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# Envariance as a Symmetry in Quantum Mechanics and Applications to Statistical Mechanics

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## Article

# Envariance as a Symmetry in Quantum Mechanics and Applications to Statistical Mechanics

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**Abstract:** A quantum symmetry called entanglement-assisted invariance, also called envariance, is introduced. It is studied with respect to the process of performing quantum measurements. An apparatus which interacts with other physical systems, which are called environments, exchanges a single state with physical states equal in number to that of the possible outcomes of the experiment. Correlations between the apparatus and environment give rise to a type of selection rule which prohibits the apparatus from appearing in a superposition corresponding to different eigenvalues of the pointer basis of the apparatus. The eigenspaces of this observable form a natural basis for the apparatus and determine the observable of the measured quantum system. It is also discussed how statistical mechanics can be formulated in terms of this symmetry.

**Keywords:** probability; correlations; operator; measurements; basis; eigenstates; envariance

## 1. Introduction

Progress in understanding the measurement process with regard to quantum mechanical systems [1–4] can be greatly increased by treating an apparatus, also thought of as an environment, quantum mechanically. The interaction of the system with the apparatus can be studied together quantum mechanically, while also including the presence of an environment or other physical systems which interact with the apparatus [5,6]. In this process, the states of the apparatus become correlated with the system and influence what is observed. Both can become correlated with an immediate environment, perhaps better thought of as a second apparatus. Thus, the apparatus should itself be described by quantum mechanical laws [7]. The von Neumann approach to the problem has a particular basis. A correlation is established between the states of the apparatus and the states of the system. Let us call the apparatus  $A$ , the system  $S$  and the environment  $E$ . If the apparatus with states  $|A_s\rangle$  is regarded as quantum mechanical, then there is nothing to prevent the state of  $A$  being presented in terms of an alternate orthonormal basis  $|A_r\rangle$  composed of superpositions of the states  $|A_s\rangle$ :

$$|A_r\rangle = \sum_s \langle A_s | A_r \rangle |A_s\rangle. \quad (1)$$

The physical state of the combined system is described by a superposition of states  $|A_s\rangle \otimes |s\rangle$ , where  $|s\rangle$  describes the states of the system. Then, the combined system can be given in terms of the new states  $|A_r\rangle$  as

$$\begin{aligned} \sum_s \tau_s |A_s\rangle \otimes |s\rangle &= \sum_r \langle A_r | A_s \rangle \sum_s \tau_s |A_s\rangle \otimes |s\rangle = \sum_r |A_r\rangle \otimes \sum_s \tau_s \langle A_r | A_s \rangle |s\rangle \\ &= \sum_r \kappa_r |A_r\rangle \otimes |r\rangle. \end{aligned} \quad (2)$$

In Equation (2), the definition  $\kappa_r = \sum_s \tau_s \langle A_r | A_s \rangle$  provides a set of relative states  $\{|r\rangle\}$ . These constitute normalized but not necessarily mutually orthogonal states of system  $S$ , relative to the arbitrarily chosen basis set  $|A_r\rangle$  of the apparatus. Is it possible for the quantum system to end up in one of the states  $|r\rangle$  rather than  $|s\rangle$ . If all the  $\tau_s$  values are of



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the same magnitude, then whenever  $|A_r\rangle$  is orthonormal, the collection of relative states  $|r\rangle$  is as well. Therefore, the apparatus, which has to be correlated with the state of the system, contains not only information about the observable  $\hat{S} = \sum_s q_s |s\rangle\langle s|$  but many other observables  $\hat{R} = \sum_r r_r |r\rangle\langle r|$  as well. These are defined in the Hilbert space of the system [8–10].

However, generally  $\hat{R}$  and  $\hat{S}$  will not commute. Quantum mechanics does not permit the simultaneous measurement of two noncommuting observables with arbitrary accuracy. What then, in a real world apparatus, determines the seemingly unique pointer basis  $|A_r\rangle$  which records the corresponding relative states  $|p\rangle$  of the system? The new feature is that interaction of the quantum apparatus with the environment also produces correlations. Correlations with the environment impose certain kinds of selection rules which prevent the apparatus from existing in a superposition of states corresponding to different eigenvalues of this special apparatus basis.

A new quantum symmetry called entanglement-assisted invariance, or enviance is introduced. It is investigated in its application to, for example, the measurement process in quantum mechanics. It also provides a new, consistent way to understand statistical mechanics, as well as the deep relationship it has with thermodynamics. Interaction with the environment is an important component of the concept. It distinguishes the model apparatus from the quantum system. The eigenspaces of the pointer observable provide a natural basis for the pointer of the quantum apparatus and determine the observable of the measured quantum system. The observation or monitoring of the apparatus by the environment terminates in the apparent reduction of the wave packet. Correlations among the states of the pointer basis and those of the relative states of the system are preserved in the end mixed-state density matrix [10]. Decay of those elements of the apparatus-system density matrix, which are off-diagonal in the apparatus observable, is a result of the natural evolution of the system-apparatus-environment combination. Selection rules need not be imposed from outside [11–13].

In statistical physics, the description of canonical thermal equilibria is usually derived from Boltzmann's  $H$  theorem, the ergodic hypothesis or maximization of the statistical entropy in equilibrium [14]. However, none of these concepts are particularly well stated for quantum systems. Statistical physics developed when the fundamental physical theory was classical mechanics. Statistical physics emerged then to develop from microstates, points in a phase space and thermodynamic macrostates. This uses, for example, concepts such as ensembles consisting of infinitely many versions of the same system. Progress on this problem has occurred by demonstrating that representations of microcanonical and canonical equilibria can be obtained from a fully quantum mechanical analysis. This means taking account of symmetry considerations such as entanglement and, consequently, enviance. After studying this symmetry in detail, it is shown how enviance can produce the microcanonical and canonical states.

## 2. Measurement Performed on Combined Two-State Systems

### 2.1. Construction of an Accurate Model

A pair of two-state systems is defined in order to create a model which emphasizes some of the main physical aspects of the model. The first two-state system goes by the name spin, and the second is referred to as atom. The spin system has a basis denoted by the states  $\{|\uparrow\rangle, |\downarrow\rangle\}$ , also written as  $\{(1,0)^t, (0,1)^t\}$ . It describes a state parallel or antiparallel to the  $z$  axis. However, there are other bases which can be formed out of linear combinations of these states, which result in orthonormal states such as

$$\begin{pmatrix} |\alpha\rangle \\ |\beta\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}, \quad \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |\alpha\rangle \\ |\beta\rangle \end{pmatrix}. \quad (3)$$

In addition, there is the basis  $\{|\rightarrow\rangle, |\leftarrow\rangle\}$ :

$$\begin{pmatrix} |\rightarrow\rangle \\ |\leftarrow\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix}, \quad \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} |\rightarrow\rangle \\ |\leftarrow\rangle \end{pmatrix}. \quad (4)$$

The second two-state system is called *atom* in order to make a connection with an object that may be present in an experiment. It is an object which consists of two states  $\{|g\rangle, |e\rangle\}$ . These states are called ground and excited. Let us assume that the atom has the same energy no matter which of these states it is in. Neither of these systems have self-Hamiltonians. As in the case of the spin system, there are alternative bases as may be expected. They are formed out of linear combinations of these two:

$$\begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |e\rangle \\ |g\rangle \end{pmatrix}, \quad \begin{pmatrix} |e\rangle \\ |g\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} |+\rangle \\ |-\rangle \end{pmatrix}. \quad (5)$$

There is also the related set

$$\begin{pmatrix} |\perp\rangle \\ |\top\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix} \begin{pmatrix} |e\rangle \\ |g\rangle \end{pmatrix}, \quad \begin{pmatrix} |e\rangle \\ |g\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix} \begin{pmatrix} |\perp\rangle \\ |\top\rangle \end{pmatrix}. \quad (6)$$

The spin system is regarded as the quantum system under observation. The role of the apparatus is taken on by the atom system. An interaction Hamiltonian is responsible for coupling the apparatus-atom group to the spin system with a coupling strength  $\alpha$ . It has the following form:

$$\hat{H}^{AS} = \alpha (|\perp\rangle\langle\perp| - |\top\rangle\langle\top|) \otimes (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|). \quad (7)$$

In terms of a Pauli matrix  $\sigma_3$ , Equation (7) can be written as

$$\hat{H}^{AS} = \alpha (|\perp\rangle\langle\perp| - |\top\rangle\langle\top|) \sigma_3 \otimes (|\uparrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|) \sigma_3. \quad (8)$$

With respect to the bases which define Equation (8), it can be written as

$$\hat{H}^{AS} = \alpha \begin{pmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{pmatrix} \quad (9)$$

The upper block works on  $|\perp\rangle$  and the lower on  $|\top\rangle$  such that  $\sigma_3$  operates on the spin part.

The evolution of a state vector expressed in terms of the basis states  $\{|\uparrow\rangle, |\downarrow\rangle, |\perp\rangle, |\top\rangle\}$  under the influence of  $\hat{H}^{AS}$  is determined by the evolution operator  $\hat{U}$ , defined by

$$\hat{U} = e^{i\hat{H}^{AS}t/\hbar} = \begin{pmatrix} e^{i\alpha\sigma_3 t/\hbar} & 0 \\ 0 & e^{-i\alpha\sigma_3 t/\hbar} \end{pmatrix}. \quad (10)$$

Let us investigate the effect of  $\hat{U}$  on a state such as an initial state defined as follows:

$$|\varphi_i\rangle = (a|\uparrow\rangle + b|\downarrow\rangle) \otimes |+\rangle. \quad (11)$$

It is to be evolved over a time interval  $[0, T]$  such that  $T > 0$ . Using Equations (3)–(6) we write Equation (11) in terms of the basis vectors that define  $\hat{H}^{AS}$ . It will be helpful to introduce a dimensionless time  $\tau = \alpha t/\hbar$  in the following:

$$\hat{U}|\varphi_i\rangle = \frac{1}{2}(1-i) \begin{pmatrix} ae^{i\tau} \\ be^{-i\tau} \end{pmatrix} \otimes |\perp\rangle + \frac{1}{2}(1+i) \begin{pmatrix} ae^{-i\tau} \\ be^{i\tau} \end{pmatrix} \otimes |\top\rangle. \quad (12)$$

Regarding the basis  $\{|e\rangle, |g\rangle\}$ , it can be clearly seen that  $|\varphi_i\rangle$  can be transformed into a correlated state

$$\begin{aligned}
\hat{U}|\varphi_i\rangle &= \frac{1}{2}(1-i)\begin{pmatrix} ae^{i\tau} \\ be^{-i\tau} \end{pmatrix} \otimes \frac{1}{\sqrt{2}}(|e\rangle + i|g\rangle) + \frac{1}{2}(1+i)\begin{pmatrix} ae^{-i\tau} \\ be^{i\tau} \end{pmatrix} \otimes \frac{1}{\sqrt{2}}(|e\rangle - i|g\rangle) \\
&= \frac{1}{2\sqrt{2}}\left[\left((1-i)\begin{pmatrix} ae^{-i\tau} \\ be^{i\tau} \end{pmatrix} + (1+i)\begin{pmatrix} ae^{-i\tau} \\ be^{i\tau} \end{pmatrix}\right)|e\rangle + \left((1+i)\begin{pmatrix} ae^{i\tau} \\ be^{-i\tau} \end{pmatrix} + (1-i)\begin{pmatrix} ae^{-i\tau} \\ be^{i\tau} \end{pmatrix}\right)|g\rangle\right] \\
&= \frac{1}{\sqrt{2}}\left[\left(\begin{pmatrix} a \\ b \end{pmatrix} \cos \tau + \begin{pmatrix} a \\ -b \end{pmatrix} \sin \tau\right)|e\rangle + \left(\begin{pmatrix} a \\ b \end{pmatrix} \cos \tau - \begin{pmatrix} a \\ b \end{pmatrix} \sin \tau\right)|g\rangle\right].
\end{aligned} \tag{13}$$

It can be stated that the interaction Hamiltonian  $\hat{H}^{AS}$  evolves the state over  $(0, \tau_1)$ , where  $\tau_1 = \pi/4$  transforms the initial product state in Equation (13) into a correlated state vector:

$$|\varphi_f\rangle = \hat{U}(\tau_1)|\varphi_i\rangle = a|\uparrow\rangle \otimes |e\rangle + b|\downarrow\rangle \otimes |g\rangle \tag{14}$$

This is still a pure state, and thus correlations between the system and apparatus have already been established. However, thus far the measurement could not have produced a definite outcome. First, the correlated apparatus-system state vector  $|\varphi_f\rangle$  in Equation (13) returns to the initial  $|\varphi_i\rangle$  if the same interaction continues for a further time  $t_2 = 3\pi/4$ :

$$\hat{U}\left(\frac{3}{4}\pi\right)|\varphi_i\rangle = -\frac{1}{\sqrt{2}}\begin{pmatrix} a \\ b \end{pmatrix} \otimes (|e\rangle + |g\rangle) = -\begin{pmatrix} a \\ b \end{pmatrix} \otimes |+\rangle. \tag{15}$$

The apparatus could not have decided at the instant characterized by Equation (14) which outcome of the measurement was the state  $|\uparrow\rangle$  and which was the state  $|\downarrow\rangle$ . If the initial direct product was to reemerge after  $\tau_1 + \tau_2 = \pi$ , then all outcomes of the measurement should have been present at  $\tau_1$ .

At the stage described by the state  $|\varphi_f\rangle$ , it is not yet determined which possible states are distinguished by the measurement of the system. We transform the state  $|\varphi_f\rangle$  to the basis  $\{|+\rangle, |-\rangle\}$  to observe that

$$|\varphi_f\rangle = a|\uparrow\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) + b|\downarrow\rangle \otimes \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) = \frac{1}{\sqrt{2}}\left[(a|\uparrow\rangle + b|\downarrow\rangle) \otimes |+\rangle + (a|\uparrow\rangle - b|\downarrow\rangle) \otimes |-\rangle\right]. \tag{16}$$

The states  $|+\rangle$  and  $|-\rangle$  for the atom system are correlated with the definite states of the spin state:

$$|S_1\rangle = a|\uparrow\rangle + b|\downarrow\rangle, \quad |S_2\rangle = a|\uparrow\rangle - b|\downarrow\rangle. \tag{17}$$

The two states  $|S_1\rangle$  and  $|S_2\rangle$  are distinct from  $|\uparrow\rangle$  and  $|\downarrow\rangle$ , which is the basis of Equation (14) as registered by the apparatus. When the spin state before the measurement is at its least certain, corresponding to  $a = b = 2^{-1/2}$ , the fixed correlated state vector using Equation (3) can be expressed as

$$|\varphi_f\rangle = \frac{1}{\sqrt{2}}(|\alpha\rangle \otimes |+\rangle + |\beta\rangle \otimes |-\rangle). \tag{18}$$

This can also be accomplished in many other equivalent ways.

The atom system at the stage of  $|\varphi_f\rangle$  in Equations (14) and (18) does not contain the information about the spin observable that was supposed to be recorded. It is not possible to claim that the measurement in the normally used sense has already happened. Although the argument following these equations applies directly to the two-state measurement interactions, it is possible to modify them. It is concluded that in a closed apparatus-system object which evolves unitarily, a reduction in the wave packet cannot be accomplished. Measurement is supposed to be a process which produces information. It is the transfer of information between the spin and atom systems that has taken place, and this information can be quantified. The pointer basis of the apparatus which eliminates ambiguity in the choice of the recorded variable has to be developed now [15].

## 2.2. Effect of the Environment

Consider the influence of the environment  $E$  consisting of  $N$  two-level atoms. Atom  $k$  has the Hilbert space spanned by the basis set  $\{|e_E\rangle_k, |g_E\rangle_k\}$ . Suppose that the self-Hamiltonians of the system is taken individually, and the interaction Hamiltonian between the atoms is zero. The only part of the Hamiltonian which remains is the apparatus–environment interaction  $H^{AE}$ , which separates as

$$\hat{H}^{AE} = \sum_k H_k^{AE}. \quad (19)$$

Suppose that the components  $\hat{H}_k^{AE}$  are assumed to have the form

$$\hat{H}_k^{AE} = g_k (|e\rangle\langle e| - |g\rangle\langle g|) \otimes (|e_E\rangle\langle e_E| - |g_E\rangle\langle g_E|)_k \prod_{j \neq k} \mathbf{1}_j. \quad (20)$$

The eigenstates have the special property that they are direct products. The components of the direct product belong to Hilbert spaces of the apparatus and the environment atoms, respectively. When the environment constructed this way interacts with the apparatus, superselection rules arise in a natural way. They make it impossible for the apparatus to be detected in a superposition of ground and excited states. Thus, let the interaction of the apparatus and environment start at  $t = 0$ . Before  $t = 0$ , no correlations with the environment exist. The combined system-apparatus-environment state vector would have the form

$$|\Psi(0)\rangle = |\varphi_f\rangle \prod_{k=1}^N \otimes [\alpha_k |e_E\rangle_k + \beta_k |g_E\rangle_k]. \quad (21)$$

The set of states  $|e\rangle, |g\rangle, |e_E\rangle_k, |g_E\rangle_k$  is the eigenstates of the interaction Hamiltonian. This is the one that acts on the combined system for  $t > 0$ . This allows the state  $|\Psi\rangle$  to be expressed at an arbitrary time  $t$  in the form

$$\begin{aligned} |\Psi(t)\rangle = & a |\uparrow\rangle \otimes |e\rangle \prod_{k=1}^N \otimes [\alpha_k e^{ig_k t/\hbar} |e_E\rangle_k + \beta_k e^{ig_k t/\hbar} |g_E\rangle_k] \\ & + b |\downarrow\rangle \otimes |g\rangle \prod_{k=1}^N \otimes [\alpha_k e^{ig_k t/\hbar} |e_E\rangle_k + \beta_k e^{ig_k t/\hbar} |g_E\rangle_k]. \end{aligned} \quad (22)$$

This follows along the same lines as the set in Equations (12) and (13) when using an evolution operator  $\hat{U}_k$  as a matrix exponential of the Hamiltonian, which breaks up into blocks of a similar exponential form. The transition between  $|\Psi(0)\rangle$  and  $|\Psi(t)\rangle$  establishes the correlation between the state of the apparatus and the state of the environment. The apparatus observable  $\hat{\Lambda}$ , which is most reliably recorded by the environment, is usually called the *pointer* observable. For this interaction,  $\hat{\Lambda}$  would have the following form, with  $\lambda_1 \neq \lambda_2$  being real:

$$\hat{\Lambda} = \lambda_1 |e\rangle\langle e| + \lambda_2 |g\rangle\langle g|, \quad (23)$$

It can be said that the pair of states  $\{|e\rangle, |g\rangle\}$  defines the pointer basis.

If the apparent reduction in the state vector is accomplished simultaneously, then the state of the apparatus-system object has to be described by the density matrix upon tracing over  $E$  while setting  $\tau = t/\hbar$ :

$$\begin{aligned}
\rho^{SA} = \text{Tr}_E |\Psi(t)\rangle\langle\Psi(t)| &= \text{Tr}_E \{ |a|^2 |\uparrow\rangle\langle\uparrow| \otimes |e\rangle\langle e| \cdot \prod_{k=1}^N [|\alpha_k|^2 + |\beta_k|^2] \\
&+ ab^* |\uparrow\rangle\langle\downarrow| \otimes |e\rangle\langle g| \prod_{k=1}^N [\alpha_k e^{ig_k\tau} |e_E\rangle_k + \beta_k e^{-ig_k\tau} |g_E\rangle_k] [\alpha_k^* e^{ig_k\tau} \langle e_E| + \beta_k^* e^{-ig_k\tau} \langle g_E|] \\
&+ a^* b |\downarrow\rangle\langle\uparrow| \otimes |g\rangle\langle e| \prod_{k=1}^N [\alpha_k e^{-ig_k\tau} |e_E\rangle_k + \beta_k e^{ig_k\tau} |g_E\rangle_k] [\alpha_k^* e^{-ig_k\tau} \langle e_E| + \beta_k^* e^{ig_k\tau} \langle g_E|] \\
&+ |b|^2 |\downarrow\rangle\langle\downarrow| \otimes |g\rangle\langle g| \prod_{k=1}^N [|\alpha_k|^2 + |\beta_k|^2] \}.
\end{aligned} \quad (24)$$

The trace of the second line for the example is

$$\begin{aligned}
&\text{Tr}_E \prod_{k=1}^N [\alpha_k e^{ig_k\tau} |e_E\rangle_k + \beta_k e^{-ig_k\tau} |g_E\rangle_k] [\alpha_k^* e^{ig_k\tau} \langle e_E| + \beta_k^* e^{-ig_k\tau} \langle g_E|] \\
&= \prod_{k=1}^N [|\alpha_k|^2 e^{2ig_k\tau} + |\beta_k|^2 e^{-2ig_k\tau}] = \prod_{k=1}^N [\cos(2g_k\tau) + i(|\alpha_k|^2 - |\beta_k|^2) \sin(2g_k\tau)].
\end{aligned} \quad (25)$$

To summarize, it has been shown that

$$\begin{aligned}
\rho^{SA} = \text{Tr}_E |\Psi(t)\rangle\langle\Psi(t)| &= |a|^2 |\uparrow\rangle\langle\uparrow| \otimes |e\rangle\langle e| + \mu(t) ab^* |\uparrow\rangle\langle\downarrow| \otimes |e\rangle\langle g| \\
&+ \mu^*(t) a^* b |\downarrow\rangle\langle\uparrow| \otimes |g\rangle\langle e| + |b|^2 |\downarrow\rangle\langle\downarrow| \otimes |g\rangle\langle g|.
\end{aligned} \quad (26)$$

In Equation (26), the function  $\mu(t)$  is the correlation amplitude

$$\mu(t) = \prod_{k=1}^N [\cos(2g_k\tau) + (|\alpha_k|^2 + |\beta_k|^2) \sin(2g_k\tau)]. \quad (27)$$

The quantity in Equation (27) depends on the initial conditions of the environment via the various probabilities of finding the system in one of the eigenstates of the interaction Hamiltonian:

$$p(|e_E\rangle_k) = |\alpha_k|^2, \quad p(|g_E\rangle_k) = |\beta_k|^2. \quad (28)$$

The property in Equation (28) is important. It gives an indication that the ability of  $\mu(t)$  to dampen correlations is the same for a mixture where only Equation (28) may be given.

In fact, the correlation amplitude  $\mu(t)$  can also be found from the scalar product

$$\mu(t) = \langle E_e(t) | E_g(t) \rangle. \quad (29)$$

The two states in Equation (29) are defined to be

$$|E_e(t)\rangle = \prod_{k=1}^N [\alpha_k e^{ig_k\tau} |e_E\rangle_k + \beta_k e^{-ig_k\tau} |g_E\rangle_k], \quad |E_g(t)\rangle = \prod_{k=1}^N [\alpha_k e^{-ig_k\tau} |e_E\rangle_k + \beta_k e^{ig_k\tau} |g_E\rangle_k]. \quad (30)$$

The two distinct records made by the environment of two alternative outcomes of the measurement are represented by Equation (30). The time dependence of  $\mu(t)$  is very important to successful damping of the off-diagonal correlation terms. It is clear that  $|\mu(t)|^2 \leq 1$ . In addition,  $\mu(0) = 1$  and

$$\langle \mu(t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \mu(t) dt = 0, \quad \langle |\mu(t)|^2 \rangle = 2^{-N} \prod_{k=1}^N (1 + (|\alpha_k|^2 - |\beta_k|^2)^2). \quad (31)$$

The last result in Equation (31) implies that unless the initial state of the environment coincides with one of the eigenstates of the Hamiltonian, the expected absolute value



of the correlation amplitude  $|\mu(t)|^2$  is much less than the initial value. Relatively small environments are quite effective in giving rise to an exact definition of the pointer variable.

As long as  $N$  is finite, a theorem from the theory of periodic functions implies that the absolute value of  $\mu(t)$  will return arbitrarily close to one. An almost periodic function attains any value in its range infinitely many times. There is a close analogy between the problem of recurring correlations exemplified by the existence of translation numbers  $T_\epsilon$  requiring  $1 - |\mu(t)|^2 < \epsilon$  for a given  $\epsilon$  at both  $t = 0$  and  $T_\epsilon$  but not in  $(0, T_\epsilon)$ .

This example shows how the interaction of the apparatus and the environment can cause an effective reduction in the state vector. Correlations established between the apparatus and the environment have taken place at the expense of the previously attained correlations between the apparatus and the system. Putting the final density matrix in a form such that the apparatus contains information about some arbitrary two states of the spin system is not possible when all off-diagonal terms in  $\rho^{AS}$  vanish, even when  $a = b = 2^{-1/2}$ . It is important to stress that through the interaction of the apparatus with the environment, both the apparent reduction in the pure state density matrix into a mixture and the determination of the observable recorded by the apparatus are achieved simultaneously. The dual role of the environment is therefore equivalent to imposition of superselection rules. In this way, they make a quite natural appearance [7].

### 3. The Pointer Basis

Observers who look at the pointer of the ideal apparatus are made aware that the system is in one of the eigenstates of the observable and not in some relative state chosen arbitrarily. Quantum mechanics alone, when applied to a composite made up of an apparatus and a system, cannot in principle determine which observable has been measured, as we have seen. It will become clear that the choice of what has been measured comes about when one realizes two things. First the apparatus interacts with its environment by means of a specific interaction Hamiltonian  $\hat{H}_{AE}$ . Secondly, the observer consults only the pointer of the apparatus and the state of the environment.

The apparatus–environment interaction may be regarded as an additional measurement in its own right which can establish measurable correlations between the apparatus and the environment. Information about the environment destroys the information about the premeasured quantum system  $S$ . Commutation of the Hamiltonian  $\hat{H}_{AE}$  with the observable of the apparatus  $\hat{I}$  ensures this variable will not be perturbed. Only the basis made up of the eigenstates of the operator  $\hat{I}$ , called the pointer basis, contains nothing but the information about the quantum system itself. The combined apparatus–system object is now represented by a mixture which is diagonal on a product basis consisting of the eigenvectors of the pointer basis of the apparatus and the corresponding relative states of the system. In fact, the pointer basis of the apparatus is chosen by the form of the apparatus–environment interaction. This is the basis which contains a reliable record of the states of the system. This has to determine the set of relative states of the system correlated with the apparatus in a unique way. In addition, the apparatus–environment correlations prevent the observation of the  $AS$  system combination in a superposition. It becomes a diagonal mixture in the basis assembled from the pointer basis eigenstates and the corresponding relative states of the system. The exact details of the environment state itself are for obtaining the pointer basis; the form of the apparatus–environment interaction suffices for that.

If the three systems are described by a combined density matrix  $\hat{\rho}_{SAE}$  of these systems, then the density matrix is the solution to the evolution

$$-i\hbar\dot{\hat{\rho}}_{SAE} = [\hat{\rho}_{SAE}, \hat{H}_S + \hat{H}_A + \hat{H}_E + \hat{H}_{SA} + \hat{H}_{AE} + \hat{H}_{SE}]. \quad (32)$$

To obtain Equation (32), it has been assumed that all the interactions are pairwise such that  $\hat{H}_{SAE} = 0$ , and the environment can be considered a quantum system. The last point should be clarified as to what is meant by the term environment; that is, which



degrees of freedom must be considered in determining the mixture to which the wave function collapses.

The environment is defined as being made up of all those degrees of freedom which contribute greatly to the evolution of the state of the apparatus. If it is agreed that the environment may in principle be regarded as isolated, then a basis  $|E\rangle$  spanning its Hilbert space can be introduced. It should be possible to formulate a criterion that excludes those degrees of freedom whose total contribution to the total apparatus–environment interaction may be ignored.

It can also be assumed that the quantum system itself remains isolated from the environment. If this is violated after the premeasurement has occurred, then the apparatus will contain the information about which state the quantum system was but is not necessarily anymore. Suppose that  $\hat{H}_{SA}$  acts only for a very short time, during which  $\hat{H}_{SA}$  dominates  $\hat{H}_{AE}$  and a correlation of the form  $|A_0\rangle \otimes |\psi\rangle$  is established, where  $|\psi\rangle$  pertains to  $S$ . After this, the interaction between the system and apparatus is nonexistent. All the vectors of the pointer basis correspond to a common degenerate energy eigenstate  $\hat{H}_A|A_p\rangle = E|A_p\rangle$ , where the eigenvalue does not depend on  $p$ . Physically, this is the case where no energy is exchanged between the system and the apparatus.

Right after the correlation between the system and apparatus has been established, the density matrix for the  $SAE$  combination evolves as follows:

$$-i\hbar\dot{\hat{\rho}}_{SAE} = [\hat{\rho}_{SAE}, \hat{H}_S + \hat{H}_A + \hat{H}_E + \hat{H}_{AE}] = [\hat{\rho}_{SAE}, \hat{H}_S + \hat{H}_A + \hat{H}_E] + [\hat{\rho}_{SAE}, \hat{H}_{AE}]. \quad (33)$$

The first commutator bracket can be ignored, which follows as the time evolution of the states  $|A_p(t)\rangle$  leaves the diagonal entries of the density matrix invariant. Therefore, evolution of the apparatus due to  $\hat{H}_A$  does not destroy information about the system.

The second commutator in Equation (33) introduces correlations between the apparatus and the environment. The diagonal terms of the density matrix remain left-invariant only if they commute with the projection operators that appear on the diagonal entries. This means that if the states  $|A_p\rangle$  are to remain correlated with the relative state of the quantum system, then the operator  $\hat{H}_{AS}$  must satisfy the commutation relation

$$[\hat{H}_{AE}, \sum_p \gamma_p |A_p\rangle\langle A_p|] = 0, \quad (34)$$

for any choice of coefficients  $\gamma_p$ . Now define the pointer observable for a real  $\gamma_p$  as

$$\hat{\Pi} = \sum_p \gamma_p |A_p\rangle\langle A_p|. \quad (35)$$

Thus, Equation (35) can be rewritten by stating that the pointer basis  $\{|A_p\rangle\}$  is a complete set of eigenfunctions of the operator  $\hat{\Pi}$  that commutes with the pointer Hamiltonian  $\hat{H}_{AE}$ :

$$[\hat{\Pi}, \hat{H}_{AE}] = 0. \quad (36)$$

The interaction Hamiltonian then depends only on one apparatus observable,  $\hat{\Pi}$ , and thus any interaction Hamiltonian which has the form

$$\hat{H}_{AE} = \sum_p \sum_{\sigma > \eta} |A_p\rangle\langle A_p| \otimes (\zeta_{\sigma\eta}^{(p)} |\eta\rangle\langle\sigma| + \zeta_{\eta\sigma}^{(p)} |\sigma\rangle\langle\eta|) \quad (37)$$

does have a form that satisfies Equation (36). Additionally, the states and  $\zeta_{\eta\sigma}^{(p)}$  in Equation (37) may depend explicitly on the time due to the interaction with the remote environment. If the interaction remains diagonal in the pointer basis, then it will not disturb correlations of the apparatus with the states of the system relative to the pointer basis.

It is then the environment–apparatus interaction that allows for the existence of the pointer basis. However, this is not sufficient for the actual successful functioning of the apparatus. The action that actually correlates the quantum state of the apparatus with

that of the system state plays an important role. Immediately after the premeasurement, the  $AS$  wave function should appear as  $\sum_p b_p |A_p\rangle \otimes |p\rangle$ , where  $p$  is an orthonormal basis composed of the eigenstates of the particular variable  $\hat{P}$ . Upon measurement, the measured system should transform into one of the mutually orthonormal eigenstates of the operator  $\hat{P}$ .

It was intended to show that when the environment, thought of as taking the additional role of an apparatus, is taken into account, a definite answer can be provided to the question of what mixture the wave packet transforms into. To describe the given world, there ought to be two distinguishable types of evolution. There is the reversible, deterministic kind, which was confirmed in 2, as well as the irreversible random one, which must be the source of what is experienced by consciousness.

#### 4. Environment-Induced Selection

The interaction Hamiltonian which couples the system  $S$  to the environment  $E$  may commute with subspaces of the Hilbert space of the system. As a result of this interaction, the state vector of the system can remain pure only if it is entirely limited to one of these subspaces  $\mathcal{H}_n$ . Arbitrary superpositions with components spanning two or more subspaces decay into mixtures which are diagonal in the state vectors belonging to the individual disjoint subspaces. The decay originates in the establishment of correlations between the quantum system  $S$  and its environment  $E$ . Moreover, as long as the environment-system coupling remains stronger than the coupling introduced by the observer conducting a measurement, the set of the observables that can be measured on  $S$  is limited to the one that leaves the subspaces invariant.

Systems which forbid the existence of groups of pure states and restrict the possible observables in a way such as the one discussed here are said to admit superselection rules. Consequently, the idea is to see how interaction with the environment can impose such rules on  $S$ . The superselection rules, once in place, make the system behave classically. Environment-induced superselection rules can be used to justify the classical nature of the apparatus reading but also apply to an even greater class of classical observables of systems which are inherently quantum.

Suppose that the combined Hilbert space of system  $S$  and environment  $E$  is of the form

$$\mathcal{H}_C = \mathcal{H}_S \otimes \mathcal{H}_E. \quad (38)$$

The evolution is given by a Hermitean operator defined on the Hilbert space. Let us suppose it breaks up into a self-Hamiltonian of the system  $\hat{H}_S$  of the environment  $\hat{H}_E$  and an interaction Hamiltonian  $\hat{H}_{SE}$ , written as

$$\begin{aligned} \hat{H}_S + \hat{H}_E + \hat{H}_{SE} = & \sum_i \chi_i |s_i\rangle \langle s_i| + \sum_j \varepsilon_j |e_j\rangle \langle e_j| + \sum_{i,j} \gamma_{ij} |s_i\rangle \langle s_i| \otimes |e_j\rangle \langle e_j| \\ & + \lambda \sum_{i,j,i',j'} \sigma_{ii',jj'} |s_i\rangle \langle s_{i'}| \otimes |e_j\rangle \langle e_{j'}|. \end{aligned} \quad (39)$$

When only the evolution of the diagonal part of the interaction Hamiltonian is considered, it is written as  $\hat{H}_{SE}^0$  and given by

$$\hat{H}_{SE}^0 = \sum_{i < j} \gamma_{ij} |s_i\rangle \langle s_i| \otimes |e_j\rangle \langle e_j|. \quad (40)$$

Setting  $\lambda = 0$  in Equation (39) is equivalent to this case and is somewhat of an idealization, where the diagonal  $\hat{H}_{SE}^0$  is much greater than the off-diagonal part of the interaction. Both  $\hat{H}_S$  and  $\hat{H}_{SE}^0$  are likely to be highly degenerate, which leaves additional freedom in the choice of the basis. Physically,  $\lambda \ll 1$  is equivalent to stating zero interactions in real-world physical systems destroy phase coherence between the system states on a time scale much shorter than the time scale of relaxation to thermal equilibrium.

Evolution of the combined system-environment state vector, which at  $t = 0$  was represented by a direct product state

$$|\Phi(0)\rangle = |\varphi_S\rangle \otimes |\psi_E\rangle = \sum_i \alpha_i |s_i\rangle \otimes \sum_j \beta_j |e_j\rangle. \quad (41)$$

is evolved by a unitary  $\hat{U}$  as demonstrated in Section 2.1, and it gives

$$|\Phi(t)\rangle = \sum_{i,j} \alpha_i \beta_j \exp[-i(\chi_i + \varepsilon_j + \gamma_{ij})t/\hbar] |s_i\rangle \otimes |e_j\rangle. \quad (42)$$

To provide an idea as to exactly how the superselection rules are manifested, we trace the density matrix over  $E$ :

$$\rho^S(t) = \text{Tr}_E |\Phi(t)\rangle \langle \Phi(t)|. \quad (43)$$

The matrix elements of  $\rho^S(t)$  given in the representation  $\rho_{ij}$  are

$$\rho^S(t) = \sum_{i,j} \rho_{ij}(t) |s_i\rangle \langle s_j|. \quad (44)$$

In Equation (44), the  $\rho_{ij}(t)$  expressions are given as

$$\begin{aligned} \rho_{ii}(t) &= |\alpha_i|^2 \sum_k |\beta_k|^2 = |\alpha_k|^2, \\ \rho_{ij}(t) &= \alpha_i \alpha_j^* \exp(-i(\chi_i - \chi_j)t/\hbar) \cdot \sum_m |\beta_m|^2 \exp(-i(\gamma_{im} - \gamma_{jm})t/\hbar). \end{aligned} \quad (45)$$

The diagonal elements are time-independent when the off-diagonal part of the perturbing Hamiltonian is not present.

They can rotate on account of the factor  $\exp(-i(\chi_i - \chi_j)t/\hbar)$ , or more importantly, they can decay as a result of a decrease in the correlation amplitude:

$$\mu_{ij}(t) = \sum_k |\beta_k|^2 \exp(-i(\gamma_{ik} - \gamma_{jk})t/\hbar). \quad (46)$$

The net result of this second type of time dependence is to lower the absolute value of the correlation amplitude from one at time zero to a value much less than this for large  $t$  values. The average of the correlation amplitude determined over a sufficiently long time interval approaches zero ( $\langle \mu_{ij} \rangle_T \rightarrow 0$ ) as  $T \rightarrow \infty$  unless the frequencies  $\omega_m^{ij} = \gamma_{im} - \gamma_{jm}$  are equal to zero.

Demanding that all  $\omega_m^{ij} = 0$  would be equivalent to the statement that the interaction Hamiltonian  $H_{SE}^0$  has a diagonal part of zero. Let the correlation amplitude be expressed by

$$\mu_{ij}(t) = \sum_k p_k \exp(-i\omega_m^{ij}t/\hbar), \quad p_k = |\beta_k|^2. \quad (47)$$

When the environment is a mixture before the interaction with the system,  $\{p_k\}$  gives the probabilities for finding the environment in the states corresponding to the distinct eigenvalues of  $\hat{H}_{SE}^0$ . Then, Equation (47) stays valid regardless of whether  $E$  is initially in a pure state or in a mixed state. Since  $\mu_{ij}(t)$  is given by Equation (46), the average absolute value is computed as follows:

$$\langle |\mu_{ij}(t)|^2 \rangle = \frac{1}{T} \int_0^T |\mu_{ij}(t)|^2 dt \rightarrow \sum_{k,m} p_k p_m \delta(\omega_k^{ij} - \omega_m^{ij}). \quad (48)$$

Assuming that all  $\omega_k^{ij}$  values are distinct, the standard derivation of the correlation amplitude from the average value is given by  $\sum_{k=1}^N p_k^2$ . Hence, environments can cause correlations to damp out between those states of the system which diagonalize  $\hat{H}_{SE}^0$ .

It has been proven that the environment will remove the correlations between states which correspond to different eigenvalues of  $\hat{H}_{SE}^0$ . Many eigenvectors may correspond to the same eigenvalues  $\gamma_{ij}$  of  $\hat{H}_{SE}^0$ , and they span a subspace  $\mathcal{H}_n$  of the system's Hilbert space. The entire Hilbert space of the system can be reconstructed from the individual subspaces. Pure states which belong to more than one subspace  $\mathcal{H}_n$  at a single instant are not admitted. This is the fundamental source of environment-induced superselection rules, and  $\mathcal{H}_S$  is a direct sum of these basic subspaces  $\oplus_n \mathcal{H}_n$ , with all the pure states in one and only one  $\mathcal{H}_n$ . As long as the coupling with some external apparatus is not in too much excess of  $\gamma$ , the system may not be prepared as measured in the state which does not remain invariant under the influence of the interaction with the environment. Only those observables which leave every  $\mathcal{H}_n$  invariant are admitted. Thus,  $\hat{B}$  is an observable with respect to a system  $S$  in interaction with the environment  $E$  if and only if  $|\psi_n\rangle \in \mathcal{H}_n$  implies that

$$\hat{B} |\psi_n\rangle \in \mathcal{H}_n. \quad (49)$$

These two conditions can be thought of as equivalent to a more formal definition of the superselection rules.

The pointer observable  $\hat{\Lambda}$  can now be defined as any observable measurement, which allows us to precisely determine the subspace  $\mathcal{H}_n$  which contains the state of the system. If  $\pi_n$  represents a set of projection operators which project onto subspaces  $\mathcal{H}_n$  and  $\lambda_n$ , and they are all real and distinct, then by the spectral theorem, the pointer observable can be expressed as

$$\hat{\Lambda} = \sum_n \lambda_n \pi_n. \quad (50)$$

The projection operators can be constructed so that they are diagonal in the basis  $|s_k\rangle$ , which diagonalizes  $\hat{H}_{SE}^0$ , and  $\hat{\Lambda}$  commutes with  $\hat{H}_{SE}^0$ :

$$[\hat{\Lambda}, \hat{H}_{SE}^0] = 0. \quad (51)$$

## 5. Envariance and Statistical Mechanics

Traditionally, thermodynamic equilibrium states are characterized by extremes of physical properties, such as maximal thermodynamic entropy or randomness. The microcanonical equilibrium can be defined as the quantum state that is maximally envariant. This means it is envariant under all unitary operators on system  $S$ . Through a theorem, a composite state  $|\Psi_{SE}\rangle$  can be written in the form of a Schmidt decomposition as follows:

$$|\Psi_{SE}\rangle = \sum_k \alpha_k |s_k\rangle \otimes |\epsilon_k\rangle, \quad (52)$$

where  $\{|s_k\rangle\}$  and  $\{|\epsilon_k\rangle\}$  are orthocomplete sets in  $S$  and  $E$ , respectively. The procedure aims to identify the special state that is maximally envariant.

In fact,  $|\Psi_{SE}\rangle$  is invariant under all unitary operators if and only if the Schmidt decomposition is even so that the coefficients satisfy  $|\alpha_m| = |\alpha_n|$  for all  $m, n$ , and thus  $|\Psi_{SE}\rangle$  can be expressed as

$$|\Psi_{SE}\rangle = C \sum_k e^{i\varphi_k} |s_k\rangle \otimes |\epsilon_k\rangle, \quad (53)$$

and  $\varphi_k$  represents the phases. The classical concept of equilibrium ensembles translates into an equilibrium state that is envariant under the maximal number; that is, all unitary operations are present.

The microcanonical state is usually identified as the state that is fully degenerate in terms of energy. We denote the Hamiltonian of the composite system by

$$H_{SE} = H \otimes I_E + I_S \otimes H_E. \quad (54)$$

The internal energy of  $S$  is given by the quantum mechanical average:

$$E = \langle \Psi_{SE} | H \otimes H_E | \Psi_{SE} \rangle = \frac{1}{Z} \sum_k \langle s_k | H | s_k \rangle. \quad (55)$$

In Equation (55),  $Z$  is the energy-dependent dimension of the Hilbert space of  $S$ . This is often called the microcanonical partition function. Since  $|\Psi_{SE}\rangle$  is envariant under all unitary maps, let us suppose, without loss of generality, that  $\{s_k\}$  represents the energy eigenbasis corresponding to  $H$ . Then, the matrix elements are given by  $\langle s_k | H | s_k \rangle = e_k$ , where  $E = e_k = e_m$  for all  $k, m \in \{1, \dots, Z\}$ . The fully quantum representation of the microcanonical state has been identified by using two conditions. The microcanonical equilibrium is not represented by a unique state but by an equivalence class of maximally envariant states all with the same energy. Thus, the microcanonical equilibrium of a system  $S$  is a state that is envariant under all unitary operations on  $S$ . It is fully degenerate in energy with respect to  $H$ . In this approach, only this quantum symmetry is required, and it is induced at a very basic level by entanglement or envariance.

The canonical state can also be determined this way. Suppose that the total system  $S$  can be separated into a smaller subsystem  $\mathcal{S}$  and its complement: a heat bath  $B$ . The Hamiltonian of  $S$  is

$$H = H_S \otimes I_B + I_S \otimes H_B + H_{SB}, \quad (56)$$

where the contribution  $H_{SB}$  is an interaction term which makes possible the energy exchange between  $\mathcal{S}$  and  $B$ . Suppose it is sufficiently small that its contribution to the total energy can be neglected ( $E = E_S + E_B$ ), and its effect on the composite equilibrium state can be neglected. This is usually called ultra-weak coupling in classical terms. In this case, every composite energy eigenstate  $|s_k\rangle$  can be written as a product

$$|s_k\rangle = |\hat{s}_k\rangle \otimes |b_k\rangle. \quad (57)$$

In (57)  $|\hat{s}_k\rangle$  and  $|b_k\rangle$  are energy eigenstates in  $\mathcal{S}$  and  $B$ , respectively. Now, all orthonormal bases are equivalent under envariance such that  $|s_k\rangle$  may be chosen as energy eigenstates of  $H$ .

For the canonical formalism, it is the number of states accessible to the total system  $S$  under the condition that the total internal energy  $E$  is constant. When  $\mathcal{S}$  happens to be in a particular energy eigenstate  $|\hat{s}_k\rangle$ , the internal energy of the subsystem is given by the corresponding energy eigenvalue  $\hat{e}_k$ . For the total energy  $E$  to be constant, the energy of the heat bath  $E_B$  has to obey

$$E_B = E - \hat{e}_k. \quad (58)$$

To satisfy this, the energy spectrum of the heat reservoir must be at least as dense as that of the subsystem. The number of states  $\mathcal{N}$  accessible to system  $\mathcal{S}$  is then

$$\mathcal{N}(\hat{e}_k) = \frac{\mathcal{N}_B(E - \hat{e}_k)}{\mathcal{N}_S(E)}. \quad (59)$$

In Equation (59),  $\mathcal{N}(E)$  is the total number of states in  $\mathcal{S}$  consistent with Equation (55), and the numerator is the number of states available to bath  $B$  as determined by Equation (58).

Suppose that  $B$  consists of  $N$  non-interacting subsystems with identical eigenvalue spectra  $\{e_j^B\}_{j=1}^m$ . The initial energy (Equation (58)) is

$$E = \hat{e}_k + n_1 e_1^B + n_2 e_2^B + \dots + n_m e_m^B, \quad \sum_{j=1}^m e_j = N. \quad (60)$$

The degeneracy takes the form of

$$\mathcal{N}(\hat{e}_k) = \frac{N!}{n_1! n_2! \dots n_m!}. \quad (61)$$

This is a quantum envariant formulation of Boltzmann's counting formula for the number of energy states involved.

To obtain the Boltzmann–Gibbs result, consider the limit  $N \gg 1$ . Then, under Stirling's formula, we have

$$\log(\mathcal{N}(\hat{e}_k)) \doteq N \log(N) - \sum_{j=1}^n n_j \log(n_j).$$

In the case of microcanonical equilibrium, this is satisfied by the state that is envariant under all unitary maps. Let us identify canonical equilibrium with the configuration of the reservoir  $B$  for which the maximal number of energy eigenvalues is occupied, subject to the constraints

$$\sum_{j=1}^m n_j = N, \quad E - \hat{e}_k = \sum_{j=1}^m n_j e_j^B. \quad (62)$$

When introducing the Lagrange multipliers  $\tau$  and  $\beta$ , it is required that

$$\delta \left( \sum_j n_j \log(n_j) + \mu \sum_j n_j + \beta \sum_j n_j e_j^B \right) = 0. \quad (63)$$

The solution to this variational constraint is the Boltzmann–Gibbs relation:

$$n_j = c e^{\beta e_j^B}. \quad (64)$$

This represents the number of states in bath  $B$  with energy  $e_j^B$  for  $S$  and  $B$  to be in thermodynamic canonical equilibrium. The temperature enters through the Lagrange multiplier  $\beta$ . This result is exact up to the use of one approximation and depends only on the fact that the total  $S$  be in microcanonical equilibrium, as defined already by using the envariance concept.

## 6. Conclusions

The interaction of the environment with the system generates a correlation similar to that between the system and the apparatus. The environment can then be thought of as a higher-order apparatus which performs a zero measurement on the state of the system, destroying coherent superpositions. The eigenbasis of the pointer observable which is determined up to the coherent degeneracy of  $\hat{\Lambda}$  is called the pointer basis.

When the eigenvalues of the self-Hamiltonian are highly degenerate, the eigenspaces  $\mathcal{H}_{\tau_i}$  which correspond to the distinct eigenvalues  $\tau_i$  may contain, be identical with or even be a subset of the eigenspaces of the observable  $\hat{\Lambda}$ ,  $\mathcal{H}_n$ . These possibilities are exhaustive as long as  $[\mathcal{H}_S, \hat{H}_{SE}^0] = 0$ . This follows from the commutation relation  $[\hat{\Lambda}, \hat{H}_{SE}^0] = 0$ . In the first instance, the interaction with the environment will remove part of the degeneracy in the spectrum of  $\hat{H}_S$ , similar to the splitting of levels observed in atomic physics. The energy levels of the system are shifted in the second case. In the last one, the state may rotate under the influence of the self-Hamiltonian in  $\mathcal{H}_n$  without loss of coherence. What is most important in this is the redundancy of the record concerning the observable which is imprinted on the correlations. The interaction with the environment forces the system to be in one particular eigenstate of the pointer observable rather than in a superposition of such states. Thus, the super selection rules need not be imposed from outside. This fact gives rise to rules which are induced by the environment [16,17]. Finally, envariance offers the possibility of establishing thermodynamics on a well-defined set of ideas originating at a fundamental quantum level.

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