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String Field Theory on the Conformal Plane
I. Kinematical Principles

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ABSTRACT

We formulate string field theory geometrically by writing each term in the field theory action as an expectation value in the 2-dimensional conformal field theory on the world-surface. We show how the symmetries of the theory can be analyzed and the gauge-invariance demonstrated from this point of view. As an application, we give a complete proof of the gauge-invariance of Witten's open-string field theory.

1. Introduction

The current interest in string theory as a unified model of the fundamental interactions has created a renewed interest in the foundations of string theory and the formulation of a field-theoretic basis for string mechanics. The original work on the quantum mechanics of strings, especially the papers of Goddard, Goldstone, Rebbi, and Thorn,[1] Mandelstam,[2] and Kaku and Kikkawa,[3] presented a complete description of interacting strings and string fields from the viewpoint of the light-cone gauge. More recently, Green and Schwarz[4] have improved this light-cone gauge field theory and generalized it to encompass their superstring theory. Siegel[5] presented a covariant formulation of the field theory of strings, and this development led to an explosion of results on gauge-invariant string field actions. At this time, there are two complete and successful formulations of the open bosonic string theory: The first of these, which generalizes the light-cone formulation, has been developed by Siegel,[5] and Hata, Itoh, Kugo, Kunitomo, and Ogawa.[6,7] The last of these groups, in particular, has constructed the theory in complete detail and has carefully studied its consistency. The second formulation, based on a suggestion of differential geometry in the space of strings, was invented by Witten[8] and has been amplified and made more concrete by several groups.[9–34] Some parts of these formalisms have been generalized to closed bosonic strings[35–44] and to superstrings.[45–54]

Despite this progress, however, the current formulations of string field theory are unsatisfactory for a number of reasons. The most obvious of these is that, while the quadratic term of the string action simplifies beautifully by the introduction of ghost variables on the world surface, the interaction terms are exceedingly complicated. In all known formulations of string field theory, it is a matter of great difficulty to even to verify gauge-invariance explicitly; it costs still more trouble to develop the rules of Feynman diagram perturbation theory. If string field theory is eventually to be a tool which can aid us in understanding string dynamics, we must find some way to simplify its internal structure so that it can become the basis for new computations rather than just a complicated way of obtaining known results. Giddings and Martinec [21] have made an important step in this direction, by formulating a set of rules which intuitively correspond to Witten's open string field theory, and showing that these rules lead to the complete perturbation series. The problem remaining is to complete their formalism by deriving their intuitive starting point from a Lagrangian. A second, equally pressing problem, is that many different formulations of the interacting theory have been proposed,[8,6,55–59] and it is not at all clear what transformations one can use to convert one of these formulations into another.

In sharp contrast to this awkward situation, the perturbation series for string interactions follows straightforwardly from considerations of conformally invariant quantum field theory on the world surface of the string. It seems quite clear that this conformal field theory is the natural setting for the calculation of string interactions.[60] In addition, the underlying structure of conformal field theory, as laid out by Belavin, Polyakov, and Zamolodchikov [61] (BPZ), is beautifully simple. We were led to apply this technology to reformulate string field theory, with the idea of representing each term in the string field theory action as a conformal field theory expectation value. In fact, this reformulation is readily accomplished, and what results is a setting for string field theory which is already highly geometrical and whose symmetries of transmutation are quite obviously displayed. In this series of papers, we will present this new formulation of string field theory from start to finish for the case of the bosonic open string. We will set up the action, prove its gauge invariance, and derive the perturbation theory rules of Giddings and Martinec, making use of intuitive geometrical constructions but also supporting these with concrete analysis.

The plan of the present paper is as follows: In Section 2, we will review the formalism presented by BPZ for transcribing between expectation values in 2-dimensional conformal field theory and Hilbert space matrix elements. We will give particular attention to the Hilbert space inner product introduced by BPZ, since this will play a central role in our analysis. In Section 3, we will review in a very schematic form the basic structural elements of string field theory and the strategy for proving gauge invariance. In Section 4, we will begin our analysis proper, presenting the basic concepts necessary to write the string field theory Lagrangian in this language, and we will display a rather general form of the string field theory vertex in an expansion in normal modes. In Section 5, we will discuss the symmetries of the string field theory vertex and show how the conformal field theory formulation makes these manifest. In Section 6, we will make these considerations more concrete by considering two explicit forms for the 3-string vertex—the dual model vertex of Sciuto and Caneschi, Schwimmer, and Veneziano[62] (CSV), and the 3-string vertex proposed by Witten.[8] We will devote considerable space in this series of papers to the CSV vertex, even though it will not prove completely satisfactory as a string field theory vertex, because this vertex is wonderfully simple and provides the most accessible illustration of the rules for analyzing and combining string vertices within our formalism.

Our discussion of the CSV and Witten vertices will introduce the idea that a contraction of string field theory vertices may be interpreted as a gluing together of pieces of string world-sheet. As an intuitive notion, this idea has motivated much of the development of string field theory, from the original work of Mandelstam[2] to the more recent developments of Giddings and Martinec.[21] The analysis of this series of papers will be aimed toward defining this notion in a precise manner. In Section 7, we will present the first segment of this argument; we will demonstrate, by explicit calculation, that the gluing operation suggested by the analysis of Section 6 actually does result from explicit operator manipulations when two CSV vertices are contracted using the BPZ inner product. This calculation gives a special case of a more general result which applies to the gluing of string field theory vertices of arbitrary form. That 'Generalized Gluing and Resmoothing Theorem' (GGRT) is the basic result which justifies the use of geometric intuition in string field theory, by making precise the way in which a given vertex acts to sew together regions of the conformal plane. The proof of the theorem in its full generality, is, however, rather involved. We will end

Section 7 with a statement of the theorem and reserve the proof to the second paper of this series.

In Section 8, we will return return to our study of the Witten open string action and apply the GGRT to give a simple but completely explicit proof of its gauge invariance. We will then argue that this vertex is the unique gauge-invariant, three-fold symmetric vertex, at least within the class of vertices built from contact interactions on the world-sheet.

The second paper of this series [63] (which we will refer to as II) will be devoted to proving the GGRT for the case of bosonic open and closed strings. The heart of the proof will be an explicit demonstration that the elements of the normal mode expansion of the vertex presented in Section 5 can be recombined into meromorphic functions which reflect directly the geometry of the glued surface. This will confirm the precise relation of the operator and geometric pictures of string field theory. The third paper of this series [64] (which we will refer to as III) presents the application of this formalism to the derivation of the string perturbation series. Here we will show how our formalism provides a coherent underpinning for the results of Giddings and Martinec.

The general idea of setting string interactions onto the conformal plane is, of course, an old one, and conformal manipulations played an important role in many of the early papers, including those of Ademollo, del Guidice, di Vecchia, and Fubini,[65] and Mandelstam,[2] as well as a remarkable paper of Lovelace [66] which we took up specifically as a source of inspiration. In the recent literature, Neveu and West [67] have emphasized the conformal relation of different string vertices. Various pieces of our explicit construction have been presented independently by a number of groups: In particular, Itoh, Ogawa and Suehiro [19] and Eastaugh and McCarthy [14] have also derived the explicit form for the Witten vertex presented here, and Di Vecchia *et al.*[68] have also found the generalized form of the CSV vertex which we will discuss. Gross and Jevicki [9] have given a different explicit proof of the gauge invariance of the Witten string field theory action (at least up to the question of overall factors), and Thorn[69] has discussed many aspects of the derivation of perturbation theory.

Our formalism subsumes and unifies these various results. Indeed, we view as our most important conclusion just this demonstration that the whole technology of string field theory can be discussed in a unified way using the language of conformal field theory. We hope that this language will prove useful in stimulating further developments in string field theory, and more generally in the study of strings.

2. Conformal Field Theory

We begin by reviewing the elements of conformal field theory needed for our analysis, using as an illustration the example of the bosonic string in a flat background space-time. In particular, we would like to review the formulation of this theory this theory within the general description of conformally-invariant quantum field theories given by Belavin, Polyakov, and Zamolodchikov (BPZ), and, especially, to recall the connections presented by BPZ between correlation functions of conformal tensors and the underlying Hilbert space structure. Our discussion of string theory within this formalism relies heavily on the work of Friedan, Martinec, and Shenker.[60]

We will study the bosonic string in orthonormal gauge, and using a Euclidean metric on the world-sheet. After gauge-fixing, the action for modes propagating on the world-sheet has the form:

$$S = \int \frac{d^2z}{\pi} \left\{ \frac{1}{2} \partial_z X^\mu \partial_{\bar{z}} X^\mu + b_{zz} \partial_{\bar{z}} c^z + b_{\bar{z}\bar{z}} \partial_z c^{\bar{z}} \right\} . \tag{2.1}$$

where X^μ is a the space-time coordinate and c^z, b_{zz} are the reparametrization ghost and antighost. The tensor structure of these parameters reflects the fact that c^z must transform as a world-sheet reparametrization $\delta z = \xi^z$; b_{zz} transforms as a metric variation δg_{zz}.

The action (2.1) is an example of a conformally-invariant quantum field theory. The action is invariant under general conformal transformations $z \to f(z)$, while the fundamental fields transform as conformal tensors. The transformation law of a tensor (or, in the language of BPZ, a *primary conformal field*) is determined by its scaling dimension. We will write the transformation law of a primary field $\phi(z)$ of dimension d as

$$f\Big[\phi(z)\Big] \;=\; \left(f'(z)\right)^d \phi(f(z)) \ . \tag{2.2}$$

The fields $\partial_z x^\mu$, c^z, b_{zz} transform as primary fields of dimension 1, -1, 2, respectively.

The correlation functions of any Euclidean field theory may be generated from a Hamiltonian evolution by slicing the Euclidean space by planes normal to a given fixed vector and then defining the Hamiltonian to be the generator of translations along that vector. In a conformally-invariant field theory, one has the additional freedom of slicing the space by any set of curves conformally equivalent to parallel planes. A particularly convenient choice is shown in Fig. 1: By mapping from a cylinder sliced by parallel lines, we can consider the conformally-invariant field theory on the plane to be generated by an evolution in which the equal-time surfaces are concentric circles. The Hamiltonian for this evolution is the dilatation generator L_0. This prescription is known as *radial quantization.* It is obvious from the figure that this evolution is closely related to the natural time evolution for (closed) strings.

In radial quantization, charges are defined as integrals around circles:

$$Q \;=\; \oint \frac{dz}{2\pi i}\, j(z) \ . \tag{2.3}$$

If, as is often the case, the charge density is an analytic function, the contour of integration may be freely deformed. Since the Hilbert space interpretation of a correlation function sets the operators in (radial) time-order, equal-time commutators of charges with operators $\phi(w)$ may be written as differences of correlations functions with the contour displaced slightly to either side of the point w. In other words,

$$\langle \cdots \; [Q, \phi(w)] \; \cdots \rangle \; = \; \langle \cdots \oint \frac{dz}{2\pi i} j(z) \, \phi(w) \; \cdots \rangle \; , \tag{2.4}$$

where the contour encircles the point w. Equal-time commutators may then be related directly to singularities of the operator-product expansion of $j(z)$ and $\phi(w)$. A particularly important set of charges are the Virasoro operators, the Fourier components of the energy-momentum tensor element T_{zz}

$$L_n \; = \; \oint \frac{dz}{2\pi i} \, z^{n+1} \, T_{zz}(z) \; . \tag{2.5}$$

In any conformally-invariant theory, T_{zz} is an analytic function of z; in the bosonic string, it has the explicit form

$$T_{zz}(z) \; = - \; \frac{1}{2} \partial_z x^\mu \partial_z x^\mu \; + \; 2 \partial_z c^z b_{zz} \; + \; c^z \partial_z b_{zz} \; . \tag{2.6}$$

The operator product of T_{zz} with a primary field has the general structure

$$T_{zz}(z) \, \phi(w) \; \sim \; \frac{d}{(z-w)^2} \, \phi(w) \; + \; + \frac{1}{(z-w)} \, \partial_w \phi(w) \; ; \tag{2.7}$$

this relation is equivalent, by the use of (2.4), to the commutator

$$[L_n, \phi(w)] \; = \; d \cdot n w^n \phi(w) \; + \; w^{n+1} \partial_w \phi(w) \; . \tag{2.8}$$

This is the infinitesimal form of (2.2), for the particular variation

$$L_n \; \Leftrightarrow \; w \rightarrow w \, + \, \epsilon w^{n+1} \; . \tag{2.9}$$

Notice that L_0 generates an infinitesimal dilatation; this operator is precisely the Hamiltonian of radial quantization introduced above.

It is convenient to define the Fourier decomposition of an arbitrary primary field by

$$\phi(z) = \sum_{n=-\infty}^{\infty} \phi_n \, z^{-n-d} \, ; \qquad \phi_n = \oint \frac{dz}{2\pi i} \, z^{n+d-1} \, \phi(z) \, . \tag{2.10}$$

The notation is arranged so that (2.7) leads to

$$[L_0, \phi_n] = -n\phi_n \, , \tag{2.11}$$

so that the ϕ_n are ladder operators for L_0. For the bosonic string, define (dropping henceforth the indices of b and c):

$$\begin{aligned} a_n^\mu &= \oint \frac{dz}{2\pi i} \, z^n \partial_z x^\mu(z) \\ b_n &= \oint \frac{dz}{2\pi i} \, z^{n+1} b(z) \\ c_n &= \oint \frac{dz}{2\pi i} \, z^{n-2} c(z) \, . \end{aligned} \tag{2.12}$$

The action (2.1) leads to the free-field propagators

$$\langle x^\mu(z) x^\nu(w) \rangle = -\delta^{\mu\nu} \log(z - w) \, ; \qquad \langle b(z) c(w) \rangle = \frac{1}{(z-w)} \tag{2.13}$$

($\langle x(z)x(w) \rangle$ represents only the part of the contraction which contributes to analytic, as opposed to anti-analytic, correlation functions.) Using (2.13) together with (2.4) and (2.12), we find the commutation relations

$$[a_n^\mu, a_m^\nu] = -\delta^{\mu\nu} n \delta(n+m) \, ; \qquad \{b_n, c_m\} = \delta(n+m) \, . \tag{2.14}$$

Thus, the operators defined in (2.12) are the usual string ladder operators*.

* A more conventional notation is $\alpha_n^\mu = i a_n^\mu$.

In radial quantization, the point $z = 0$ (or, more generally, the center of the concentric circles) represents $t = -\infty$. In quantum field theory, one usually defines the vacuum as the state which develops from $t = -\infty$; the generalization to radial quantization is to define the vacuum $|0\rangle$ to be the state which develops from the point $z = 0$ when we put no operator there. More concretely,

$$\langle \cdots | \, \phi_n \, |0\rangle \;=\; \left\langle \cdots \oint \frac{dz}{2\pi i} \, z^{n+d-1} \, \phi(z) \right\rangle \,, \tag{2.15}$$

with the contour enclosing no other operators. From this it follows that ϕ_n annihilates $|0\rangle$ if $n \geq (1-d)$. For the bosonic string, this tells us that

$$\begin{aligned} a_n^\mu \, |0\rangle &= 0 \quad \text{for } n \geq 0 \\ c_n \, |0\rangle &= 0 \quad \text{for } n \geq 2 \\ b_n \, |0\rangle &= 0 \quad \text{for } n \geq -1 \\ L_n \, |0\rangle &= 0 \quad \text{for } n \geq -1 \, . \end{aligned} \tag{2.16}$$

In string theory, the operator a_0^μ is proportional to the center-of-mass momentum of the string. We can see that our a_0^μ has that property, and identify the states of definite momentum, by using the contraction (2.13) to compute

$$\begin{aligned} a_0^\mu \, e^{ip\cdot X(0)} \, |0\rangle &= \oint \frac{dz}{2\pi i} \, \partial_z x^\mu(z) \, e^{ip\cdot X(0)} \, |0\rangle \\ &= (-ip^\mu) \, e^{ip\cdot X(0)} \, |0\rangle \, . \end{aligned} \tag{2.17}$$

It will be useful to note that $\exp(ip \cdot X(z))$ is a primary conformal field with dimension $d = p^2/2$; the reader may verify this by using the propagator (2.13) to check the operator product (2.7).

Now that we have identified the vacuum on the right, we still need to identify the vacuum on the left and define a suitable inner product. It is natural to define the left vacuum as the state which develops by evolving backward from $t = \infty$, that is, from $z = \infty$. Then, following BPZ, we can set up an inner product as follows: Let $I(z)$ be the conformal transformation which interchanges the interior and exterior of the unit circle, while taking the upper half plane to itself:

$$I(z) \;=\; -\frac{1}{z}\,. \tag{2.18}$$

Each state in the Hilbert space of the conformal field theory may be created from $|0\rangle$ by a corresponding operator

$$|A\rangle \;=\; \mathcal{O}_A\;|0\rangle\;. \tag{2.19}$$

$\mathcal{O}_A$ may or may not be a primary field. In any case, define the dual of the state $|A\rangle$ created by $\mathcal{O}_A$ as the state created by the operator formed by acting on $\mathcal{O}_A$ by the inversion R according to the conformal transformation law (2.2):

$$\langle A|B\rangle \;=\; \Big\langle I\Big[\mathcal{O}_A\Big]\;\mathcal{O}_B\Big\rangle\;. \tag{2.20}$$

It is straightforward to compute, by a change of variables

$$\begin{aligned} I\Big[\phi_n\Big] \;&=\; \oint \frac{dz}{2\pi i}\, z^{n+d-1}\cdot\Big(\frac{1}{z^2}\Big)^d \phi\Big(-\frac{1}{z}\Big) \\ &= (-1)^{n+d}\,\phi_{-n}\;; \end{aligned} \tag{2.21}$$

thus, the adjoint operation with respect to this inner product does carry ϕ_n to ϕ_{-n}, as expected. The inner product (2.21) is linear in both arguments, rather than antilinear in one and linear in the other, but, because the two-dimensional fields X^μ, b, and c are real, this will be not be a serious difficulty. However, our string component fields will be forced to obey a somewhat complicated reality condition, to be discussed in Section 4. Because b and c are Grassmann fields, whose ordering is crucial, it is important to note that a conformal transformation such as I does not change the formal ordering of operators with respect to Grassmann multiplication.

Eq. (2.20) has the following physical interpretation: The states $|A\rangle$ and $|B\rangle$ may be thought of as superpositions of field configurations on the unit circle, with $|B\rangle$ defined just inside and $|A\rangle$ defined just outside. The inner product should overlap these states with a contact delta-function. The right-hand side of (2.20) indicates that we can use the evolution generated by L_0 to retract $|B\rangle$ and $|A\rangle$ to states created by operators acting in the vicinity of 0 and ∞. Thus, a rigid constraint along a single line can be transformed into a more flexible expression involving dynamics on the whole conformal plane. That transformation is shown graphically in Fig. 2. This observation will be the key to our analysis of string field theory.

The adjoint of the last relation in (2.16) tells us that

$$\langle 0| L_n = 0 \quad \text{for } n \leq +1 \ . \tag{2.22}$$

Thus, the three generators L_{-1}, L_0, L_1 annihilate both of $|0\rangle$ and $\langle 0|$ and thus are symmetries of all conformal field theory matrix elements. These three charges, plus the corresponding charges built of anti-analytic fields, generate the $SL(2,C)$ subgroup of conformal transformations

$$z \rightarrow \frac{az+b}{cz+d} \ , \qquad ad - bc = 1 \ . \tag{2.23}$$

These transformations will be manifest symmetries of string field theory, in a sense that will become clear in Section 5. We will refer to $|0\rangle$ henceforth as the $SL(2,C)$-invariant vacuum.

Since b has the same dimension as T_{zz}, the operators b_{-1}, b_0, b_1 also annihilate both $|0\rangle$ and $\langle 0|$. On the other hand, the operators c_{-1}, c_0, c_1 annihilate *neither* of these states. We may interpret this by saying that the three $SL(2,C)$ transformations of the conformal plane are zero modes of the field $c(z)$, and that these must be saturated in order to obtain a nonzero matrix element. If this interpretation were correct, we would expect:

$$\langle c(z_1)c(z_2)c(z_3)\rangle = \det |Z_i(z_j)| , \tag{2.24}$$

where the $Z_i(z)$ are the three zero modes: $Z_i(z) = (1, z, z^2)$ for $i = -1, 0, 1$, and $j = 1, 2, 3$. Indeed, if we choose a normalization by writing

$$\langle 0| c_{-1}c_0c_1 |0\rangle = 1 \tag{2.25}$$

and use this together with the Fourier expansion (2.10) to evaluate (2.24), we find

$$\langle c(z_1)c(z_2)c(z_3)\rangle = \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_1^2 & z_2^2 & z_3^2 \end{vmatrix} , \tag{2.26}$$

as required. In general, conformal field theory matrix elements will be nonvanishing only if they contain 3 more c operators than b operators. If we define a *ghost charge* G such that $c(z)$ raises G by 1, and $b(z)$ lowers G by 1, then the conformal field theory matrix element annihilates $G = 3$. It will be convenient to define a left vacuum which has a nonzero overlap with the vacuum $|0\rangle$ and can give nonzero matrix elements to operators with $G = 0$. Let us, then, define

$$\langle 3| = \langle 0| c_{-1}c_0c_1 , \quad \text{so that } \langle 3|0\rangle = 1 . \tag{2.27}$$

The ghost charge nonconservation which we have found explicitly here reflects a more general law which follows from the gravitational anomaly of the ghost number current: $\Delta G = -3(g-1)$ for conformal field theory on a Riemann surface of genus g. We will see this more general conservation law realized in the analysis in Section 5 of III.

3. A Precis of (Open) String Field Theory

Having now reviewed the basics of conformal field theory, we should turn next to the elements of string field theory. In this section, we will review the basic algebra of the field theory of open strings and the strategy for constructing a gauge invariant action first presented by Witten.[8] These basic ingredients have since been applied by many groups; in particular, Hata, Itoh, Kugo, Kunitomo, and Ogawa[6] have shown in complete detail how a slight generalization of this algebra appears for their light-cone-like vertex and guarantees the gauge-invariance of their action. Our discussion in this section is intentionally schematic; it will be the task of the next several sections to make precise the various operations introduced here.

A string field Φ is a functional of string embeddings $X^{\mu}(\sigma)$. It has proven useful to consider Φ also as a functional of the configuration of reparametrization ghosts $b(\sigma)$, $c(\sigma)$. A convenient basis for expanding such functionals is provided by the L_0 eigenfunctions of the first-quantized string theory. Keeping the dependence on the center-of-mass coordinate x of the string in the coefficient functions, we may expand:

$$\Phi[x(\sigma), b(\sigma), c(\sigma)] \;=\; \left[\phi(x) + A_{\mu}(x) a^{\mu}_{-1} + T_{\mu\nu}(x) a^{\mu}_{-1} a^{\nu}_{-1} + \ldots\right] |\Omega\rangle \ , \tag{3.1}$$

where $|\Omega\rangle$ is a suitable vacuum state. The string field then contains an infinite number of local fields, including states of arbitrarily high spin. These local fields should be matrix-valued to incorporate quantum numbers via the Chan-Paton prescription. The vacuum $|\Omega\rangle$ should be chosen to be the string state of lowest L_0. Since, as we saw in the previous section, the L_0 lowering operator c_1 does not annihilate $|0\rangle$, it makes sense to choose $|\Omega\rangle = c_1 |0\rangle$. Then, if its component fields $\phi(x)$, $A_{\mu}(x)$, etc. are of bosonic character, the string field Φ will be an anticommuting number. We will be somewhat cavalier about the commutativity or anticommutativity of string fields in the remainder of this section; however, we will take care to treat this issue correctly in our later discussion.

Let us now sketch the construction of an action for Φ, of the form

$$S = \mathcal{K} + g\mathcal{V} , \tag{3.2}$$

where $\mathcal{K}$ is the free-field action, quadratic in Φ, $\mathcal{V}$ is a 3-string interaction term, and g is the coupling constant. A suitable form for $\mathcal{K}$ can be constructed by making use of the BRST charge $\mathbf{Q}$ associated with the action (2.1):

$$\mathbf{Q} = \oint \frac{dz}{2\pi i} \mathrm{j} , \qquad \mathrm{j} = : c\{ -\frac{1}{2}\partial_z x^\mu \partial_z x^\mu + \partial_z c^z b_{zz} \} . \tag{3.3}$$

In the critical dimensionality, $d = 26$, $\mathbf{Q}$ satisfies

$$\mathbf{Q}^2 = 0 . \tag{3.4}$$

Thus, if we write

$$\mathcal{K} = \langle \Phi | \, \mathbf{Q} \, | \Phi \rangle , \tag{3.5}$$

this term will have the gauge-invariance

$$\delta \, | \Phi \rangle = \mathbf{Q} \, | \Lambda \rangle . \tag{3.6}$$

It has been checked with care that, if Φ is restricted to states of ghost number $G = 1$—that is, to the ghost number of the state $|\Omega\rangle$—and Λ is correspondingly restricted to $G = 0$, this gauge symmetry is exactly what is required to eliminate all spurious degrees of freedom, and that the resulting gauge-fixed free-field action coincides with the standard string theory in the transverse gauge. If the gauge transformation law (3.6) is written in terms of local component fields, the first component relation is exactly the linearized gauge transformation law $\delta A_\mu = \partial_\mu \lambda$, where $\lambda(x)$ is the leading scalar field in Λ.

We now wish to extend this structure to the interacting theory. Since this theory should contain non-Abelian gauge bosons, the gauge variation of the vector field should be generalized to contain the inhomogeneous term

$$\delta A_\mu \;=\; \partial_\mu \lambda \;+\; g[A_\mu, \lambda] \; . \tag{3.7}$$

Thus, the transformation law (3.6) should be generalized to include an inhomogeneous term

$$\delta \Phi \;=\; \mathbf{Q}\Phi \;+\; g\,[\Phi, \Lambda] \; . \tag{3.8}$$

where

$$[\Phi, \Lambda] \;=\; \Phi * \Lambda \;-\; \Lambda * \Phi \; , \tag{3.9}$$

and $*$ is a suitable product on the space of single-string states. $*$ can be represented as an operation joining the Fock spaces of two strings to that of a third

$$|A * B\rangle_1 \;=\; \langle V_{1\dagger 23}|\; |A\rangle_2 \otimes |B\rangle_3 \; ; \tag{3.10}$$

the indices $1, 2, 3$ refer to the three Hilbert spaces. It is instructive to think of the coefficients $\langle V_{1\dagger 23}|$ as the structure constants of the string gauge algebra.

The requirement that the string gauge algebra close,

$$(\delta_\Lambda \delta_{\tilde\Lambda} \;-\; \delta_{\tilde\Lambda}\delta_\Lambda)\, \Phi \;=\; \delta_{[\Lambda, \tilde\Lambda]}\, \Phi \; , \tag{3.11}$$

produces some non-trivial conditions on $|V_{1\dagger 23}\rangle$. In Yang-Mills theory, closure of the gauge algebra depends on two properties: the Leibnitz rule for differentiation

$$\partial_\mu [\lambda, \tilde\lambda] \;=\; [\partial_\mu \lambda, \tilde\lambda] \;+\; [\lambda, \partial_\mu \tilde\lambda] \tag{3.12}$$

and the Jacobi identity

$$\big[[A_\mu, \lambda], \tilde\lambda\big] \;+\; (\text{cyclic}) \;=\; 0 \; . \tag{3.13}$$

These requirements generalize straightforwardly to the string algebra. The generalization of (3.12) will be satisfied if $\mathbf{Q}$ is a derivation of the $*$ algebra

$$\mathbf{Q}(A * B) = (\mathbf{Q}A) * B + A * (\mathbf{Q}B) . \tag{3.14}$$

Since the Jacobi identity is satisfied for any associative multiplication operation, such as matrix multiplication in the case of Yang-Mills theory, the string generalization of (3.13) will be satisfied if the $*$ algebra is associative

$$(A * B) * C = A * (B * C) . \tag{3.15}$$

These requirements are easily translated into conditions on $\langle V_{1\dagger 23}|$. Let $\langle V_{123}|$ be the operator obtained from $\langle V_{1\dagger 23}|$ by taking the adjoint of the states in the Hilbert space 1. Then the derivation property is equivalent to the condition

$$\langle V_{123}| \left(\mathbf{Q}_1 + \mathbf{Q}_2 + \mathbf{Q}_3\right) = 0 . \tag{3.16}$$

The associative property requires that $\langle V_{123}|$ be cyclically symmetric, and also that the 4-point vertex obtained by contracting two of these objects be cyclically symmetric:

$$\langle V_{123}| = \langle V_{231}| = \langle V_{312}| . \tag{3.17}$$

$$\langle V_{125}| \langle V_{5\dagger 34}| = \langle V_{235}| \langle V_{5\dagger 41}| = \cdots \tag{3.18}$$

It is straightforward to check (ignoring Grassmann minus signs) that the conditions (3.16), (3.17), and (3.18) provide exactly the information required to guarantee the gauge invariance of the action

$$S = \langle\Phi| \mathbf{Q} |\Phi\rangle + \frac{2}{3} g \langle V_{123}| \, |\Phi\rangle_1 \otimes |\Phi\rangle_2 \otimes |\Phi\rangle_3 . \tag{3.19}$$

The derivation property and the three-fold cyclicity suffice to prove gauge invariance to order g. We will see in the next section that these properties are trivial to insure in a large class of string field theory vertices, including vertices which are nonlocal on the world sheet. Thus, the demonstration that a string field theory action is gauge-invariant to order g should not be taken to be a stringent test of its validity. On the other hand, the order g^2 terms in the proof of gauge invariance require the condition of four-fold cyclicity, and this condition is quite nontrivial. We will argue in Section 8 that this criterion is satisfied only for Witten's specific choice of the vertex and a class of its generalizations. (Other choices for the vertex require more complicated string actions, involving four- and possibly higher-string interactions.) The condition of four-fold cyclicity is actually closely related to the duality of string scattering amplitudes, in a sense that we will make clear in Sections 3 and 4 of III.

4. Construction of String Field Actions

Now that we have described the operations needed to construct a gauge-invariant string field theory, let us try to define these operations precisely. Our strategy will be to represent string fields in terms of operators in conformal field theory, and then define operations on string fields as conformal field theory matrix elements of those operators.

Let us first set up a precise prescription for the decomposition of a string field into local component fields. Off mass shell, this decomposition is a matter of convention and can be altered freely by conformal transformations. Still, it will eliminate considerable confusion, especially in our discussion of the symmetries of string field theory, to define a canonical decomposition once and for all. Let us, then, consider the L_0 eigenstates of a first-quantized string theory to be represented as boundary conditions on the unit circle of a conformal plane, with each state evolving from a given collection of operators applied inside the unit circle. For the closed string, this procedure is conceptually straightforward: A given state may be represented as

$$|A\rangle \;=\; \cdots c_{-k}c_{-\ell}\bar{b}_{-p}a^{\mu}_{-m}\bar{a}^{\nu}_{-n}e^{ip_A\cdot X(0)}\,|0\rangle \;\equiv\; \mathcal{O}_A\,|0\rangle \ , \tag{4.1}$$

where a_{-n}, b_{-n}, c_{-n} are defined in (2.12) and $\bar{a}_{-n}$, $\bar{b}_{-n}$, $\bar{c}_{-n}$ are the corresponding quantities built from anti-analytic fields. To discuss open strings, we should properly construct conformal field theory on the upper half plane with Neumann boundary conditions; however, we will use the shorthand of considering conformal field theory on the full plane, but using the analytic sector only. This prescription leads to the same algebra of ladder operators; after we cancel one awkward phase in the correlation function of $\exp(ip\cdot X)$ operators (noted explicitly below), it gives the correct result for all string amplitudes. Most of our explicit analysis will be done for the analytic sector alone. These results will apply equally well to open and closed strings. Using one of these canonical decompositions, then, we can write (3.1) in the form

$$\Phi = \sum_A \phi_A |A\rangle = \sum_A \phi_A O_A |0\rangle . \tag{4.2}$$

The label A represents both the ghost and matter oscillator excitations in a given string state and the center-of-mass momentum p^μ_A. Note that here, in contrast to (3.1), we consider our reference vacuum to be the $SL(2,C)$-invariant vacuum. Then the component fields of the classical string field theory (including those fields written explicitly in (3.1)) multiply operators of ghost number $G = 1$. It will be useful to give a name to the sum of operators appearing in (4.2):

$$\hat{\Phi} = \sum_A \phi_A O_A . \tag{4.3}$$

We can now define operations on the string field Φ by mapping the unit circle described in the previous paragraph into the conformal plane and computing the joint expectation value with other fields. This operation is shown in Fig. 3. For open strings, in which the real axis of the original circle represents the boundary of the string and thus plays a preferred role, a conformal mapping of the circle into the plane is uniquely specified in terms of the image of the boundary of the unit circle, the image of the point $z = 0$, and the orientation of the image of the real axis. For closed strings, the real axis is no longer preferred, but one normally considers only states which are rotationally invariant in the unit circle.

It is simplest to begin by constructing the 3-string interaction. This can be done by the following simple prescription: map 3 string states into the plane by three arbtrarily chosen conformal mappings $h_1(z)$, $h_2(z)$, $h_3(z)$, analytic for z inside the unit circle, by defining

$$\begin{aligned} V(A,B,C) &= \langle V_{123}| \, |A\rangle_1 \otimes |B\rangle_2 \otimes |C\rangle_3 \\ &= \Big\langle h_1\Big[O_A\Big] \, h_2\Big[O_B\Big] \, h_3\Big[O_C\Big]\Big\rangle . \end{aligned} \tag{4.4}$$

The action of the conformal mappings on the string operators can be reconstructed from (2.2); we will make this transformation explicitly in a moment.

For our general geometrical arguments, we must require that the $h_i(z)$ are analytic and invertible for $|z| < 1$. The condition of gauge-invariance will impose stronger restrictions, which we will set out in Sections 5 and 8.

Notice that the ghost counting in the vertex works out precisely right: each string operator has ghost number $G = 1$, and the conformal field theory matrix element annihilates ghost number $G = 3$. This counting makes it slightly problematical to construct a kinetic energy term, since we apparently must include together with the two string fields an additional operator with $G = 1$. But we have a natural candidate for this operator; define

$$\begin{aligned} K(A,B) &= \langle K_{12}|\, |A\rangle_1 \otimes |B\rangle_2 \\ &= \left\langle I\left[\mathcal{O}_A\right] \mathrm{Q}\, \mathcal{O}_B \right\rangle . \end{aligned} \tag{4.5}$$

where I is the inversion (2.18) and Q is the BRST charge. Assembling the pieces, we propose the following form for the open string field theory action:

$$S = \left\langle I\left[\hat{\Phi}\right] \mathrm{Q}\, \hat{\Phi} \right\rangle + \frac{2}{3} g \left\langle h_1\left[\hat{\Phi}\right] h_2\left[\hat{\Phi}\right] h_3\left[\hat{\Phi}\right] \right\rangle . \tag{4.6}$$

To check the validity of (4.6), we must verify that the kinetic energy term has the correct form (3.5) and that the three string product satisfies the identities listed in the previous section. To do this, it will be helpful to recast each term of S in terms of more explicit operations on string states. Let us first discuss the kinetic energy term. Before discussing this expression using the full Q, let us evaluate this term with Q replaced by $\mathrm{Q}_0 = c_0 L_0$. When we quantize the string theory in Section 2 of III and derive the perturbation theory, this will be the gauge-fixed form of the kinetic energy term. Denote the simplified form of $K(A,B)$ as $K_0(A,B)$.

Our first step is to evaluate $K_0(A,B)$ for A and B tachyon states of the form:

$$|A\rangle = c_1 e^{ip_A \cdot X(0)} |0\rangle . \tag{4.7}$$

This entails a few subtleties. Before acting I on the operators in (4.7), we should move them slightly away from $z = 0$. Put the exponential, then, at $z = \epsilon$, with the limit $\epsilon \to 0$ to be taken after the matrix element is evaluated. With this prescription:

$$\begin{aligned} K_0(A,B) &= \left\langle I\left[c_1 e^{ip_A\cdot X(\epsilon)}\right] c_0 L_0\, c_1 e^{ip_B\cdot X(\epsilon)}\right\rangle \\ &= \left\langle c_{-1}\big(\frac{1}{\epsilon^2}\big)^{\frac{1}{2}p_A^2} e^{ip_A\cdot X(-\frac{1}{\epsilon})}\, c_0\big(\frac{1}{2}p_B^2 - 1\big)\, c_1 e^{ip_B\cdot X(\epsilon)}\right\rangle \qquad (4.8) \\ &= \langle c_{-1}c_0c_1\rangle \cdot \big(\frac{1}{2}p_B^2 - 1\big)\cdot\big(\frac{1}{\epsilon^2}\big)^{\frac{1}{2}p_A^2} e^{p_A\cdot p_B \log((-1/\epsilon)-\epsilon)} \ . \end{aligned}$$

To finish this calculation, we must introduce two additional rules: First, it is well known that matrix elements of exponentials of free fields in 2 dimensions are forced to 0 by infrared factors unless the coefficients in the exponentials sum to zero. Let us implement that requirement by associating with every conformal field theory matrix element of exponentials a momentum-conserving delta function:

$$\left\langle \prod_I \epsilon^{ip_I\cdot X(z_I)}\right\rangle \sim (2\pi)^d\delta^d(\sum_I p_I) \ . \qquad (4.9)$$

Imposing momentum conservation on (4.8) causes the factors of $(1/\epsilon)$ to cancel; then we can smoothly take the limit $\epsilon \to 0$. Note that this cancellation does not require the tachyon states to be on shell. The second prescription corrects for our shorthand treatment of the open string in terms of analytic fields: We must replace $\log(z - w)$ by $\log|z - w|$ in the matrix elements of exponentials. Evaluating (4.8) with these rules, we find

$$K_0(A,B) = \big(\frac{1}{2}p_B^2 - 1\big)\cdot(2\pi)^d\delta^d(p_A + p_B) \ ; \qquad (4.10)$$

this is exactly the desired form for the tachyon kinetic energy term.

From this point, it is straightforward to evaluate K_0 for all other string states. Any other string state may be written as a string of L_0-raising ladder operators applied to a tachyon state

$$|A\rangle \;=\; \left[\cdots b_{-m}c_{-\ell}c_{-p}a_{-m}a_{-n}\cdots\right] c_1 e^{ip_A\cdot X(0)}\,|0\rangle \;. \qquad (4.11)$$

Let us define a phase $(-1)^{r(A)}$ by

$$I\Big[[\cdots b_{-m}c_{-\ell}c_{-p}a_{-m}a_{-n}\cdots]c_1\Big]\,c_0 \;=\; (c_{-1}c_0)\cdot(-1)^{r(A)}\cdot[\cdots a_n a_m c_p c_\ell b_m\cdots]\;; \qquad (4.12)$$

the factors of (-1) arise both from the action of I, eq. (2.21), and from the indicated reordering of Grassmann operators. For a physical string state, the product of ladder operators in brackets in (4.11) has $G = 0$ and so contains the same number of b and c operators. Let us define $\tilde{A}$ to be the state obtained from A by interchanging the labels b and c and sending $p_A \to -p_A$. Using these definitions,

$$K_0(A,B) \;=\; \left(\frac{1}{2}p_B^2 \;+\; \frac{1}{2}\mathcal{M}^2\right)\cdot(-1)^{r(A)}\cdot\delta(A,\tilde{B})\;, \qquad (4.13)$$

where $\frac{1}{2}\mathcal{M}^2$ is the usual bosonic string mass operator, equal to the sum of the excitation numbers minus 1.

To make contact with earlier treatments of the kinetic term, we would like to write $K_0(\hat{\Phi},\hat{\Phi})$ as a Hilbert space matrix element. To do this, we must impose the following reality condition on the component fields:

$$\phi_A(p_A) \;=\; (-1)^{r(A)}\big(\phi_A(-p_A)\big)^\dagger\;, \qquad (4.14)$$

where, in this relation, the dependence of the component field on the string momentum is indicated explicitly. This condition is essentially the condition that the string field be real under the combined operations of Hermitian conjugation and reversing the direction of the string coordinate σ. The condition looks less strange if we note that it guarantees the reality of

$$\left\langle I\left[\hat{\Phi}\right] c_0 \, \hat{\Phi}\right\rangle \tag{4.15}$$

and thus reinforces the BPZ inner product. The reader should note that we have not yet defined $r(A)$ for states annihilated by c_0; to do this, replace c_0 in (4.12) by a simple Grassmann number. With all of these conventions,

$$\left\langle I\left[\hat{\Phi}\right] \mathbf{Q}_0 \, \hat{\Phi}\right\rangle \;=\; \langle\Phi|\, \mathbf{Q}_0 \,|\Phi\rangle \;. \tag{4.16}$$

Having now determined the structure of $K_0(A,B)$, it is a simple matter to see that $K(A,B)$ has the correct form. First note that if we write

$$\mathbf{Q} \;=\; \mathbf{Q}_0 \;+\; \Delta\mathbf{Q} \;, \tag{4.17}$$

$\Delta\mathbf{Q}$ annihilates the tachyon state. Thus, the desired equivalence

$$\left\langle I\left[\hat{\Phi}\right] \mathbf{Q} \, \hat{\Phi}\right\rangle \;=\; \langle\Phi|\, \mathbf{Q} \,|\Phi\rangle \tag{4.18}$$

holds for the tachyon components. From here we need only note that our reality condition converts the string of ladder operators in $I[\mathcal{O}_A]$ into that in $\langle A|$, for each higher string state A. The algebraic manipulations required to reduce the higher matrix element to a tachyon matrix element are then identical on the two sides of (4.18). This establishes (4.18) for all components of Φ.

As a check on this identification, it is instructive to show that the expression $\langle I[\mathcal{O}_A]\mathbf{Q}\mathcal{O}_B\rangle$ is symmetric under interchange of A and B, if A and B are states with $G = 1$. This argument will make use of the $SL(2,C)$ invariance of the conformal field theory matrix element and the Grassmann nature of the three operators. It will also use an important property of the BRST charge:

$$[\mathbf{Q}, L_n] \;=\; 0 \;, \text{for all } n \;; \tag{4.19}$$

in other words, $\mathbf{Q}$ commutes with all conformal transformations. This relation, which will play a crucial role later in our analysis, is true only in the critical dimension; it is easily proved by noting that, in that case, the operator product of the BRST current $\mathrm{j}(w)$ with T_{zz} is a total derivative in w. We may apply these ingredients as follows:

$$
\begin{aligned}
\left\langle I\left[\mathcal{O}_A\right] \mathbf{Q}\, \mathcal{O}_B \right\rangle &= \left\langle I \cdot I\left[\mathcal{O}_A\right] I\left[\mathbf{Q}\mathcal{O}_B\right]\right\rangle \\
&= \left\langle \mathcal{O}_A\, \mathbf{Q} I\left[\mathcal{O}_B\right]\right\rangle \\
&= \left\langle (-1)^2 \mathbf{Q}\mathcal{O}_A\, I\left[\mathcal{O}_B\right]\right\rangle \\
&= \left\langle I\left[\mathcal{O}_B\right] \mathbf{Q}\, \mathcal{O}_A \right\rangle ,
\end{aligned}
\tag{4.20}
$$

as required. The two minus signs in the third line have the following origin: one comes from reversing the order of two Grassmann operators, the second comes from reversing the direction of the contour of integration for $\mathbf{Q}$. The origin of this second sign is shown graphically in Fig. 4.

The evaluation of the 3-string vertex (4.5) in terms of Fock space states is even more straightforward; this evaluation can be carried out explicitly for the most general conformal mappings h_1, h_2, h_3. For future reference, we note that the analysis to follow is left unchanged if we relax both the restriction to 3 states and the restriction to states associated with operators of $G = 1$, as long as we continue to insist that the ghost charges of all the operators in the matrix element sum to $G = 3$. To begin this evaluation, write explicitly the conformal transforms of the various components of an operator $\mathcal{O}_A$:

$$
\begin{aligned}
h_1\left[e^{ip\cdot X(0)}\right] &= \left|h_1'(z)\right|^{p^2/2} e^{ip_A\cdot X(h_1(0)} \\
h_1\left[a_{-n}\right] &= \oint \frac{dz}{2\pi i}\, z^{-n} \left(h_1'(z)\right) \partial_z x\left(h_1(z)\right) \\
h_1\left[b_{-n}\right] &= \oint \frac{dz}{2\pi i}\, z^{-n+1} \left(h_1'(z)\right)^2 b\left(h_1(z)\right) \\
h_1\left[c_{-n}\right] &= \oint \frac{dz}{2\pi i}\, z^{-n-2} \left(h_1'(z)\right)^{-1} c\left(h_1(z)\right)
\end{aligned}
\tag{4.21}
$$

The transforms of $\mathcal{O}_B$ and $\mathcal{O}_C$ are computed similarly. Once all operators are placed on the same plane, we can compute their joint expectation value by making use of the contractions (2.13). For example,

$$\begin{aligned} & h_1\Big[\cdots a_{-n}\cdots\Big]h_2\Big[\cdots a_{-m}\cdots\Big] \\ &\quad = \oint \frac{dz}{2\pi i}\, z^{-n}\left(h_1'(z)\right) \oint \frac{dw}{2\pi i}\, w^{-m}\left(h_2'(w)\right) \frac{-1}{(h_1(z)-h_2(w))^2}\,. \end{aligned} \tag{4.22}$$

The factors of $\partial_z x$ in the a_{-n} operators can also contract with factors of $p\cdot X$ in the exponentials. To complete the evaluation of the correlation function of operators built from $X(z)$, we must compute the matrix element of exponentials:

$$\left\langle \prod_I e^{ip_I\cdot X(h_I(0))} \right\rangle = \exp\Big(\sum_{I<J} p_I\cdot p_J \log\left|h_I(0)-h_J(0)\right|\Big)\cdot (2\pi)^d\delta^d\Big(\sum_I p_I\Big) \tag{4.23}$$

The absolute values appearing in this expression and in the first line of (4.21) correct our analytic-fields shorthand for the open string. To evaluate the correlation function of ghost operators, we need contractions of the form

$$\begin{aligned} & h_1\Big[\cdots b_{-n}\cdots\Big]h_2\Big[\cdots c_{-m}\cdots\Big] \\ &\quad = \oint \frac{dz}{2\pi i}\, z^{-n+1}\left(h_1'(z)\right)^2 \oint \frac{dw}{2\pi i}\, w^{-m-2}\left(h_2'(w)\right)^{-1} \frac{1}{(h_1(z)-h_2(w))}\,. \end{aligned} \tag{4.24}$$

We must also remember that three operators $c(z)$ must be used to saturate the three ghost zero modes, as was indicated in eq. (2.24).

The result of this evaluation can be represented compactly, in just the form of (4.5), as an operator acting on three single-string Hilbert spaces:

$$\Big\langle h_1\Big[\mathcal{O}_A\Big]\, h_2\Big[\mathcal{O}_B\Big]\, h_3\Big[\mathcal{O}_C\Big]\Big\rangle = \langle V_{123}|\, |A\rangle_1 \otimes |B\rangle_2 \otimes |C\rangle_3\,. \tag{4.25}$$

Let us simply write the final expresion for $\langle V_{123}|$; the reader can check that it reproduces the results of the calculation just defined for all possible states

$|A\rangle_1 \otimes |B\rangle_2 \otimes |C\rangle_3$ on which it might act. We find

$$
\begin{aligned}
\langle V_{123}| &= \langle 3|_1 \otimes \langle 3|_2 \otimes \langle 3|_3 \int_{p_A,p_B,p_C} (2\pi)^d \delta^d(p_A+p_B+p_C) \int_{\varsigma_1,\varsigma_0,\varsigma_{-1}} \\
&\cdot \exp\Big(-\frac{1}{2}\sum_{I,J}\sum_{n,m\geq 0} a_n^I N_{nm}^{IJ} a_m^J + \sum_{I,J}\sum_{\substack{n\geq 2\\ m\geq -1}} c_n^I \tilde{N}_{nm}^{IJ} b_m^J - \sum_{J}\sum_{\substack{i=\pm 1,0\\ m\geq -1}} \varsigma_i M_{i\ m}^{\ J} b_m^J\Big) ,
\end{aligned}
\tag{4.26}
$$

where I, J label strings $1,2,3$, the subscripts n, m are mode numbers, and $i = -1,0,1$ labels zero modes. The values of the *Neumann coefficients* N_{nm}^{IJ} are given below. The ς_i are a set of classical Grassmann variables whose significance will be explained in a moment. For the variables a_n^I, n runs over the values $0,1,2,\ldots$, with $a_0^I = -ip_I$. The Neumann coefficients with nonzero indices represent the contraction (4.22). The coefficients with indices 0 represent contractions involving the exponentials and the prefactor in the first line of (4.21). These coefficients are given explicitly by:

$$
\begin{aligned}
N_{00}^{IJ} &= \begin{cases} \log|h_I'(0)| & I=J\ , \\ \log|h_I(0)-h_J(0)| & I\neq J\ , \end{cases} \\
N_{0m}^{IJ} &= \frac{1}{m}\oint \frac{dw}{2\pi i} w^{-m} \left(h_J'(w)\right) \frac{-1}{(h_I(0)-h_J(w))}\ , \\
N_{nm}^{IJ} &= \frac{1}{n}\oint \frac{dz}{2\pi i} z^{-n} \left(h_I'(z)\right) \frac{1}{m}\oint \frac{dw}{2\pi i} w^{-m} \left(h_J'(w)\right) \frac{1}{(h_I(z)-h_J(w))^2}\ .
\end{aligned}
\tag{4.27}
$$

For the ghost variables, we should recall that $\mathcal{O}_A$ contains only ghost operators which do not annihilate $|0\rangle$, and so the vertex should contain only operators which do not annihilate $\langle 3|$. This implies that, for b_n^I, n runs over $-1,0,1,2,\ldots$, while for c_n^I, n runs over $2,3,\ldots$. The Neumann coefficients may be read from (4.24):

$$
\tilde{N}_{nm}^{IJ} = \oint \frac{dz}{2\pi i} z^{-n+1} \left(h_I'(z)\right)^2 \oint \frac{dw}{2\pi i} w^{-m-2} \left(h_J'(w)\right)^{-1} \frac{-1}{(h_I(z)-h_J(w))}\ .
\tag{4.28}
$$

The extra minus sign relative to (4.24) compensates an extra Grassmann-interchange minus sign which appears when the term containing this N^{IJ}_{nm} is applied to a pair of ghost ladder operators $b_{-n}\cdots|0\rangle_1\otimes\cdots c_{-m}\cdots|0\rangle_2\otimes\cdots$. Since the ordering of Grassmann operators is the same on both sides of (4.25), this is the only correction needed to account for all of the Grassmann minus signs from b-c contractions The zero modes of $c(z)$ are taken into account in (4.26) by the introduction of the classical Grassmann variables ς_i. We define

$$M_i{}^J{}_m = \oint \frac{dw}{2\pi i}\, w^{-m-2}\left(h'_J(w)\right)^{-1} Z_i(h_J(w)) \;, \tag{4.29}$$

where $Z_i(z) = z^{i+1}$. Then if we set

$$\int_{\varsigma_1,\varsigma_0,\varsigma_{-1}} \varsigma_1\varsigma_0\varsigma_{-1} \;=\; 1 \;, \tag{4.30}$$

the integral over the three ς_i will pick out three operators $c(z)$ and assign them to the three zero modes. The reader can check that, again, all the Grassmann minus signs are accounted for.

We have now constructed a string field theory vertex of a very general form and given its explicit representation as an operator on three single-string Fock spaces. The general form of this representation, in which the Neumann coefficients are defined in terms of contour integrals, has been seen many times in discussions of specific string vertex functions, beginning with the work of Ademollo, del Guidice, di Vecchia, and Fubini,[65] Mandelstam,[2] Kaku and Kikkawa,[70] and Cremmer and Gervais[71] on the light-cone vertex. The fact that this decomposition holds for arbitrary conformal mappings which link the strings is, however, a very powerful observation, and we will make strong use of this observation in the course of our analysis.

5. Symmetries of the 3-String Vertex

We have now introduced a proposal for the 3-string vertex of a very general form, built from three conformal transformations chosen completely arbitrarily. We have learned in the discussion just concluded that this general vertex is still a very simple object. It is, for example, technically much more transparent than the kinetic energy term. We might bolster this conclusion by recalling a deep speculation of of Hata, Itoh, Kugo, Kunitomo, and Ogawa[41] and Horowitz, Lykken, Rohm, and Strominger.[28] These authors have argued that in string field theory only the 3-string vertex is fundamental, and that the quadratic terms in the string field action are generated dynamically by replacing one field by its nontrivial vacuum expectation value. Whether one finds this speculation compelling or not,* it is certainly worth digressing to work out the symmetries of the vertex we have defined and to ask what restrictions must be imposed on the general form to build in higher symmetries.

Before beginning this study, let us remark that the vertex we have proposed

$$V(A,B,C) \;=\; \Big\langle h_1\Big[\mathcal{O}_A\Big]\, h_2\Big[\mathcal{O}_B\Big]\, h_3\Big[\mathcal{O}_C\Big]\Big\rangle \tag{5.1}$$

is essentially a realization of a deep and, for us, quite mysterious speculation of Friedan[72] that the 3-string vertex should be identified with the operator product coefficient of the vertex operators corresponding to the three string states. The success of our construction might, then, shed light on other aspects of Friedan's geometrical intuition.

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* We note that the equation $\{\mathbf{Q}, \mathbf{Q}_L\}$, which is necessary for the proof of equivalence between the cubic action and Witten's, does not hold with the naive definition of the BRST current. The autors of refs. [33] and [34] justify it by adding normal-ordering terms to the BRST current:

$$j_{BRST} = cT^{z} + c\partial_z cb + z^{-2}c - 2z^{-1}\partial_z c + 3/2\partial_z^2 c\ .$$

The additional terms of course do not transform like a dimension 1 conformal field. Since this property of the BRST current is of vital importance in our approach to string field theory, we have reservations about this procedure.

Let us now discuss the effect on $V(A,B,C)$ of conformal transformations. We consider in turn the two possibilities:

$$h_I(z) \;\to f \circ h_I\,(z) \;, \tag{5.2}$$

$$h_I(z) \;\to h_I \circ f\,(z) \;, \tag{5.3}$$

The operation (5.2) transforms V according to

$$V(A,B,C) \;\to\; \Big\langle f \circ h_1\Big[\mathcal{O}_A\Big]\, f \circ h_2\Big[\mathcal{O}_B\Big]\, f \circ h_3\Big[\mathcal{O}_C\Big]\Big\rangle \;. \tag{5.4}$$

It is sometime useful to think of such conformal transformations as being generated by the action of the Virasoro operators: Let $v(z) = v_n z^{-n+1}$ be the vector field generating the conformal transformation $z \to f(z)$. Then

$$f\big[\phi(z)\big] = [f'(z)]^d \phi\,(f(z)) = U_f \phi(z) U_f^{-1} \;, \tag{5.5}$$

where $U_f = \exp(v_{-n}L_n)$. The reader should note that our various definitions are consistent in their operator ordering:

$$f \circ h\big[\phi(z)\big] = f\Big[h\Big[\phi\Big]\Big] = U_f U_h\, \phi(z)\, U_h^{-1} U_f^{-1} \;. \tag{5.6}$$

Since conformal field theory matrix elements are $SL(2,C)$-invariant, we expect that $V(A,B,C)$ will be unchanged by the transformation (5.2) if $f \in SL(2,C)$. Thus, our vertex, even in its most general form, addresses a conjecture of Banks[73] that the 3-string vertex can be cast into an $SL(2,C)$-invariant form. The exercise of checking this invariance explicitly provides a simple and appealing confirmation of the Fock space form for the vertex given in the previous section. Let us carry out that analysis, and afterward discuss the situation for more general functions $f(z)$.

Consider, then, the transformation (5.2), for

$$f(w) = \frac{aw+b}{cw+d}\,, \qquad ad - bc = 1\,. \tag{5.7}$$

From this form, it follows that

$$(f\circ h_I)'(z) = \frac{h_I'(z)}{(ch_I(z)+d)^2}\,,\; f\circ h_I(z) - f\circ h_J(z) = \frac{h_I(z)-h_J(z)}{(ch_I(z)+d)(ch_J(z)+d)}\,. \tag{5.8}$$

Inserting these identities into the transformed Neumann coefficients (4.27), we see that the extra factors of $(ch_I + d)$ neatly cancel out. (In N^{IJ}_{nm}, this cancellation is automatic. In the first two terms, which involve the zero modes, one must also make use of the fact that $\sum_I p_I = i\sum_I a_0^I = 0$.) For the ghosts the situation is somewhat more subtle. The $SL(2,C)$ invariance is not yet manifest in (4.26), though it can be made manifest by the following set of rearrangements: First, consider the action of (4.26) on three states of the form of physical string excitations:

$$\mathcal{O}_A = \cdots a_{-n}a_{-m}a_{-p}\cdots c_1 e^{ip_A\cdot X(0)}\,, \tag{5.9}$$

where the string of ladder operators contains only a_ns. Using $c_1 = c(0) \rightarrow (h_I'(z))^{-1}c(h_I(z))$ to represent the ghost operator on each state, and using the three ghost operators which result to saturate the three zero modes, we find for the ghost part of the evaluation of $V(A,B,C)$:

$$\left(h_1'(0)h_2'(0)h_3'(0)\right)^{-1}\left[(h_1(0)-h_2(0))(h_1(0)-h_3(0))(h_2(0)-h_3(0))\right]\,. \tag{5.10}$$

Now consider acting (4.26) on string states with a more general ghost structure. It can be shown that the the result can be written as a product of (5.10) with antisymmetrized contractions of the other b_{-n} and c_{-n} operators, provided that the contraction is altered slightly from (4.24) to

$$\oint \frac{dz}{2\pi i} z^{-n+1} \left(h_1'(z)\right)^2 \oint \frac{dw}{2\pi i} w^{-m-2} \left(h_2'(w)\right)^{-1} \frac{1}{(h_1(z) - h_2(w))} \cdot \prod_K \frac{(h_2(w) - h_K(0))}{(h_1(z) - h_K(0))}, \tag{5.11}$$

where K runs over the three strings. The form of (5.11) is easy to understand: It is a Fourier transform of the b–c propagator in the presence of the three $c(h_K(0))$ zero mode operators; these extra ghost operators require the propagator to contain extra poles and zeros. The proof of this rearrangement is straightforward; we give it in the Appendix. The reader can now verify that the expressions (5.10), (5.11) return to their original form when we substitute (5.2) and then invoke the identities (5.8). Thus, both the coordinate and ghost parts of our vertex are explicitly $SL(2,C)$-invariant, even before we restrict the mappings $h_I(z)$.

Is our vertex invariant to more general conformal transformations? This is a subtle question, for the following reason: $SL(2,C)$ transformations are the most general conformal transformations which map the complex plane onto itself in a single-valued way. Any more general choice for $f(z)$ will carry the plane into a Riemann surface with branch points. That brings us outside the class vertices defined by (4.4), and we need to extend our prescription to treat this case. The most natural way to evaluate conformal field theory expectation value on this Riemann surface would be to evaluate the $\langle XX\rangle$ and $\langle bc\rangle$ propagators by mapping the surface back into a plane. With this definition, the form of (4.26) remains unchanged. It is true that the evaluation of expectation values for a conformal field on a Riemann surface will also produce a factor of the determinant of the Laplacian for that field. However, for Riemann surfaces of the topology of a plane, and assuming that we work in the critical dimension, the determinant factors should cancel between coordinate and ghost fields. In the analysis which we will present in II, we will see that the formalism dictates that the branched surfaces which arise from gluing together two vertices should be treated in exactly this way, and that the determinant factors cancel explicitly in that case. With this prescription, we find that the vertex (5.1), in the critical dimension, is invariant to all conformal transformations of the form (5.2).

Transformations of the form (5.3) are conformal transformations of the unit circle used to define the canonical set of string states. These transformations may thus be viewed as field redefinitions. Normally, however, we will wish to leave the kinetic energy term in the action unchanged; then we must ask whether these transformations leave the form of the vertex invariant. This will be true only if the transformations of the form (5.3) have appropriate commutation relations with the h_I to be converted into the form (5.2). The question of which transformations have this property can only be discussed case by case for particular forms of the vertex. The important special case of the reparametrization (K_n) symmetries of Witten's vertex[8,9] will be discussed in Section 8.

One often wishes to answer the question of whether two different 3-string vertices are equivalent on-shell. In our formalism, this question is very easy to address. On-shell states correspond to vertex operators $\mathcal{O}_A$ which are primary conformal fields of dimension 0, evaluated at $z = 0$. For three such operators, our general vertex takes the form

$$\left\langle h_1\Big[\mathcal{O}_A(0)\Big]\, h_2\Big[\mathcal{O}_B(0)\Big]\, h_3\Big[\mathcal{O}_C(0)\Big]\right\rangle \;=\; \Big\langle \mathcal{O}_A(h_1(0))\;\mathcal{O}_B(h_2(0))\;\mathcal{O}_C(h_3(0))\Big\rangle \;. \tag{5.12}$$

By $SL(2,C)$, this correlator is a pure number, independent of the three points $h_I(0)$. Thus, a vertex defined by any other triple of mappings $\hat{h}_1(z)$, $\hat{h}_2(z)$, $\hat{h}_3(z)$ will be equivalent on shell.

Let us now turn to the properties of the vertex needed to establish the gauge-invariance of the action, according to the logic of Section 4. Consider first the property (3.16), which is just the requirement that the vertex have a BRST symmetry. We will now show that our vertex is BRST invariant in the general form (5.1), even before any specific choice is made for the h_I. To see this, note that, for our vertex, (3.16) is equivalent to the condition

$$0 = \left\langle h_1\left[\mathrm{Q}\mathcal{O}_A\right] h_2\left[\mathcal{O}_B\right] h_3\left[\mathcal{O}_C\right]\right\rangle + (-1)^A \left\langle h_1\left[\mathcal{O}_A\right] h_2\left[\mathrm{Q}\mathcal{O}_B\right] h_3\left[\mathcal{O}_C\right]\right\rangle + (-1)^{A+B} \left\langle h_1\left[\mathcal{O}_A\right] h_2\left[\mathcal{O}_B\right] h_3\left[\mathrm{Q}\mathcal{O}_C\right]\right\rangle , \tag{5.13}$$

where $(-1)^A$ is the Grassmann parity of $\mathcal{O}_A$. But this is manifestly true: Since Q commutes with conformal transformations, each factor Q can be carried outside the corresponding h_I:

$$h_I\left[\mathrm{Q}\mathcal{O}_A\right] = \mathrm{Q}\, h_I\left[\mathcal{O}_A\right] . \tag{5.14}$$

The three Qs can be brought to the front in Grassmann ordering (cancelling the factors of $(-1)^A$) and their contours joined. The resulting contour can then be pushed to infinity, as indicated in Fig. 5. The vertex (5.1) is thus manifestly BRST invariant in the critical dimensionality, for all choices of the h_I. A similar argument has been given, specifically for the Witten vertex, by Itoh *et al.*.[19] The second property which we must require is the threefold cyclic symmetry. This property places restrictions on the h_I, but there is a simple condition which insures this symmetry for a large class of vertices. Let us impose this condition as follows:

$$h_1 = T^2 \circ h , \qquad h_2 = T \circ h , \qquad h_3 = h , \tag{5.15}$$

where $T \in SL(2,C)$, $T^3 = 1$. This criterion is slightly too restrictive to include the vertex of the light-cone type, which, in any event, does not lead to an open-string theory with only cubic interactions. (The application of our formalism to light-cone field theories will be discussed elsewhere[74]) This form does apply to the Witten vertex[8] and the Caneschi-Schwimmer-Veneziano[62] dual model vertex. It also encompasses many other possible vertices, in which the three regions which are the images of the unit circle under h_1, h_2, h_3 overlap arbitrarily on the complex plane. The proof that (5.15) suffices to make $V(A, B, C)$ cyclically symmetric is quite similar to the proof of the symmetry of the kinetic energy term given in the previous section (eq. (4.20)). Using the $SL(2,C)$ invariance of the conformal field theory matrix element, we can write

$$\begin{aligned}\left\langle T^2h\left[\mathcal{O}_A\right]Th\left[\mathcal{O}_B\right]h\left[\mathcal{O}_C\right]\right\rangle &= \left\langle T^3h\left[\mathcal{O}_A\right]T^2h\left[\mathcal{O}_B\right]Th\left[\mathcal{O}_C\right]\right\rangle \\ &= (-1)^{A\cdot(B+C)}\left\langle T^2h\left[\mathcal{O}_B\right]Th\left[\mathcal{O}_C\right]h\left[\mathcal{O}_A\right]\right\rangle\,. \end{aligned} \tag{5.16}$$

One or all three of the operators creating A, B, C will be Grassmann-odd. This implies that the prefactor in the last line is $(+1)$, and so the vertex is cyclically symmetric.

A similar argument shows that the structure

$$\left\langle S^3h\left[\mathcal{O}_A\right]S^2h\left[\mathcal{O}_B\right]Sh\left[\mathcal{O}_C\right]h\left[\mathcal{O}_D\right]\right\rangle \tag{5.17}$$

has a fourfold cyclic symmetry if $S \in SL(2,C)$, $S^4 = 1$, and A, B, C, D comprise three states of $G = 1$ and one of $G = 0$, corresponding to the gauge parameter string field Λ. A vertex passes all of the requirements needed to form a gauge-invariant action with only three-string interactions if the contraction of two vertices indicated schematically in eq. (3.18) has the structure of (5.17). The main purpose of Section 7 will be to define this contraction operation precisely and reduce this operation to geometry, so that a precise comparison with (5.17) can be made. Before beginning this analysis, however, it will be worthwhile to illustrate our construction at its present state of development by considering some interesting specific choices for the h_I.

6. Examples of String Vertices

At this point in our development, we admit 3-string vertices of the general form

$$\left\langle T^2 h\left[\mathcal{O}_A\right] Th\left[\mathcal{O}_B\right] h\left[\mathcal{O}_C\right]\right\rangle \tag{6.1}$$

with T an $SL(2,C)$ transformation such that $T^3 = 1$ but with $h(z)$ a completely arbitrary map of the unit circle into the complex plane. Two choices for h are especially simple. The first is $h(z) = z$, which carries the unit circle into the complex plane unchanged. The second is the map which carries the unit circle into a wedge covering a 120° angle, so that the images of three unit circles cover the plane and abut one another neatly along their boundaries. These two possibilities are illustrated in Fig. 6. In this section, we would like to work out the consequences of these two choices for $h(z)$. The first of these possibilities will lead to a string field theory vertex which generalizes the Caneschi-Schwimmer-Veneziano[62] dual model vertex. This generalization has recently been presented, from another viewpoint, by Di Vecchia *et al.*[68] and by Watamura and Watamura.[75] The second choice will be seen to represent the Witten vertex.

Consider first the case $h(z) = z$. To construct (6.1), we must supplement this choice with a suitable T which acts nontrivially on the unit circle. Since $T \in SL(2,C)$, it is specified by its action on three points. Choose, then,

$$\begin{array}{lcccc} T & : & 0 & \rightarrow & 1 \\ & & 1 & \rightarrow & \infty \\ & & \infty & \rightarrow & 0 \end{array} \qquad\qquad \begin{array}{lcccc} T^2 & : & 0 & \rightarrow & \infty \\ & & 1 & \rightarrow & 0 \\ & & \infty & \rightarrow & 1 \end{array} \tag{6.2}$$

that is,

$$Tz = \frac{1}{1-z}\,, \qquad T^2 z = \frac{z-1}{z}\,. \tag{6.3}$$

The three unit circles corresponding to the three coupled strings are shown in

Fig. 7. The $SL(2,C)$ transformation

$$f(z) = -i\left(\frac{1-e^{i\pi/3}z}{z+e^{i\pi/3}}\right) \tag{6.4}$$

carries the unit circle into the right half-plane, sending the real axis to the new unit circle. Applying this transformation to the vertex according to (5.4), we find a second picture of this vertex in which the three strings are mapped to three half-planes rotated from one another by 120° (and thus overlapping in 60° sectors). This picture is shown in Fig. 8.

It is illuminating and also quite straightforward to compute the Neumann coefficients N^{IJ}_{nm} associated with the coordinate degrees of freedom. The factors N^{IJ}_{00} all vanish; thus, the expression (4.26) contains factors of the center-of-mass momenta p^I only in conjunction with oscillator creation and annihiliation operators. In this case, then, all 3-field couplings derived from the string field theory vertex are polynomial in momenta, that is, local in space-time. The remaining coefficients are nonzero is a cyclically symmetric pattern. Representative coefficients are:

$$\begin{aligned} N^{IC}_{0m} &= \begin{cases} -\frac{1}{m} & \text{if } I = A \\ 0 & \text{otherwise.} \end{cases} \\ N^{CC}_{mn} &= 0\ . \\ N^{BC}_{mn} &= \begin{cases} (-1)^{m+1}\frac{(n-1)!}{m!(n-m)!} & n \geq m \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{6.5}$$

The remaining coefficients can be found by cycling. Inserting these values into (4.26), one finds a dependence on the coordinate oscillators which is exactly that of the CSV vertex. This close connection between the CSV vertex and the cyclic group (6.2) is not surprising; this connection was set out clearly in an early paper of Lovelace.[66] The full expression (4.26) provides a manifestly BRST-invariant generalization of the CSV vertex, in which all couplings, including those of auxiliary fields, are local in space time.

Is this very simple vertex an acceptable vertex for a string field theory? Since we are assured that it is 3-fold cyclic and BRST-invariant, only the condition of 4-fold cyclicity remains to be checked. The computation suggested by (3.18) is in fact straightforward to carry out explicitly for the coordinate oscillators. What is needed is the product

$$\langle V_{ABC}|\,\langle V_{DEF}|\;|I_{CD}\rangle\ , \tag{6.6}$$

where $|I_{CD}\rangle$ is a suitable inner product. The natural choice for this product is the BPZ inner product described in Section 2. Considering the coordinate oscillators only, we may write (2.20) in the explicit form

$$\langle A|B\rangle \;=\; \langle I_{AB}|\;|A\rangle\otimes|B\rangle\ , \tag{6.7}$$

where

$$\langle I_{AB}| \;=\; \langle 3|_A\otimes\langle 3|_B\cdot\exp\Big(\sum_n a_n^A\frac{(-1)^n}{n}a_n^B\Big)\ . \tag{6.8}$$

We may define the right inner product as the inverse of this operator

$$\langle I_{AB}|\;|I_{CB}\rangle \;=\; \mathbf{1}_{CA}\ . \tag{6.9}$$

From (6.8), we obtain

$$|I_{AB}\rangle \;=\; \exp\Big(\sum_n a_{-n}^A\frac{(-1)^n}{n}a_{-n}^B\Big)\;|0\rangle_A\otimes|0\rangle_B\ . \tag{6.10}$$

This expression will suffice for our immediate purposes; a complete representation of $|I_{AB}\rangle$, including the ghost pieces, will be given in the next section.

Combining two CSV vertices according to (6.6) using the explicit inner product (6.10), we obtain the following result: The fused vertex, which is now a

4-string vertex, has again the general form

$$\langle V_{ABEF}| \;=\; \langle 3|_A \otimes \langle 3|_B \otimes \langle 3|_E \otimes \langle 3|_F \;\exp\Big(-\frac{1}{2}\sum_{I,J}\sum_{n,m} a^I_n N^{IJ}_{nm} a^J_m\Big)\;. \qquad (6.11)$$

The Neumann coefficients linking A and B and those linking E and F are unchanged from the values indicated in (6.5). The remaining coefficients are given by contractions of the coefficients in (6.5); for example,

$$N^{AF}_{mn} \;=\; \sum_{k=1}^{\infty} N^{AC}_{mk}\,(-1)^{k+1}k\,N^{DF}_{kn}\;. \qquad (6.12)$$

Carrying out these summations, we eventually find

$$\begin{aligned} N^{AF}_{mn} &= N^{EB}_{mn} \;=\; \begin{cases} 2^{m-n}\cdot(-1)^{n+1}\frac{(m-1)!}{n!(m-n)!} & m \geq n \\ 0 & \text{otherwise.} \end{cases} \\ N^{BF}_{mn} &= F(n+1, m+1, 2; -1) \\ N^{AE}_{mn} &= 2^{n+m}\cdot F(n+1, m+1, 2; -1)\;. \end{aligned} \qquad (6.13)$$

These Neumann coefficients do not form a cyclic structure; the result is tantalizingly close to cyclic but is ruined by the factors of 2^n. Still, (6.13) has a simple physical interpretation. It is not difficult to check that this set of Neumann coefficients is exactly that obtained from the vertex

$$\Big\langle T^2\Big[\mathcal{O}_A\Big]\,T\Big[\mathcal{O}_B\Big]\;IT^2\Big[\mathcal{O}_E\Big]IT\Big[\mathcal{O}_F\Big]\Big\rangle\;, \qquad (6.14)$$

where T is still given by (6.3) and I is the inversion. The contraction (6.6) has apparently carried out the operation shown in Fig. 9: The two complex planes representing the two vertices have been cut along the circles representing the states C and D and then glued together by conformally mapping the exterior of the D circle via the inversion I into the interior of the C circle. The failure of 4-fold cyclic symmetry is apparent in (6.14): This vertex cannot be of the form (5.17) because it contains the threefold elements T, T^2. One can observe, however, that the four points $h_I(0)$—the four points $-1, 0, 1, \infty$—have a fourfold cyclic symmetry in the sense that they are linked by an $SL(2, C)$ transformation

$$S(z) = \frac{1+z}{1-z} \tag{6.15}$$

satisfying $S^4 = 1$. The transformation S bear no relation to any of the transformations appearing in (6.14). Nevertheless, the logic of eq. (5.12) leads us to state that the CSV vertex satisfies fourfold cyclicity at least on shell. This fact has an amusing physical consequence which we will present in Section 3 of III.

If the CSV vertex does not give an appropriate string field theory vertex, how then do we find one? As a guide in thinking about this question, let us assume that the geometrical representation of the BPZ inner product that we have partially derived in the previous paragraph is correct and general. Then we must search for a way to map three unit circles onto the complex plane in a fashion which is threefold symmetric and which, after a cutting and gluing operation such as that defined above, produces a figure which is fourfold symmetric. It was exactly this geometrical requirement which motivated Witten in his choice of vertex. The second possibility shown in Fig. 6 is indeed manifestly threefold symmetric. Further, the operation of cutting out one of the three regions on each of two planes and then gluing the two planes together produces a fourfold symmetric diagram. This diagram is shown in three views in Fig. 10. If the second (EF) plane is mapped conformally into the hole left in the first, the resulting figure has an angle of $4 \cdot 120°$ at the joining point. We may represent this by drawing a plane with a branch cut, as shown in Fig. 10(a); the figure is the covering surface of this cut plane. Rotation of the branch cut reveals the whole of region F and, in turn, conceals a part of region B. Alternatively, we may use the (formal) conformal invariance of the vertex to map the twisted surface into a simple plane by applying

$$f(z) = z^{3/4} \tag{6.16}$$

The effect of this transformation is to smooth out the fold at the joining point and convert the surface into Fig. 10(c).

To see that the construction shown in Fig. 6(b) is indeed equivalent to Witten's prescription, we may proceed in stages. First, as we have discussed in our arguments below eq. (2.21), the conformal field theory matrix element involving two unit circles which have been mapped so as to abut one another has the operator interpretation of a contact delta-function on the world-sheet. In each unit circle, the boundary condition is determined from the operators placed inside by evolution using the Hamiltonian of radial quantization. If these unit circles are mapped so that their boundaries coincide, this adds the constraint that these generated boundary conditions be identical. To give the diagram an interpretation in terms of open strings, we must specify where the string boundary appears on the conformal plane. Throughout our discussion of the Witten vertex, we will map unit circles in such a way that the real axis of the original circle is mapped to the unit circle of the conformal plane. Then the images of open strings can be viewed in the interior of the unit circle on the plane. In Fig. 6(b), the interior of the unit circle is neatly trisected into three regions which overlap at their boundaries. The string endpoints are mapped to this unit circle; the string midpoints are mapped to $z = 0$.

To solidify this interpretation, let us work out the operator decomposition of our vertex somewhat more explicitly and compare it to other forms for the Witten vertex which have been given in the literature. The Neumann coefficients for coordinate oscillators are given by the general relation (4.27) in terms of the conformal mappings $h_I(z)$ which carry unit circles into the plane. The mapping which carries a unit circle into a 120° wedge in the right half-plane has been written down by a number of authors;[76,9,12,19,11] it is easy to construct by following the steps shown in Fig. 11. The result is

$$h(z) = \left(\frac{1-iz}{1+iz}\right)^{2/3} \tag{6.17}$$

The three h_I are then given as h, Th and T^2h, with T given by the rotation $Tz = e^{2\pi i/3}z$. Since $|h'_I(0)| = 4/3$ and $|h_I(0) - h_J(0)| = \sqrt{3}$, the piece of V quadratic in momenta is given by

$$\prod_I \left|\frac{4}{3}\right|^{\frac{1}{2}p_I^2} \prod_{I \neq J} |\sqrt{3}|^{\frac{1}{2}p_I p_J} = \prod_I \left|\frac{4}{3\sqrt{3}}\right|^{\frac{1}{2}p_I^2} , \tag{6.18}$$

where we have used momentum conservation. The remaining Neumann coefficients are given by integrals of the form

$$N^{BC}_{n\,m} = \frac{1}{nm} \oint_{h_I(0)} \frac{dz}{2\pi i} \oint_{h_J(0)} \frac{dw}{2\pi i} \frac{1}{(z-w)^2} [h_I^{-1}(z)]^{-n} [h_J^{-1}(w)]^{-m} . \tag{6.19}$$

This is exactly the form of the Neumann coefficients Gross and Jevicki[9] used as the starting point for their analysis of Witten's vertex. An elegant proof of equivalence of the various formulations of this vertex[12,9,11,14,19] that includes the bc-sector has been given by Suehiro.[48] The explicit expressions for the Neumann coefficients are necessary to study the local field decomposition of a string field theory vertex. However, they are complex and unwieldy, and one might hope that they are not actually needed to prove the general properties of the vertex. We have seen already that our geometrical method allows one to circumvent completely the use of the explicit Neumann coefficients in verifying the BRST-invariance of the vertex. We have seen above that the geometrical intuition which our method supplies allows one to understand the cyclicity conditions as well. It is less clear, though, that the fourfold cyclicity condition can actually be proved along these lines. It is possible to construct such a proof, however, by giving a general proof of the correspondence suggested earlier in this section between the contraction with the BPZ inner product and the geometrical operation of gluing. We devote the next two sections to the presentation of that proof.

7. Gluing Circles to Circles

In this section, we will present a more precise connection between the operation of contraction with the BPZ inner product, eq. (6.6), and the gluing operation indicated schematically in Fig. 9. For simplicity, we will restrict ourselves here to the case where the two states which are contracted are represented on the plane by circles; equivalently, we assume that the mappings $h_I(z)$ corresponding to the contracted states C and D are elements of $SL(2, C)$. At the end of this section, we will generalize the statement of the result to the more complicated case in which these $h_I(z)$ are general conformal mappings.

In some sense, the identity we are trying to establish is obvious from the beginning. We can view the functional integral defining $\langle V_{ABC}|$, for fixed state C, as a functional integral with definite boundary conditions on the boundary of the region into which C is mapped. The contraction of C and D identifies the boundaries of their respective regions and sums over possible boundary conditions. This should produce a single functional integral over the full fused region. On the other hand, it would clearly be valuable to make this argument more precise. In addition, there are several subtle points which can be settled only by careful analysis: What happens to the ghost zero modes on the fused planes? If the fusion introduces folds or singular points, as we saw in our examination of the Witten vertex, how should these singularities be treated? We will indeed see the answers to these questions emerge from our analysis.

In the literature, a form of the the gluing relation has been assumed in Witten's work [8,45] and in the papers of Giddings and Martinec.[21,22] Mandelstam's original work [2] on the light-cone vertex actually proved a form of this relation specific to that case as a set of explicitly derived identities for the Neumann coefficients. This proof was developed and clarified in the work of Cremmer and Gervais [71] and Green and Schwarz.[4] The work of Gross and Jevicki [9] gives an explicit, though not quite complete proof of the gluing relation for the Witten vertex in the specific configuration needed to prove gauge-invariance. We are not aware of a previous attempt to formulate this relation as a mathematical identity satisfied by a general class of string vertices.

In this section, we will prove the following identity, which we call the *gluing theorem for circles*: Let $h_{Ai}(z)$ $h_{Bj}(z)$ be some number of conformal mappings bringing an arbitrary number of unit circles into two conformal plane. In each conformal plane, we include also one unit circle which has been brought in using the identity operator. (For each vertex, any $h(z)$ which belongs to $SL(2,C)$ can be reduced to the identity by a tranformation (5.2).) Let $\langle V_{\{Ai\}C}|$ be defined by

$$\langle V_{\{Ai\}C}| \prod_i |Ai\rangle \otimes |C\rangle \equiv \left\langle \prod_i \left(h_{Ai}\left[\mathcal{O}_{Ai}\right]\right)\ \mathcal{O}_C \right\rangle \tag{7.1}$$

and let $\langle V_{\{Bj\}D}|$ be defined similarly. Then, if $|I_{CD}\rangle$ is the BPZ inner product, the fused vertex

$$\langle V_{\{Ai\}\{Bj\}}| = \langle V_{\{Ai\}C}|\, \langle V_{\{Bj\}D}|\ |I_{CD}\rangle \tag{7.2}$$

is given by

$$\langle V_{\{Ai\}\{Bj\}}| \prod_i |Ai\rangle \otimes \prod_j |Bj\rangle = \left\langle \prod_i \left(h_{Ai}\left[\mathcal{O}_{Ai}\right]\right)\ \prod_j \left(Ih_{Bj}\left[\mathcal{O}_{Bj}\right]\right) \right\rangle . \tag{7.3}$$

That is, the operators defining the states Bj are carried into the interior of the unit circle in the A plane by the inversion I. (Since $I \in SL(2,C)$, this result is symmetric under interchange of the Ai and the Bj.) The geometry of this construction is illustrated in Fig. 12. The states Ai and Bj may have arbitrary ghost number, as long as the conservation law at each vertex is satisfied; the Grassmann minus signs are properly accounted for if we choose the same ordering of the Ai and the Bj in (7.1) and (7.3).

We cannot begin a proof of the gluing theorem without a more explicit representation of the BPZ inner product. We have already discussed the coordinate oscillator part of the inner product, for which we obtained the representation (6.10). We must still analyze the ghost part of the inner product. Let us proceed along the route which led to (6.10). Consider, then, the left inner product (6.7), defined from the conformal field theory matrix element (2.20). The part of this operator which depends on the ghost nonzero modes is reproduced by

$$\langle I_{AB}| = \langle 3|_A \otimes \langle 3|_B \cdot \exp\Big(\sum_{n\geq 2} \{b_n^A(-1)^n c_n^B - c_n^A(-1)^n b_n^B\}\Big) . \tag{7.4}$$

This expression properly accounts for all Grassmann minus signs, as the reader may readily discover by trying a few cases. The most straightforward way to determine the dependence on ghost zero modes is simply to consider all possible way of assigning the operators c_{-1}, c_0, c_1 to the operators creating A and B in (2.20). The action of the conformal field theory matrix element, determined by (2.21) and (2.25), is reproduced by writing

$$\begin{aligned}\langle I_{AB}| &= \langle 3|_A \otimes \langle 3|_B \ (b_{-1}^A + b_1^B)\,(b_0^A - b_0^B)\,(b_1^A + b_{-1}^B) \\ &= \langle 3|_A \otimes \langle 3|_B \int_{\varsigma_1,\varsigma_0,\varsigma_{-1}} \exp\Big(\varsigma_i\big((-)^i b_{-i}^A - b_i^B\big)\Big) .\end{aligned} \tag{7.5}$$

The full expression for $\langle I_{AB}|$ is then given by combining (6.8), (7.4), (7.5), and a delta function $\delta(p_A + p_B)$. The inverse of this operator defined by (6.9) is then given by

$$\begin{aligned}|I_{CD}\rangle &= \exp\left(\sum_{n=1}^{\infty}\{a_{-n}^C\frac{(-1)^n}{n}a_{-n}^D\} + \sum_{n=2}^{\infty}\{c_{-n}^C(-1)^n b_{-n}^D - b_{-n}^C(-1)^n c_{-n}^D\}\right) \\ &\cdot \int_{p_C,p_D}(2\pi)^d\delta^d(p_C+p_D)\cdot(-1)(c_1^C - c_{-1}^D)(c_0^C + c_0^D)(c_{-1}^C - c_1^D)\,|p_C\rangle_C\otimes|p_D\rangle_D\ .\end{aligned} \tag{7.6}$$

This specification of the inner product completes the formulation of the gluing theorem.

To prove the gluing theorem, we must contract two copies of (4.26) using the inner product (7.6). We may consider the coordinate and ghost pieces separately. Since the analysis is more straightforward for the coordinate pieces, let us begin with them.

In order to take advantage of the special form we have chosen for the contracted states, let us work out the detailed form of the Neumann coefficients (4.27) in the case where one of the $h_I(z)$ is simply $h_C(z) = z$. In this case, we can carry out the contour integrals involving $h_C(z)$ explicitly, to find:

$$\begin{aligned}
N^{AC}_{0\,0} &= \log|h_A(0)| \\
N^{AC}_{0\,m} &= -\frac{1}{m}(h_A(0))^{-m} \\
N^{CA}_{0\,m} &= \frac{1}{m}\oint\frac{dw}{2\pi i}\,w^{-m}\left(h'_A(w)\right)\frac{1}{h_A(w)}\,, \\
N^{AC}_{n\,m} &= \frac{1}{n}\oint\frac{dz}{2\pi i}\,z^{-n}\left(h'_A(z)\right)\cdot\frac{m}{(h_A(z))^{m+1}}\cdot\frac{1}{m}\,,
\end{aligned} \tag{7.7}$$

for $A \neq C$. For the diagonal Neumann coefficients, we find

$$N^{CC}_{n\,m} = 0\,, \tag{7.8}$$

and the same result if n, m, or both are zero. The vanishing of $N^{CC}_{n\,m}$ is actually obvious from the definition (4.27): if $n, m > 0$, the indicated integrals converge well enough that we may push them to infinity, provided that the functions $h_I(z)$ and their derivatives are analytic outside the unit circle. If $h_I(z)$ is the identity, or, more generally, if $h_I(z)$ is an element of $SL(2,C)$, that analyticity is assured. The vanishing of (7.8) is connected to the geometrical simplicity of gluing circles to circles. This point will become more clear when we display the significance of $N^{CC}_{n\,m}$ in more general situations of gluing, in the analysis of Section 3 of II.

The relation (7.8) implies that the coordinate ladder operators a^C_n appear only linearly in the exponent of (4.26). This makes it easy to contract those operators with the operators a^C_{-n} in (7.6). This contraction produced new quadratic terms in the exponential which link operators $a^{A_i}_n$ to operators $a^{B_j}_m$. The general form of these terms is:

$$a^{A_i}_n \mathcal{N}^{A_i\,B_j}_{n\;\;m} a^{B_j}_m \;, \tag{7.9}$$

where

$$\mathcal{N}^{A_i\,B_j}_{n\;\;m} \;=\; \sum_{k>0} N^{A_i\,C}_{n\;\;k} \cdot (-1)^{k+1}k \cdot N^{D\,B_j}_{k\;\;m} \;. \tag{7.10}$$

This can be explicitly evaluated by inserting the expressions (7.7). It will obviously be convenient to define

$$\mathcal{F}^{A}_{n,z}[\omega_z(z)] \;=\; \frac{1}{n}\oint \frac{dz}{2\pi i}\, z^{-n}\left(h'_A(z)\right)\, \omega_z\!\left(h_A(z)\right) \tag{7.11}$$

as a standard form of the Fourier transform of a one-form $\omega_z(z)$ with respect to the string A. Using this notation, the nonzero components of $\mathcal{N}^{A_i\,B_j}_{n\;\;m}$ take the form

$$\mathcal{N}^{A_i\,B_j}_{n\;\;m} \;=\; \mathcal{F}^{A_i}_{n,z}\,\mathcal{F}^{B_j}_{m,w}\left[\sum_{k>0}(-1)^{k+1}\,k\,\frac{1}{(z)^{k+1}}\frac{1}{(w)^{k+1}}\right] , \tag{7.12}$$

where the Fourier transform for A_i is applied to the variable z and that for B_j is applied to w. The series is easy to sum:

$$\begin{aligned}
\mathcal{N}^{A_i\,B_j}_{n\;\;m} \;&=\; \mathcal{F}^{A_i}_{n,z}\,\mathcal{F}^{B_j}_{m,w}\left[\frac{1}{(zw+1)^2}\right] \\
&=\; \mathcal{F}^{A_i}_{n,z}\left[\frac{1}{m}\oint\frac{dw}{2\pi i}\,w^{-m}\left(h'_{B_j}(w)\right)\frac{1}{(h_{B_j}(w))^2}\cdot\frac{1}{\left(z-(-1/(h_{B_j}(w))\right)}\right] \\
&=\; \mathcal{F}^{A_i}_{n,z}\,\mathcal{F}^{IB_j}_{m,w}\left[\frac{1}{(z-w)^2}\;,\right]
\end{aligned} \tag{7.13}$$

where we have used the action of the inversion: $h_{IB_j}(w) = (-1/h_{B_j}(w))$. The final result is exactly

$$\mathcal{N}^{A_i B_j}_{n\ m} = N^{A_i (IB_j)}_{n\ m} ; \tag{7.14}$$

the new quadratic terms which are generated are exactly the Neumann coefficients that one would find linking A_i and B_j after the exterior of the unit circle on the B plane is inverted by R and glued into the A plane as shown in Fig. 12. Notice that, because the formula for N^{IJ}_{nm} is invariant to $SL(2,C)$ transformations, our result is identical to the one we would have obtained by inverting the A plane.

This result generalizes straightforwardly to the terms containing zero modes $a_0^{A_i}$, $a_0^{B_j}$. For the case of one zero mode, $\mathcal{N}^{A_i B_j}_{n\ 0}$ receives contributions from two sources: the sum (7.10) and an additional term which arises, using the momentum conservation of the BPZ inner product, by replacing p_C by $\sum_j p_{B_j}$ in

$$a_n^{A_i} N^{A_i C}_{n\ 0} a_0^C . \tag{7.15}$$

These two terms assemble to form

$$\begin{aligned} \mathcal{N}^{A_i B_j}_{n\ 0} &= \mathcal{F}^{A_i}_{n,z}\left[\sum_{k=0}^{\infty} (-1)^k \cdot \frac{1}{(z)^{k+1}} \cdot \frac{1}{(h_{B_j}(0))^k}\right] \\ &= \mathcal{F}^{A_i}_{n,z}\left[\frac{1}{\Big(z-(-1/h_{B_j}(0))\Big)}\right] \\ &= N^{A_i (IB_j)}_{n\ 0} . \end{aligned} \tag{7.16}$$

For the case of two zero modes, there are two such additional terms.

$$\mathcal{N}^{A_i B_j}_{0\ 0} = N^{A_i C}_{0\ 0} + N^{D B_j}_{0\ 0} + \sum_{k>0} N^{A_i C}_{0\ k} \cdot (-1)^{k+1} k \cdot N^{D B_j}_{k\ 0} . \tag{7.17}$$

Inserting (7.7), we can evaluate the indicated sum; then

$$\begin{aligned} \mathcal{N}^{A_i B_j}_{0\ 0} &= +\ \log h_{A_i}(0) \ +\ \log h_{B_j}(0) \ +\ \log\Big(1 \ +\ \frac{1}{h_{A_i}(0) h_{B_j}(0)}\Big) \\ &= +\log h_{B_j}(0) \ +\ \log\Big[h_{A_i}(0) \ -\ (-1/h_{B_j}(0))\Big] \qquad\qquad eqn(7.17) \\ &= \log h_{A_i}(0) \ +\ N^{A_i (IB_j)}_{0\ \ 0} \ . \end{aligned}$$

There is an extra term left over. This can be rewritten

$$\frac{1}{2}\sum_{i,j} a_0^{A_i}\Big(\log h_{B_j}(0)\Big) a_0^{B_j} \ = -\ \frac{1}{2}\sum_{k,j} a_0^{B_k}\Big(\log h_{B_j}(0)\Big) a_0^{B_j} \qquad (7.18)$$

and added to the B_j–B_k zero mode terms:

$$N^{B_j B_k}_{0\ 0} \ -\ \log h_{B_j}(0) \ -\ \log h_{B_k}(0) \ =\ N^{(IB_j)(IB_k)}_{0\ \ \ \ 0} \ . \qquad (7.19)$$

Now the coordinate operator part of the glued product of vertices has been rearranged exactly into the form of the right-hand side of (7.3).

The computation of the C and D ghost matrix element indicated in (7.2) is only slightly more difficult. The main complication comes from the treatment of zero modes. One should keep in mind, though, that the ghost Neumann functions (4.28) are not themselves $SL(2,C)$-invariant, so that the fact that the gluing operation is independent of which plane is inverted will not be obvious until the end of the calculation. For definiteness, I will always consider the inversion to be applied to the states B_j.

Let us begin by computing the matrix element of the nonzero modes. Specializing (4.28) to the situation where one of the $h_I(z)$ equal the identity, we find

$$\begin{aligned} \tilde{N}^{A_i C}_{n\ m} &= \mathcal{F}^{(b) A_i}_{n,\ z}\left[\frac{-1}{(z)^{m+2}}\right] \\ \tilde{N}^{D B_j}_{n\ m} &= \mathcal{F}^{(c) B_j}_{m,\ w}\left[\frac{+1}{(w)^{n-1}}\right] , \end{aligned} \qquad (7.20)$$

where we have defined two more canonical Fourier transforms,

$$\begin{aligned}
\mathcal{F}^{(b)A}_{n,\ z}[f(z)] &= \oint \frac{dz}{2\pi i} z^{-n+1} \left(h'_A(z)\right)^2 f\left(h_A(z)\right) , \\
\mathcal{F}^{(c)A}_{m,\ z}[f(z)] &= \oint \frac{dz}{2\pi i} z^{-m-2} \left(h'_A(z)\right)^{-1} f\left(h_A(z)\right) .
\end{aligned} \tag{7.21}$$

Again we find

$$\tilde{N}^{CC}_{n\,m} = 0 , \tag{7.22}$$

as long as $n \geq 2$, $m \geq -1$, since, with these restrictions, we can freely push the contours to infinity. One further simplification comes from the evaluation of (4.29):

$$M_i{}^C_m = \delta_{im} ; \tag{7.23}$$

this is nonzero only for the case of the zero modes $m = -1, 0, 1$. Then it is straightforward to the contract the ghost exponential in (7.6) with the ghost exponentials in the two copies of (4.26). The result is an exponential with new quadratic terms coupling the c operators of each A_i to the b operators of each B_j, and vice versa. These new terms have the form $(c^{A_i}_n \tilde{\mathcal{N}}^{A_i\,B_j}_{n\,m} b^{B_j}_m)$, and the corresponding terms with A_i and B_j interchanged, where

$$\begin{aligned}
\tilde{\mathcal{N}}^{A_i\,B_j}_{n\,m} &= \sum_{k=2}^{\infty} N^{A_i\,C}_{n\,k} \cdot (-1)^k \cdot N^{D\,B_j}_{k\,m} \\
&= \mathcal{F}^{(b)A_i}_{n,\ z} \mathcal{F}^{(c)B_j}_{m,\ w} \left[\sum_{k=2}^{\infty} (-1)^k \frac{1}{(z)^{k+2}(w)^{k-1}} \right] \\
&= \mathcal{F}^{(b)A_i}_{n,\ z} \mathcal{F}^{(c)B_j}_{m,\ w} \left[\frac{1}{(z)^4} \cdot \frac{-1}{\left((-1/z) - w\right)} \right] .
\end{aligned} \tag{7.24}$$

If we combine the prefactor inside the bracket— $(h_{A_i}(z))^{-4}$ —with the integration measure in $\mathcal{F}^{(b)A_i}_{n,\ z}$, we can see this rearranges into the formula

$$\tilde{\mathcal{N}}^{A_i B_j}_{n\ m} = \tilde{N}^{(IA_i)B_j}_{n\ \ m} . \tag{7.25}$$

The analogous relation for the term linking $c^{B_j}_n$ with $b^{A_i}_m$ is

$$\tilde{\mathcal{N}}^{B_j A_i}_{n\ m} = \tilde{N}^{(IB_j)A_i}_{n\ \ m} . \tag{7.26}$$

Eq. (7.26) is exactly what we wanted to find. However, (7.25) is not equivalent to the same formula with the I applied to B_j. Something is missing.

Leave this result aside for the moment and turn to the ghost zero mode contributions. The zero mode operators b^C_{-1}, b^C_0, b^C_1 appear in (4.26) both in terms involving $\tilde{N}^{A_i C}_{n\ m}$ and in terms involving $M_{i\ m}^{\ C}$. The terms with $M_{i\ m}^{\ C}$ involve the supplementary Grassmann variables; we will distinguish the variables associated with the A and B planes by denoting them as ς^A_i, ς^B_i. Now operate on the two vertices with the first zero mode factor $(c^C_1 - c^D_{-1})$ from the second line of (7.6). The action of this factor brings down from the exponentials in (4.26) the quantity

$$\mathcal{E}_1 = -\varsigma^A_{-1} + c^{A_i}_n \tilde{N}^{A_i C}_{n\ -1} + \varsigma^B_1 - c^{B_j}_n \tilde{N}^{B_j D}_{n\ 1} . \tag{7.27}$$

Integrate this over ς^B_1, taking into account also the exponential factor in the B plane vertex which depends on ς^B_1. We may, however, now ignore the term $\varsigma^B_1 M_{1\ m}^{\ D} b^D_m = \varsigma^B_1 b^D_1$, since the b^D_1 may now be moved to the right to annihilate the D vacuum of (7.6). The integral then gives

$$\begin{aligned}
&\int_{\varsigma^B_1} \exp\Big(-\varsigma^B_1 \cdot \big[M_{1\ m}^{\ B_j} b^{B_j}_m\big]\Big) \cdot \mathcal{E}_1 \\
&= \cdot\Big(1 - c^{B_j}_n \tilde{N}^{B_j D}_{n\ 1} M_{1\ m}^{\ B_k} b^{B_k}_m + \varsigma^A_{-1} M_{1\ m}^{\ B_j} b^{B_j}_m - c^{A_i}_n \tilde{N}^{A_i C}_{n\ -1} M_{1\ m}^{\ B_j} b^{B_j}_m\Big) \\
&= \cdot\exp\Big(- c^{B_j}_n \tilde{N}^{B_j D}_{n\ 1} M_{1\ m}^{\ B_k} b^{B_k}_m + \varsigma^A_{-1} M_{1\ m}^{\ B_j} b^{B_j}_m - c^{A_i}_n \tilde{N}^{A_i C}_{n\ -1} M_{1\ m}^{\ B_j} b^{B_j}_m\Big) ,
\end{aligned} \tag{7.28}$$

using, in the last line, the Grassmann property of the ghost operators. The other two ghost zero mode factors in (7.6), and the remaining two ς^B integrals, may be carried through a similar reduction procedure.

We must now interpret the three terms in the exponent in the last line of (7.28). The easiest to understand is the middle term. Because of the relation $h_{RB_j}(z) = (-1/h_{B_j}(z))$, (4.29) obeys the inversion formula

$$M_i{}^{B_j}_{m} = (-1)^{i-1} M_{(-i)}{}^{(IB_j)}_{m} . \tag{7.29}$$

This term then gives the coupling of the inverted B_j ghost operators to ς^A_{-1} required to implement the zero modes properly in the expectation value displayed in (7.3).

The first term in the exponential modifies the Neumann coefficient linking two B_j ghost operators. We must add this term, and its counterparts in the other two factors arising from the ghost zero mode reduction, to the expression (4.28). Using (7.20) and the explicit formulae for the $M_i{}^{B_k}_{m}$, we can combine these four terms as follows:

$$\begin{aligned}
\tilde{N}^{B_j}_{n}{}^{B_k}_{m} &= \mathcal{F}^{(b)B_j}_{n,\ z}\ \mathcal{F}^{(b)B_k}_{m,\ w}\left[\frac{-1}{(z-w)} + \frac{1}{z} + \frac{w}{z^2} + \frac{w^2}{z^3}\right] \\
&= \mathcal{F}^{(b)B_j}_{n,\ z}\ \mathcal{F}^{(b)B_k}_{m,\ w}\left[\frac{w^2}{z^4}\cdot\frac{1}{\Big((-1/z)-(-1/w)\Big)}\right] \\
&= \tilde{N}^{(IB_j)}_{n}{}^{(IB_k)}_{m} ,
\end{aligned} \tag{7.30}$$

where in the last line we have combined the prefactors $w^2 \cdot z^{-4}$ with the measures for the two Fourier integrals. This sets the quadratic term in B ghosts into the right form to account the action of the inversion on the states B_j.

Finally, consider the last term in the exponential of (7.28), and its two counterparts. These are the terms which must combine with (7.24) to give the full quadratic term involving one $c^{A_i}_n$ and one $b^{B_j}_m$. Using (7.20) once again, we find that the last line of (7.24) is replaced by

$$\begin{aligned}
&\mathcal{F}^{(b)A_i}_{n,\ z}\,\mathcal{F}^{(c)B_j}_{m,\ w}\left[\frac{1}{z^3}\cdot\frac{1}{\Big(wz+1\Big)}\ -\ \frac{1}{z^3}\ +\ \frac{w}{z^2}\ -\ \frac{w^2}{z}\right]\\
&\qquad = \mathcal{F}^{(b)A_i}_{n,\ z}\,\mathcal{F}^{(c)B_j}_{m,\ w}\left[w^2\cdot\frac{-1}{\Big(z-(-1/w)\Big)}\right]\\
&\qquad = \tilde{N}^{A_i\,(IB_j)}_{n\ \ m}\ .
\end{aligned}\tag{7.31}$$

Again, the prefactor w^2 supplies a factor of $(h_{B_j}(z))^2$ necessary to rearrange the Fourier integral. Now the ghost factors have also come into the form required to represent the right-hand side of (7.3). This completes the proof of the gluing theorem for circles.

This simplest version of the gluing theorem is sufficient to discuss (and disprove) the gauge-invariance of the string field theory constructed using the CSV vertex. However, it is not sufficient for more general situations in which the mappings $h_I(z)$ which carry string into the plane are not elements of $SL(2,C)$. Let us conclude this section by discussing the limitations of the proof we have just given and stating a more generally applicable form of this result. The proof of the more general theorem is somewhat involved; our paper II will be devoted to presenting it in detail.

One particular circumstance in which a more powerful result is needed is the discussion of the gauge-invariance of the string field theory built from the Witten vertex. In that case, the gluing procedure has the complication shown in Fig. 10: The two regions two be glued together have different shapes, so that joining them produces a fold in the world-sheet. Since the only conformal transformations that map the plane to itself one-to-one are $SL(2,C)$ transformations, we should expect always to find branch cuts and nonanalyticity after we identify and glue together two strings which have been embedded through general transformations. These branch cuts should naturally appear, and should naturally be removed, in a more general formulation of gluing.

This issue of the appearance of cuts and folds in the world-surface is a crucial part of the result that we will prove in II. The geometry of this more general gluing is shown in Fig. 13. Let C and D be the states contracted by the BPZ inner product, as in the analysis of the previous section. However, let us drop the requirement that the mappings which embed the states C and D in the A and B planes be $SL(2,C)$. Then if we glue C to D by mapping C and D to the interior and exterior, respectively, of the unit circle on a third plane F, the A and B planes will be carried onto this third plane in a manner which is not, in general, single-valued. The image of the A plane in the F plane will have branch cuts outside the unit circle, and the image of B in the F plane will have branch cuts inside the unit circle. However, the Riemann covering surface for F formed by joining these exterior regions will have the topology of a plane. Thus, there exists a conformal mapping $g(z)$ which maps this covering surface into a final plane G in a single-valued manner. (This mapping is unique up to $SL(2,C)$.) The mapping $g(z)$ is a smoothing operation which irons out the branch cuts.

With this geometrical picture in mind, we may state our main result, the *Generalized Gluing and Resmoothing Theorem* (GGRT): Let $\langle V_{\{Ai\}C}|$ be defined by

$$\langle V_{\{Ai\}C}| \prod_i |A_i\rangle \otimes |C\rangle = \left\langle \prod_i \left(h_{A_i}\left[\mathcal{O}_{A_i}\right]\right) h_C\left[\mathcal{O}_C\right] \right\rangle \tag{7.32}$$

and let $\langle V_{\{Bj\}D}|$ be defined similarly. Then, if $|I_{CD}\rangle$ is the BPZ inner product, the fused vertex

$$\langle V_{\{Ai\}\{Bj\}}| = \langle V_{\{Ai\}C}| \, \langle V_{\{Bj\}D}| \; |I_{CD}\rangle \tag{7.33}$$

is given by

$$\langle V_{\{Ai\}\{Bj\}}| \prod_i |Ai\rangle \otimes \prod_j |Bj\rangle = \left\langle \prod_i \left(\hat{h}_{A_i}\left[\mathcal{O}_{Ai}\right]\right) \prod_j \left(\hat{h}_{Bj}\left[\mathcal{O}_{Bj}\right]\right) \right\rangle , \tag{7.34}$$

where

$$\begin{aligned}\hat{h}_{A_i} &= g \circ h_C^{-1} \circ h_{A_i} \\ \hat{h}_{B_j} &= g \circ I \circ h_D^{-1} \circ h_{B_j} \,,\end{aligned} \tag{7.35}$$

and g in these definitions is the smoothing transformation described in the previous paragraph. The presence of this element in our final result treats the singularities created by the general gluing process and insures that we are free to map our world surface onto a simple cover of the complex plane in order to assess its symmetries. In particular, it precisely justifies the argument for the symmetry of the Witten vertex displayed in Fig. 10. This theorem will be proved in II for the open and closed bosonic string.

8. Gauge Invariance of Witten's String Field Theory

Now that we have in hand the general relation between the Hilbert-space manipulation of contraction with the BPZ inner product and the geometrical operation of gluing and smoothing, we have all the formalism we need to complete the proof the gauge-invariance of Witten's string field theory action. In this section, we will present that proof, and exhibit some generalizations of it.

Following the notations of Section 4, we write the Witten action as

$$
\begin{aligned}
S &= \left\langle I\left[\hat{\Phi}\right] \mathbf{Q}\,\hat{\Phi} \right\rangle + \frac{2}{3} g \left\langle T^2 h\left[\hat{\Phi}\right] T h\left[\hat{\Phi}\right] h\left[\hat{\Phi}\right] \right\rangle , \\
&= \langle I_{12}| \; |\Phi\rangle_1 \otimes \mathbf{Q}\,|\Phi\rangle_2 + \frac{2}{3} g \; \langle V_{123}| \; |\Phi\rangle_1 \otimes |\Phi\rangle_2 \otimes |\Phi\rangle_3
\end{aligned} \quad , \tag{8.1}
$$

where $h(z)$ is the mapping given in (6.17) and the vertex in the second line is that following from this choice. The subscripts label distinct single-string Hilbert spaces. We claim that this action is invariant under the transformation

$$
\delta\,|\Phi\rangle_1 = \mathbf{Q}\,|\Lambda\rangle_1 + g\,\langle V_{245}|\,|I_{12}\rangle \left[|\Phi\rangle_4 \otimes |\Lambda\rangle_5 - |\Lambda\rangle_4 \otimes |\Phi\rangle_5 \right] , \tag{8.2}
$$

where $|\Lambda\rangle$ is a state of ghost number (-1) and even Grassmann parity. To check this, insert (8.2) into (8.1) and study the terms at each order in g. At order g^0, we can use the result (4.20) that the kinetic term is symmetric in its two arguments to write the variation as

$$
\delta S^{(0)} = 2\,\langle I_{12}| \; |\Phi\rangle_1 \otimes \mathbf{Q}^2\,|\Lambda\rangle_2 = 0 \,. \tag{8.3}
$$

Using the cylic symmetry of the vertex, we can rewrite the order g^1 variation in the form

$$
\begin{aligned}
\delta S^{(1)} &= 2g\,\langle V_{145}|\ \mathbf{Q}\,|\Phi\rangle_1 \otimes \Big[|\Phi\rangle_4 \otimes |\Lambda\rangle_5 \ - \ |\Lambda\rangle_4 \otimes |\Phi\rangle_5\Big] \\
&\qquad + \ 2g\,\langle V_{123}|\ \mathbf{Q}\,|\Lambda\rangle \otimes |\Phi\rangle_2 \otimes |\Phi\rangle_3 \\
&= 2g\,\langle V_{123}|\ \Big[\mathbf{Q}\,|\Phi\rangle_1 \otimes |\Phi\rangle_2 \otimes |\Lambda\rangle_3 \ - \ |\Phi\rangle_1 \otimes \mathbf{Q}\,|\Phi\rangle_2 \otimes |\Lambda\rangle_3 \\
&\qquad + \ |\Phi\rangle_1 \otimes |\Phi\rangle_2 \otimes \mathbf{Q}\,|\Lambda\rangle_3\Big] \ ,
\end{aligned}
\tag{8.4}
$$

which vanishes by virtue of the identity (5.13). Finally, the order g^2 variation is

$$
\delta S^{(2)} \ = \ \frac{2}{3} g^2\,\langle V_{2345}|\,|\Phi\rangle_2 \otimes |\Phi\rangle_3 \otimes \Big[|\Phi\rangle_4 \otimes |\Lambda\rangle_5 \ - \ |\Lambda\rangle_4 \otimes |\Phi\rangle_5\Big] \ , \tag{8.5}
$$

where the four-string vertex which appears here is that obtained by gluing and smoothing two three-string vertices:

$$
\langle V_{2345}| \ = \ \langle V_{123}|\ \langle V_{645}|\,|I_{16}\rangle \ . \tag{8.6}
$$

If this object is cyclically invariant, (8.5) is a difference of two terms which cancel exactly. That the four-string vertex built from Witten vertices is cyclically invariant follows from our geometrical interpretation of gluing (backed by the analytical work which we will present in II) and the physical picture of the glued vertex shown in Fig. 10. This completes the proof of gauge invariance. The argument we have just given is essentially identical to that given in Witten's original paper; here, however, all ingredients of the formalism have been defined precisely and all postulated relations of these objects have been verified.

Now that we can claim to understand fully the Witten open string field theory, we should inquire to what extent this theory is unique. We have already noted that an arbitrary vertex of the form (6.1) leads to an action which is gauge-invariant through order g^1. Presumably, given any such vertex, it is possible to add four- and higher-string vertices, order by order in g, to achieve any desired level of gauge-invariance. We have not pursued this idea; rather, we will concentrate on the question of whether there exist open-string actions inequivalent to Witten's which terminate after the three-string vertex.

What constraints must the vertex in such an action satisfy? Obviously, it must link open strings; this implies that the mapping h which defines the vertex must satisfy an appropriate symmetry condition. If the real axis is taken to represent the open-string boundary, h must be symmetric with respect to a reflection in the real axis. We have found it convenient to represent the string boundary as the unit circle, so that T is simply the rotation $Tz = e^{2\pi i/3}z$. Then h must convert this reflection to an inversion in the unit circle. Indeed, (6.17) satisfies

$$h(\bar{z}) \;=\; \overline{1/h(z)}\;. \tag{8.7}$$

A second constraint is much less trivial but follows straightforwardly from the logic of eqs. (6.14)–(6.16). Let us formally represent the gluing of two vertices of the form (6.1) in the following way: Apply the conformal transformation h^{-1} to bring the third region to the unit circle. (This transformation does not act in a single-valued way on the whole plane; we will soon see the consequences of that fact.) Then glue to produce a 4-string vertex and restore the original geometry by acting with h. This gives:

$$\left\langle T^2h\Big[\mathcal{O}_A\Big]\,Th\Big[\mathcal{O}_B\Big]\,hIh^{-1}T^2h\Big[\mathcal{O}_E\Big]\,hIh^{-1}Th\Big[\mathcal{O}_F\Big]\right\rangle\;. \tag{8.8}$$

To check the cyclic symmetry of this construction, we would cycle A into B by acting on each state with the $SL(2,C)$ transformation T. Then A cycles into F and E into B only if the following conditions hold:

$$h \;=\; hIh^{-1}Th \;=\; hIh^{-1}T^2h\;. \tag{8.9}$$

It seems that we have already reached a contradiction: We apparently require $T = T^2$, while $T^3 = 1$ but T is nontrivial. However, our discussion of Fig.

10 explained that this contradiction is in fact avoided for the Witten vertex. The mapping $h(z)$ used in the Witten vertex is nonanalytic at two points on the image of the unit circle. A $z^{2/3}$ branch cut runs across the complex plane from one of these points to the other. Then the mappings $ITh(z)$ and $IT^2h(z)$ carry the unit circle into two regions with the same coordinates on this plane but standing on opposite sides of the branch cut. Precisely because this branch cut corresponds to a $z^{2/3}$ singularity, multiplication by T carries the first region into the second. The singularity structure of the function $h(z)$ given in (6.17) is thus not accidental but required for gauge invariance. To put it succinctly, we need

$$hI = T^{-1}h \tag{8.10}$$

in Fig. 11: the transformation $z \to I(z)$ corresponds to $w \to e^{i\pi}w$, and we have cut the w-plane in such a way that this rotation gets transformed into $y \to e^{-2\pi i/3}y$.

A more general gauge-invariant 3-string vertex must also satisfy (8.7) and (8.10). Let us investigate whether theere are any additional solutions to these constraints which have the form of contact delta-functions on the world-sheet. Any such vertex will carry the unit circle to some conformal transform of the region which produces the Witten vertex. We should have, then,

$$\hat{h} = h\hat{g} \ . \tag{8.11}$$

A mapping of this form satisfies (8.7) and (8.10) if $\hat{g}$ commutes with I and if $\hat{g}(\bar{z}) = \overline{\hat{g}(z)}$. However, mappings satsifying these condition belong to a very special class. If we consider $\hat{g}(z)$ as generated by a vector field $v(z) = \sum v_n z^{-n+1}$, the condition that $\hat{g}$ commute with I implies $Iv(z) = v(z)$ and hence there is a vector field $w(z)$ such that

$$\sum_n v_{-n} L_n = \sum_n w_{-n}\left(L_n - (-)^n L_{-n}\right) = \sum_n w_{-n} K_n \quad , \tag{8.12}$$

where the K_n are well known to generate the reparametrization symmetries of Witten's vertex.

The explicit proof that the K_n are symmetries of the vertex is quite straightforward in our formulation and provides a nice application of the result (8.10). We have to show

$$\langle V_{123}| \left(K_n^{(1)} + K_n^{(2)} + K_n^{(3)}\right) = 0 \quad , \tag{8.13}$$

for all $n > 0$, or equivalently, for arbitrary states $\mathcal{O}_A$, $\mathcal{O}_B$ and $\mathcal{O}_C$,

$$\begin{aligned} 0 = \oint \frac{dz}{2\pi i} z^{n+1} \Big\{ & \left\langle T^2 h\Big[(T_{zz}(z) - IT_{zz}(z))\, \mathcal{O}_A\Big] Th\Big[\mathcal{O}_B\Big] h\Big[\mathcal{O}_C\Big]\right\rangle \\ & + \left\langle T^2 h\Big[\mathcal{O}_A\Big] Th\Big[(T_{zz}(z) - IT_{zz}(z))\, \mathcal{O}_B\Big] h\Big[\mathcal{O}_C\Big]\right\rangle \\ & + \left\langle T^2 h\Big[\mathcal{O}_A\Big] Th\Big[\mathcal{O}_B\Big] h\Big[(T_{zz}(z) - IT_{zz}(z))\, \mathcal{O}_C\Big]\right\rangle \Big\} . \end{aligned} \tag{8.14}$$

But by $hI = T^{-1}h$ the terms on the right hand side cancel pairwise, and this completes the proof. Notice that (8.14) holds for *any* conformal field, including the ghost fields. By a similar argument and $[\mathbf{Q}, L_n] = 0$, the K_n also leave the kinetic term of the action invariant.

How can we reconcile the fact that the K_n generate nontrivial transformations of the form (8.11) with the identity that the K_n leave the Witten vertex unchanged? It is instructive to write explicitly the finite conformal transformation generated by the operator K_n; this is

$$g_n(z) = \left(\frac{z^n + (-1)^n a_n}{1 + a_n z^n}\right)^{1/n} . \tag{8.15}$$

This mapping is analytic in the vicinity of the unit circle, but for a_n small it has a complicated branch structure near the origin, with branch points at $z = -(a_n)^{1/n}$. The cancellation shown in (8.14) assumes that the branch cuts which emerge from these points are disposed symmetrically with branch cuts at large z ($z = (-a_n^{-1})^{1/n}$), and that none of the branch cuts cross the unit circle. The original conformal plane is then mapped to a branched Riemann surface. To define the conformal field theory matrix elements on this Riemann surface, we smooth out the branch cuts by mapping it back to a plane. This map restores the original configuration of the Witten vertex. The transformations (8.11), then, produce no new gauge-invariant vertices.

We would like to point out, however, that our formalism allows a wide class of BRST-invariant vertices which are not contact delta-functions but are, rather, nonlocal overlaps on the world-surface. We consider it likely that, among this class of vertices, additional configurations can be found which satisfy the criteria for gauge-invariance. We consider this a promising avenue for further investigation.

APPENDIX : The $SL(2,C)$-Covariant Ghost Propagator

In Section 5, we needed to rearrange the system of ghost propagators and zero modes in order to give an economical proof of the $SL(2,C)$-invariance of our vertex. In this appendix, we will prove the validity of that rearrangement.

The strategy of our argument was the following: In the formulation of the theory used in most of the paper, nonzero correlation functions of b and c operators were evaluated by summing all possible contractions of c operators to the b operators and to the three zero modes. However, in the proof of $SL(2,C)$-invariance, we found it more convenient to consider three particular ghost operators at the three points $h_I(0)$ to be contracted to the zero modes. To justify this change in the calculational rules, we needed to find a modified ghost propagator which, in conjunction with the new rules, reproduces the results of the first formulation.

To solve this problem, let us generalize and abstract it a bit. Consider a system of ghosts with the propagator

$$\langle b(z)c(w)\rangle \;=\; G(z,w) \tag{A.1}$$

and the c zero modes $Z_1(w),\ldots,Z_n(w)$. A correlation function of m b's and $(m+n)$ c's is then calculated by contracting the c's to b's and zero modes in all possible ways. Now let the locations of the c's be divided into two classes: $x_1,\ldots,x_n$, $z_1,\ldots,z_m$. We would like to rewrite the formula for the correlation function in such a way that the fields $c(x_i)$ are contracted only to zero modes and the fields $c(z_j)$ are contracted only to b fields.

If we compute with the standard rules, a general correlation function has the value (up to obvious minus signs which are accounted systematically in our main development)

$$\langle b(z_1)b(z_2)\cdots b(z_m)c(x_1)\cdots c(x_n)c(w_1)\cdots c(w_m)\rangle$$

$$\begin{vmatrix} Z_1(x_1) & \dots & Z_1(x_n) & Z_1(w_1) & \dots & Z_1(w_m) \\ Z_2(x_1) & \dots & Z_2(x_n) & Z_2(w_1) & \dots & Z_2(w_m) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_n(x_1) & \dots & Z_n(x_n) & Z_n(w_1) & \dots & Z_n(w_m) \\ G(z_1,x_1) & \dots & G(z_1,x_n) & G(z_1,w_1) & \dots & G(z_1,w_m) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ G(z_m,x_1) & \dots & G(z_m,x_n) & G(z_m,w_1) & \dots & G(z_m,w_m) \end{vmatrix} . \tag{A.2}$$

But we can rearrange this determinent to isolate the upper left-hand corner. To do this, subtract a multiple of the first column from all successive columns to set all elements after the first in the top row equal to zero. Now subtract a multiple of the new second column from all successive columns to set all elements of the second row after the second equal to zero. Proceed through m steps. Then the above determinant takes the form:

$$\begin{aligned} &\begin{vmatrix} Z_1(x_1) & 0 & \dots & 0 & 0 & \dots & 0 \\ Z_2(x_1) & \tilde{Z}_{2,2} & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ Z_n(x_1) & \tilde{Z}_{m,2} & \dots & \tilde{Z}_{m,m} & 0 & \dots & 0 \\ G(z_1,x_1) & \tilde{G}_{m+1,2} & \dots & \tilde{G}_{m+1,m} & \mathrm{G}(z_1,w_1) & \dots & \mathrm{G}(z_1,w_m) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ G(z_m,x_1) & \tilde{G}_{m+n,2} & \dots & \tilde{G}_{m+n,m} & \mathrm{G}(z_m,w_1) & \dots & \mathrm{G}(z_m,w_m) \end{vmatrix} \\ &= Z_1(x_1)\tilde{Z}_{2,2}\cdots\tilde{Z}_{m,m}\cdot\det\left|\mathrm{G}(z_j,w_k)\right| . \end{aligned} \tag{A.3}$$

This procedure for reducing the original determinant depends mainly on the values of the $Z(x_i)$; the final result for any other matrix element depends only on the original matrix elements in the same column or the same row. Thus, we can recognize the ingredients of the last line of (A.3) in smaller matrices of similar structure. For example, the product of $\tilde{Z}$'s in (A.3) appears in the same way in the reduction of the $m \times m$ determinant of zero modes alone:

$$Z_1(x_1)\tilde{Z}_{2,2}\cdots\tilde{Z}_{m,m} = \det\left|Z_i(x_j)\right| . \tag{A.4}$$

Similarly, the same $\mathrm{G}(z_j, w_k)$ would appear from an $(m+1)\times(m+1)$ matrix:

$$\det\left|Z_i(x_j)\right| \cdot \mathrm{G}(z,w) = \left|\begin{matrix} Z_1(x_1) & Z_1(x_2) & \ldots & Z_1(x_m) & Z_1(w) \\ Z_2(x_1) & Z_2(x_2) & \ldots & Z_2(x_m) & Z_2(w) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ Z_m(x_1) & Z_m(x_2) & \ldots & Z_m(x_m) & Z_m(w) \\ G(z,x_1) & G(z,x_2) & \ldots & G(z,x_m) & G(z,w) \end{matrix}\right| . \tag{A.5}$$

With the identification (A.4), we write the result of (A.3) as follows:

$$\begin{aligned} &\langle b(z_1)b(z_2)\cdots b(z_m)c(x_1)\cdots c(x_n)c(w_1)\cdots c(w_m)\rangle \\ &\quad = \det\left|Z_i(x_j)\right| \cdot \det\left|\mathrm{G}(z_j,w_k)\right| . \end{aligned} \tag{A.6}$$

This is exactly the result we had sought. Eq. (A.6) instructs us to saturate the zero modes of c with the fields $c(x_i)$ and then contract the $b(z_j)$ with the $c(w_k)$ using the propagator $\mathrm{G}(z,w)$. This new propagator depends, of course, on the locations of the x_i. It may be evaluated explicitly using the formula (A.5).

For the case of interest to us, set $Z_i(x) = x^{i+1}$, for $i = -1, 0, 1$, and $G(z,w) = 1/(z-w)$. Then it is not difficult to work out:

$$\begin{aligned} \det\left|Z_i(x_j)\right| &= (x_1-x_2)(x_1-x_3)(x_2-x_3) \\ \mathrm{G}(z,w) &= \frac{1}{(z-w)}\prod_{j=1,2,3}\frac{(w-x_i)}{(z-x_i)} . \end{aligned} \tag{A.7}$$

This is the Green function which was presented in eq. (5.11).

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FIGURE CAPTIONS

1) Hamiltonian evolution on the Euclidean plane, viewed as the conformal image of time evolution on a cylinder or string.

2) Equivalence of a contact delta-function to an operator expectation value.

3) Construction of string interactions by mapping canonically defined string states into the conformal plane.

4) The conformal transformation which proves the symmetry of the string kinetic energy term.

5) The contour deformation which proves the BRST invariance of the vertex in its most general form.

6) Two simple choices for the conformal mappings which define the 3-string vertex.

7) A view of the CSV vertex, using T given by (6.3).

8) A second view of the CSV vertex, using $T = e^{2\pi i/3}$.

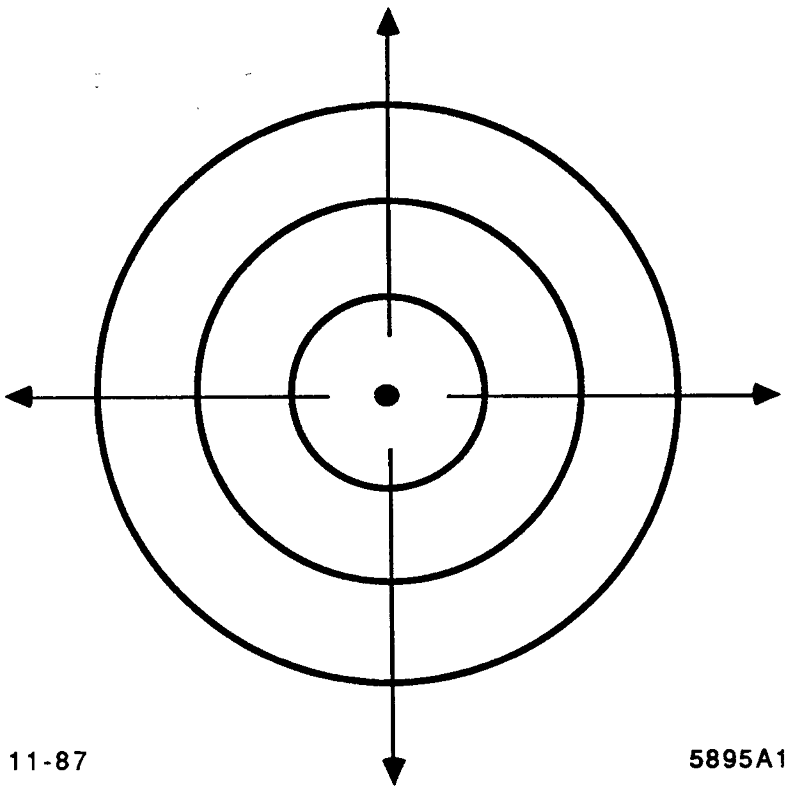
9) The geometrical operation which corresponds to the contraction of two CSV vertices.

10) The figure which results from the contraction of two Witten vertices, in shown in three views.

11) Derivation of the conformal mapping required for the construction of the Witten vertex.

12) Geometry of the gluing resulting from the contraction of circles by the BPZ inner product.

13) Gluing and subsequent smoothing in the contraction of two string field vertices.



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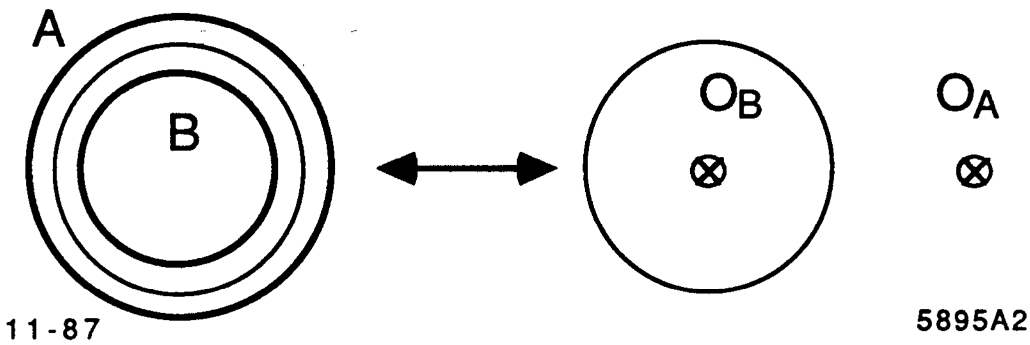

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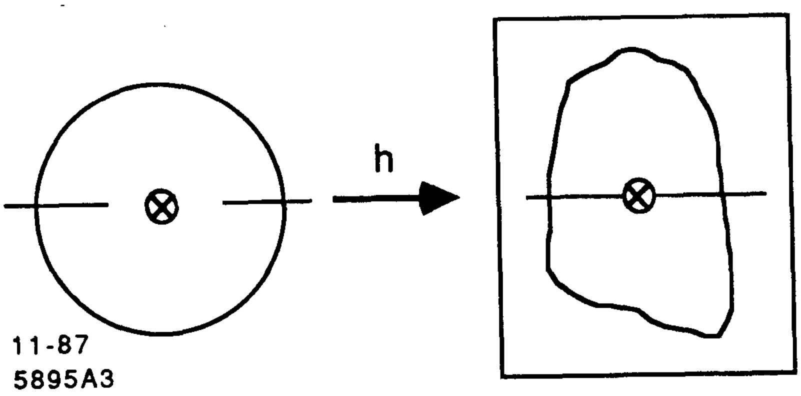

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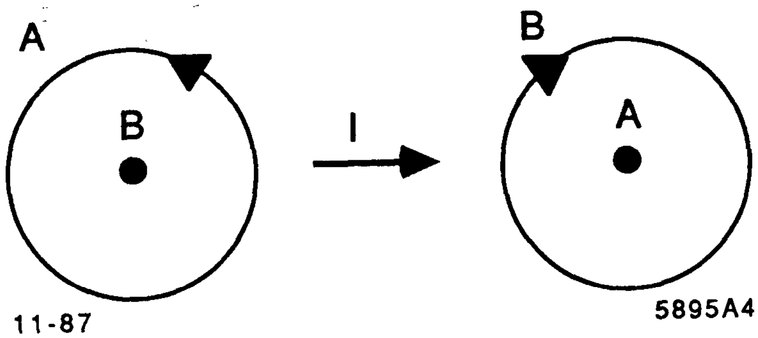

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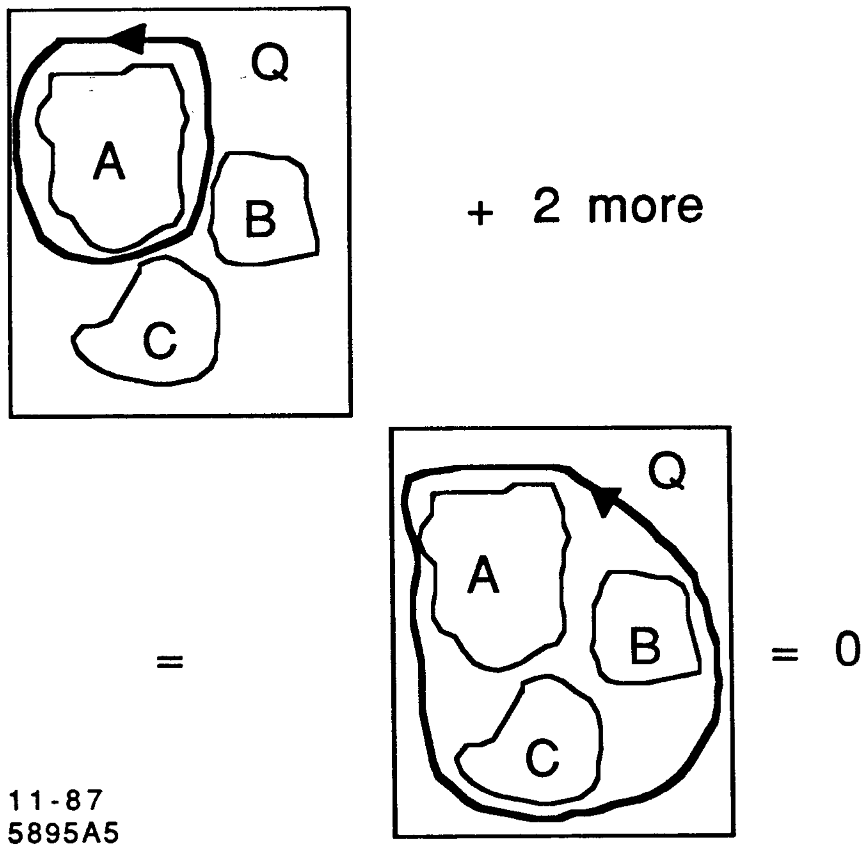

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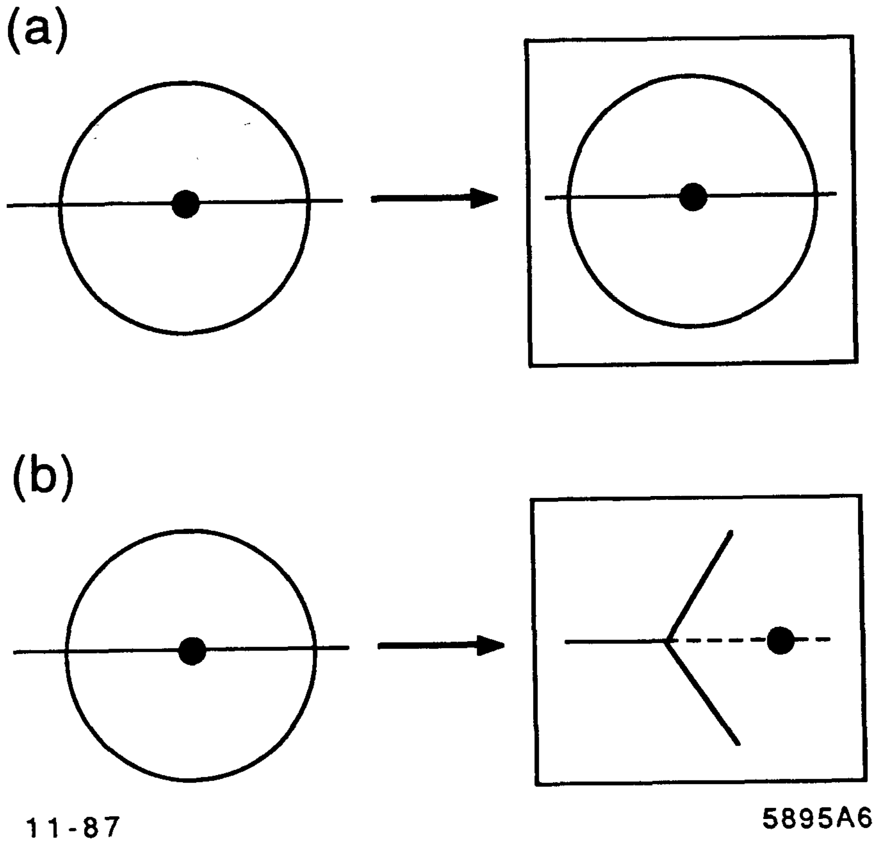

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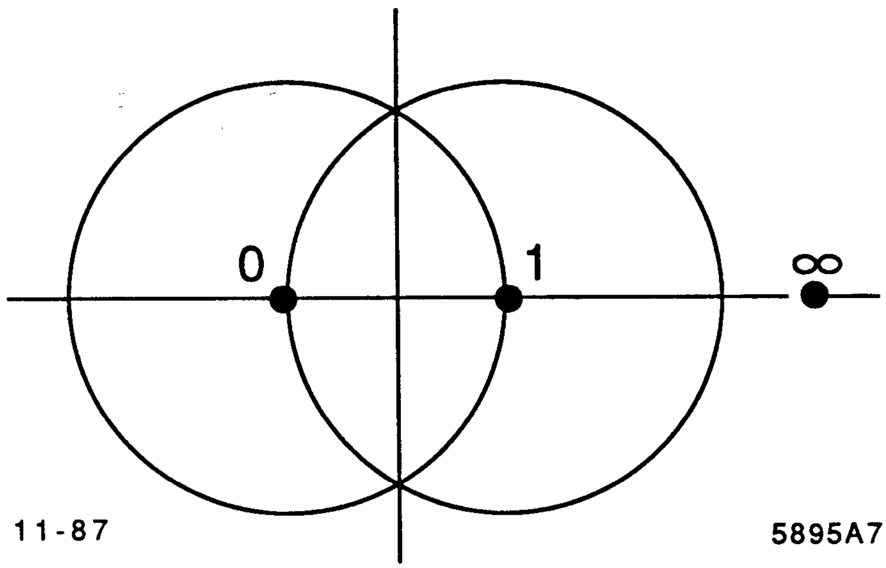

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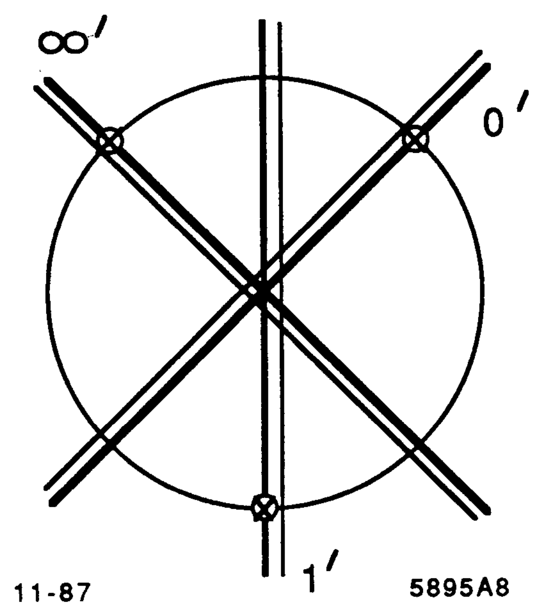

Fig. 8

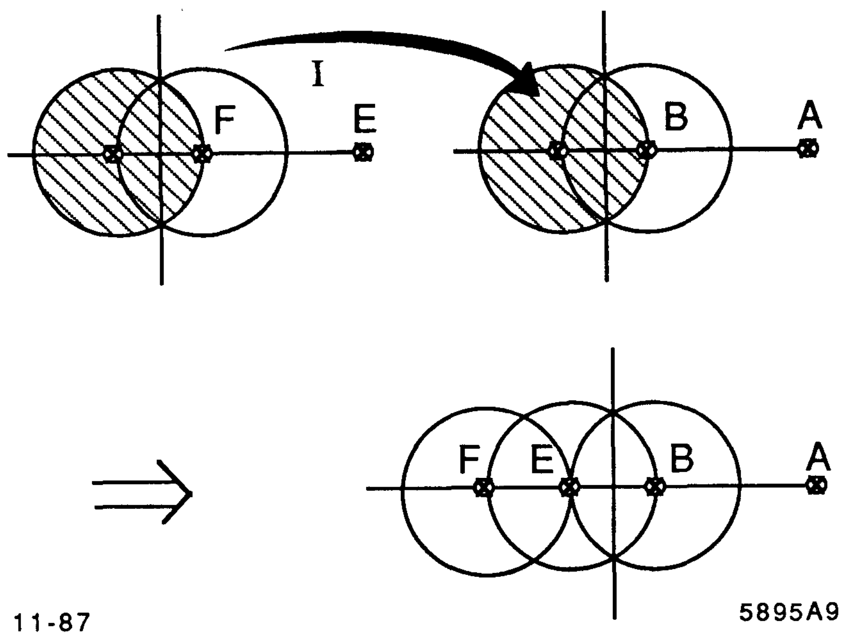

Fig. 9

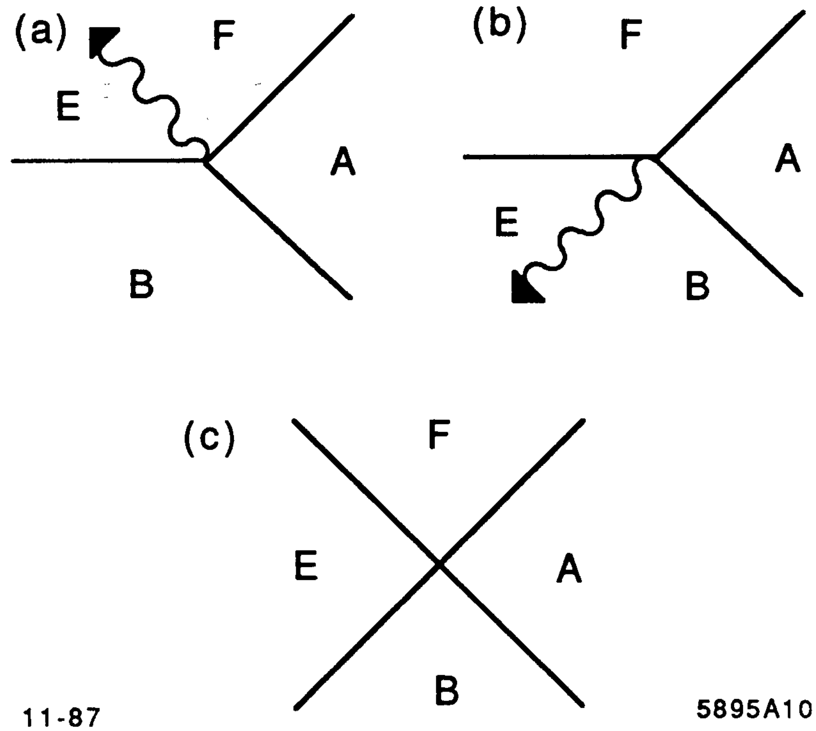

Fig. 10

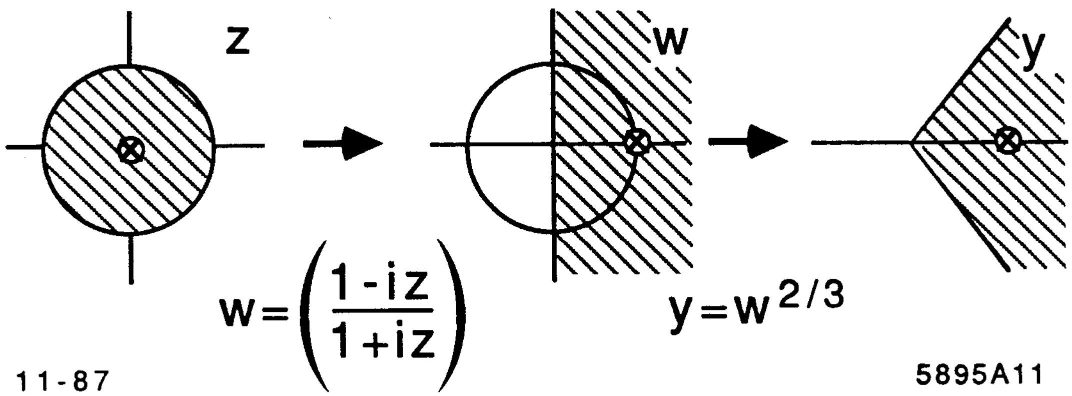

Fig. 11

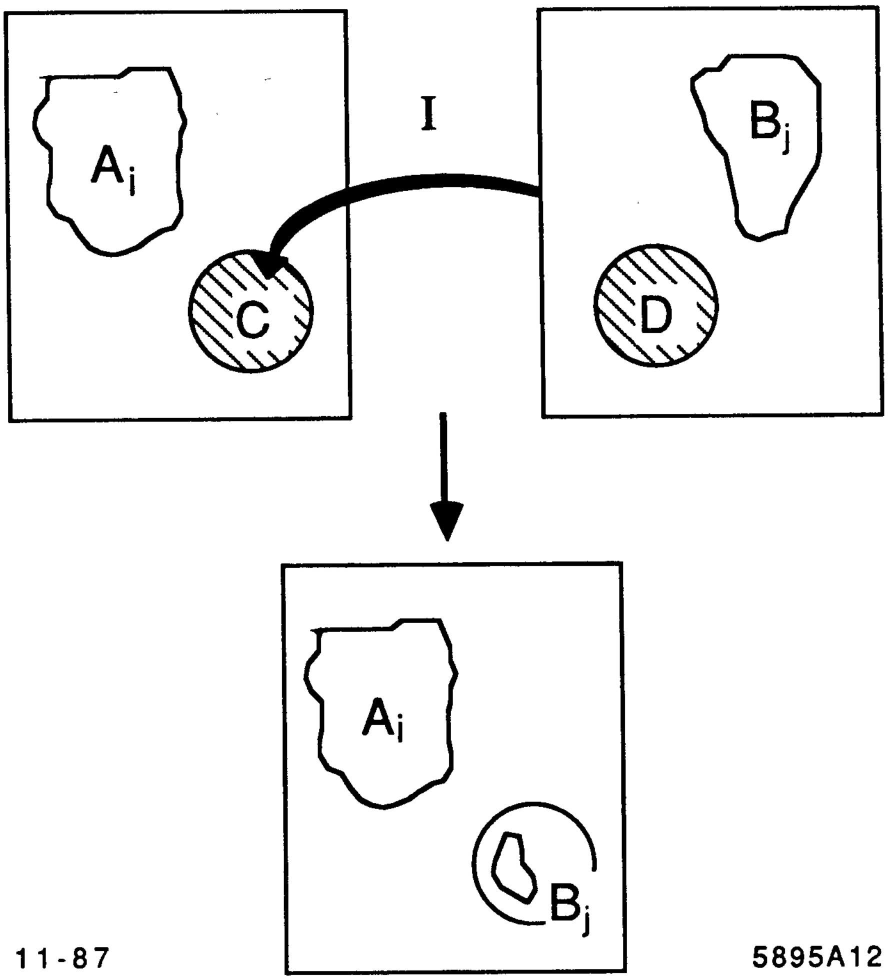

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Fig. 12

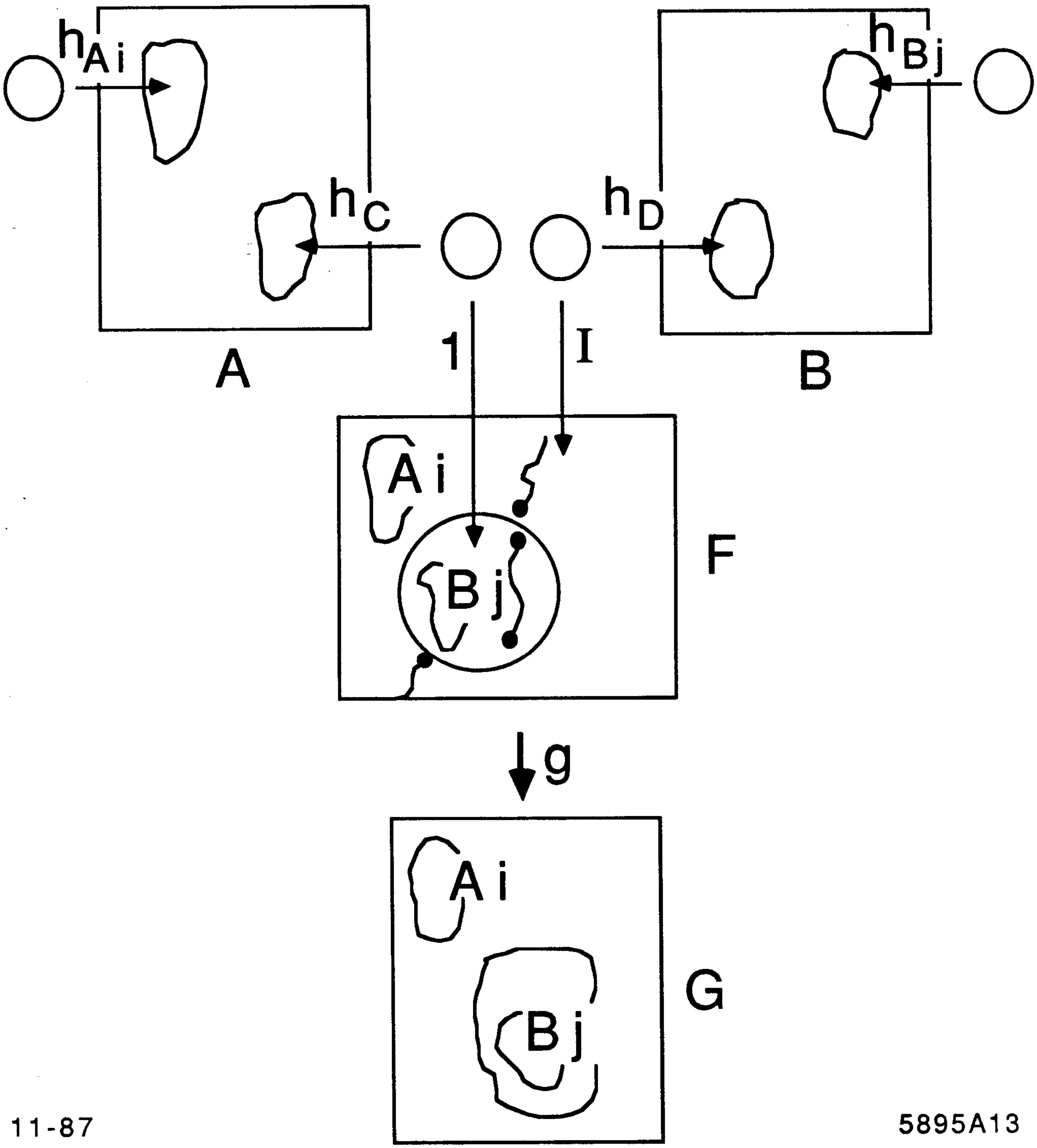

Fig. 13