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On formal integrability of PDEs in general relativity

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Abstract. Partial differential equations (PDEs) remain notoriously difficult to work with and, in many cases, are unsolvable. An alternative approach to studying PDEs, known as the formal theory of PDEs, treats PDEs as submanifolds of a specific vector bundle called the jet bundle. This perspective allows the extensive machinery of differential geometry to be applied to the analysis of PDEs. Einstein's Equation, the cornerstone of General Relativity, is known to be formally integrable, a property that can be demonstrated using Spencer cohomology. In this work, I will discuss obstacles for setting up the Einstein–Vlasov equation in this framework.

1 Introduction and background

Formal integrability of a partial differential equation (PDE) essentially guarantees that any solution of the PDE can be extended to a solution of higher order. For many equations in mathematical physics, formal integrability is known; in particular, the Einstein equation [4], the Einstein–Maxwell equation [8], the Yang–Mills equation [5], and scalar PDEs with a regularity assumption [7] are all formally integrable. The Einstein–Vlasov equation, which is an integro-differential equation, cannot be readily assessed for formal integrability in the same way as the aforementioned PDEs, as it is not immediately expressible as a formal PDE, i.e. it is not immediately expressible in the language of jet bundles.

For a detailed introduction to jet bundles, as well as notation conventions followed in this paper, we refer to [13]. In particular, since notation regarding jet bundles is not universally established, see the Glossary of Symbols in [13] for notation not explicitly defined in this paper.

Let X be an n-dimensional differentiable manifold, let T^*X be the cotangent bundle of X, let $\pi : E \to X$ be a vector bundle of rank m over X, and let $J^k\pi$ be the k-order jet bundle of π, for $k \geq 1$. It is well known that there is an affine bundle structure of $J^k\pi$ over $J^{k-1}\pi$, due to the split-exact sequence

$$0 \longrightarrow S^k(T^*X) \otimes E \xrightarrow{\ \varepsilon\ } J^k\pi \xrightarrow{\ \pi^k_{k-1}\ } J^{k-1}\pi \longrightarrow 0, \tag{1.1}$$

where $S^k(T^*X)$ denotes the k-fold symmetric product of T^*X, and π^k_{k-1} is the natural projection map from $J^k\pi$ to $J^{k-1}\pi$. (The explicit definition of ε is not essential for our purposes, but can be found in many sources, including [4] and [9].) This sequence gives us the tools to view the pure k^{th}-order jets in $J^k\pi$ as a bundle over $J^{k-1}\pi$ with fiber $S^k(T^*X) \otimes E$, and it is this structure that allows us to transition from PDE analysis by differential geometry to PDE analysis by algebra and cohomology.

It should be noted that all of these methods can still be applied if E is a fiber bundle as opposed to a vector bundle, but the notation then becomes extremely involved. Consequently, we will stick to vector bundles for this paper, but the interested reader may refer to Chapter 6 of [13] for more details.

Definition 1.1. A *PDE* $\mathcal{R}_k$ *of order* k is simply a fibered submanifold of $J^k\pi$ over X.

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For the purposes of this paper, we assume that there exists another vector bundle $\pi' : F \to X$ and a vector bundle morphism $\Phi : J^k\pi \to F$ of locally constant rank such that $\mathcal{R}_k = \ker \Phi$. This assumption restricts $\mathcal{R}_k$ topologically [6, p295] but is sufficient for our purposes.

Definition 1.2. A *solution* of $\mathcal{R}_k$ is a local section ψ of π such that the k-jet $j^k_x\psi \in \mathcal{R}_k$ for all x in the domain of ψ.

When studying a PDE $\mathcal{R}_k$, it becomes pertinent to examine the higher-order derivative consequences of $\mathcal{R}_k$. If $\Phi : J^k\pi \to F$ is the bundle morphism determining $\mathcal{R}_k$ (i.e. $\mathcal{R}_k = \ker \Phi$), then we may define the ℓ^{th} *prolongation of* $\mathcal{R}_k$, for $\ell \geq 0$, by

$$\mathcal{R}_{k+\ell} = \{ j^{k+\ell}_x\psi \in J^{k+\ell}\pi \ : \ j^\ell_x(\Phi(j^k_x\psi)) = 0 \}. \tag{1.2}$$

$\mathcal{R}_k$ is *formally integrable* if $\pi^{k+\ell+1}_{k+\ell} : \mathcal{R}_{k+\ell+1} \to \mathcal{R}_{k+\ell}$ is surjective for all $\ell \geq 0$.

Because formal integrability requires an infinite number of surjectivity checks, it is difficult to show directly from the definition that a PDE is formally integrable. Instead, we may define $g_{k+\ell} \subset S^{k+\ell}(T^*X) \otimes E$ so that the following sequence, which is the sequence (1.1) restricted to $\mathcal{R}_k$ and its prolongations, is exact:

$$0 \longrightarrow g_{k+\ell} \xrightarrow{\ \varepsilon\ } \mathcal{R}_{k+\ell} \xrightarrow{\pi^{k+\ell}_{k+\ell-1}} \mathcal{R}_{k+\ell-1} \tag{1.3}$$

The family of vector spaces $(g_{k+\ell})_{\ell \geq 0}$ forms a *symbolic system* (see [14]), whose Spencer cohomology can be computed and used to determine formal integrability.[6, Theorem 8.1] This turns the problem of formal integrability, an infinite problem by nature, into a finite number of cohomology group computations (see, for example, [14, Theorem 6.1.21]), and it is for this reason that we wish to view the Einstein–Vlasov Equation as a PDE in the sense of Definition 1.1.

2 The Einstein–Vlasov equation

We assume that the reader is familiar with the mathematical construction of a spacetime M. For the reader who needs an introduction or a refresher, see [3] or [11]. We will take the Lorentz signature (1,3) for the metric, and choose units so that the speed of light and the gravitational constant are both equal to 1.

The Vlasov Equation describes a collection of particles that are only dynamically affected by the collection of all particles as a whole. In particular, particles do not affect each other directly, and travel along geodesics in M. If we take "particles" to be stars, galaxies, or galaxy clusters, and assume that any internal structures thereof are insignificant to the large scale dynamics, we may apply the Vlasov Equation in a cosmological context.[2, 10] We assume that all particles have the same rest mass $m = 1$.

Let (M, g) be a spacetime, and let P be the mass-shell defined by g, which is the hypersurface of future-pointing, timelike, unit vectors in TM. Unless otherwise stated, all Greek indices will run from 0 to 3 and all Latin indices will run from 1 to 3. Given local coordinates (x^μ) on M, we denote the induced coordinates on P by (x^μ, p^a). (We assume that p^0 is written in terms of the other p^a via $g_{\mu\nu}p^\mu p^\nu = -1$.) The Einstein–Vlasov system is then

$$\frac{1}{8\pi}\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) = -\int_{\mathbb{R}^3} f p_\alpha p_\beta |\det g|^{1/2} \frac{dp^1 dp^2 dp^3}{p^0}, \tag{2.1}$$

$$\partial_{x^0} f + \frac{p^i}{p^0}\partial_{x^i} f - \frac{1}{p^0}\Gamma^j_{\alpha\beta}p^\alpha p^\beta \partial_{p^j} f = 0, \tag{2.2}$$

where $R_{\alpha\beta}$ is the Ricci curvature tensor of g, R is the scalar curvature of g, f is a nonnegative real-valued function on P, and $\Gamma^j_{\alpha\beta}$ are the Christoffel symbols of the Levi-Civita connection. For more details on the Einstein–Vlasov system, see [1], [2], or [10]. The energy momentum tensor $T_{\alpha\beta}$ is given by the right side of equation (2.1), and the geodesic spray vector field applied to f is given by the left side of (2.2).

Our original goal was to translate these equations into a submanifold $\mathcal{R}_k$ of an appropriate jet bundle, but several problems arise immediately.

(i) In the language of jet bundles, a solution must be a section of the original bundle $E \to X$. A solution to the Einstein–Vlasov system consists of a metric g and a Vlasov field f, which are sections of $S^2(T^*M) \to M$ and $P \times \mathbb{R} \to P$, respectively. In particular, they do not share the same base manifold.

(ii) The domain of f, the hypersurface $P \subset TM$, is dependent on g, as P is given by $P = \{v \in TM : g(v,v) = -1\}$.

(iii) The most obvious problem with this system is the integral that has made itself welcome in the otherwise differential equation via the energy momentum tensor. This does not seem to have an easy fix, and in fact it is our expectation that we will have to define a notion of "weak" formal integrability in order to say anything about it.

We address issue (i) for the remainder of this paper, but defer the resolutions of issues (ii) and (iii) to future work. For issue (i), we note that the two different base spaces, M and P, are very closely related: $P \subset TM$. Therefore we may pull back g along the projection map $\tau : TM \to M$, to get a form $\mathfrak{g}$ on TM, which can then be restricted to P:

$$\begin{array}{ccccc} S^2(T^*P) & & S^2(T^*(TM)) & & S^2(T^*M) \\ {\scriptstyle \underline{\mathfrak{g}}}\uparrow\ \downarrow{\scriptstyle \underline{\widetilde{\pi}}} & & {\scriptstyle \mathfrak{g}}\uparrow\ \downarrow{\scriptstyle \widetilde{\pi}} & & {\scriptstyle g}\uparrow\ \downarrow{\scriptstyle \pi} \\ P & \hookrightarrow & TM & \xrightarrow{\ \tau\ } & M \end{array} \tag{2.3}$$

Here, π, $\widetilde{\pi}$, and $\underline{\widetilde{\pi}}$ are the canonical projections, and $\underline{\mathfrak{g}}$ denotes the restriction of $\mathfrak{g}$ to the subbundle $\underline{\widetilde{\pi}}$ of $\widetilde{\pi}$.

Let's write this out in coordinates to give a fuller picture of $\mathfrak{g}$. Let (x^μ) be coordinates on M. Let (x^μ, ξ^ν) be induced coordinates on TM. Then we have, in turn, induced coordinates $(x^\mu, \xi^\nu, y^\alpha, \zeta^\beta)$ on $T(TM)$, and a generic vector in each of the aforementioned spaces can be written as follows:

$$\xi = \xi^\mu \frac{\partial}{\partial x^\mu}\bigg|_x \in T_xM, \qquad \Xi = y^\alpha \frac{\partial}{\partial x^\alpha}\bigg|_\xi + \zeta^\nu \frac{\partial}{\partial \xi^\nu}\bigg|_\xi \in T_\xi(TM). \tag{2.4}$$

Next, we have coordinates on $S^2(T^*M)$ and $S^2(T^*(TM))$ given by $(x^\mu, h_{\eta\gamma})$ and $(x^\mu, \xi^\nu, \mathfrak{h}_{\mathfrak{r}\mathfrak{s}})_{1\le \mathfrak{r},\mathfrak{s},\le 8}$, respectively, so that a generic vector in each of these spaces can be written as

$$h = h_{\mu\nu}(dx^\mu \otimes dx^\nu)\big|_x \in S^2_x(T^*M), \tag{2.5}$$

$$\mathfrak{h} = h^1_{\mu\nu}(dx^\mu \otimes dx^\nu)\big|_\xi + h^2_{\mu\nu}(dx^\mu \otimes d\xi^\nu)\big|_\xi + h^3_{\mu\nu}(d\xi^\mu \otimes d\xi^\nu)\big|_\xi \in S^2_\xi(T^*(TM)). \tag{2.6}$$

The coefficient matrix $(\mathfrak{h}_{\mathfrak{r}\mathfrak{s}})$ is given by the symmetric block matrix $\begin{pmatrix} (h^1_{\mu\nu}) (h^2_{\mu\nu}) \\ (h^2_{\nu\mu}) (h^3_{\mu\nu}) \end{pmatrix}$. Given $\Xi_1, \Xi_2 \in T_\xi(TM)$, with coordinate representations as in (2.4), and a Lorentzian metric g on M, the pulled-back $\mathfrak{g} := \tau^* g$ on TM is given, at $\xi \in T_xM$, by

$$\mathfrak{g}|_\xi(\Xi_1, \Xi_2) = (\tau^* g)|_\xi(\Xi_1, \Xi_2) = g|_{\tau(\xi)}(d\tau_\xi(\Xi_1), d\tau_\xi(\Xi_2)) = g|_x\left(y_1^\alpha \tfrac{\partial}{\partial x^\alpha}\big|_x, y_2^\beta \tfrac{\partial}{\partial x^\beta}\big|_x\right) = g_{\alpha\beta}(x) y_1^\alpha y_2^\beta. \tag{2.7}$$

Since the restriction of TM to P happens in the vertical coordinates (ξ^ν) of TM, and our expression for $\mathfrak{g}$ is independent of the vertical coordinates, the coordinate description of $\underline{\mathfrak{g}}$ acting on vectors in TP is identical to the one above.

With this new section $\underline{\mathfrak{g}} : P \to S^2(T^*P)$, we can construct the fibered product bundle (see [13], Definition 1.4.4) from $\underline{\widetilde{\pi}} : S^2(T^*P) \to P$ and $\rho : P \times \mathbb{R} \to P$. A section of this product bundle, $(\underline{\mathfrak{g}}, f)$, may then be interpreted as a solution to the Einstein–Vlasov equation in the sense of Definition 1.2, provided its 2-jet satisfies equations (2.1) and (2.2) at every point in P:

$$\begin{array}{ccc} S^2(T^*P) & \xleftarrow{\ \mathrm{pr}_1\ } & S^2(T^*P) \times_P (P \times \mathbb{R}) \\ {\scriptstyle \underline{\mathfrak{g}}}\uparrow\ \downarrow{\scriptstyle \underline{\widetilde{\pi}}} & \nearrow{\scriptstyle (\underline{\mathfrak{g}}, f)} & \downarrow{\scriptstyle \mathrm{pr}_2} \\ P & \xleftarrow{\ \rho\ } & P \times \mathbb{R}. \\ & \xrightarrow[f]{} & \end{array} \tag{2.8}$$

The previous discussion provides just one possibility for lifting g to the mass shell. Another possibility would be to restrict the Sasaki metric determined by g (see [12]) to the mass shell; however this might be more structure than we need for the Einstein–Vlasov system, and might further complicate the situation by introducing nonzero vertical derivatives of $\mathfrak{g}$.

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