mathcal{Y}_{n,k}$ defined in the following way:

$$\widetilde{\mathcal{Y}}_{n,k} = \left\{y\in\mathcal{Y}_{n,k}\middle|\, l_i(y) = 1 \Rightarrow \lambda_i(y)\neq -1,\ l_i(y) = 2\Rightarrow\lambda_i(y)\neq 0\right\}. \qquad (5.26)$$

If we exclude all unstable terms, we obtain the following expression.

Proposition 5.1.22. *We have:*

$$\begin{aligned}&\langle \mathcal{A}_+(z_1, uz_1)\dots\mathcal{A}_+(z_n, uz_n)\rangle_k \qquad (5.27)\\ &= \sum_{y\in\widetilde{\mathcal{Y}}_{n,k}}\prod_{i=1}^{h(y)}\left\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\dots\mathcal{A}(z_{c_{i,l_i(y)}(y)}, uz_{c_{i,l_i(y)}(y)})\right\rangle^\circ_{\lambda_i(y)}.\end{aligned}$$

In other words, $\langle\mathcal{A}_+(z_1, uz_1)\dots\mathcal{A}_+(z_n, uz_n)\rangle_k$ is equal to $\langle\mathcal{A}(z_1, uz_1)\dots\mathcal{A}(z_n, uz_n)\rangle_k$ with all the unstable terms dropped.

Proof. Let us first compute the unstable factors, i. e. the genus-zero one- and two-point connected correlators.

Note that

$$\langle\mathcal{A}(z, uz)\rangle = \frac{1}{uz} + \frac{z(z-1)}{24}u + O(u^2). \qquad (5.28)$$

This directly implies the following formula for the genus-zero one-point correlator:

$$\langle\mathcal{A}(z, uz)\rangle^\circ_{-1} = \frac{1}{z}. \qquad (5.29)$$

The definition of the operator $\mathcal{A}$ implies that

$$\langle 0|\mathcal{A}(z, uz) = \frac{1}{uz}\langle 0| + \langle 0|\mathcal{A}_+(z, uz). \qquad (5.30)$$

Using the definition of the two-point connected correlators together with formulas (5.28), (5.30) and (5.23) we can derive the following formula for the genus-zero two-point connected correlator (we expand it in the region $|z_1| < |z_2|$):

$$\begin{aligned}
\langle \mathcal{A}(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle_0^\circ &= \langle \mathcal{A}(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle_0 - \langle \mathcal{A}(z_1,uz_1)\rangle_{-1}\langle \mathcal{A}(z_2,uz_2)\rangle_1 \\
&\quad - \langle \mathcal{A}(z_1,uz_1)\rangle_1\langle \mathcal{A}(z_2,uz_2)\rangle_{-1} \\
&= \langle \mathcal{A}_+(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle_0 - \langle \mathcal{A}(z_1,uz_1)\rangle_1\langle \mathcal{A}(z_2,uz_2)\rangle_{-1} \\
&= \langle \mathcal{A}(z_2,uz_2)\mathcal{A}_+(z_1,uz_1)\rangle_0 + z_1\sum_{k=0}^{\infty}(-1)^k\left(\frac{z_1}{z_2}\right)^k \\
&\quad - \langle \mathcal{A}(z_2,uz_2)\rangle_{-1}\langle \mathcal{A}(z_1,uz_1)\rangle_1 \\
&= \langle \mathcal{A}_+(z_2,uz_2)\mathcal{A}_+(z_1,uz_1)\rangle_0 + z_1\sum_{k=0}^{\infty}(-1)^k\left(\frac{z_1}{z_2}\right)^k \\
&= z_1\sum_{k=0}^{\infty}(-1)^k\left(\frac{z_1}{z_2}\right)^k .
\end{aligned} \tag{5.31}$$

Since the same arguments are used many times in the computations below, let us go through all the steps of this computation. The first equality,

$$\begin{aligned}
&\langle \mathcal{A}(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle_0^\circ = \langle \mathcal{A}(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle_0 \\
&- \langle \mathcal{A}(z_1,uz_1)\rangle_{-1}\langle \mathcal{A}(z_2,uz_2)\rangle_1 - \langle \mathcal{A}(z_1,uz_1)\rangle_1\langle \mathcal{A}(z_2,uz_2)\rangle_{-1},
\end{aligned}$$

is a special case of the definition of a connected correlator. In the next equality we first replace $\langle \mathcal{A}(z_1,uz_1)\rangle_{-1}\langle \mathcal{A}(z_2,uz_2)\rangle_1$ by $(1/z_1)\langle \mathcal{A}(z_2,uz_2)\rangle_1$ and then use the following consequence of Equation (5.30):

$$\langle \mathcal{A}(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle_0 - \frac{1}{z_1}\langle \mathcal{A}(z_2,uz_2)\rangle_1 = \langle \mathcal{A}_+(z_1,uz_1)\mathcal{A}(z_2,uz_2)\rangle_0 .$$

For the third equality we use the commutation relation (5.23). For the fourth equality we first replace $\langle \mathcal{A}(z_2,uz_2)\rangle_{-1}\langle \mathcal{A}(z_1,uz_1)\rangle_1$ by $(1/z_2)\langle \mathcal{A}_+(z_1,uz_1)\rangle_1$ and then use the following consequence of Equation (5.30):

$$\langle \mathcal{A}(z_2,uz_2)\mathcal{A}_+(z_1,uz_1)\rangle_0 - \frac{1}{z_2}\langle \mathcal{A}_+(z_1,uz_1)\rangle_1 = \langle \mathcal{A}_+(z_2,uz_2)\mathcal{A}_+(z_1,uz_1)\rangle_0 .$$

Finally, we observe that $\langle \mathcal{A}_+(z_2,uz_2)\mathcal{A}_+(z_1,uz_1)\rangle_0 = 0$.

Now we prove the statement of the proposition by induction over the number of operators n in the correlator on the left hand side. From the definition of the operator $\mathcal{A}$ it is easy to see that the statement holds for $n = 1$. Suppose that it holds for the correlator of any number of operators less than n. We will prove that it holds for n operators.

Taking into account (5.30), (5.23), (5.29) and (5.31) and using the same arguments as above we see with that

$$\begin{aligned}
&\langle \mathcal{A}(z_1, uz_1) \ldots \mathcal{A}(z_n, uz_n)\rangle_k \qquad (5.32)\\
&= \frac{1}{z_1} \langle \mathcal{A}(z_2, uz_2) \ldots \mathcal{A}(z_n, uz_n)\rangle_{k+1}\\
&\quad + \langle \mathcal{A}_+(z_1, uz_1)\mathcal{A}(z_2, uz_2) \ldots \mathcal{A}(z_n, uz_n)\rangle_k\\
&= \langle \mathcal{A}(z_1, uz_1)\rangle^\circ_{-1} \langle \mathcal{A}(z_2, uz_2) \ldots \mathcal{A}(z_n, uz_n)\rangle_{k+1}\\
&\quad + z_1 \sum_{k=0}^{\infty} (-1)^k \left(\frac{z_1}{z_2}\right)^k \langle \mathcal{A}(z_3, uz_3) \ldots \mathcal{A}(z_n, uz_n)\rangle_k\\
&\quad + \langle \mathcal{A}(z_2, uz_2)\mathcal{A}_+(z_1, uz_1)\mathcal{A}(z_3, uz_3) \ldots \mathcal{A}(z_n, uz_n)\rangle_k\\
&= \langle \mathcal{A}(z_1, uz_1)\rangle^\circ_{-1} \langle \mathcal{A}(z_2, uz_2) \ldots \mathcal{A}(z_n, uz_n)\rangle_{k+1}\\
&\quad + \langle \mathcal{A}(z_1, uz_1)\mathcal{A}(z_2, uz_2)\rangle^\circ_0 \langle \mathcal{A}(z_3, uz_3) \ldots \mathcal{A}(z_n, uz_n)\rangle_k\\
&\quad + \langle \mathcal{A}(z_2, uz_2)\rangle^\circ_{-1} \langle \mathcal{A}_+(z_1, uz_1)\mathcal{A}(z_3, uz_3) \ldots \mathcal{A}(z_n, uz_n)\rangle_{k+1}\\
&\quad + \langle \mathcal{A}_+(z_1, uz_1)\mathcal{A}_+(z_2, uz_2)\mathcal{A}(z_3, uz_3) \ldots \mathcal{A}(z_n, uz_n)\rangle_k .
\end{aligned}$$

We continue with the same computation (replacing the leftmost operator $\mathcal{A}$ with $\mathcal{A}_+$ and commuting it to the right, collecting the emerging coefficients in the unstable correlators), finally arriving at the following expression.

$$\begin{aligned}
&\langle \mathcal{A}(z_1, uz_1) \ldots \mathcal{A}(z_n, uz_n)\rangle_k = \langle \mathcal{A}_+(z_1, uz_1) \ldots \mathcal{A}_+(z_n, uz_n)\rangle_k \qquad (5.33)\\
&+ \sum_{p=3}^{n-1} \sum_{q=0}^{[\frac{n-p}{2}]} \sum_{y\in\widehat{\mathcal{Y}}^{p,q}_{n,k}} \big\langle \mathcal{A}_+(z_{c_{1,1}(y)}, uz_{c_{1,1}(y)}) \ldots \mathcal{A}_+(z_{c_{1,p}(y)}, uz_{c_{1,p}(y)})\big\rangle_{k+h(y)-q-1}\\
&\quad \times \prod_{i=2}^{q+1} \big\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\mathcal{A}(z_{c_{i,2}(y)}, uz_{c_{i,2}(y)})\big\rangle^\circ_0 \prod_{i=q+2}^{h(y)} \big\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\big\rangle^\circ_{-1}\\
&+ \sum_{q=0}^{[\frac{n-2}{2}]} \sum_{y\in\widehat{\mathcal{Y}}^{2,q}_{n,k}} \Big\langle \mathcal{A}_+(z_{c_{s(y),1}(y)}, uz_{c_{s(y),1}(y)})\mathcal{A}_+(z_{c_{s(y),2}(y)}, uz_{c_{s(y),2}(y)})\Big\rangle_{k+h(y)-q-1}\\
&\quad \times \prod_{\substack{i=1\\ i\neq s(y)}}^{q+1} \big\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\mathcal{A}(z_{c_{i,2}(y)}, uz_{c_{i,2}(y)})\big\rangle^\circ_0 \prod_{i=q+2}^{h(y)} \big\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\big\rangle^\circ_{-1}\\
&+ \sum_{q=0}^{[\frac{n-1}{2}]} \sum_{y\in\widehat{\mathcal{Y}}^{1,q}_{n,k}} \Big\langle \mathcal{A}_+(z_{c_{s(y),1}(y)}, uz_{c(y)_{s(y),1}(y)})\Big\rangle_{k+h(y)-q-1}\\
&\quad \times \prod_{i=1}^{q} \big\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\mathcal{A}(z_{c_{i,2}(y)}, uz_{c_{i,2}(y)})\big\rangle^\circ_0 \prod_{\substack{i=q+1\\ i\neq s(y)}}^{h(y)} \big\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\big\rangle^\circ_{-1}
\end{aligned}$$

Here $\widehat{\mathcal{Y}}^{p,q}_{n,k}$ contains all elements y of $\mathcal{Y}_{n,k}$ such that there is precisely one row of length p labeled by $k+h(y)-q-1$, q rows of length 2 labeled by 0, and all other rows are of length 1 and labeled by -1. $s(y)$ stands for the position of the row with p elements labeled by

$k+h(y)-q-1$. If $p=2$ and $k+h(y)-q-1=0$ or $p=1$ and $k+h(y)-q-1=-1$ one cannot determine $s(y)$ in this way, but this is not a problem since, due to the fact that $\langle\mathcal{A}_+(z,uz)\rangle_{-1}=0$ and $\langle\mathcal{A}_+(z_1,uz_1)\mathcal{A}_+(z_2,uz_2)\rangle_0=0$, the corresponding term vanishes in any case. Also note that, obviously, for $p\geq 3$ we have $s(y)=1$.

Note that the right hand side of formula (5.33) is equal to the correlator

$$\langle\mathcal{A}_+(z_1,uz_1)\dots\mathcal{A}_+(z_n,uz_n)\rangle_k \tag{5.34}$$

plus all possible unstable terms entering exactly once, since, by the induction hypothesis, the correlators of less than n operators $\mathcal{A}_+$ are equal to sums of all possible stable terms. This means that upon moving these terms to the left hand side and subtracting them from $\langle\mathcal{A}(z_1,uz_1)\dots\mathcal{A}(z_n,uz_n)\rangle_k$ we get precisely all possible stable terms. This proves the proposition. □

5.1.5 Polynomiality

In this subsection we establish polynomiality of some correlators, and this allows us to complete the proof of Theorem 5.1.1.

Proposition 5.1.23. *The series*

$$\frac{\langle\mathcal{A}_+(z_1,uz_1)\dots\mathcal{A}_+(z_n,uz_n)\rangle_k}{z_1\cdots z_n} \tag{5.35}$$

for $(n,k)\notin\{(1,-1),(2,0)\}$ *is a symmetric polynomial in* $z_1,\dots,z_n$.

Proof. From the definition of $\mathcal{A}_+$ it is easy to see that for every i the power of z_i in the series $\langle\mathcal{A}_+(z_1,uz_1)\dots\mathcal{A}_+(z_n,uz_n)\rangle_k$ is bounded from below by 1. From (5.23) it is clear that this series is symmetric in $z_1,\dots,z_n$. Let us prove that, for fixed k, the power of z_n in this series is bounded from above, following the proof of Proposition 9 of [83].

Note that

$$\langle\mathcal{E}_{k_1}(uz_1)\dots\mathcal{E}_{k_n}(uz_n)\rangle=\left\langle\frac{\mathcal{E}_{k_1}(uz_1)}{u^{k_1}}\dots\frac{\mathcal{E}_{k_n}(uz_n)}{u^{k_n}}\right\rangle, \tag{5.36}$$

holds since the correlator vanishes unless $\sum k_i=0$.

Let us apply this transformation to the correlator of $\mathcal{A}$ operators:

$$\langle\mathcal{A}(z_1,uz_1)\dots\mathcal{A}(z_n,uz_n)\rangle=\left\langle\widetilde{\mathcal{A}}(z_1,uz_1)\dots\widetilde{\mathcal{A}}(z_n,uz_n)\right\rangle. \tag{5.37}$$

Here $\widetilde{\mathcal{A}}$ stands for the operator $\mathcal{A}$ where the substitution $\mathcal{E}_k\mapsto u^{-k}\mathcal{E}_k$ was made. Note that each term in each $\widetilde{\mathcal{A}}$ is then regular and non-vanishing at $u=0$, except for the term

$\frac{1}{\zeta(uz)}$ coming from $\mathcal{E}_0$, which has a simple pole. Let us write the following:

$$\begin{aligned}
&\widetilde{\mathcal{A}}(z_n, uz_n)\,\big|0\big\rangle \\
&= \left(\frac{\zeta(uz_n)}{uz_n}\right)^{z_n} \sum_{k\in\mathbb{Z}} \frac{\zeta(uz_n)^k}{(z_n+1)_k}\,\frac{\mathcal{E}_k(uz_n)}{u^k}\big|0\big\rangle \\
&= \left(\frac{\zeta(uz_n)}{uz_n}\right)^{z_n} \sum_{k=0}^{\infty} \frac{u^k}{\zeta(uz_n)^k} z_n \dots (z_n-k+1)\,\mathcal{E}_{-k}(uz_n)\big|0\big\rangle \\
&= \left(\frac{\zeta(uz_n)}{uz_n}\right)^{z_n} \sum_{k=0}^{\infty} \left(\frac{uz_n}{\zeta(uz_n)}\right)^k \left(1-\frac{1}{z_n}\right)\dots\left(1-\frac{k-1}{z_n}\right)\,\mathcal{E}_{-k}(uz_n)\big|0\big\rangle.
\end{aligned} \tag{5.38}$$

It is easy to see that z_n and u enter this expression in such a way that for all terms with a fixed power of u the power of z_n is bounded from above. Since in (5.37) this expression is multiplied by operators $\widetilde{\mathcal{A}}(z_i, uz_i)$, $i \in \{1,\dots,n-1\}$, which have at most simple poles in u, the whole correlator (5.37) is bounded from above in powers of z_n, for a fixed power of u.

From the definition of $\mathcal{A}_+$ it is clear that the fact that the power of z_n in

$$\langle \mathcal{A}(z_1, uz_1)\dots\mathcal{A}(z_n, uz_n)\rangle_k$$

is bounded from above for a fixed k immediately implies that the power of z_n in

$$\langle \mathcal{A}_+(z_1, uz_1)\dots\mathcal{A}_+(z_n, uz_n)\rangle_k$$

is bounded from above for a fixed k as well.

The symmetricity of $\langle \mathcal{A}_+(z_1, uz_1)\dots\mathcal{A}_+(z_n, uz_n)\rangle_k$ then implies that for fixed k the power of z_i in this expression is bounded from above for any i, which implies that the series

$$\langle \mathcal{A}_+(z_1, uz_1)\dots\mathcal{A}_+(z_n, uz_n)\rangle_k \tag{5.39}$$

is polynomial in $z_1,\dots,z_n$, which, in turn, leads to the fact that the series

$$\frac{\langle \mathcal{A}_+(z_1, uz_1)\dots\mathcal{A}_+(z_n, uz_n)\rangle_k}{z_1\cdots z_n} \tag{5.40}$$

is polynomial in $z_1,\dots,z_n$. □

Proposition 5.1.24. *For $(n,k) \notin \{(1,-1),(2,0)\}$ the series*

$$\frac{\langle \mathcal{A}(z_1, uz_1)\dots\mathcal{A}(z_n, uz_n)\rangle_k^\circ}{z_1\cdots z_n} \tag{5.41}$$

is a symmetric polynomial in $z_1,\dots,z_n$.

Proof. Let us prove the statement of this proposition by induction in n, the number of operators in the correlator. It is clear that for $n=1$ the statement holds. Suppose that it holds for any number of operators less than n. We will prove that it then holds for n operators as well.

Formula (5.27) can be rewritten as

$$\frac{\langle \mathcal{A}(z_1, uz_1)\dots\mathcal{A}(z_n, uz_n)\rangle_k^\circ}{z_1\cdots z_n} = \frac{\langle \mathcal{A}_+(z_1, uz_1)\dots\mathcal{A}_+(z_n, uz_n)\rangle_k}{z_1\cdots z_n} \tag{5.42}$$
$$- \sum_{y\in\widetilde{\mathcal{Y}}'_{n,k}} \prod_{i=1}^{h(y)} \frac{\left\langle \mathcal{A}(z_{c_{i,1}(y)}, uz_{c_{i,1}(y)})\dots\mathcal{A}(z_{c_{i,l_i(y)}(y)}, uz_{c_{i,l_i(y)}(y)})\right\rangle^\circ_{\lambda_i(y)}}{z_{c_{i,1}(y)}\cdots z_{c_{i,l_i(y)}(y)}}.$$

Here, naturally, $\widetilde{\mathcal{Y}}'_{n,k}$ is equal to $\widetilde{\mathcal{Y}}_{n,k}$ with the single-row Young tableau thrown away.

By Proposition 5.1.23, the first term on the right hand side of (5.42) is polynomial in $z_1,\dots,z_n$. By induction hypothesis, all the terms in the sum on the right hand side of (5.42) are polynomial as well, since they are finite products of connected correlators of the lower number of operators (and, by definition of $\widetilde{\mathcal{Y}}'_{n,k}$, correlators with $(n_i,k_i)\in\{(1,-1),(2,0)\}$ never appear). This implies the statement of the proposition. □

Taking into account formula (5.16), we see that Proposition 5.1.24 directly implies the statement of Theorem 5.1.1.

5.2 Proof of the Bouchard-Mariño conjecture

In the present section we give a new proof of the Bouchard-Mariño conjecture using the polynomiality result from the previous section and not using the ELSV formula.

This conjecture was already proved in [11] and [43]. The first of these papers provides a "physical" proof through the study of the corresponding matrix model. Unfortunately, we were not able to attribute precise mathematical meaning to all of the statements of that paper (see [89] for a related discussion). In the second paper the Bouchard-Mariño formula is derived directly from the known cut-and-join recursion relation for Hurwitz numbers, with the help of the ELSV formula.

Here we follow the ideas of the proof of [43] presenting them in a simplified way, with one essential modification: we do not use the ELSV formula in this proof, using instead just the polynomiality property.

5.2.1 The Lambert curve

The *Lambert curve* is a curve in $\mathbb{C}^2$ defined by the equation

$$x = y\,e^{-y}. \tag{5.43}$$

We consider this affine curve as an open part of its compactification $C = \mathbb{C}P^1$. We regard y as a rational coordinate on C and the projection to the x-line as a holomorphic function with an essential singularity at the point $y=\infty$. In addition to y we use other convenient rational coordinates on C. In particular, we keep the notations z and t for the rational coordinates related to y by

$$y = 1 + z = 1 + \frac{1}{t}, \quad t = \frac{1}{z}.$$

There are two points on C of special interest for us: the *origin* O corresponding to the coordinates $y = 0$, $z = -1$, $t = -1$, and the *branching point* P with the coordinates $y = 1$, $z = 0$, $t = \infty$. The point P is a Morse critical point for the function x. It means that the projection to the x-line considered as a branched cover has ramification of order two at P.

Consider also the function $w = \log x$. It is multi-valued, however, its differential is a well-defined meromorphic differential on C,

$$dw = \frac{dx}{x} = \frac{1-y}{y}dy = -\frac{z}{z+1}dz = \frac{dt}{t^2(t+1)}. \tag{5.44}$$

Denote also by D the vector field dual to this 1-form,

$$D = x\partial_x = \frac{y}{1-y}\partial_y = -\frac{z+1}{z}\partial_z = t^2(t+1)\partial_t. \tag{5.45}$$

We regard (5.44) and (5.45) as a single meromorphic form and a single vector field on C respectively, whose coordinate presentation depends on the chosen local coordinate. Remark that the form dw vanishes at P, while the field D has a simple pole at this point.

The inversion of (5.43) near the origin is given [19, 31] by the expansion

$$y = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu-1}}{\mu!} x^\mu.$$

It follows from (5.45) that for any integer k the series

$$\rho_k = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu+k}}{\mu!} x^\mu = D^{k+1} y$$

is a rational function on C. More explicitly, in the t-coordinate it is given for $k \geq 0$ by the recursion

$$\rho_0(t) = -1 - t, \qquad \rho_{k+1}(t) = t^2(t+1)\frac{d}{dt}(\rho_k(t)).$$

It is a polynomial in t:

$$\rho_k(t) = -k!\, t^{k+1} - \cdots - (2k-1)!!\, t^{2k+1}.$$

The degree of this polynomial is $2k+1$. Equivalently, one can say that ρ_k considered as a meromorphic function on C has pole of order $2k+1$ at P. It follows that the linear span of the polynomials ρ_k form a subspace of 'approximately half' dimension in the space of all polynomials in t. This subspace has a nice characterization that we describe now.

Denote by σ the involution interchanging the sheets of the ramification defined by the function x near the point P. The function σ is holomorphic in a neighborhood of P and its Taylor expansion can be computed from the equation

$$(1+z)\, e^{-z} = (1+\sigma(z))\, e^{-\sigma(z)}. \tag{5.46}$$

Here are the first few terms of this expansion written in the coordinates z and t, respectively:

$$\begin{aligned}\sigma(z) &= -z + \frac{2}{3}z^2 - \frac{4}{9}z^3 + \frac{44}{135}z^4 - \frac{104}{405}x^5 + \frac{40}{189}z^6 + \dots \\ \tilde{\sigma}(t) &= \frac{1}{\sigma(1/t)} = -t - \frac{2}{3} - \frac{4}{135t^2} + \frac{8}{405t^3} - \frac{8}{567t^4} + \dots\end{aligned}$$

Lemma 5.2.1. *For any $k \geq 0$ the principal part of the pole of $\rho_k(t)$ at the point P is odd with respect to the involution σ. In other words, the function $\rho_k(t) + \rho_k(\tilde{\sigma}(t))$ is holomorphic at P.*

Proof. For $k = 0$ the assertion is obvious since the principal part of any simple pole is odd. Now, arguing by induction, we assume that ρ_k is represented in the form

$$\rho_k(t) = \eta_k(t) + F_k(t)$$

where $\eta_k(t) = \frac{1}{2}\big(\rho_k(t) - \rho_k(\tilde{\sigma}(t))\big)$ is odd and $F_k(t) = \frac{1}{2}\big(\rho_k(t) + \rho_k(\tilde{\sigma}(t))\big)$ is even and holomorphic at P. Then, by definition,

$$\rho_{k+1}(t) = D(\rho_k(t)) = D(\eta_k(t)) + D(F_k(t)).$$

The field D is invariant with respect to the involution, therefore, it preserves the parity. It follows that $D(\eta_k(t))$ is odd, and $D(F_k(t))$ is even and the order of its pole at P is at most 1. It follows that $D(F_k(t))$ is, in fact, holomorphic at P, which proves the lemma. □

5.2.2 Generating function for Hurwitz numbers

Let us introduce the generating function for the connected Hurwitz numbers $h^{\circ}_{g;\mu}$ in the following way:

$$H^{\circ}_{g,n} := \sum_{\mu_1,\dots,\mu_n \in \{1,2,\dots\}} \frac{h^{\circ}_{g;\mu_1,\dots,\mu_n}}{\mathfrak{b}(g,\mu)!} x_1^{\mu_1} \dots x_n^{\mu_n}. \tag{5.47}$$

Theorem 5.1.1 implies that, for $(g, n) \notin \{(0, 1), (0, 2)\}$,

$$H^{\circ}_{g,n} = \sum_{\substack{k_1,\dots,k_n \in \\ \{0,1,\dots,K_{g,n}\}}} c_{k_1\dots k_n} \prod_{i=1}^{n} \sum_{\mu_i=1}^{\infty} \frac{\mu_i^{\mu_i+k_i}}{\mu_i!} x_i^{\mu_i}, \tag{5.48}$$

where $c_{k_1\dots k_n}$ are the coefficients of the polynomials $P_{g,n}$ from Theorem 5.1.1, and $K_{g,n}$ is the highest power appearing in $P_{g,n}$.

Define

$$\rho_k(x) := \sum_{m=1}^{\infty} \frac{m^{m+k}}{m!} x^m. \tag{5.49}$$

Now we can rewrite (5.48) as

$$H^{\circ}_{g,n} = \sum_{\substack{k_1,\dots,k_n \in \\ \{0,1,\dots,K_{g,n}\}}} c_{k_1\dots k_n} \prod_{i=1}^{n} \rho_{k_i}(x_i). \tag{5.50}$$

Consider the following change of variables:

$$x_i = \left(1 + \frac{1}{t_i}\right) e^{-1-\frac{1}{t_i}}. \tag{5.51}$$

We see that the generating function $H_{g,n}$ is a polynomial in variables t_i (in all but two 'unstable' cases when $g = 0$ and $n \leq 2$) after the above substitution (we treat this substitution as a power series expansion at the point $t_i = -1$). For the unstable cases we have

$$\begin{aligned} H^\circ_{0,1} &= \sum_{a=1}^{\infty} \frac{a^{a-2}}{a!} x_1^a = \rho_{-2}(x_1) = \frac{1}{2} - \frac{1}{2t_1^2}, \\ H^\circ_{0,2} &= \sum_{a,b} \frac{a^a}{a!} \frac{b^b}{b!} \frac{x_1^a x_2^b}{a+b} = \log\left(\frac{\frac{1}{t_2+1} - \frac{1}{t_1+1}}{\frac{1}{x_1} - \frac{1}{x_2}} \right). \end{aligned} \tag{5.52}$$

The formula of Bouchard and Mariño is a recursion relation for these polynomials. In order to present it in a more closed form it is convenient to introduce another family of polynomials $W_{g,n}(t_1, \ldots, t_n)$ obtained by the above substitution from the series

$$\Big(\prod x_k \partial_{x_k}\Big) H^\circ_{g,n} = \sum_{\mu_1,\ldots,\mu_n} \frac{h^\circ_{g;\mu_1,\ldots,\mu_n}}{\mathfrak{b}(g,\mu)!}\, \mu_1 \ldots \mu_n\, x_1^{\mu_1} \ldots x_n^{\mu_n},$$

i. e., for $(g, n) \notin \{(0, 1), (0, 2)\}$,

$$W_{g,n}(t_1, \ldots, t_n) = \sum_{\substack{k_1,\ldots,k_n \in \\ \{0,1,\ldots,K_{g,n}\}}} c_{k_1 \ldots k_n} \prod_{i=1}^{n} \rho_{k_i+1}(t_i). \tag{5.53}$$

In the unstable cases we define the functions $W_{g,n}$ by setting explicitly

$$W_{0,1}(t_1) = 0, \tag{5.54}$$

$$W_{0,2}(t_1, t_2) = \frac{t_1^2(t_1+1)t_2^2(t_2+1)}{(t_2-t_1)^2}. \tag{5.55}$$

Define also auxiliary functions $\widetilde{W}_{g,n}(u, v; t_2, \ldots, t_n)$ by

$$\begin{aligned} \widetilde{W}_{g,n}(u, v; t_{L'}) :=& W_{g-1,n+1}(u, v, t_{L'}) \\ &+ \sum_{g_1+g_2=g} \sum_{A \sqcup B = L'} W_{g_1,|A|+1}(u, t_A) W_{g_2,|B|+1}(v, t_B). \end{aligned}$$

We denote here by $L' = \{2, \ldots, n\}$ the index set, $t_{L'} = (t_2, \ldots, t_n)$; the summation is taken over the set of all possible partitions of the index set into a disjoint union of two subsets, A and B.

Theorem 5.2.2 (Bouchard-Mariño conjecture)**.** *The polynomials $W_{g,n}$ can be determined by the either of the following recursive formulas*

$$\begin{aligned} W_{g,n}(t_1, t_{L'}) &= -\operatorname*{res}_{z=0} \Big(K(z, t_1)\, \widetilde{W}_{g,n}\big(\frac{1}{z}, \frac{1}{z}; t_{L'}\big) \Big) \\ &= \operatorname*{res}_{z=0} \Big(K(z, t_1)\, \widetilde{W}_{g,n}\big(\frac{1}{z}, \frac{1}{\sigma(z)}; t_{L'}\big) \Big) \\ &= -\operatorname*{res}_{z=0} \Big(K(z, t_1)\, \widetilde{W}_{g,n}\big(\frac{1}{\sigma(z)}, \frac{1}{\sigma(z)}; t_{L'}\big) \Big) \end{aligned}$$

where

$$K(z,t_1) = \frac{t_1^2(1+t_1)}{2(1-z\,t_1)(1-\sigma(z)\,t_1)}\,\frac{z\,dz}{z+1}$$

and the series $\sigma(z) = -z + \frac{2}{3}z^2 - \frac{4}{9}z^3 + \dots$ *is defined in the previous subsection.*

The second equality is a reformulation of the Bouchard-Mariño conjecture. Experiments show, however, that the first formula is more efficient for practical computations.

Analytically, the meaning of this theorem is as follows. The function $H_{g,n}$ is defined originally as a formal power series expansion at $x_i = 0$. It turns out, however, that this series has a finite radius of convergence with respect to each variable x_i (to be precise, the radius of convergence is e^{-1}). An attempt to extend it beyond the radius of convergence meets difficulties: the function becomes multi-valued with ramification at $x_i = e^{-1}$. Therefore, it is more natural to consider $H_{g,n}$ as a function on the product $C \times \dots \times C$ where C is the curve given by the equation $x = \left(1+\frac{1}{t}\right)e^{-1-\frac{1}{t}}$. When treated in this way, it becomes single-valued and even rational. The recursive relation of the theorem is formulated in terms of the analysis of the behavior of the function $H_{g,n}$ (and closely related to it function $W_{g,n}$) in a neighborhood of the ramification point $x_1 = e^{-1}$ which is different from the origin.

5.2.3 The cut-and-join equation

Yet another way to collect Hurwitz numbers into a generating series is given by the expansion

$$G_{g,n}(p_1,p_2,\dots) = \frac{1}{n!}\sum_{\mu_1,\dots,\mu_n} \frac{h^\circ_{g;\mu_1,\dots,\mu_n}}{\mathfrak{b}(g,\mu)!}\,p_{\mu_1}\dots p_{\mu_n}.$$

The series $G_{g,n}$ involves an infinite collection of variables $p_1, p_2, \dots$ and all its terms are homogeneous of degree n. The relation between the two series $G_{g,n}$ and $H^\circ_{g,n}$ is obvious. In particular, $G_{g,n}$ can be obtained from $\frac{1}{n!}H^\circ_{g,n}$ by replacing every monomial $x_1^{\mu_1}\dots x_n^{\mu_n}$ by the corresponding monomial $p_{\mu_1}\dots p_{\mu_n}$.

The cut-and-join equation is a recursion on Hurwitz numbers obtained through the analysis of the cyclic type of the result of multiplication of a given permutation by a single transposition. In its original form [55], it is written as

$$2u\frac{\partial e^G}{\partial u} + \sum_{i=1}^{\infty}(i+1)\,p_i\frac{\partial e^G}{\partial p_i} = \frac{1}{2}\sum_{a,b}\Big((a+b)p_a p_b\frac{\partial e^G}{\partial p_{a+b}} + u\,ab\,p_{a+b}\frac{\partial^2 e^G}{\partial p_a \mathrm{d}p_b}\Big)$$

where

$$G = \sum_{g,n} u^{g-1}G_{g,n}.$$

The same equation written in terms of the individual components $G_{g,n}$ is

$$(2g-2+n)G_{g,n}+\sum_{i=1}^{\infty} i\, p_i \frac{\partial G_{g,n}}{\partial p_i} \tag{5.56}$$
$$=\frac{1}{2}\sum_{a,b}\Big((a+b)p_a p_b \frac{\partial G_{g,n-1}}{\partial p_{a+b}}$$
$$+ab\, p_{a+b}\Big(\frac{\partial^2 G_{g-1,n+1}}{\partial p_a \partial p_b}+\sum_{\substack{g_1+g_2=g\\ n_1+n_2=n+1}} \frac{\partial G_{g_1,n_1}}{\partial p_a}\frac{\partial G_{g_2,n_2}}{\partial p_b}\Big)\Big).$$

Let us rewrite this equation in terms of the functions $H^\circ_{g,n}$. The operator $\sum i\, p_i \partial_{p_i}$ from the left hand side of the equation corresponds to the operator $\sum_{i=1}^n D_i$ acting on $H^\circ_{g,n}$ where

$$D_i = x_i \partial_{x_i} = t_i^2(t_i+1)\partial_{t_i}.$$

The action of the 'cut' operator $\sum (a+b)p_a p_b \mathrm{d}_{p_{a+b}}$ in terms of the series $H^\circ_{g,n}$ results in the replacement of any monomial x_m^ℓ by the sum

$$\sum_{a+b=\ell}(a+b)x_j^a x_k^b = \ell\frac{x_k x_j(x_k^{\ell-1}-x_j^{\ell-1})}{x_k-x_j}$$
$$=\frac{x_j}{x_k-x_j}x_k\frac{\partial(x_k^\ell)}{\partial x_k}+\frac{x_k}{x_j-x_k}x_j\frac{\partial(x_j^\ell)}{\partial x_j}$$
$$=\frac{x_j}{x_k-x_j}D_k(x_k^\ell)+\frac{x_k}{x_j-x_k}D_j(x_j^\ell).$$

In a similar way, the action of the 'join' operator $ab\, p_{a+b}\dfrac{\partial^2}{\partial p_a \mathrm{d} p_b}$ results in the replacement of any monomial $x_j^a x_k^b$ by the monomial

$$ab\, x_m^{a+b}=\Big(x_m\frac{\partial(x_m^a)}{\partial x_m}\Big)\Big(x_m\frac{\partial(x_m^b)}{\partial x_m}\Big)=D_m(x_m^a)D_m(x_m^b).$$

The relation between the indices k, j, and m in the above considerations is not essential. One should only take care that the result is symmetric with respect to the permutations of the variables $x_1,\dots,x_n$.

The relation obtained from (5.56) in this way is presented below. In this relation L denotes the collection of indices $L=\{1,2,\dots,n\}$, and $t_L=(t_1,\dots,t_n)$.

$$(2g-2+n)H^\circ_{g,n}(t_L)+\sum_{k=1}^n D_k H^\circ_{g,n}(t_L) \tag{5.57}$$
$$=\frac{1}{2}\sum_{k\neq j}2\frac{x_j}{x_k-x_j}D_k H^\circ_{g,n-1}(t_{L\setminus\{j\}})$$
$$+\frac{1}{2}\sum_{k=1}^n\Big(D_k D_{n+1}H^\circ_{g-1,n+1}(t_L,t_{n+1})\,\big|_{t_{n+1}=t_k}$$
$$+\sum_{g_1+g_2=g}\ \sum_{A\sqcup B=L\setminus\{k\}} D_k H^\circ_{g_1,|A|+1}(t_k,t_A)D_k H^\circ_{g_2,|B|+1}(t_k,t_B)\Big),$$

where the last summation is taken over the set of all possible partitions of the index set $L \setminus \{k\} = \{1, \dots, k-1, k+1, \dots, n\}$ into a disjoint union of two subsets, A and B.

This relation can be regarded as a relation on the functions in either x or t-variables, where x_i and t_i are related by (5.51). We consider this relation as the 'preliminary form' of the required cut-and-join equation. The final form is obtained by extracting unstable terms from the last summation corresponding to the functions $H^{\circ}_{0,1}$ and $H^{\circ}_{0,2}$ and combining these terms with the corresponding terms of the previous sums. Using (5.52), we find the coefficients of the recombined terms

$$1 - D_1 H^{\circ}_{0,1}(t_1) = -\frac{1}{t_1},$$
$$\frac{x_2}{x_1 - x_2} + D_1 H^{\circ}_{0,2}(t_1, t_2) = \frac{t_1^2(1+t_2)}{t_1 - t_2}.$$

We obtain thus the final form of the cut-and-join equation in the t-coordinates, see more details in [79]:

$$\begin{aligned}
&(2g-2+n) H^{\circ}_{g,n}(t_L) + \sum_{k=1}^{n} \Big(-\frac{1}{t_k}\Big) D_k H^{\circ}_{g,n}(t_L) \\
&= \sum_{k \neq j} \frac{t_k^2(1+t_j)}{t_k - t_j} D_k H^{\circ}_{g,n-1}(t_{L \setminus \{j\}}) \\
&\quad + \frac{1}{2} \sum_{k=1}^{n} \Big(D_k D_{n+1} H^{\circ}_{g-1,n+1}(t_L, t_{n+1}) \,\big|_{t_{n+1}=t_k} \\
&\quad + \sum_{g_1+g_2=g} \sum_{A \sqcup B = L \setminus \{k\}}^{\text{stable}} D_k H^{\circ}_{g_1,|A|+1}(t_k, t_A) D_k H^{\circ}_{g_2,|B|+1}(t_k, t_B) \Big).
\end{aligned} \tag{5.58}$$

It is remarkable that the 'non-polynomial' summands are canceled out, and both sides of the relation proved to be polynomial in t-variables. As it is pointed out in [58, 79], selecting the highest and the lowest degree terms of this formula one gets immediately the Virasoro constrains for the intersection numbers of ψ-classes on the moduli spaces of curves (see e.g. [20, 67, 98, 6, 90]) and the relation of the λ_g-formula [49, 50], respectively.

5.2.4 Reduction by symmetrization

The cut-and-join equation (5.58) can be used to determine $H^{\circ}_{g,n}$ inductively. However, in the presented form it is not very convenient since it is not clear how to invert the operator on the left hand side of the equation. It is not even obvious that the function $H^{\circ}_{g,n}$ obtained by this recursion is polynomial in t-variables. The following two key observations of [43] lead to a considerable simplification of (5.58):

1. The function $H^{\circ}_{g,n}$ is polynomial in each variable t_i, therefore, the whole information about this function is contained in the principal part of its pole at the point P with respect to t_i. Let us stress that in [43] this polynomiality was derived from the ELSV formula, while here we have it independently due to the results of Section 5.1, as noted above.

2. The principal part of the pole of $H_{g,n}$ is odd with respect to the involution σ on each t_i-line (as it follows from Lemma 5.2.1).

Consider the *even* summand of the principal part of the pole at P of each term in (5.58) *with respect to the first variable* t_1. It follows that most of the terms will give trivial contribution to the result so that the whole equation will be considerably simplified.

It is more convenient for us to use a slight modification of this idea. Namely, set

$$\eta(t_1) = \sigma\Big(\frac{1}{t_1}\Big) - \frac{1}{t_1}.$$

This function is holomorphic at P and odd with respect to the involution. Now, consider an arbitrary meromorphic function $f(t_1)$.

Notation 5.2.3. We denote by

$$\left\lfloor \frac{f(t_1)}{\eta(t_1)} \right\rfloor_1^-$$

the *odd residueless principal part* of the pole of the quotient f/η at the point P. More explicitly, if we write the Laurent expansion

$$\frac{f(t_1) + f(\tilde\sigma(t_1))}{2\,\eta(t_1)} = \sum_{-\infty<i\le N} a_i\, t_1^i$$

at P, then we set, by definition,

$$\lfloor f/\eta \rfloor_1^- = \sum_{i=2}^{N} a_i\, t_1^i.$$

From this definition we see that $\lfloor f/\eta \rfloor_1^-$ is a polynomial in t_1 divisible by t_1^2.

We apply the transformation $f(t_1) \mapsto \lfloor 2f/\eta \rfloor_1^-$ to both sides of (5.58). This transformation annihilates any function in t_1 whose pole at P has odd principal part. In particular, it annihilates $H^\circ_{g,n}(t_L)$ on the left hand side as well as all terms on both sides of the equality corresponding to the summation index k different from 1.

Let us compute the action of this transformation on the term $-\frac{1}{t_1} D_1 H^\circ_{g,n}$ on the left hand side. For any meromorphic function $f(t_1)$ which is odd with respect to the involution we have

$$\frac{\frac{-f(t_1)}{t_1} - \frac{f(\tilde\sigma(t_1))}{\tilde\sigma(t_1)}}{\eta(t_1)} = f(t_1)\,\frac{-\frac{1}{t_1} + \frac{1}{\tilde\sigma(t_1)}}{\eta(t_1)} = f(t_1).$$

Therefore, $\left\lfloor -\frac{2f(t_1)}{\eta(t_1)\,t_1} \right\rfloor_1^- = \lfloor f(t_1) \rfloor_1^-$. The function $D_1 H^\circ_{g,n}$ differs from such a function by a holomorphic summand that gives trivial contribution to the transformation. This implies

$$\left\lfloor -\frac{2}{\eta(t_1)\,t_1} D_1 H^\circ_{g,n} \right\rfloor_1^- = \lfloor D_1 H^\circ_{g,n} \rfloor_1^- = D_1 H^\circ_{g,n}.$$

We obtain finally the equation

$$D_1 H^\circ_{g,n} = \left\lfloor \frac{1}{\eta} \left(\begin{array}{l} \displaystyle\sum_{j=2}^{n} \frac{t_1^2(1+t_j)}{t_1 - t_j} D_1 H^\circ_{g,n-1}(t_1, t_{L'\setminus\{j\}}) \\ + D_1 D_{n+1} H^\circ_{g-1,n+1}(t_1, t_{L'}, t_{n+1}) \big|_{t_{n+1}=t_1} \\ + \displaystyle\sum_{g_1+g_2=g} \sum_{A\sqcup B = L'}^{\text{stable}} D_1 H^\circ_{g_1,|A|+1}(t_1, t_A) D_1 H^\circ_{g_2,|B|+1}(t_1, t_B) \end{array} \right) \right\rfloor_1^- \tag{5.59}$$

where $L' = L \setminus \{1\} = \{2, \dots, n\}$.

In order to represent this equation in a more readable form, let us apply $\prod_{k=2}^n D_k$ to both its sides and observe that the expression inside the square brackets becomes algebraic with respect to the functions $W_{g,k}$ defined by (5.53). Moreover, the first term on the right hand side can be formally included into the last since we defined the contribution of the unstable terms as in Equations (5.54) and (5.55). With this notation, the result of the application of $\prod_{k=2}^n D_k$ to both sides of Equation (5.59) takes the form of the following recursive relation on $W_{g,n}$.

Proposition 5.2.4. *The function $W_{g,n}$ defined by* (5.53)–(5.55) *satisfies the recursive equation*

$$W_{g,n}(t_1, t_{L'}) = \left\lfloor \frac{1}{\eta(t_1)} \widetilde{W}(t_1, t_1; t_{L'}) \right\rfloor_1^-,$$

where $L' = \{2, \dots, n\}$, $t_{L'} = (t_2, \dots, t_n)$, and

$$\begin{aligned} \widetilde{W}_{g,n}(u, v; t_{L'}) &= W_{g-1,n+1}(u, v, t_{L'}) \\ &+ \sum_{g_1+g_2=g} \sum_{A\sqcup B = L'} W_{g_1,|A|+1}(u, t_A) W_{g_2,|B|+1}(v, t_B). \end{aligned} \tag{5.60}$$

Remark 5.2.5. If $f(t_1)$ is a meromorphic function whose pole at P has odd principal part then for any other function g we have

$$\left\lfloor \frac{f(t_1)g(t_1)}{\eta(t_1)} \right\rfloor_1^- + \left\lfloor \frac{f(t_1)g(\tilde\sigma(t_1))}{\eta(t_1)} \right\rfloor_1^- = \left\lfloor f(t_1) \frac{g(t_1) + g(\tilde\sigma(t_1))}{\eta(t_1)} \right\rfloor_1^- = 0$$

since $(g(t_1) + g(\tilde\sigma(t_1)))/\eta(t_1)$ is odd. Therefore, $W_{g,n}$ can equivalently be obtained by the either of the following relations

$$\begin{aligned} W_{g,n}(t_1, t_{L'}) &= - \left\lfloor \frac{1}{\eta(t_1)} \widetilde{W}_{g,n}(t_1, \tilde\sigma(t_1); t_{L'}) \right\rfloor_1^- \\ &= \left\lfloor \frac{1}{\eta(t_1)} \widetilde{W}_{g,n}(\tilde\sigma(t_1), \tilde\sigma(t_1); t_{L'}) \right\rfloor_1^-. \end{aligned}$$

5.2.5 Residual formalism

The coefficient f_k of the meromorphic function $f(t_1) = \sum_{-\infty < i \le N} f_i t_1^i$ can be extracted by taking the residue

$$f_k = \operatorname*{Res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) z^{k-1}\, dz \Big).$$

It follows that the whole residueless principal part of the pole of f is given by

$$\sum_{k=2}^{N} f_k t_1^k = \operatorname*{Res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) \sum_{k=2}^{\infty} t_1^k z^{k-1}\, dz \Big) = \operatorname*{Res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) \frac{t_1^2 z}{1-t_1 z}\, dz \Big). \tag{5.61}$$

Similarly, for the function $\bar{f}(t_1) = f(\tilde{\sigma}(t_1)) = \sum_{-\infty<i\le N} \bar{f}_i t_1^i$ we get

$$\begin{aligned} \sum_{k=2}^{N} \bar{f}_k t_1^k &= \operatorname*{Res}_{z=0} \Big(f\big(\tfrac{1}{s(z)}\big) \frac{t_1^2 z}{1-t_1 z}\, dz \Big) \\ &= \operatorname*{Res}_{z=0} \Big(f\big(\tfrac{1}{z}\big) \frac{t_1^2 \sigma(z)}{1-t_1\sigma(z)} \frac{z}{1+z} \frac{1+\sigma(z)}{\sigma(z)}\, dz \Big). \end{aligned} \tag{5.62}$$

We used here the equality

$$\frac{z\, dz}{1+z} = \frac{\sigma(z)\, d\sigma(z)}{1+\sigma(z)}$$

that follows from Equation (5.46).

Combining (5.61) and (5.62) we obtain a residual formula for the odd residueless principal part of the pole of a function:

$$\lfloor f(t_1)/\eta(t_1) \rfloor_1^- = - \operatorname*{Res}_{z=0} \big(K(z,t_1)\, f(1/z) \big)$$

where

$$\begin{aligned} K(z,t_1) &= \frac{1}{2\eta(1/z)} \Big(\frac{t_1^2 z}{1-t_1 z} - \frac{t_1^2 \sigma(z)}{1-t_1\sigma(z)} \frac{z}{1+z} \frac{1+\sigma(z)}{\sigma(z)} \Big)\, dz \\ &= \frac{t_1^2(1+t_1)}{2(1-z\,t_1)(1-\sigma(z)\,t_1)}\ \frac{z\, dz}{z+1}. \end{aligned} \tag{5.63}$$

This, substituted into the recursive formulas of Proposition 5.2.4 and Remark 5.2.5, directly gives Theorem 5.2.2.

5.3 Spectral curve topological recursion / Givental correspondence revisited

In this section we review and reformulate the correspondence between spectral curve topological recursion and Givental theory established in Chapter 3. We use it in the next section to prove the equivalence between the Bouchard-Mariño conjecture and the ELSV formula. This way we obtain a new proof of the ELSV formula, using the new independent proof of the Bouchard-Mariño conjecture from the previous section.

5.3.1 Givental formula

Let H be a Frobenius algebra, that is, a finite-dimensional commutative associative algebra over $\mathbb{C}$ with a unit denoted by $\mathbb{1} \in H$, equipped with a linear function $\ell : H \to \mathbb{C}$ such that the symmetric bilinear form given by $\langle a, b\rangle = \ell(a\, b)$ is non-degenerate. A typical example is the (even part of the) cohomology ring of a complex compact manifold. The dimension of H will be denoted by $N = \dim H$. Fix a basis $e_1, \ldots, e_N$ in H.

Consider also an element of the *Givental upper triangular twisted loop group*, that is, a formal series of the form

$$R(z) = 1 + \sum_{k=1}^{\infty} R_k z^k, \quad R_k \in \mathrm{End}(H),$$

satisfying

$$R(z)\, R^*(-z) = 1.$$

In terms of the Lie algebra element $r(z) = \log(R(z))$, $R(z) = \exp r(z)$, the last relation can be equivalently rewritten as $r(z) + r^*(-z) = 0$.

To this data (a Frobenius algebra and an element R of the upper triangular group) Givental associates a *formal Gromov-Witten potential* F, a formal series in an infinite number of variables $t_{k\nu}$, $k = 0, 1, 2, \ldots$, $\nu = 1, 2, \ldots, N$, and one extra variable $\hbar$, defined by the formula

$$e^{\frac{1}{\hbar}F} = \widehat{R}\, e^{\frac{1}{\hbar}F^{\mathrm{top}}}, \quad \widehat{R} = e^{\widehat{r}}, \tag{5.64}$$

where F^{top} is the potential of the topological field theory associated with the Frobenius algebra H, and $\widehat{r}$ is a second-order differential operator obtained from $r(z)$ by a procedure of 'quantization of quadratic Hamiltonians', see details in [52].

A choice of basis in H is not essential. A change of the basis leads to a linear change of variables in the potential of the form $t_{k\nu} \longrightarrow \sum_{\mu=1}^{N} \Psi^{\mu}_{\nu} t_{k\mu}$ where Ψ is the matrix of the change of basis. In other words, we can treat F as a formal function on $H \oplus H \oplus \ldots$.

It was observed in [51, 65] that the potential F constructed this way is, in fact, a descendant potential of a certain cohomological field theory. Moreover, it is proved in [93] that the descendant potential of any semi-simple cohomological field theory can be represented in such form.

5.3.2 Spectral curve topological recursion

Spectral curve topological recursion is a formal procedure leading to a family of certain differentials $w_{g,n}$ associated with a plane complex curve. They were introduced originally for particular curves in relation to matrix models in mathematical physics, then the procedure was formalized for arbitrary abstract curves (see [16, 18, 17, 4, 3, 48, 85]).

Let $C \subset \mathbb{C}^2$ be a smooth complex curve on the plane with coordinates x, y. Let $a_1, \ldots, a_N \in C$ be the critical points of the coordinate function x. The construction of the differentials $w_{g,n}$ requires the study of the curve in a neighborhood of these points, therefore, it is sufficient to assume that instead of C we have a union of N small discs centered at the points a_i, $i = 1, \ldots, N$, or even the union of formal neighborhoods of these points. Respectively, by a function or differential form (holomorphic or meromorphic) on C we mean a collection of germs of functions or differential forms at the points a_i or even a collection of formal Laurent series at these points.

Assume that each point a_i is a Morse critical point of the function x, that is, x is a ramified covering with a ramification of order 2 at a_i. Let σ be the holomorphic involution on C interchanging the branches of the function x near a_i. In order to simplify notations, for any function or differential form α we denote $\overline{\alpha} = \sigma^*\alpha$. With this notation the involution is given by $\sigma\colon (x,y)\mapsto (x,\overline{y})$. Remark that this bar sign has nothing to do with the complex conjugation in the present context. Remark also that the form $\overline{\alpha}$ is defined in a neighborhood of the point a_i only, even if the form α is globally defined.

On top of that, assume that we are given a 2-*point differential* $B(z_1,z_2)$ (referred to as *Bergman kernel* in some papers), that is, a meromorphic symmetric 2-differential on $C\times C$ representable near $a_i\times a_j\in C\times C$ in the form

$$B(z_1,z_2)=\delta_{i,j}\frac{dz_1^{(i)}dz_2^{(j)}}{(z_1^{(i)}-z_2^{(j)})^2}+B^{(ij)}_{\mathrm{reg}}(z_1^{(i)},z_2^{(j)})$$

where $z^{(i)}$ is a local coordinate on C near a_i and where $B^{(ij)}_{\mathrm{reg}}(z_1^{(i)},z_2^{(j)})$ is holomorphic at $a_i\times a_j$.

The spectral curve n-point functions $w_{g,n}$, $g\geq 0$, $n\geq 1$, are meromorphic n-differentials on $C^{\times n}$ defined inductively by the following formulas:

$$w_{0,1}(z)=0,\qquad w_{0,2}(z_1,z_2)=B(z_1,z_2),$$

and for $2g-2+n>0$,

$$w_{g,n}(z,z_2,\dots,z_n)=-\sum_{i=1}^{N}\operatorname*{res}_{z'=a_i}\left(\frac{\widetilde{w}_{g,n}(z',\bar{z}',z_2,\dots,z_n)}{2\,\mu(z')}\int_{z'}^{\bar{z}'}B(z,\cdot)\right), \tag{5.65}$$

where μ is the 1-form $\mu:=y\,dx-\overline{y}\,dx$ defined in a neighborhood of the union of points a_i, and where

$$\widetilde{w}_{g,n}(z',z'',z_K)=w_{g-1,n+1}(z',z'',z_K)+\sum_{\substack{g_1+g_2=g\\ I\sqcup J=K}}w_{g_1,|I|+1}(z',z_I)\,w_{g_2,|J|+1}(z'',z_J). \tag{5.66}$$

We used here notation $K=\{2,\dots,n\}$, and $u_I=(u_{i_1},\dots,u_{i_{|I|}})$ for any subset $I=\{i_1,\dots,i_{|I|}\}\subset K$.

Remark 5.3.1. We collect here several important remarks clarifying the meaning of all these formulas.

1. Consider the following operator $\alpha\mapsto P\alpha$ acting on the space of meromorphic 1-forms,
$$(P\alpha)(z)=\sum_{i=1}^{N}\operatorname*{res}_{z'=a_i}\left(\frac{\alpha(z')}{2}\int_{z'}^{\bar{z}'}B(z,\cdot)\right).$$
Denote by L the image of this operator. Then *the operator P is the projection to the subspace L*, that is, it is identical on L. The kernel of P is generated by holomorphic and by even (in the sense of local automorphism σ at each critical point) meromorphic 1-forms.

2. It follows that the 1-form in z on the right hand side of (5.65) belongs to L. In other words, the invariants $w_{g,n}$ can be regarded as tensors $w_{g,n}\in L^{\otimes n}$ (for $(g,n)\neq(0,2)$). These tensors are *symmetric* and *polynomial*. The last property means that $w_{g,n}$ belongs to the corresponding tensor product space itself, not just to its completion.

3. The data contained in the collection of invariants $w_{g,n}$ can be collected in a single *potential* $F = \sum \hbar^g F_g$ such that the symmetric tensor $w_{g,n}$ is identified with the nth homogeneous term of the Taylor expansion of F_g,

$$w_{g,n} = \sum_{\alpha_1,\dots,\alpha_n} \frac{\partial^n F_g}{\partial t_{\alpha_1} \dots \partial t_{\alpha_n}} \Big|_{t=0} d\xi_{\alpha_1} \otimes \dots \otimes d\xi_{\alpha_n}.$$

Here $\{d\xi_a\}_{\alpha\in\mathcal{A}}$ is some chosen basis in L, and $t = \{t_\alpha\}_{\alpha\in\mathcal{A}}$ is the set of formal variables labeled by the same set of indices. The coordinate expression of the potential F depends on a choice of the basis in L. A different choice of the basis leads to the corresponding linear change of coordinates in F. In other words, F can be regarded as a formal function on the infinite dimensional space L^*; with this treatment of the potential, it is invariantly defined and independent of any basis.

4. The dual space $V = L^*$ can be identified with the space of *odd holomorphic* 1-forms defined in a neighborhood of the union of points a_i. The pairing is given by

$$(\alpha, \beta) = \sum_{\nu=1}^{N} \operatorname*{res}_{z=a_\nu} (\alpha \int \beta), \quad \alpha \in L. \quad \beta \in V.$$

If $\{d\xi_\alpha\}_{\alpha\in\mathcal{A}}$ is any basis in L and $\{d\xi^\alpha\}_{\alpha\in\mathcal{A}}$ is the dual basis in $V = L^*$, then there is an asymptotic expansion

$$\frac{1}{2}(B(z,w) - B(z,\overline{w})) = \sum_{\alpha\in\mathcal{A}} d\xi_\alpha(z) d\xi^\alpha(w).$$

This expansion takes place as $w \to a_i$, $|w - w(a_i)| \ll |z - z(a_i)|$.

5. It follows, in particular, that the subspace L is spanned by the coefficients of the Taylor expansion of the antisymmetrized Bergman kernel $\frac{1}{2}(B(z,w) - B(z,\overline{w}))$ with respect to the second argument w at the points a_i.

5.3.3 Givental action as spectral curve topological recursion

Here we formulate in a refined way the result of Chapter 3 in the case $N = 1$.

Let C be a curve on the (x, y)-plane as above. Consider the following operator acting in the space of meromorphic 1-forms,

$$\mathcal{D} : \alpha \mapsto d\Big(\frac{\alpha}{dx}\Big).$$

This operator commutes with the action of the involution σ, $\mathcal{D}\overline{\alpha} = \overline{\mathcal{D}\alpha}$. Set

$$d\xi^k := \mathcal{D}^{-k} dy, \quad k = 0, 1, 2, \dots.$$

The forms $d\xi^k$ are holomorphic in a neighborhood of the point a_1. There is an ambiguity in the choice of integration constants appearing in the inversion of D. Different choices of these constants lead to forms that differ by a holomorphic and *even* (with respect to the involution σ) summand. It follows that the odd parts of these forms

$$\frac{1}{2}\big(d\xi^k - d\overline{\xi}^k\big), \quad k = 0, 1, 2, \dots$$

are independent of any choice. Moreover, these odd forms form a basis in the space of odd holomorphic forms. Let us take the antisymmetrized Bergman kernel $\frac{1}{2}\big(B(z,w) - B(z,\overline{w})\big)$, develop it over the obtained basis, and denote by $d\xi_k$ the coefficients of this expansion:

$$\frac{1}{2}\big(B(z,w) - B(z,\overline{w})\big) = \sum_{k=0}^{\infty} d\xi_k(z) \frac{d\xi^k(w) - d\overline{\xi}^k(w)}{2}. \tag{5.67}$$

This asymptotic expansion takes place as $w \to 0$, $|w| \ll |z|$ where z is a local holomorphic coordinate on C near the point a_1. The form $d\xi_k$ defined by this expansion is meromorphic with a pole of order $2k+1$ at $z = 0$.

Definition 5.3.2. The Bergman kernel is said to be *compatible* with the operator $\mathcal{D}$ if the introduced meromorphic forms $d\xi_k$ are given explicitly by $d\xi_k = (-1)^{k+1}\mathcal{D}^{k+1}d\xi^0$.

The following criterion simplifies the verification of the compatibility condition.

Lemma 5.3.3. *Assume that the Bergman kernel satisfies the identity*

$$(\mathcal{D}_z + \mathcal{D}_w)B(z,w) = -\mathcal{D}_z d\xi^0(z)\, \mathcal{D}_w d\xi^0(w).$$

Then it is compatible with $\mathcal{D}$.

Proof. Applying the expansion (5.67) we get

$$\begin{aligned} 0 &= (\mathcal{D}_z + \mathcal{D}_w)\frac{B(z,w) - B(z,\overline{w})}{2} + \mathcal{D}_z d\xi^0(z)\, \mathcal{D}_w \frac{d\xi^0(w) - d\overline{\xi}^0(w)}{2} \\ &= \sum_{k=0}^{\infty} (\mathcal{D}_z d\xi_k(z) + d\xi_{k+1}(z)) \frac{d\xi^k(w) - d\overline{\xi}^k(w)}{2} \\ &\quad + (\mathcal{D}_z d\xi^0(z) + d\xi_0(z))\, \mathcal{D}_w \frac{d\xi^0(w) - d\overline{\xi}^0(w)}{2}. \end{aligned}$$

This equality is equivalent to the system of equations $d\xi_0 = -\mathcal{D}d\xi^0$, $d\xi_{k+1} = -\mathcal{D}d\xi_k$, that is, $d\xi_k = (-1)^{k+1}\mathcal{D}^{k+1}d\xi^0$, as required. □

Now, assume that the Bergman kernel is compatible with $\mathcal{D}$. Introduce the local coordinate s on the curve near the point a_1 from the relation $dx = s\,ds$, that is,

$$s = \sqrt{2(x - x(a_1))}.$$

This coordinate is defined up to a sign, and the involution in this coordinate is given simply by $\overline{s} = -s$. Consider the expansion of the odd part of the form dy in this coordinate,

$$\frac{1}{2}(dy - d\overline{y}) = ds + \sum_{k=1}^{\infty} R_k \frac{s^{2k}\, ds}{(2k-1)!!}. \tag{5.68}$$

We can now formulate the main result of Chapter 3 for the case of $N = 1$.

Theorem 5.3.4. *If the Bergman kernel is compatible with the operator $\mathcal{D}$, then the spectral curve n-point functions are the n-point correlator functions of a certain formal GW potential $F(t_0, t_1, \dots) = \sum \hbar^g F_g$,*

$$w_{g,n} = \sum_{k_1,\dots,k_n} \frac{\partial^n F_g}{\partial t_{k_1} \dots dt_{k_n}} \Big|_{t=0} d\xi_{k_1} \otimes \dots \otimes d\xi_{k_n}.$$

Moreover, this GW potential is given by the Givental formula (5.64) *with the Witten-Kontsevich potential for F^{top} and with the element $R(z) = 1 + R_1 z + R_2 z^2 + \dots$ of the upper triangular group whose components R_k are determined by the expansion* (5.68).

5.4 New proof of the ELSV formula

In the present section we prove the equivalence of the Bouchard-Mariño formula and the ELSV formula with the help of the Givental-topological recursion correspondence reviewed in the previous section. Note that this equivalence was already proved by Eynard in [36], see also [89].

From this equivalence, using our new proof of the Bouchard-Mariño conjecture (Theorem 5.2.2), we obtain a new proof of the ELSV formula.

5.4.1 Hodge class

The total Hodge class $\Lambda_g = 1 - \lambda_1 + \dots + (-1)^g \lambda_g \in H^*(\mathcal{M}_{g,n})$ provides the simplest non-trivial example of a cohomological field theory (of dimension $N = 1$). It follows that its potential, the generating function for Hodge integrals,

$$F(\hbar, t_0, t_1, \dots) = \sum_{g,n} \frac{\hbar^g}{n!} \sum_{k_1,\dots,k_n} \int_{\mathcal{M}_{g,n}} \Lambda_g \, \psi_1^{k_1} \dots \psi_n^{k_n} \; t_{k_1} \dots t_{k_n}$$

is a formal GW potential. Indeed, Mumford's formula [80] for the Chern characters of the Hodge bundle rewritten in terms of intersection numbers has exactly the form (5.64) with the Witten-Kontsevich potential for the series F^{top} and the following element of the upper triangular group

$$R(z) = \exp\left(\sum_{n=1}^{\infty} \frac{B_{2n}}{2n\,(2n-1)} z^{2n-1}\right) = 1 + \frac{1}{12} z + \frac{1}{288} z^2 - \frac{139}{51840} z^3 + \dots, \tag{5.69}$$

where B_n is the nth Bernoulli number. The operator $\widehat{R} = \exp\left(\sum_{n=1}^{\infty} \frac{B_{2k}}{2n\,(2n-1)} \widehat{z^{2n-1}}\right)$ corresponding to this element acts by

$$\widehat{z^{2n-1}} = -\frac{\partial}{\partial t_{2n}} + \sum_{i=0}^{\infty} t_i \frac{\partial}{\partial t_{i+2n-1}} - \frac{1}{2} \sum_{i+j=2n-2} (-1)^i \frac{\partial^2}{\partial t_i \mathrm{d} t_j}.$$

In the definition of this operator, we use a convention which differs by the sign from that of the original paper [52].

5.4.2 BM-ELSV equivalence

Consider the Lambert curve (5.43)

$$\tilde{x} = \tilde{y} - \log(1+\tilde{y}), \qquad d\tilde{x} = \frac{\tilde{y}\, d\tilde{y}}{1+\tilde{y}},$$

which is given here in logarithmic coordinates

$$\tilde{x} = -1 - \log x,$$
$$\tilde{y} = -1 + y.$$

For this curve, the standard Bergman kernel $B(\tilde{y}_1, \tilde{y}_2) = \frac{d\tilde{y}_1 d\tilde{y}_2}{(\tilde{y}_1 - \tilde{y}_2)^2}$ is compatible with the operator $\mathcal{D}$. Indeed, we have

$$\begin{aligned}(\mathcal{D}_{\tilde{y}_1} + \mathcal{D}_{\tilde{y}_2}) \frac{d\tilde{y}_1 d\tilde{y}_2}{(\tilde{y}_1 - \tilde{y}_2)^2} &= d_{\tilde{y}_1} \frac{(1+\tilde{y}_1)\, d\tilde{y}_2}{\tilde{y}_1(\tilde{y}_1 - \tilde{y}_2)^2} + d_{\tilde{y}_2} \frac{(1+\tilde{y}_2)\, d\tilde{y}_1}{\tilde{y}_2(\tilde{y}_1 - \tilde{y}_2)^2} \\ &= -\frac{d\tilde{y}_1 d\tilde{y}_2}{\tilde{y}_1^2 \tilde{y}_2^2} \\ &= -\mathcal{D}_{\tilde{y}_1} d\tilde{y}_1\ \mathcal{D}_{\tilde{y}_2} d\tilde{y}_2.\end{aligned}$$

Therefore, by Lemma 5.3.3 and Theorem 5.3.4, the spectral curve n-point functions in this case are the correlation functions of a certain formal GW potential. Moreover, this potential is obtained from the Kontsevich-Witten potential by the action of the element $R(z) = 1 + \sum R_k z^k$ of the Givental group whose coefficients are determined by the expansion

$$\frac{d}{ds} \frac{\tilde{y}(s) - \tilde{y}(-s)}{2} = 1 + \sum_{k=1}^{\infty} R_k \frac{s^{2k}}{(2k-1)!!},$$

where the function $\tilde{y}(s)$ is given by the implicit equation

$$s = \sqrt{2\,(\tilde{y} - \log(1+\tilde{y}))}.$$

It is proved in [13] that these coefficients are the same as those given by the expansion (5.69).

This means that for our spectral curve we have

$$\begin{aligned} w_{g,n} &= \sum_{k_1,\dots,k_n} \frac{\partial^n F_g}{\partial t_{k_1} \dots \mathrm{d}t_{k_n}} \Big|_{t=0} (d\xi_{k_1})_1 \dots (d\xi_{k_n})_n \\ &= \sum_{k_1,\dots,k_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \Lambda_g \psi_1^{k_1} \dots \psi_n^{k_n} \right) \prod_{i=1}^n \sum_{\mu_i=1}^{\infty} \frac{\mu_i^{\mu_i+k_i+1}}{\mu_i!} x_i^{\mu_i-1} dx_i \\ &= \sum_{\mu_1,\dots,\mu_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g}{\prod_{i=1}^n (1-\mu_i\psi_i)} \right) \prod_{i=1}^n \frac{\mu_i^{\mu_i+1}}{\mu_i!} x_i^{\mu_i-1} dx_i. \end{aligned} \tag{5.70}$$

Here we used the fact that in our case

$$\begin{aligned} d\xi_k &= (-1)^{k+1} \mathcal{D}^{k+1} d\tilde{y} = d\left(\left(x\frac{d}{dx} \right)^{k+1} y \right) \\ &= d\left(\left(x\frac{d}{dx} \right)^{k+1} \sum_{\mu=1}^{\infty} \frac{\mu^{\mu-1}}{\mu!} x^{\mu} \right) = \sum_{\mu=1}^{\infty} \frac{\mu^{\mu+k+1}}{\mu!} x^{\mu-1} dx. \end{aligned}$$

Note that the Bouchard-Mariño conjecture may be written as

$$w_{g,n} = \sum_{\mu_1,\dots,\mu_n} \frac{h^\circ_{g;\mu_1,\dots,\mu_n}}{\mathfrak{b}(g,\mu)!}\, \mu_1 \dots \mu_n\, x_1^{\mu_1-1} \dots x_n^{\mu_n-1} dx_1 \dots dx_n, \tag{5.71}$$

while the ELSV formula states that

$$h^\circ_{g;\mu_1,\dots,\mu_n} = \mathfrak{b}(g,\mu)! \left(\int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda_g}{\prod_{i=1}^n (1-\mu_i\psi_i)} \right) \prod_{i=1}^n \frac{\mu_i^{\mu_i}}{\mu_i!}. \tag{5.72}$$

We immediately see that formula (5.70) directly implies the following

Theorem 5.4.1. *The Bouchard-Mariño conjecture and the ELSV formula are equivalent.*

This means that we have a new proof of the ELSV formula, since we proved the Bouchard-Mariño conjecture independently in Section 5.2. Note that the Bouchard-Mariño conjecture as given in Theorem 5.2.2 is equivalent to formula (5.71), if one takes into account the topological recursion formula for $w_{g,n}$, given by Equations (5.65) and (5.66).

Chapter 6

Combinatorics of loop equations for branched covers of sphere

This chapter is based on paper [102], joint work with N. Orantin, A. Popolitov and S. Shadrin. In this chapter we prove the spectral curve topological recursion for the problem of enumeration of bi-colored maps.

The chapter is organized as follows.

In Section 6.1 we recall the definitions of hypermaps and discuss generating functions corresponding to hypermap enumeration problems. In Section 6.2 we reformulate the definition of hypermaps in terms of bi-colored maps and discuss the 2-matrix model which gives rise to enumeration of bi-colored maps. In Section 6.3 we recall the form of the loop equations for the 2-matrix model and then we show that using purely combinatorial argument to prove the basic building blocks of loop equations, we can obtain a purely combinatorial proof of the spectral curve for the enumeration of bi-colored maps In Section 6.4 we review the problem of finding the quantum curve for enumeration of hypermaps. In Section 6.5 we outline the proof of the spectral curve topological recursion for the even further generalization of our problem: the case of 4-colored maps, which corresponds to 4-matrix models.

6.1 Branched covers of $\mathbb{P}^1$

6.1.1 Definitions

We are interested in the enumeration of covers of $\mathbb{P}^1$ branched over three points. These covers are defined as follows.

Definition 6.1.1. Consider m positive integers $a_1, \dots, a_m$ and n positive integers $b_1, \dots, b_n$. We denote by $\mathcal{M}_{g,m,n}(a_1, \dots, a_m | b_1, \dots, b_n)$ the weighted count of branched covers of $\mathbb{P}^1$ by a genus g surface with $m+n$ marked points $f\colon (\mathcal{S}; q_1, \dots, q_m; p_1, \dots, p_n) \to \mathbb{P}^1$ such that

- f is unramified over $\mathbb{P}^1 \backslash \{0, 1, \infty\}$;
- the preimage divisor $f^{-1}(\infty)$ is $a_1 q_1 + \dots a_m q_m$;
- the preimage divisor $f^{-1}(1)$ is $b_1 p_1 + \dots b_n p_n$;

Of course, a cover f can exist only if $a_1+\cdots+a_m = b_1+\cdots+b_n$. In this case $d = b_1+\cdots+b_n$ is called the degree of a cover.

These covers are counted up to isomorphisms preserving the marked points $p_1,\ldots,p_n$ pointwise and covering the identity on $\mathbb{P}^1$. The weight of a cover is equal to the inverse order of its automorphism group.

Example 6.1.2. In [23] the authors consider the case of

$$\mathcal{M}_{g,d/a,n}(a,\ldots,a|b_1,\ldots,b_n),$$

and relate this enumeration problem to the existence of a quantum curve.

Since such a branched cover can be recovered just from its monodromy around 0, 1 and ∞, it is convenient to reformulate this enumeration problem in different terms.

Definition 6.1.3. Let us fix $d \geq 1$, $g \geq 0$, $m \geq 1$, and $n \geq 1$. A hypermap of type (g,m,n) is a triple of permutations $(\sigma_0,\sigma_1,\sigma_\infty) \in S_d^3$ such that

- $\sigma_0\sigma_1\sigma_\infty = Id$;
- σ_1 is composed of n cycles;
- σ_∞ is composed of m cycles.

A hypermap is called *connected* if the permutations σ_0, σ_1, σ_∞ generate a transitive subgroup of S_d. A hypermap is called *labelled* if the disjoint cycles of σ_1 are labelled from 1 to n.

Two hypermaps $(\sigma_0,\sigma_1,\sigma_\infty)$ and $(\tau_0,\tau_1,\tau_\infty)$ are equivalent if one can conjugate all the σ_i's to obtain the τ_i's. Two labelled hypermaps are equivalent if in addition the conjugation preserves the labelling.

By Riemann existence theorem, one has

Lemma 6.1.4. *The number $\mathcal{M}_{g,m,n}(a_1,\ldots,a_m|b_1,\ldots,b_n)$ is equal to the weighted count of labelled hypermaps of type (g,m,n) where the cycles of σ_∞ have lengths $a_1,\ldots,a_m$ and the cycles of σ_1 have length $b_1,\ldots,b_n$. Here the weight of a labelled hypermap is the inverse order of its automorphism group.*

6.1.2 Generating functions

In order to compute these numbers, it is very useful to collect them in generating functions. For this purpose, we define:

Definition 6.1.5. Let us fix integer $g \geq 0$ and $n \geq 1$ such that $2g-2+n > 0$. We also fix one more integer $a \geq 1$ that will be used to restrict the possible length of cycle in σ_∞.

The n-point correlation function is defined by

$$\Omega^{(a)}_{g,n}(x_1,\ldots,x_n) := \sum_{m=0}^{\infty} \sum_{\substack{1\leq a_1,\ldots,a_m\leq a \\ 0\leq b_1,\ldots,b_n}} \mathcal{M}_{g,m,n}(a_1,\ldots,a_m|b_1,\ldots,b_n) \prod_{i=1}^{m} t_{a_i} \prod_{j=1}^{n} b_j x_i^{-b_j-1}. \quad (6.1)$$

It is a function of the variables $x_1,\ldots,x_n$ that depends on formal parameters $t_1,\ldots,t_a$.

Remark 6.1.6. Note that the product

$$\mathcal{M}_{g,m,n}(a_1,\dots,a_m|b_1,\dots,b_n)\ \prod_{j=1}^{n} b_j$$

counts the same covers as in Definition 6.1.1, but with an additional choice, for each i, of one of the possible b_i preimages of a path from 1 to 0 starting at point p_i.

For later convenience in the definition of the quantum curve, we define the symmetric counterpart of the n-point correlation function by (for $(g,n)\neq(0,1)$)

$$\mathcal{F}^{(a)}_{g,n}(x) := \int^x \dots \int^x \Omega^{(a)}_{g,n}(x_1,\dots,x_n)dx_1\dots dx_n \tag{6.2}$$

The special case $(g,n)=(0,1)$, as usual, includes a logarithmic term:

$$\mathcal{F}^{(a)}_{0,1}(x) := \log(x) + \int^x \Omega^{(a)}_{0,1}(x_1)dx_1 \tag{6.3}$$

Then we define the wave function by

$$Z^{(a)}(x,\hbar) := \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g+n-2}}{n!}\mathcal{F}^{(a)}_{g,n}(x)\right]. \tag{6.4}$$

Remark 6.1.7. Note that in Do and Manescu's paper [23] a different definition of $F_{g,n}$ was used, differing by $(-1)^n$, which leads to a different definition of $Z^{(a)}$, and, in turn, to a slightly different quantum spectral curve equation. See more on this in Section 6.4.

6.2 Maps and matrix models

In the present section we discuss the definition of bi-colored maps and a formal matrix model argument on the existence of a topological recursion for them. However, it is possible to prove the topological recursion in a purely combinatorial way without even mentioning any formal matrix model integral representation of the problem[1] . This is done in the subsequent section, Section 6.3.

6.2.1 Covers branched over 3 points and maps

There exists a natural graphical representation of hypermaps (which generalizes the notion of dessin d'enfant [69] and the construction of [23]).

Let us now describe how to associate a colored map[2] to any labelled hypermap.

[1]It is important to note that the combinatorial derivation is equivalent to the matrix model derivation as it is only a translation of the latter in purely combinatorial arguments. However, it might be easier to understand for some readers not used to the matrix model formalism.

[2]In the following, when referring to a map, we refer to a combinatorial object corresponding to a polygonalisation of a surface. These objects appear naturally in the literature in the context of random matrices and were introduced in physics as part of various attempts to quantize gravity in 2 dimensions and to approach string theory from a discrete point of view.

Each independent cycle ρ_i in the decomposition of $\sigma_1 = \rho_1\rho_2\dots\rho_n$ is represented by a black $|\rho_i|$-gon whose corners are cyclically ordered and labelled by the numbers composing ρ_i. We glue these black polygons by their corners following σ_0. Namely, for each disjoint cycle $\rho = (\alpha_1, \dots, \alpha_k)$ of σ_0, one attachs the corners of black faces labelled by $\alpha_1, \dots \alpha_k$ to a $2k$-valent vertex such that:

- Turning around the vertex, one encounters alternatively white and black sectors (k of each) separated by the edges adjacent to the vertex;
- when turning counterclockwise around the vertex starting from the corner labelled by α_1, the labels of the corner corresponding to the black sectors adjacent to the vertex form the sequence $\alpha_1, \alpha_2, \dots, \alpha_k$.

Example 6.2.1. Let us give an example of a bi-colored map. Consider a hypermap corresponding to $d = 7$, $g = 0$,

$$\begin{aligned}
\sigma_0 &= (1,5,7)(4,6), \\
\sigma_1 &= (1,2,3,4)(5,6,7), \\
\sigma_\infty &= (1,6,3,2)(4,5)
\end{aligned}$$

Then the corresponding bi-colored map can be seen in Figure 6.1.

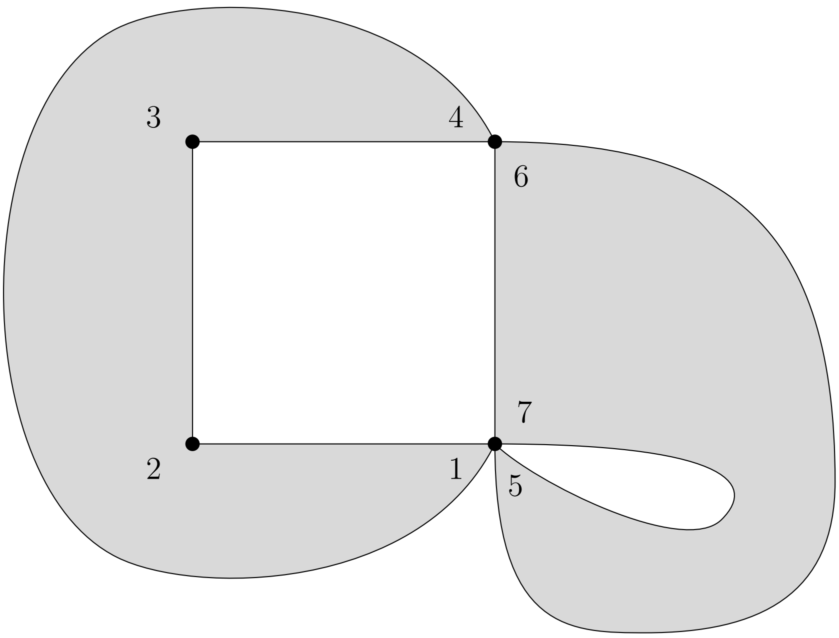


Figure 6.1: Bi-colored map

In this figure we see two black polygons corresponding to cycles $(1,2,3,4)$ and $(5,6,7)$ of σ_1; they are glued according to σ_0.

Let us fix $a \geq 1$. We denote by $G^{(a)}_{g,m,n}$ the set of bi-colored maps, where m is the number of white polygons, n is the number of black polygons, and g is the genus of the surface we get by gluing the polygons and a is the maximum perimeter of a white polygon. We assume that the black polygons are labelled, and we consider the maps up to combinatorial isomorphisms preserving this labelling. For a particular map $M \in G^{(a)}_{g,m,n}$ we denote by $\mathrm{Aut}(M)$ its automorphism group.

One can restate the problem of enumerating covers of $\mathbb{P}^1$ as counting bi-colored maps as follows.

Lemma 6.2.2. *The function $\Omega^{(a)}_{g,n}(x_1,\dots,x_n)$ is the generating function of bi-colored maps with an arbitrary number $m \geq 1$ of white faces whose perimeters are less or equal to a and n marked black faces with perimeters $b_1,\dots,b_n$. That is,*

$$\Omega^{(a)}_{g,n}(x_1,\dots,x_n) = \sum_{m=1}^{\infty} \sum_{M\in G^{(a)}_{g,m,n}} \frac{\prod_{i=1}^{a} t_i^{n_i(M)}}{|\mathrm{Aut}(M)|} \prod_{j=1}^{n} b_j(M) x_j^{-b_j(M)-1}. \tag{6.5}$$

Here by $n_i(M)$ we denote the number of white polygons of perimeter i in M, and $b_1(M),\dots,b_n(M)$ are the perimeters of the black polygons in M.

6.2.2 Matrix model and topological recursion

The enumeration of bi-colored maps is a classical problem of random matrix theory which is equivalent to the computation of formal matrix integrals. One can state this equivalence in the following way.

Lemma 6.2.3. *(see, e. g. [35]) Consider the partition function of a formal Hermitian two-matrix model*

$$\mathcal{Z}\left(\vec{t}^{(1)},\vec{t}^{(2)}\right) := \int_{H_N}^{formal} dM_1\, dM_2\, e^{-N[\mathrm{Tr}(M_1M_2)-\mathrm{Tr}\,V_1(M_1)-\mathrm{Tr}\,V_2(M_2)]} \tag{6.6}$$

where the potentials $V_i(x)$, $i=1,2$, are polynomials of degree d_i,

$$V_i(x) = \sum_{d=1}^{d_i} \frac{t_d^{(i)}}{d} x^d. \tag{6.7}$$

This partition function is a generating function of bi-colored maps, that is,

$$\mathcal{Z}\left(\vec{t}^{(1)},\vec{t}^{(2)}\right) = \sum_{g,m,n=0}^{\infty} \sum_{M\in\mathcal{S}^{\bullet}_{g,m,n}} \frac{\prod_{i=1}^{d_1}\left[t_i^{(1)}\right]^{n_i^{(1)}(M)} \prod_{i=1}^{d_2}\left[t_i^{(2)}\right]^{n_i^{(2)}(M)}}{|\mathrm{Aut}(M)|} \tag{6.8}$$

where

- *$\mathcal{S}^{\bullet}_{g,m,n}$ is the set of bi-colored maps, possibly disconnected, of genus g composed of n black polygons and m white polygons glued by their edges, such that black polygons are glued only to white polygons and vice versa. Neither black nor white polygons are marked.*

- *By $n_i^{(1)}(M)$ (resp. $n_i^{(2)}(M)$) we denote the number of black (resp. white) polygons of perimeter i in M;*

It is also possible to enumerate connected maps with some specific marked faces by computing certain correlation functions of this formal matrix model.

Definition 6.2.4. For any set of words (non-commutative monomials) $\{f_i(x,y)\}_{i=1}^s$ in two variables, we define the correlator of the formal matrix model by

$$\left\langle \prod_{i=1}^s \operatorname{Tr} f_i(M_1,M_2) \right\rangle := \frac{\int_{H_N}^{formal} d\mu_N(M_1,M_2) \prod_{i=1}^s \operatorname{Tr} f_i(M_1,M_2)}{\mathcal{Z}\left(\vec{t}^{(1)},\vec{t}^{(2)}\right)},$$

where the measure of integration $\mu(M_1,M_2)$ is the same as before,

$$d\mu_N(M_1,M_2) := dM_1\, dM_2\, e^{-N[\operatorname{Tr}(M_1M_2) - \operatorname{Tr} V_1(M_1) - \operatorname{Tr} V_2(M_2)]}.$$

We denote by $\left\langle \prod_{i=1}^s \operatorname{Tr} f_i(M_1,M_2) \right\rangle_c$ its connected part.

In matrix models, one classically works with generating series of such correlators (named correlation functions) defined by

$$W_{k,l}(x_1,\ldots,x_k;y_1,\ldots,y_l) := \left\langle \prod_{i=1}^k \operatorname{Tr}\frac{1}{x_i - M_1} \prod_{j=1}^l \operatorname{Tr}\frac{1}{y_j - M_2} \right\rangle_c .$$

These correlation functions have to be understood as series expansions around $x_i, y_i \to \infty$:

$$W_{k,l}(x_1,\ldots,x_k;y_1,\ldots,y_l) := \sum_{\vec{n}\in\mathbb{N}^k}\sum_{\vec{m}\in\mathbb{N}^l} \left\langle \prod_{i=1}^k \frac{\operatorname{Tr} M_1^{n_i}}{x_i^{n_i+1}} \prod_{j=1}^l \frac{\operatorname{Tr} M_2^{m_j}}{y_j^{m_j+1}} \right\rangle_c . \tag{6.9}$$

These correlation functions admit a topological expansion, i. e. they can be written as

$$W_{k,l}(x_1,\ldots,x_k;y_1,\ldots,y_l) = \sum_{g=0}^{\infty} N^{2-2g-k-l} W_{k,l}^{(g)}(x_1,\ldots,x_k;y_1,\ldots,y_l)$$

where each of $W_{k,l}^{(g)}$ does not depend on N.

With this notation,

$$W_{k,l}^{(g)}(x_1,\ldots,x_k;y_1,\ldots,y_l) = \sum_{m,n=0}^{\infty} \sum_{\substack{\vec{\alpha}\in\mathbb{N}^k \\ \vec{\beta}\in\mathbb{N}^l}} \sum_{M\in\mathcal{S}^\circ_{g,m,n|\vec{\alpha},\vec{\beta}}} \frac{\prod_{i=1}^{d_1}\left[t_i^{(1)}\right]^{n_i^{(1)}(M)} \prod_{i=1}^{d_2}\left[t_i^{(2)}\right]^{n_i^{(2)}(M)}}{|\operatorname{Aut}(M)| \prod_{i=1}^k x_i^{\alpha_i+1} \prod_{j=1}^l y_j^{\beta_j+1}},$$

where $\mathcal{S}^\circ_{g,m,n|\vec{\alpha},\vec{\beta}}$ is the set of connected bi-colored maps of genus g composed of n unmarked black faces, m unmarked white faces, k marked black faces of perimeters $\alpha_1,\ldots,\alpha_k$, each having one marked edge, and l marked white faces of perimeter $\beta_1, \ldots, \beta_n$, each having one marked edge too; black faces are only glued to white faces and vice versa, as above.

Such a model admits a spectral curve. This means that there exists a polynomial $P(x, y)$ of degree $d_1 - 1$ in x and $d_2 - 1$ in y such that the generating function for discs $W_{1,0}^{(0)}(x)$ satisfies an algebraic equation:

$$E_{2MM}(x, W_{1,0}^{(0)}(x)) = 0, \qquad x \in \mathbb{C},$$

where

$$E_{2MM}(x, y) = (V_1'(x) - y)(V_2'(y) - x) - P(x, y) + 1.$$

In [47, 18], it was proved that the correlation functions $W_{k,0}^{(g)}$ can be computed by topological recursion on this spectral curve:

Theorem 6.2.5. *[47, 18] The correlation functions of the 2-matrix models can be computed by the topological recursion procedure of [48] with the genus* 0 *spectral curve*

$$E_{2MM}(x, y) = (V_1'(x) - y)(V_2'(y) - x) - P(x, y) + 1$$

and the genus 0*, 2-point function defined by the bilinear differential*

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

for a global coordinate z on the spectral curve.

The proof of this theorem consists in three steps:

- First, find a set of equations satisfied by the correlation functions of the matrix model.
- Second, show that these equations admit a unique solution admitting a topological expansion.
- Third, exhibit a solution which immediately implies the topological recursion.

6.2.3 A matrix model for branched covers

Since the problem of enumerating branched covers can be rephrased in terms of bi-colored maps, one can find a matrix model representation for it.

Using the definition of the preceding section together with the hypermap representation of section 6.2.1, one immediately finds that

Lemma 6.2.6. *The correlation functions of the formal two matrix model with potentials $V_1(x) = 0$ and $V_2(x) = \sum_{i=1}^{a} \frac{t_i}{i} x^i$ coincide with the generating series of covers of $\mathbb{P}^1$ branched over 3 points defined in* (6.5)*, for $(g, k) \neq (0, 1)$:*

$$W_{k,0}^{(g)}(x_1, \ldots, x_k) = \Omega_{g,k}^{(a)}(x_1, \ldots, x_k). \tag{6.10}$$

For $(g, k) = (0, 1)$ we have

$$W_{1,0}^{(0)}(x_1) = \frac{1}{x} + \Omega_{0,1}^{(a)}(x_1). \tag{6.11}$$

Applying [48, 18], one can thus compute the generating series using topological recursion. We have:

Corollary 6.2.7. *The generating series $\Omega^{(a)}_{g,k}(x_1,\dots,x_k)$ can be computed by topological recursion with a genus 0 spectral curve*

$$E^{(a)}(x,y) = y\left(\sum_{i=1}^{a} t_i y^{i-1} - x\right) + 1 = 0 \tag{6.12}$$

and the genus 0 2-point function defined by the corresponding Bergmann kernel, i. e.

$$\omega_{0,2}(z_1,z_2) = \frac{dz_1 \otimes dz_2}{(z_1-z_2)^2} \tag{6.13}$$

for a global coordinate z on the genus 0 spectral curve.

Remark 6.2.8. This corollary proves in particular the conjecture made by Do and Manescu [23] considering such covers with only type a ramifications above 1. The spectral curve is indeed,

$$E^{(a)}(x,y) = y\left(y^{a-1} - x\right) + 1 = 0 \tag{6.14}$$

coinciding with the classical limit of their quantum curve.

6.3 Loop equations and combinatorics

The proof of Corollary 6.2.7 relies on the representation of our combinatorial under the form of a formal matrix integral. Actually, the only input from the formal matrix model is the existence of loop equations satisfied by the correlation functions of the model. These loop equations are of combinatorial nature and should reflect some cut-and-join procedure satisfied by the hypermaps enumerated. However, a simple combinatorial interpretation of these precise 2-matrix model loop equations could not be found in the literature, even if some similar and probably equivalent equations have been derived combinatorially in some particular cases [9, 95]. In this section, we derive such an interpretation, allowing to bypass the necessity to use any integral (matrix model) representation and thus getting a completely combinatorial proof of the results of the preceding section.

Remark 6.3.1. We have been informed that such a direct derivation of the loop equations for the 2-matrix model is performed in chapter 8 [34] which is in preparation and whose preliminary version can be found online.

6.3.1 Loop equations

In order to produce the hierarchy of loop equations whose solution gives rise to the topological recursion, one combines two set of equations which can be written as follows:

- The first one corresponds to the change of variable

$$M_2 \to M_2 + \epsilon \frac{1}{x - M_1} \prod_{i=1}^{n} \mathrm{Tr}\,\frac{1}{x_i - M_1}$$

in the formal matrix integral defining the partition function. To first order in ϵ, the compensation of the Jacobian (which is vanishing here) with the variation of the action gives rise to the equation:

$$\left\langle \operatorname{Tr}\left(\frac{M_1}{x-M_1}\right)\prod_{i=1}^{n}\operatorname{Tr}\frac{1}{x_i-M_1}\right\rangle \tag{6.15}$$

$$=\left\langle \operatorname{Tr}\left(\frac{1}{x-M_1}V_2'(M_2)\right)\prod_{i=1}^{n}\operatorname{Tr}\frac{1}{x_i-M_1}\right\rangle$$

- The second one corresponds to the change of variable

$$M_1 \to M_1 + \epsilon\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\prod_{i=1}^{n}\operatorname{Tr}\frac{1}{x_i-M_1} \tag{6.16}$$

and reads

$$\begin{aligned}
&\left\langle \operatorname{Tr}\left(\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}M_2\right)\prod_{i=1}^{n}\operatorname{Tr}\frac{1}{x_i-M_1}\right\rangle \qquad (6.17)\\
&=\left\langle \operatorname{Tr}\left(\frac{V_1'(M_1)}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{i=1}^{n}\operatorname{Tr}\frac{1}{x_i-M_1}\right\rangle\\
&+\frac{1}{N}\left\langle \operatorname{Tr}\left(\frac{1}{x-M_1}\right)\operatorname{Tr}\left(\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{i=1}^{n}\operatorname{Tr}\frac{1}{x_i-M_1}\right\rangle\\
&+\frac{1}{N}\sum_{i=1}^{n}\left\langle \operatorname{Tr}\left(\frac{1}{(x_i-M_1)^2}\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{j\neq i}\operatorname{Tr}\frac{1}{x_j-M_1}\right\rangle
\end{aligned}$$

Note that in these equations the correlators are not the connected ones, but they are generating functions of possibly disconnected maps of arbitrary genus.

6.3.2 Combinatorial interpretation

The loop equations (6.15), (6.17) make sense only in their $x, x_i, y \to \infty$ series expansions. These expansions generate a set of equations for the correlators of the matrix models which can be interpreted as relations between the number of bi-colored maps with different boundary conditions. In this section, we give a combinatorial derivation of these relations.

Definition of boundary conditions

In order to derive the loop equations, we have to deal with bi-colored maps with boundaries (or marked faces) of general type. A map with n boundaries is a map with n marked faces (polygons), each carrying a marked edge. The boundary conditions are defined as the color of the marked face. However, in the following, we need to introduce mixed-type boundary conditions obtained by considering marked faces with sides colored differently.

The boundary conditions of a marked face (or a boundary) are then given by the sequence of colors of the edges of this face starting from the marked edge and going clockwise from it.

Considering a set of n sequences of non-negative integers

$$S_i = b_{i,1}, a_{i,1}, b_{i,2}, a_{i,2} \dots b_{i,l_i}, a_{i,l_i}, \qquad i = 1, \dots n. \tag{6.18}$$

Here $b_{i,1}$ is the number of consecutive black edges starting from the marked edge and going clockwise (it is equal to zero if the marked edge is white), $a_{i,1}$ is the number of the following consecutive white edges, and so on.

We define $\mathcal{T}^{(g)}_{S_1,\dots,S_n}$ to be the number of connected bi-colored maps of genus g with n boundaries with the boundary conditions $S_1, \dots, S_n$.

Remark 6.3.2. In terms of correlators of a two matrix model, one can write

$$\mathcal{T}^{(g)}_{S_1,\dots,S_n} = N^{n+2g-2} \left\langle \prod_{i=1}^{n} \mathrm{Tr}\left(M_1^{b_{i,1}} M_2^{a_{i,1}} M_1^{b_{i,2}} M_2^{a_{i,2}} \dots M_1^{b_{i,l_i}} M_2^{a_{i,l_i}} \right) \right\rangle_c^g \tag{6.19}$$

where the superscript g means that we only consider the g'th term of the expansion in N^{-2} of this correlator.

Cut-and-join equations

With these definitions, we are ready to derive the loop equations (6.15) and (6.17).

Namely, we can generalize to the two matrix model the procedure developed by Tutte for the enumeration of maps [94] and then extensively developed in the study of formal random matrices. Let us consider a connected genus g map with $n+1$ boundaries with boundary conditions

$$S_0 = k+1, 0; \qquad S_i = k_i, 0, \quad i = 1, \dots, n. \tag{6.20}$$

This means that all the edges of the marked faces are black. This surface contributes to $\mathcal{T}^{(g)}_{k+1,0;k_1,0;\dots;k_n,0}$. Let us remove the marked edge from the boundary 0. Since one can only glue together polygons of different colors, on the other side of the marked edge one can find only a white (unmarked) l-gon with $1 \le l \le d_2$. After removing the edge, let us mark in the resulting joint polygon the edge which is located clockwise from the origin of the removed edge (the origin of an edge is the vertex located on the counterclockwise side of the edge). We end up with a map that contributes to $\mathcal{T}^{(g)}_{0,l-1,k,0;k_1,0;\dots;k_n,0}$. This procedure is bijective between the sets considered. We take the sum over all possibilities, taking into account the weight of the edge and l-gon removed, and we see that

$$\mathcal{T}^{(g)}_{k+1,0;k_1,0;\dots;k_n,0} = \sum_{l=1}^{d_2} t_l^{(2)} \mathcal{T}^{(g)}_{0,l-1,k,0;k_1,0;\dots;k_n,0}. \tag{6.21}$$

Multiplying by $x^{-k-1} x^{-k_1-1} \dots x^{-k_n-1}$ and taking the sum over $k, k_1, \dots, k_n$, one recovers the loop equation (6.15).

This first equation produces mixed boundary condition out of homogeneous black conditions. Let us now proceed one step further and apply Tutte's method to the maps produced in this way.

Let us consider a map contributing to $\mathcal{T}^{(g)}_{0,l+1,k,0;k_1,0;\dots;k_n,0}$, i. e. a genus g connected map with boundary condition:

$$S_0 = 0, l+1, k, 0; \qquad S_i = k_i, 0, \quad i = 1, \dots, n. \tag{6.22}$$

Note that it follows from our definition that the marked edge is of white boundary condition type. When we remove it, we can produce different types of maps, namely, one of the following cases:

- Either the other side of the edge is an unmarked black m-gon. We remove the edge and this gives a map that contributes to $\mathcal{T}^{(g)}_{0,l,k+m-1,0;k_1,0;\dots;k_n,0}$
- Or the other side of the edge belongs to the black boundary condition type of the same marked face. Then two possible cases occur. The resulting surface can still be connected, giving rise to a map contributing to $\mathcal{T}^{(g-1)}_{m,0;0,l,k-m,0;k_1,0;\dots;k_n,0}$ for some $1 \leq m \leq k$, i. e. with one more boundary but a genus decreased by one. Or removing the marked edge can disconnect the map into two connected component giving contributions to $\mathcal{T}^{(h)}_{m,0;k_{\alpha_1},0;\dots;k_{\alpha_j},0}$ and $\mathcal{T}^{(g-h)}_{0,l,k-m,0;k_{\beta_1},0;\dots;k_{\beta_{n-j}},0}$ respectively, where $0 \leq h \leq g$ and $\{\alpha_1, \dots, \alpha_j\} \cup \{\beta_1, \dots, \beta_{n-j}\} = \{1, \dots, n\}$. This type of behavior can be thought of as a "cut" move.
- Finally, the other side of the edge can be another marked black face with boundary condition $(k_i, 0)$. Removing the edge, one gets a contribution to $\mathcal{T}^{(g)}_{0,l,k+k_i-1,0;k_1,0;\dots;k_{i-1},0;k_i-m,0;k_{i+1},0\dots;k_n,0}$. This type of behavior can be thought of as a "join" move.

Once again, this procedure is bijective, if we take the sum over all cases. Taking into account the weight of the elements removed, we end up with an equation relating the number of bi-colored maps with different boundary conditions:

$$\begin{aligned}
&\mathcal{T}^{(g)}_{0,l+1,k,0;k_1,0;\dots;k_n,0} \\
&= \sum_{m=0}^{d_2} t^{(2)}_m \mathcal{T}^{(g)}_{0,l,k+m-1,0;k_1,0;\dots;k_n,0} \\
&+ \sum_{m=0}^{k} \mathcal{T}^{(g-1)}_{m,0;0,l,k-m,0;k_1,0;\dots;k_n,0} \\
&+ \sum_{m=0}^{k} \sum_{h=0}^{g} \sum_{\vec{\alpha}\cup\vec{\beta}=\{1,\dots,n\}} \mathcal{T}^{(h)}_{m,0;k_{\alpha_1},0;\dots;k_{\alpha_j},0} \mathcal{T}^{(g-h)}_{0,l,k-m,0;k_{\beta_1},0;\dots;k_{\beta_{n-j}},0} \\
&+ \sum_{i=1}^{n} \mathcal{T}^{(g)}_{0,l,k+k_i-1,0;k_1,0;\dots;k_{i-1},0;k_i-m,0;k_{i+1},0\dots;k_n,0}
\end{aligned} \tag{6.23}$$

where $\vec{\alpha} = \{\alpha_1, \dots, \alpha_j\}$ and $\vec{\beta} = \{\beta_1, \dots, \beta_{n-j}\}$. This equation is the genus g contribution to the expansion of the loop equation (6.17) when all its variables are large.

This concludes the fully combinatorial proof of the two matrix model's loop equations. The latter can be seen as some particular cut-and-join equations. One can now apply the procedure used in [18] for solving them (without having to introduce any matrix model consideration!) and derive the topological recursion for the generating functions of bi-colored maps with homogenous boundary conditions.

6.4 Quantum curve

In this section we prove a generalization of the theorem of Do and Manescu from [23] on the quantum spectral curve equation for enumeration of hypermaps.

Theorem 6.4.1. *The wave function* $Z^{(a)}(x)$*, defined in* (6.4)*, satisfies the ODE:*

$$\left(-\hbar x\frac{\partial}{\partial x}+1+\sum_{i=1}^{a}t_i\left(\hbar\frac{\partial}{\partial x}\right)^i\right)Z^{(a)}(x)=0 \tag{6.24}$$

Remark 6.4.2. The differential operator in the previous theorem is given by the naive quantization of the classical spectral curve (6.12), $y\leftrightarrow\hbar\frac{\partial}{\partial x}$. Note that in Do and Manescu's paper [23] a different definition of $Z^{(a)}$ was used, as noted above, and a different convention $y\leftrightarrow-\hbar\frac{\partial}{\partial x}$.

6.4.1 Wave functions

In the proof we use the notations coming from the formal matrix model formalism for simplicity, but as usual in the formal matrix model setup, they just represent well defined combinatorial objects which satisfy the loop equations derived in the preceding sections.

In what follows we identify N with $1/\hbar$.

From the definition of the wave function $Z^{(a)}$, given in formulas (6.2)-(6.4), from the identification between $W^{(g)}_{k,0}$ and $\Omega^{(a)}_{g,k}$ given by Equation (6.10) and from the definition of $W_{k,l}$ (6.9), we have

$$Z^{(a)}(x)=\exp\left(\frac{1}{\hbar}\log(x)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{b_1,\dots b_n=1}^{\infty}\frac{\left\langle\mathrm{Tr}(M_1^{b_1})\dots\mathrm{Tr}(M_1^{b_n})\right\rangle_c}{b_1\dots b_n x^{b_1}\dots x^{b_n}}\right). \tag{6.25}$$

The standard relation between connected and disconnected correlators imply

$$Z^{(a)}(x)=x^{1/\hbar}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{b_1,\dots b_n=1}^{\infty}\frac{\left\langle\mathrm{Tr}(M_1^{b_1})\dots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_1\dots b_n x^{b_1}\dots x^{b_n}}. \tag{6.26}$$

In order to simplify the notation, we introduce functions $Z^r_n(x_1,\dots,x_n)$ and $Z^r(y,x)$ for integers $n\geq 1$, $r\geq 0$ (we call these functions non-principally-specialized wave functions).

Definition 6.4.3. The n-point wave function Z^r_n of level r is defined as

$$\begin{aligned}
&Z^r_n(x_1,\dots,x_n):= \\
&\log(x_1)\,(-1)^{n-1}\sum_{b_2\dots b_n=1}^{\infty}\frac{\left\langle\mathrm{Tr}(M_2^r)\,\mathrm{Tr}(M_1^{b_2})\dots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_2\dots b_n\;x_2^{b_2}\dots x_n^{b_n}} \\
&+(-1)^n\sum_{b_1,b_2\dots b_n=1}^{\infty}\frac{\left\langle\mathrm{Tr}(M_2^rM_1^{b_1})\,\mathrm{Tr}(M_1^{b_2})\dots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_1\dots b_n\;x_1^{b_1}\dots x_n^{b_n}},\quad r>0 \\
&Z^0_n(x_1,\dots,x_n):=(-1)^n\sum_{b_1,b_2\dots b_n=1}^{\infty}\frac{\left\langle\mathrm{Tr}(M_1^{b_1})\dots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_1\dots b_n\;x_1^{b_1}\dots x_n^{b_n}}
\end{aligned} \tag{6.27}$$

and the almost-fully principally-specialized wave function of level r is

$$Z^r(y,x) = \sum_{n=0}^{\infty} \frac{1}{n!} Z^r_n(y,x,\dots x) \tag{6.28}$$

Note that with these definitions

$$Z^{(a)}(x) = x^{1/\hbar} Z^0(x,x) \tag{6.29}$$

6.4.2 Loop equations in terms of Z^r_n

Considering the coefficient in front of particular powers of $1/x$ and $1/x_i$'s in loop equations (6.15) and (6.17) we get the following equations relating particular formal matrix model correlators

$$\begin{aligned}
&\left\langle \mathrm{Tr}(M_1^{b_1}) \dots \mathrm{Tr}(M_1^{b_n}) \right\rangle = \\
&\sum_{i=1}^{a} t_i \left\langle \mathrm{Tr}\left(M_1^{b_1-1} M_2^{i-1}\right) \mathrm{Tr}(M_1^{b_2}) \dots \mathrm{Tr}(M_1^{b_n}) \right\rangle \\
&\left\langle \mathrm{Tr}\left(M_2^{r} M_1^{b_1}\right) \mathrm{Tr}(M_1^{b_2}) \dots \mathrm{Tr}(M_1^{b_n}) \right\rangle = \\
&\hbar \sum_{j=2}^{n} b_j \left\langle \mathrm{Tr}\left(M_2^{r-1} M_1^{b_1+b_j-1}\right) \mathrm{Tr}(M_1^{b_2}) \dots \widehat{\mathrm{Tr}(M_1^{b_j})} \dots \mathrm{Tr}(M_1^{b_n}) \right\rangle \\
&+ \hbar \sum_{p+q=b_1-1} \left\langle \mathrm{Tr}(M_2^{r-1} M_1^{p}) \, \mathrm{Tr}(M_1^{q}) \, \mathrm{Tr}(M_1^{b_2}) \dots \mathrm{Tr}(M_1^{b_n}) \right\rangle,
\end{aligned} \tag{6.30}$$

Here the hat above $\mathrm{Tr}(M_1^{b_j})$ means that it is excluded from the correlator.

Let us sum the above equations over all $b_1, \dots, b_n$ from 1 to ∞ with the coefficient

$$\frac{(-1)^n}{x_1^{b_1} b_2 \dots b_n x_2^{b_2} \dots x_n^{b_n}}.$$

(note the absence of the $1/b_1$ factor). We get:

Lemma 6.4.4. *Loop equations, written in terms of Z^r_n, read*

$$\begin{aligned}
&(-x_1 \frac{\partial}{\partial x_1}) Z^0_n(x_1,\dots,x_n) = \\
&\sum_{i=1}^{a} t_i (-\frac{\partial}{\partial x_1}) Z^{i-1}_n(x_1,\dots,x_n) - \frac{1}{\hbar x_1} Z^0_{n-1}(x_2,\dots,x_n),
\end{aligned} \tag{6.31}$$

$$
\begin{aligned}
&\frac{1}{\hbar}(-x_1\frac{\partial}{\partial x_1})Z_n^1(x_1,\dots,x_n) = \qquad (6.32)\\
&-\sum_{j=2}^{n}\left[(-\frac{\partial}{\partial x_j})Z_{n-1}^0(x_j,x_2,\dots\widehat{x_j}\dots x_n)+\frac{1}{\hbar x_j}Z_{n-2}^0(x_2,\dots\widehat{x_j}\dots x_n)\right]\\
&+\frac{2}{\hbar}(-\frac{\partial}{\partial x_1})Z_n^0(x_1,\dots,x_n)-\frac{1}{\hbar^2 x_1}Z_{n-1}^0(x_2,\dots,x_n)\\
&-x_1\frac{\partial^2}{\partial u_1\partial u_2}\Big|_{u_1=u_2=x_1}Z_{n+1}^0(u_1,u_2,x_2,\dots,x_n)\\
&-\sum_{j=2}^{n}\frac{1}{(x_1-x_j)}\left[x_1\frac{\partial}{\partial x_1}Z_{n-1}^0(x_1,\dots\widehat{x_j}\dots x_n)-x_j\frac{\partial}{\partial x_j}Z_{n-1}^0(x_j,x_2,\dots\widehat{x_j}\dots x_n)\right],
\end{aligned}
$$

and, for all $r>1$,

$$
\begin{aligned}
&\frac{1}{\hbar}(-x_1\frac{\partial}{\partial x_1})Z_n^r(x_1,\dots,x_n) = \qquad (6.33)\\
&-\sum_{j=2}^{n}(-\frac{\partial}{\partial x_j})Z_{n-1}^{r-1}(x_j,x_2,\dots\widehat{x_j}\dots x_n)+\frac{1}{\hbar}(-\frac{\partial}{\partial x_1})Z_n^{r-1}(x_1,\dots,x_n)\\
&-x_1\frac{\partial^2}{\partial u_1\partial u_2}\Big|_{u_1=u_2=x_1}Z_{n+1}^{r-1}(u_1,u_2,x_2,\dots,x_n)\\
&-\sum_{j=2}^{n}\frac{1}{(x_1-x_j)}\left[x_1\frac{\partial}{\partial x_1}Z_{n-1}^{r-1}(x_1,x_2,\dots\widehat{x_j}\dots x_n)-x_j\frac{\partial}{\partial x_j}Z_{n-1}^{r-1}(x_j,x_2,\dots\widehat{x_j}\dots x_n)\right].
\end{aligned}
$$

6.4.3 Symmetrization of loop equations

Last step to obtain quantum curve equation is to put all equations (6.31)–(6.33) into principal specialization: put all x_i's equal to x.

The following obvious statement plays a crucial role in the induction:

Lemma 6.4.5. *Let $f(x_1|x_2,\dots,x_n)$ be a symmetric function in the variables $x_2,\dots,x_n$ (so, x_1 is treated specially here). Then we have the following formula for the derivative in the principal specialization.*

$$
\frac{\partial}{\partial x}f(x|x,\dots,x)=\frac{\partial}{\partial u}\Big|_{u=x}f(u|x,\dots,x)+(n-1)\frac{\partial}{\partial u}\Big|_{u=x}f(x|u,x,\dots,x). \qquad (6.34)
$$

In particular, if $f(x_1,x_2,\dots,x_n)=\frac{\partial}{\partial x_1}g(x_1,x_2,\dots,x_n)$, then

$$
\begin{aligned}
&\frac{\partial}{\partial x}\frac{\partial}{\partial y}\Big|_{y=x}g(y|x,\dots,x)= \qquad (6.35)\\
&\frac{\partial^2}{\partial u^2}\Big|_{u=x}g(u|x,\dots,x)+(n-1)\frac{\partial^2}{\partial u_1\partial u_2}\Big|_{u_1=x,u_2=x}g(u_1|u_2,x,\dots,x).
\end{aligned}
$$

Since Z_n^0 is symmetric in all its arguments, the first equation of (6.31) is equivalent to

$$\begin{aligned}
&\frac{1}{n}(-x\frac{\partial}{\partial x})Z_n^0(x,\dots,x) = \\
&-t_1\frac{1}{\hbar x}Z_{n-1}^0(x,\dots,x) - \frac{1}{n}\frac{\partial}{\partial x}Z_n^0(x,\dots,x) \\
&+\sum_{i=2}^{a} t_i(-\frac{\partial}{\partial y})\Big|_{y=x} Z_n^{i-1}(y,x,\dots,x)
\end{aligned} \tag{6.36}$$

We multiply this by $\frac{1}{(n-1)!}$ and take the sum over $n \geq 0$. We have:

$$\begin{aligned}
&(-x\frac{\partial}{\partial x})Z^0(x,\dots,x) = \\
&-t_1\left(\frac{\partial}{\partial x} + \frac{1}{\hbar x}\right)Z^0(x,\dots,x) \\
&+\sum_{i=2}^{a} t_i \sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z_n^{i-1}(y,x,\dots,x).
\end{aligned} \tag{6.37}$$

Then, the existence of a quantum curve equation relies on two observations:

Lemma 6.4.6. *We have:*

$$\begin{aligned}
i>1:\ &\sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z_n^{i}(y,x,\dots,x) \\
&= \left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right)\sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z_n^{i-1}(y,x,\dots,x) \\
i=1:\ &\sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z_n^{i}(y,x,\dots,x) \\
&= \hbar\left[-\frac{\partial^2}{\partial x^2} - \frac{2}{\hbar x}\frac{\partial}{\partial x} - \frac{1/\hbar(1/\hbar-1)}{x^2}\right]Z^0 \\
&= -\frac{1}{\hbar}\left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right)^2 Z^0
\end{aligned} \tag{6.38}$$

Proof. These equations are direct corollaries of Equations (6.31), we just have to put them into principal specialization and apply Lemma 6.4.5. □

We combine Equation (6.37) and Lemma (6.4.6), and we obtain the following equation:

$$(-\hbar x\frac{\partial}{\partial x})Z^0 = -\sum_{i=1}^{a} t_i\left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right)^i Z^0. \tag{6.39}$$

which, with help of commutation relation

$$x^{1/\hbar}\left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right) = \hbar\frac{\partial}{\partial x}\circ x^{1/\hbar}, \tag{6.40}$$

leads directly to the statement of Theorem 6.4.1.

6.5 4-colored maps and 4-matrix models

It turns out that the ideas above can be applied not only to bi-colored maps (which correspond to the 2-matrix model case), but also to 4-colored maps. In the current section we outline the proof of a spectral curve topological recursion for the enumeration of 4-colored maps.

4-colored maps arise as a natural generalization of bi-colored maps. Instead of considering partitions of surfaces into black and white polygons, we consider partitions into polygons of four colors c_1, c_2, c_3, c_4, such that polygons of color c_1 are glued only to polygons of color c_2, polygons of color c_2 are glued only to polygons of colors c_1 and c_3, polygons of color c_3 are only glued to those of color c_2 and c_4 and finally polygons of color c_4 are only glued to polygons of color c_3. This can be represented in terms of the following color incidency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.41}$$

Applying considerations similar to the ones in the above sections, it's easy to see that the problem of enumeration of such 4-colored maps is governed by a 4-matrix model with the interaction part of the potential being equal to

$$-N\,\mathrm{Tr}(M_1M_2 - M_1M_4 + M_3M_4), \tag{6.42}$$

since the inverse of the above incidency matrix is equal to

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \tag{6.43}$$

We see that in the 4-colored maps case, after a renumeration of matrices and a certain change of signs, this still gives us the matrix model for a chain of matrices (which is no longer true for, e.g., 6-colored maps). Fortunately, the case of matrix model for a chain of matrices was studied by Eynard in [39], and the master loop equation obtained there gives rise to the spectral curve topological recursion for this problem. Again, it's easy to see in the analogous way to what was discussed in the previous sections that the individual building blocks of loop equations can be proved to hold by purely combinatorial means.

Chapter 7

Populaire samenvatting

Dit proefschrift gaat over problemen in de algebraïsche meetkunde en mathematische fysica die verband hebben met Gromov-Witten-theorie, topologische recursie van spectrale krommen en Hurwitzgetallen.

Een van de centrale objecten in de algebraïsche meetkunde is een algebraïsche kromme. Ik het kader van dit proefschrift werken we met algebraïsche krommen gedefinieerd over het lichaam van complexe getallen. In dit geval kunnen we deze krommen beschouwen als twee-dimesionale oppervlakken.

De *Gromov-Witten-theorie* bestudeert afbeeldingen van algebraïsche krommen naar een gegeven complexe variëteit (in andere woorden, naar een zekere meer-dimensionale ruimte). Deze theorie komt uit snaartheorie, een vakgebied in de theoretische natuurkunde. *Gromov-Witten-invarianten* zijn getallen die het aantal mogelijke inbeddingen geven van krommen van een bepaald type in een gegeven complexe variëteit. Het belang van de Gromov-Witten-invarianten is duidelijk door hun gebruik in snaartheorie en algebraïsche meetkunde. Ze zijn echter ook verbonden met een heel ander vakgebied van mathematische fysica, namelijk de theorie van integreerbare systemen.

Topologische recursie van de spectrale krommen is een algemene methode die toepassingen heeft in veel verschillende takken van de wiskunde en natuurkunde. Deze methode heeft als input een spectrale kromme die een algebraïsche kromme is met het aantal extra structuren, en geeft als output zogenoemde n-punt functies op de spectrale kromme. Het blijkt dat voor een enorm groot aantal problemen in algebraïsche meetkunde, mathematische fysica, topologie, en combinatoriek deze n-punt functies de genererende functies voor de oplossingen van deze problemen zijn (de input is dan bepaald door een probleem). Hier bedoelen we met een genererende functie een functie waarvan de coëfficiënten van de machtreeksonwikkeling de getallen zijn, die deze problemen oplossen.

De topologische recursie van de spectrale krommen is heel interessant, omdat het een universele procedure is die op een universele manier de antwoorden geeft op een groot aantal problemen, die onderling geen verband hebben.

Hurwitzgetallen berekenen de *overdekkingen* van een sfeer door twee-dimensionale compacte oppervlakken. Een overdekking is een afbeelding van een oppervlak naar de sfeer. Voor bijna alle punten van de sfeer (met uitzondering van een eindig aantal zogenoemde *vertakkingspunten*) hebben we hetzelfde aantal punten op het oppervlak die daarop afgebeeld worden. Dit getal noemt men de *graad* van de overdekking. Als we het type vertakkingspunten vastleggen, dan hebben we een eindig aantal mogelijke overdekkingen met deze voorwaarden. Dit getal heet het Hurwitzgetal. Hurwitzgetallen zijn belangrijk vanwege hun meerdere interpretaties in combinatoriek en topologie, en hebben tevens een belangrijke rol in representatietheorie.

Dit proefschrift onderzocht een aantal verbindingen tussen de bovengenoemde objecten en theoriëen. Een van de hoofdstellingen van het proefschrift geeft een toepassing van de lokale versie van de topologische recursie op een willekeurige Gromov-Witten-theorie. Namelijk, we bewijzen dat we altijd de juiste input kunnen kiezen zo dat de n-punt functies de genererende functies van de Gromov-Witten-invarianten worden.

Een andere stelling houdt verband met de vergelijking van de *kwantum*-spectrale kromme. In een aantal gevallen kan men bewijzen dat een bepaalde genererende functie, de zogenoemde golffunctie, voldoet aan een kwantum-versie van de vergelijking voor de spectrale kromme. We bewijzen dit voor de golffunctie van de Gromow-Witten-theorie van een sfeer.

Een ander resultaat van dit proefschrift is een nieuw, zuiver combinatorisch bewijs van de beroemde ELSV formule. Deze formule verbindt Hurwitzgetallen en de integralen van Hodge, die in vele aspecten lijken op Gromov-Witten-invarianten. Het oorspronkelijke bewijs en alle andere bewijzen van de ELSV formule gebruiken lastige meetkundige redeneringen. In plaats daarvan bewijzen we eerst op een zuiver combinatorische manier dat deze Hurwitzgetallen polynomen in de vertakkingsindexen zijn. Dan gebruiken we dit resultaat om een spectrale kromme voor de Hurwitzgetallen af te leiden en vervolgens gebruiken we ons resultaat over de correspondentie tussen topologische recursie en Gromov-Witten-theorie om een verbinding met de Hodgeintegralen te zien.

Ten slotte geven we een zuiver combinatorisch bewijs van het feit dat het probleem van de berekening van zogenoemde twee-kleuren kaarten op oppervlakken ook door de topologische recursie opgelost kan worden. Met een twee-kleuren kaart bedoelen we een verdeling van een twee-dimensionaal oppervlak in polygonen gekleurd in twee kleuren, zwart en wit, zodat de witte polygonen alleen aan zwarte polygonen mogen grenzen, en omgekeerd. We bewijzen dat het aantal van dit soort kaarten is gegeven door de topologische recursie van een bepaalde spetrale kromme.

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