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Lagrangian and Hamiltonian Formalisms for Relativistic Mechanics with Lorentz-Invariant Evolution Parameters in 1 + 1 Dimensions

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Abstract: This article presents alternative Hamiltonian and Lagrangian formalisms for relativistic mechanics using proper time and proper Lagrangian coordinates in 1 + 1 dimensions as parameters of evolution. The Lagrangian and Hamiltonian formalisms for a hypothetical particle with and without charge are considered based on the relativistic equation for the dynamics and integrals of particle motion. A relativistic invariant law for the conservation of energy and momentum in the Lorentz representation is given. To select various generalized coordinates and momenta, it is possible to modify the Lagrange equations of the second kind due to the relativistic laws of conservation of energy and momentum. An action function is obtained with an explicit dependence on the velocity of the relativistic particles. The angular integral of the particle motion is derived from Hamiltonian mechanics, and the displacement Hamiltonian is obtained from the Hamilton–Jacobi equation. The angular integral of the particle motion θ is an invariant form of the conservation law. It appears only at relativistic intensities and is constant only in a specific case. The Hamilton–Jacobi–Lagrange equation is derived from the Hamilton–Jacobi equation and the Lagrange equation of the second kind. Using relativistic Hamiltonian mechanics, the Euler–Hamilton equation is obtained by expressing the energy balance through the angular integral of the particle motion θ. The given conservation laws show that the angular integral of the particle motion reflects the relativistic Doppler effect for particles in 1 + 1 dimensions. The connection between the integrals of the particle motion and the doubly special theory of relativity is shown. As an example of the applicability of the proposed invariant method, analyses of the motion of relativistic particles in circularly polarized, monochromatic, spatially modulated electromagnetic plane waves and plane laser pulses are given, and comparisons are made with calculations based on the Landau and Lifshitz method. To allow for the analysis of the oscillation of a particle in various fields, a phase-plane method is presented.

Keywords: Lagrangian coordinate; relativistic Hamiltonian; action function; Hamilton–Jacobi–Lagrange equation; angular integral of particle motion; double special relativity; Euler–Hamilton equation; phase-plane method

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1. Introduction

The motion of relativistic particles can be determined using Lagrangian and Hamiltonian formalisms through various powerful approaches, including relativistic Newtonian mechanics. The motion of relativistic particles, such as a particle moving along a geodesic, is described well by the Lagrangian and Hamiltonian formalisms. This approach considers the relativistic laws of conservation of energy and momentum. However, relativistic Newtonian mechanics can only provide information on particle dynamics based on initial data and energy characteristics, such as energy, momentum, speed, and position [1–4].

An alternative Lagrangian and Hamiltonian formalism that is distinct from the classical standard formulations of relativistic mechanics has previously been presented [5–8]. An attempt to describe the dynamics of a particle that explicitly and implicitly differs from the Lorentz covariant form for the non-relativistic case of particle motion with a Lorentz-invariant evolutionary parameter has also been presented [9].

The principle of relativity proposed by Einstein in 1905 states that the speed of light in any inertial frame of reference is constant and all physical processes in inertial frames of reference proceed in the same way [10]. Additionally, Einstein showed that the symmetry principle holds in an isotropic space, and that the function

$$Q_t Q_\xi = 1 \tag{1}$$

is equivalent to the symmetry principle. With the rapid development of theories such as doubly special relativity and gravitomagnetism [11–17], it has become desirable to apply the Hamiltonian and Lagrangian formalisms [18] to generalize the conservation laws into an invariant form and to study the dynamics of relativistic hypothetical particles that are not charged and are not affected by external forces, electric and magnetic fields, or scalar and vector potentials. In this work, based on the form advanced by Lorentz [19], $\xi = t - f\mathbf{nr}/c$, where $\mathbf{n}$ is a gyronormal vector [20] and $f = \pm 1$, the direct Q_t and inverse Q_ξ integrals of particle motion are introduced. All the characteristics of the energy of a particle in 1 + 1 dimensions are obtained. The identical transformations given in ref. [10] are generalized in the inertial coordinate system of the particle.

The goal of this work is to obtain the energy characteristics of a particle in 1 + 1 dimensions and to derive the angular relativistic integral of the particle motion θ using the Lagrangian and Hamiltonian formalisms, which generalize the identical transformations given in refs. [1–20]. This work presents alternative formalisms for Lagrangian and Hamiltonian mechanics. These formalisms differ in both their explicit and implicit Lorentz-covariant interpretations of the Lagrangian and Hamiltonian formalisms in relativistic mechanics [1–4]. The formalisms are presented in the form of integrals of particle motion that incorporate relativistic physical parameters, such as the energy, momentum, velocity, and space coordinates of a particle in its own inertial reference frame. By applying algebraic operations to the integrals of particle motion, one can obtain other physical properties, such as the radiative friction force and the radiation intensity of a relativistic particle in the far field [19].

Starting from the properties of closure for the Lorentz group and the established one-to-one correspondences between the time $t = t(q)$ and coordinate $q = q(t)$ of a hypothetical particle, this work uses the Hamiltonian and Lagrangian formalisms to search for new invariant transformations based on symmetry properties and mutually invariant transformations for the integrals of particle motion $Q_t = Q_t(\gamma)$ and $\gamma = \gamma(Q_t)$. The classical integral of particle motion is given as:

$$E - \mathbf{nP} = \gamma^+,\ \gamma^+ = \frac{(1 - \mathbf{n}\boldsymbol{\beta})}{\sqrt{1-\beta^2}},\ \boldsymbol{\beta} = \frac{\mathbf{V}}{c},\ \gamma^+ = \frac{\gamma_0^+}{mc}, \tag{2}$$

where

$$\mathbf{nP} = \mathbf{np}/mc = \sqrt{Q_\xi^+ Q_\xi^- - 1},\ E = \varepsilon/mc^2 = \sqrt{Q_\xi^+ Q_\xi^-}, \tag{3}$$

$$\mathbf{p} = \frac{m\mathbf{V}}{\sqrt{1-\beta^2}},\ \varepsilon = \frac{mc^2}{\sqrt{1-\beta^2}} = \sqrt{m^2c^4 + c^2p^2}, \tag{4}$$

and where m is the mass of the particle, $\mathbf{V}$ is the speed of the particle, c is the speed of light, $\mathbf{n}$ is a gyrovector, $Q_{\xi}^{+} = \frac{1}{1-\mathbf{n\beta}}$ and $Q_{\xi}^{-} = \frac{1}{1+\mathbf{n\beta}}$ are the integrals of particle motion from refs. [10,19,21], and $\check{E} \geq 1$.

These equations appear in many works, such as those on analytical calculations of motion for a relativistic charged particle in an electromagnetic field [22–30], or numerical calculations for the field of a Gaussian beam [31–34]. Many works on the acceleration of particles by electromagnetic fields (see, for example, ref. [35]), have considered the integral of particle motion $\gamma_0 \equiv \text{const}$ as:

$$\left(p^2 + m^2c^2\right)^{\frac{1}{2}} - \mathbf{np} \equiv \text{const}. \tag{5}$$

2. The Landau–Lifshitz Method for Calculating the Energy Characteristics of a Particle in an Electromagnetic Field

In the literature, the description of the dynamics of relativistic particles in electromagnetic fields, in electric and magnetic static or alternating fields with different directions, and in vacuums has become very widespread, using the concept of the dynamics of a relativistic particle from the book by Landau and Lifshitz, *The Classical Theory of Fields* [21].

As an example, Rukhadze and others [22–31] applied the concept of the motion of a relativistic particle in an electromagnetic field from ref. [21] to carry out numerical and analytical calculations of the dynamics of a relativistic particle for different phenomena, including various types of waves, wave packets, Gaussian laser pulses, and configurations of electric and magnetic fields, using a component-wise representation of the relativistic momentum of a particle in the absence of a scalar field potential. This method is referred to here as the Landau–Lifshitz method.

As is well known, the Landau–Lifshitz method is based on the four-vector law of the conservation of energy-momentum, which can be represented in a component-by-component form as

$$E^2 = P_{\perp}^2 + P_{\|}^2 + 1, \tag{6}$$

where $P_{\perp}$ and $P_{\|}$ are the perpendicular and longitudinal components of the particle momentum in the Euclidean geometry representation.

Furthermore, from the change in the energy of a particle and the equation of motion of a relativistic particle in an electromagnetic field, taking into account the Lorentz force, the following equations can be derived:

$$\frac{dE}{dt} = \frac{q}{mc}\mathbf{E\beta},\ \frac{d\mathbf{P}}{dt} = \frac{q}{mc}(\mathbf{E} + [\boldsymbol{\beta} \times \mathbf{H}]). \tag{7}$$

Here, $\mathbf{E}$ and $\mathbf{H}$ are the strengths of the electric and magnetic fields.

From Equation (7), for the longitudinal component of the momentum of the particle, the integral of particle motion $\gamma^+ = E - P_{\|}$ can be found, which is defined as a constant.

Substituting the integral of particle motion γ^+ into Equation (6) gives the value of the longitudinal component of the particle momentum and the total energy of the particle (see, for example, refs. [22–30]):

$$P_{\|} = \gamma^+ g,\ E = \gamma^+ g_{LL};\ g_{LL} = \frac{P_{\perp}^2 + 1}{2(\gamma^+)^2} - \frac{1}{2}. \tag{8}$$

The perpendicular component of the particle momentum can be found directly from the equation of motion using the Lorentz force (7), which has the form

$$\mathbf{P}_{\perp} = \frac{q}{mc}\int \mathbf{E}_{\perp}dt + \frac{q}{mc}\int [\boldsymbol{\beta} \times \mathbf{H}]_{\perp}dt + \boldsymbol{\chi}_{\perp}, \tag{9}$$

where $\boldsymbol{\chi}_{\perp}$ is a constant determined by the initial condition $\mathbf{P}_{\perp}(t=0) \equiv \boldsymbol{\chi}_{\perp}$.

Due to the fact that in works [22–30], it was assumed that the integral of particle motion is a constant $\gamma^+ = const$, the total radiation power of the particle can be determined using the expression below:

$$W = -\frac{dE}{dt} = -\gamma^+ \frac{dg_{LL}}{dt} = -\frac{\mathbf{P}_\perp}{\gamma^+} \frac{d\mathbf{P}_\perp}{dt}. \tag{10}$$

As can be seen from Equations (9) and (10), in the absence of an external electromagnetic field, the total radiation power is identically equal to zero.

The disadvantage of calculations that use the method of Landau and Lifshitz is that the calculations are carried out only on the integral of the particle motion γ^+ and the integral of particle motion is a constant. Furthermore, calculations of the trajectory of motion and the dynamic characteristics of the particle, such as the momentum, velocity, energy, and coordinates of the particle, are applicable only to the electromagnetic field. For cases in which the influence of an electromagnetic field on a particle can be neglected, the dynamic characteristics of the particle are understood as some kind of constant. In addition, if the particle has a dynamic functional characteristic that changes with time, and then falls into an external electromagnetic field, then the initial characteristics of the particle are not taken into account.

3. The Method of Coupled Parameters

The method of coupled parameters can be applied in relativistic hydrodynamics [36], when all available thermodynamic quantities depend on one variable, for example, the temperature. In this paper, the laws of relativistic hydrodynamics are not considered, but a relationship is introduced between the coordinates $q = q(t)$ and $t = t(q)$ from ref. [19], expressing the related parameters in two ways: (1) through the integral of particle motion Q_t, i.e., $q = q(t) = q(Q_t)$ and $t = t(q) = t(Q_t)$, and (2) through the angular integral of particle motion or angular rapidity θ [37], $q = q(t) = q(Q_t) = q(\theta)$ and $t = t(q) = q(Q_t) = t(\theta)$, depending on which the subsequent analysis considers, with the relationship between the integrals of particle motion $\theta = \theta(Q_t)$ and $\theta = \theta(\gamma)$ being derived.

4. The Invariant form Description of the Motion of a Relativistic Particle through the Space–Time Coordinate ξ

To describe the dynamics of a relativistic particle from first principles, as stated earlier, the space–time Lorentz coordinate ξ is used:

$$\xi = t - fq, \; Q_t = \frac{d\xi}{dt} = 1 - f\mathbf{n}\boldsymbol{\beta}, \tag{11}$$

where t is the proper time, $q = \mathbf{nr}/c = \ln(1 - Q_t^+ Q_t^-)/2$ is the Lagrange coordinate satisfying the relativistic equation for a free particle [19], and Q_t is the integral of particle motion of a relativistic particle.

The choice of $f = 1$ or $f = -1$ is determined by the principle of relativity for an inertial system K and characterizes the direction of motion of a relativistic particle relative to the initial position (q_0, t_0). It is furthermore assumed that $\xi^+ = t - q$ represents movement to the right relative to the initial position and $\xi^- = t + q$ represents movement to the left.

For an inertial frame of reference K' in 1 + 1 dimensions, the representation of the coordinates q' and t' in terms of the proper coordinates q and t has the form

$$q' = \frac{q - \mathbf{n}\boldsymbol{\beta}t}{\sqrt{1-\beta^2}} = qE - \mathbf{nP}t, \; t' = \frac{t - \mathbf{n}\boldsymbol{\beta}q}{\sqrt{1-\beta^2}} = tE - \mathbf{nP}q. \tag{12}$$

From Equation (12), the Lorentz invariant coordinate ξ' in the inertial frame K' has the following representations:

$$\xi' = t' - fq' = \xi\gamma^{-f}, \; \gamma^{-f} = E + f\mathbf{nP}; \; \xi'^+ = \xi^+\gamma^-, \; \xi'^- = \xi^-\gamma^+, \tag{13}$$

where the multiplication of ξ'^{+} and ξ'^{-} represents an invariant action for any inertial frame of reference:

$$s^2 = \xi'^{+}\xi'^{-} = \xi^{+}\xi^{-} = t^2 - q^2 = inv. \tag{14}$$

From the principle of invariance in Equation (14), it can be seen that the second postulate of Einstein from the special theory of relativity is observed, which states that in different inertial frames of reference all physical processes proceed in the same way.

5. The Trajectory of a Relativistic Particle in a General Electromagnetic Field

To describe the trajectory of a relativistic particle in a more general form, for example, in an electromagnetic wave, it can be imagined that the coordinate of the particle has the following form:

$$\frac{\mathbf{nR}(\mathbf{r},t)}{c} = \frac{\mathbf{nR}(\xi)}{c} = \frac{q_{EM}}{2} + \frac{f}{2}\int \mathbf{n\beta}_{EM}dt, \tag{15}$$

where q_{EM} is the dynamic parameter of the Doppler shift of a relativistic particle in an electromagnetic wave, which plays the role of the shift coordinate, $\mathbf{n\beta}_{EM} = \sqrt{1-\left(Q_t^{+}Q_t^{-}\right)_{EM}}$ is the speed of the relativistic particle in the electromagnetic field, $f = 1$ represents the motion of the particle along the direction of propagation of the electromagnetic wave, $f = -1$ represents the motion of the particle against the direction of propagation of the electromagnetic wave, and $\left(Q_t^{+}Q_t^{-}\right)_{EM}$ are the integrals of the motion of the particle in the electromagnetic wave, which are related to the integrals of the motion of the momentum and the energy of the particle, as follows: $\left(Q_t^{+}Q_t^{-}\right)_{EM} = 1/\left(Q_\xi^{+}Q_\xi^{-}\right)_{EM}$.

In the absence of an electromagnetic wave, when the field strength is zero, the coordinate of the Doppler shift in the electromagnetic field becomes the Lagrangian coordinate:

$$q_{EM} = \frac{1}{2}\ln\left(1-\left(Q_t^{+}Q_t^{-}\right)_{EM}\right) \to q = \frac{1}{2}\ln\left(1-Q_t^{+}Q_t^{-}\right). \tag{16}$$

The main task is to find the connection and functional dependence between the integrals of particle motion $\left(Q_t^{+}Q_t^{-}\right)_{EM}$ in an electromagnetic field and the integrals of particle motion $Q_t^{+}Q_t^{-}$ in the absence of an electromagnetic field, in order to describe the work undertaken by the field in moving the particle. To search for a connection between the integrals of particle motion, it is necessary to find a mechanism for describing the dynamics of a relativistic particle both in the absence and in the presence of external forces or electromagnetic fields. It seems that the most optimal methods are the use of relativistic Lagrangian and Hamiltonian mechanics with the use of proper coordinates $q = q(t)$ and $t = t(q)$ with coupled parameters.

6. The Laws of Conservation of Energy and Momentum in Relativistic Dynamics in a Differential Form

The relation between energy and momentum in a relativistic form in 1 + 1 dimensions is given by the following equation:

$$E^2 = \mathbf{P}^2 + 1. \tag{17}$$

Differentiating Equation (17) with respect to the integrals of particle motion $Q_\xi^{+}Q_\xi^{-}$ gives

$$E\frac{dE}{d\left(Q_\xi^{+}Q_\xi^{-}\right)} = \vec{P}\frac{d\vec{P}}{d\left(Q_\xi^{+}Q_\xi^{-}\right)}. \tag{18}$$

Taking the derivative of Equation (18) with respect to $Q_\xi^+ Q_\xi^-$ gives the law of conservation of energy and the law of conservation of momentum in a differential form:

$$\left(\frac{dE}{d\left(Q_\xi^+ Q_\xi^+\right)}\right)^2 + E\frac{d^2E}{d\left(Q_\xi^+ Q_\xi^+\right)^2} = 0, \quad \left(\frac{d\mathbf{P}}{d\left(Q_\xi^+ Q_\xi^+\right)}\right)^2 + \mathbf{P}\frac{d^2\mathbf{P}}{d\left(Q_\xi^+ Q_\xi^+\right)^2} = 0, \tag{19}$$

where the law of conservation of energy and the law of conservation of momentum in Equation (19) use an invariant form of Q_ξ in the Lorentz representation.

Differentiating Equation (3) with respect to $Q_\xi^+ Q_\xi^-$ and substituting the result into Equation (19), it is easy to verify that the conservation laws hold. This law of conservation of energy and momentum is valid for a free particle in the absence of external fields and dissipative forces acting on the particle. In the presence of influences on the particle, the law of conservation of momentum and energy in Equation (19) takes a certain distribution function $g\left(Q_\xi^+ Q_\xi^+\right)$ with respect to the integral of particle motion $Q_\xi^+ Q_\xi^+$.

$$\left(\frac{dE_g}{d\left(Q_\xi^+ Q_\xi^+\right)}\right)^2 + E_g\frac{d^2E_g}{d\left(Q_\xi^+ Q_\xi^+\right)^2} = g_E, \tag{20}$$

$$\left(\frac{d\mathbf{P}_g}{d\left(Q_\xi^+ Q_\xi^+\right)}\right)^2 + \mathbf{P}_g\frac{d^2\mathbf{P}_g}{d\left(Q_\xi^+ Q_\xi^+\right)^2} = g_{\mathbf{nP}}, \tag{21}$$

where $E_g = \sqrt{Q_{\xi g}^+ Q_{\xi g}^+} = E_g\left(Q_\xi^+ Q_\xi^+\right)$ is some energy and $\mathbf{nP}_g = \sqrt{Q_{\xi g}^+ Q_{\xi g}^+ - 1} = \mathbf{nP}_g\left(Q_\xi^+ Q_\xi^+\right)$ is some momentum under the action of external or dissipative forces, $g_{\mathbf{nP}}$ is a certain momentum distribution function $\mathbf{nP}_g$, and g_E is a certain energy distribution function E_g.

7. The Lagrangian Formalism and the Lagrange Equation of the Second Kind in 1 + 1 Dimensions

As is well known, the Lagrangian $L' = L/mc^2$ for a particle without charge has the following form [21]:

$$L' = -\sqrt{1-\beta^2} = -\sqrt{Q_t^+ Q_t^-}, \tag{22}$$

where $Q_t^+ = 1 - \mathbf{n\beta}$ and $Q_t^- = 1 + \mathbf{n\beta}$ are integrals of the particle motion in the Lorentz representation.

Taking into account the differential connection in the Lorentz representation $\sqrt{1 - Q_t^+ Q_t^-} = f(1 - Q_t)$ from ref. [19] gives

$$L' = -\sqrt{Q_t^+ Q_t^-} = -\sqrt{1-(1-Q_t)^2}. \tag{23}$$

Differentiating the Lagrangian from Equation (23) with respect to Q_t gives the Euler–Lagrange equation in 1 + 1 dimensions:

$$\frac{\partial L'}{\partial Q_t} = -\frac{1-Q_t}{\sqrt{1-(1-Q_t)^2}} = -\frac{1-Q_t}{\sqrt{Q_t^+ Q_t^-}}. \tag{24}$$

Using the representations for Q_t^+ gives

$$\frac{\partial L'}{\partial Q_t^+} = -\frac{(1-Q_t^+)}{\sqrt{Q_t^+ Q_t^-}} = -\sqrt{Q_\xi^+ Q_\xi^-} + \frac{Q_t^+}{\sqrt{Q_t^+ Q_t^-}} = -\frac{\mathbf{n\beta}}{\sqrt{1-\beta^2}} = -\mathbf{nP}, \tag{25}$$

where $E = \sqrt{Q_\xi^+ Q_\xi^-}$, $\gamma^+ = \sqrt{Q_t^+ / Q_t^-}$, and

$$\frac{\partial L'}{\partial Q_t^+} = -E + \gamma^+ = -\mathbf{nP}. \tag{26}$$

Equation (26) is the integral of particle motion γ^+ from ref. [19]:

$$E - \mathbf{nP} = \gamma^+. \tag{27}$$

Similarly, for Q_t^-:

$$\frac{\partial L'}{\partial Q_t^-} = -\frac{(1 - Q_t^-)}{\sqrt{Q_t^+ Q_t^-}} = -\sqrt{Q_\xi^+ Q_\xi^-} + \frac{Q_t^-}{\sqrt{Q_t^+ Q_t^-}} = \frac{\mathbf{n\beta}}{\sqrt{1 - \beta^2}} = \mathbf{nP}, \tag{28}$$

where $E = \sqrt{Q_\xi^+ Q_\xi^-}$, $\gamma^- = \sqrt{Q_t^- / Q_t^+}$, and

$$\frac{\partial L'}{\partial Q_t^-} = -E + \gamma^- = \mathbf{nP}. \tag{29}$$

Equation (29) is also the integral of particle motion from ref. [19]:

$$E + \mathbf{nP} = \gamma^-. \tag{30}$$

Differentiating Equation (24) with respect to time gives

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t}\right) = \frac{\dot{Q}_t}{\left(1 - (1 - Q_t)^2\right)^{3/2}}, \tag{31}$$

where $\dot{Q}_t = dQ_t/dt$.

Substituting the differential forms $\dot{Q}_t = -f(1 - Q_t)^2$ into Equation (31) gives

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t}\right) = -\frac{f(1 - Q_t)^2}{\left(1 - (1 - Q_t)^2\right)^{3/2}}. \tag{32}$$

Substituting the Lorentz representation $\sqrt{1 - Q_t^+ Q_t^-} = f(1 - Q_t)$ into Equation (32) gives

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t}\right) = -f\left(\sqrt{Q_\xi^+ Q_\xi^-} - \sqrt{Q_t^+ Q_t^-}\right). \tag{33}$$

The particle coordinate in 1 + 1 dimensions from ref. [19] has the form:

$$-2fq = 2\mathbf{nr}/c = \ln\left(1 - Q_t^+ Q_t^-\right), \tag{34}$$

where q is the Lagrangian coordinate.

Expressing $Q_t^+ Q_t^-$ using Equation (34) and substituting the result into the Lagrangian in Equation (23) gives

$$L' = -\sqrt{Q_t^+ Q_t^-} = -\sqrt{1 - \exp(-2fq)}. \tag{35}$$

Differentiating the Lagrangian in Equation (35) with respect to q gives:

$$\frac{\partial L'}{\partial q} = -f\frac{\exp(-2fq)}{\sqrt{1 - \exp(-2fq)}} = -f\left(\sqrt{Q_\xi^+ Q_\xi^-} - \sqrt{Q_t^+ Q_t^-}\right). \tag{36}$$

Substituting Equations (33) and (36) into the Lagrange equation of the second kind, it is found that the Lagrange equation is satisfied for q and Q_t:

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t}\right) - \frac{\partial L'}{\partial q} = 0. \quad (37)$$

Multiplying Equations (26) and (29) together and using $E^2 = P^2 + 1$, it is easy to show that

$$E = \frac{\gamma^+ + \gamma^-}{2}, \quad (38)$$

where γ^+ and γ^- are the integrals of particle motion.

Substituting the energy in Equation (38) into the integrals of particle motion in Equations (27) and (30) gives the momentum of the particle:

$$\mathbf{nP} = \frac{\gamma^- - \gamma^+}{2}. \quad (39)$$

as can be seen from Equations (22)–(39). The integrals of particle motion have an explicit connection, and are mutually expressed as $\gamma = \gamma(Q_t, q, t)$ and $Q_t = Q_t(\gamma, q, t)$.

From the equations in this section, it can be seen that the integrals of particle motion γ^+ and γ^- are in an invariant form of the conservation law. It is only in the particular case that $\mathbf{n\beta} =$ const that they are constant for uniform and rectilinear motion ($\mathbf{nP} =$ const) or for a particle at rest ($\mathbf{n\beta} = 0$) relative to its own coordinate system ($\gamma_0 = mc$).

8. Invariant Forms of the Decomposition of the Integrals of Particle Motion of a Free Relativistic Particle into Positive and Negative Representations

The energy and momentum from Equation (3) can be represented in the invariant form:

$$E = Q_\xi \gamma,\ \mathbf{nP} = f(Q_\xi - 1)\gamma, \quad (40)$$

where the decomposition into positive and negative representations have the form

$$E^+ = Q_\xi^+ \gamma^+,\ E^- = Q_\xi^- \gamma^-,\ \mathbf{nP}^+ = \left(Q_\xi^+ - 1\right)\gamma^+,\ \mathbf{nP}^- = -\left(Q_\xi^- - 1\right)\gamma^-. \quad (41)$$

Representations of positive and negative energies can be easily represented in the form below:

$$E^+E^- = Q_\xi^+ Q_\xi^- = E^2,\ \mathbf{P}^+\mathbf{P}^- = Q_\xi^+ Q_\xi^- - 1 = P^2, \quad (42)$$

for the total energy and momentum of the particle. It is assumed that $E \geq 1$ and $\mathbf{nP} \geq 0$, and then Equation (42) provides the representations in Equation (3).

Note that the integrals of particle motion γ^+ and γ^- from Equation (41) can also be represented as a decomposition into positive and negative representations, which then have the form

$$\gamma^+ = Q_t^+ E,\ \gamma^- = Q_t^- E, \quad (43)$$

where $\gamma^+\gamma^- = 1$.

For the integral of particle motion γ^+, using a positive expansion for energy and momentum, it can be shown that

$$\gamma^+ = E - \mathbf{nP} \Leftrightarrow E^+ - \mathbf{nP}^+ = \gamma^+. \quad (44)$$

Similarly, for γ^+, using the negative representations of energy and momentum gives

$$\gamma^+ = E - \mathbf{nP} \Leftrightarrow E^- - \mathbf{nP}^- = \left(2Q_\xi^- - 1\right)\gamma^-. \quad (45)$$

Putting the connection of the integrals of particle motion $Q_\xi^- = \left((\gamma^+)^2 + 1\right)/2$ from ref. [19] into Equation (45) shows that the identical equality $\gamma^+ \Leftrightarrow \gamma^+$ is fulfilled.

It can be seen from Equation (45) that the relationship between the integrals of particle motion γ and Q_ξ can be represented in a more general form:

$$\gamma = \sqrt{2Q_\xi^{-f} - 1} = \frac{1}{\sqrt{2Q_\xi - 1}}, \tag{46}$$

that is, for the positive and negative representation of the energy and momentum of a free particle, the invariance property is satisfied:

$$E \equiv E^+ \equiv E^- \equiv inv,\ \mathbf{nP} \equiv \mathbf{nP}^+ \equiv \mathbf{nP}^- \equiv inv. \tag{47}$$

9. The Relativistic Hamiltonian in 1 + 1 Dimensions

The Hamiltonian has the following form:

$$H' = \mathbf{P}\boldsymbol{\beta} - L', \tag{48}$$

where $L' = -\sqrt{Q_t^+ Q_t^-}$, $\mathbf{P} = \mathbf{n}\sqrt{Q_\xi^+ Q_\xi^- - 1}$, and $\boldsymbol{\beta} = \mathbf{n}\sqrt{1 - Q_t^+ Q_t^-}$.

Substituting L', $\mathbf{P}$, and $\boldsymbol{\beta}$ into Equation (48) gives

$$H' = \sqrt{Q_\xi^+ Q_\xi^-}. \tag{49}$$

Using the property of the Lorentz group $Q_\xi^+ Q_\xi^- = \left(Q_\xi^+ + Q_\xi^-\right)/2$, Equation (49) takes the form:

$$H' = \sqrt{\left(Q_\xi^+ + Q_\xi^-\right)/2}. \tag{50}$$

From the properties of the Lorentz group, a connection is also found between Q_ξ^+ and Q_ξ^-:

$$Q_\xi^+ = \frac{Q_\xi^-}{2Q_\xi^- - 1},\ Q_\xi^- = \frac{Q_\xi^+}{\left(2Q_\xi^+ - 1\right)}. \tag{51}$$

10. The Action Function for a Particle in 1 + 1 Dimensions

The action function $S' = S/mc^2$ for a particle without charge takes the following form [38]:

$$S' = -\int \sqrt{Q_t^+ Q_t^-}\, dt. \tag{52}$$

Taking into account the property of the total time differential form $dt = d(1 - Q_t^+ Q_t^-)/\left(2(1 - Q_t^+ Q_t^-)^{3/2}\right)$ and integrating from $Q_t^+ Q_t^-$ to 0 gives

$$S' = \sqrt{\frac{Q_t^+ Q_t^-}{(1 - Q_t^+ Q_t^-)}} - \arcsin\left(\sqrt{Q_t^+ Q_t^-}\right). \tag{53}$$

Using the time form [19] and substituting into Equation (53), the partial time derivative of the action function can be found:

$$\frac{\partial S'}{\partial t} = -\sqrt{Q_t^+ Q_t^-}, \tag{54}$$

where the action function S' is invariant under Lorentz transformations.

11. The Hamilton–Jacobi Equation and the Hamilton–Jacobi–Lagrange Equation

The Hamilton–Jacobi equation has the following form [38]:

$$H' + \frac{\partial S'}{\partial t} = \widetilde{H}'. \tag{55}$$

Substituting in the Hamiltonian H' from Equation (49) and the partial derivative of the action function from Equation (54) gives the displacement Hamiltonian as:

$$\widetilde{H}' = \sqrt{Q_\xi^+ Q_\xi^-} - \sqrt{Q_t^+ Q_t^-}. \tag{56}$$

Thus, if the particle has no velocity (it is at rest), the displacement Hamiltonian is zero, $\widetilde{H}'(\boldsymbol{\beta} = 0) \equiv 0$, and the Hamiltonian of Equation (49) is equal to 1 or mc^2, $H'(\boldsymbol{\beta} = 0) \equiv 1$.

Comparing the obtained solutions in Equations (33), (36), (55) and (56) gives the Hamilton–Jacobi–Lagrange equation:

$$H' + \frac{\partial S'}{\partial t} = \widetilde{H}' = -f\frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t}\right) = -f\frac{\partial L'}{\partial q}. \tag{57}$$

Considering that $q^+ = q$, $q^- = -q$, and $q^+ = -q^-$, the Hamilton–Jacobi–Lagrange equations take the forms

$$H' + \frac{\partial S'}{\partial t} = \widetilde{H}' = -\frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t^+}\right) = -\frac{\partial L'}{\partial q^+} = -\frac{\partial L'}{\partial q}, \tag{58}$$

and

$$H' + \frac{\partial S'}{\partial t} = \widetilde{H}' = \frac{d}{dt}\left(\frac{\partial L'}{\partial Q_t^-}\right) = \frac{\partial L'}{\partial q^-} = -\frac{\partial L'}{\partial q}. \tag{59}$$

Substituting the values of the momenta for particles from Equations (26) and (29) into Equations (58) and (59) gives:

$$H' + \frac{\partial S'}{\partial t} = \widetilde{H}' = \mathbf{n}\frac{d\mathbf{P}}{dt} = -\frac{\partial L'}{\partial q}. \tag{60}$$

12. The Angular Integral of Particle Motion (Angular Rapidity)

Consider a partial derivative of the Hamiltonian H' with respect to the particle momentum $P = \mathbf{nP} = \sqrt{Q_\xi^+ Q_\xi^- - 1}$ as:

$$\frac{\partial H'}{\partial P} = \sqrt{1 - Q_t^+ Q_t^-} = \beta = \mathbf{n}\boldsymbol{\beta}. \tag{61}$$

Suppose that the derivative of the Hamiltonian of the system with respect to some of the parameter θ is equal to the momentum of the system as:

$$\frac{\partial H'}{\partial \theta} = \sqrt{Q_\xi^+ Q_\xi^- - 1} = P = \mathbf{nP}. \tag{62}$$

θ can be expressed as:

$$\theta = \cosh^{-1}\left(\sqrt{Q_\xi^+ Q_\xi^-}\right), \tag{63}$$

where θ is the angular integral of the particle motion or angular rapidity [37] and $E = \sqrt{Q_\xi^+ Q_\xi^-}$, $E \geq 1$. Thus, the energy and momentum of the system take the forms:

$$E = \sqrt{Q_\xi^+ Q_\xi^-} = \cosh\theta,\ \mathbf{nP} = \sqrt{Q_\xi^+ Q_\xi^- - 1} = \sinh\theta, \tag{64}$$

which correspond to Equation (17). That is, the conservation law is satisfied for θ. As seen from Equations (61)–(64), the integral angle θ has a functional dependence on $\theta = \theta(\mathbf{nr}, t, \gamma, Q_t)$. Note that the integrals of particle motion γ and Q_t are also mutually expressed through the angular integral of the particle motion θ. That is, $\gamma = \gamma(\mathbf{nr}, t, Q_t, \theta)$ and $Q_t = Q_t(\mathbf{nr}, t, \gamma, \theta)$.

Next, these results are generalized. Consider electrons accelerated by the transverse electromagnetic field of incident pulsed laser radiation. The temperature of the fast electrons on the frontal surface of the target is estimated in the same way as in ref. [39]. The expressions for the amplitude of an oscillating electron, which presumably increases in the field of a linearly polarized electromagnetic plane wave, are then substituted in:

$$P^2 = Q_\xi^+ Q_\xi^- - 1 = \sinh^2\theta = \frac{e^2(\boldsymbol{\chi}\mathbf{E_0})^2}{m_e^2 c^2 \omega^2} = \frac{I}{I_{rel}} = a, \tag{65}$$

where a is the dimensionless amplitude, e is the electron charge, m_e is the electron mass, c is the speed of light, $\mathbf{E_0}$ is the amplitude vector for the electric field of the incident electromagnetic wave, $\boldsymbol{\chi}$ is the polarization gyrovector, ω is the oscillation frequency of the electromagnetic wave, $I = cE_0^2/(8\pi)$ is the intensity of the incident linearly polarized electromagnetic wave (in W/cm^2), and I_{rel} is the relativistic intensity:

$$I_r = \frac{m_e^2 c^3 \omega^2}{8\pi e^2} = 1.37 \times 10^{18} \lambda^{-2}, \tag{66}$$

where λ is the wavelength (in µm). In Equation (65), the relation between the dimensionless momentum and the dimensionless field amplitude is expressed in terms of the dimensionless intensity I/I_{rel}. There are several expressions for I_{rel} that differ by a numerical factor. As in ref. [31], the most correct criterion for such a determination is based on a comparison between the maximum total energy of an electron oscillating in the field of a laser pulse and its rest energy $m_e c^2$.

Substituting Equation (65) into Equation (64) gives:

$$\theta = \operatorname{arcsinh}\left(f\sqrt{\frac{I}{I_{rel}}}\right), \tag{67}$$

where $f = \pm 1$. This equation expresses the laws of conservation and the law of deviation for the particles, or the so-called Doppler effect for a force or electromagnetic field in 1 + 1 dimensions. The choice of the particle trajectory as $f = 1$ or $f = -1$ under the influence of an electromagnetic wave or acting forces is determined by the principle of relativity. The conservation laws for particles of mass m show that the Doppler effect is observed at $I/I_{rel} \geq 1$ and is determined by the law of motion $\mathbf{nr}_{\boldsymbol{\theta}}(t) = c\int \theta dt$. When $I/I_{rel} < 1$, the invariant form of the motion is $\mathbf{nr}_{\boldsymbol{\gamma}}(t) = c\int \gamma dt$. However, $\mathbf{r}_{\boldsymbol{\theta}}(t)$ and $\mathbf{r}_{\boldsymbol{\gamma}}(t)$ are special cases of the Lorentz invariant form $\mathbf{nr}(t) = c\int(1 - Q_t)dt$, since $\mathbf{r} = \mathbf{r}(t, \theta, \gamma, Q_t)$.

In general terms, the integral of particle motion γ depending on the angular integral of particle motion θ has the form

$$\gamma = \cosh\theta - f\sinh\theta = \exp(-f\theta), \tag{68}$$

where the invariant form of the integrals of particle motion has the form

$$\gamma^+ = \exp(-\theta),\ \gamma^- = \exp(\theta),\ \theta^+ = \theta^- = \theta. \tag{69}$$

It is also convenient to express in terms of the angular integral of particle motion θ the projection of the particle velocity $\mathbf{n}\boldsymbol{\beta}$ and the integrals of particle motion Q_t and Q_ξ, which have the form

$$\mathbf{n}\boldsymbol{\beta} = \tanh\theta,\ Q_t = 1 - f\tanh\theta,\ Q_\xi = \frac{1}{2}(\exp(2f\theta) + 1), \tag{70}$$

where the choice of $f = 1$ and $f = -1$ is also determined by the principle of relativity, and the invariant form is

$$\mathbf{n\beta} = \mathbf{n\beta}^{+} = \mathbf{n\beta}^{-} = \tanh\theta, \tag{71}$$

$$Q_t^{+} = 1 - \tanh\theta,\ Q_t^{-} = 1 - \tanh\theta,\ Q_t^{+} Q_t^{-} = 1 - \tanh^2\theta, \tag{72}$$

$$Q_\xi^{+} = \frac{1}{2}(\exp(2\theta) + 1),\ Q_\xi^{-} = \frac{1}{2}(\exp(-2\theta) + 1),\ Q_\xi^{+} Q_\xi^{-} = \cosh^2\theta. \tag{73}$$

Furthermore, from the presented relationships, it is of interest to express the Lorentz-invariant coordinates t and $q = \mathbf{nr}/c$ from ref. [19] in terms of the angular integral of particle motion θ,

$$t = 1 - \frac{f}{(1 - Q_t)} = 1 - \coth\theta,\ q = \frac{1}{2}\ln\left((1 - Q_t)^2\right) = \frac{1}{2}\ln\left(\tanh^2\theta\right). \tag{74}$$

Differentiating the proper coordinates t and q from Equation (74) with respect to θ gives

$$\frac{dt}{d\theta} = \frac{1}{\sinh^2\theta},\ \frac{dq}{d\theta} = \frac{2}{\sinh(2\theta)},\ \text{and}\ \dot{\theta} = \frac{d\theta}{dt} = \sinh^2\theta,\ \frac{d\theta}{dq} = \frac{\sinh(2\theta)}{2}. \tag{75}$$

From Equations (71) and (75), it is easy to show that the resulting differential forms are correct:

$$\mathbf{n\beta} = \frac{dq}{d\theta}\frac{d\theta}{dt} = \tanh\theta. \tag{76}$$

13. The Integrals of Particle Motion in a General Form and the Relation between the Lagrangian and the Hamiltonian through the Angular Integral of Particle Motion $\dot{\theta}$

In the general case, the integral of particle motion γ can be expressed in terms of the Hamiltonian, where the relation $\gamma = \gamma(H)$ has the form

$$\gamma = H' - f\frac{\partial H'}{\partial\theta}, \tag{77}$$

and the invariant forms of the integrals of particle motion γ^{+} and γ^{-} are

$$\gamma^{+} = H' - \frac{\partial H'}{\partial\theta},\ \gamma^{-} = H' + \frac{\partial H'}{\partial\theta}, \tag{78}$$

$$\gamma^{+}\gamma^{-} = (H')^2 - \left(\frac{\partial H'}{\partial\theta}\right)^2 = 1, \tag{79}$$

with Equation (79) expressing the energy conservation law in the Hamiltonian form.

By determining the integral of particle motion γ from the Hamiltonian of the system from Equation (77), it is easy to show that θ is invariant:

$$\theta = \operatorname{arctanh}\left(f\frac{(1 - \gamma^2)}{1 + \gamma^2}\right) = \operatorname{arctanh}\left(\frac{\gamma^{-} - \gamma^{+}}{\gamma^{-} + \gamma^{+}}\right). \tag{80}$$

Using the inter-functional dependence of the Hamiltonian H' and the angular integral of particle motion $\dot{\theta}$ from Equation (75), the relationship between H' and L' has the following form:

$$H' = \frac{1}{2}\int_0^{\dot{\theta}} L' d\dot{\theta}. \tag{81}$$

14. The Hamiltonian Formalism and the Euler–Hamilton Equation

The momentum of the system from the Hamiltonian in Equation (49), depending on the angular integral of particle motion θ, has the following form:

$$\mathbf{nP} = \frac{\partial H'}{\partial \theta} = \sinh\theta. \tag{82}$$

From Equations (75) and (82) for the Hamiltonian of a free relativistic particle, the following equalities are valid:

$$\left(\frac{\partial H'}{\partial \theta}\right)^2 - \dot{\theta} = 0,\ \frac{\partial^2 H'}{\partial \theta^2} - H' = 0, \tag{83}$$

where the Hamiltonian with an arbitrary given function $H_g = \sqrt{Q^+_{\xi g} Q^-_{\xi g}} = Q_{\xi g}\gamma_g$ for Equation (83) expresses the energy distribution functions G and $G_{\dot{\theta}}$ of the particle in the system:

$$\left(\frac{\partial H'_g}{\partial \theta}\right)^2 - \dot{\theta} = G_{\dot{\theta}},\ \frac{\partial^2 H'_g}{\partial \theta^2} - H'_g = G. \tag{84}$$

Differentiating the particle momentum in Equation (82) with respect to time t gives the following expression:

$$\frac{d(\mathbf{nP})}{dt} = \frac{d}{dt}\left(\frac{\partial H'}{\partial \theta}\right) = \cosh\theta \sinh^2\theta. \tag{85}$$

Similarly, differentiating the Hamiltonian H' with respect to the coordinate gives

$$\frac{\partial H'}{\partial q} = \cosh\theta \sinh^2\theta. \tag{86}$$

Balancing Equations (85) and (86) gives the relativistic Euler–Hamilton equation:

$$\frac{d}{dt}\left(\frac{\partial H'}{\partial \theta}\right) - \frac{\partial H'}{\partial q} = 0. \tag{87}$$

Comparing the relativistic Euler–Hamilton equation in Equation (87) with the Lagrange equation of the second kind in Equation (37) shows that

$$\frac{\partial H'}{\partial q} \neq \frac{\partial L'}{\partial q}, \tag{88}$$

$$\mathbf{nP} = \frac{\partial H'}{\partial \theta} = \sqrt{\mathbf{P}^+\mathbf{P}^-} = \sqrt{-\frac{\partial L'}{\partial Q_t^+}\frac{\partial L'}{\partial Q_t^-}} \geq 0, \tag{89}$$

and thus, due to the invariance of the angular integral of particle motion $\theta = \theta^+ = \theta^-$, the relativistic Euler–Hamilton equation gives a more compact solution.

Differentiating the left equation from Equation (83) with respect to time t and comparing it with the Euler–Hamilton equation (87) gives

$$\frac{\partial H'}{\partial q}\frac{\partial H'}{\partial \theta} = \frac{1}{2}\frac{d\dot{\theta}}{dt} = \frac{\ddot{\theta}}{2}. \tag{90}$$

It follows from the above equations that, if the arbitrary Lagrangian and Hamiltonian of the system are known, one can determine the force acting on the particle:

$$F_g = \frac{d}{dt}\left(\frac{\partial H'_g}{\partial \theta}\right) = \frac{d}{dt}\left(\sqrt{-\frac{\partial L'_g}{\partial Q_t^+}\frac{\partial L'_g}{\partial Q_t^-}}\right) \geq 0. \quad (91)$$

In general, it is possible to describe the energy balance of the system as

$$\frac{d}{dt}\left(\frac{\partial H'_g}{\partial \theta}\right) - \frac{\partial H'_g}{\partial q} = D_{H'_g}, \quad (92)$$

$$\frac{d}{dt}\left(\frac{\partial L'_g}{\partial Q_t}\right) - \frac{\partial L'_g}{\partial q} = D_{L'_g}, \quad (93)$$

where $D_{L'_g}$ and $D_{H'_g}$ are the Lagrangian and Hamiltonian dissipative forces, and $D_{L'_g} \neq D_{H'_g}$.

15. A Description of the Dynamics of a Relativistic Particle in an Electromagnetic Field

The most important advantage of the approach adopted in this work is that the Lagrangian for the electromagnetic field takes the following form:

$$L_{EM} = -\sqrt{(Q_t^+ Q_t^-)_{EM}} = -\sqrt{Q_t^+ Q_t^-} + \frac{q}{mc^2}(\boldsymbol{\beta}\mathbf{A}) - \frac{q}{mc^2}\varphi, \quad (94)$$

where $\mathbf{A}$ and φ are the scalar and vector potentials, respectively, which provide the momentum of a particle in an external electromagnetic field as:

$$-f\mathbf{n}\mathbf{P_{EM}} = \frac{\partial L_{EM}}{\partial Q_t}, \quad (95)$$

$$\vec{n}\vec{P}_{EM} = \frac{\partial H'_{EM}}{\partial \theta} = \sqrt{\mathbf{P^+_{EM}}\mathbf{P^-_{EM}}} = \sqrt{\left(Q_\xi^+ Q_\xi^-\right)_{EM} - 1} = \sqrt{-\frac{\partial L_{EM}}{\partial Q_t^-}\frac{\partial L_{EM}}{\partial Q_t^+}}. \quad (96)$$

Thus, the integral of the motion in the electromagnetic field is found to be $\left(Q_\xi^+ Q_\xi^-\right)_{EM}$. To describe the particle dynamics from these equations, one can use the transition $Q_\xi^+ Q_\xi^- \to \left(Q_\xi^+ Q_\xi^-\right)_{EM}$ into the Lorentz-invariant symmetry of the electromagnetic field. This transition from the Lagrangian of a free particle in Equation (22) to the Lagrangian in the field of an electromagnetic wave in Equation (94) is valid when the particle enters an external electromagnetic field. In this work, more complex scenarios are not considered; for example, the resulting stochastic instabilities for a trapped particle during the transition from a free state to an electromagnetic field. Only the transition itself and limiting cases are discussed, such as whether the particle is in an electromagnetic field $\left(Q_\xi^+ Q_\xi^-\right)_{EM}$ or outside the field $Q_\xi^+ Q_\xi^-$.

16. The Relation of the Integrals of Particle Motion to the Doubly Special Theory of Relativity in 1 + 1 Dimensions

In the doubly special theory of relativity, the laws of conservation of energy and momentum in Equation (4) can be written in a dimensionless form as [12]:

$$E^2 = P^2 + f(\mathbf{r}, t, \mathbf{P}, E), \quad (97)$$

where $f(\mathbf{r}, t, \mathbf{P}, E)$ is a function of the particle coordinate $\mathbf{r}$, the particle's own time t, the energy E, and the momentum $\mathbf{P}$ of the particle. For example, in refs. [11–17,40–43], the function $f(\mathbf{r}, t, \mathbf{P}, E)$ contains the Planck length L_p, and the laws of conservation of energy and momentum have the form:

$$E^2 - c^2p^2 = f(\mathbf{r}, t, \mathbf{P}, E, L_p) = \eta L_p c E p^2. \quad (98)$$

The authors of refs. [12,40–43] derived a fixed real number for η that can be either positive or negative and is approximately of the order 1: $|\eta| \sim 1$.

The above representations for the laws of conservation of energy and momentum in 1 + 1 dimensions show that in the dimensionless form, the function is normalized to 1 as:

$$f(\mathbf{r}, t, \mathbf{P}, E) = \gamma^+\gamma^- = 1. \tag{99}$$

In Planck-dimensional quantities, the function in Equation (97) has the form

$$f'\left(\mathbf{r}, t, \mathbf{P}, E, L_p\right) = \gamma^+\gamma^- = \frac{m^2c^2}{\hbar^2}L_p^2, \tag{100}$$

where $\hbar$ is the Planck constant. The classical integral of particle motion in Equation (2) is quantized by the quantum-dimensional quantity:

$$\gamma_\hbar \to \gamma\frac{mc}{\hbar}L_p. \tag{101}$$

Due to this normalization, it is of interest to search for an invariant form for the integrals of particle motion, or, in other words, to search for various representations with the parameter X when the integrals of particle motion are described in terms of their own inertial system as:

$$\frac{\partial\gamma^+}{\partial X} = X_{\gamma^+}\gamma^+, \ \frac{\partial\gamma^-}{\partial X} = X_{\gamma^-}\gamma^-, \tag{102}$$

where X_{γ^+} and X_{γ^-} are the eigenvalues of the integrals of particle motion.

17. Analysis Using the Lorentz-Invariant Method for the Motion of a Charged Particle in the Field of a Circularly Polarized Monochromatic Plane Wave

Based on the available integrals of particle motion, the motion of a relativistic particle in the field of a monochromatic electromagnetic plane wave is analyzed. The obtained solutions are comparable to the analytical solutions provided by the Landau and Lifshitz method [21,24]. It is assumed that the plane wave and relativistic particle are colinear with the direction of propagation for the normal gyrovector. Then, the four-vector potential in the form

$$[\varphi, \mathbf{A}] = \left[0, \frac{cE}{\omega}\cos(\omega\xi)\mathbf{e}_x, \frac{cE}{\omega}\sin(\omega\xi)\mathbf{e}_y\right] \tag{103}$$

is chosen, where $E \equiv const$ is the wave amplitude, $\omega \equiv const$ is the particle oscillation frequency, and $\mathbf{e}_x$ and $\mathbf{e}_y$ are orthogonal basis vectors with respect to the normal gyrovector $\mathbf{n}$. Then, the square of the vector potential $\mathbf{A}$ is determined in terms of the polarization gyrovector $\boldsymbol{\chi}$ as

$$A^2 = \frac{c^2E^2}{\omega^2} \text{ or } \boldsymbol{\chi}\mathbf{A} = \frac{cE}{\omega}, \tag{104}$$

where $(E, \omega) = const$ and $E/\omega \geq 0$.

The relationship between the gyrovectors $\boldsymbol{\chi}$ and $\mathbf{n}$ through the angular integral of the particle motion θ from geometric considerations is determined in the form

$$\boldsymbol{\chi} = \mathbf{n}\tanh\theta. \tag{105}$$

Then, using Equations (64) and (103)–(105), the Lagrangian in Equation (94) for a relativistic particle in a circularly polarized plane wave has the form

$$L_{cir} = -\sqrt{Q_t^+Q_t^-} + \sqrt{a}\left(1 - Q_t^+Q_t^-\right), \tag{106}$$

where a is the dimensionless invariant field amplitude, as seen in Equation (65). Applying the following invariant forms

$$Q_t^+ = 1 - f(1 - Q_t),\ Q_t^- = 1 + f(1 - Q_t), \tag{107}$$

substituting them into the Lagrangian of Equation (106), and differentiating with respect to the integral of particle motion gives

$$-f\mathbf{n}\mathbf{P}_{cir} = \frac{\partial L_{cir}}{\partial Q_t} = -(1 - Q_t)\left(\sqrt{Q_\xi^+ Q_\xi^-} + 2\sqrt{a}\right). \tag{108}$$

Using the invariant form of Equation (96) provides the squared momentum of the relativistic particle as:

$$P_{cir}^2 = \left(Q_\xi^+ Q_\xi^-\right)_{cir} - 1 = -\frac{\partial L_{cir}}{\partial Q_t^+}\frac{\partial L_{cir}}{\partial Q_t^-} = \left(1 - Q_t^+ Q_t^-\right)\left(\sqrt{Q_\xi^+ Q_\xi^-} + 2\sqrt{a}\right)^2, \tag{109}$$

which expresses the relationship between the Lorentz groups $\left(Q_\xi^+ Q_\xi^-\right)_{cir}$ and $Q_\xi^+ Q_\xi^-$.

Equation (109) gives the kinetic energy of the particle minus the rest energy as:

$$E_{kin} = E_{cir} - 1 = \sqrt{\left(Q_\xi^+ Q_\xi^+\right)_{cir}} - 1 = \sqrt{\left(1 - Q_t^+ Q_t^-\right)\left(\sqrt{Q_\xi^+ Q_\xi^-} + 2\sqrt{a}\right)^2 + 1} - 1, \tag{110}$$

where $E_{cir} \geq 1$. Considering that the square of the dimensionless momentum of a free particle is equal to the dimensionless field amplitude in Equation (65), these solutions are compared with those obtained via the Landau and Lifshitz method [24]. Thus, a normalization is introduced for the wave field amplitude, $E \to E/\sqrt{2}$, so that Equation (110) takes the following form:

$$E_{kin} = \sqrt{\frac{a}{a+2}\left(\sqrt{\frac{a}{2} + 1} + \sqrt{2a}\right)^2 + 1} - 1. \tag{111}$$

18. The Dynamics of a Particle in the Field of a Spatially Modulated Circularly Polarized Flat Laser Pulse

For a spatially modulated wave, it is assumed that the dimensionless field amplitude a is a function of the coordinate and time as $a = a(\mathbf{r}, t)$. Changes in the field amplitude are determined through the integrals of particle motion Q_ξ^+ and Q_ξ^- from Equation (65). Then, the Lagrangian of Equation (106) has the following form:

$$L_{SM} = -\sqrt{Q_t^+ Q_t^-} + \sqrt{Q_\xi^+ Q_\xi^- - 1}\left(1 - Q_t^+ Q_t^-\right). \tag{112}$$

Similarly, differentiating Equation (112) with respect to the integral of particle motion Q_t provides the invariant form of the particle momentum as:

$$-f\mathbf{n}\mathbf{P}_{SM} = \frac{\partial L_{SM}}{\partial Q_t} = -(1 - Q_t)\left(\sqrt{a+1} + a^{\frac{3}{2}} + 3\sqrt{a}\right). \tag{113}$$

Using the invariant form in Equation (96) gives the momentum of a relativistic particle as:

$$P_{SM}^2 = \left(Q_\xi^+ Q_\xi^-\right)_{SM} - 1 = -\frac{\partial L_{SM}}{\partial Q_t^+}\frac{\partial L_{SM}}{\partial Q_t^-} = \frac{a}{a+1}\left(\sqrt{a+1} + a^{\frac{3}{2}} + 3\sqrt{a}\right)^2. \tag{114}$$

Expressing the velocity of a relativistic particle from Equations (109) and (114) gives

$$\mathbf{n}\boldsymbol{\beta}_{cir} = \sqrt{1 - \left(Q_t^+ Q_t^-\right)_{cir}},\ \mathbf{n}\boldsymbol{\beta}_{SM} = \sqrt{1 - \left(Q_t^+ Q_t^-\right)_{SM}}. \tag{115}$$

Taking the connection between the integrals of motion $(Q_t^+ Q_t^-)_{cir} = 1/\left(Q_\xi^+ Q_\xi^+\right)_{cir}$, $(Q_t^+ Q_t^-)_{SM} = 1/\left(Q_\xi^+ Q_\xi^-\right)_{SM}$ and the angular integral of motion θ from Equation (65), Figure 1 shows a graph of the dependence of velocities on the angle θ for a free particle, for a particle in the field of a plane wave, and for a particle in the field of a laser plane pulse.

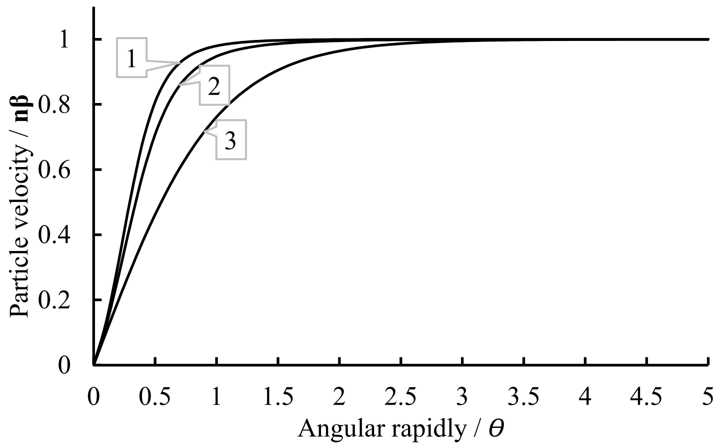

Figure 1. A graph of the dependence of the particle velocity on the angular integral of motion θ for (1) a free particle, (2) a particle in the field of a plane wave, and (3) a particle in the field of a plane laser pulse.

As can be seen from Figure 1, in the fields of a plane wave and a plane laser pulse for an angle $\theta \in [0;\ 2.5]$, a Doppler shift of the relativistic particle is observed, where at $\theta \to \infty$ all velocities satisfy Einstein's first postulate from the special theory of relativity, which states that in vacuum the group velocity of a relativistic particle does not exceed the speed of light, that is, $0 \leq \mathbf{n\beta} \leq 1$, $0 \leq \mathbf{n\beta}_{cir} \leq 1$, and $0 \leq \mathbf{n\beta}_{SM} \leq 1$.

Considering the normalization of a circularly polarized wave from Equation (114), the kinetic energy of the particle is the total energy minus the particle resting energy:

$$E_{(SM)kin} = \sqrt{\frac{a}{a+2}\left(\sqrt{\frac{a}{2}+1}+\left(\frac{a}{2}\right)^{\frac{3}{2}}+3\sqrt{\frac{a}{2}}\right)^2+1}-1. \quad (116)$$

Figure 2 compares the kinetic energies of an electron from Equations (111) and (116). An analytic calculation $E_{(Andreev)kin}$ from the Landau and Lifshitz method [24] gives the kinetic energy of a particle in the non-relativistic case as $E_{(CLASS)kin} = a/4$. The formula $E_{(Wilks)kin} = \sqrt{a/2+1} - 1$ estimates the kinetic energy of electrons on the frontal surface of a target in a field of an intense circularly polarized electromagnetic wave [39], where the rest energy of the electron is $mc^2 \approx 0.51$ MeV. All formulas are given by considering the normalization of the wave field amplitude $E \to E/\sqrt{2}$.

In a monochromatic, circularly polarized plane wave model, when $(E, \omega) = const$, the kinetic energy of the particle is very close to those for the non-relativistic and sub-relativistic cases. From Equation (111) and ref. [24], for sufficiently high intensities such as $I \to \infty$, the kinetic energy increases monotonically and the ratio between the kinetic energies is $\delta_1 = E_{kin}/E_{(Wilks)kin} = 3$, with $\delta_2 = E_{(Andreev)kin}/E_{(Class)kin} = 1.53$. Comparing the obtained Equation (11) and ref. [22] shows that these formulas can describe the energy characteristics of relativistic particles well in the sub-relativistic regime when $I \in \left[0 \div 10^{19}\right]$ W/cm^2. This is similar to the model of classical non-relativistic electron acceleration [43]. The model of acceleration for a monochromatic electromagnetic plane wave of $(E, \omega) = const$ gives classical solutions to the non-relativistic and sub-relativistic domains and is not suitable for describing the dynamics of a relativistic particle in the ultra-relativistic regime.

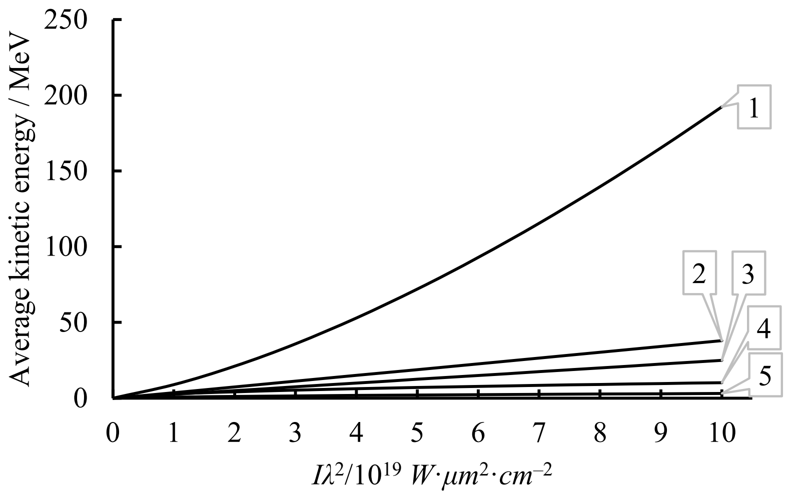

Figure 2. The dependence of the kinetic energies of an electron on the intensity of a circularly polarized wave: (1) Equation (116), (2) analytical calculations based on the Landau and Lifshitz method [24], (3) the kinetic energy for the non-relativistic case, (4) Equation (111), and (5) the formula from ref. [39].

To describe the particle dynamics in the ultra-relativistic regime, it is necessary to consider the dimensionless field amplitude a as a dynamic parameter $a = a(\mathbf{r}, t)$. This is demonstrated in Equations (112)–(116) and describes the particle dynamics in the non-relativistic, sub-relativistic, and ultra-relativistic regimes well. Figure 1 shows that the increased electromagnetic field intensity also increases the kinetic energy. For example, a field intensity that reaches 10^{20} W/cm^2 gives kinetic energies for the plane wave model and spatially modulated wave model of 400 and 500 MeV, respectively.

19. The Acceleration of a Particle by a Modulated Electromagnetic Wave and a Flat Modulated Laser Pulse

The kinetic energies of a particle in a modulated electromagnetic field obtained from refs. [28–30,44] are compared with the proposed invariant method. For simplicity, a modulated electromagnetic wave is considered that includes all types of modulations. According to the Landau and Lifshitz method, the kinetic energy of a particle in a circularly polarized and modulated electromagnetic field has the following form [28–30,44]:

$$E_{(Kopytov)kin} = \frac{\widetilde{a}}{2}\left(1 + \frac{\widetilde{a}}{4 + 2\widetilde{a}}\right). \tag{117}$$

The dimensionless amplitudes of the $\widetilde{a}_{AM}$ amplitude-modulated (AM), $\widetilde{a}_{FM}$ frequency-modulated (FM), $\widetilde{a}_{AM-FM}$ amplitude-frequency-modulated (AM-FM), and $\widetilde{a}_{PM}$ polarization-modulated (PM) wave fields are determined as:

$$\widetilde{a} \to \widetilde{a}_{AM} = a\sum_{n=0}^{N} \frac{(1+\delta^2/2)}{(1+n\hat{\alpha})^2}\,,\ \widetilde{a} \to \widetilde{a}_{AM-FM} = a\sum_{n=0}^{N} \frac{(1+\delta^2/2)}{(1+n\hat{\alpha})^2} \sum_{k=0}^{K} \frac{J_k^2(\delta)}{(1+k\hat{\alpha})^2}\,,$$

$$\widetilde{a} \to \widetilde{a}_{FM} = a\sum_{n=0}^{N} \frac{J_n^2(\delta)}{(1+n\hat{\alpha})^2}\,,\ \widetilde{a} \to \widetilde{a}_{PM} = a\sum_{k=0}^{K} \frac{J_k^2(\delta)}{(1+k\hat{\alpha})^2} \sum_{n=0}^{N} \frac{J_n^2(\delta)}{(1+n\hat{\alpha})^2}\,, \tag{118}$$

where $J_n(\delta)$ and $J_k(\delta)$ are Bessel functions of the first kind of orders n and k, $N = n + 1$ and $K = k + 1$ are integers characterizing the number of modes, δ is the modulation depth, and $\alpha = \hat{\omega}'/\omega$ is the ratio of the modulated frequency $\hat{\omega}'$ to the carrier frequency ω.

For the invariant method, the kinetic energies of a particle in the AM, FM, AM-FM, and PM electromagnetic waves are determined by the transitions $a \to \widetilde{a} \to \widetilde{a}_{AM}$, $a \to \widetilde{a} \to \widetilde{a}_{FM}$, $a \to \widetilde{a} \to \widetilde{a}_{AM-FM}$, and $a \to \widetilde{a} \to \widetilde{a}_{PM}$. Then, Equation (67) takes the form:

$$E_{SM(Mod)kin} = \sqrt{\frac{\widetilde{a}}{\widetilde{a}+2}\left(\sqrt{\frac{\widetilde{a}}{2}+1} + \left(\frac{\widetilde{a}}{2}\right)^{\frac{3}{2}} + 3\sqrt{\frac{\widetilde{a}}{2}}\right)^2 + 1} - 1. \quad (119)$$

The frequency deviation $\hat{\alpha}$ and modulation depth δ are defined as the ratio of the kinetic energy to the total energy in the absence of modulation (see definitions of $\hat{\alpha}$ and δ in reference [44]). Figure 3 plots the dependence of the kinetic energy of a particle on the intensity of the modulated electromagnetic field. Due to the relatively small amplitudes for the lateral modes of the wave, only K and N values of approximately 5 are considered.

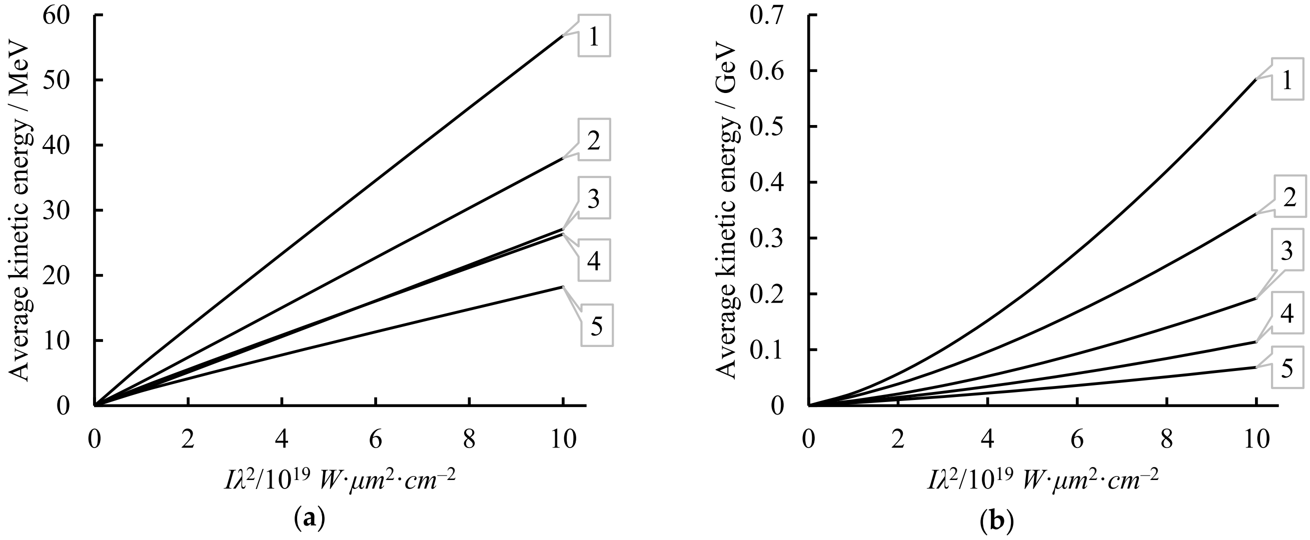

Figure 3. The dependence of the electron kinetic energy on the intensity of the electromagnetic wave: (**a**) an analytical calculation using Landau and Lifshitz method for AM-FM (1), for a circularly polarized plane wave in the absence of modulation (2), AM (3) and FM (4), and PM (5); (**b**) the proposed invariant method for AM (1), AM-FM (2), in the absence of electromagnetic field modulation (3), FM (4), and PM (5).

The modulation gain is equal to the ratio of the kinetic energy in the modulated field to the total energy of the particle in the absence of modulation with a similar configuration of the wave field amplitude. This gives

$$\kappa_{(Kopytov)} = \frac{E_{(Kopytov)kin}}{E_{(Andreev)kin}+1},\ \kappa_{SM(Mod)} = \frac{E_{SM(Mod)kin}}{E_{SMkin}+1}. \quad (120)$$

The ranges of the wave intensities $I \in \left[0 \div 10^{20}\right]$ W/cm^2 in Figure 4 give the following amplification factors: $\kappa_{(Kopytov)}$ and $\kappa_{SM(Mod)}$. The results from the Landau and Lifshitz method in Figures 3 and 4 show that the spectrum of the energy characteristics of a particle under AM and FM waves practically coincides, which is true for the invariant method only in the non-relativistic and sub-relativistic limits.

The above solutions indicate that the most optimal wave packets to accelerate relativistic particles are packets with spatial amplitudes and AM-FM waveforms. Wave packets with FM and PM are less effective for particle accelerations due to limitations in the energy spectrum of the wave caused by the modulation depth parameters. The solutions of the proposed invariant method show that when using AM waves under various modifications, it is possible to accelerate electrons with a wave intensity $I = 10^{22}$ W/cm^2 of up to 600 MeV.

The advantage of this method is that it can describe the dynamics of a relativistic particle well in various modes of interaction between an electromagnetic field and a relativistic charged particle, such as non-relativistic ($0 \leq a < 1$), sub-relativistic ($a \approx 1$), and ultra-relativistic ($a > 1$) modes. For non-relativistic and sub-relativistic regimes and the interaction between an electromagnetic field and a relativistic charged particle, the particle energy calculations are fully consistent with the Landau–Lifshitz method.

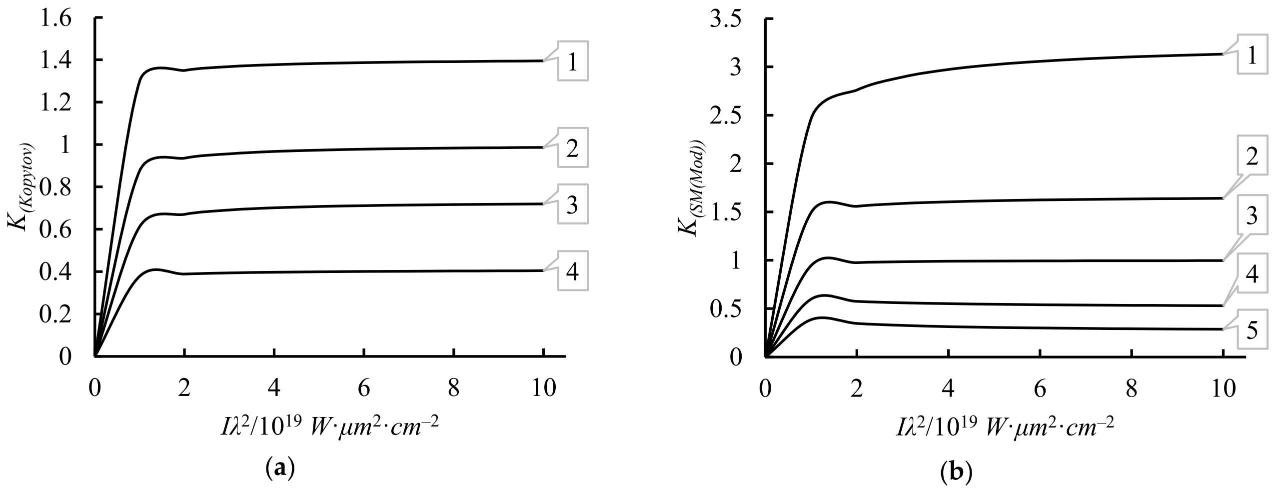

Figure 4. The dependence of the modulation gain on the intensity of the electromagnetic wave: (**a**) an analytic calculation from the Landau and Lifshitz method for AM-FM (1), in the absence of modulation (2), AM and FM (3), and PM (4); (**b**) the proposed invariant method for AM (1), AM-FM (2), in the absence of electromagnetic field modulation (3), FM (4), and PM (5).

20. The Phase Plane Method for the Analysis of Phase Oscillations of a Classical Relativistic Particle in the Field of a Plane Wave and in the Field of a Plane Laser Pulse

From the above solutions, the total radiation power is derived from the total energies by differentiation with respect to time t, and the obtained solutions depend on the angular integral of particle motion θ:

$$W_{free} = -\frac{dE}{dt} = -\sinh^3(\theta),\ W_{cir} = W_{cir}(\theta) = -\frac{dE_{cir}(\theta)}{dt}, \quad W_{SM} = W_{SM}(\theta) = -\frac{dE_{SM}(\theta)}{dt}. \tag{121}$$

The phase plane method relies on the fact that the angular integral of motion θ, which is a hyperbolic angle in the Lobachevsky plane, is translated into the Euclidean plane by the transformation $\theta \to i\theta'$.

As can be seen, for the total power, the oscillation of a free particle in the Euclidean phase plane is equal to zero, since there are no external forces acting on the particle:

$$\mathrm{Re}\left(W'_{free}\right) = 0. \tag{122}$$

Graphs of the dependence of the total radiation power of a relativistic particle in the Euclidean phase plane for $W'_{cir} = W'_{cir}(\theta')$ and $W'_{SM} = W'_{SM}(\theta')$ are shown in Figure 5.

It can be seen from Figure 5 that in the Euclidean phase plane, the total radiation power of a particle W'_{cir} in the field of a circularly polarized wave calculated by this method completely coincides with the calculations from Equation (10) given in ref. [45]. In the field of a plane laser pulse, there is no phase synchronization for $\theta' = -3\pi/2, -\pi/2, \pi/2, 3\pi/2$, etc., so $W'_{SM} \to \pm\infty$. In addition to the existing problems [46], one of the main problems concerning the possibility of creating sources of synchrotron radiation from laser pulses is the problem of the synchronization of a laser pulse and a relativistic particle with respect to the oscillation phase θ'. Therefore, a future goal should be to find a configuration of fields with different sources of laser pulses for which the phase diagram of particle oscillations would be close or similar to the oscillation diagram of a particle in the field of a circularly polarized wave.

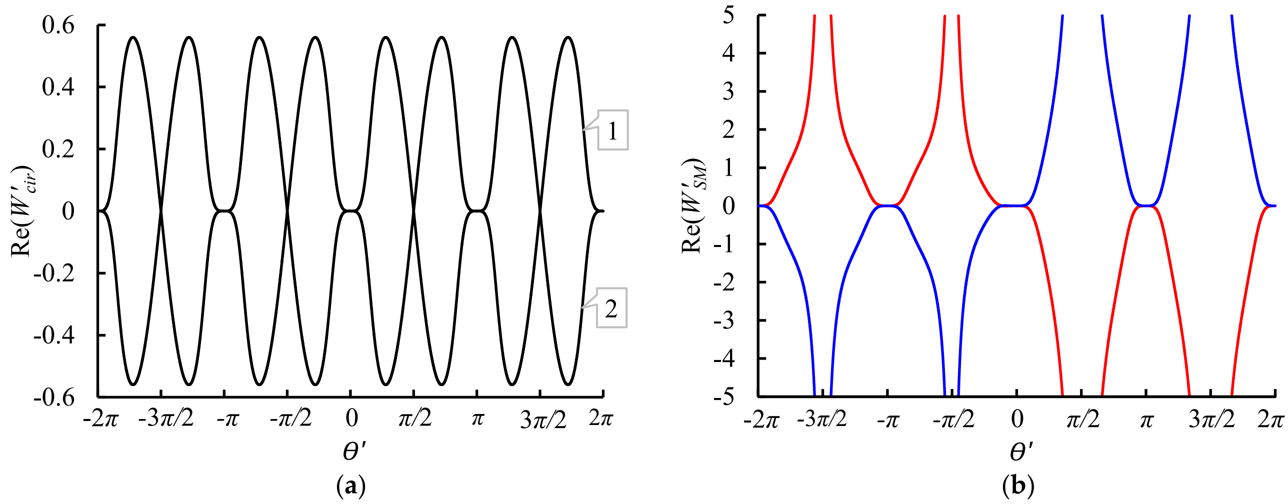

Figure 5. (**a**) The Euclidean phase plane $W'_{cir} = W'_{cir}(\theta')$ of the total radiation power of a particle in the field of a circularly polarized electromagnetic wave. The upper phase oscillation trajectory 1 – (f = 1) corresponds to the particle oscillation in the right circularly polarized wave, and the lower trajectory of the phase oscillation 2 – (f = -1) corresponds to the left circularly polarized wave. (**b**) The Euclidean phase plane $W'_{SM} = W'_{SM}(\theta')$ of the total radiation power of a particle in the field of a circularly polarized laser pulse, where the trajectory of phase oscillations marked by the red line (f = 1) corresponds to particle oscillation in the field of right circular polarization. The blue line represents the trajectory of phase oscillations of a particle in the field of a plane laser pulse with left circular polarization (f = -1).

21. Conclusions

The Hamiltonian and Lagrangian formalisms of a hypothetical chargeless relativistic particle in the absence of external forces, electromagnetic fields, or scalar and vector potentials have been studied in this work. Equations have been presented outside of the typical tensor form, which helps with the application of conservation laws. The angular integral of particle motion (angular rapidity) θ was obtained and expressed in a dimensionless form as $\theta = \theta(\gamma)$, $\theta = \theta(Q_t)$, and $\gamma = \gamma(\theta)$. In 1 + 1 dimensions, it is convenient in the Lagrangian formalism to introduce the generalized particle momentum as the integral of the particle motion Q_t. For the Hamiltonian formalism, the angular integral of the motion for a hypothetical relativistic particle was obtained. This explicitly corresponds to the Pythagorean theorem in Lobachevsky geometry as $\cosh^2\theta - \sinh^2\theta = 1$ and has the form of a dimensionless conservation law as $E^2 - P^2 = 1$. A connection was obtained between the Hamilton–Jacobi equations and the Lagrange equation of the second kind.

A combined Hamilton–Jacobi–Lagrange equation was introduced, where the action function was obtained in an explicit dimensionless form that depends on the integrals of the particle motion $S' = S'(\mathbf{nr}, t, Q_t, \gamma)$. The displacement Hamiltonian was introduced in the Hamilton–Jacobi equation, and is equal to zero for a particle without an initial velocity. These equations were presented for the system K, in which there is a conditional observer with proper time t and proper coordinate $\mathbf{nr}$ in 1 + 1 dimensions. With the use of the related parameters $t = t(q)$ and $q = q(t)$, a connection was made between the inertial reference systems K and K' through the space-time coordinate $\xi' = \xi\gamma$. For a particle located in various intense fields, for example in a plane wave or in a plane laser pulse when $I/I_{rel} \geq 1$, in the interval $\theta \in [0; 2.5]$, the Doppler effect was observed relative to the trajectory of a free particle in Figure 1. The laws of conservation of energy and momentum in Equations (20) and (21) are also important with the substitution $P \to P_g$ and $E \to E_g$, i.e., $E_g^2 = P_g^2 + 1$. Further developments of this approach could help advance the doubly special theory of relativity. This may also help with considering quantum gravity from a different perspective and to develop new research methods in areas such as gravitomagnetism. Moreover, the motivation and advantage of the proposed method is that using the Lorentz-invariant form expressed as integrals of particle motion also makes it possible to solve a number of analytic problems related to the motion and radiation of relativistic charged particles in electromagnetic fields. This includes fields for standing waves when one of the waves has a Lorentz invariant form $\xi^+ = t - \mathbf{nr}/c$ and the other has the form $\xi^- = t + \mathbf{nr}/c$.

Through an analogy with the Euler–Lagrange equation, the Euler–Hamilton equation was obtained, which expresses the energy balance of the system in terms of the angular integral of particle motion. It was also shown that if the Hamiltonian and Lagrangian of relativistic systems are given, then to calculate the dynamic characteristics of the system, it is more compact to use the Euler–Hamilton equation than the Euler–Lagrange equation.

As an example of the applicability of the invariant method, good convergence with the Landau–Lifshitz method was demonstrated for the energy characteristics of a particle in the field of a plane wave and in the field of a plane laser pulse of various modulations, in both the non-relativistic and sub-relativistic modes of interaction between an electromagnetic field and a relativistic charged particle.

In the described solutions for the ultra-relativistic interactions of an electron with an intense AM wave of the order of $I \approx 10^{20}$ W/cm^2, the Landau and Lifshitz method gives solutions for the sub-relativistic region where the kinetic energy of the particle is ≈60 MeV, while the invariant method has a solution in the ultra-relativistic regime that is one order of magnitude greater at ≈0.6 GeV (Figure 3). Finally, for the considered electromagnetic field configurations, the phase plane method in Euclidean geometry is given, which allows one to determine whether the interaction of a particle with an electromagnetic field is synchronized or not.

Of further interest is the application of this method of coupled parameters to the Landau–Khalatnikov problem in relativistic fluid dynamics, which considers the hydrodynamic theory of the production of multiple particles in collisions between fast nucleons and nuclei in 1 + 1 dimensions.

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