OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **55** (2022) 215302 (22pp) https://doi.org/10.1088/1751-8121/ac6402

Dimension towers of SICS: II. Some constructions

Ingemar Bengtsson[1,*] **and Basudha Srivastava**[2]

[1] Stockholms Universitet, AlbaNova, Fysikum, SE-10691 Stockholm, Sverige
[2] Göteborgs Universitet, Institutionen för fysik, SE-41296 Göteborg, Sverige

E-mail: ibeng@fysik.su.su

Received 9 February 2022
Accepted for publication 4 April 2022
Published 11 May 2022

Abstract

A SIC is a maximal equiangular tight frame in a finite dimensional Hilbert space. Given a SIC in dimension d, there is good evidence that there always exists an aligned SIC in dimension $d(d-2)$, having predictable symmetries and smaller equiangular tight frames embedded in them. We provide a recipe for how to calculate sets of vectors in dimension $d(d-2)$ that share these properties. They consist of maximally entangled vectors in certain subspaces defined by the numbers entering the d dimensional SIC. However, the construction contains free parameters and we have not proven that they can always be chosen so that one of these sets of vectors is a SIC. We give some worked examples that, we hope, may suggest to the reader how our construction can be improved. For simplicity we restrict ourselves to the case of odd dimensions.

Keywords: SIC-POVMs, equiangular tight frames, Weyl–Heisenberg orbits

1. Introduction

A tight frame in a complex d dimensional Hilbert space is a collection of N unit vectors that provide a resolution of the identity. In quantum information theory tight frames are known as POVMs. If in addition we require the absolute values of all the mutual overlaps between the vectors to be the same, then the tight frame is said to be equiangular (and the POVM is said to be symmetric). Translating this into equations one easily finds that the vectors $|\psi_I\rangle$ in an equiangular tight frame (abbreviated ETF) must obey

*Author to whom any correspondence should be addressed.

$$\sum_{I=1}^{N} |\psi_I\rangle\langle\psi_I| = \frac{N}{d}\mathbf{1} , \qquad |\langle\psi_I|\psi_J\rangle|^2 = \begin{cases} 1 & \text{if } I = J \\ \dfrac{N-d}{d(N-1)} & \text{if } I \neq J. \end{cases} \tag{1}$$

It is also easy to prove that $d \leqslant N \leqslant d^2$, but it is not at all easy to see for what numbers N of vectors an ETF exists [1]. When written as multivariate polynomial equations for the components of the vectors the defining equations have a distinctly intimidating look. Anyway the minimal case $N = d$ is that of an orthonormal basis, which is a very important ETF that exists for any d. It takes only a small amount of curiousity to ask whether the maximal case $N = d^2$ is important too. Maximal ETFs are known as SICs—which we treat as a name, not as an abbreviation of anything.

It was conjectured, by the authors who first brought them to the attention of the world, that SICs do exist in all dimensions. Moreover they were conjectured to exist as orbits under the Weyl–Heisenberg group [2, 3]. Since this is a group with a very large number of applications in quantum theory and in classical signal processing, the conjecture does strengthen the belief that SICs will prove important at least in these two areas. Another early conjecture was that when formed in this way SICs are always left invariant by a unitary symmetry of order three [2, 4]. This became known as the Zauner symmetry, and was an early indication that something interesting is going on in the depths below.

For the purposes of this paper it is convenient to include the requirement that a SIC be an orbit of the Weyl–Heisenberg group in the definition of a SIC, and we will do so from now on. This has the further advantage that we can fix the standard clock-and-shift representation of the group [5]. Once this has been done we can meaningfully ask for the kind of numbers that are needed to form the components of the vectors in a SIC. In this direction, a surprising addition to the existing conjectures was provided a few years ago.

Based on the solutions that had been found at the time, Appleby *et al* conjectured that, in a standard basis provided by the Weyl–Heisenberg group, the number field generated by a SIC in dimension d is, or includes, a ray class field over a real quadratic base field with conductor d [6]. Without going into detailed explanations we observe that the problem of finding generators for these number fields was raised by Hilbert in his so far unsolved 12th problem [7]. This suggests another angle from which SIC existence is important: to borrow a phrase, it 'reshapes the boundary between physics and pure mathematics' [8].

Judging from numerical searches and exact solutions, all these conjectures are true [9–13]. And the testing has been extensive: so far unpublished work by Grassl includes exact solutions in more than one hundred different dimensions [14], and the connection to number theory has enabled us to construct SICs in several four and five digit dimensions [15]. But it is not clear how close we are to a general existence proof for SICs—and still less what implications a proof would have for quantum theory, signal processing, number theory, or the boundary between physics and pure mathematics.

The ray class conjecture does, however, suggest an intermediate aim because it implies that there are infinite sequences of dimensions in which the number field needed to construct a SIC in one dimension is a subfield of the number field needed to construct a SIC in the next dimension [6]. Here we will focus on special subsequences of dimensions $\{d_n\}_{n=0}^{\infty}$ obtained recursively by

$$d_{n+1} = d_n(d_n - 2) \ . \tag{2}$$

They are called *dimension ladders*. We assume that $d_0 > 3$. Our main conjecture is that for every SIC in dimension d_n there exists an *aligned* SIC in dimension d_{n+1} such that the number field needed to construct the latter contains the number field needed to construct the former. Moreover the phase factors in the overlaps of the SIC vector pairs (which are not stipulated by the definition of a SIC) in dimension d_n reappear in squared form among the overlap phases for the aligned SIC in dimension d_{n+1} [16]. This gained additional interest when it was observed that (Galois conjugates of) the squared overlap phases play a particularly interesting number theoretical role [17]. Closer investigation gave us a much more detailed (if still largely conjectural) picture, and provided more than twenty explicit examples of this alignment phenomenon. In particular it was noticed that aligned SICs in dimensions of the form $d(d-2)$ have to contain Weyl–Heisenberg orbits of smaller equiangular tight frames sitting inside them [18, 19].

In this paper we will go the other way: using only information provided by a SIC in dimension d we will explicitly construct these orbits of equiangular tight frames. In fact we will find a continuous family of such orbits, and are then left with the problem of locating the aligned SICs within them. We do not solve this problem in general, but we take steps in the right direction.

We are thus led to introduce a new conjecture, which is that it should be possible to climb the dimension ladders rung by rung, in the sense that it should be possible to prove the existence of a SIC in dimension $d(d-2)$ given a SIC in dimension d. This would prove SIC existence in many infinite sequences of dimensions. However, while we find the evidence for all the other conjectures we have mentioned to be overwhelming, we are not at all convinced of the truth of this last one. Our purpose here is to present a few theorems to pinpoint exactly where missing ingredients are encountered.

Section 2 provides a sketch of the group theoretical background, with some useful technicalities relegated to the appendix A. At the end of section 2 we also recall some definitions from frame theory. Section 3 introduces the ladders formed by aligned SICs, and gives the key properties of the latter. Sections 4 and 5 present the theorems that are our main results. Given a SIC in dimension d they allow us to calculate Weyl–Heisenberg orbits of equiangular tight frames embedded in dimension $d(d-2)$, and it is shown how one can impose symmetry requirements on these orbits. Since the details are a little involved, section 6 condenses sections 4 and 5 into a few easy-to-follow steps. The following sections describe explicit examples in some detail. We summarize our findings in section 10, and allow the reader to draw her own conclusions.

2. Background material

To prevent too much overlap with previous papers we will rely freely on several facts concerning the Weyl–Heisenberg group, its symplectic automorphism group, and the Chinese remainder theorem as applied to these groups. If the reader is unfamiliar with these ingredients we refer to reference [18] for a short introduction, to the references therein for long ones, and to the appendix A for a few useful formulas. But we do need to fix some notation. First

$$\omega_d = e^{\frac{2\pi i}{d}} , \quad \tau_d = -e^{\frac{i\pi}{d}} . \tag{3}$$

Sub- and superscripts referring to the dimension will be dropped if they are not needed. Second, the displacement operators $D_{\mathbf{p}}$ are defined by

$$D_{\mathbf{p}} = D_{i,j} = \tau^{ij} X^i Z^j . \tag{4}$$

Here Z and X obey $ZX = \omega XZ$ and $X^d = Z^d = \mathbf{1}$. They are the clock and shift operators in the standard Weyl representation, which takes the operator Z to be diagonal [5]. The labelling 'vector' $\mathbf{p}$ has two components i, j that are integers counted modulo d (in odd dimensions). The displacement operators form a unitary operator basis acting on $\mathbf{C}^d$. The SIC is constructed by applying all of them to a fiducial vector $|\Psi_\mathbf{0}\rangle$, and we define the SIC overlap phases by

$$e^{i\theta_\mathbf{p}} = \begin{cases} 1 & \text{if } \mathbf{p} = 0 \\ \sqrt{d+1}\langle\Psi_\mathbf{0}|D_\mathbf{p}|\Psi_\mathbf{0}\rangle & \text{if } \mathbf{p} \neq \mathbf{0}. \end{cases} \tag{5}$$

They are phase factors by the definition (1) of the SIC, but they are not fixed by the definition. Orbits of unitarily equivalent SICs are created by $SL(2, \mathbf{Z}_d)$, the symplectic group of two-by-two matrices with entries in the set of integers modulo d. Given a matrix $F \in SL(2, \mathbf{Z}_d)$ its unitary representative U_F is determined up to an overall phase factor by the representation of the Weyl–Heisenberg group, and its matrix elements are dth roots of unity again up to a harmless normalisation factor [4]. This group also gives rise to unitary symmetries of individual SICs, while anti-symplectic matrices give rise to anti-unitary symmetries (if any). The symplectic group and the Weyl–Heisenberg group taken together generate the Clifford group.

When the dimension is composite with relatively prime factors, $d = n_1n_2$ with $(n_1, n_2) = 1$, the Weyl–Heisenberg group and its symplectic automorphism group splits into a direct product in a canonical way. Thus, effectively,

$$D_\mathbf{p}^{(n_1n_2)} = D_{\mathbf{p}'}^{(n_1)} \otimes D_{\mathbf{p}''}^{(n_2)} \ , \qquad U_F^{(n_1n_2)} = U_{F'}^{(n_1)} \otimes U_{F''}^{(n_2)} \ . \tag{6}$$

We use superscript to denote the dimension on which the operators act only when this is necessary to avoid confusion. The labelling 'vectors' and matrices $\mathbf{p}'$, F', $\mathbf{p}''$, F'' are computed from $\mathbf{p}$ and F through a simple application of the Chinese remainder theorem [18].

Symplectic unitaries worth special mention are the order 2 parity operator U_P, and the order 3 Zauner unitary $U_\mathcal{Z}$. The parity operator is obtained from the $SL(2, \mathbf{Z}_d)$ matrix

$$P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7}$$

Its unitary representative is a permutation matrix. This is actually true for every diagonal symplectic matrix [4]. In prime dimensions the Zauner unitary $U_\mathcal{Z}$ belongs to a unique conjugacy class of the group. A standard choice for the corresponding $SL(2, \mathbf{Z}_d)$ matrix is

$$\mathcal{Z} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \ . \tag{8}$$

A point worth noticing is that whenever $d = 3$ or $d = p = 1$ modulo 3, where p is a prime, the conjugacy class to which the Zauner matrix $\mathcal{Z}$ belongs contains a representative whose corresponding unitary is a permutation matrix [4]. This is particularly relevant here because, with the possible exceptions of d_0 and d_1, all the dimensions in the sequences (2) are divisible by 3 so that the dimensions we are factoring in, of the form $d_n - 2$, are equal to 1 modulo 3. In at least one case, the ladder starting at $d_0 = 5$, this is also true for all the prime factors of $d_n - 2$, for all $n > 1$ [20].

We will be interested in the eigenspaces of the unitary operators U_P and $U_\mathcal{Z}$ [2]. Their dimensions are given in table 1. The eigenvalues are convention dependent, but the sizes of the eigenspaces are not.

A number theoretical comment is called for here, even though number theory will be kept in the background throughout this paper. We are interested in vectors whose components lie

Table 1. Dimensions of the eigenspaces for the parity and Zauner unitaries [2], with $\omega_3 = e^{2\pi i/3}$ and $d = 2n + 1$ odd. For the Zauner eigenspaces the dimensions depend on the value of d modulo 3. An eigenspace is in boldface if standard conjectures [9, 10] imply that it contains a SIC vector. The brackets that occur in one case signify that (as far as we know [10]) this eigenspace contains a SIC vector only if $d = 8$ modulo 9. The eigenvalues are convention dependent.

U_P	d	1	-1
	$2n+1$	$n+1$	n

$U_{\mathcal{Z}}$	d	1	ω_3	ω_3^2
	$6k+1$	$\mathbf{2k+1}$	$2k$	$2k$
	$6k+3$	$\mathbf{2k+2}$	$2k+1$	$2k$
	$6k+5$	$(\mathbf{2k+1})$	$\mathbf{2k+2}$	$\mathbf{2k+2}$

in a specified algebraic number field. Since we think of all our algebraic numbers as embedded in the complex field we can work in a projective space using the embedded number field as its field of scalars. The displacement operators and the symplectic group elements can be represented by unitary matrices all of whose matrix elements belong to the cyclotomic field, and this number field is a subfield of any number field that houses a SIC [6]. But consider a Zauner unitary $U_{\mathcal{Z}}$ of order 3 as an example. Then we get the eigenvectors by choosing any vector $|\psi\rangle$, and performing the projections

$$\begin{aligned} &\frac{1}{3}(\mathbf{1} + U_{\mathcal{Z}} + U_{\mathcal{Z}}^2)|\psi\rangle \\ &\frac{1}{3}(\mathbf{1} + \omega_3^2 U_{\mathcal{Z}} + \omega_3 U_{\mathcal{Z}}^2)|\psi\rangle \\ &\frac{1}{3}(\mathbf{1} + \omega_3 U_{\mathcal{Z}} + \omega_3^2 U_{\mathcal{Z}}^2)|\psi\rangle. \end{aligned} \tag{9}$$

Assume that the components of $|\psi\rangle$ lie in the specified number field. If degenerate eigenvalues occur (as they will in this case) we can resort to Gram–Schmidt orthogonalization without leaving that number field. However, a complication arises for the symplectic unitaries because their eigenvalues may or may not belong to the desired field. In particular the third root of unity, which is an eigenvalue of the Zauner unitary, will belong to the relevant field in some cases (say if we start the ladder at $d_0 = 5$) but not in others (say if we start from $d_0 = 7$). This is a complication. Another complication arises if we normalize the eigenvectors, since the square root of the norm squared will typically not lie in the field. However, this does not mean that we cannot use normalized eigenvectors at intermediate stages in the construction of a SIC. If the ray class conjecture is true all the complications will cancel themselves out in the end. Examples can be found in reference [21].

We end this section with a few definitions from frame theory. A *frame* of type (d, N) is a collection of N vectors $\{|u_I\rangle\}_{I=1}^N$ that span $\mathbf{C}^d$. The frame is *covariant* if the vectors form an orbit of a group. The frame is *tight* if there exists a real number α such that

$$\sum_{I=1}^{N} |u_I\rangle\langle u_I| = \alpha \mathbf{1}_d. \tag{10}$$

The *generator matrix* of a frame is a $d \times N$ matrix M_1 whose columns are the vectors that make up the frame. Thus

$$M_1 = \begin{pmatrix} |u_1\rangle & |u_2\rangle & \dots & |u_N\rangle \end{pmatrix}. \tag{11}$$

The condition that the frame be tight can then be written as $M_1 M_1^\dagger = \alpha \mathbf{1}$, which implies that the rows of the matrix be orthogonal to each other. The generator matrix can always be extended to a unitary matrix by adding $N - d$ additional rows, and by a suitable rescaling. Thus we have

$$U = \left(\frac{M_1}{M_2} \right)_{N \times N}, \qquad UU^\dagger = \mathbf{1}_N \quad \Leftrightarrow \quad \begin{cases} M_1 M_1^\dagger = \mathbf{1}_d \\ \text{and} \\ M_2 M_2^\dagger = \mathbf{1}_{N-d} \ . \end{cases} \tag{12}$$

We have then constructed a tight frame of type $(N - d, N)$ with generator matrix M_2, known as the *Naimark complement* of the frame we started out with. Note however that the complement is by no means uniquely defined. Insisting that a tight frame be equiangular leads to a difficult problem. But since the columns of a unitary matrix are orthonormal to each other the mutual overlaps among the vectors in the Naimark complement are determined by the mutual overlaps in the frame we started out with. In particular, if M_1 is the generator matrix of an ETF then so is M_2.

3. Aligned SICs and proto-SICs

We now slow down a little since the ladders themselves do deserve a more detailed introduction. The number theoretical reasoning that suggests the existence of the dimension ladders begins with the observation that the number field needed to construct a SIC in dimension d is an extension of the number field $\mathbf{Q}(\sqrt{D})$, where D is the square free part of the integer $(d + 1)(d - 3)$ [22]. But it is easy to see that

$$(d(d-2)+1)(d(d-2)-3) = (d-1)^2(d+1)(d-3). \tag{13}$$

It follows that the square free part of $(d + 1)(d - 3)$ is unchanged under the substitution $d \to d(d - 2)$. The ray class conjecture then implies that the minimal number field needed to construct a SIC in dimension d is a subfield of the minimal number field needed to construct a SIC in dimension $d(d - 2)$, essentially because the conductor d divides the conductor $d(d - 2)$. To avoid misunderstanding we should add that the sequences $d_{n+1} = d_n(d_n - 2)$ are only subsequences of the towers of dimensions that are related to a given real quadratic field [6].

If d is odd then d and $d - 2$ are relatively prime and the Weyl–Heisenberg group splits as a direct product. We focus on the displacement operators, and find following equation (6) that

$$D_{\mathbf{p}}^{d(d-2)} = D_{\mathbf{p}'}^{(d-2)} \otimes D_{\mathbf{p}''}^{(d)} \ . \tag{14}$$

The symplectic unitaries behave similarly. Hence we can write

$$\mathbf{C}^{d(d-2)} = \mathbf{C}^{d-2} \otimes \mathbf{C}^{d} \tag{15}$$

in a meaningful way. In even dimensions complications arise at this point. With appropriate measures taken the basic logic is unchanged [19], but these measures are subtle and we have decided to postpone a discussion of even dimensions to a later occasion.

From now on we work in $\mathbf{C}^{d-2} \otimes \mathbf{C}^{d}$ with d odd. Paraphrasing the definition in a previous paper [18] we define aligned SICs as follows:

Definition 1. A SIC with fiducial vector $|\Psi_{\mathbf{0}}\rangle$ in dimension $d(d-2)$ is *aligned* to a SIC with overlap phases $e^{i\theta_{\mathbf{p}}}$ in an odd dimension d if

$$(d-1)\langle\Psi_0|\mathbf{1}_{d-2} \otimes D_{\mathbf{p}}^{(d)}|\Psi_0\rangle = -e^{2i\theta_{M\mathbf{p}}} \tag{16}$$

and

$$(d-1)\langle\Psi_0|D_{\mathbf{p}}^{(d-2)} \otimes \mathbf{1}_d|\Psi_0\rangle = 1\ , \tag{17}$$

for a matrix M with determinant ± 1 modulo d.

Recall that $d-1 = \sqrt{d(d-2)+1}$. As we will show below, the key condition is equation (16). The other follows as a corollary. The precise linear relation between $\mathbf{p}$ and $M\mathbf{p}$ is slightly complicated by the symplectic automorphism group, which is why M is not completely fixed by the definition. The available evidence [18] suggests that *every* SIC in dimension d has an aligned counterpart in dimension $d(d-2)$.

We make one more definition:

Definition 2. A proto-SIC in dimension $d(d-2)$ is a Weyl–Heisenberg orbit of a vector $|\Psi_0\rangle$ obeying equations (16) and (17).

An aligned SIC is a proto-SIC, but the converse need not hold. In this paper we will provide a recipe for how to calculate proto-SICs, given a SIC in dimension d to start from. In fact we will obtain a continuous family of proto-SICs. We will *not* provide a proof that the parameters can be chosen so that the SIC condition

$$(d-1)^2|\langle\Psi_0|D_{\mathbf{p}'}^{(d-2)} \otimes D_{\mathbf{p}''}^{(d)}|\Psi_0\rangle|^2 = 1 \tag{18}$$

holds as well.

Still it is interesting to pause and consider the geometric meaning of proto-SICs. It is known [18] that the d^2 vectors obtained by acting with $\mathbf{1}_{d-2} \otimes D_{\mathbf{p}}^{(d)}$ on $|\Psi_0\rangle$ span a subspace of dimension $d(d-1)/2$ in $\mathbf{C}^{d(d-2)}$. Hence they form an equiangular tight frame in this subspace. As a consistency check on the proof we examine equation (1). By replacing $d \to d(d-1)/2$, and setting $N = d^2$, we obtain

$$|\langle\psi_I|\psi_J\rangle|^2 = \frac{d^2 - \frac{d(d-1)}{2}}{\frac{d(d-1)}{2}(d^2-1)} = \frac{1}{(d-1)^2} = \frac{1}{d(d-2)+1}. \tag{19}$$

The right-hand side is just right for a pair of SIC vectors in dimension $d(d-2)$. Acting on the vectors in this ETF with group elements of the form $D_{\mathbf{p}}^{(d-2)} \otimes \mathbf{1}_d$ we obtain a set of $(d-2)^2$ equiangular tight frames, all of them spanning some $d(d-1)/2$ dimensional subspace. We can regard the resulting set of subspaces as so many points in a Grassmannian, and then it can be shown that these points are equidistant with respect to the natural metric on the Grassmannian [23].

An aligned SIC therefore has such a structure sitting inside it. There is an alternative partitioning of the aligned SIC into d^2 sets of $(d-2)^2$ vectors forming ETFs in subspaces of dimension $(d-2)(d-1)/2$, but this will not play a role in the arguments to follow.

Aligned SICs have found some use. For instance, the exact solution in dimension $323 = 19 \cdot 17$ was found using the guidance of equations (16) and (17) [13]. But we want more—more, in fact, than we can provide at the moment. In sections 4 and 5 we will present a chain of theorems that allows us to calculate a fiducial vector $|\Psi_0\rangle$ obeying equations (16) and (17), given any SIC in dimension d to start from. The key to the construction is the calculation of the equiangular tight frames that are to be found embedded in the higher dimensional SIC. Unfortunately no chain is stronger than its weakest link. The weak link in our chain will turn out to be that the construction contains ambiguities and leads to a continuous family of fiducial vectors. We repeat that we have not proven that there is a SIC present in every such family of proto-SICs.

4. General theorems

In the discovery papers it was proved that a SIC is a complex projective two-design [2, 3]. We can restate this as a theorem that starts our chain:

Theorem 1. *It holds that*

$$\{|\psi_I\rangle\}_{I=1}^{d^2} \quad \text{is a SIC} \quad \Leftrightarrow \quad \{|\psi_I\rangle \otimes |\psi_I\rangle\}_{I=1}^{d^2} \quad \text{a tight frame in } \mathbf{C}^d \otimes_S \mathbf{C}^d. \tag{20}$$

Proof. This is a restatement of the 2-design property. We are using the formulation from section 2 in reference [24]. □

If the d-dimensional SIC is an orbit under the Weyl–Heisenberg group so is the ETF in the symmetric subspace, although in the latter case the representation is by necessity reducible. We will need a definite prescription for it. To find one we introduce the generators of the group, and observe that

$$(Z \otimes Z)(X \otimes X) = \omega^2 (X \otimes X)(Z \otimes Z) \ . \tag{21}$$

There is a clear dichotomy between odd and even dimensions. In odd dimensions ω_d^2 remains a dth root of unity, and the symmetric product group is isomorphic to the Weyl–Heisenberg group we start out with. This is not so in even dimensions, but we have already made the restriction to odd d. We can then define the operators

$$\mathbf{X} = X \otimes X \quad \text{and} \quad \mathbf{Z} = Z^{\frac{d+1}{2}} \otimes Z^{\frac{d+1}{2}} \tag{22}$$

to generate the group when acting on $\mathbf{C}^d \otimes \mathbf{C}^d$. Then there holds

$$\mathbf{ZX} = \omega \mathbf{XZ}. \tag{23}$$

We define the displacement operators

$$\tilde{D}_{i,j} = \tau^{ij} \mathbf{X}^i \mathbf{Z}^j = D_{i,2^{-1}j} \otimes D_{i,2^{-1}j} \ . \tag{24}$$

Recall that the arithmetic of the labelling integers is modulo d. It will be convenient to introduce the matrix

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 2^{-1} \end{pmatrix} \ , \tag{25}$$

and write this as

$$\tilde{D}_{\mathbf{p}} = D_{H\mathbf{p}} \otimes D_{H\mathbf{p}}. \tag{26}$$

We will decompose this reducible representation as a direct sum of d-dimensional Weyl representations. First we introduce the basis vectors

$$|(i,i)\rangle = |e_i\rangle \otimes |e_i\rangle \tag{27}$$

$$|(i,j)\rangle = \frac{1}{\sqrt{2}}\left(|e_i\rangle \otimes |e_j\rangle + |e_j\rangle \otimes |e_i\rangle\right) \tag{28}$$

$$|[i,j]\rangle = \frac{1}{\sqrt{2}}\left(|e_i\rangle \otimes |e_j\rangle - |e_j\rangle \otimes |e_i\rangle\right). \tag{29}$$

Then we can decompose $\mathbf{C}^d \otimes \mathbf{C}^d$ into the symmetric and anti-symmetric subspaces at our convenience. If the standard representation of the displacement operators is being used we can take the representation to be carried by the d subspaces spanned by

$$\left|\left(x\frac{d-1}{2} + i, x\frac{d+1}{2} + i\right)\right\rangle, \qquad 0 \leqslant x \leqslant \frac{d-1}{2} \tag{30}$$

$$\left|\left[x\frac{d-1}{2} + i, x\frac{d+1}{2} + i\right]\right\rangle, \qquad 1 \leqslant x \leqslant \frac{d-1}{2}. \tag{31}$$

To check that this works it is enough to check that the action of the generators $\mathbf{X}$ and $\mathbf{Z}$ works as expected, in particular that they leave the label x invariant.

At this point we may wish to forget about the tensor product structure we started out with. Instead we replace it with a new one, so that the representation can be written

$$\tilde{D}_{\mathbf{p}} = \mathbf{1}_d \otimes D_{\mathbf{p}}. \tag{32}$$

In this way the group provides a natural decomposition of $\mathbf{C}^{d^2}$ as a direct sum of d copies of $\mathbf{C}^d$. We can restrict ourselves to the symmetric subspace by letting the unit operator act on a space of dimension $(d+1)/2$.

We are interested in covariant frames made from vectors of the form $\tilde{D}_{\mathbf{p}}\mathbf{u}$, where the fiducial vector $\mathbf{u}$ is a direct sum of k vectors $\mathbf{x}_r \in \mathbf{C}^d$. This is to say that the generator matrices are d$k \times d^2$ matrices formed according to

$$M_1 = \begin{pmatrix} \mathbf{x}_0 & D_{0,1}\mathbf{x}_0 & \dots & D_{d-1,d-1}\mathbf{x}_0 \\ \vdots & \vdots & & \vdots \\ \mathbf{x}_{k-1} & D_{0,1}\mathbf{x}_{k-1} & \dots & D_{d-1,d-1}\mathbf{x}_{k-1} \end{pmatrix}_{dk\times d^2}. \tag{33}$$

Recall from section 2 that the columns of M_1 form a tight frame in $\mathbf{C}^{kd}$ if and only if $M_1M_1^\dagger = \mathrm{d}\mathbf{1}_{kd}$.

Theorem 2. *The frame with generator matrix M_1 is tight if and only if the vectors $\mathbf{x}_i$ form an orthonormal set in $\mathbf{C}^d$,*

$$M_1 M_1^\dagger = \mathrm{d}\mathbf{1}_{kd} \quad \Leftrightarrow \quad \langle \mathbf{x}_s, \mathbf{x}_r \rangle = \delta_{rs}. \tag{34}$$

Proof. For any unitary operator basis $\{U_I\}_{I=1}^{d^2}$, for any operator A, and for any pair of vectors $\mathbf{x}_i, \mathbf{x}_j$ there holds

$$\sum_I U_I A U_I^\dagger = \mathrm{dTr}A\mathbf{1}_d \quad \text{and} \quad \sum_I U_I \mathbf{x}_r \mathbf{x}_s^\dagger U_I^\dagger = \langle \mathbf{x}_s, \mathbf{x}_r \rangle \mathrm{d}\mathbf{1}_d \ . \tag{35}$$

Because the displacement operators form a unitary operator basis the proof of the theorem follows by inspection. □

Following section 2 we can find the Naimark complement by adding an additional set of $d(d-k)$ rows to M_1, obtaining a matrix that is unitary up to an overall factor. Explicitly

$$U = \begin{pmatrix} M_1 \\ M_2 \end{pmatrix} = \left(\begin{array}{cccc} \mathbf{x}_0 & D_{0,1}\mathbf{x}_0 & \dots & D_{d-1,d-1}\mathbf{x}_0 \\ \vdots & \vdots & & \vdots \\ \mathbf{x}_{k-1} & D_{0,1}\mathbf{x}_{k-1} & \dots & D_{d-1,d-1}\mathbf{x}_{k-1} \\ \hline \mathbf{x}_k & D_{0,1}\mathbf{x}_k & \dots & D_{d-1,d-1}\mathbf{x}_k \\ \vdots & \vdots & & \vdots \\ \mathbf{x}_{d-1} & D_{0,1}\mathbf{x}_{d-1} & \dots & D_{d-1,d-1}\mathbf{x}_{d-1} \end{array} \right)_{d^2 \times d^2} . \tag{36}$$

The frame whose generator matrix is M_2 will be tight provided only that the vectors $\{\mathbf{x}_r\}_{r=k}^{d-1}$ fill out an orthonormal basis $\{\mathbf{x}_r\}_{r=0}^{d-1}$ in $\mathbf{C}^d$. This can be arranged with Gram–Schmidt, if no better alternative offers itself.

5. General theorems, continued

We begin a new section here because from now on we will specialize the vectors $\mathbf{x}_r$ from which our tight frames are constructed.

In theorem 1 we obtained not only a tight frame but a group covariant equiangular tight frame in the symmetric subspace of $\mathbf{C}^d \otimes \mathbf{C}^d$. Its fiducial vector is

$$|\tilde{\Psi}\rangle = |\psi_0\rangle \otimes |\psi_0\rangle, \tag{37}$$

where $|\psi_0\rangle$ is the fiducial vector of a SIC in dimension d. Its overlaps are

$$\langle \tilde{\Psi} | \tilde{D}_{\mathbf{p}} | \tilde{\Psi} \rangle = \left(\langle \psi_0 | D_{H\mathbf{p}} | \psi_0 \rangle \right)^2 = \frac{e^{2i\theta_{H\mathbf{p}}}}{d+1} \ . \tag{38}$$

Using the basis (30) and rewriting $\tilde{D}_{\mathbf{p}}$ as $\mathbf{1} \otimes D_{\mathbf{p}}$ we obtain an ETF of the form presented in theorem 2, with $k = (d+1)/2$. Hence, from now on,

$$\begin{pmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_{k-1} \end{pmatrix} = \sqrt{\frac{d+1}{2}} |\psi_0\rangle \otimes |\psi_0\rangle \ , \qquad k = \frac{d+1}{2}, \tag{39}$$

where it is understood that $|\psi_0\rangle \otimes |\psi_0\rangle$ has been expressed in the basis (30). Hence the components of the $(d+1)/2$ vectors $\{\mathbf{x}_r\}_{r=0}^{k-1}$ are known quadratic functions of the components of the SIC fiducial vector $|\psi_0\rangle$.

The overlaps are not of the right size for this ETF to be embedded in a $d(d-2)$ dimensional SIC. This is easily remedied by taking the Naimark complement. This gives us an ETF in dimension $d(d-1)/2$, with fiducial vector

$$|\Psi_{\mathbf{0}}\rangle = \sqrt{\frac{2}{d-1}}\begin{pmatrix}\mathbf{x}_k\\ \mathbf{x}_{k+1}\\ \vdots\\ \mathbf{x}_{d-1}\end{pmatrix}, \qquad k = \frac{d+1}{2}. \tag{40}$$

By construction the columns of the matrix U defined in equation (36) are orthogonal. Given equation (38), and keeping track of the normalizing factors, we conclude that

$$\langle\Psi_{\mathbf{0}}|\tilde{D}_{\mathbf{p}}|\Psi_{\mathbf{0}}\rangle = -\frac{e^{2i\theta_{H\mathbf{p}}}}{d-1}\ . \tag{41}$$

We have arrived at

Theorem 3. *Given a Weyl–Heisenberg covariant SIC in dimension d, with overlap phases $e^{i\theta_{\mathbf{p}}}$, an ETF with d^2 vectors in dimension $d(d-1)/2$ and overlap phases given by equation (41) can be obtained by direct calculation.*

Proof. Given already. It only remains to add that the argument was inspired by Ostrovskyi and Yakymenko [24]. □

The construction of the Naimark complement is not unique. We have the freedom to choose an orthonormal basis in a subspace of dimension $d(d-1)/2$. Ignoring an overall phase this leaves us with a family of solutions parameterized by the group $SU((d-1)/2)$. Theorem 5 below will provide a nice interpretation of this freedom.

We begin by collecting all the available information about the subspace of $\mathbf{C}^d$ where the basis is to be chosen. Also, following the best of examples [25], we switch to bra-ket notation for all vectors.

Theorem 4. *There holds*

$$\sum_{r=0}^{\frac{d-1}{2}} |x_r\rangle\langle x_r| = \frac{1}{2}(\mathbf{1} + P_\theta), \tag{42}$$

where P_θ obeys $P_\theta^2 = \mathbf{1}$ and is given by

$$P_\theta = \frac{1}{d}\sum_{\mathbf{p}} e^{2i\theta_{H\mathbf{p}}} D_{-\mathbf{p}}\ . \tag{43}$$

Proof. Expand the projection operator as

$$\sum_{r=0}^{\frac{d-1}{2}} |x_r\rangle\langle x_r| = \frac{1}{d}\sum_{\mathbf{p}}\sum_{r}\langle x_r|D_{\mathbf{p}}|x_r\rangle D_{-\mathbf{p}}. \tag{44}$$

The squared SIC phases enter when we use the fact that

$$\sum_{r=0}^{\frac{d-1}{2}} \langle x_r|D_{\mathbf{p}}|x_r\rangle = \frac{d+1}{2}\langle\psi_0|\langle\psi_0|\tilde{D}_{\mathbf{p}}|\psi_0\rangle|\psi_0\rangle. \tag{45}$$

Full details are given by Ostrovskyi and Yakymenko, in their section 7 [24]. □

The operator P_θ is known as the generalised parity operator [18]. It played a significant role in reference [26], where it was used to construct ETFs at the Gram matrix level. Equation (42) provides a convenient way to calculate it from the components of a d dimensional SIC vector.

The important point is that we have divided $\mathbf{C}^d$ into a direct sum,

$$\mathbf{C}^d = \mathbf{C}^{\frac{d+1}{2}} \oplus \mathbf{C}^{\frac{d-1}{2}} \equiv \mathcal{H}^{(+)}_{P_\theta} \oplus \mathcal{H}^{(-)}_{P_\theta} \tag{46}$$

(where we took the opportunity of introducing some new notation), and that this decomposition into eigenspaces carries information about the SIC overlap phases from dimension d.

In dimension $d-2$ there is a similar decomposition into

$$\mathbf{C}^{d-2} = \mathbf{C}^{\frac{d-1}{2}} \oplus \mathbf{C}^{\frac{d-3}{2}} \equiv \mathcal{H}^{(+)}_{P} \oplus \mathcal{H}^{(-)}_{P} \tag{47}$$

defined by the standard parity operator U_P. See table 1.

We now have a family of ETFs with overlaps of the desired size. Moreover they carry some information about the SIC in dimension d. The orthonormal vectors from which their fiducial vector is built span $\mathcal{H}^{(-)}_{P_\theta} \in \mathbf{C}^d$, an eigenspace of the generalised parity operator P_θ. The next step is to embed the fiducial vector in the Hilbert space $\mathbf{C}^{d-2} \otimes \mathbf{C}^d$.

For convenience we rename the basis vectors in $\mathcal{H}^{(-)}_{P_\theta}$ according to

$$\mathbf{x}_{\frac{d+1}{2}} \to |e_0\rangle\ ,\ \ldots\ ,\ \mathbf{x}_{d-1} \to |e_{\frac{d-1}{2}-1}\rangle\ . \tag{48}$$

When we introduce orthonormal basis vectors in $\mathcal{H}^{(+)}_{P}$ (which has the same dimension as $\mathcal{H}^{(-)}_{P_\theta}$) we will denote them by $|f_r\rangle$. Should we wish to climb more than one rung of the ladder, and work in a Hilbert space built from more than two factors, we will introduce further basis vectors $|g_r\rangle$, $|h_r\rangle$, and so on. We do not believe that explicit calculations for more than three rungs will be performed any time soon, and hence we foresee no notational problems in the near future.

Now consider the tensor product Hilbert space

$$\mathcal{H}^{(+)}_{P} \otimes \mathcal{H}^{(-)}_{P_\theta} \in \mathbf{C}^{d-2} \otimes \mathbf{C}^d. \tag{49}$$

In section 5 of reference [18] it was proved that every aligned SIC has a fiducial vector that is a maximally entangled vector in this subspace. Here we are more interested in the converse:

Theorem 5. *Every maximally entangled unit vector* $|\Psi_{\mathbf{0}}\rangle \in \mathcal{H}^{(+)}_{P} \otimes \mathcal{H}^{(-)}_{P_\theta}$ *obeys*

$$(d-1)\langle\Psi_{\mathbf{0}}|\mathbf{1}_{d-2} \otimes D^{(d)}_{\mathbf{p}}|\Psi_{\mathbf{0}}\rangle = -e^{2i\theta_{\mathbf{p}'}} \tag{50}$$

$$(d-1)\langle\Psi_{\mathbf{0}}|D^{(d-2)}_{\mathbf{p}} \otimes \mathbf{1}_d|\Psi_{\mathbf{0}}\rangle = 1 \tag{51}$$

for $\mathbf{p} \neq \mathbf{0}$, *where* $\mathbf{p}'$ *is linearly related to* $\mathbf{p}$.

Proof. The proof uses the Schmidt decomposition of the state. We introduce the adapted basis (48) and write the state on the form

$$|\Psi_{\mathbf{0}}\rangle = \sqrt{\frac{2}{d-1}} \sum_{r=0}^{(d-1)/2-1} |f_r\rangle \otimes |e_r\rangle. \tag{52}$$

By definition of 'maximally entangled' the basis vectors $|f_r\rangle$ can always be found. We find

$$(d-1)\langle\Psi_{\mathbf{0}}|\mathbf{1}_{d-2} \otimes D_{\mathbf{p}}^{(d)}|\Psi_{\mathbf{0}}\rangle = \sum_r \langle e_r|D_{\mathbf{p}}^{(d)}|e_r\rangle = -e^{2i\theta_{\mathbf{p}'}} \tag{53}$$

by preceding results. We also find

$$\begin{aligned}(d-1)\langle\Psi_{\mathbf{0}}|D_{\mathbf{p}}^{(d-2)} \otimes \mathbf{1}_d|\Psi_{\mathbf{0}}\rangle &= \sum_r \langle f_r|D_{\mathbf{p}}^{(d-2)}|f_r\rangle \\ &= \frac{1}{2}\mathrm{Tr}\, D_{\mathbf{p}}^{(d-2)}(\mathbf{1}_{d-2} + U_P) = 1\ .\end{aligned} \tag{54}$$

This holds because the basis vectors are positive parity eigenstates, and because $\mathrm{Tr}\, D_{\mathbf{p}} U_P = 1$ for all $\mathbf{p}$ in all odd dimensions. □

It is well known that the set of maximally entangled states in a Hilbert space of the form $\mathbf{C}^N \otimes \mathbf{C}^N$ can be obtained by choosing orthonormal bases in the two factors, and writing the states in the form

$$|\Phi\rangle = \frac{1}{\sqrt{N}} \sum_{r,s} U_{r,s}|r\rangle \otimes |s\rangle, \tag{55}$$

where U is any unitary matrix. Hence the set of maximally entangled states is isomorphic to the group manifold $SU(N)/Z_N$, where an overall phase factor was removed by factoring out the discrete subgroup Z_N. By adapting the bases to the state the matrix U can be written in diagonal form, which is how the Schmidt form of the state arises. In this way theorem 5 provides us with an alternative view of the freedom observed in theorem 3.

We are now in a position to calculate a set of vectors in dimension $d(d-2)$ that obey equations (16) and (17), given a SIC in dimension d to start with. But the freedom we have in doing this is large while the number of SICs is expected to be finite. To cut down on the ambiguities we will appeal to the symmetries of the d-dimensional SIC. This symmetry survives (and is in fact enhanced) when we go to the next rung of the ladder [18, 19]. Here we will rely on:

Theorem 6. *For a symplectic unitary U_F it holds that*

$$U_F|\psi_0\rangle \sim |\psi_0\rangle \quad \Rightarrow \quad [U_{F'}, P_\theta] = 0, \tag{56}$$

where $\sim$ means equal up to a phase factor, and $F' = H^{-1}FH$ where H is the matrix defined in equation (25).

Proof. Suppose that $|\psi_0\rangle$ is an eigenvector of U_F. We then find that

$$\langle\psi_0|\langle\psi_0|\tilde{D}_{\mathbf{p}}|\psi_0\rangle|\psi_0\rangle = \langle\psi_0|\langle\psi_0|U_F D_{H\mathbf{p}} U_F^{-1} \otimes U_F D_{H\mathbf{p}} U_F^{-1}|\psi_0\rangle|\psi_0\rangle$$

$$= \langle\psi_0|\langle\psi_0|D_{FH\mathbf{p}} \otimes D_{FH\mathbf{p}}|\psi_0\rangle|\psi_0\rangle \tag{57}$$

$$= \langle\psi_0|\langle\psi_0|D_{HH^{-1}FH\mathbf{p}} \otimes D_{HH^{-1}FH\mathbf{p}}|\psi_0\rangle|\psi_0\rangle.$$

We define

$$F' = H^{-1}FH, \tag{58}$$

and rewrite the equation (using the second tensor product structure) as

$$\langle u|\mathbf{1} \otimes D_{\mathbf{p}}|u\rangle = \langle u|\mathbf{1} \otimes D_{F'\mathbf{p}}|u\rangle = \langle u|\mathbf{1} \otimes U_{F'} D_{\mathbf{p}} U_{F'}^{-1}|u\rangle. \tag{59}$$

Given this, we can proceed to show that

$$U_{F'}^{-1}\frac{1}{2}(\mathbf{1} + P_\theta)U_{F'} = U_{F'}^{-1}\sum_r |x_r\rangle\langle x_r|U_{F'}$$

$$= \frac{1}{d}\sum_{\mathbf{p}}\sum_r \langle x_r|U_{F'} D_{\mathbf{p}} U_{F'}^{-1}|x_r\rangle D_{-\mathbf{p}} = \frac{1}{d}\sum_{\mathbf{p}} \langle u|\mathbf{1} \otimes U_{F'} D_{\mathbf{p}} U_{F'}^{-1}|u\rangle D_{-\mathbf{p}}$$

$$= \frac{1}{d}\sum_{\mathbf{p}} \langle u|\mathbf{1} \otimes D_{\mathbf{p}}|u\rangle D_{-\mathbf{p}} = \frac{1}{d}\sum_{\mathbf{p}}\sum_r \langle x_r|D_{\mathbf{p}}|x_r\rangle D_{-\mathbf{p}} = \frac{1}{2}(\mathbf{1} + P_\theta)\ . \tag{60}$$

Hence $U_{F'}$ commutes with P_θ. □

The ambiguities in the construction of the ETF will be cut down considerably if we insist that the vectors $|e_r\rangle$ from which it is built are eigenvectors of $U_{F'}$. In the Schmidt form of the fiducial vector to be built in $\mathbf{C}^{d-2} \otimes \mathbf{C}^d$ we are free to use basis vectors in $\mathcal{H}_P^{(+)}$ in such a way that the fiducial vector becomes an eigenvector of a symplectic unitary $U_{F''} \otimes U_{F'}$. In this way we can arrange that its symmetry group has the same order as that of the SIC fiducial in $\mathbf{C}^d$. In fact its order will be twice that because by construction the fiducial is invariant under $U_P \otimes \mathbf{1}_d$. (This is the F_b symmetry in the notation of Scott and Grassl [9].) The upshot is that the construction will use quite special maximal entangled states for which the unitary matrix in equation (55) becomes block-diagonal.

6. Instructions for how to climb a ladder

We are ready to climb the ladders. Regardless of the somewhat lengthy argument that we have gone through the procedure is quite simple, so we spell it out here:

(1). Choose a SIC in dimension d invariant under a symplectic unitary $U_F^{(d)}$. Compute the generalised parity operator P_θ as well as the Zauner unitary $U_{F'}^{(d)}$ defined in theorem 6.

(2). Use the parity operators U_P and P_θ to define the positive parity subspace of $\mathbf{C}^{d-2}$ and the negative generalised parity subspace of $\mathbf{C}^d$. Their dimensions are equal to $(d-1)/2$ in both cases.

(3). Compute bases for the subspaces defined in the previous step, as eigenvectors of $(\mathbf{1} + U_P)U_{F''}^{(d-2)}$ and of $(\mathbf{1} - P_\theta)U_{F'}^{(d)}$, where $U_{F''}^{(d-2)}$ is some chosen Zauner unitary having the same order as $U_{F'}^{(d)}$.

(4). Use these bases to form maximally entangled states in $\mathbf{C}^{(d-1)/2} \otimes \mathbf{C}^{(d-1)/2}$ considered as a subspace of $\mathbf{C}^{d-2} \otimes \mathbf{C}^{d}$, and arrange them so that they lie in an eigenspace of $U_{F''}^{(d-2)} \otimes U_{F'}^{(d)}$.

(5). The resulting family of proto-SICs is parameterized by a block diagonal unitary matrix. Search for a SIC within this family.

There are a few useful hints that should be digested before one applies this recipe. Rather than spelling them all out at once we turn to examples.

7. A simple example

We first choose $d_0 = 5$. There is a unique Clifford orbit of SICs labelled 5a [9], having a symmetry of order 3. Each eigenspace defined by the standard Zauner matrix with eigenvalues chosen according to table 1 contains four distinct but Clifford equivalent SIC fiducial vectors. We choose the one given in equation (4) in reference [21]. It is invariant under the standard Zauner unitary corresponding to the Zauner matrix (8), and it determines the $(d+1)/2$ vectors $|x_r\rangle$ appearing in equation (42). We use this equation to calculate the generalised parity operator P_θ. We also calculate the Zauner unitary $U_{\mathcal{Z}'}^{(5)}$ appearing in theorem 6. We then calculate the eigenvectors of $U_{\mathcal{Z}'}^{(5)}(\mathbf{1} - P_\theta)$. When we have discarded the ones that correspond to eigenvalue zero there are two such vectors left, and we label them with the eigenvalues of $U_{\mathcal{Z}'}^{(5)}$. That is

$$\begin{aligned} |e_0\rangle &= |\omega_3\rangle_5 \\ |e_1\rangle &= |\omega_3^2\rangle_5. \end{aligned} \tag{61}$$

Since this step follows a standard procedure we do not give the details. It is however of importance to ensure that no spurious phase factors arise here, especially if one intends to convert a numerical solution to an exact one at the end. For this reason we insist that the first components of all the basis vectors we introduce are real. (Should the first component of some eigenvector vanish we set the second components of all the vectors in that basis to be real.)

We then factor in the Hilbert space $\mathbf{C}^3$, and choose a Zauner unitary there. If possible we choose one that is represented by a monomial matrix. An appropriate choice is [21]

$$\mathcal{Z}'' = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_3 \quad \Rightarrow \quad U_{\mathcal{Z}}^{(3)} = \begin{pmatrix} \omega_3^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{62}$$

There is a further choice made here, since we could replace $\mathcal{Z}''$ with its square. We have tried both possibilities and we have chosen the one that works. We calculate the eigenvectors of $U_{\mathcal{Z}''}^{(3)}(\mathbf{1} + U_P)$ and again label the eigenvectors by their eigenvalues,

$$\begin{aligned} |f_0\rangle &= |1\rangle_3 \\ |f_1\rangle &= |\omega_3^2\rangle_3. \end{aligned} \tag{63}$$

From these basis vectors we can construct maximally entangled states in $\mathbf{C}^2 \otimes \mathbf{C}^2 \subset \mathbf{C}^3 \otimes \mathbf{C}^5$, and hence families of proto-SICs.

We want a family of proto-SICs that contains a SIC. For this purpose we choose the maximally entangled states so that they belong to an eigenspace of the 15 dimensional Zauner unitary $U_{\mathcal{Z}''} \otimes U_{\mathcal{Z}'}$ in $\mathbf{C}^3 \otimes \mathbf{C}^5$. Hence we choose

$$|\Psi_{\mathbf{0}}(\sigma)\rangle = \frac{1}{\sqrt{2}} \left(|f_0\rangle|e_0\rangle + \mathrm{e}^{\mathrm{i}\sigma}|f_1\rangle|e_1\rangle\right) \ . \tag{64}$$

The question now arises if this family contains a SIC. Thus we are looking for values of σ such that

$$\sum_{\mathbf{p}\neq\mathbf{0}} \left(|\langle\Psi_{\mathbf{0}}(\sigma)|D^{(d)}_{\mathbf{p}}|\Psi_{\mathbf{0}}(\sigma)\rangle|^2 - \frac{1}{d+1} \right)^2 = 0, \tag{65}$$

where $d = 15$ in this case. These values are quickly found by numerically minimizing the function on the left-hand side. In practice, the number of terms can be taken to be one more than the number of parameters to be determined, and the full SIC condition can be checked afterwards. When the precision is high enough the exact phase factors $\mathrm{e}^{\mathrm{i}\sigma}$ can be determined using Mathematica's 'RootApproximant' command. In this case we find that there are three choices of the so far undetermined phase factor that give a SIC, namely

$$\mathrm{e}^{\mathrm{i}\sigma} = \omega_3^n P_5^{\frac{1}{3}} \ , \quad P_5 = -\frac{4}{5} - \frac{3i}{5}, \quad n \in \{0, 1, 2\}. \tag{66}$$

Indeed, adding the phase factor $P_5^{\frac{1}{3}}$ to the generators of the number field holding the SIC 5a gives the number field holding the SIC 15d [21].

Note that the symmetry of the 15 dimensional SIC that we constructed is of order 6. Its symmetry group is generated by the unitaries corresponding to the symplectic matrices

$$\mathcal{Z}_{15} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}_3 \times \begin{pmatrix} 0 & 2 \\ 2 & -1 \end{pmatrix}_5 \sim \begin{pmatrix} 10 & 9 \\ 1 & 4 \end{pmatrix}_{15} \tag{67}$$

$$\mathcal{S}_{15} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_3 \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_5 \sim \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}_{15} . \tag{68}$$

In fact the symmetry groups of the aligned SICs grow with a factor of two for each rung of the ladder [18, 19]. That SICs with these symmetries appear in dimensions of the form $d(d-2)$ was first noticed by Scott and Grassl [9, 10]. For later convenience we also gave the symplectic matrices obtained from the Chinese remainder theorem when we express the Hilbert space globally as $\mathbf{C}^{15}$ and use the standard representation of the Weyl–Heisenberg group there (see the appendix A).

It is interesting to count the number of aligned SICs obtained. In dimension five the symplectic group contains ten distinct Zauner subgroups of order 3, and each of them have two eigenspaces containing four SIC vectors each, so in total there are 80 distinct but equivalent SICs in the Clifford orbit labelled 5a. In $\mathbf{C}^3$ there are four distinct Zauner subgroups, which (given that each one has two generators) means that there are $10 \cdot 4 \cdot 2 = 80$ different Zauner subgroups in dimension 15. They are generated by matrices of the form $\mathcal{Z}_3 \times \mathcal{Z}_5$. Having fixed the Zauner matrix $\mathcal{Z}_5$ we have $4 \cdot 2$ Zauner matrices $\mathcal{Z}_3$ to choose from, and eight SIC fiducials to start from. For each of the latter we have checked that the proto-SIC vector family contains a SIC vector in $\mathbf{C}^{15}$ for exactly four choices of $\mathcal{Z}_3$. If a given $\mathcal{Z}_3$ works its square does not. Thus there are four 'empty branches' consisting of families of proto-SICs with the expected symmetry but without any SIC vectors in them. Keeping in mind that a proto-SIC family contains either three or no SICs we end up with $10 \cdot 8 \cdot 4 \cdot 3 = 960$ aligned SICs in dimension 15, which is precisely the number of distinct but equivalent SICs in the Clifford orbit 15d [4].

It is also interesting to consider the triplets of aligned SIC fiducial vectors somewhat further. The family of proto-SIC vectors that we defined in equation (64) defines a closed curve in the set of maximally entangled states in $\mathbf{C}^2 \otimes \mathbf{C}^2 \subset \mathbf{C}^{15}$. Using the Fubini–Study metric to define the geometry we find that it is a circle of maximal length, and hence it forms the equator of an embedded Bloch sphere (or complex projective line, for readers who prefer this language). On this equator there are three equidistant points that represent SIC fiducial states. In fact they form an orbit under $U^{(3)}_{\mathcal{Z}''} \otimes \mathbf{1}_5$.

8. Two more examples and a parameter count

We move on to $d_0 = 7$. There are two inequivalent SICs in this dimension, and we choose to work with the one that has acquired the label 7b [9]. This time it will pay to spend some thought on step **1** of the recipe (in section 6). Since the dimension equals 1 modulo 3 there exists a diagonal Zauner matrix, namely

$$\mathcal{Z} = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}. \tag{69}$$

Then $\mathcal{Z}' = \mathcal{Z}$. More importantly the unitary operator $U_{\mathcal{Z}}$ is a permutation matrix [4]. The large eigenspace (of dimension three) contains four distinct SIC fiducial vectors, two of which are real due to an extra anti-unitary symmetry that is present in this case. We choose to work with one of these, and calculate the resulting P_θ. We choose $\mathcal{Z}''$ to be the standard Zauner matrix, go through the first three steps of the recipe, and obtain the eigenbases

$$\begin{aligned} |f_0\rangle &= |1\rangle_5 \qquad & |e_0\rangle &= |1\rangle_7 \\ |f_1\rangle &= |\omega_3\rangle_5 \qquad & |e_1\rangle &= |\omega_3\rangle_7 \\ |f_2\rangle &= |\omega_3^2\rangle_5 \qquad & |e_2\rangle &= |\omega_3^2\rangle_7. \end{aligned} \tag{70}$$

From these basis vectors we want to construct maximally entangled states that are eigenvectors of $U^{(5)}_{\mathcal{Z}''} \otimes U^{(7)}_{\mathcal{Z}}$. This can be done in more than one way, but we expect (in fact we know [18]) that there is an aligned SIC in the smallest of the three eigenspaces of this operator. We have chosen the phase factors of the Zauner unitaries in dimensions d and $d-2$ in conformity with the conventions of table 1, and are thus led to propose the proto-SIC family

$$|\Psi_{\mathbf{0}}(\sigma_1, \sigma_2)\rangle = \frac{1}{\sqrt{3}} \left(|f_0\rangle|e_0\rangle + \mathrm{e}^{\mathrm{i}\sigma}|f_1\rangle|e_2\rangle + \mathrm{e}^{-\mathrm{i}\sigma}|f_2\rangle|e_1\rangle \right) \ . \tag{71}$$

We use only one free phase factor for the proto-SIC because the anti-unitary symmetry of the SIC 7b is inherited by the aligned SIC. A numerical search reveals that there are three mutually orthogonal SIC fiducial vectors hidden here. They are obtained by letting the sixth power $(\mathrm{e}^{\mathrm{i}\sigma})^6$ of the phase factor be a root of the minimal polynomial

$$\begin{aligned} p(t) = 4375t^8 - 35\,000t^7 + 19\,222\,300t^6 + 70\,190\,980t^5 + 102\,366\,979t^4 \\ + 70\,190\,980t^3 + 19\,222\,300t^2 - 35\,000 + 4375. \end{aligned} \tag{72}$$

This is no more complicated than it has to be and can be dealt with if one wants an exact solution, in this case for the SIC that Scott and Grassl labelled as 35j [9].

So far we did not encounter any degenerate eigenspaces when we defined the bases in the factor Hilbert spaces. When $d_0 = 9$ we do. We choose the standard Zauner matrix for the dimension 9 factor and a diagonal Zauner matrix for the other factor. At the end of step 3 we arrive at the bases

$$\begin{aligned} |f_0\rangle &= |1a\rangle_7 \qquad |e_0\rangle = |\omega_3 a\rangle_9 \\ |f_1\rangle &= |1b\rangle_7 \qquad |e_1\rangle = |\omega_3 b\rangle_9 \\ |f_2\rangle &= |\omega_3^2\rangle_7 \qquad |e_2\rangle = |\omega_3^2\rangle_9 \\ |f_3\rangle &= |\omega_3\rangle_7 \qquad |e_3\rangle = |1\rangle_9. \end{aligned} \tag{73}$$

We must now form a maximally entangled state that is an eigenvector of the Zauner unitary $U^{(7)}_{\mathcal{Z}''} \otimes U^{(9)}_{\mathcal{Z}'}$. Moreover we expect the aligned SIC to lie in the largest of the three eigenspaces, which in this case (in disagreement with the conventions adopted in table 1) means that the eigenvalue should be ω_3. To deal with the degenerate eigenspace we recall equation (55), and introduce an $SU(2)$ matrix

$$U = \begin{pmatrix} \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\sigma_0} & \sin\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\sigma_1} \\ -\sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_1} & \cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_0} \end{pmatrix}. \tag{74}$$

A family of proto-SIC vectors invariant under the Zauner unitary $U^{(7)}_{\mathcal{Z}''} \otimes U^{(9)}_{\mathcal{Z}'}$ is therefore given by

$$\begin{aligned} |\Psi_{\mathbf{0}}\rangle = \frac{1}{\sqrt{4}} \Bigg(& \cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\sigma_0}|f_0\rangle|e_0\rangle + \sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_1}|f_0\rangle|e_1\rangle \\ & - \sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_1}|f_1\rangle|e_0\rangle + \cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_0}|f_1\rangle|e_1\rangle \\ & + \mathrm{e}^{\mathrm{i}\sigma_2}|f_2\rangle|e_2\rangle + \mathrm{e}^{\mathrm{i}\sigma_3}|f_3\rangle|e_3\rangle \Bigg). \end{aligned} \tag{75}$$

There are five real parameters in this expression. We thus have a function of five variables to minimize, but Mathematica's 'NMinimize command' handles this easily. In this way we identify an aligned SIC. Starting from the SIC labelled 9a by Scott and Grassl [9] we end up with the SIC that they label 63b. It was originally found by a numerical search through the 22 complex dimensional Zauner subspace in dimension 63.

It is clear that numerical searches for aligned SICs can be greatly facilitated by our recipe. Let us count the number of free parameters that need to be fixed, assuming that the SIC from which we start in dimension d_0 has only the Zauner symmetry. The details depend a little on the value of d_0 modulo 3, and for definiteness we assume that $d_0 = 6k + 3$. The dimension of the Zauner subspaces will be $(2k + 2, 2k + 1, 2k)$ in $\mathbf{C}^{d_0}$ and $(2k + 1, 2k, 2k)$ in $\mathbf{C}^{d_0-2}$. When we restrict ourselves to the parity eigenspaces the dimensions drop to $(k + 1, k, k)$ in both cases. A maximally entangled proto-SIC sitting in the largest Zauner subspace of $\mathbf{C}^{d_0-2} \otimes \mathbf{C}^{d_0}$ will then be given by a block diagonal unitary matrix with blocks of size $k + 1$, k, and k. Subtracting an overall phase we find that the number of real parameters that must be fixed in order to obtain a SIC is

$$(k+1)^2 + k^2 + k^2 - 1 = 3k^2 + 2k. \tag{76}$$

The complex dimension of the relevant Zauner subspace in $\mathbf{C}^{d_0(d_0-2)}$ is $12k^2 + 8k + 2$, so this is an improvement even though the quadratic dependence on k is still there. For other ways to reduce the dimension of the search space see reference [27].

Going beyond this reduction in the number of parameters presumably needs ideas from number theory.

9. Higher rungs of the ladder

The complexity of the calculations, and the size of the proto-SIC family that we will have to search through in step **5** of the recipe (in section 6), evidently grows as we try to reach beyond the first rung of the ladders. Fortunately this problem is somewhat mitigated by the extra symmetry that we pick up each time we reach a new rung on a ladder. As our example we choose the ladder $5 \to 15 \to 195 \to 37\,635 \to \ldots$. The first three entries are known in exact form already [21]. Let us see how we can obtain a SIC in dimension 195, given the $d_1 = 15$ SIC that we constructed in section 7. To avoid modifications of the representation given in section 4 we express the latter in 'global' form.

The SIC we are about to construct will have a symmetry group of order 12, generated by

$$\mathcal{Z}_{195} = \begin{pmatrix} 3 & 0 \\ 0 & 9 \end{pmatrix}_{13} \times \begin{pmatrix} 10 & 9 \\ 1 & 4 \end{pmatrix}_{15} \tag{77}$$

$$\mathcal{S}_{195} = \begin{pmatrix} 5 & 0 \\ 0 & -5 \end{pmatrix}_{13} \times \begin{pmatrix} 11 & 0 \\ 0 & 11 \end{pmatrix}_{15}. \tag{78}$$

The right hand factors are as given in equations (67) and (68). The Zauner matrix in the $\mathbf{C}^{13}$ factor was chosen so that its unitary representative is a permutation matrix. But note that

$$\mathcal{S}_{195}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}_{13} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{15}. \tag{79}$$

The parity matrix appears in the left hand factor, and gives rise to the extra symmetry of the aligned SIC. This works only because -1 is a quadratic residue modulo 13. At the next rung of the ladder -1 has to be a quartic residue modulo 193, and so on. This part of the construction is actually known to work all the way up the ladder [20], but we leave this aside here.

We define the two $(15-1)/2$-dimensional subspaces as the positive parity eigenspace in $\mathbf{C}^{13}$, and the negative θ-parity eigenspace in $\mathbf{C}^{15}$. We span these eigenspaces with bases consisting of eigenvectors of the relevant factors of the unitaries that generate the symmetry group. We end up with the preferred basis

$$\begin{aligned}
|f_0\rangle &= |1a\rangle_{13} \quad & |e_0\rangle &= |1a\rangle_{15} \\
|f_1\rangle &= |1b\rangle_{13} \quad & |e_1\rangle &= |1b\rangle_{15} \\
|f_2\rangle &= |\omega_6\rangle_{13} \quad & |e_2\rangle &= |\omega_6\rangle_{15} \\
|f_3\rangle &= |\omega_6^2\rangle_{13} \quad & |e_3\rangle &= |\omega_6^2\rangle_{15} \\
|f_4\rangle &= |\omega_6^3\rangle_{13} \quad & |e_4\rangle &= |\omega_6^3\rangle_{15}
\end{aligned}$$

$$|f_5\rangle = |\omega_6^4\rangle_{13} \quad |e_5\rangle = |\omega_6^4\rangle_{15}$$
$$|f_6\rangle = |\omega_6^5\rangle_{13} \quad |e_6\rangle = |\omega_6^5\rangle_{15}. \tag{80}$$

A family of proto-SICs is now formed as maximally entangled states having the appropriate symmetry (eigenvalue 1, in this case). It is given by

$$|\Psi_{\mathbf{0}}\rangle = \frac{1}{\sqrt{7}}\left(\cos\frac{\theta}{2}\mathrm{e}^{\mathrm{i}\sigma_0}|f_0\rangle|e_0\rangle + \sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_1}|f_0\rangle|e_1\rangle \right.$$
$$- \sin\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_1}|f_1\rangle|e_0\rangle + \cos\frac{\theta}{2}\mathrm{e}^{-\mathrm{i}\sigma_0}|f_1\rangle|e_1\rangle$$
$$+ \mathrm{e}^{\mathrm{i}\sigma_2}|f_2\rangle|e_6\rangle + \mathrm{e}^{\mathrm{i}\sigma_3}|f_3\rangle|e_5\rangle + \mathrm{e}^{\mathrm{i}\sigma_3}|f_4\rangle|e_4\rangle$$
$$\left. + \mathrm{e}^{\mathrm{i}\sigma_5}|f_5\rangle|e_3\rangle + \mathrm{e}^{\mathrm{i}\sigma_6}|f_6\rangle|e_2\rangle \right). \tag{81}$$

It is an eight real dimensional family, small enough so that we can use standard Mathematica routines for the numerical search that determines the 195 dimensional SIC. By construction it has a symmetry of order 12, and is equivalent to the SIC labelled 195d [18]. We again used standard Mathematica routines to determine that $\cos\theta/2 = 3/\sqrt{13}$ and that the sixth powers of the phase factors can be obtained by finding the roots of seven polynomials of degree 32 and large coefficients.[3] This is encouraging given that the ray class field that houses the SIC has degree $2^8 \cdot 3^3$. The solution is easily brought to the more elegant form given in reference [21].

The logic remains the same as we continue up the ladder. To reach the next rung we consider $\mathbf{C}^{193} \otimes \mathbf{C}^{195}$. The symmetry operators in the $\mathbf{C}^{193}$ factor can again be chosen as permutation matrices, and the family of proto-SICs with the expected symmetry has 98 real parameters. However, here we encounter the problem that the present authors have no expertise in numerical optimization. We have not solved it, and hence we have not found the $d = 37\,635$ SIC. This is a pity, because there are some reasons to believe that all the SICs on this ladder can be written in an especially appealing exact form [20, 21]. If this is true we would need only modest numerical precision in order to obtain the exact solution through an integer relation algorithm, and the result might suggest how to go even higher.

10. Summary

We believe that we have clarified how aligned SICs, as defined in reference [18], arise. By a suitable arrangement of known results we have shown that the set of Naimark complements of the equiangular tight frame appearing in theorem 1, which is parameterized by the group manifold of $SU((d-1)/2)$, can be lifted to $\mathbf{C}^{d(d-2)}$ where they form maximally entangled states in a subspace that is defined by numbers entering the SIC in $\mathbf{C}^d$. This leads to a straightforward calculational procedure for producing a family of proto-SIC vectors in dimension $d(d-2)$, given a SIC in dimension d to start with. By definition, a proto-SIC vector obeys the alignment conditions (16) and (17), but it is not necessarily a fiducial vector for a SIC. We have also explained how the ambiguities that we have encountered can be partly removed by imposing symmetry conditions. Still the full SIC condition eludes us, so we have not proved that aligned SICs

[3] To reach this conclusion we increased the precision of the numerical vector from 200 to 3400 digits using a Mathematica file kindly provided by Marcus Appleby. This calculation took 773 s.

must exist. What we do have is a remarkably convenient way to conduct numerical searches for aligned SICs. Even so we have not calculated any new SICs, because the easy targets have already been calculated with other methods. For $d = 195 \cdot 193 = 37\,635$ we would have to optimize a function of 98 real variables. While this is not an insuperable task, it is insuperable for the present authors.

We end by echoing a remark of Appleby's [4]: the crucial discovery may lie just round the corner.

Acknowledgments

We thank Irina Dumitru for discussions in the course of this work. We also thank Markus Grassl, Danylo Yakymenko, and Ole Sönnerborn (one of the authors of reference [19]) for very useful comments on a draft. BS acknowledges the financial support from the Knut and Alice Wallenberg Foundation through the Wallenberg Centre for Quantum Technology.

Data availability statement

The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Useful formulas

For the convenience of the reader we collect some useful formulas here. For the displacement operators we use the representation

$$(D_{i,j})_{r,s} = \tau^{ij+2js}\delta_{r,s+i}. \tag{82}$$

For the symplectic unitaries the requirement that $U_F D_{\mathbf{p}} U_F^{-1} = D_{F\mathbf{p}}$ implies

$$F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_d \quad \Rightarrow \quad (U_F)_{r,s} = \frac{\mathrm{e}^{\mathrm{i}\varphi}}{\sqrt{d}} \tau^{\beta^{-1}(\delta r^2 - 2rs + \alpha s^2)} \tag{83}$$

whenever the integer β is invertible modulo d. If not, the symplectic matrix can be written as a product of two matrices with an invertible integer in the upper right hand corner, and the symplectic unitary as a product of the two corresponding symplectic unitaries. The overall phase factor $\mathrm{e}^{\mathrm{i}\varphi}$ is not determined at this stage. We chose it so that table 1 applies. For the Chinese remaindering we use

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}_{d(d-2)} \sim \begin{pmatrix} \alpha & \kappa^{-1}\beta \\ \kappa\gamma & \delta \end{pmatrix}_{d-2} \times \begin{pmatrix} \alpha & \kappa^{-1}\beta \\ \kappa\gamma & \delta \end{pmatrix}_{d}, \tag{84}$$

where $\kappa = (d-1)/2$.

ORCID iDs

Ingemar Bengtsson https://orcid.org/0000-0002-4203-3180
Basudha Srivastava https://orcid.org/0000-0002-4972-4216

References

[1] Fickus M and Mixon D G 2015 Tables of the existence of equiangular tight frames (arXiv:1504.00253)
[2] Zauner G 2011 Quantum designs: foundations of a noncommutative design theory **vol 9** *PhD Thesis* University of Wien 1999 (Also published as Quantum designs: Foundations of a noncommutative design theory, Int. J. Quant. Inf. 9) p 445
[3] Renes J M, Blume-Kohout R, Scott A J and Caves C M 2004 Symmetric informationally complete quantum measurements *J. Math. Phys.* **45** 2171
[4] Appleby D M 2005 SIC-POVMs and the extended Clifford group *J. Math. Phys.* **46** 052107
[5] Weyl H 1928 *Gruppentheorie und Quantenmechanik* (Leipzig: Hirzel) also published as Theory of Groups and Quantum Mechanics, Dutton, New York 1932
[6] Appleby M, Flammia S, McConnell G and Yard J 2020 Generating ray class fields of real quadratic fields via complex equiangular lines *Acta Arith.* **192** 211
Appleby M, Flammia S, McConnell G and Yard J 2020 arXiv:1604.06098
[7] Hilbert D 1900 *Matematische Probleme* (Göttinger: Nachrichten) p 253 also published as Mathematical problems, Bull. AMS 8 (1902) 437
[8] DeBrota J B and Stacey B C 2018 FAQBism (arXiv:1810.13401)
[9] Scott A J and Grassl M 2010 Symmetric informationally complete positive-operator-valued measures: a new computer study *J. Math. Phys.* **51** 042203
[10] Scott A J 2017 SICs: Extending the list of solutions (arXiv:1703.03993)
[11] Appleby M, Chien T-Y, Flammia S and Waldron S 2018 Constructing exact symmetric informationally complete measurements from numerical solutions *J. Phys. A: Math. Theor.* **51** 165302
[12] Fuchs C, Hoang M and Stacey B 2017 The SIC question: history and state of play *Axioms* **6** 21
[13] Grassl M and Scott A J 2017 Fibonacci–Lucas SIC-POVMs *J. Math. Phys.* **58** 122201
[14] Grassl M unpublished
[15] Appleby M, Bengtsson I, Grassl M, Harrison M and McConnell G 2021 SIC-POVMs from Stark units (arXiv:2112.0552)
[16] McConnell G unpublished
[17] Kopp G S 2021 SIC-POVMs and the Stark conjectures *Int. Math. Res. Not.* **18** 13812–38
[18] Appleby M, Bengtsson I, Dumitru I and Flammia S 2017 Dimension towers of SICs: I. Aligned SICs and embedded tight frames *J. Math. Phys.* **58** 112201
[19] Andersson O and Dumitru I 2019 Aligned SICs and embedded tight frames in even dimensions *J. Phys. A: Math. Theor.* **52** 425302
[20] Bengtsson I and McConnell G unpublished
[21] Appleby M and Bengtsson I 2019 Simplified exact SICs *J. Math. Phys.* **60** 062203
[22] Appleby D M, Yadsan-Appleby H and Zauner G 2013 Galois automorphisms of symmetric measurements *Quantum Inf. Comput.* **13** 672
[23] Bengtsson I 2020 SICs: some explanations *Found. Phys.* **50** 1794
[24] Ostrovskyi V and Yakymenko D 2022 Geometric properties of SIC-POVM tensor square *Lett. Math. Phys.* **112** 7
[25] Dirac P A M 1939 A new notation for quantum mechanics *Math. Proc. Camb. Phil. Soc.* **35** 416
[26] Appleby M, Bengtsson I, Flammia S and Goyeneche D 2019 Tight frames, Hadamard matrices and Zauner's conjecture *J. Phys. A: Math. Theor.* **52** 295301
[27] Caro Pérez F, González Avella V and Goyeneche D 2021 Mutually unbiased frames (arXiv:2110.08293)