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Twisting it up the Quantum Way

On Matrix Models, q-deformations and Supersymmetric Gauge Theories

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Abstract

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The mathematical framework which quantum field theory constitutes has been very successful in describing nature. As an extension of such a framework, the idea of supersymmetry was introduced. This greatly simplified the mathematical description of the theories, making them more tractable. Recently, the method of supersymmetric localisation, in which one can compute infinite dimensional integrals exactly, enabled computations of partition functions for different supersymmetric gauge theories in various dimensions. Such partition functions sometimes resulted in the form of matrix models or even q-deformed matrix models, where the latter are not very well-studied. Classical, or un-deformed, matrix models on the other hand are studied in much greater detail. One particular tool that is used in the study of classical matrix models is the Ward identities called Virasoro constraints. Motivated by firstly the desire to understand q-deformed matrix models better and secondly the gauge theory applications of the results, we studied the derivation of and solution to such q-deformed Virasoro constraints. We also explored the implications of partition functions taking the form of q-deformed matrix models in the case of three and four dimensional supersymmetric gauge theories. Furthermore, we studied various generalisations of the classical matrix model, such as having different limits of integration and different potentials, in order to see how the Virasoro constraints and its solution changed. Finally, we made a connection with the area of integrability and investigated how classical matrix model satisfying the Virasoro constraints could be related to certain tau-functions satisfying the Hirota equations.

Keywords: Matrix models, Virasoro constraints, Virasoro algebra, W algebra, q-deformations, Supersymmetric gauge theories

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List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I R. Lodin, F. Nieri and M. Zabzine, *Elliptic modular double and 4d partition functions*, *J. Phys. A* **51** (2018) 045402, [1703.04614].

II R. Lodin, A. Popolitov, S. Shakirov and M. Zabzine, *Solving q-Virasoro constraints*, *Lett. Math. Phys.* **110** (2020) 179-210, [1810.00761].

III L. Cassia, R. Lodin, A. Popolitov and M. Zabzine, *Exact SUSY Wilson loops on S^3 from q-Virasoro constraints*, *JHEP* **12** (2019) 121, [1909.10352].

IV L. Cassia, R. Lodin and M. Zabzine, *On matrix models and their q-deformations*, *JHEP* **10** (2020) 126, [2007.10354].

V L. Cassia, R. Lodin and M. Zabzine, *Virasoro constraints revisited*, submitted to *Commun. Math. Phys.*, [2102.05682].

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Contents

1. Introduction

Quantum field theory (QFT) has been a very successful mathematical framework used in describing nature as it combines quantum mechanics with special relativity. The most prominent example of a QFT is the Standard Model which currently is the best description of the world around us. The Standard Model is formulated via gauge theories in which Lagrangians possess redundant degrees of freedom resulting in a so-called gauge symmetry. Such gauge theories were later generalised in order to include supersymmetry; a symmetry imposed between bosons and fermions through the addition of super-partners to the Standard Model. Supersymmetry was first introduced in the context of QFT's in [1, 2] to correct for problems such as the hierarchy problem in the Standard Model. This problem addressed the large discrepancy between the strength of the weak and the gravitational interactions. However, supersymmetry also has the benefit of making gauge theories more tractable and explicit computations easier. In supersymmetric gauge theories one is typically interested in computing observables of the theory, including partition functions and expectation values of Wilson loops. In the last ten years, the evaluation of partition functions in supersymmetric gauge theories has seen a great advancement. This is due to the development of a method called supersymmetric localisation, an area initiated by the work [3]. For a review on the method and a collection of various results in the area we refer to [4]. Sometimes such localisation computations of partition functions results in expressions known as matrix models and it is such models which will be of interest here. In order to motivate this interest, we can view our endeavours in the light of what is called the BPS/conformal field theory (CFT) correspondence [5–9]. In this correspondence, expectation values of certain protected BPS observables in supersymmetric gauge theories are shown to have an interpretation through correlation functions in two dimensional CFT's. In our case, the observables are typically partition functions expressed as matrix models and on the CFT side we find that we can generate the same type of object from a free field realisation of the Virasoro algebra.

Matrix models first arose in the context of statistical distributions [10–12] and later as examples of gauge and string theories which were exactly solvable [13]. In certain cases, the matrix models could alternatively be recast as eigenvalue models. The canonical example of a 1-matrix model is the Hermitean 1-matrix model depending on an infinite set of variables called time variables. In [14, 15] the Hermitean matrix model was shown to satisfy a set of Ward identities known as the Virasoro constraints. The name originates from the fact that the constraints are given in terms of (a subset of) the generators of the Virasoro algebra. In this sense, the Virasoro constraints provide a link between matrix models on the one hand and CFT's on the other hand. Conversely, one can also ask the question that if a model satisfies the Virasoro constraints, what does the model look like. This has been referred to as solving the Virasoro constraints. The answer to this question is typically given in the form of either the compact notation of the W-representation for the matrix model [16], or through explicitly determining the correlators (i.e. the coefficients when expanding the model in the time variables). In papers IV and V we explored such solutions to the Virasoro constraints.

In parallel with the developments within classical matrix models, there also existed the concept of q-deformations as a method of generalising functions and operations by introducing a complex deformation parameter q. One example where such deformations were investigated, was in the correspondence between the Calogero-Sutherland model, whose excited states are given using Jack polynomials, and CFT in the form of the Virasoro algebra [17–20]. Since there existed a q-deformation of the Jack polynomials given by the Macdonald polynomials [21], the authors of [22] wanted to see if this correspondence could be q-deformed. They then searched for and found, a q-deformation of the Virasoro algebra (as reviewed in [23]). Later, it was observed that also the matrix models could be q-deformed in [24]. Additionally, the q-deformed version of the Virasoro constraints could be considered, for instance as given in [25, 26]. This was the motivation for paper II. There we explored how the q-deformed Virasoro constraints could be derived by the insertion of a specific operator under the integral, mirroring the usual insertion of a derivative to obtain Ward identities. This q-deformation has sometimes been referred to as a trigonometric deformation.

Let us now come back to observables in gauge theories. As discussed above, the programme of supersymmetric localisation resulted in partition functions for some gauge theories taking the form of matrix models. There are both classical and q-deformed examples of such matrix models arising from localisation, but it is the q-deformed such examples that will be of main interest here. We then introduce a dependence on the time variables of the partition function, after which we refer to it as a generating function. These q-deformed matrix models or generating functions then satisfy q-Virasoro constraints. To be concrete, let us consider a supersymmetric gauge theory on the squashed three-sphere S_b^3 whose generating function is given in the form of a q-deformed matrix model. It can be observed on geometric grounds that S_b^3 can be obtained from gluing together two copies of $D^2 \times_q S^1$ in a particular way. What is more remarkable is that the S_b^3

partition function has been shown to respect such a decomposition into two partition functions on $D^2 \times_q S^1$, where the $D^2 \times_q S^1$ partition functions then has been referred to as half-index or 3d holomorphic blocks [27–29]. Inspired by this factorisation, the authors of [26] then investigated if this also holds at the level of the S_b^3 q-Virasoro constraints and found an affirmative answer. The generators of the q-Virasoro constraints for S_b^3 was then given by two commuting copies of the q-Virasoro algebra, which was given the name of the modular double. In paper III, we employed this modular double construction in order to obtain results for $\mathcal{N} = 2$ Yang-Mills Chern-Simons gauge theory on S_b^3 and in particular expectation values of supersymmetric Wilson loops.

The q-deformation introduced above can also be taken a step further, by introducing yet another deformation labelled by q'. This deformation has been referred to as the elliptic deformation, where the elliptic Virasoro algebra was introduced in [30]. As opposed to the q-deformed case described above, in the elliptic case we instead have the generating functions of gauge theories living in 4d. However, just as in the 3d case, it has been found that 4d gauge theory observables allow for a factorisation [28,31–33]. In paper I we explored how the modular double construction carried over to the elliptic case and also applied this construction to the study of 4d supersymmetric gauge theories on compact manifolds of the form $\mathcal{M}_3 \times S^1$. This was motivated by the existence of the elliptic deformation of the Virasoro algebra together with this factorisation of 4d observables. Finally, it is worth highlighting that the results of papers I, II, III and IV might appear centred around and motivated by their applications in gauge theories. However, the obtained results are also valid from a pure matrix model perspective, as they can be viewed as part of the search for a better understanding of the structure of matrix models. For instance, the results can be used to understand connections to integrable models and also provide applications within the area of special functions, such as the integral representation of Macdonald polynomials [34].

Another area which deals with solvable systems is the area of integrability, where with integrable system we in broad terms think of a system where all dynamical characteristics can be determined exactly. There, the object of interest is the τ-function which can be thought of as the generating function for correlation functions in a theory with free particles. One such example is when the particles are free fermions, where for a review we refer to for instance [35,36]. Then, the τ-functions satisfy a set of bilinear equations called the Hirota equations [37,38]. One particular class of τ-functions are classical matrix models (with $\beta = 1$) see for example [39,40], which therefore satisfy both Virasoro constraints and Hirota equations. However, the relation between the two sets of conditions and their corresponding solutions is not fully understood and the purpose of paper V was to explore this.

1.1 Outline of thesis

This thesis is divided into three parts. In the first part we review classical matrix models, where the standard example of the Hermitean 1-matrix model is introduced. We then explore the main tool accessible when studying matrix models, namely the Virasoro constraints. We review the derivation of these constraints together with recalling the details of the Virasoro algebra satisfied by the generators of the constraints. We then consider solutions of such matrix models, available to us via the Virasoro constraints. One way to solve such matrix models is by determining the correlators, or the coefficients when expanding the generating function in time variables. However, the solution can also be given through a compact form known as the W-representation. Finally, we close the first part of the thesis by considering matrix models which have a non-trivial boundary as discussed in paper V. In this case, the boundary generates contributions to the Virasoro constraints, rendering them non-homogeneous.

In the second part of the thesis we explore quantum matrix models. We begin with reviewing some concepts within q-calculus to then provide a summary of q-functions which are used throughout the thesis. Within this part, we investigate two deformations of the classical model. Firstly, the q- (or sometimes q, t-) deformation which has also been referred to as the trigonometric deformation. Here we start by giving the q-Virasoro algebra together with the q-deformed matrix model. Using the findings of paper II, we then solve the constraints which are now q-Virasoro constraints. As a second, and yet further, deformation we introduce the q, t, q'-deformation which has also been called the elliptic deformation. Similarly to before, we then introduce the elliptic Virasoro algebra and the elliptic matrix model.

Finally, in the third and last part we investigate how the knowledge from the classical and deformed matrix models can be applied to other areas. In the case of the deformed models, the applications are different kinds of supersymmetric gauge theories on various backgrounds. We therefore begin the part with reviewing how to obtain observables in such theories using the method of supersymmetric localisation, together with recalling some results which this method has produced. For the q, t-deformed model we find suitable applications in three dimensional supersymmetric gauge theories and to be precise on the backgrounds $D^2 \times_q S^1$ and S^3_b, as found in papers III and IV. On the other hand, for the q, t, q'-deformed models we instead find an application in four dimensional theories on compact backgrounds of the form $\mathcal{M}_3 \times S^1$, as explored in paper I. We then close this part by considering an application of the classical matrix models. More specifically, we explore the connection between classical Virasoro constraints and a concept from integrability, namely Hirota equations as investigated in paper V.

Part I:
Classical matrix models

In this first part we introduce classical matrix models and the tools available when studying them. For reviews on the subject we refer to [35, 41–43]. The reasons to why one considers matrix models are because they are typically easy to study and can serve as a playground for more elaborate theories. For instance, as we will see later, deformed versions of these classical matrix models appear in the partition functions of supersymmetric gauge theories. Consequently, understanding the classical theories at a more fundamental level can therefore help us in understanding such applications better. Matrix models early established their importance when they appeared in string theory and the description of two dimensional quantum gravity as the generating function for random triangulations of surfaces. This was useful as such summations could then replace the integral over all possible geometries when evaluating partition functions [13, 44–47]. (For a review see [48].) More recent applications of classical matrix models include for instance topological strings and applications within large N expansions as reviewed in [49].

2. Matrix models

Let us start the first chapter with exploring the world of matrix models. To begin with, we consider the matrix models themselves, both formulated using matrices and also using the eigenvalues of the matrices. We then review some details about integer partitions and certain special polynomials which satisfies some particularly nice properties related to formulas for expectation values. Finally, we introduce the notion of correlators of the matrix model.

2.1 Matrices and eigenvalues

The simplest example of a matrix model is the Hermitean 1-matrix model, as reviewed for instance in [35],

$$Z(t) = \int_{N\times N} [DM]\, \mathrm{e}^{-\mathrm{Tr}V(M)+\sum_{s=1}^{\infty} t_s \mathrm{Tr}(M^s)} \ , \tag{2.1}$$

where the complex function $V(M)$ is called the *potential* of the matrix model. Additionally, the model depends on an infinite set of parameters usually called *time variables* which we denote collectively by $t = \{t_1, t_2, \dots\}$. The integration in (2.1) is over all $N \times N$ Hermitean matrices M, such that $M = M^\dagger$, and the measure is [14]

$$[DM] = \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{1\le i<j\le N} \mathrm{d}\left(\mathrm{Re}M_{ij}\right) \mathrm{d}\left(\mathrm{Im}M_{ij}\right) \ . \tag{2.2}$$

The measure $[DM]$ is special in the sense that it is invariant under the adjoint action of the Lie group $U(N)$, under which $M \to U^\dagger M U$. This enables us to diagonalise the matrix, provided that also the potential $V(M)$ is invariant. In what follows we will therefore assume such an invariant potential. Consequently, one can rewrite the above using the eigenvalues of the matrix M denoted by $\{\lambda_1, \dots, \lambda_N\}$. More explicitly, a Hermitean matrix M can be written as

$$M = U^\dagger D U \tag{2.3}$$

for some unitary matrix U and its adjoint $U^\dagger$, and diagonal matrix D consisting of the eigenvalues of M, so that

$$\mathrm{Tr}(M^s) = \mathrm{Tr}(D^s) = \sum_{i=1}^{N} \lambda_i^s \ . \tag{2.4}$$

Then, in order to determine the measure $[DM]$ in terms of the eigenvalues, one can consider the square of the line element to find the Jacobian. Following [41] we find

$$\mathrm{Tr}\left[(\mathrm{d}M)^2\right] = \mathrm{Tr}\left[\left((\mathrm{d}U^\dagger)DU + U^\dagger(\mathrm{d}D)U + U^\dagger D(\mathrm{d}U)\right)^2\right] \ . \tag{2.5}$$

Expressing $\mathrm{d}U = \mathrm{i}\,(\mathrm{d}T)\,U$ for a Hermitean matrix T (such that unitarity $\mathrm{d}U\mathrm{d}U^\dagger = \mathbb{I}$ is preserved), we find

$$\begin{aligned}\mathrm{Tr}\left[(\mathrm{d}M)^2\right] &= \mathrm{Tr}\left[\left(U^\dagger\left(\mathrm{d}D + \mathrm{i}\,[D,\mathrm{d}T]\right)U\right)^2\right] = \\ &= \mathrm{Tr}\left[(\mathrm{d}D)^2\right] + \mathrm{Tr}\left[\left(-\left([D,\mathrm{d}T]\right)^2\right)\right] = \\ &= \sum_{i=1}^{N}(\mathrm{d}\lambda_i)^2 + \sum_{i,j=1}^{N}(\lambda_i-\lambda_j)^2\,|\mathrm{d}T_{ij}|^2 \ .\end{aligned} \tag{2.6}$$

Here, the second line follows from that D and $\mathrm{d}D$ commute and that the trace is cyclic and the second term in the last line follows from

$$\begin{aligned}-\mathrm{Tr}\left[\left([D,\mathrm{d}T]\right)^2\right] &= \sum_{i,j=1}^{N}[D,\mathrm{d}T]_{ij}\,[\mathrm{d}T,D]_{ji} = \\ &= \sum_{i,j=1}^{N}(\lambda_i-\lambda_j)\,|\mathrm{d}T|_{ij}\,(\lambda_i-\lambda_j)\,|\mathrm{d}T|_{ji}\end{aligned} \tag{2.7}$$

and that T is Hermitean. Thus, the determinant of the metric tensor becomes $\prod_{1\leq i\neq j\leq N}(\lambda_i-\lambda_j)^2$ and the Jacobian is then the (square of the) *Vandermonde determinant* $\Delta(\underline{\lambda})$, in other words

$$\Delta(\underline{\lambda})^2 = \prod_{1\leq i<j\leq N}(\lambda_i-\lambda_j)^2 \ . \tag{2.8}$$

Thus, the measure becomes

$$[DM] = \prod_{i=1}^{N}\mathrm{d}\lambda_i\,\Delta(\underline{\lambda})^2 \ . \tag{2.9}$$

Finally, as earlier mentioned, we assume that the potential $V(M)$ is invariant under the adjoint action. Thus

$$\mathrm{Tr}\left[V(M)\right] = \mathrm{Tr}\left[V(D)\right] = \sum_{i=1}^{N}V(\lambda_i) \ , \tag{2.10}$$

which is the case for instance for polynomial potentials as will be introduced later. Using this, one can recast the Hermitean matrix model in (2.1) as the *eigenvalue matrix model*

$$Z(t) = \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \Delta(\underline{\lambda})^2 \, \mathrm{e}^{-\sum_{i=1}^{N} V(\lambda_i) + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \ , \tag{2.11}$$

where all the N integrations are over the real line $\mathbb{R}$.

The potential $V(\lambda)$ introduced above is of the form

$$V(\lambda) = \sum_{k=1}^{\mathsf{p}} \frac{a_k}{k} \lambda^k \ , \tag{2.12}$$

where the parameters $\{a_k\}$ with $a_k \in \mathbb{C}$ sometimes are referred to as *coupling constants*. To clarify the dependence of the generating function on the coupling constants, we use the notation $Z(t; a)$ where $a = \{a_1, \ldots, a_{\mathsf{p}}\}$. Instead of introducing a potential $V(\lambda)$, an equivalent interpretation is to view the potential as being generated by shifting the first p times as

$$t_s \mapsto t_s - a_s/s, \qquad s = 1, \ldots, \mathsf{p} \ . \tag{2.13}$$

One example of the above matrix model is the case $\mathsf{p} = 1$, which is related to the complex 1-matrix model as studied in [50, 51] given by

$$\int_{N \times N} [DM] \, \mathrm{e}^{-\mathrm{Tr}\left[V(M, M^\dagger)\right] + \sum_{s=1}^{\infty} t_s \mathrm{Tr}\left[(MM^\dagger)^s\right]} \ , \tag{2.14}$$

where the integral is now over the space of all complex $N \times N$ matrices M. We then make the change of variables $\Phi = MM^\dagger$ and assume that the potential V has a single quadratic term. If we denote the eigenvalues of M by θ_i and the eigenvalues of Φ by λ_i, then the potential will be quadratic in θ_i but linear in $\lambda_i = |\theta_i|^2$. The integration over λ_i will then be over the positive real line. Using this, one can write this model in its most general form as

$$Z(t; a_1) = \int_{\mathbb{R}^N_{>0}} \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \Delta(\underline{\lambda})^2 \prod_{i=1}^{N} \lambda_i^\nu \, \mathrm{e}^{-a_1 \sum_{i=1}^{N} \lambda_i + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \tag{2.15}$$

which is the *Wishart-Laguerre eigenvalue model* reviewed for instance in [52]. Here, we have allowed for a term parametrised by ν where this term at the level of matrices M corresponds to the determinant insertion $\left(\det(MM^\dagger)\right)^\nu$. To have convergence of the integrals we require $\mathrm{Re}(\nu) > -1$. This term can also be introduced via modifying the potential in (2.12) as

$$V(\lambda) = -\delta_{\mathsf{p},1} \nu \ln(\lambda) + \sum_{k=1}^{\mathsf{p}} \frac{a_k}{k} \lambda^k \ . \tag{2.16}$$

Another well-known example of the matrix model given above with $\mathsf{p} = 2$ is the *Gaussian Hermitean matrix model*, which is obtained by a potential with coupling constants

$$a_k = \delta_{k,2} \tag{2.17}$$

or alternatively the shift in times

$$t_2 \mapsto t_2 - \frac{1}{2} \; . \tag{2.18}$$

In other words

$$Z(t; a_1 = 0, a_2 = 1) = \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \Delta(\underline{\lambda})^2 \mathrm{e}^{-\frac{1}{2}\sum_{i=1}^{N}\lambda_i^2 + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \; . \tag{2.19}$$

We will return to this example later in the context of the W-representations.

Finally, in order to generalise the above matrix model, one might introduce what is called the *β-deformation*. This is a 1-parameter deformation of the model in (2.11) in which the square of the Vandermonde instead takes the form

$$\Delta(\underline{\lambda})^2 = \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^2 \xrightarrow{\beta-\text{deformation}} \Delta(\underline{\lambda})^{2\beta} = \prod_{1\le i<j\le N} (\lambda_i - \lambda_j)^{2\beta} \tag{2.20}$$

for a deformation parameter β. For the model to be well-defined, one has to take $\beta \in \mathbb{Z}_{>0}$. However, from the point of view of the constraints that this model satisfies (to be discussed in Chapter 3) there are no restrictions on β, and we therefore analytically continue β to the complex plane. The β-deformed matrix model then becomes

$$Z_\beta(t; a) = \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \Delta(\underline{\lambda})^{2\beta} \mathrm{e}^{-\sum_{i=1}^{N} V(\lambda_i) + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \tag{2.21}$$

(where we use the restricted contours $\mathbb{R}^N_{>0}$ in the case of $\mathsf{p} = 1$ as discussed above). From now on we assume this β-deformation so in what follows we simply denote this matrix model by $Z(t; a)$ to ease notation. It should be noted here that the β-deformation is not on the same footing as the q-deformations that will be introduced later, in particular the β-deformation is considered a classical feature.

2.2 Partitions

To ease later discussions, we now introduce concepts related to integer partitions. An *integer partition* γ is denoted by $\gamma = \{\gamma_1, \dots, \gamma_k\}$, where $\gamma_1 \ge \cdots \ge \gamma_k > 0$ are positive integers where each $\gamma_i \in \gamma$ is called a *part* of γ. For a given partition γ one can define the following properties. Firstly, the *degree* (also called *weight*) of the partition γ, denoted by $|\gamma|$, is

$$|\gamma| = \sum_{a\in\gamma} a \; . \tag{2.22}$$

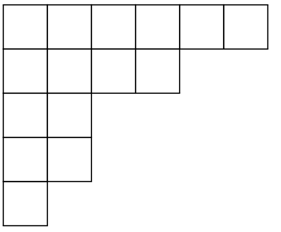

Figure 2.1. The Young diagram corresponding to the partition $\gamma = \{6, 4, 2, 2, 1\}$.

Secondly, the *length* of the partition, $l(\gamma)$, is given by

$$l(\gamma) = \sum_{a \in \gamma} 1 \ . \tag{2.23}$$

Thirdly, one can define $\#_\gamma j$ as the number of parts j in the partition γ,

$$\#_\gamma j = \sum_{a \in \gamma} \delta_{a,j} \ . \tag{2.24}$$

Finally, $|\mathrm{Aut}(\gamma)|$ is the *order of the automorphism group* of the partition

$$|\mathrm{Aut}(\gamma)| = \prod_{j=\gamma_k}^{\gamma_1} (\#_\gamma j)! \ , \tag{2.25}$$

or alternatively

$$|\mathrm{Aut}(\gamma)| = \prod_{a \in \gamma} \frac{\partial}{\partial t_a} \prod_{b \in \gamma} t_b \ . \tag{2.26}$$

Integer partitions such as γ above, can also be described using *Young diagrams*. These diagrams consist of left-aligned rows of boxes, where the number of boxes on each row is non-increasing as you move down the rows. The number of boxes on the i-th row, counting from the top of the diagram, is equal to the part γ_i of the partition γ. Thus the number of rows of boxes is equal to the length of the partition and the total number of boxes in the Young diagram is the degree of the partition. For instance, the partition $\gamma = \{6, 4, 2, 2, 1\}$ corresponds to the Young diagram illustrated in Figure 2.1. This partition has degree $|\gamma| = 15$ and length $l(\gamma) = 5$. The automorphism group of γ is then equivalent to the group of permutations of rows which has the same number of boxes, in other words $|\mathrm{Aut}(\gamma)| = 2$.

2.3 Special polynomials

Let us now review some properties of the multivariate Schur polynomials $\mathsf{Schur}_\gamma(\lambda_k)$, where for details we refer to [21]. Schur polynomials can be defined as irreducible characters of $U(N)$. For an irreducible representation $\mathcal{R}_\gamma$ and partition γ, then for a given group element Φ the *Schur polynomial* $\mathsf{Schur}_\gamma(\lambda_k)$ satisfies

$$\mathrm{ch}_{\mathcal{R}_\gamma}(\Phi) = \mathrm{Tr}_{\mathcal{R}_\gamma}(\Phi) = \mathsf{Schur}_\gamma(\lambda_k = \mathrm{Tr}(\Phi^k)) \ . \tag{2.27}$$

Consequently they form a basis for the space of all polynomial characters, which in turn can be extended to all symmetric functions using that characters are invariant under the Weyl group S_N.

The Schur polynomials are indexed by an integer partition γ and are symmetric functions of the set of variables $\lambda_k = \{\lambda_1, \ldots, \lambda_N\}$. They are homogeneous in degree, where with degree we mean $d = \sum_{k=1}^N \mu_k$ for a monomial $\lambda_1^{\mu_1} \ldots \lambda_N^{\mu_N}$. For a generic partition $\gamma = \{\gamma_1, \ldots, \gamma_N\}$, the general expression for a Schur polynomial in terms of determinants is

$$\mathsf{Schur}_\gamma(\lambda_k) = \frac{\det\limits_{1\le i,j\le N} \ \lambda_i^{N+\gamma_j-j}}{\det\limits_{1\le i,j\le N} \ \lambda_i^{N-j}} \ . \tag{2.28}$$

An alternative way of expressing the Schur polynomials is through

$$\mathsf{Schur}_\gamma(\lambda_k) = m_\gamma(\lambda_k) + \sum_{\mu<\gamma} K_{\gamma\mu} m_\mu(\lambda_k) \tag{2.29}$$

where $K_{\gamma\mu}$ are the *Kostka numbers* which are non-negative integer coefficients and $m_\mu(\lambda_k)$ are the *monomial symmetric functions* defined by

$$m_\mu(\lambda_k) = \sum_\alpha \lambda_1^{\alpha_1} \ldots \lambda_N^{\alpha_N} \ , \tag{2.30}$$

where the summation is over all distinct permutations $\alpha = \{\alpha_1, \ldots, \alpha_N\}$ of $\mu = \{\mu_1, \ldots, \mu_N\}$. Furthermore, we can define an inner product

$$(f(\underline{\lambda}) \,|\, g(\underline{\lambda})) = \frac{1}{(2\pi \mathrm{i})^N N!} \oint_{|\lambda|=1} \prod_{i=1}^N \frac{\mathrm{d}\lambda_i}{\lambda_i} \Delta(\underline{\lambda})\Delta(\underline{\lambda}^{-1}) f(\underline{\lambda}) g(\underline{\lambda}^{-1}) \ , \tag{2.31}$$

where we recall the Vandermonde determinant $\Delta(\underline{\lambda})$ in (2.8) and with $\underline{\lambda}^{-1}$ we mean the inverse of each argument $\{\lambda_1^{-1}, \ldots, \lambda_N^{-1}\}$. Then, with respect to this inner product the Schur polynomials are orthonormal, i.e.

$$(\mathsf{Schur}_\gamma(\lambda_k) \,|\, \mathsf{Schur}_\rho(\lambda_k)) = \delta_{\gamma,\rho} \ . \tag{2.32}$$

Let us now introduce another set of variables $\{p_k\}$, sometimes referred to as *power sum symmetric polynomials*. These typically encode traces of powers of $N \times N$ matrices M according to

$$p_k = \mathrm{Tr}(M^k) \ , \tag{2.33}$$

or alternatively using the eigenvalues $\{\lambda_1, \ldots, \lambda_N\}$ of the matrix M in other words the first set of variables,

$$p_k = \sum_{i=1}^{N} \lambda_i^k \ . \tag{2.34}$$

Using these power sum variables, the Schur polynomials then satisfy what is called the *Cauchy identity* [21],

$$\exp\left(\sum_{k=1}^{\infty} \frac{t_k p_k}{k}\right) = \sum_{\gamma} \mathsf{Schur}_{\gamma}\left(t_k\right) \mathsf{Schur}_{\gamma}\left(p_k\right) \ , \tag{2.35}$$

where the summation on the right hand side is over all partitions γ. The Cauchy identity takes a simpler form when one uses the plethystic substitution $t_k = z^k$. In particular, the summation on the right hand side then only contains symmetric partitions $\gamma = \{m\}$, in other words partitions of length 1. Using the formula for symmetric Schur polynomials

$$\mathsf{Schur}_{\{m\}}(p_1, \ldots, p_m) = \sum_{\{\gamma \text{ s.t. } |\gamma| = m\}} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{i=1}^{l(\gamma)} \frac{p_i}{i} \ , \tag{2.36}$$

then $\mathsf{Schur}_{\{m\}}(t_k = z^k) = z^m$ and the Cauchy identity in (2.35) becomes

$$\exp\left(\sum_{k=1}^{\infty} \frac{z^k p_k}{k}\right) = \sum_{m=0}^{\infty} z^m \mathsf{Schur}_{\{m\}}(p_1, \ldots, p_m) \ . \tag{2.37}$$

Similarly, one can find that for antisymmetric Schur polynomials, $\mathsf{Schur}_{\{1,\ldots,1\}}(p_1, \ldots, p_m)$, that the Cauchy identity takes the form

$$\exp\left(-\sum_{k=1}^{\infty} \frac{z^k p_k}{k}\right) = \sum_{m=0}^{\infty} z^m \left(-1\right)^m \mathsf{Schur}_{\underbrace{\{1,\ldots,1\}}_{m}}(p_1, \ldots, p_m) \ . \tag{2.38}$$

To give an example, the Schur polynomials up to degree three are

$$\begin{aligned}
\mathsf{Schur}_{\emptyset}\left(p_k\right) &= \mathsf{Schur}_{\{\}}\left(p_k\right) = 1 \\
\mathsf{Schur}_{\square}\left(p_k\right) &= \mathsf{Schur}_{\{1\}}\left(p_k\right) = p_1 \\
\mathsf{Schur}_{\square\square}\left(p_k\right) &= \mathsf{Schur}_{\{2\}}\left(p_k\right) = \frac{p_1^2 + p_2}{2} \\
\mathsf{Schur}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}\left(p_k\right) &= \mathsf{Schur}_{\{1,1\}}\left(p_k\right) = \frac{p_1^2 - p_2}{2} \\
\mathsf{Schur}_{\square\square\square}\left(p_k\right) &= \mathsf{Schur}_{\{3\}}\left(p_k\right) = \frac{p_1^3 + 3p_1p_2 + 2p_3}{6} \\
\mathsf{Schur}_{\begin{smallmatrix}\square\square\\\square\end{smallmatrix}}\left(p_k\right) &= \mathsf{Schur}_{\{2,1\}}\left(p_k\right) = \frac{p_1^3 - p_3}{3} \\
\mathsf{Schur}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}\left(p_k\right) &= \mathsf{Schur}_{\{1,1,1\}}\left(p_k\right) = \frac{p_1^3 - 3p_1p_2 + 2p_3}{6} \ .
\end{aligned} \tag{2.39}$$

Another family of symmetric functions which we now review are the *Jack polynomials* $\mathsf{Jack}_\gamma(\lambda_k)$. The Jack polynomials are a 1-parameter deformation of the Schur polynomials, since they in addition to depending on the integer partition γ and the set of variables $\lambda_k = \{\lambda_1, \ldots, \lambda_N\}$, also depend on a real positive parameter β. Let us introduce the inner product

$$(f(\underline{\lambda}) \mid g(\underline{\lambda}))_\beta = \frac{1}{(2\pi \mathrm{i})^N N!} \oint_{|\lambda|=1} \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\lambda_i} \left(\Delta(\underline{\lambda})\Delta(\underline{\lambda}^{-1})\right)^\beta f(\underline{\lambda}) g(\underline{\lambda}^{-1}) \ , \tag{2.40}$$

which the Jack polynomials by definition are orthogonal to,

$$(\mathsf{Jack}_\gamma(\lambda_k) \mid \mathsf{Jack}_\mu(\lambda_k))_\beta = \beta^{l(\gamma)} \delta_{\gamma,\mu} \prod_{j\geq 1} j^{(\#_\gamma j)} (\#_\gamma j)! \ , \tag{2.41}$$

recalling definitions related to partitions in Section 2.2. Using the power sum variables in (2.34), the first Jack polynomials are given by

$$\begin{aligned}
\mathsf{Jack}_{\{\}}(p_k) &= 1 \\
\mathsf{Jack}_{\{1\}}(p_k) &= p_1 \\
\mathsf{Jack}_{\{2\}}(p_k) &= \frac{\beta p_1^2 + p_2}{1+\beta} \\
\mathsf{Jack}_{\{1,1\}}(p_k) &= \frac{p_1^2 - p_2}{2} \\
\mathsf{Jack}_{\{3\}}(p_k) &= \frac{\beta^2 p_1^3 + 3\beta p_1 p_2 + 2p_3}{(1+\beta)(2+\beta)} \\
\mathsf{Jack}_{\{2,1\}}(p_k) &= \frac{\beta p_1^3 + (1-\beta) p_1 p_2 - p_3}{1+2\beta} \\
\mathsf{Jack}_{\{1,1,1\}}(p_k) &= \frac{p_1^3 - 3p_1 p_2 + 2p_3}{6} \ .
\end{aligned} \tag{2.42}$$

One can observe that in the limit $\beta = 1$, also called the *Schur limit*,

$$\mathsf{Jack}_\gamma(p_k)\big|_{\beta=1} = \mathsf{Schur}_\gamma(p_k) \ . \tag{2.43}$$

Thus, the Jack polynomials can be considered the β-deformed version of the Schur polynomials. We will discuss a further deformation of the above symmetric polynomials in Chapter 5. In what follows we will almost exclusively use the power sum variables for the Schur and Jack polynomials.

2.4 Expectation values and correlators

Using the β-deformed eigenvalue matrix model defined in (2.21), one can also define the *expectation value* or *average*. For an operator $\mathcal{O}(\underline{\lambda})$ we have the average

$$\langle \mathcal{O}(\underline{\lambda}) \rangle = \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \mathcal{O}(\underline{\lambda}) \, \Delta(\underline{\lambda})^{2\beta} \mathrm{e}^{-\sum_{i=1}^{N} V(\lambda_i)} \ . \tag{2.44}$$

Similarly, we can define a *time-dependent average* according to

$$\langle \mathcal{O}(\underline{\lambda}) \rangle_t = \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \, \mathcal{O}(\underline{\lambda}) \, \Delta(\underline{\lambda})^{2\beta} \mathrm{e}^{-\sum_{i=1}^{N} V(\lambda_i) + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \tag{2.45}$$

where we use the subscript t to highlight the dependence on the time variables. The two averages are then related by

$$\langle \mathcal{O}(\underline{\lambda}) \rangle_{t=0} = \langle \mathcal{O}(\underline{\lambda}) \rangle \ . \tag{2.46}$$

Thus we note that the matrix model is given by

$$Z(t;a) = \langle 1 \rangle_t \ . \tag{2.47}$$

Next, one can expand the matrix model in terms of the time variables as

$$\begin{aligned} Z(t;a) = & \left\langle \exp\left(\sum_{s=1}^{\infty} t_s \mathrm{Tr}\left(M^s\right) \right) \right\rangle = \\ = & \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{s_1=1}^{\infty} \cdots \sum_{s_k=1}^{\infty} \Big\langle \mathrm{Tr}\left(M^{s_1}\right) \ldots \mathrm{Tr}\left(M^{s_k}\right) \Big\rangle t_{s_1} \ldots t_{s_k} = \\ = & \sum_{\rho} \frac{1}{|\mathrm{Aut}(\rho)|} c_\rho(a) \prod_{\mu \in \rho} t_\mu \ , \end{aligned} \tag{2.48}$$

where the last summation is over all integer partitions $\rho = \{\rho_1, \ldots, \rho_n\}$ and where the order of the automorphism group is defined in (2.25). Equation (2.48) then defines the *correlators* $c_\rho(a)$ of the model in question, in other words the expectation values of the multi-trace operators

$$c_\rho(a) = \langle \mathrm{Tr}\left(M^{\rho_1}\right) \ldots \mathrm{Tr}\left(M^{\rho_n}\right) \rangle \ . \tag{2.49}$$

Alternatively, the correlators can also be expressed as

$$c_\rho(a) = \left[\frac{\partial \ldots \partial}{\partial t_{\rho_1} \ldots \partial t_{\rho_n}} Z(t;a) \right] \Bigg|_{t=0} \tag{2.50}$$

for a partition $\rho = \{\rho_1, \ldots, \rho_n\}$. Consequently one can view $Z(t;a)$ as the *generating function* for the correlators $c_\rho(a)$, which is a name we will employ frequently. We also note that the correlators depend on the coupling constants $a = \{a_1, \ldots, a_\mathsf{p}\}$ appearing in the potential and that the empty correlator $c_\emptyset(a)$ can be written as

$$c_\emptyset(a) = \langle 1 \rangle \ . \tag{2.51}$$

This is by definition equal to the *partition function* $Z(a) = Z(0; a) = c_\emptyset(a)$. It should then be noted that if one can find all the correlators for a given model, this information is enough to completely determine the model.

Returning to the averages, one particular average that can be computed is that of the Schur polynomials in Section 2.3. Such averages reduce to an unexpectedly simple form in the case of a potential of the form in (2.16) where $\mathsf{p} = 2$, $a_1 = 0$ and where $\beta = 1$ [53]

$$\begin{aligned} &\langle \mathsf{Schur}_\gamma(p_k)\rangle\big|_{\mathsf{p}=2,a_1=0,\beta=1} = \\ &= \frac{1}{a_2^{|\gamma|/2}} \frac{\mathsf{Schur}_\gamma(p_k = \delta_{k,2})}{\mathsf{Schur}_\gamma(p_k = \delta_{k,1})} \mathsf{Schur}_\gamma(p_k = N) \;\; c_\emptyset(a_1 = 0, a_2)\big|_{\beta=1} \;\; , \end{aligned} \tag{2.52}$$

which is valid for a generic partition γ. This is a phenomenon called *superintegrability*, as first observed in [54, 55]. In paper IV we explored this property further and found firstly in the case of a potential with $\mathsf{p} = 1$ that for generic β the average of a Jack polynomial takes the simple form

$$\begin{aligned} &\langle \mathsf{Jack}_\gamma(p_k)\rangle\big|_{\mathsf{p}=1} = \\ &= \frac{\mathsf{Jack}_\gamma(p_k = \beta^{-1}(\nu + \beta(N-1)+1))}{\mathsf{Jack}_\gamma(p_k = \beta^{-1} a_1 \delta_{k,1})} \mathsf{Jack}_\gamma(p_k = N)\, c_\emptyset(a_1) \; . \end{aligned} \tag{2.53}$$

Upon letting $\beta = 1$ one instead obtains the average of a Schur polynomial,

$$\langle \mathsf{Schur}_\gamma(p_k)\rangle\big|_{\mathsf{p}=1,\beta=1} = \frac{\mathsf{Schur}_\gamma(p_k = N + \nu)}{\mathsf{Schur}_\gamma(p_k = a_1 \delta_{k,1})} \mathsf{Schur}_\gamma(p_k = N) \;\; c_\emptyset(a_1)\big|_{\beta=1} \;\; . \tag{2.54}$$

One can also observe this superintegrability property in the case of $\mathsf{p} = 2$, where one finds for generic β

$$\begin{aligned} &\langle \mathsf{Jack}_\gamma(p_k)\rangle\big|_{\mathsf{p}=2} = \\ &= \frac{\mathsf{Jack}_\gamma\left(p_k = (-1)^k \beta^{-1}(a_1\delta_{k,1} + a_2\delta_{k,2})\right)}{\mathsf{Jack}_\gamma(p_k = \beta^{-1} a_2 \delta_{k,1})} \mathsf{Jack}_\gamma(p_k = N)\, c_\emptyset(a_1, a_2) \; . \end{aligned} \tag{2.55}$$

In the limit $\beta = 1$ we then have

$$\begin{aligned} &\langle \mathsf{Schur}_\gamma(p_k)\rangle\big|_{\mathsf{p}=2,\beta=1} = \\ &= \frac{\mathsf{Schur}_\gamma\left(p_k = (-1)^k(a_1\delta_{k,1} + a_2\delta_{k,2})\right)}{\mathsf{Schur}_\gamma(p_k = a_2 \delta_{k,1})} \mathsf{Schur}_\gamma(p_k = N) \;\; c_\emptyset(a_1, a_2)\big|_{\beta=1} \;\; , \end{aligned} \tag{2.56}$$

generalising the result of [53] (as given in (2.52)) for arbitrary a_1.

3. Virasoro constraints

We now move on to the main tool when studying matrix models which will be employed to a large extent throughout, namely the Ward identities or the Virasoro constraints. We investigate how these constraints are derived, but more importantly how the constraints can be used to completely determine the model in terms of the correlators. Sometimes we also find an even more compact and neat way to express the solution, in which we re-cast the matrix model using what has been called a W-representation. Lastly, we consider an extension of the Virasoro constraints. We review the analysis in paper V on how the solution to the Virasoro constraints can be generalised to the case when the constraints receive boundary contributions and are effectively rendered non-homogeneous.

3.1 Deriving the Virasoro constraints

Starting from the matrix model given in (2.1), it can be shown that this model satisfies constraints known as the Virasoro constraints, as first shown in [14, 15]. Following the derivation in [14] (also reviewed in [25])[1], we begin with rewriting the eigenvalue matrix model in (2.21) as

$$\begin{aligned} Z(t;a) = \int_{\mathbb{R}^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \, (-1)^{\frac{N(N-1)}{2}} \prod_{1\leq i\neq j\leq N} (\lambda_i - \lambda_j)^\beta \times \\ \times \, \mathrm{e}^{-\sum_{i=1}^N V(\lambda_i) + \sum_{s=1}^\infty t_s \sum_{i=1}^N \lambda_i^s} \, . \end{aligned} \tag{3.1}$$

Then, one can consider the saddle point equations for the model, where for each $i = 1, \ldots, N$ we have

$$\sum_{s=1}^{\infty} s t_s \lambda_i^{s-1} + 2\beta \sum_{\substack{1\leq j\leq N \\ j\neq i}} \frac{1}{\lambda_i - \lambda_j} - \frac{\partial}{\partial \lambda_i} V(\lambda_i) = 0 \, . \tag{3.2}$$

This means that there is an invariance with respect to the shift

$$\lambda_i \to \lambda_i + \delta\lambda_i = \lambda_i + \epsilon_n \lambda_i^{n+1} \, , \qquad n \geq -1 \, , \tag{3.3}$$

[1] It should be noted that with respect to [25], we use $t_{s,\text{there}} = s!\, t_{s,\text{here}}$ and we also allow for a generic parameter β and a generic potential $V(\lambda)$.

under which the effective action of the eigenvalue model in (3.1) becomes

$$
\begin{aligned}
&\delta\left(\sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s + \frac{\beta}{2} \sum_{1\le i\neq j\le N} \ln(\lambda_i-\lambda_j)^2 - \sum_{i=1}^{N} V(\lambda_i)\right) = \\
&= \epsilon_n \left(\sum_{s=1}^{\infty} s t_s \sum_{i=1}^{N} \lambda_i^{s+n} + \beta \sum_{1\le i\neq j\le N} \frac{\lambda_i^{n+1}-\lambda_j^{n+1}}{\lambda_i-\lambda_j} - \sum_{i=1}^{N} \lambda_i^{n+1} \frac{\partial}{\partial\lambda_i} V(\lambda_i)\right) = \\
&= \epsilon_n \sum_{i=1}^{N} \lambda_i^{n+1} \left(\sum_{s=1}^{\infty} s t_s \lambda_i^{s-1} + 2\beta \sum_{\substack{1\le j\le N\\ j\neq i}} \frac{1}{\lambda_i-\lambda_j} - \frac{\partial}{\partial\lambda_i} V(\lambda_i)\right) = 0 \ ,
\end{aligned}
\tag{3.4}
$$

upon using the saddle point equations in (3.2) in the last step. This implies that the shift in (3.3) is a symmetry of the matrix model, and this symmetry can furthermore be rewritten as a constraint. Thus, applying the shift in (3.3) to the model in (3.1) we find for $n \ge -1$,

$$
\begin{aligned}
&Z(t;a) \to \\
&\to \int_{\mathbb{R}^N} \left[\prod_{i=1}^{N} (1+(n+1)\epsilon_n \lambda_i^n)\, \mathrm{d}\lambda_i\right] (-1)^{\frac{N(N-1)}{2}} \prod_{1\le i\neq j\le N} (\lambda_i-\lambda_j)^{\beta} \times \\
&\times \mathrm{e}^{-\sum_{i=1}^{N} V(\lambda_i) + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \times \\
&\times \exp\left(\epsilon_n \sum_{i=1}^{N} \lambda_i^{n+1} \left\{\sum_{s=1}^{\infty} s t_s \lambda_i^{s-1} + 2\beta \sum_{\substack{1\le j\le N\\ j\neq i}} \frac{1}{\lambda_i-\lambda_j} - \frac{\partial}{\partial\lambda_i} V(\lambda_i)\right\}\right) .
\end{aligned}
\tag{3.5}
$$

We then use the explicit form of the potential given in equation (2.16), such that

$$
\frac{\partial}{\partial\lambda_i} V(\lambda_i) = -\delta_{\mathsf{p},1} \nu \lambda_i^{-1} + \sum_{k=1}^{\mathsf{p}} a_k \lambda_i^{k-1} \ . \tag{3.6}
$$

Next, one can use the time-dependent averages in (2.45) to extract the variations which are linear in ϵ_n for $n \ge -1$. Starting with the $n=-1$ constraint (where as we will comment on later this constraint is only valid when $\mathsf{p} \neq 1$ to avoid expectation values of negative powers of λ_i),

$$
\left\langle \sum_{s=1}^{\infty} s t_s \sum_{i=1}^{N} \lambda_i^{s-1} - \sum_{k=1}^{\mathsf{p}} a_k \sum_{i=1}^{N} \lambda_i^{k-1} \right\rangle_t = 0 \ . \tag{3.7}
$$

Then the $n=0$ constraint becomes

$$
\left\langle \beta N^2 + (1-\beta)N + \sum_{s=1}^{\infty} s t_s \sum_{i=1}^{N} \lambda_i^s + \delta_{\mathsf{p},1} \nu N - \sum_{k=1}^{\mathsf{p}} a_k \sum_{i=1}^{N} \lambda_i^k \right\rangle_t = 0 \ , \tag{3.8}
$$

and for $n \geq 1$ we have

$$\left\langle (1-\beta)(n+1)\sum_{i=1}^{N}\lambda_i^n + \sum_{s=1}^{\infty} s t_s \sum_{i=1}^{N}\lambda_i^{s+n} + \beta \sum_{i,j=1}^{N}\sum_{k=0}^{n}\lambda_i^k\lambda_j^{n-k} + \right.$$
$$\left. + \delta_{\mathsf{p},1}\nu \sum_{i=1}^{N}\lambda_i^n - \sum_{k=1}^{\mathsf{p}} a_k \sum_{i=1}^{N}\lambda_i^{n+k} \right\rangle_t = 0 \ , \tag{3.9}$$

thus recovering (2.24) of paper IV. One might then trade the expectation value for a differential operator in the time variables according to

$$\left\langle \sum_{i_1=1}^{N}\cdots\sum_{i_k=1}^{N}\lambda_{i_1}^{s_1}\dots\lambda_{i_k}^{s_k}\right\rangle_t = \frac{\partial}{\partial t_{s_1}}\cdots\frac{\partial}{\partial t_{s_k}} Z(t;a) \ , \tag{3.10}$$

where the powers $\{s_1,\dots,s_k\}$ are non-negative integers and in case it is zero we replace the corresponding derivative with multiplication by N. Firstly, the $n=-1$ constraint in (3.7) can be written as the differential equation

$$\left(N t_1 + \sum_{s=2}^{\infty} s t_s \frac{\partial}{\partial t_{s-1}} - \sum_{k=2}^{\mathsf{p}} a_k \frac{\partial}{\partial t_{k-1}} - a_1 N \right) Z(t;a) = 0 \ . \tag{3.11}$$

Secondly, the $n=0$ constraint in (3.8) takes the form

$$\left(\beta N^2 + (1-\beta)N + \sum_{s=1}^{\infty} s t_s \frac{\partial}{\partial t_s} + \delta_{\mathsf{p},1}\nu N - \sum_{k=1}^{\mathsf{p}} a_k \frac{\partial}{\partial t_k}\right) Z(t;a) = 0 \ , \tag{3.12}$$

and thirdly the $n \geq 1$ constraint in (3.9) becomes

$$\left((1-\beta)(n+1)\frac{\partial}{\partial t_n} + \sum_{s=1}^{\infty} s t_s \frac{\partial}{\partial t_{s+n}} + \beta \sum_{\substack{\ell+m=n \\ \ell,m>0}} \frac{\partial^2}{\partial t_\ell \partial t_m} + \right.$$
$$\left. + \delta_{\mathsf{p},1}\nu \frac{\partial}{\partial t_n} - \sum_{k=1}^{\mathsf{p}} a_k \frac{\partial}{\partial t_{n+k}} \right) Z(t;a) = 0 \ . \tag{3.13}$$

The above can then collectively be written as the *Virasoro constraints*

$$\left(L_n + \delta_{\mathsf{p},1}\nu \left(\frac{\partial}{\partial t_n} + \delta_{n,0} N \right) - \sum_{k=1}^{\mathsf{p}} a_k \frac{\partial}{\partial t_{n+k}} - a_1 N \delta_{n,-1} \right) Z(t;a) = 0 \ , \tag{3.14}$$

valid for $n \geq -1$ where the differential operators L_n are given by

$$
\begin{aligned}
L_{-1} &= Nt_1 + \sum_{s=2}^{\infty} st_s \frac{\partial}{\partial t_{s-1}} \\
L_0 &= \beta N^2 + (1-\beta)N + \sum_{s=1}^{\infty} st_s \frac{\partial}{\partial t_s} \\
L_{n>0} &= 2\beta N \frac{\partial}{\partial t_n} + \beta \sum_{\substack{\ell+m=n \\ \ell,m>0}} \frac{\partial^2}{\partial t_\ell \partial t_m} + (1-\beta)(n+1)\frac{\partial}{\partial t_n} + \sum_{s=1}^{\infty} st_s \frac{\partial}{\partial t_{s+n}} \ .
\end{aligned}
\tag{3.15}
$$

It can be noted that from the point of view of the constraints in (3.14) only, i.e. disregarding the matrix model picture, N can be a generic complex variable. One can then observe that the Virasoro constraints in (3.14) posses a symmetry, as they are invariant under the simultaneous shifts

$$
\sqrt{\beta} \to -\frac{1}{\sqrt{\beta}}, \quad N \to -\beta N, \quad t_s \to -\frac{1}{\beta} t_s, \quad a_k \to -\frac{1}{\beta} a_k, \quad \nu \to -\frac{1}{\beta}\nu \ . \tag{3.16}
$$

Upon including also generators L_n for $n < -1$ (whose explicit form can be deduced from the free boson realisation to be introduced shortly), the generators L_n for $n \in \mathbb{Z}$ together with a central charge c, generate the *Virasoro algebra*

$$
[L_n,\, L_m] = (n-m)L_{n+m} + \frac{c}{12} n(n^2-1)\delta_{n+m,0}\,, \qquad [L_n, c] = 0\ , \tag{3.17}
$$

for $n, m \in \mathbb{Z}$. It can be shown that the central charge is given by

$$
c = 1 - 6Q_\beta^2 \tag{3.18}
$$

with

$$
Q_\beta = \sqrt{\beta} - \frac{1}{\sqrt{\beta}}\ , \tag{3.19}
$$

where one can note that the central charge is invariant under $\sqrt{\beta} \to -1/\sqrt{\beta}$, consistent with the symmetry in (3.16) above.

Before moving on, let us remark here that starting from the matrix model in (2.1), we have only included time variables $\{t_k\}$ for $k > 0$ in order to be consistent with papers IV and V. However, in the literature sometimes also the variable t_0 is included. This appears through the inclusion of a factor e^{Nt_0} in the β-deformed eigenvalue matrix model in (2.21). Consequently, derivatives with respect to t_0 can be traded for multiplication by factors of N and vice versa. By considering the Virasoro constraints written using derivatives of t_0 instead of factors of N, the most generic form of a generating function which satisfies the Virasoro constraints is that constructed from some linear combination

$$
\mathcal{Z}(t;a) = \sum_N Z(t;a)\,\mathrm{e}^{Nt_0} \tag{3.20}
$$

as each $Z(t;a)$ depends on N. However, because the operator $\frac{\partial}{\partial t_0}$ commutes with the Virasoro constraints (i.e. the combination of operators appearing in the brackets in (3.14)), $\frac{\partial}{\partial t_0}$ and the Virasoro constraints can be simultaneously diagonalised. Therefore we can consider a specific value of N without loss of generality, which is what we will do. In the following derivation using the free boson realisation, we will however include t_0 to be consistent with literature.

An alternative way to see that the eigenvalue matrix model in (2.21) satisfies the Virasoro constraints, is by constructing the model using the *free boson realisation* of the Virasoro algebra [23, 26]. To do so, we introduce the free boson oscillator a_n with $n \in \mathbb{Z}\backslash\{0\}$ and zero mode P together with Q, which satisfy the Heisenberg algebra given by

$$[\mathsf{a}_n, \mathsf{a}_m] = 2n\delta_{n+m,0} \ , \qquad [\mathsf{P}, \mathsf{Q}] = 2 \tag{3.21}$$

for $n, m \in \mathbb{Z}\backslash\{0\}$. We also have the Fock module $\mathcal{F}_\alpha$ generated by

$$\mathcal{F}_\alpha = \left\{ \prod_{k=1}^{l(\gamma)} \mathsf{a}_{-\gamma_k} |\alpha\rangle \right\} \ , \tag{3.22}$$

for any partition $\gamma = \{\gamma_1, \ldots, \gamma_{l(\gamma)}\}$ of length $l(\gamma)$. The state, or charged vacuum, $|\alpha\rangle$ is then characterised by

$$\prod_{k=1}^{l(\gamma)} \mathsf{a}_{\gamma_k} |\alpha\rangle = 0 \tag{3.23}$$

a condition which is called the *highest weight condition*, together with

$$|\alpha\rangle = \mathrm{e}^{\frac{\alpha}{2}\mathsf{Q}}|0\rangle \ , \qquad \mathsf{P}|\alpha\rangle = \alpha|\alpha\rangle \ , \tag{3.24}$$

for a given charge or momentum $\alpha \in \mathbb{C}$. Using the free boson oscillators, a boson field $\phi(z)$ is then given by

$$\phi(z) = \mathsf{Q} + \mathsf{P}\ln(z) - \sum_{k \in \mathbb{Z}\backslash\{0\}} \frac{\mathsf{a}_k}{k} z^{-k} \tag{3.25}$$

such that

$$\partial\phi(z) = \mathsf{P}z^{-1} + \sum_{k \in \mathbb{Z}\backslash\{0\}} \mathsf{a}_k z^{-k-1} \ . \tag{3.26}$$

Using this free boson, one might write the Virasoro current $L(z)$ as

$$L(z) = \frac{1}{4} : \partial\phi(z)\partial\phi(z) : + Q_\beta \frac{1}{2}\partial^2\phi(z) \tag{3.27}$$

with Q_β as given in (3.19). Here we introduced the normal ordering prescription $: \ :$ in which we move positive modes $\mathsf{a}_{n>0}$ to the right of negative modes $\mathsf{a}_{n<0}$ and P to the right of Q. Using the expansion of the Virasoro current in terms of modes,

$$L(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{3.28}$$

one finds

$$\begin{aligned} L_n =& \frac{1}{4}\sum_{k\neq 0,n} :\mathsf{a}_{n-k}\mathsf{a}_k: + \frac{1}{2}\mathsf{a}_n\mathsf{P} - \frac{1}{2}Q_\beta(n+1)\mathsf{a}_n\ , \qquad n\neq 0 \\ L_0 =& \frac{1}{2}\sum_{k>0}\mathsf{a}_{-k}\mathsf{a}_k + \frac{\mathsf{P}^2}{4} - \frac{1}{2}\mathsf{P}Q_\beta\ . \end{aligned} \tag{3.29}$$

The Virasoro current above is then realising the Virasoro algebra with the central charge as given in (3.19) and $|\alpha\rangle$ is the highest weight state of the algebra with conformal weight $h(\alpha)$,

$$h(\alpha) = \frac{1}{4}\Big((\alpha - Q_\beta)^2 - Q_\beta^2\Big)\ . \tag{3.30}$$

Using a differential representation of the Heisenberg oscillators given by[2]

$$\begin{aligned} \mathsf{a}_{-n} \simeq \frac{n}{\sqrt{\beta}}\,t_n\,, \qquad \mathsf{a}_n \simeq 2\sqrt{\beta}\frac{\partial}{\partial t_n}\,, \\ \mathsf{Q} \simeq \frac{t_0}{\sqrt{\beta}}\,, \qquad \mathsf{P} \simeq 2\sqrt{\beta}N\,, \qquad |\alpha\rangle = \mathrm{e}^{\frac{\alpha}{2}\mathsf{Q}}|0\rangle \simeq \mathrm{e}^{t_0\frac{\alpha}{2}}\cdot 1\ , \end{aligned} \tag{3.31}$$

with $n>0$, one can recover the differential operators in (3.15) from the ones in (3.29). Then, the problem of finding a matrix model which satisfies the Virasoro constraints in (3.14) can be solved using the construction of a *screening current* $\mathsf{S}(\lambda)$. This is defined by

$$[L_n, \mathsf{S}(\lambda)] = \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathsf{O}(\lambda) \tag{3.32}$$

for some operator $\mathsf{O}(\lambda)$ whose explicit form will not be of importance. We now use this screening current to construct the matrix model by means of

$$\mathsf{Z}|\alpha\rangle = \int\prod_{i=1}^N \mathsf{S}(\lambda_i)\,\mathrm{d}\lambda_i\,|\alpha\rangle\ . \tag{3.33}$$

This satisfies the Virasoro constraints $L_{n>0}\mathsf{Z}|\alpha\rangle = 0$ (excluding the potential) by construction, since

$$L_{n>0}\mathsf{Z}|\alpha\rangle = [L_{n>0},\mathsf{Z}]|\alpha\rangle = [L_{n>0},\int\prod_{i=1}^N\mathsf{S}(\lambda_i)\,\mathrm{d}\lambda_i]|\alpha\rangle = 0 \tag{3.34}$$

[2]With respect to [26] we use the normalisation of time variables $\sqrt{\beta}\,t_{s,\text{there}} = t_{s,\text{here}}$.

for an appropriately chosen contour and where we used the highest weight condition $L_{n>0}|\alpha\rangle = 0$ on the charged vacuum (in other words (3.23)). We then take as ansatz that the screening current is given by

$$\mathsf{S}(\lambda) =: \mathrm{e}^{\sqrt{\beta}\phi(\lambda)} :=: \mathrm{e}^{-\sqrt{\beta}\sum_{n\neq 0}\frac{\lambda^{-n}}{n}\mathsf{a}_n} : \mathrm{e}^{\sqrt{\beta}\mathsf{Q}}\lambda^{\sqrt{\beta}\mathsf{P}} \;, \tag{3.35}$$

using the boson field $\phi(z)$ in (3.25). As a side note, one can observe that there is another screening current which also solves the Virasoro constraints. It is the screening current obtained by performing the shift [23]

$$\sqrt{\beta} \to -\frac{1}{\sqrt{\beta}} \tag{3.36}$$

in (3.35). This screening current can be treated similarly and for concreteness we consider (3.35) here. To then obtain the matrix model, we rewrite the integrand of (3.33) as the normal ordered product

$$\prod_{i=1}^{N}\mathsf{S}(\lambda_i) =: \prod_{i=1}^{N}\mathsf{S}(\lambda_i) : \Delta(\underline{\lambda})^{2\beta} \;, \tag{3.37}$$

recalling the β-deformed Vandermonde determinant $\Delta(\underline{\lambda})^{2\beta}$ in (2.20). Thus, we finally obtain the matrix model

$$\mathsf{Z}|\alpha\rangle = \int\prod_{i=1}^{N}\mathrm{d}\lambda_i\,\Delta(\underline{\lambda})^{2\beta}\mathrm{e}^{\sqrt{\beta}\sum_{j=1}^{N}\sum_{n=1}^{\infty}\frac{\lambda_j^n}{n}\mathsf{a}_{-n}}\mathrm{e}^{N\sqrt{\beta}\mathsf{Q}}\prod_{k=1}^{N}\lambda_k^{\sqrt{\beta}\mathsf{P}}|\alpha\rangle \tag{3.38}$$

which, using the time representation of the free bosons in (3.31) becomes

$$\mathsf{Z}|\alpha\rangle \simeq \mathrm{e}^{t_0\left(N+\frac{\alpha}{2}\right)}\int\prod_{i=1}^{N}\mathrm{d}\lambda_i\,\Delta(\underline{\lambda})^{2\beta}\mathrm{e}^{\sum_{j=1}^{N}\sqrt{\beta}\alpha\ln(\lambda_j)+\sum_{n=1}^{\infty}t_n\sum_{j=1}^{N}\lambda_j^n} \;, \tag{3.39}$$

where we can note the appearance of the factor e^{Nt_0} mentioned earlier together with the charged vacuum $|\alpha\rangle \simeq \mathrm{e}^{t_0\frac{\alpha}{2}}$. Upon disregarding these prefactors and performing the shift of times in (2.13) to obtain the potential, one can recover the β-deformed eigenvalue matrix model in (2.21) (when identifying $\sqrt{\beta}\alpha = \nu$).

3.2 Solving the Virasoro constraints

Let us now discuss how to solve the Virasoro constraints in (3.14). There are two ways to solve them. Either we express the generating function using its W-representation or we use the Virasoro constraints to derive a recursion relation allowing us to evaluate the correlators of the model. Let us now discuss the methods in more detail, starting with the former.

3.2.1 W-representations

Instead of expressing the eigenvalue matrix model as an integral, one can also derive something which is called the *W-representation*. Schematically, in this representation the generating function is given in terms of the exponent of a *W-operator* acting on a simple function,

$$Z(t;a) \sim \mathrm{e}^{W} \cdot (\text{simple function}) \ , \tag{3.40}$$

where we will shortly exemplify what "simple" means together with giving explicit expressions for the W-operators in question.

W-representation for the Gaussian potential

The W-representation of a matrix model can be illustrated in the case of the Gaussian Hermitean matrix model in (2.19), as was first shown in [16]. For simplicity, in this subsection we use the notation

$$Z_G(t) = Z(t; a_1 = 0, a_2 = 1) \tag{3.41}$$

to denote the Gaussian Hermitean generating function. $Z_G(t)$ then satisfies the Virasoro constraints in (3.14),

$$L_n Z_G(t) = \frac{\partial}{\partial t_{n+2}} Z_G(t) \,, \qquad n \geq -1 \ . \tag{3.42}$$

Next, we shift $n \to n-2$ and take a summation over $\sum_{n=1}^{\infty} n t_n$ to find

$$\sum_{n=1}^{\infty} n t_n L_{n-2} Z_G(t) = \sum_{n=1}^{\infty} n t_n \frac{\partial}{\partial t_n} Z_G(t) \ . \tag{3.43}$$

We then identify the right hand side of (3.43) as the *dilatation operator* D

$$D = \sum_{n=1}^{\infty} n t_n \frac{\partial}{\partial t_n} \ , \tag{3.44}$$

and we define the left hand side to be the *W-operator* W_{-2},

$$\begin{aligned} W_{-2} = &\sum_{n=1}^{\infty} n t_n L_{n-2} = \\ =&\beta \sum_{n,m=1}^{\infty} (n+m+2) t_{n+m+2} \frac{\partial^2}{\partial t_n \partial t_m} + \\ &+ (1-\beta) \sum_{n=1}^{\infty} (n+1)(n+2) t_{n+2} \frac{\partial}{\partial t_n} + \\ &+ \sum_{n,m=1}^{\infty} n m t_n t_m \frac{\partial}{\partial t_{n+m-2}} + 2\beta N \sum_{n=1}^{\infty} (n+2) t_{n+2} \frac{\partial}{\partial t_n} + \\ &+ (\beta N^2 + (1-\beta) N) 2 t_2 + t_1^2 N \ , \end{aligned} \tag{3.45}$$

such that (3.43) becomes

$$W_{-2}Z_G(t) = DZ_G(t)\ . \tag{3.46}$$

To find the W-representation of the generating function, we note that

$$[D, W_{-2}] = 2W_{-2}\ , \tag{3.47}$$

together with the fact that D counts the *degree* in times i.e.

$$D\prod_{\mu\in\rho} t_\mu = \Big(\sum_{\mu\in\rho}\mu\Big)\prod_{\mu\in\rho} t_\mu = |\rho|\prod_{\mu\in\rho} t_\mu \tag{3.48}$$

for an integer partition ρ. In other words, the degree is $\deg(t_s) = s$ and $\deg(t_a\, t_b) = \deg(t_a) + \deg(t_b)$. Then, one can arrange the generating function according to components of degree d in times

$$Z_G(t) = \sum_{d=0}^{\infty} Z_G^{(d)}(t)\ . \tag{3.49}$$

We now recall the matrices M in the original matrix model (2.1) together with the expansion of the generating function given in (2.48), such that

$$Z_G^{(d)}(t) = \sum_{m=0}^{\infty}\frac{1}{m!}\sum_{k_1+\cdots+k_m=d}\langle \mathrm{Tr}(M^{k_1})\ldots\mathrm{Tr}(M^{k_m})\rangle\, t_{k_1}\ldots t_{k_m}\ . \tag{3.50}$$

The generating function then respects the grading with respect to the operator D, in the sense that

$$D\, Z_G^{(d)}(t) = d\, Z_G^{(d)}(t)\ . \tag{3.51}$$

Then using the commutation relation in (3.47), one might deduce that

$$\begin{gathered} D\left(W_{-2}Z_G^{(d)}(t)\right) = (d+2)\left(W_{-2}Z_G^{(d)}(t)\right) \\ W_{-2}Z_G^{(d)}(t) = C(d)Z_G^{(d+2)}(t) \end{gathered} \tag{3.52}$$

for a coefficient $C(d)$ depending on the degree d. Then (3.46) implies that

$$W_{-2}Z_G(t) = \sum_{d=0}^{\infty} C(d)Z_G^{(d+2)}(t) = DZ_G(t) = \sum_{d=0}^{\infty} d\, Z_G^{(d)}(t) \tag{3.53}$$

so that we must have $C(d) = (d+2)$. Finally we find that the operator W_{-2} respects the grading in the sense that

$$W_{-2}Z_G^{(d)}(t) = (d+2)Z_G^{(d+2)}(t)\ . \tag{3.54}$$

Thus each $Z_G^{(d)}(t)$ is constructed by acting with W_{-2} on lower degrees. Using this and the observation that odd degrees vanishes one finds

$$\begin{aligned}
d = 0 \Rightarrow \qquad & Z_G^{(2)}(t) = \frac{W_{-2}}{2} Z_G^{(0)}(t) \\
d = 2 \Rightarrow \qquad & Z_G^{(4)}(t) = \frac{W_{-2}}{4} Z_G^{(2)}(t) = \frac{1}{2}\left(\frac{W_{-2}}{2}\right)^2 Z_G^{(0)}(t) \\
d = 4 \Rightarrow \qquad & Z_G^{(6)}(t) = \frac{W_{-2}}{6} Z_G^{(4)}(t) = \frac{1}{6}\left(\frac{W_{-2}}{2}\right)^3 Z_G^{(0)}(t) \ ,
\end{aligned} \tag{3.55}$$

for some normalisation $Z_G^{(0)}(t)$ that is to be determined. Thus we can re-sum the degrees to find the W-representation of the generating function,

$$\begin{aligned}
Z_G(t) =& Z_G^{(0)}(t) + \frac{W_{-2}}{2} Z_G^{(0)}(t) + \frac{1}{2}\left(\frac{W_{-2}}{2}\right)^2 Z_G^{(0)}(t) + \cdots = \\
=& \, \mathrm{e}^{W_{-2}/2} Z_G^{(0)}(t) \ .
\end{aligned} \tag{3.56}$$

We then make the observation that

$$Z_G^{(0)}(t) = \left. Z_G(t) \right|_{t=0} = c_\emptyset(a_1 = 0, a_2 = 1) \ . \tag{3.57}$$

Summarising the above findings, the *W-representation* of the Gaussian Hermitean matrix model is then

$$Z_G(t) = \mathrm{e}^{W_{-2}/2} c_\emptyset(a_1 = 0, a_2 = 1) \tag{3.58}$$

with W_{-2} as given in (3.45). Let us now discuss other potentials.

Other choices of the potential $V(\lambda)$

The above construction can be generalised to other choices of the potential $V(\lambda)$ in (2.16) than the Gaussian one, as shown in paper IV. For instance, in the case of $\mathsf{p} = 1$ where $V(\lambda) = -\nu \ln(\lambda) + a_1\lambda$, one can derive the W-representation for the generating function using the Virasoro constraints starting from $n = 0$. This is because the $n = -1$ constraint in (3.7) is not valid for $\mathsf{p} = 1$ since there are expectation values of negative powers of $\{\lambda_i\}$ appearing. Thus, we have the Virasoro constraints valid for $n \geq 0$

$$\begin{aligned}
a_1 \frac{\partial}{\partial t_{n+1}} Z(t; a_1) =& \Bigg[\beta \sum_{\substack{a+b=n \\ a,b>0}} \frac{\partial^2}{\partial t_a \partial t_b} + ((1-\beta)(n+1) + \nu + 2\beta N)\frac{\partial}{\partial t_n} + \\
& + \sum_{s=1}^{\infty} s t_s \frac{\partial}{\partial t_{s+n}} + \delta_{n,0} N\left(\nu + \beta(N-1) + 1\right)\Bigg] Z(t; a_1) \ ,
\end{aligned} \tag{3.59}$$

which upon shifting $n \to n - 1$ and summing over $\sum_{n=1}^{\infty} n t_n$ becomes

$$a_1 D Z(t; a_1) = W_{-1} Z(t; a_1) \ . \tag{3.60}$$

The dilatation operator D is given in (3.44) and the W_{-1} operator is

$$\begin{aligned} W_{-1} =&\beta \sum_{n,m=1}^{\infty} (n+m+1) t_{n+m+1} \frac{\partial^2}{\partial t_n \partial t_m} + \sum_{n,m=1}^{\infty} n m t_n t_m \frac{\partial}{\partial t_{n+m-1}} + \\ &+ t_1 N(\nu + \beta(N-1)+1) + \\ &+ \sum_{n=1}^{\infty} \left(\nu + (1-\beta)(n+1) + 2\beta N\right)(n+1) t_{n+1} \frac{\partial}{\partial t_n} \ . \end{aligned} \tag{3.61}$$

Finally, we use that

$$[D, W_{-1}] = W_{-1} \tag{3.62}$$

to obtain the W-representation for $\mathsf{p} = 1$

$$Z(t; a_1) = \sum_{d=0}^{\infty} \frac{W_{-1}^d}{a_1^d d!} \cdot c_\emptyset(a_1) = \exp\left(\frac{W_{-1}}{a_1}\right) \cdot c_\emptyset(a_1) \ . \tag{3.63}$$

This generalises the results in [54,56], as the above allows for a determinant insertion parametrised by ν.

Another example is when $\mathsf{p} = 2$, where the potential is given by $V(\lambda) = a_1 \lambda + \frac{a_2}{2}\lambda^2$. The Gaussian potential (i.e. $a_1 = 0$ and $a_2 = 1$) as mentioned above, was originally solved in [16] and the solution can be generalised for arbitrary a_2 to

$$Z(t; a_1 = 0, a_2) = \exp\left(\frac{1}{2a_2} W_{-2}\right) \cdot c_\emptyset(a_1 = 0, a_2) \ . \tag{3.64}$$

To then find the solution for arbitrary a_1, we use the above together with

$$[L_{-1}, W_{-2}] = 0 \ , \tag{3.65}$$

and the Virasoro constraint for $n = -1$ and $a_1 = 0$

$$a_2 \frac{\partial}{\partial t_1} Z(t; a_1 = 0, a_2) = L_{-1} Z(t; a_1 = 0, a_2) \ . \tag{3.66}$$

One might then write the W-representation for $\mathsf{p} = 2$ as

$$\begin{aligned} Z(t; a_1, a_2) &= \exp\left(-a_1 \frac{\partial}{\partial t_1}\right) Z(t; a_1 = 0, a_2) = \\ &= \exp\left(-\frac{a_1}{a_2} L_{-1}\right) Z(t; a_1 = 0, a_2) = \\ &= \exp\left(\frac{1}{2a_2} W_{-2} - \frac{a_1}{a_2} L_{-1}\right) \cdot c_\emptyset(a_1, a_2) \ . \end{aligned} \tag{3.67}$$

For $\mathsf{p} \geq 3$ we are not able to obtain the W-representation of the generating function (see for instance [57–59]). This can be seen from noting that the Virasoro constraints in (3.14) can be re-summed to

$$
\begin{aligned}
&a_{\mathsf{p}}\left(D-\sum_{k=1}^{\mathsf{p}-2} k t_k \frac{\partial}{\partial t_k}\right) Z(t;a) = \\
&\quad = \left[W_{-\mathsf{p}} - \sum_{k=1}^{\mathsf{p}-1} a_{\mathsf{p}-k}\left(\sum_{n=\mathsf{p}-1}^{\infty} n t_n \frac{\partial}{\partial t_{n-k}} + \delta_{k,\mathsf{p}-1}(\mathsf{p}-1)t_{\mathsf{p}-1}N\right)\right] Z(t;a)
\end{aligned}
\tag{3.68}
$$

with

$$
\begin{aligned}
W_{-\mathsf{p}} =&(\mathsf{p}-1)t_1 t_{\mathsf{p}-1}N + (\beta N^2 + (1-\beta)N)\mathsf{p} t_{\mathsf{p}} + \\
&+\sum_{n=1}^{\infty}\sum_{m=\mathsf{p}-1}^{\infty} nm t_n t_m \frac{\partial}{\partial t_{n+m-\mathsf{p}}} + \\
&+\sum_{n=1}^{\infty}\left[2\beta N + (1-\beta)(n+1)\right](n+\mathsf{p})t_{n+\mathsf{p}}\frac{\partial}{\partial t_n} + \\
&+\beta\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(n+m+\mathsf{p})t_{n+m+\mathsf{p}}\frac{\partial^2}{\partial t_n \partial t_m}\ .
\end{aligned}
\tag{3.69}
$$

We can now observe that the kernel of the operator $D - \sum_{k=1}^{\mathsf{p}-2} k t_k \frac{\partial}{\partial t_k}$ is infinite dimensional. To be more precise, in the case of $\mathsf{p} = 3$ for instance, all the monomials t_1^{ℓ} for some positive integer power ℓ are annihilated by $D - \sum_{k=1}^{\mathsf{p}-2} k t_k \frac{\partial}{\partial t_k}$ with the result that all correlators of the form $c_{\{1,\dots,1\}}$ cannot be determined. Such correlators could then be considered as additional initial data which would need to be provided in order to give a complete solution of the model.

We would like to end this subsection with a final remark. As the name suggests, the W-operators appearing in the W-representations above are the generators of a W algebra. More specifically, they are the spin-3 generators of the $W^{(3)}$ algebra which is a generalisation of the Virasoro algebra in (3.17) [16].

3.2.2 Determining correlators

Another way to solve the Virasoro constraints in (3.14), in addition to deducing the W-representation, is by determining all the correlators $c_\rho(a)$ as defined in (2.48). These two methods are equivalent since knowing all the correlators uniquely determines the generating function. Here, we consider it enough to be able to determine any correlator in a finite number of steps of a recursion, since any correlator can then in principle be established with enough computing power.

Let us give a few examples of correlators for the models discussed so far, for a potential as in (2.16). For $\mathsf{p} = 1$ one finds, as shown in paper IV,

$$
\begin{aligned}
c_{\{1\}}(a_1) &= \frac{N(\nu+\beta(N-1)+1)}{a_1} c_{\emptyset}(a_1) \\
c_{\{1,1\}}(a_1) &= \frac{N(\nu+\beta(N-1)+1)(N(\nu+\beta(N-1)+1)+1)}{a_1^2} c_{\emptyset}(a_1) \\
c_{\{2\}}(a_1) &= \frac{N(\nu+\beta(N-1)+1)(\nu+2\beta(N-1)+2)}{a_1^2} c_{\emptyset}(a_1) \\
c_{\{1,1,1\}}(a_1) &= \frac{N(\nu+\beta(N-1)+1)(N(\nu+\beta(N-1)+1)+1)}{a_1^3} \times \\
&\quad \times (N(\nu+\beta(N-1)+1)+2) c_{\emptyset}(a_1) \\
c_{\{2,1\}}(a_1) &= \frac{N(\nu+\beta(N-1)+1)(\nu+2\beta(N-1)+2)}{a_1^3} \times \\
&\quad \times (N(\nu+\beta(N-1)+1)+2) c_{\emptyset}(a_1) \\
c_{\{3\}}(a_1) &= \frac{N(\nu+\beta(N-1)+1)}{a_1^3} \Big(\nu^2+5\nu+5\beta\nu(N-1)+ \\
&\qquad + \beta(N-1)(\beta(5N-6)+11)+6\Big) c_{\emptyset}(a_1) \; .
\end{aligned}
\tag{3.70}
$$

Here it can also be noted that all correlators in degree 2 and higher are proportional to the correlator in degree 1, namely $c_{\{1\}}(a_1)$. This is because W_{-1} is of degree 1 in the time variables, such that all correlators are built up starting from the single correlator of degree 1. As another example, the first correlators for $\mathsf{p} = 2$ are given by

$$
\begin{aligned}
c_{\{1\}}(a_1,a_2) &= -\frac{a_1 N}{a_2} c_{\emptyset}(a_1,a_2) \\
c_{\{1,1\}}(a_1,a_2) &= \frac{N\left(a_1^2 N + a_2\right)}{a_2^2} c_{\emptyset}(a_1,a_2) \\
c_{\{2\}}(a_1,a_2) &= \frac{N\left(a_2(\beta(N-1)+1)+a_1^2\right)}{a_2^2} c_{\emptyset}(a_1,a_2) \\
c_{\{1,1,1\}}(a_1,a_2) &= -\frac{a_1 N^2\left(a_1^2 N + 3a_2\right)}{a_2^3} c_{\emptyset}(a_1,a_2) \\
c_{\{2,1\}}(a_1,a_2) &= -\frac{a_1 N\left(a_2\left(\beta N^2-\beta N+N+2\right)+a_1^2 N\right)}{a_2^3} c_{\emptyset}(a_1,a_2) \\
c_{\{3\}}(a_1,a_2) &= -\frac{a_1 N\left(3a_2(\beta(N-1)+1)+a_1^2\right)}{a_2^3} c_{\emptyset}(a_1,a_2) \; .
\end{aligned}
\tag{3.71}
$$

Then, in the case $\mathsf{p} = 3$ we find the correlators

$$\begin{aligned}
c_{\{1\}}(a_1, a_2, a_3) &= -\frac{\partial}{\partial a_1} c_\emptyset(a_1, a_2, a_3) \\
c_{\{1,1\}}(a_1, a_2, a_3) &= \left(-\frac{\partial}{\partial a_1}\right)^2 c_\emptyset(a_1, a_2, a_3) \\
c_{\{2\}}(a_1, a_2, a_3) &= -a_1 N c_\emptyset(a_1, a_2, a_3) \\
c_{\{1,1,1\}}(a_1, a_2, a_3) &= \left(-\frac{\partial}{\partial a_1}\right)^3 c_\emptyset(a_1, a_2, a_3) \\
c_{\{2,1\}}(a_1, a_2, a_3) &= N\left(1 + a_1 \frac{\partial}{\partial a_1}\right) c_\emptyset(a_1, a_2, a_3) \\
c_{\{3\}}(a_1, a_2, a_3) &= \left((\beta(N-1)N + N) + a_1 \frac{\partial}{\partial a_1}\right) c_\emptyset(a_1, a_2, a_3) \ , \quad (3.72)
\end{aligned}$$

where it can be noted that we can only determine the correlators up to a_1-derivatives of the empty correlator. This corresponds to the additional initial data required for models with $\mathsf{p} \geq 3$, as mentioned earlier.

3.2.3 Determining normalisations

In order to determine the normalisations or initial data – in other words the empty correlators $c_\emptyset(a)$ – one can use the correlators given above together with an additional constraint which the model in (2.21) satisfies,

$$\left(\frac{\partial}{\partial t_k} + k\frac{\partial}{\partial a_k}\right) Z(t; a) = 0 \, , \qquad k = 1, \ldots, \mathsf{p} \ , \tag{3.73}$$

as discussed in paper V. This constraint follows from the particular form of the potential in (2.16), since

$$\frac{\partial}{\partial t_k} Z(t; a) = \left\langle \sum_{i=1}^{N} \lambda_i^k \right\rangle_t = -k\frac{\partial}{\partial a_k} Z(t; a) \ . \tag{3.74}$$

We begin with the providing the example of $\mathsf{p} = 1$, where we recall from (3.70) the correlator $c_{\{1\}}(a_1)$,

$$c_{\{1\}}(a_1) = \frac{N(\nu + \beta(N-1) + 1)}{a_1} c_\emptyset(a_1) \ . \tag{3.75}$$

The additional constraint (3.73) then results in the condition

$$\frac{\partial}{\partial t_1} Z(t; a_1) = -\frac{\partial}{\partial a_1} Z(t; a_1) \ , \tag{3.76}$$

which can be translated to an infinite number of relations between correlators and their a_1-derivatives. However, due to the recursive solution to the Virasoro constraints we only need the first such relation, obtained by

$$\left[\frac{\partial}{\partial t_1}Z(t;a_1)\right]\Bigg|_{t=0} = \left[-\frac{\partial}{\partial a_1}Z(t;a_1)\right]\Bigg|_{t=0} . \tag{3.77}$$

Recalling the expression for correlators in terms of derivatives in times acting on the generating function in (2.50), the above constraint becomes

$$c_{\{1\}}(a_1) = -\frac{\partial}{\partial a_1}c_{\emptyset}(a_1) . \tag{3.78}$$

Equating the expression for $c_{\{1\}}(a_1)$ in (3.75) with the one above, we get

$$-\frac{\partial}{\partial a_1}c_{\emptyset}(a_1) = \frac{N(\nu+\beta(N-1)+1)}{a_1}c_{\emptyset}(a_1) \tag{3.79}$$

which can then be viewed as a differential equation for the empty correlator $c_{\emptyset}(a_1)$. This has a solution

$$c_{\emptyset}(a_1) = k_1^{N,\beta,\nu}a_1^{-N(\nu+\beta(N-1)+1)} \tag{3.80}$$

for some integration constant $k_1^{N,\beta,\nu}$ depending on N, β and ν but independent of a_1.

Another example is $\mathsf{p}=2$, in which case the empty correlator is $c_{\emptyset}(a_1,a_2)$ and we require two equations to determine it. Using the same logic as above where we equate the expression for the correlator from Virasoro constraints with that obtained from the additional constraint in (3.73), we then find one equation for $c_{\{1\}}(a_1,a_2)$ and one equation for $c_{\{2\}}(a_1,a_2)$. Starting with the former, we have

$$-\frac{\partial}{\partial a_1}c_{\emptyset}(a_1,a_2) = -\frac{a_1N}{a_2}c_{\emptyset}(a_1,a_2) , \tag{3.81}$$

and for the latter we get the equation

$$-2\frac{\partial}{\partial a_2}c_{\emptyset}(a_1,a_2) = \frac{N(a_2(\beta(N-1)+1)+a_1^2)}{a_2^2}c_{\emptyset}(a_1,a_2) . \tag{3.82}$$

We therefore find the solution

$$c_{\emptyset}(a_1,a_2) = k_2^{N,\beta}a_2^{-\frac{1}{2}N(\beta(N-1)+1)}\exp\left(\frac{Na_1^2}{2a_2}\right) \tag{3.83}$$

up to an integration constant $k_2^{N,\beta}$ depending on N and β but independent of a_1 and a_2.

The last example we consider is that of $\mathsf{p}=3$, in other words we wish to determine the empty correlator $c_{\emptyset}(a_1,a_2,a_3)$. Similarly to the previous examples, we therefore want to use the conditions on the correlators $c_{\{1\}}(a_1,a_2,a_3)$, $c_{\{2\}}(a_1,a_2,a_3)$ and $c_{\{3\}}(a_1,a_2,a_3)$ to determine this. However, as can be seen in (3.72) the Virasoro constraints imply

$$c_{\{1\}}(a_1,a_2,a_3) = -\frac{\partial}{\partial a_1} c_{\emptyset}(a_1,a_2,a_3), \tag{3.84}$$

which is equivalent to the condition from the additional constraint in (3.73). It turns out that $c_{\{1\}}(a_1,a_2,a_3)$ is in fact part of the initial data, in addition to the empty correlator $c_{\emptyset}(a_1,a_2,a_3)$. Thus we can only use the equations for $c_{\{2\}}(a_1,a_2,a_3)$ and $c_{\{3\}}(a_1,a_2,a_3)$ and we therefore cannot uniquely determine the empty correlator $c_{\emptyset}(a_1,a_2,a_3)$. Then, starting with the constraint from considering $c_{\{2\}}(a_1,a_2,a_3)$ we find the condition

$$-2\frac{\partial}{\partial a_2} c_{\emptyset}(a_1,a_2,a_3) = \left(-\frac{a_1 N}{a_3} + \frac{a_2}{a_3}\frac{\partial}{\partial a_1}\right) c_{\emptyset}(a_1,a_2,a_3)\ , \tag{3.85}$$

with solution

$$c_{\emptyset}(a_1,a_2,a_3) = \exp\left(\frac{N a_1 a_2}{2a_3} - \frac{N a_2^3}{12 a_3^2}\right) h\left(a_3, \frac{a_2^2}{2} - 2a_1 a_3\right) \tag{3.86}$$

for an undetermined function $h\left(a_3, \frac{a_2^2}{2} - 2a_1a_3\right)$. Upon considering $c_{\{3\}}(a_1,a_2,a_3)$ we instead have

$$\begin{aligned} -3\frac{\partial}{\partial a_3} c_{\emptyset}(a_1,a_2,a_3) &= \\ &= \left(\frac{(1-\beta(N-1))N}{a_3} - \frac{a_2^2}{a_3^2}\frac{\partial}{\partial a_1} + \frac{a_1 a_2 N}{a_3^2} + \frac{a_1}{a_3}\frac{\partial}{\partial a_1}\right) c_{\emptyset}(a_1,a_2,a_3)\ . \end{aligned} \tag{3.87}$$

We then insert the solution in (3.86) into the condition above, and upon defining the auxiliary variable $y = \frac{a_2^2}{2} - 2a_1a_3$ we find the following condition on $h\left(a_3, y\right)$,

$$-(1-\beta(N-1))N h\left(a_3,y\right) - 4y\frac{\partial}{\partial y} h\left(a_3,y\right) - 3a_3\frac{\partial}{\partial a_3} h\left(a_3,y\right) = 0\ . \tag{3.88}$$

This can be solved to find

$$h\left(a_3,y\right) = a_3^{-\frac{1}{3}(1-\beta(N-1))N} g\left(a_3^{-4/3} y\right) \tag{3.89}$$

for some function $g\left(a_3^{-4/3}y\right)$. The $\mathsf{p}=3$ solution is then

$$c_{\emptyset}(a_1,a_2,a_3) = \exp\left(\frac{N a_1 a_2}{2a_3} - \frac{N a_2^3}{12 a_3^2}\right) a_3^{-\frac{1}{3}(1-\beta(N-1))N} g\left(\frac{a_2^2 - 4a_1a_3}{2a_3^{4/3}}\right)\ , \tag{3.90}$$

where it can be remarked that this is just a solution to the Virasoro constraints in (3.14) without imposing it is of matrix model form. The undetermined function g is therefore a consequence of the fact that we only had two non-trivial conditions to impose on $c_\emptyset(a_1, a_2, a_3)$.

In order to specify the $\mathsf{p} = 3$ solution further, we can impose that the solution is of matrix integral form. If we for concreteness consider the case of $N = 1$, one constraint this matrix model assumption results in is that taking the trace and taking powers of matrices commutes, which at the level of the power sum variables in (2.33) implies

$$p_s = p_1^s \ . \tag{3.91}$$

Thus correlators of the same size must be equal when $N = 1$, i.e.

$$\frac{\partial}{\partial t_k} Z(t; a_1, a_2, a_3)|_{N=1} = \left(\frac{\partial}{\partial t_1}\right)^k Z(t; a_1, a_2, a_3)|_{N=1} \ , \tag{3.92}$$

and the additional constraint in (3.73) takes the form

$$\left(\frac{\partial}{\partial t_k} - \left(-\frac{\partial}{\partial a_1}\right)^k\right) Z(t; a_1, a_2, a_3)|_{N=1} = 0 \,, \qquad k = 1, 2, 3 \ . \tag{3.93}$$

Then the constraint for $c_{\{2\}}(a_1, a_2, a_3)$ in (3.85) becomes

$$\left(-\frac{\partial}{\partial a_1}\right)^2 c_\emptyset(a_1, a_2, a_3)|_{N=1} = \left(-\frac{a_1}{a_3} + \frac{a_2}{a_3}\frac{\partial}{\partial a_1}\right) c_\emptyset(a_1, a_2, a_3)|_{N=1} \ , \tag{3.94}$$

which results in the differential equation

$$z g(z) - 8 g''(z) = 0 \ , \qquad z = a_3^{-4/3} y = \frac{a_2^2 - 4 a_1 a_3}{2 a_3^{4/3}} \tag{3.95}$$

on the function $g(z)$ introduced in (3.89). Up to a rescaling of z this is the Airy equation and we therefore find

$$g(z)|_{N=1} = k_A \mathrm{Ai}(z/2) + k_B \mathrm{Bi}(z/2) \ , \tag{3.96}$$

for integration constants k_A, k_B and Airy functions $\mathrm{Ai}(z)$, $\mathrm{Bi}(z)$ given by

$$\begin{aligned} \mathrm{Ai}(z) &= \frac{1}{2\pi \mathrm{i}} \int_{\infty \mathrm{e}^{-\frac{\pi \mathrm{i}}{3}}}^{\infty \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}} \mathrm{e}^{\frac{t^3}{3} - tz} \, \mathrm{d}t \\ \mathrm{Bi}(z) &= \frac{1}{2\pi} \int_{-\infty}^{\infty \mathrm{e}^{-\frac{\pi \mathrm{i}}{3}}} \mathrm{e}^{\frac{t^3}{3} - tz} \, \mathrm{d}t \, + \, \frac{1}{2\pi} \int_{-\infty}^{\infty \mathrm{e}^{\frac{\pi \mathrm{i}}{3}}} \mathrm{e}^{\frac{t^3}{3} - tz} \, \mathrm{d}t \ . \end{aligned} \tag{3.97}$$

Thus the solution for $\mathsf{p} = 3$ and $N = 1$ is given by

$$c_\emptyset(a_1, a_2, a_3)|_{N=1} = \exp\left(\frac{a_1 a_2}{2 a_3} - \frac{a_2^3}{12 a_3^2}\right) a_3^{-1/3} \Big[k_A \mathrm{Ai}(z/2) + k_B \mathrm{Bi}(z/2)\Big] \ . \tag{3.98}$$

We will return to the above observations in Chapter 11.

3.3 The non-homogeneous Virasoro constraints

As a final topic in this chapter on Virasoro constraints, we wish to generalise the Virasoro constraints in (3.14). In particular, we allow for the matrix model to have non-trivial boundaries, generating boundary contributions to the Virasoro constraints. Consequently we can therefore view the Virasoro constraints as now being *non-homogeneous*. However, as shown in paper V, such constraints can sometimes be solved and we will now review their derivation and solution.

3.3.1 Deriving the non-homogeneous Virasoro constraints

The above derivation of the Virasoro constraints and their solutions can be generalised to include models with boundaries i.e.

$$Z(t;a) = \int_{[\mathfrak{a},\mathfrak{b}]^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^{2\beta} \mathrm{e}^{-\sum_{i=1}^{N} V(\lambda_i) + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \tag{3.99}$$

where the boundary contributions arise from the generic limits of integration $\mathfrak{a}$ and $\mathfrak{b}$ as shown in paper V. We can view $[\mathfrak{a},\mathfrak{b}]$ as a finite interval on the real line and thus $[\mathfrak{a},\mathfrak{b}]^N$ can be thought of as a hypercube inside $\mathbb{R}^N$. Although the generating function does depend on the integration domain parametrised by $\mathfrak{a}$ and $\mathfrak{b}$, we do not write out this dependence for ease of notation. For concreteness, we consider the particular case of a potential with $\mathsf{p} = 2$, $a_1 = 0$ and $a_2 = 1$, in other words a Gaussian potential $V_G(\lambda) = \frac{\lambda^2}{2}$. Other potentials can be analysed similarly. In this subsection we therefore drop the dependence of the partition function on the coupling constants and simply denote it by $Z_G(t)$. The Virasoro constraints in (3.14) are then modified according to

$$\left(\frac{\partial}{\partial t_{n+2}} - L_n\right) Z_G(t) = B_n(t), \qquad n \geq -1\ , \tag{3.100}$$

where the boundary contributions are collected in the term $B_n(t)$. To elaborate more on the boundary contribution $B_n(t)$, this is obtained by evaluating the integrand at the boundaries for one direction λ_k with $k = 1, \ldots, N$ at the time, while the integrals over the other $N-1$ directions remain. The boundary contribution can therefore be associated to a matrix model living on the $(N-1)$-dimensional faces of the N-dimensional hypercube. Explicitly, we get

$$B_n(t) = -\left[\sum_{k=1}^{N} \lambda_k^{n+1} \exp\left(-V_G(\lambda_k) + \sum_{s=1}^{\infty} t_s \lambda_k^s\right) \times \right.$$
$$\times \int_{[\mathfrak{a},\mathfrak{b}]^{N-1}} \prod_{\substack{i=1\\ i\neq k}}^{N-1} \mathrm{d}\lambda_i \prod_{\substack{1\leq i\leq N\\ i\neq k}} (\lambda_i - \lambda_k)^{2\beta} \prod_{\substack{1\leq i<j\leq N\\ i,j\neq k}} (\lambda_i - \lambda_j)^{2\beta} \times$$
$$\left.\left.\times \exp\left(-\sum_{\substack{i=1\\ i\neq k}}^{N} V_G(\lambda_i) + \sum_{s=1}^{\infty} t_s \sum_{\substack{i=1\\ i\neq k}}^{N} \lambda_i^s\right)\right]\right|_{\lambda_k=\mathfrak{a}}^{\lambda_k=\mathfrak{b}}, \tag{3.101}$$

using the notation

$$f(z)|_{z=\mathfrak{a}}^{z=\mathfrak{b}} = f(\mathfrak{b}) - f(\mathfrak{a}) . \tag{3.102}$$

Since all the N directions are equivalent, we can simply call the evaluated variable for $\lambda_k = z$ and we get N equal contributions to $B_n(t)$. Thus,

$$B_n(t) = -N \left\langle \prod_{i=1}^{N-1} (\lambda_i - z)^{2\beta} \right\rangle_t^{N-1} z^{n+1} \exp\left(-V_G(z) + \sum_{s=1}^{\infty} t_s z^s\right)\Bigg|_{z=\mathfrak{a}}^{z=\mathfrak{b}}, \tag{3.103}$$

where we introduced the notation

$$\left\langle \prod_{i=1}^{N} (\lambda_i - z)^{2\beta} \right\rangle_t^{N} = \int_{[\mathfrak{a},\mathfrak{b}]^N} \prod_{i=1}^{N} \mathrm{d}\lambda_i \prod_{i=1}^{N} (\lambda_i - z)^{2\beta} \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)^{2\beta} \times$$
$$\times \exp\left(-\sum_{i=1}^{N} V_G(\lambda_i) + \sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s\right) \tag{3.104}$$

to stress the dependence on the rank N.

3.3.2 Solving the non-homogeneous Virasoro constraints

Let us now solve the non-homogeneous Virasoro constraints in (3.100). Following the procedure outlined in Section 3.2.1, we re-sum the constraints with weight $(n+2)t_{n+2}$ from $n = -1$ to $n = \infty$ to find

$$DZ_G(t) = W_{-2} Z_G(t) + B(t) \tag{3.105}$$

where D and W_{-2} are given in (3.44) and (3.45) respectively, and $B(t)$ is the re-summed contribution from the boundary,

$$B(t) = \sum_{n=-1}^{\infty} (n+2) t_{n+2} B_n(t) . \tag{3.106}$$

Then, the recursive solution is given by

$$Z_G^{(d)}(t) = \frac{1}{d}\left(W_{-2} Z_G^{(d-2)}(t) + B^{(d)}(t)\right) \tag{3.107}$$

for a degree d with respect to the dilatation operator D. Upon assuming

$$B(t) = \sum_{d=1}^{\infty} B^{(d)}(t)\,, \qquad [D, B^{(d)}(t)] = d\, B^{(d)}(t) \tag{3.108}$$

with $B(0) = 0$, we find

$$Z_G(t) = \exp(W_{-2}/2) c_{\emptyset}(a_1 = 0, a_2 = 1) + \sum_{s=0}^{\infty}\sum_{d=1}^{\infty} \frac{W_{-2}^s B^{(d)}(t)}{2^s d \left(\frac{d}{2}+1\right)_s}\,. \tag{3.109}$$

In the above we can observe that the non-homogeneous solution factorises into a homogeneous contribution and the contribution from the boundary given by $B(t)$. Here, $(x)_n$ is the Pochhammer symbol defined by

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \tag{3.110}$$

using the Gamma function $\Gamma(x)$,

$$\Gamma(x) = \int_0^{\infty} t^{x-1} \mathrm{e}^{-t} \mathrm{d}t \tag{3.111}$$

for $\mathrm{Re}(x) > 0$. Similarly to the homogeneous Virasoro constraints, one can use the expression for the generating function in (3.109) in order to determine the correlators for the model. Summarising the results of paper V, we find for $N = 1$ the correlators

$$\begin{aligned}
c_{\emptyset}^{N=1} &= Z_G(0)|_{N=1} \\
c_{\{1\}}^{N=1} &= \mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} - \mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} \\
c_{\{2\}}^{N=1} &= Z_G(0)|_{N=1} + \mathfrak{a}\mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} - \mathfrak{b}\mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} \\
c_{\{3\}}^{N=1} &= (\mathfrak{a}^2+2)\mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} - (\mathfrak{b}^2+2)\mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} \\
c_{\{4\}}^{N=1} &= 3\, Z_G(0)|_{N=1} + \mathfrak{a}(\mathfrak{a}^2+3)\mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} - \mathfrak{b}(\mathfrak{b}^2+3)\mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} \\
c_{\{5\}}^{N=1} &= (\mathfrak{a}^4+4\mathfrak{a}^2+8)\mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} - (\mathfrak{b}^4+4\mathfrak{b}^2+8)\mathrm{e}^{-\frac{\mathfrak{b}^2}{2}}\,,
\end{aligned} \tag{3.112}$$

upon letting $c_{\lambda}^{N=1} = c_{\lambda}^{N=1}(a_1 = 0, a_2 = 1)$. We remark that for $N = 1$ the correlators only depend on the size and not the shape of the partition. Upon explicitly evaluating the empty correlator

$$Z_G(0)|_{N=1} = \int_{\mathfrak{a}}^{\mathfrak{b}} \mathrm{e}^{-\lambda^2/2} \mathrm{d}\lambda = \sqrt{\frac{\pi}{2}}\left(\mathrm{erf}\left(\frac{\mathfrak{b}}{\sqrt{2}}\right) - \mathrm{erf}\left(\frac{\mathfrak{a}}{\sqrt{2}}\right)\right)\,, \tag{3.113}$$

where the error function $\mathrm{erf}(z)$ is defined by the series

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} \ , \tag{3.114}$$

we can obtain a closed formula for the $N=1$ correlators given by

$$c_{\rho}^{N=1} = F_{|\rho|} \ . \tag{3.115}$$

Here we introduced the auxiliary function F_s,

$$\begin{aligned} F_s =& \frac{(1+(-1)^s)}{2}(s-1)!! \ Z_G(0)|_{N=1} + \\ &+ \left(\sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \frac{(s-1)!!}{(s-1-2k)!!} \mathfrak{a}^{s-1-2k} \right) \mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} + \\ &- \left(\sum_{k=0}^{\lfloor \frac{s-1}{2} \rfloor} \frac{(s-1)!!}{(s-1-2k)!!} \mathfrak{b}^{s-1-2k} \right) \mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} \ . \end{aligned} \tag{3.116}$$

For $N=2$, we use the form of the boundary contribution in (3.103) to observe that we require integer β in order to have expectation values of polynomials which can be evaluated combinatorially. As the simplest example, we consider $\beta=1$ using which we find the boundary contribution

$$\begin{aligned} B^{(d)}(t) =& 2 \sum_{n=-1}^{d-2} (n+2) t_{n+2} \sum_{\substack{\lambda,\rho \\ |\lambda|+|\rho|=d-(n+2)}} \frac{1}{|\mathrm{Aut}(\lambda)|} \frac{1}{|\mathrm{Aut}(\rho)|} \prod_{\ell\in\lambda} t_\ell \prod_{r\in\rho} t_r \times \\ & \times \Big[\Big(F_{2+|\lambda|} - 2\mathfrak{a} F_{1+|\lambda|} + \mathfrak{a}^2 F_{|\lambda|} \Big) \mathfrak{a}^{n+1+|\rho|} \mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} + \\ & - \Big(F_{2+|\lambda|} - 2\mathfrak{b} F_{1+|\lambda|} + \mathfrak{b}^2 F_{|\lambda|} \Big) \mathfrak{b}^{n+1+|\rho|} \mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} \Big] \end{aligned} \tag{3.117}$$

such that the correlators take the form

$$\begin{aligned} c_{\emptyset}^{N=2} &= Z_G(0)|_{N=2} \\ c_{\{1\}}^{N=2} =& 2\mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} (F_0 \mathfrak{a}^2 - 2F_1 \mathfrak{a} + F_2) - 2\mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} (F_0 \mathfrak{b}^2 - 2F_1 \mathfrak{b} + F_2) \\ c_{\{1,1\}}^{N=2} =& 2\Big(\ Z_G(0)|_{N=2} + \\ & + \mathfrak{a}\mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} (F_0 \mathfrak{a}^2 - 2F_1 \mathfrak{a} + F_2) - \mathfrak{b}\mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} (F_0 \mathfrak{b}^2 - 2F_1 \mathfrak{b} + F_2) + \\ & + \mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} (F_1 \mathfrak{a}^2 - 2F_2 \mathfrak{a} + F_3) - \mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} (F_1 \mathfrak{b}^2 - 2F_2 \mathfrak{b} + F_3) \Big) \\ c_{\{2\}}^{N=2} =& 4 \ Z_G(0)|_{N=2} + 2\mathfrak{a}\mathrm{e}^{-\frac{\mathfrak{a}^2}{2}} (F_0 \mathfrak{a}^2 - 2F_1 \mathfrak{a} + F_2) + \\ & - 2\mathfrak{b}\mathrm{e}^{-\frac{\mathfrak{b}^2}{2}} (F_0 \mathfrak{b}^2 - 2F_1 \mathfrak{b} + F_2) \ . \end{aligned} \tag{3.118}$$

As a final comment on the solution to the non-homogeneous Virasoro constraints, we consider the example of a boundary where $\mathfrak{a} \to 0$ and $\mathfrak{b} \to \infty$, i.e. the orthant $[0, \infty)^N$. As before, we restrict to the Gaussian potential $V_G(\lambda) = \frac{\lambda^2}{2}$. In this case the boundary term in (3.103) simplifies to

$$B_n(t) = N \left\langle \prod_{i=1}^{N-1} \lambda_i^{2\beta} \right\rangle_t^{N-1} \delta_{n,-1} \ , \tag{3.119}$$

as there is only a contribution from the boundary at zero. Therefore only the $n = -1$ constraint in (3.100) is non-homogeneous,

$$\left(\frac{\partial}{\partial t_1} - L_{-1} \right) Z_G(t) = B_{-1}(t) \ , \tag{3.120}$$

while the Virasoro constraints for $n > -1$ are given by the homogeneous Virasoro constraints for a Gaussian potential in (3.14). In the case of rank $N = 1$, we then employ (3.109) to find that the correlators are given by

$$c_\rho^{N=1} = 2^{\frac{|\rho|-1}{2}} \Gamma \left(\frac{|\rho| + 1}{2} \right) \tag{3.121}$$

for a partition ρ and with the Γ-function in (3.111). Then in the case of $N = 2$ we instead get for the first few correlators

$$\begin{aligned}
c_\emptyset^{N=2} &= Z_G(0)|_{N=2} \\
c_{\{1\}}^{N=2} &= 2^{\beta+\frac{1}{2}} \Gamma \left(\beta + \frac{1}{2} \right) \\
c_{\{1,1\}}^{N=2} &= 2 \left(2^\beta \Gamma(\beta + 1) + Z_G(0)|_{N=2} \right) \\
c_{\{2\}}^{N=2} &= 2(\beta + 1) \ Z_G(0)|_{N=2} \\
c_{\{1,1,1\}}^{N=2} &= 2^{\beta+\frac{1}{2}} (2\beta + 5) \Gamma \left(\beta + \frac{1}{2} \right) \\
c_{\{2,1\}}^{N=2} &= \frac{2^{\beta+\frac{5}{2}} \Gamma \left(\beta + \frac{5}{2} \right)}{2\beta + 1} \\
c_{\{3\}}^{N=2} &= 2^{\beta+\frac{3}{2}} (\beta + 1) \Gamma \left(\beta + \frac{1}{2} \right) \ .
\end{aligned} \tag{3.122}$$

Part II: Quantum matrix models

Let us now consider quantum, or q-deformed, versions of the classical matrix model introduced earlier. The motivation for introducing this deformation at the level of the Virasoro algebra was to explore the correspondence between the Calogero-Sutherland model – a many body quantum mechanical system – and CFT [22]. This was because excited states of the Calogero-Sutherland model are given in terms of Jack polynomials, who have a known q-deformation given by Macdonald polynomials [21]. The question was then if the Virasoro algebra on the CFT side could also be q-deformed, where the answer was found to be affirmative [22]. At the level of matrix models on the other hand, q-deformed matrix models first appeared in [24], where they were used in trying to extend the AGT correspondence between 4d $\mathcal{N} = 2$ theories and 2d CFT's into a 5d setting. More recently, quantum matrix models have appeared in the results of localisation computations in which supersymmetric gauge theory observables are computed exactly. (See for instance [4] and references therein for examples of such results.) With this localisation application in mind, we now wish to explore how the matrix model and the Virasoro constraints introduced in the first part, generalises under a q-deformation.

4. Introduction to q-calculus

In this chapter we introduce the basics of q-calculus as given in [60]. Firstly, we have the *q-shift* operator $\hat{M}_q$, which shifts the argument of a function $F(x)$ according to

$$\hat{M}_q F(x) = F(qx) \ . \tag{4.1}$$

The q-shift operator can also be generalised to act on functions with multiple variables, $\hat{M}_{q,i}$. This acts as

$$\hat{M}_{q,i} F(x_1, \ldots, x_m) = F(x_1, \ldots, qx_i, \ldots, x_m) \ , \tag{4.2}$$

where the q-shift is on the i-th variable. Next, the *q-differential* is

$$\mathrm{d}_q F(x) = F(qx) - F(x) = (\hat{M}_q - 1)F(x) \tag{4.3}$$

so that in particular $\mathrm{d}_q x = (q-1)x$. Generalising the q-differential similarly to the q-shift operator, we define $\mathrm{d}_{q,i}$ as

$$\begin{aligned} \mathrm{d}_{q,i} F(x) =& F(x_1, \ldots, qx_i, \ldots, x_m) - F(x_1, \ldots, x_m) = \\ =& (\hat{M}_{q,i} - 1)F(x_1, \ldots, x_m) \ . \end{aligned} \tag{4.4}$$

Using the q-differential, the *q-derivative* D_q is then

$$D_q F(x) = \frac{\mathrm{d}_q F(x)}{\mathrm{d}_q x} = \frac{F(qx) - F(x)}{(q-1)x} = \frac{1}{(q-1)x}(\hat{M}_q - 1)F(x) \tag{4.5}$$

valid for $|q| < 1$. In order to recover the standard derivative, the limit to take is $q \to 1$ upon which $D_q F(x) \to \frac{\mathrm{d}}{\mathrm{d}x} F(x)$. One can also define the *q-number* generalising a positive integer n through

$$[n]_q = \frac{q^n - 1}{q - 1} \ , \tag{4.6}$$

which is a q-analogue in the sense that $[n]_q \to n$ as $q \to 1$. Thus, the q-derivative as given in (4.5) of x^n is

$$D_q x^n = \frac{(qx)^n - x^n}{(q-1)x} = [n]_q \, x^{n-1} \ . \tag{4.7}$$

Again, in the limit $q \to 1$ we recover the standard result $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n x^{n-1}$. To end this chapter, we would also like to mention the concept of a *q-constant*. This is a function which is invariant under q-shifts $\hat{M}_q$ defined in (4.1), i.e. for a function $g(x)$,

$$\hat{M}_q \, g(x) = g(qx) = g(x) \ . \tag{4.8}$$

5. Special q-functions

As an extension of the previous chapter, we now introduce various q-functions that are used in the coming chapters when discussing quantum generating functions. To begin with, the multiple *q-Pochhammer symbol* $(z; q_1, \ldots, q_N)_\infty$ is given by

$$
\begin{aligned}
(z; q_1, \ldots, q_N)_\infty =& \exp\left(-\sum_{n=1}^{\infty} \frac{z^n}{n \prod_{k=1}^{N}(1-q_k^n)}\right) = \\
=& \prod_{n_1,\ldots,n_N=0}^{\infty} (1 - z q_1^{n_1} \ldots q_N^{n_N}) \ ,
\end{aligned}
\tag{5.1}
$$

where $z \in \mathbb{C}$ and $|q_k| < 1$. In the case of a single argument, one can define the function in the region $|q| > 1$ by means of

$$
(z; q)_\infty = (q^{-1} z; q^{-1})_\infty^{-1} \ . \tag{5.2}
$$

The next function we wish to consider is the *theta function* $\Theta(z; q)$,

$$
\Theta(z; q) = (z; q)_\infty \, (q z^{-1}; q)_\infty \ . \tag{5.3}
$$

Denoting $\underline{\omega} = \{\omega_1, \omega_2\} \in \mathbb{C}^2$ and $\omega = \omega_1 + \omega_2$ with $\mathrm{Re}(\omega_1) > 0$, $\mathrm{Re}(\omega_2) > 0$ and $X \in \mathbb{C}$, the theta function satisfies the modular property

$$
\Theta\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\omega_1} X}; \mathrm{e}^{2\pi \mathrm{i} \frac{\omega}{\omega_1}}\right) \Theta\left(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\omega_2} X}; \mathrm{e}^{2\pi \mathrm{i} \frac{\omega}{\omega_2}}\right) = \mathrm{e}^{-\mathrm{i}\pi B_{22}(X|\underline{\omega})} \tag{5.4}
$$

where $B_{22}(X|\underline{\omega})$ is the quadratic Bernoulli polynomial given by

$$
B_{22}(X|\underline{\omega}) = \frac{1}{\omega_1 \omega_2} \left(\left(X - \frac{\omega}{2}\right)^2 - \frac{\omega_1^2 + \omega_2^2}{12}\right) \ . \tag{5.5}
$$

Then, the *elliptic Gamma function* $\Gamma(z; p, q)$ [61], is defined as

$$
\Gamma(z; p, q) = \frac{(p q z^{-1}; p, q)_\infty}{(z; p, q)_\infty} = \exp\left(\sum_{k \neq 0} \frac{z^k}{k(1-p^k)(1-q^k)}\right) \tag{5.6}
$$

using the q-Pochhammer symbol in (5.1). Letting now $\underline{\omega} = \{\omega_1, \omega_2, \omega_3\}$, the elliptic Gamma function satisfies the modular properties

$$
\begin{aligned}
\mathrm{e}^{-\frac{\mathrm{i}\pi}{3} B_{33}(X|\underline{\omega})} =& \Gamma(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\omega_1} X}; \mathrm{e}^{2\pi \mathrm{i} \frac{\omega_2}{\omega_1}}, \mathrm{e}^{2\pi \mathrm{i} \frac{\omega_3}{\omega_1}}) \Gamma(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\omega_2} X}; \mathrm{e}^{2\pi \mathrm{i} \frac{\omega_1}{\omega_2}}, \mathrm{e}^{2\pi \mathrm{i} \frac{\omega_3}{\omega_2}}) \times \\
& \times \Gamma(\mathrm{e}^{\frac{2\pi \mathrm{i}}{\omega_3} X}; \mathrm{e}^{2\pi \mathrm{i} \frac{\omega_1}{\omega_3}}, \mathrm{e}^{2\pi \mathrm{i} \frac{\omega_2}{\omega_3}}) \ ,
\end{aligned}
\tag{5.7}
$$

using this time the cubic Bernoulli polynomial $B_{33}(X|\underline{\omega})$

$$\begin{aligned} B_{33}(X|\underline{\omega}) =& \frac{1}{\omega_1\omega_2\omega_3}\left(X - \frac{\omega}{2} - \frac{\omega_3}{2}\right)\times \\ & \times\left(\left(X-\frac{\omega}{2}\right)^2 - \omega_3\left(X-\frac{\omega}{2}\right) - \frac{\omega_1^2+\omega_2^2}{4}\right) , \end{aligned} \tag{5.8}$$

where $\omega = \omega_1 + \omega_2$ as before.

Next, the *double sine function* $S_2\left(X|\underline{\omega}\right)$ is defined by [62,63],

$$S_2\left(X|\underline{\omega}\right) = \prod_{n_1,n_2=0}^{\infty} \frac{n_1\omega_1 + n_2\omega_2 + X}{n_1\omega_1 + n_2\omega_2 + \omega - X} . \tag{5.9}$$

This infinite product is ζ-regularised meaning that we define the double sine through the combination of elliptic Gamma functions

$$S_2\left(X|\underline{\omega}\right) = \Gamma\left(X;\omega_1,\omega_2\right)^{-1}\Gamma\left(\omega - X;\omega_1,\omega_2\right) , \tag{5.10}$$

where the elliptic Gamma is given by

$$\Gamma\left(X;\omega_1,\omega_2\right) = \exp\left(\frac{\partial}{\partial s}\zeta_2(s,X|\underline{\omega})\bigg|_{s=0}\right) \tag{5.11}$$

with the double ζ-function being

$$\zeta_2(s,X|\underline{\omega}) = \sum_{n_1,n_2=0}^{\infty}\frac{1}{(n_1\omega_1+n_2\omega_2+X)^s} . \tag{5.12}$$

The double sine function also satisfies an inversion property given by

$$S_2\left(X|\underline{\omega}\right)S_2\left(\omega - X|\underline{\omega}\right) = 1 , \tag{5.13}$$

which can be seen from the definition in (5.9). Then, imposing that the parameters satisfies $\mathrm{Im}(\frac{\omega_2}{\omega_1}) \neq 0$, the double sine function factorises into q-Pochhammer symbols according to

$$S_2(X|\underline{\omega}) = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}B_{22}(X|\underline{\omega})}\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_1}X};\mathrm{e}^{2\pi\mathrm{i}\frac{\omega}{\omega_1}}\right)_\infty\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_2}X};\mathrm{e}^{2\pi\mathrm{i}\frac{\omega}{\omega_2}}\right)_\infty , \tag{5.14}$$

with the quadratic Bernoulli polynomial defined in (5.5). Another remark about the double sine function, is regarding a comparison with the notation used here and the literature, for instance [64]. One finds that

$$S_2\left(\omega/2 - \mathrm{i}X|\underline{\omega}\right) = s_b\left(X\right) \tag{5.15}$$

with $\omega_1 = \omega_2^{-1} = b$ and $\omega = Q$.

Finally, we have the *Macdonald polynomials* $\mathsf{Macdonald}_\gamma\left(\lambda_k\right)$ which are a 1- and 2-parameter deformation of the Jack and Schur polynomials in

Section 2.3 respectively [21]. They are uniquely defined by two conditions. Firstly, they are of the form

$$\mathsf{Macdonald}_\gamma(\lambda_k) = \sum_{\mu\leq\gamma} u_{\gamma\mu} m_\mu(\lambda_k) \tag{5.16}$$

where $u_{\gamma\mu}$ are rational functions of the deformation parameters q and t with $u_{\gamma\gamma}=1$ and $m_\mu(\lambda_k)$ are the monomial symmetric functions in (2.30). Secondly, they are orthogonal with respect to the inner product

$$(f(\underline{\lambda})\,|\,g(\underline{\lambda}))_{q,t} = \frac{1}{(2\pi\mathrm{i})^N N!}\oint_{|\lambda|=1}\prod_{i=1}^{N}\frac{\mathrm{d}\lambda_i}{\lambda_i}\Delta_{q,t}(\underline{\lambda})f(\underline{\lambda})g(\underline{\lambda}^{-1})\ , \tag{5.17}$$

where the measure $\Delta_{q,t}(\underline{\lambda})$ is the q-deformed Vandermonde determinant to be defined in (6.16). Thus,

$$\left(\mathsf{Macdonald}_\gamma(\lambda_k)\ |\ \mathsf{Macdonald}_\mu(\lambda_k)\right)_{q,t} = 0\,, \qquad \text{if } \gamma\neq\mu\ . \tag{5.18}$$

Using instead the power sum variables in (2.34), the first Macdonald polynomials are explicitly given by

$$\begin{aligned}
\mathsf{Macdonald}_{\{\}}(p_k) &= 1\\
\mathsf{Macdonald}_{\{1\}}(p_k) &= p_1\\
\mathsf{Macdonald}_{\{2\}}(p_k) &= \frac{p_1^2(q+1)(t-1)+p_2(q-1)(t+1)}{2(qt-1)}\\
\mathsf{Macdonald}_{\{1,1\}}(p_k) &= \frac{p_1^2-p_2}{2}\\
\mathsf{Macdonald}_{\{3\}}(p_k) &= \frac{1}{6(qt-1)(q^2t-1)}\Big[3p_2p_1(q^3-1)(t^2-1)+\\
&\quad + p_1^3(q^3+2q^2+2q+1)(t-1)^2+\\
&\quad + 2p_3(q-1)^2(q+1)(t^2+t+1)\Big]\\
\mathsf{Macdonald}_{\{2,1\}}(p_k) &= \frac{1}{6(qt^2-1)}\Big[-2p_3(q-1)(t^2+t+1)+\\
&\quad + p_1^3(t-1)(2qt+q+t+2)+3p_2p_1(t+1)(q-t)\Big]\\
\mathsf{Macdonald}_{\{1,1,1\}}(p_k) &= \frac{p_1^3-3p_1p_2+2p_3}{6}\ .
\end{aligned} \tag{5.19}$$

We can then take the limit $t=q$ of the Macdonald polynomials to obtain the Schur polynomials,[1]

$$\mathsf{Macdonald}_\gamma(p_k)\big|_{t=q} = \mathsf{Schur}_\gamma(p_k)\ . \tag{5.20}$$

[1] It should be noted that $t=q$ is enough to recover Schur from Macdonald polynomials, but at the level of operators and other functions we also need to take $q\to 1$.

Additionally we can take the *semi-classical limit* where we let $t = q^\beta$ with $q \to 1$, to obtain the Jack polynomials

$$\mathsf{Macdonald}_\gamma(p_k)\big|_{t=q^\beta,\, q\to 1} = \mathsf{Jack}_\gamma(p_k)\ . \tag{5.21}$$

In other words, the Macdonald polynomials are the q, t-deformed version of the Schur polynomials and the q-deformed version of the Jack polynomials. The relations between the special polynomials are illustrated in Figure 5.1.

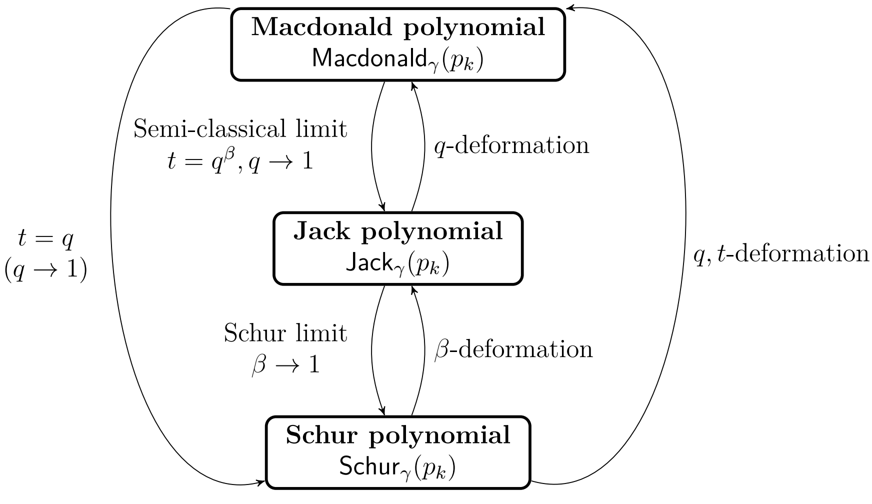

Figure 5.1. Illustration of relations between Schur, Jack and Macdonald polynomials.

6. The q, t-deformation

One particular example of a quantum model is the q, t-deformation, where the deformed algebra was discovered in [22] whereas the deformed matrix model was introduced in [24]. Here, the deformation is parametrised by the variables q and t, or sometimes q and β, and is consequently a 2-parameter deformation of the un-deformed classical case. This deformation is sometimes referred to as a *trigonometric* deformation, which is a name we will at times employ. We begin with exploring the q-analogue of the Virasoro algebra, together with the q-deformed matrix model and its corresponding q-Virasoro constraints. Similarly to the classical case, we then review how to solve the q-Virasoro constraints. Finally, we summarise how to recover Virasoro from q-Virasoro.

6.1 The q-Virasoro algebra

Let us now explore the q-analogue of both the generating function and also the Virasoro constraints that the generating function satisfies. The q-deformed version of the Virasoro algebra is given by the q-Virasoro algebra first introduced in [22]. Before we explore the constraints that the q-deformed generating functions satisfy, we first review the construction of the q-Virasoro algebra which will be crucial in defining the constraints. The generators of the *q-Virasoro algebra*, T_n, satisfy the associative algebra [22]

$$\begin{aligned}[T_n, T_m] = &-\sum_{l=1}^{\infty} f_l (T_{n-l}T_{m+l} - T_{m-l}T_{n+l}) + \\ &- \frac{(1-q)(1-t^{-1})}{(1-p)}(p^n - p^{-n})\delta_{n+m,0} \ .\end{aligned} \tag{6.1}$$

The parameters here are $p, q, t \in \mathbb{C}$ with $p = qt^{-1}$. It should be noted that the deformation parameter t is not to be confused with the time variables $\{t_s\}$. Then, the coefficients of the structure function $f(z) = \sum_{l=0}^{\infty} f_l z^l$ are given by the series expansion of

$$f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{(1-q^n)(1-t^{-n})}{n(1+p^n)} z^n\right) \ . \tag{6.2}$$

Collecting the generators into the stress tensor current $T(z)$,

$$T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n} \ , \tag{6.3}$$

one can also cast the commutation relation in (6.1) as

$$\begin{aligned} & f\left(\frac{w}{z}\right) T(z)T(w) - f\left(\frac{z}{w}\right) T(w)T(z) = \\ & = -\frac{(1-q)(1-t^{-1})}{(1-p)}\left(\delta\left(p\frac{w}{z}\right) - \delta\left(p^{-1}\frac{w}{z}\right)\right) \ . \end{aligned} \tag{6.4}$$

Here $\delta(z)$ is the multiplicative δ-function

$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n \ , \tag{6.5}$$

which acts as $\delta(z)\phi(z) = \delta(z)\phi(1)$ for a Laurent series $\phi(z)$.

In order to give explicit expressions for the generators of the q-Virasoro algebra, one first needs to introduce the free boson oscillators a_n generalising the classical oscillators in (3.21),[1]

$$\begin{aligned} & [\mathsf{a}_n, \mathsf{a}_m] = \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\delta_{n+m,0} \ , \qquad n,m \in \mathbb{Z}\backslash\{0\} \ , \\ & [\mathsf{P}, \mathsf{Q}] = 2 \ . \end{aligned} \tag{6.6}$$

The stress tensor current $T(z)$ then takes form

$$T(z) = \sum_{\sigma=\pm 1} \Lambda_\sigma(z) = \sum_{\sigma=\pm 1} : \mathrm{e}^{\sigma \sum_{n\neq 0} \frac{z^{-n}}{(1+p^{-\sigma n})}\mathsf{a}_n} : q^{\sigma\frac{\sqrt{\beta}}{2}\mathsf{P}} p^{\frac{\sigma}{2}} \ . \tag{6.7}$$

Thus the explicit expressions for the generators become[2]

$$\begin{aligned} T_{n\geq 0} &= \sum_{\sigma=\pm 1} q^{\sigma\frac{\sqrt{\beta}}{2}\mathsf{P}} p^{\frac{\sigma}{2}} \sum_{m\geq 0} \mathsf{Schur}_{\{m\}}\left(p_k = A^{(\sigma)}_{-k}\right) \mathsf{Schur}_{\{n+m\}}\left(p_k = A^{(\sigma)}_{k}\right) \\ T_{n<0} &= \sum_{\sigma=\pm 1} q^{\sigma\frac{\sqrt{\beta}}{2}\mathsf{P}} p^{\frac{\sigma}{2}} \sum_{m\geq 0} \mathsf{Schur}_{\{m-n\}}\left(p_k = A^{(\sigma)}_{-k}\right) \mathsf{Schur}_{\{m\}}\left(p_k = A^{(\sigma)}_{k}\right) \end{aligned} \tag{6.8}$$

where

$$A^{(\sigma)}_n = \sigma \frac{\mathsf{a}_n |n|}{(1+p^{-\sigma n})} \tag{6.9}$$

and $\mathsf{Schur}_{\{n\}}(p_k)$ is the Schur polynomial in symmetric representation $\{n\}$, as given in (2.36). Similarly to the classical case, we also use a differential representation of the free boson algebra in terms of the time variables

[1] We will here use the same notation for the free boson oscillators as in the classical case, but we hope it will be clear from the context which ones we are referring to.

[2] Here it should be noted that p is a complex deformation parameter which should not be confused with the argument of the symmetric polynomials typically denoted by p_k.

$\{t_k\}$, given by

$$\mathsf{a}_{-n} \simeq (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) t_n \ , \qquad \mathsf{a}_n \simeq \frac{1}{n}(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}}) \frac{\partial}{\partial t_n} \ , \qquad n \in \mathbb{Z}_{>0}$$
$$\mathsf{Q} \simeq \sqrt{\beta} t_0 \ , \qquad \mathsf{P} \simeq 2\frac{1}{\sqrt{\beta}} \frac{\partial}{\partial t_0} \ , \qquad |\alpha\rangle = \mathrm{e}^{\frac{\alpha}{2}\mathsf{Q}}|0\rangle \simeq \mathrm{e}^{\sqrt{\beta} t_0 \frac{\alpha}{2}} \cdot 1 \ . \qquad (6.10)$$

This again generalises the representation in the classical case in (3.31). Finally, we note that the q-Virasoro algebra is invariant under

$$\sqrt{\beta} \to -\frac{1}{\sqrt{\beta}} \, , \qquad q \to t^{-1} \ , \qquad (6.11)$$

similar to the classical symmetry observed in (3.16).

6.2 The q-deformed matrix model

Mirroring the classical case, we now wish to construct a generating function which satisfies a q-deformed version of the Virasoro constraints in (3.14) following [26]. To do so, we use the screening current $\mathsf{S}(x)$,

$$\mathsf{S}(x) =: \mathrm{e}^{-\sum_{n\neq 0} \frac{x^{-n}}{(q^{n/2} - q^{-n/2})} \mathsf{a}_n} : \mathrm{e}^{\sqrt{\beta}\mathsf{Q}} x^{\sqrt{\beta}\mathsf{P}} \ , \qquad (6.12)$$

which is defined by the property that

$$[T_n, \mathsf{S}(x)] = D_q \mathsf{O}_n(x) = \frac{\mathsf{O}_n(qx) - \mathsf{O}_n(x)}{(q-1)x} \qquad (6.13)$$

for some operator $\mathsf{O}_n(x)$, recalling the q-derivative D_q defined in (4.5). Just as in the classical case in (3.36), there is another screening current which can be used. This is the one obtained by performing the shifts in (6.11) [26]. Again, this screening current can be treated similarly and we here consider the current in (6.12). Next, we follow the classical construction in (3.33) and employ this screening current to build up the generating function (where the x_i^{-1} in the measure is for consistency with paper II)

$$\mathsf{Z}|\alpha\rangle = \oint \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{2\pi \mathrm{i} x_i} \mathsf{S}(x_i)|\alpha\rangle \ . \qquad (6.14)$$

In order to obtain the generating function, we rewrite the integrand as

$$\prod_{i=1}^{N} \mathsf{S}(x_i) = : \prod_{i=1}^{N} \mathsf{S}(x_i) : \ \Delta_{q,t}(\underline{x}) c_\beta(\underline{x}; q) \prod_{j=1}^{N} x_j^{\beta(N-1)} \ . \qquad (6.15)$$

Here we defined the *q-deformed Vandermonde determinant* $\Delta_{q,t}(\underline{x})$,

$$\Delta_{q,t}(\underline{x}) = \prod_{1\leq k\neq j\leq N} \frac{(x_k x_j^{-1};q)_\infty}{(t x_k x_j^{-1};q)_\infty} , \tag{6.16}$$

together with the function $c_\beta(\underline{x};q)$,

$$c_\beta(\underline{x};q) = \prod_{1\leq k<j\leq N} \left(x_k x_j^{-1}\right)^\beta \frac{\Theta(t x_k x_j^{-1};q)}{\Theta(x_k x_j^{-1};q)} . \tag{6.17}$$

Then, with the choice of the differential representation of the free boson algebra in (6.10), we find that the generating function $Z(t) \simeq \mathsf{Z}|\alpha\rangle$ is[3]

$$\begin{aligned} Z(t) = \mathrm{e}^{t_0\left(N+\frac{\alpha}{2\sqrt{\beta}}\right)} \oint \prod_{i=1}^N \frac{\mathrm{d}x_i}{2\pi\mathrm{i}x_i} \Delta_{q,t}(\underline{x}) c_\beta(\underline{x};q) \times \\ \times\, \mathrm{e}^{\sum_{k=1}^N \sqrt{\beta}\left(\alpha+\sqrt{\beta}N-Q_\beta\right)\ln(x_k)+\sum_{s=1}^\infty t_s \sum_{j=1}^N x_j^s} . \end{aligned} \tag{6.18}$$

In what follows we refer to such q-deformed matrix model as the *quantum* matrix model. $Z(t)$ then satisfies the *q-Virasoro constraints*

$$T_n Z(t) = 0 , \qquad n > 0 , \tag{6.19}$$

for generators T_n of the q-Virasoro algebra in (6.1), which follows from

$$T_n \mathsf{Z}|\alpha\rangle = [T_n, \mathsf{Z}]|\alpha\rangle = [T_n, \oint \prod_{i=1}^N \frac{\mathrm{d}x_i}{2\pi\mathrm{i}x_i} \mathsf{S}(x_i)]|\alpha\rangle = 0 , \tag{6.20}$$

for a suitable contour. Alternatively, using the stress tensor current $T(z)$ in (6.7), the Virasoro constraints can also be written as

$$T(z)Z(t) = P(z) , \tag{6.21}$$

where $P(z)$ is a function which is holomorphic as $z \to 0$. Furthermore, the charged vacuum $|\alpha\rangle$ is an eigenstate of the generator T_0 as shown in [22],

$$T_0|\alpha\rangle = \lambda_\alpha|\alpha\rangle \tag{6.22}$$

with eigenvalue λ_α for momentum α given by

$$\lambda_\alpha = p^{\frac{1}{2}} q^{\frac{\sqrt{\beta}\alpha}{2}} + p^{-\frac{1}{2}} q^{-\frac{\sqrt{\beta}\alpha}{2}} . \tag{6.23}$$

The generating function is an eigenstate of T_0 with the same eigenvalue,

$$T_0 Z(t) = \lambda_\alpha Z(t) . \tag{6.24}$$

[3]Here we also use the same symbol for the generating function as in the classical case, i.e. Z, although it should be clear from the context which one we are referring to.

Additionally, in paper II we find that for the particular momentum $\alpha = 0$,

$$T_{-1} Z(t) = 0 \ . \tag{6.25}$$

Similarly to the classical case we can then introduce the time dependent expectation value

$$\langle \mathcal{O}(\underline{x}) \rangle_t = \oint \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{2\pi \mathrm{i} x_i} \mathcal{O}(\underline{x}) \prod_{i=1}^{N} \mathsf{S}(x_i) \ , \tag{6.26}$$

for an operator $\mathcal{O}(\underline{x})$, again using the subscript t to stress the dependence on the time variables. Furthermore we can expand the generating function in the time variables to find the correlators c_λ,

$$Z(t) = \sum_{\lambda} \frac{1}{|\mathrm{Aut}(\lambda)|} c_\lambda \prod_{\mu \in \lambda} t_\mu \ , \tag{6.27}$$

where the summation is over all integer partitions λ.

6.3 Deriving the q-Virasoro constraints

We now wish to solve the q-Virasoro constraints in (6.19), where with solve we again mean to determine a model in terms of its correlators. The alternative of solving the model in terms of its W-operator representation has not yet been very tractable in the q-deformed case, although attempts at partial such solutions have been made in for instance [65].

One way to derive the constraints is via insertion of an operator under the integral as investigated in papers II, III and IV. We will now review the derivation in the case of the q-deformed matrix model from paper II. The generating function under consideration is then

$$Z(t) = \oint_{C_1} \cdots \oint_{C_N} \prod_{i=1}^{N} \frac{\mathrm{d}x_i}{x_i} \ F(\underline{x}) \ , \tag{6.28}$$

for contours $\{C_1, \ldots, C_N\}$ which will be specified later. The integrand $F(\underline{x})$, where $\underline{x}$ is the integration variables $\underline{x} = \{x_1, \ldots, x_N\}$, is

$$F(\underline{x}) = \prod_{1 \leq k \neq l \leq N} \frac{(x_k/x_l; q)_\infty}{(t x_k/x_l; q)_\infty} \mathrm{e}^{\sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} x_i^s} \prod_{i=1}^{N} c_q(x_i) \ . \tag{6.29}$$

Here we recognise the q-deformed Vandermonde determinant $\Delta_{q,t}(\underline{x})$ in (6.16) together with the usual exponent with the time variables. $(x; q)_\infty$ is the q-Pochhammer symbol in (5.1) and $c_q(x)$ is of the form

$$c_q(x) = x^{\sqrt{\beta}(\alpha + \sqrt{\beta} N - Q_\beta)} \lambda_q(x) \prod_{k=1}^{N_f} \frac{(q x \overline{m}_k; q)_\infty}{(x m_k; q)_\infty} \ , \tag{6.30}$$

for a q-constant $\lambda_q(x)$, with the property in (4.8). At this stage we can think of N_f as an additional parameter on which the generating function depends, but we will later see its gauge theory interpretation in Chapter 9 (which is also the motivation for the form of $c_q(x)$). The parameters $t, q, \beta \in \mathbb{C}$ are related via $t = q^\beta$, and Q_β is defined in (3.19). Here we interpret the parameter α similar to the momentum of a vacuum state as introduced earlier. We will later discuss the restrictions on $c_q(x)$.

Let us now consider the insertion of a particular q-operator under the integral in the generating function in (6.28) with the goal of obtaining the q-Virasoro constraints. To define this operator, we recall the definition of the q-differential $\mathrm{d}_{q,i}$ in (4.4) but with an argument q^{-1}, i.e.

$$\mathrm{d}_{q^{-1},i} = \hat{M}_{q^{-1},i} - 1 \tag{6.31}$$

using the q-shift $\hat{M}_{q,i}$ in (4.2). The operator we then wish to insert under the integral is given by

$$\sum_{i=1}^{N} \mathrm{d}_{q^{-1},i} \left[\sum_{n\in\mathbb{Z}} (zx_i)^n \, G_i(\underline{x}) \ldots \right] , \tag{6.32}$$

with $\ldots$ denoting the integrand and where

$$G_i(\underline{x}) = \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{x_j - tx_i}{x_j - x_i} . \tag{6.33}$$

In other words, we are considering the constraints encoded in the equation

$$\oint_{C_1} \cdots \oint_{C_N} \prod_{j=1}^{N} \frac{\mathrm{d}x_j}{x_j} \sum_{i=1}^{N} \mathrm{d}_{q^{-1},i} \left[\sum_{n\in\mathbb{Z}} (zx_i)^n \, G_i(\underline{x}) F(\underline{x}) \right] = 0 , \tag{6.34}$$

with $F(\underline{x})$ as given in (6.29). It should be noted that the variable z appearing in the insertion in (6.32) is a formal variable in the sense that we will use its expansion to obtain one constraint for each power z^m (and later we will restrict to $m \geq -1$).

To further specify the contours $\{C_i\}$ and the functions $c_q(x)$ appearing in the generating function in (6.28), they are chosen such that the constraints in (6.34) are satisfied. Said differently, the integral is required to be invariant when acting with the q-shift operator $\hat{M}_{q^{-1},i}$ on the integrand. Rewriting (6.34) this implies that we impose

$$\begin{aligned} (\mathrm{LHS}) = \oint_{C_1} \cdots \oint_{C_N} \prod_{j=1}^{N} \frac{\mathrm{d}x_j}{x_j} \sum_{i=1}^{N} \sum_{n\in\mathbb{Z}} (zx_i)^n \; G_i(\underline{x}) F(\underline{x}) \stackrel{!}{=} \qquad\qquad (6.35) \\ \stackrel{!}{=} \oint_{C_1} \cdots \oint_{C_N} \prod_{j=1}^{N} \frac{\mathrm{d}x_j}{x_j} \sum_{i=1}^{N} \hat{M}_{q^{-1},i} \left[\sum_{n\in\mathbb{Z}} (zx_i)^n \; G_i(\underline{x}) F(\underline{x}) \right] = (\mathrm{RHS}) , \end{aligned}$$

where we have introduced the notation (LHS) and (RHS) to denote the left and right hand side of the above equation respectively to ease the following discussion. Now, the (RHS) can be written as

$$\begin{aligned}
&\oint_{C_1}\cdots\oint_{C_N}\prod_{j=1}^{N}\frac{\mathrm{d}x_j}{x_j}\sum_{i=1}^{N}\hat{M}_{q^{-1},i}\left[\sum_{n\in\mathbb{Z}}(zx_i)^n\;G_i(\underline{x})F(\underline{x})\right]= \\
&=\sum_{i=1}^{N}\oint_{C_1}\cdots\oint_{C_N}\prod_{j=1}^{N}\frac{\mathrm{d}x_j}{x_j}\sum_{n\in\mathbb{Z}}(zx_iq^{-1})^n\,G_i(x_iq^{-1})F(x_iq^{-1})= \\
&=\sum_{i=1}^{N}\oint_{C_1}\cdots\oint_{C_iq^{-1}}\cdots\oint_{C_N}\prod_{j=1}^{N}\frac{\mathrm{d}x_j}{x_j}\sum_{n\in\mathbb{Z}}(zx_i)^n\;G_i(\underline{x})F(\underline{x})\;,
\end{aligned}\tag{6.36}$$

using the shorthand notations $G_i(x_iq^{-1})=G_i(x_1,\ldots,x_iq^{-1},\ldots,x_N)$ and $F(x_iq^{-1})=F(x_1,\ldots,x_iq^{-1},\ldots,x_N)$ where only the i-th argument is shifted. Thus, to have the equality in (6.35) we find that the contours $\{C_i\}$ and functions $c_q(x)$ must be invariant under the shift of contours

$$C_iq^{-1}\to C_i\;,\qquad i=1,\ldots,N\;.\tag{6.37}$$

This means that there cannot be any singularities in the region between the two contours C_iq^{-1} and C_i so that the shift can be performed without any new contributions to the contour integral. Let us now examine this. In the case of the insertion in (6.32), we treat z as a formal expansion variable, thus ignoring the singularity at $x_i=1/z$. Then, the other singularities at $x_i=x_j$ of $G_i(\underline{x})$ are cancelled by the zeros of $F(\underline{x})$ in (6.29). Similarly, the singularities of $F(\underline{x})$ at $x_i=tx_j$ are cancelled by the zeros of $G_i(\underline{x})$. Thus, we are left with the requirement that the function $c_q(x)$ cannot have any singularities in the region between the two contours. Assuming $|q|<1$, the contours can be schematically illustrated as in Figure 6.1.

To continue the discussion on the derivation of the q-Virasoro constraints from imposing (6.35), we employ the identity

$$\sum_{i=1}^{N}\sum_{n\in\mathbb{Z}}(zx_i)^n\;G_i(\underline{x})=\frac{1}{1-t}\prod_{i=1}^{N}\frac{(1-tz^{-1}x_i^{-1})}{(1-z^{-1}x_i^{-1})}-\frac{t^N}{1-t}\prod_{i=1}^{N}\frac{1-t^{-1}zx_i}{1-zx_i}\;,\tag{6.38}$$

in order to rewrite the (LHS) in (6.35) as

$$\begin{aligned}
\text{(LHS)}&=\left\langle\frac{1}{1-t}\prod_{i=1}^{N}\frac{(1-tz^{-1}x_i^{-1})}{(1-z^{-1}x_i^{-1})}-\frac{t^N}{1-t}\prod_{i=1}^{N}\frac{1-t^{-1}zx_i}{1-zx_i}\right\rangle_t= \\
&=\frac{1}{1-t}\left\langle\exp\left(\sum_{s=1}^{\infty}z^{-s}\frac{(1-t^s)}{s}\sum_{i=1}^{N}x_i^{-s}\right)\right\rangle_t+ \\
&\qquad-\frac{t^N}{1-t}\exp\left(\sum_{s=1}^{\infty}z^s\frac{(1-t^{-s})}{s}\frac{\partial}{\partial t_s}\right)Z(t)\;,
\end{aligned}\tag{6.39}$$

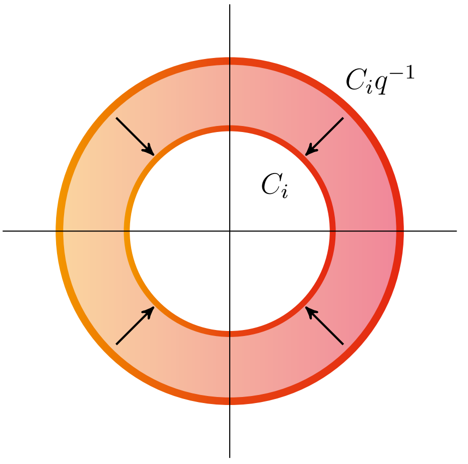

Figure 6.1. Illustration of change of integration contours from $C_i q^{-1}$ to C_i.

recalling the time-dependent average in (6.26). In the last term we used the coupling of the times to the integration variables in (6.29), to rewrite

$$\left\langle \sum_{i=1}^{N} x_i^s \right\rangle_t = \frac{\partial}{\partial t_s} Z(t) \ . \tag{6.40}$$

We also note that the first term involves an expectation value of a negative power of x_i, which we cannot rewrite as a differential operator on the generating function $Z(t)$. Therefore, we require this term to be cancelled by a term originating from the (RHS).

Moving on to the (RHS) of (6.35), we begin with evaluating the action of the q-shift operator $\hat{M}_{q^{-1},i}$ on the integrand,

$$\hat{M}_{q^{-1},i} \left[\sum_{n\in\mathbb{Z}} (zx_i)^n \ G_i(\underline{x}) F(\underline{x}) \right] \ . \tag{6.41}$$

We now consider the three factors separately. Firstly,

$$\hat{M}_{q^{-1},i} \sum_{n\in\mathbb{Z}} (zx_i)^n = \sum_{n\in\mathbb{Z}} (zx_i q^{-1})^n \tag{6.42}$$

and secondly

$$\hat{M}_{q^{-1},i} G_i(\underline{x}) = \hat{M}_{q^{-1},i} \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{x_j - t x_i}{x_j - x_i} = \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{x_j - t x_i q^{-1}}{x_j - x_i q^{-1}} \ . \tag{6.43}$$

Thirdly, the q-shifted integrand $F(\underline{x})$ becomes

$$\begin{aligned}
\hat{M}_{q^{-1},i}F(\underline{x}) =& \hat{M}_{q^{-1},i}\left[\prod_{1\leq k\neq l\leq N}\frac{(x_k/x_l;q)_\infty}{(tx_k/x_l;q)_\infty}\mathrm{e}^{\sum_{s=1}^{\infty}t_s\sum_{i=1}^{N}x_i^s}\prod_{i=1}^{N}c_q(x_i)\right]= \\
=&\prod_{\substack{l=1\\l\neq i}}^{N}\frac{(1-x_iq^{-1}/x_l)}{(1-tx_iq^{-1}/x_l)}\prod_{\substack{k=1\\k\neq i}}^{N}\frac{(1-tx_k/x_i)}{(1-x_k/x_i)}\times \\
&\times\,\mathrm{e}^{\sum_{s=1}^{\infty}t_sx_i^s(q^{-s}-1)}\frac{c_q\left(x_iq^{-1}\right)}{c_q(x_i)}F(\underline{x})\ ,
\end{aligned} \tag{6.44}$$

using the explicit form of the q-Pochhammer given in (5.1). Overall, we then find that the q-shift in (6.41) becomes

$$\begin{aligned}
&\hat{M}_{q^{-1},i}\left[\sum_{n\in\mathbb{Z}}(zx_i)^n\ G_i(\underline{x})F(\underline{x})\right]= \\
&=\sum_{n\in\mathbb{Z}}(zx_iq^{-1})^n\prod_{\substack{j=1\\j\neq i}}^{N}\frac{x_j-tx_iq^{-1}}{x_j-x_iq^{-1}}\prod_{\substack{l=1\\l\neq i}}^{N}\frac{(1-x_iq^{-1}/x_l)}{(1-tx_iq^{-1}/x_l)}\times \\
&\qquad\times\prod_{\substack{k=1\\k\neq i}}^{N}\frac{(1-tx_k/x_i)}{(1-x_k/x_i)}\mathrm{e}^{\sum_{s=1}^{\infty}t_sx_i^s(q^{-s}-1)}\frac{c_q\left(x_iq^{-1}\right)}{c_q(x_i)}F(\underline{x})= \\
&=\sum_{n\in\mathbb{Z}}(zx_iq^{-1})^n\prod_{\substack{k=1\\k\neq i}}^{N}\frac{(1-tx_k/x_i)}{(1-x_k/x_i)}\mathrm{e}^{\sum_{s=1}^{\infty}t_sx_i^s(q^{-s}-1)}\frac{c_q\left(x_iq^{-1}\right)}{c_q(x_i)}F(\underline{x})
\end{aligned} \tag{6.45}$$

noting in the last line the cancellation between the q-shifted $G_i(\underline{x})$ and the first factor of $F(\underline{x})$ in accordance with the discussion about the shift in contour after equation (6.37). Thus, the (RHS) takes the form

$$\begin{aligned}
&(\mathrm{RHS})=\oint_{C_1}\cdots\oint_{C_N}\prod_{j=1}^{N}\frac{\mathrm{d}x_j}{x_j}\sum_{i=1}^{N}\hat{M}_{q^{-1},i}\left[\sum_{n\in\mathbb{Z}}(zx_i)^n\,G_i(\underline{x})F(\underline{x})\right]= \\
&=\left\langle\sum_{i=1}^{N}\sum_{n\in\mathbb{Z}}(zx_iq^{-1})^n\prod_{\substack{k=1\\k\neq i}}^{N}\frac{(1-tx_k/x_i)}{(1-x_k/x_i)}\mathrm{e}^{\sum_{s=1}^{\infty}t_sx_i^s(q^{-s}-1)}\frac{c_q\left(x_iq^{-1}\right)}{c_q(x_i)}\right\rangle_t\ .
\end{aligned} \tag{6.46}$$

The next step is to rewrite the expression inside the average above as a summation over residues using the relation

$$\sum_{i=1}^{N}\hat{M}_{q^{-1},i}\left[\sum_{n\in\mathbb{Z}}(zx_i)^n\ G_i(\underline{x})F(\underline{x})\right]=$$
$$=\frac{F(\underline{x})}{t-1}\sum_{i=1}^{N}\mathrm{Res}_{w=x_i^{-1}}\frac{\mathrm{d}w}{w}\left[\sum_{n\in\mathbb{Z}}\left(\frac{z}{qw}\right)^n\right]\prod_{k=1}^{N}\frac{(1-tx_kw)}{(1-x_kw)}\times$$
$$\times\,\mathrm{e}^{\sum_{s=1}^{\infty}t_sw^{-s}(q^{-s}-1)}\frac{c_q(w^{-1}q^{-1})}{c_q(w^{-1})}\ ,\tag{6.47}$$

for the auxiliary complex variable $w=x_i^{-1}$. Using the above, one can re-express the (RHS) in (6.46) as

$$(\mathrm{RHS})=\frac{1}{t-1}\left\langle\sum_{i=1}^{N}\frac{1}{2\pi\mathrm{i}}\oint_{w=x_i^{-1}}\frac{\mathrm{d}w}{w}\left[\sum_{n\in\mathbb{Z}}\left(\frac{z}{qw}\right)^n\right]\prod_{k=1}^{N}\frac{(1-tx_kw)}{(1-x_kw)}\times\right.$$
$$\left.\times\,\mathrm{e}^{\sum_{s=1}^{\infty}t_sw^{-s}(q^{-s}-1)}\frac{c_q(w^{-1}q^{-1})}{c_q(w^{-1})}\right\rangle_t\ .\tag{6.48}$$

The aim is to be able to write the (RHS) as differential operators acting on the generating function as desired for the final constraint. To do so, we want to change the order of integration between the integration over the eigenvalues x_i, encoded in the expectation value, and the contour integrals in w. However, this cannot immediately be done as the contours explicitly depend on the eigenvalues as we are integrating around the N points $w=x_i^{-1}$ with $i=1,\ldots,N$. It can be noted that the integrand also have singularities at $w=0$ and $w=\infty$. Using this together with that w is a point on Riemann sphere such that the sum over residues is vanishing, we can move the integration contour away from the points $w=x_i^{-1}$ to instead encircle $w=0$ and $w=\infty$ where the contours are now in the opposite orientation thus contributing with the opposite sign. This is illustrated in Figures 6.2 and 6.3.

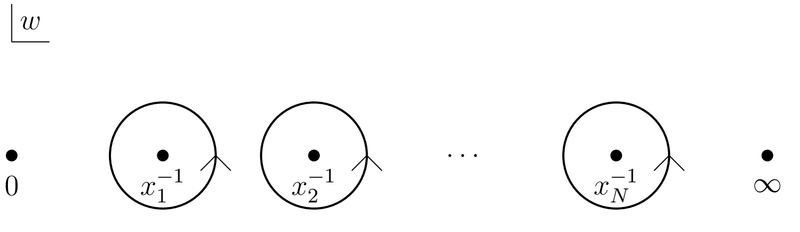

Figure 6.2. Schematic illustration of integration contours in the w plane *before* the change of contours.

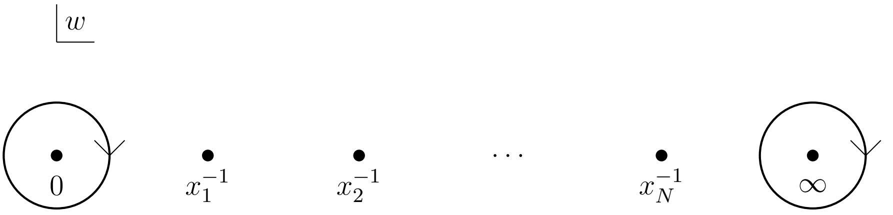

Figure 6.3. Schematic illustration of integration contours in the w plane *after* the change of contours.

Using this change of contours, we can rewrite (6.48) as

$$\begin{aligned}(\mathrm{RHS}) = &-\frac{1}{t-1}\frac{1}{2\pi\mathrm{i}}\oint_{w=\{0,\infty\}}\frac{\mathrm{d}w}{w}\left[\sum_{n\in\mathbb{Z}}\left(\frac{z}{qw}\right)^n\right]\times\\ &\times \underbrace{\mathrm{e}^{\sum_{s=1}^{\infty}t_s w^{-s}(q^{-s}-1)}\frac{c_q(w^{-1}q^{-1})}{c_q(w^{-1})}\left\langle\prod_{k=1}^{N}\frac{(1-tx_k w)}{(1-x_k w)}\right\rangle_t}_{\mathcal{F}(w)}\ ,\end{aligned} \tag{6.49}$$

bringing the expectation value inside the w integral and introducing the notation $\mathcal{F}(w)$ with the corresponding power series expansion

$$\mathcal{F}(w) = \sum_{m\in\mathbb{Z}}\mathcal{F}_m w^m\ . \tag{6.50}$$

Let us now consider the two contributions $w = \{0,\infty\}$ one at a time. Starting with the residue at $w=\infty$, we find the contribution

$$-\frac{1}{t-1}\frac{1}{2\pi\mathrm{i}}\oint_{w=\infty}\frac{\mathrm{d}w}{w}\left[\sum_{n\in\mathbb{Z}}\left(\frac{z}{qw}\right)^n\right]\mathcal{F}(w) = \frac{1}{t-1}\mathcal{F}\left(\frac{z}{q}\right)\ , \tag{6.51}$$

which can be seen from making the change of variables $w = 1/\alpha$ with $\mathrm{d}w = -\frac{1}{\alpha^2}\mathrm{d}\alpha$ and integrating around $\alpha = 0$. Next, we use the fact that we are in a neighbourhood of $w=\infty$ such that we can rewrite

$$\prod_{k=1}^{N}\frac{1-tx_k w}{1-x_k w} = t^N\exp\left(\sum_{s=1}^{\infty}w^{-s}\frac{(1-t^{-s})}{s}\sum_{i=1}^{N}x_i^{-s}\right)\ , \tag{6.52}$$

to recast the $w=\infty$ contribution as

$$\begin{aligned}\frac{1}{t-1}\mathcal{F}\left(\frac{z}{q}\right) = &\frac{t^N}{t-1}\exp\left(\sum_{k=1}^{\infty}t_k z^{-k}(1-q^k)\right)\frac{c_q(z^{-1})}{c_q(z^{-1}q)}\times\\ &\times\left\langle\exp\left(\sum_{s=1}^{\infty}z^{-s}q^s\frac{(1-t^{-s})}{s}\sum_{i=1}^{N}x_i^{-s}\right)\right\rangle_t\ .\end{aligned} \tag{6.53}$$

Then, the contribution from $w = 0$ is

$$-\frac{1}{t-1}\frac{1}{2\pi\mathrm{i}}\oint_{w=0}\frac{\mathrm{d}w}{w}\left[\sum_{n\in\mathbb{Z}}\left(\frac{z}{qw}\right)^n\right]\mathcal{F}(w) = -\frac{1}{t-1}\mathcal{F}\left(\frac{z}{q}\right) . \tag{6.54}$$

This time we are in the region near $w = 0$ and instead have that

$$\prod_{k=1}^{N}\frac{1-tx_k w}{1-x_k w} = \exp\left(\sum_{s=1}^{\infty}w^s\frac{(1-t^s)}{s}\sum_{i=1}^{N}x_i^s\right) . \tag{6.55}$$

The contribution from $w = 0$ then becomes

$$\begin{aligned}-\frac{1}{t-1}\mathcal{F}\left(\frac{z}{q}\right) = -\frac{1}{t-1}\frac{c_q(z^{-1})}{c_q(z^{-1}q)}\exp\left(\sum_{k=1}^{\infty}\frac{(1-q^k)}{z^k}t_k\right)\times \\ \times\exp\left(\sum_{s=1}^{\infty}z^s\frac{(1-t^s)}{sq^s}\frac{\partial}{\partial t_s}\right)Z(t) .\end{aligned} \tag{6.56}$$

Using this result for $w = 0$ together with that for $w = \infty$ we find that (RHS) in (6.49) becomes

$$\begin{aligned}(\mathrm{RHS}) = \frac{t^N}{t-1}\exp\left(\sum_{k=1}^{\infty}t_k z^{-k}(1-q^k)\right)\frac{c_q(z^{-1})}{c_q(z^{-1}q)}\times \\ \times\left\langle\exp\left(\sum_{s=1}^{\infty}z^{-s}q^s\frac{(1-t^{-s})}{s}\sum_{i=1}^{N}x_i^{-s}\right)\right\rangle_t + \\ -\frac{1}{t-1}\frac{c_q(z^{-1})}{c_q(z^{-1}q)}\exp\left(\sum_{k=1}^{\infty}\frac{(1-q^k)}{z^k}t_k\right)\exp\left(\sum_{s=1}^{\infty}z^s\frac{(1-t^s)}{sq^s}\frac{\partial}{\partial t_s}\right)Z(t) .\end{aligned} \tag{6.57}$$

Upon equating the results of the (LHS) in (6.39) and the (RHS) in (6.57) above, we finally obtain the *q-Virasoro constraints*

$$\begin{aligned}&t^N\exp\left(\sum_{s=1}^{\infty}z^s\frac{(1-t^{-s})}{s}\frac{\partial}{\partial t_s}\right)Z(t)+ \\ &\quad+\frac{c_q(z^{-1})}{c_q(z^{-1}q)}\exp\left(\sum_{k=1}^{\infty}\frac{(1-q^k)}{z^k}t_k\right)\exp\left(\sum_{s=1}^{\infty}z^s\frac{(1-t^s)}{sq^s}\frac{\partial}{\partial t_s}\right)Z(t) = \\ &=\left\langle\exp\left(\sum_{s=1}^{\infty}z^{-s}\frac{(1-t^s)}{s}\sum_{i=1}^{N}x_i^{-s}\right)\right\rangle_t + \\ &+t^N\frac{c_q(z^{-1})}{c_q(z^{-1}q)}\exp\left(\sum_{k=1}^{\infty}\frac{(1-q^k)}{z^k}t_k\right)\left\langle\exp\left(\sum_{s=1}^{\infty}q^s\frac{(1-t^{-s})}{sz^s}\sum_{i=1}^{N}x_i^{-s}\right)\right\rangle_t\end{aligned} \tag{6.58}$$

with $c_q(x)$ as in (6.30). As explained around (6.34), it is the expansion in z which then provides each q-Virasoro constraint separately. Comparing to [22] together with using the generator current in (6.3), we can make the above statement more precise. The above constraints should be interpreted as q-Virasoro constraints in the sense that it corresponds to

$$\psi(1/z)T(1/z)Z(t) = P(1/z) \tag{6.59}$$

with $P(1/z)$ containing (an infinite number of) terms with zero or negative powers in z and where

$$\psi(1/z) = p^{-\frac{1}{2}} \exp\left(\sum_{k=1}^{\infty} \frac{(1-q^k)t_k}{(1+p^k)z^k}\right) . \tag{6.60}$$

In particular we are interested in the constraints z^m with $m \geq -1$. However, at the order $m = -1$ there is a contribution to the constraints of the form

$$\frac{(1-t)}{z}(1-q^{-\sqrt{\beta}\alpha})\left\langle \sum_{i=1}^{N} \frac{1}{x_i} \right\rangle_t , \tag{6.61}$$

using the explicit form of $c_q(x)$ in (6.30). As this cannot be written as a differential operator in the times acting on the generating function, we require it to vanish. Therefore we need to choose the momentum $\alpha = 0$ which we restrict to from now on. We then specialise to the case of what is known as the q,t-*Gaussian model* as discussed in [53], where the model was expressed via its Macdonald averages. This corresponds to selecting the parameter $N_f = 2$ with parameters m_k having values $m_1 = q(1-q)^{\frac{1}{2}}$ and $m_2 = -q(1-q)^{\frac{1}{2}}$. The name q,t-Gaussian comes from the fact that the q-Pochhammer symbols in $c_q(x)$ in (6.30) in this case becomes

$$\prod_{k=1}^{N_f=2} \frac{(qx\overline{m}_k;q)_\infty}{(xm_k;q)_\infty} = \frac{1}{(-xq(1-q)^{\frac{1}{2}};q)_\infty} \frac{1}{(xq(1-q)^{\frac{1}{2}};q)_\infty} \underset{q\to 1}{=} \mathrm{e}^{-\frac{x^2}{2}} . \tag{6.62}$$

In the last step we took the semi-classical limit $q \to 1$, such that the q-Pochhammer symbols can be viewed as the simplest q-deformation of the standard Gaussian exponent. With this choice of parameters m_k, the function $c_q(x)$ in (6.30) takes the form (upon choosing $\alpha = 0$)

$$\begin{aligned} c_{q,G}(x) = {} & x^{\sqrt{\beta}(\sqrt{\beta}N - Q_\beta)}(x^2q^2(1-q);q^2)_\infty(1-q)\times \\ & \times\left\{-\frac{(q;q)_\infty^2}{\Theta(q^\lambda;q)}\left[f_\lambda(x(1-q)^{\frac{1}{2}};q) - f_\lambda(-x(1-q)^{\frac{1}{2}};q)\right]\right\} , \end{aligned} \tag{6.63}$$

where $\Theta(x;q)$ is the theta function in (5.3) and $f_\lambda(x;q)$ is given by the q-constant (with the property in (4.8)) for a parameter λ,

$$f_\lambda(x;q) = x^\lambda \, \frac{\Theta(q^\lambda x;q)}{\Theta(x;q)} . \tag{6.64}$$

Using this example of the q, t-Gaussian model, we consider the constraints in (6.58). For the purposes of q-Virasoro constraints we are only interested in constraints z^m for $m \geq -1$. We therefore expand the expectation values of the exponents containing negative powers of x_i, and collect any contributions to z^m for $m \leq -2$ (which we are not interested in) into the expression `remainder`. Recalling the choice $\alpha = 0$, we obtain

$$\begin{aligned}
&t^N \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^{-s})}{s} \frac{\partial}{\partial t_s}\right) Z(t) + \\
&+ t^{1-N} q^{-1} \Big(1 - z^{-2} q^2 (1-q)\Big) \exp\left(\sum_{k=1}^{\infty} \frac{(1-q^k)}{z^k} t_k\right) \times \\
&\qquad \times \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^s)}{s q^s} \frac{\partial}{\partial t_s}\right) Z(t) = \\
&= (1 + q^{-1} t) Z(t) + \frac{t(q^{-1}-1)t_1}{z} Z(t) - \texttt{remainder} \, .
\end{aligned} \tag{6.65}$$

Even though usual integrals are used above, Jackson q-integrals could be used instead. As the insertion of a q-differential under a q-integral is also vanishing, the above derivation should in principle hold for such q-integrals although the details remain to be verified.

6.4 Solving the q-Virasoro constraints

Let us now continue by solving the constraints in (6.65) specialised to the case of the q, t-Gaussian model. As discussed around (6.34), the purpose of the parameter z is to serve as an expansion parameter using which we obtain a separate constraint for each power z^m for $m \geq -1$. With this goal in mind, we start from (6.65) to rewrite this as

$$\begin{aligned}
&t^N \exp\left(-\sum_{k=1}^{\infty} z^{-k} (1-q^k) t_k\right) \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^{-s})}{s} \frac{\partial}{\partial t_s}\right) Z(t) + \\
&\qquad + t^{1-N} q^{-1} \Big(1 - z^{-2} q^2 (1-q)\Big) \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^s)}{s q^s} \frac{\partial}{\partial t_s}\right) Z(t) = \\
&= (1 + q^{-1} t) \exp\left(-\sum_{k=1}^{\infty} z^{-k} (1-q^k) t_k\right) Z(t) + \\
&\qquad + \frac{t(q^{-1}-1)t_1}{z} \exp\left(-\sum_{k=1}^{\infty} z^{-k} (1-q^k) t_k\right) Z(t) - \texttt{remainder} \, .
\end{aligned} \tag{6.66}$$

We then replace the exponents with symmetric Schur polynomials $\mathsf{Schur}_{\{m\}}(p_k)$ using the Cauchy identity in (2.37), such that

$$\begin{aligned} t^N \sum_{\ell=0}^{\infty} z^{-\ell} \mathsf{Schur}_{\{\ell\}}(p_s = -s\,(1-q^s)\,t_s)\times \\ \times \sum_{n=0}^{\infty} z^n \mathsf{Schur}_{\{n\}} \left(p_s = (1-t^{-s})\,\frac{\partial}{\partial t_s}\right) Z(t) + \\ + t^{1-N} q^{-1} \left(1 - z^{-2} q^2 (1-q)\right) \sum_{\ell=0}^{\infty} z^{\ell} \mathsf{Schur}_{\{\ell\}} \left(p_s = \frac{1-t^s}{q^s}\frac{\partial}{\partial t_s}\right) Z(t) = \\ = (1+q^{-1}t) \sum_{\ell=0}^{\infty} z^{-\ell} \mathsf{Schur}_{\{\ell\}}(p_s = -s\,(1-q^s)\,t_s) Z(t) + \\ + \frac{t(q^{-1}-1)t_1}{z} \sum_{\ell=0}^{\infty} z^{-\ell} \mathsf{Schur}_{\{\ell\}}(p_s = -s\,(1-q^s)\,t_s) Z(t) - \texttt{remainder}\ . \end{aligned} \tag{6.67}$$

Next, we replace the generating function with its expansion

$$Z(t) = \sum_{d=0}^{\infty} Z^{(d)}(t)\ , \tag{6.68}$$

where $Z^{(d)}(t)$ is the component of $Z(t)$ of degree d with respect to the dilatation operator D in (3.44). We now wish to extract the coefficient of z^m for $m \geq -1$ in (6.67) and when doing so, considering an overall degree d in times $\{t_k\}$. We also need to note that a symmetric Schur polynomial of the form $\mathsf{Schur}_{\{m\}}(p_s \propto t_s)$ has degree m in times, whereas $\mathsf{Schur}_{\{m\}}(p_s \propto \partial/\partial t_s)$ has degree $-m$. Thus by considering the first line of (6.67) we require by matching powers of z that $-\ell + n = m$ and by selecting the degree d in $\{t_k\}$ we have that $\ell - n = d$. For the last two lines of (6.67), the first term contributes to z^m with $m = -1, 0$ and the second only contributes to $m = -1$. This is because only z^m for $m \geq -1$ are relevant for the q-Virasoro constraints in (6.19). We then obtain

$$\begin{aligned} t^N \sum_{\ell=0}^{d} \mathsf{Schur}_{\{\ell\}}\left(p_s = -s(1-q^s)t_s\right) \times \\ \times\, \mathsf{Schur}_{\{\ell+m\}} \left(p_s = (1-t^{-s})\frac{\partial}{\partial t_s}\right) Z^{(d+m)}(t) + \\ + t^{1-N} q^{-1} \mathsf{Schur}_{\{m\}} \left(p_s = \frac{1-t^s}{q^s}\frac{\partial}{\partial t_s}\right) Z^{(d+m)}(t) + \\ - (1-q) q t^{1-N} \mathsf{Schur}_{\{m+2\}} \left(p_s = \frac{1-t^s}{q^s}\frac{\partial}{\partial t_s}\right) Z^{(d+m+2)}(t) = \\ = \delta_{m,0}(1+q^{-1}t) Z^{(d)}(t) - \delta_{m,-1}(1-q) t_1 Z^{(d-1)}(t)\ . \end{aligned} \tag{6.69}$$

We then rewrite the symmetric Schur polynomials using (2.36),

$$
\begin{aligned}
&t^N \sum_{\ell=0}^{d} \prod_{\{\gamma\,\text{s.t.}\,|\gamma|=\ell\}} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{i=1}^{l(\gamma)} \frac{-i(1-q^i)t_i}{i} \times \\
&\qquad\qquad \times \prod_{\{\gamma\,\text{s.t.}\,|\gamma|=\ell+m\}} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{i=1}^{l(\gamma)} \frac{(1-t^{-i})\frac{\partial}{\partial t_i}}{i} Z^{(d+m)}(t) + \\
&+ t^{1-N} q^{-1} \prod_{\{\gamma\,\text{s.t.}\,|\gamma|=m\}} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{i=1}^{l(\gamma)} \frac{\frac{1-t^i}{q^i}\frac{\partial}{\partial t_i}}{i} Z^{(d+m)}(t) + \\
&- (1-q) q t^{1-N} \prod_{\{\gamma\,\text{s.t.}\,|\gamma|=m+2\}} \frac{1}{|\mathrm{Aut}(\gamma)|} \prod_{i=1}^{l(\gamma)} \frac{\frac{1-t^i}{q^i}\frac{\partial}{\partial t_i}}{i} Z^{(d+m+2)}(t) = \\
=&\delta_{m,0}(1+q^{-1}t) Z^{(d)}(t) - \delta_{m,-1}(1-q) t_1 Z^{(d-1)}(t)\ .
\end{aligned}
\tag{6.70}
$$

Finally, we wish to rewrite the above into a recursion for the correlator c_λ for a partition $\lambda = \{\lambda_1, \ldots, \lambda_{\bullet-1}, \lambda_\bullet\}$, with $\lambda_\bullet$ being the last part of the partition λ. We then recall the expansion of the generating function in (6.27), together with that correlators can be obtained by acting with derivatives in times, as given in (2.50) for the classical case. Upon identifying $m+2 = \lambda_\bullet$ and $d + \lambda_\bullet = |\lambda|$ so that the term on the fourth line in (6.70) contains the desired correlator $c_{\lambda_1 \ldots \lambda_{\bullet-1}\lambda_\bullet}$, we act with the operator

$$
\left[\frac{\partial \ldots \partial}{\partial t_{\lambda_1} \ldots t_{\lambda_{\bullet-1}}} (\ldots) \right]\Bigg|_{t=0}
\tag{6.71}
$$

with $(\ldots)$ denoting each term in (6.70). Upon rearranging we then get the final recursion relation

$$
\begin{aligned}
&(1-q) q t^{1-N} \frac{(1-t^{\lambda_\bullet})}{q^{\lambda_\bullet}\lambda_\bullet} c_{\lambda_1\ldots\lambda_\bullet} = \\
&= -(1-q) q t^{1-N} \sum_{\substack{|\gamma|=\lambda_\bullet \\ l(\gamma)\geq 2}} \frac{1}{|\mathrm{Aut}(\gamma)|} \left(\prod_{a\in\gamma} \frac{(1-t^a)}{q^a a} \right) c_{\lambda_1\ldots\lambda_{\bullet-1}\gamma_1\ldots\gamma_\bullet} + \\
&\quad + q^{-1} t^{1-N} \sum_{|\gamma|=\lambda_\bullet-2} \frac{1}{|\mathrm{Aut}(\gamma)|} \left(\prod_{a\in\gamma} \frac{(1-t^a)}{q^a a} \right) c_{\lambda_1\ldots\lambda_{\bullet-1}\gamma_1\ldots\gamma_\bullet} + \\
&\quad + t^N \sum_{\nu\subseteq\lambda\setminus\lambda_\bullet} \left(\prod_{a\in\nu} (-1)(1-q^a) \right) \times \\
&\qquad \times \sum_{|\gamma|=|\nu|+\lambda_\bullet-2} \frac{1}{|\mathrm{Aut}(\gamma)|} \left(\prod_{a\in\gamma} \frac{(1-t^{-a})}{a} \right) c_{\lambda\setminus\{\lambda_\bullet,\nu\}\gamma_1\ldots\gamma_\bullet} + \\
&\quad - \delta_{\lambda_\bullet,2}(1+q^{-1}t) c_{\lambda_1\ldots\lambda_{\bullet-1}} + \delta_{\lambda_\bullet,1}(1-q)((\#_\lambda 1)-1) c_{\lambda_1\ldots\lambda_{\bullet-2}}\ ,
\end{aligned}
\tag{6.72}
$$

recalling the notation $\#_\lambda j$ given in (2.24). Also, $\lambda \setminus \lambda_\bullet$ is used to denote a partition λ without part $\lambda_\bullet$ and $\lambda \setminus \{\lambda_\bullet, \nu\}$ is used to denote a partition λ where $\lambda_\bullet$ and all parts of the partition ν are removed. It can be noted that the above is indeed a recursion for c_λ since all the terms on the right hand side except the first have a sum of indices $|\lambda| - 2$. The first term on the right hand side instead has a sum of indices $|\lambda|$, but a minimal index $\gamma_\bullet < \lambda_\bullet$. Upon applying the initial condition $c_{\{1\}} = 0$ we obtain for the first correlators,

$$c_{\{1,1\}} = \frac{t^N\left(1-t^N\right)}{t(1-t)} c_\emptyset \tag{6.73}$$

and

$$c_{\{2\}} = \frac{\left(1-t^N\right)\left(2t - t^N - qt^N + t^{N+1} - qt^{N+1}\right)}{t(1-q)(1-t^2)} c_\emptyset \ . \tag{6.74}$$

6.5 Recovering Virasoro from q-Virasoro

As a sanity check, we can also verify that the Virasoro algebra, the Virasoro constraints and also the classical matrix model can be obtained from their corresponding q-analogues. To see this, we return to the semi-classical limit introduced in the context of the Macdonald polynomials in (5.21), in which we let $t = q^\beta$ with $q \to 1$. More specifically, we introduce a parameter $\hbar$, such that

$$q = \mathrm{e}^{\hbar} \,, \qquad \hbar \to 0 \ . \tag{6.75}$$

Starting with the free boson algebra in (6.6), the semi-classical limit is

$$\begin{aligned} [\mathsf{a}_n, \mathsf{a}_m] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\delta_{n+m,0} \\ &\to 2\beta n \hbar^2 \delta_{n+m,0} + \mathcal{O}(\hbar^3) \ , \end{aligned} \tag{6.76}$$

so that the classical free boson algebra in (3.21) is recovered at the second order of the $\hbar$-expansion (up to a factor of β). Taking this limit in (6.10) we instead obtain for the free boson oscillators

$$\begin{aligned} \mathsf{a}_{-n} &\simeq (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) t_n \ \to \ n\hbar t_n + \mathcal{O}(\hbar^3) \\ \mathsf{a}_n &\simeq \frac{1}{n}(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(p^{\frac{n}{2}} + p^{-\frac{n}{2}})\frac{\partial}{\partial t_n} \ \to \ 2\beta\hbar\frac{\partial}{\partial t_n} + \mathcal{O}(\hbar^3) \ , \end{aligned} \tag{6.77}$$

as expected. At the level of the q-Virasoro constraints, one can find that the q-Virasoro generators has the expansion [25]

$$T_n = 2\delta_{n,0} + \hbar^2\beta\Big(L_n + \frac{Q_\beta^2}{4}\delta_{n,0}\Big) + \mathcal{O}(\hbar^4) \ , \tag{6.78}$$

where we recover the Virasoro generators L_n at second order in the $\hbar$ expansion and with Q_β given in (3.19).

Let us now turn to the q-deformed matrix model in (6.18) and more specifically the resulting functions due to normal ordering in (6.15). Starting with the q-deformed Vandermonde determinant in (6.16), we find

$$\Delta_{q,t}(\underline{x}) = \prod_{1\leq k\neq j\leq N} \frac{(x_k x_j^{-1};q)_\infty}{(t x_k x_j^{-1};q)_\infty} \to \prod_{1\leq k\neq j\leq N} \left(1 - x_k x_j^{-1}\right)^\beta \ , \tag{6.79}$$

using the definition of the q-Pochhammer in (5.1). Next, we consider the other function appearing in the integrand, $c_\beta(\underline{x};q)$, which becomes

$$\begin{aligned} c_\beta(\underline{x};q) &= \prod_{1\leq k<j\leq N} \left(x_k x_j^{-1}\right)^\beta \frac{\Theta(t x_k x_j^{-1};q)}{\Theta(x_k x_j^{-1};q)} \\ &\to \prod_{1\leq k<j\leq N} \left(x_k x_j^{-1}\right)^\beta \left(1 - x_k x_j^{-1}\right)^{-\beta} \left(1 - x_j x_k^{-1}\right)^\beta \ , \end{aligned} \tag{6.80}$$

using the definition of the Θ-function in (5.3). Thus, the combination in (6.15) then becomes,

$$\Delta_{q,t}(\underline{x}) c_\beta(\underline{x};q) \prod_{j=1}^{N} x_j^{\beta(N-1)} \to \prod_{1\leq k<j\leq N} (x_k - x_j)^{2\beta} \tag{6.81}$$

since we can rewrite $\prod_{j=1}^N x_j^{\beta(N-1)} = \prod_{1\leq k<j\leq N} x_j^\beta x_k^\beta$. We thus recover the classical β-deformed Vandermonde determinant in (2.20).

Let us also observe the semi-classical limit at the level of the insertion which generates the q-Virasoro constraints in (6.32). There we find that the function $G_i(\underline{x})$ in (6.33) becomes 1 while the q-differential (together with the factor of $\frac{1}{x_i}$ from the measure) becomes

$$\lim_{q\to 1} \frac{\mathrm{d}_{q^{-1},i}}{x_i(q^{-1}-1)} = \frac{\partial}{\partial x_i} \ . \tag{6.82}$$

In other words we recover the usual derivative, which can also be seen as the motivation for the form of the insertion.

7. The q, t, q'-deformation

We now take the deformation a step further, by introducing another deformation labelled by the parameter q'. We will in other words study the q, t, q'-deformation of the algebra and the matrix model construction, which therefore is a 3-parameter deformation of the un-deformed classical case. Similarly to the q, t-deformation being referred to as the trigonometric deformation, we will sometimes refer to the q, t, q'-deformation as the *elliptic* deformation.

7.1 The elliptic Virasoro algebra

Let us begin with discussing the deformation of the Virasoro algebra. The elliptic generalisation of the Virasoro algebra, also called the $W_{q,t;q'}$ *algebra*, was first given in [30]. This algebra is generated by operators T_n satisfying the associative algebra which can by given in terms of stress tensor currents $T(z) = \sum_{n\in\mathbb{Z}} T_n z^{-n}$ as[1]

$$\begin{aligned} f\left(\frac{w}{z}\right)T(z)T(w) - T(w)T(z)f\left(\frac{z}{w}\right) = \\ = -\frac{\Theta(q;q')\Theta(t^{-1};q')}{(q';q')_\infty^2\Theta(p;q')}\left(\delta\left(p\frac{w}{z}\right) - \delta\left(p^{-1}\frac{w}{z}\right)\right) \ . \end{aligned} \tag{7.1}$$

In addition to the previously introduced parameters $q, t, p = qt^{-1} \in \mathbb{C}$, we have $q' \in \mathbb{C}$ parametrising the elliptic deformation. The q-Pochhammer symbol $(z; q)_\infty$ is given in (5.1), the theta function $\Theta(z; q)$ is defined in (5.3) and we recall the multiplicative delta function $\delta(x)$ given in (6.5). The structure function $f(x) = \sum_{n\in\mathbb{Z}} f_n x^n$ is defined via the expansion of

$$f(x) = \frac{\Gamma(x;p^2,q')\Gamma(p^2q^{-1}x;p^2,q')\Gamma(pqx;p^2,q')}{\Gamma(p^2x;p^2,q')\Gamma(pq^{-1}x;p^2,q')\Gamma(qx;p^2,q')} \tag{7.2}$$

where the elliptic Gamma function is defined in (5.6) in the region $|p^2|, |q| < 1$. Alternatively, one can express the commutation relation using the generators T_n as

$$\sum_{\ell\in\mathbb{Z}} f_\ell \left(T_{n-\ell}T_{m+\ell} - T_{m-\ell}T_{n+\ell}\right) = -\frac{\Theta(q;q')\Theta(t^{-1};q')}{(q';q')_\infty^2\Theta(p;q')}\left(p^n - p^{-n}\right)\delta_{n+m,0} \ . \tag{7.3}$$

[1] We hope that although the trigonometric and elliptic generators are both represented by T_n, it will be clear from the context which one we have in mind.

This construction was later generalised for arbitrary quiver diagrams in [66]. To see this, we now briefly introduce the notion of quivers. A quiver, typically denoted by Γ, consists of a set of nodes Γ_0 and arrows Γ_1. For instance, a quiver with two nodes $a, b \in \Gamma_0$ and two arrows $e, f \in \Gamma_1$ is illustrated in Figure 7.1. Then, one can associate a deformed Cartan

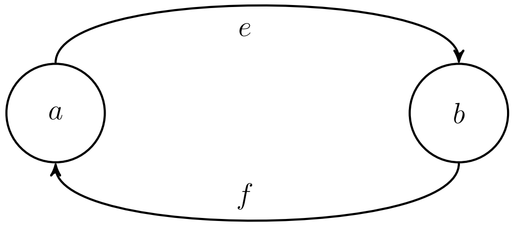

Figure 7.1. Illustration of a quiver Γ given by two nodes $a, b \in \Gamma_0$ and two arrows $e : a \to b, f : b \to a \in \Gamma_1$.

matrix $C_{ab} \in |\Gamma_0| \times |\Gamma_0|$ to the quiver Γ. This matrix is given by

$$C_{ab} = (1 + p^{-1})\delta_{a,b} - \sum_{e:b\to a} \mu_e^{-1} - p^{-1} \sum_{e:a\to b} \mu_e \ . \tag{7.4}$$

We here introduce the parameter $\mu_e \in \mathbb{C}$ which will be interpreted as masses for the bifundamental fields in the gauge theory picture [67], as shown in Chapter 10. We would now like to give a free boson realisation of the above algebra. For each node in the quiver Γ we introduce the free boson oscillators $\{\mathsf{s}_{a,n}^{(\pm)}\}$ which satisfy the Heisenberg algebra

$$\left[\mathsf{s}_{a,n}^{(\pm)}, \mathsf{s}_{b,m}^{(\pm)}\right] = \mp \frac{1 - t^{\mp n}}{n(1 - q^{\mp n})(1 - q'^{\pm n})} C_{ab}^{[\pm n]} \delta_{n+m,0} \ , \qquad n, m \in \mathbb{Z}\backslash\{0\} \tag{7.5}$$

for $a, b \in \Gamma_0$ and

$$[\mathsf{s}_{a,0}, \tilde{\mathsf{s}}_{b,0}] = \beta C_{ab}^{[0]} \ . \tag{7.6}$$

Here we use the notation ${}^{[n]}$, which means that we replace each parameter with its n-th power. In particular,

$$C_{ab}^{[n]} = (1 + p^{-n})\delta_{a,b} - \sum_{e:b\to a} \mu_e^{-n} - p^{-n} \sum_{e:a\to b} \mu_e^{n} \ . \tag{7.7}$$

Generalising the generators in the commutation relation (7.3) for a generic quiver, the elliptic $W_{q,t;q'}(\Gamma)$ algebra is then generated by $\{T_n^a, a \in \Gamma_0, n \in \mathbb{Z}\}$. We then wish to give the generator T_n^a in terms of free boson oscillators. In order do so, we consider the stress tensor current given by

$$T^a(w) = \sum_{n\in\mathbb{Z}} T_n^a w^{-n} \tag{7.8}$$

which can be expressed via the auxiliary operator $\mathsf{Y}^a(w)$

$$\mathsf{Y}^a(w) = :\mathrm{e}^{\sum_{n\neq 0}\left(\mathsf{y}^{(+)}_{a,n}w^{-n}+\mathsf{y}^{(-)}_{a,n}w^{n}\right)} : t^{\mathsf{y}_{a,0}-\tilde{\rho}_a} \ , \quad \tilde{\rho}_a = \sum_{b\in\Gamma_0}\left(C^{-1}\right)^{[0]}_{ab} \ . \tag{7.9}$$

Here the free boson oscillators $\{\mathsf{y}^{(\pm)}_{a,n}\}$ satisfy

$$\left[\mathsf{y}^{(\pm)}_{a,n}, \mathsf{s}^{(\pm)}_{b,m}\right] = \mp\frac{1-t^{\mp n}}{n(1-q'^{\pm n})}\delta_{n+m,0}\delta_{a,b} \ , \qquad [\mathsf{y}_{a,0}, \tilde{\mathsf{s}}_{b,0}] = [\mathsf{s}_{a,0}, \tilde{\mathsf{y}}_{b,0}] = \delta_{a,b} \ , \tag{7.10}$$

where

$$\begin{gathered} \mathsf{y}^{(\pm)}_{a,n} = (1-q^{\mp n})\left(C^{-1}\right)^{[\pm n]}_{ab}\mathsf{s}^{(\pm)}_{b,n} \ , \qquad \mathsf{y}_{a,0} = \beta^{-1}\left(C^{-1}\right)^{[0]}_{ab}\mathsf{s}_{b,0} \ , \\ \tilde{s}_{a,0} = \tilde{y}_{b,0}\beta C^{[0]}_{ba} \ . \end{gathered} \tag{7.11}$$

In order to give the stress tensor current $T^a(w)$ in terms of the operator $\mathsf{Y}^a(w)$, we need to specify the quiver. As outlined in [66], the simplest example is the quiver given by the single node, i.e. $\Gamma_0 = \{a\}$ and $\Gamma_1 = \emptyset$. We therefore drop the label of the node and simply use $T(w)$ and $\mathsf{Y}(w)$. In this particular case, the stress tensor current is

$$T(w) = \mathsf{Y}(w) + \mathsf{Y}(p^{-1}w)^{-1} \ . \tag{7.12}$$

As a final remark, we introduce the time representation of the free boson algebra,

$$\begin{gathered} \mathsf{s}^{(\pm)}_{a,-n} \simeq \mp\frac{t^a_{\pm n}}{n(1-q'^{\pm n})} \ , \quad \mathsf{s}^{(\pm)}_{a,n} \simeq \frac{1-t^{\mp n}}{1-q^{\mp n}}C^{[\pm n]}_{ab}\frac{\partial}{\partial t^b_{\pm n}} \ , \quad n>0 \\ \tilde{\mathsf{y}}_{a,0} \simeq t^a_0 \ , \quad \mathsf{s}_{a,0} \simeq \frac{\partial}{\partial t^a_0} \ , \quad |0\rangle \simeq 1 \ , \end{gathered} \tag{7.13}$$

which we will make use of shortly.

7.2 The elliptically deformed matrix model

Similarly to the Virasoro and q-Virasoro cases described earlier, we then construct the screening current. Here it takes the form

$$\mathsf{S}^a(w) = :\mathrm{e}^{\sum_{n\neq 0}\left(\mathsf{s}^{(+)}_{a,n}w^{-n}+\mathsf{s}^{(-)}_{a,n}w^{n}\right)} : w^{\mathsf{s}_{a,0}}\mathrm{e}^{\tilde{\mathsf{s}}_{a,0}} \ , \tag{7.14}$$

and it is defined by the property

$$\left[T^a_n, \mathsf{S}^b(w)\right] = \delta_{a,b}D_q\mathsf{O}^b_n(w) = \delta_{a,b}\frac{(\mathsf{O}^b_n(qw)-\mathsf{O}^b_n(w))}{(q-1)w} \tag{7.15}$$

for $a, b \in \Gamma_0$, using the q-derivative in (4.5) for some operators $\mathsf{O}_n^b(w)$. For an explicit example of such an operator $\mathsf{O}_n^b(w)$ see [66].

Next, we wish to explore the matrix model which can be constructed in the elliptic case. Similarly to the classical and trigonometric cases, we build up the generating function using the screening current in (7.14)

$$\mathsf{Z}|\underline{\alpha}\rangle = \oint \prod_{a=1}^{|\Gamma_0|} \prod_{j=1}^{N_a} \frac{\mathrm{d}w_{a,j}}{2\pi \mathrm{i} w_{a,j}} \mathsf{S}^a(w_{a,j})|\underline{\alpha}\rangle \ , \tag{7.16}$$

where we used the Fock module $\mathcal{F}_{\underline{\alpha}}$ with vacuum $|\underline{\alpha}\rangle$ which satisfies

$$\mathsf{s}_{a,n}^{(\pm)}|\underline{\alpha}\rangle = 0 \ , \quad n > 0 \ , \quad |\underline{\alpha}\rangle = \mathrm{e}^{\sum_{a=1}^{|\Gamma_0|} \alpha_a \tilde{\mathsf{y}}_{0,a}}|0\rangle \ , \quad \mathsf{s}_{a,0}|\underline{\alpha}\rangle = \alpha_a|\underline{\alpha}\rangle \ . \tag{7.17}$$

Upon normal ordering of two screening currents one finds

$$\begin{aligned} \mathsf{S}^a(w_{a,j})\mathsf{S}^b(w_{b,k}) = \, :\mathsf{S}^a(w_{a,j})\mathsf{S}^b(w_{b,k}): \Delta_{\mathrm{node}}^{(a)}\left(\frac{w_{a,k}}{w_{a,j}}\right)^{\delta_{a,b}} \Delta_{\mathrm{self}}^{(a)}\left(\frac{w_{a,k}}{w_{a,j}}\right)^{\delta_{a,b}} \times \\ \times \Delta_{\mathrm{off}}^{(ab)}\left(\frac{w_{b,k}}{w_{a,j}}\right)^{1-\delta_{a,b}} w_{b,k}^{-\beta C_{ba}^{[0]}} \ , \end{aligned} \tag{7.18}$$

where we introduced the three functions

$$\Delta_{\mathrm{node}}^{(a)}(w) = \frac{\Gamma(tw;q,q')\Gamma(tw^{-1};q,q')}{\Gamma(w;q,q')\Gamma(w^{-1};q,q')} \frac{\Theta(tw^{-1};q)}{\Theta(w^{-1};q)} \ , \tag{7.19}$$

$$\Delta_{\mathrm{self}}^{(a)}(w) = \prod_{e:a\to a} \frac{\Gamma(\mu_e w;q,q')\Gamma(\mu_e w^{-1};q,q')}{\Gamma(t\mu_e w;q,q')\Gamma(t\mu_e w^{-1};q,q')} \frac{\Theta(\mu_e w^{-1};q)}{\Theta(t\mu_e w^{-1};q)} \tag{7.20}$$

and finally

$$\Delta_{\mathrm{off}}^{(ab)}(w) = \prod_{e:a\to b} \frac{\Gamma(\mu_e w;q,q')}{\Gamma(t\mu_e w;q,q')} \prod_{e:b\to a} \frac{\Gamma(qt^{-1}\mu_e^{-1}w;q,q')}{\Gamma(q\mu_e^{-1}w;q,q')} \ . \tag{7.21}$$

We can recast the above functions using instead

$$\Delta_E^{(a)}(\underline{w_a}) = \prod_{1\le j\ne k\le N_a} \frac{\Gamma(tw_{a,k}/w_{a,j};q,q')}{\Gamma(w_{a,k}/w_{a,j};q,q')} \tag{7.22}$$

and

$$\Delta_{\mathrm{loop}}^{(a)}(\underline{w_a}) = \prod_{e:a\to a} \prod_{1\le j\ne k\le N_a} \frac{\Gamma(\mu_e w_{a,k}/w_{a,j};q,q')}{\Gamma(t\mu_e w_{a,k}/w_{a,j};q,q')} \ , \tag{7.23}$$

together with the two q-constants (recalling their definition in (4.8))

$$c_\beta^{(a)}(\underline{w_a},\mu;q) = \prod_{1\le j<k\le N_a} \left(\frac{w_{a,j}}{w_{a,k}}\right)^\beta \frac{\Theta(t\mu w_{a,j}/w_{a,k};q)}{\Theta(\mu w_{a,j}/w_{a,k};q)} \tag{7.24}$$

and

$$C_\beta^{(a)}(\underline{w_a};q)=\frac{c_\beta^{(a)}(\underline{w_a},1;q)}{\prod_{e:a\to a}c_\beta^{(a)}(\underline{w_a},\mu_e;q)}\ . \tag{7.25}$$

Using these functions and q-constants, the state $\mathsf{Z}|\underline{\alpha}\rangle$ in (7.16) becomes

$$\begin{aligned}\mathsf{Z}|\underline{\alpha}\rangle = &\oint\prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a}\frac{\mathrm{d}w_{a,j}}{2\pi\mathrm{i}w_{a,j}}\Delta_{q,t,q'}(\underline{w})\times\\ &\times\prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a}w_{a,j}^{-\beta((N_a-1)\frac{C_{aa}^{[0]}}{2}+\sum_{a>b,b=1}^{|\Gamma_0|}C_{ab}^{[0]}N_b)}:\prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a}\mathsf{S}^a(w_{a,j}):|\underline{\alpha}\rangle\ ,\end{aligned} \tag{7.26}$$

where $\Delta_{q,t,q'}(\underline{w})$ is the *elliptic Vandermonde determinant* given by

$$\Delta_{q,t,q'}(\underline{w})=\prod_{a=1}^{|\Gamma_0|}C_\beta^{(a)}(\underline{w_a};q)\Delta_E^{(a)}(\underline{w_a})\Delta_{\mathrm{loop}}^{(a)}(\underline{w_a})\prod_{a<b,b=1}^{|\Gamma_0|}\prod_{j,k=1}^{N_a,N_b}\Delta_{\mathrm{off}}^{(ab)}\left(\frac{w_{b,k}}{w_{a,j}}\right)\ . \tag{7.27}$$

Using the time representation of the free boson algebra in (7.13), we obtain the generating function for the q,t,q'-model

$$\begin{aligned}\mathsf{Z}|\underline{\alpha}\rangle\simeq Z(\{t_0^a,t^a\}) =&\mathrm{e}^{\mathcal{N}_0(\{t_0^a\})}\oint\prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a}\frac{\mathrm{d}w_{a,j}}{2\pi\mathrm{i}w_{a,j}}\Delta_{q,t,q'}(\underline{w})\times\\ &\times\exp\left\{\sum_{a=1}^{|\Gamma_0|}\sum_{j=1}^{N_a}\left(V^{(a)}(w_{a,j})-\sum_{n>0}\frac{t_n^aw_{a,j}^n}{n(1-q'^n)}\right)\right\}\ .\end{aligned} \tag{7.28}$$

Here we have interpreted the state

$$\prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a}w_{a,j}^{-\beta((N_a-1)\frac{C_{aa}^{[0]}}{2}+\sum_{a>b,b=1}^{|\Gamma_0|}C_{ab}^{[0]}N_b)}|\underline{\alpha}\rangle \tag{7.29}$$

as the potential

$$V^{(a)}(w)=\left[\alpha_a+\beta\left(\sum_{b=1}^{|\Gamma_0|}C_{ab}^{[0]}N_b-(N_a-1)\frac{C_{aa}^{[0]}}{2}-\sum_{a>b,b=1}^{|\Gamma_0|}C_{ab}^{[0]}N_b\right)\right]\ln w\ . \tag{7.30}$$

We also have the overall normalisation factor

$$\mathcal{N}_0(\{t_0^a\})=\sum_{a=1}^{|\Gamma_0|}t_0^a\left(\alpha_a+\beta\sum_{b=1}^{|\Gamma_0|}C_{ab}^{[0]}N_b\right)\ . \tag{7.31}$$

To close this discussion on the elliptically deformed matrix model, we consider the Ward identities for the q,t,q'-deformed matrix model in (7.28). To do so, we again specialise to the case of the quiver A_1, or a single node, where $|\Gamma_0|=1$ and we can drop the dependence on the node. The set of constraints the model has to satisfy is then encoded in

$$\begin{aligned} &T(p^{1/2}z)Z(t)= \\ &=\mathrm{e}^{\mathcal{N}_0(t_0)}\oint\prod_{j=1}^{N}\frac{\mathrm{d}w_j}{2\pi\mathrm{i}w_j}\Delta_{q,t,q'}(\underline{w})\left(\sum_{\sigma=\pm}\mathrm{e}^{Y_\sigma(z|t_0,t)}\prod_{j=1}^{N}f_\sigma(w_j/z)\right)\times \\ &\qquad\times\mathrm{e}^{\sum_{j=1}^{N}V(w_j)-\sum_{j=1}^{N}\sum_{n>0}\frac{t_nw_j^n}{n(1-q'^n)}}\ , \end{aligned} \tag{7.32}$$

similar to the q-Virasoro constraints in equation (6.21). $T(p^{1/2}z)$ is then the elliptic Virasoro current in terms of the Y operators in (7.9)

$$T(p^{1/2}z)=\mathsf{Y}(p^{1/2}z)+\mathsf{Y}(p^{-1/2}z)^{-1}\ , \tag{7.33}$$

$Y_\sigma(z|t_0,t)$ is the contribution from the non-positive modes of $\mathsf{Y}(p^{\sigma/2}z)^\sigma$ for $\sigma=\pm$ and $f_\sigma(w_j/z)$ is

$$f_\sigma(w_j/z)=\frac{\Theta(p^{-\sigma/2}w_j/z;q')^\sigma}{\Theta(t^{-1}p^{-\sigma/2}w_j/z;q')^\sigma}\ . \tag{7.34}$$

7.3 Recovering q-Virasoro from elliptic Virasoro

In order to see the q-Virasoro construction arising from the $W_{q,t;q'}$ one, the limit to take is $q'\to 0$ and restricting to an A_1 quiver. In this limit, the deformed Cartan matrix in (7.4) takes the form

$$C_{ab}\to(1+p^{-1})\delta_{a,b}\ . \tag{7.35}$$

The free boson commutation relations restricted to $\{\mathsf{s}^{(+)}_{a,n}\}$ becomes

$$\left[\mathsf{s}^{(+)}_{a,n},\mathsf{s}^{(+)}_{b,m}\right]\to-\frac{(t^{\frac{n}{2}}-t^{-\frac{n}{2}})(p^{\frac{n}{2}}+p^{-\frac{n}{2}})}{n(q^{\frac{n}{2}}-q^{-\frac{n}{2}})}\delta_{n+m,0}\delta_{a,b} \tag{7.36}$$

which recovers (6.6) as we can find by comparing screening currents that $\mathsf{s}^{(+)}_{a,n}\to-\mathsf{a}_n(q^{\frac{n}{2}}-q^{-\frac{n}{2}})^{-1}$. Next, the time representation becomes

$$\mathsf{s}^{(+)}_{a,-n}\to-\frac{t^a_n}{n}\ ,\qquad \mathsf{s}^{(+)}_{a,n}\to\frac{(t^{\frac{n}{2}}-t^{-\frac{n}{2}})(p^{\frac{n}{2}}+p^{-\frac{n}{2}})}{(q^{\frac{n}{2}}-q^{-\frac{n}{2}})}\frac{\partial}{\partial t^b_n}\delta_{a,b}\ , \tag{7.37}$$

which recovers (6.10) upon noting that the time variables in the quantum case are equal to the elliptic time variables up to a factor $1/n$. Then, to see the limit at the level of the algebra in (7.1) we find

$$\frac{\Theta(q;q')\Theta(t^{-1};q')}{(q';q')^2_\infty\Theta(p;q')}\to\frac{(1-q)(1-t^{-1})}{(1-p)} \tag{7.38}$$

and

$$f(x) \to \ \mathrm{e}^{\sum_{n>0} \frac{(1-q^n)(1-t^{-n})}{n(1+p^n)} x^n} \tag{7.39}$$

using (5.1), (5.3) and (5.6), so that one recovers the q-Virasoro current in (6.4) with the structure function $f(x)$ in (6.2). Finally, the elliptically deformed Vandermonde determinant in (7.27) becomes as desired

$$\Delta_{q,t,q'}(\underline{w}) \to \Delta_{q,t}(\underline{w}) c_\beta(\underline{w}; q) \ , \tag{7.40}$$

with $\Delta_{q,t}(\underline{w})$ as in (6.16) and $c_\beta(\underline{w}; q)$ as in (6.17).

Part III: Applications

It should be noted that our results – and particularly those of papers II, IV and V – can be viewed as interesting findings solely within the area of matrix models. However, the results can also be viewed in a different light. The purpose of this part is therefore to explore applications of the classical and quantum algebras and matrix models introduced earlier. We begin with introducing the necessary background required for the first two applications we have in mind. In particular, we review how to obtain gauge theory observables on curved spaces while preserving some supersymmetry through applying the method of supersymmetric localisation. We then move on to discuss physical examples where we can make use of the deformed algebras and matrix models. As a final application, we introduce the concept of τ-functions in order to investigate the connection between Virasoro constraints and Hirota equations.

8. Exact results in supersymmetric gauge theories

In this chapter we review the relevant background needed for the applications of the quantum algebras and matrix models. To be more specific, the applications we consider are firstly a three dimensional $\mathcal{N}=2$ gauge theory on $D^2 \times_q S^1$ and S_b^3 and secondly a four dimensional $\mathcal{N}=1$ gauge theory on compact spaces $\mathcal{M}_3 \times S^1$. To motivate why these applications are worthwhile, one can take a wider perspective in which they can be considered part of a programme called the BPS/CFT correspondence [5–9]. In this programme, exact results for expectation values of certain protected BPS observables are expressed via two dimensional CFT's. There are several applications of this, for instance chiral algebras and superconformal field theories [68, 69], the gauge/Bethe correspondence [70–72] and the AGT correspondence [73, 74]. In the AGT correspondence, 4d $\mathcal{N}=2$ partition functions computed via localisation [3, 75] are identified with 2d CFT correlators. In paper I, we investigated possible extensions of this AGT correspondence in the form of elliptic deformations, in the particular case of compact surfaces $\mathcal{M}_3 \times S^1$. In paper III and IV we instead focused on applications of the BPS/CFT correspondence in the case of 3d supersymmetric gauge theories and more specifically on the evaluation of expectation values of Wilson loops.

In order to study supersymmetric gauge theories on compact spaces, one needs to have a consistent formalism in which such theories can be placed on our backgrounds of interest. This has been explored in [76–82]. In the 3d case it was observed that $\mathcal{N}=2$ theories are supported on several backgrounds with two supercharges of opposite R-charge being preserved. However, in the 4d case we can have $\mathcal{N}=1$ if we again wish to preserve at least two supercharges of opposite R-charge and the backgrounds are restricted to being T^2 fibrations over a Riemann surface.

Let us now review the method of supersymmetric localisation which can be used as a tool in evaluating gauge theory observables and in particular partition functions. We will then end the chapter with collecting various results which will be useful in the discussions in later chapters.

8.1 Supersymmetric localisation

We now review the method of *supersymmetric localisation*. Loosely speaking, the essence of the method is to reduce the domain of integration and consequently enable the exact evaluation of infinite dimensional integrals, typically in terms of finite dimensional ones. Since about ten years ago, there has been a plethora of results using localisation in dimensions ranging from 2 to 7 and for a review on the subject we refer to [83]. Upon applying the localisation method, the resulting models are often given in the form of classical or quantum matrix models. This enables us to use the matrix model tools outlined in earlier parts and in particular the property that matrix models satisfy Virasoro or q-Virasoro constraints. Then, by clever use of these constraints we are in some cases able to solve the gauge theories in terms of determining its generating function. We now briefly review the derivation of the localisation formula following [83].

The key relation in the localisation procedure is the Berline-Vergne-Atiyah-Bott formula [84, 85] given by

$$\int_{\mathcal{M}} \alpha = \sum_i \frac{(2\pi)^n \alpha_0(x_i)}{\sqrt{\det\left(\partial_\mu V^\nu(x_i)\right)}} \ , \tag{8.1}$$

for a compact manifold $\mathcal{M}$ of complex dimension n with a $U(1)$ action described by a vector field $V(x)$ and en equivariantly closed form α satisfying $(\mathrm{d} + \iota_V)\alpha = 0$. Here it is assumed that the fixed points $\{x_i\}$ of the $U(1)$ action are isolated and α_0 is the zero-component of the form α. We now wish to briefly recall how this formula arises.

For an odd tangent bundle $\Pi T\mathcal{M}$ with coordinates x^μ on $\mathcal{M}$ and Grassmann coordinates ψ^μ on the fiber, the functions $F(x, \psi)$ are differential forms and the vector field of the $U(1)$ action is $V^\mu(x)\partial_\mu$. We define the transformations

$$\delta x^\mu = \psi^\mu \, , \qquad \delta\psi^\mu = V^\mu(x) \tag{8.2}$$

corresponding to $(\mathrm{d} + \iota_V)\alpha = 0$. We then want to compute the integral

$$Z(0) = \int_{\Pi T\mathcal{M}} \alpha(x, \psi) \mathrm{d}^n x \, \mathrm{d}^n \psi \tag{8.3}$$

where $\alpha(x, \psi)$ can be thought of as the supersymmetric observable which is equivariantly closed, i.e. $\delta\alpha(x, \psi) = 0$. The trick is now to deform this integral to include a dependence on a parameter $t \in \mathbb{R}$, hence the suggestive notation $Z(0)$, such that

$$Z(t) = \int_{\Pi T\mathcal{M}} \alpha(x, \psi) \mathrm{e}^{-t\delta W(x,\psi)} \mathrm{d}^n x \, \mathrm{d}^n \psi \ . \tag{8.4}$$

Here we also introduced a function $W(x, \psi)$ which we require to be δ-exact so $\delta^2 W(x, \psi) = 0$. Using this, we can consider

$$\frac{\mathrm{d}}{\mathrm{d}t} Z(t) = -\int_{\Pi T\mathcal{M}} [\delta W(x, \psi)] \alpha(x, \psi) \mathrm{e}^{-t\delta W(x,\psi)} \mathrm{d}^n x \, \mathrm{d}^n \psi \ . \tag{8.5}$$

Upon integrating by parts

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}Z(t) = &- \left.[W(x,\psi)\alpha(x,\psi)\mathrm{e}^{-t\delta W(x,\psi)}]\right|_{\partial\mathcal{M}} + \\ &+ \int_{\Pi T\mathcal{M}} W(x,\psi)[\delta\alpha(x,\psi)]\mathrm{e}^{-t\delta W(x,\psi)}\mathrm{d}^n x\,\mathrm{d}^n\psi = 0 \ ,\end{aligned} \tag{8.6}$$

where we used the Stokes theorem i.e. $\int_{\partial\Omega} w = \int_{\Omega} \mathrm{d}w$ on the first term and the fact that $\delta\alpha(x,\psi) = 0$ on the second term. Thus, $Z(t)$ is independent of the parameter t. This means that the original integral $Z(0)$ can be computed at any value of t. For instance, we could let $t \to \infty$ to find

$$Z(0) = Z(t) = \lim_{t\to\infty} Z(t) = \lim_{t\to\infty} \int_{\Pi T\mathcal{M}} \alpha(x,\psi)\mathrm{e}^{-t\delta W(x,\psi)}\mathrm{d}^n x\,\mathrm{d}^n\psi \ , \tag{8.7}$$

and then use saddle-point methods in order to evaluate $Z(0)$. In order to recover the formula in equation (8.1), we need to make the choice $W(x,\psi) = V^\mu(x)g_{\mu\nu}\psi^\nu$ with the metric $g_{\mu\nu}$ being $U(1)$ invariant. This implies that the exponent in (8.4) is given by

$$\delta W(x,\psi) = \delta(V^\mu(x)g_{\mu\nu}\psi^\nu) = V^\mu(x)g_{\mu\nu}V^\nu(x) + \partial_\rho(V^\mu(x)g_{\mu\nu})\psi^\rho\psi^\nu \ , \tag{8.8}$$

recalling the transformations in (8.2). Therefore, at $t \to \infty$ the critical points x_i of the $U(1)$ action dominates, in other words where $V(x_i) = 0$. Let us now consider one such isolated point x_i and we assume for simplicity that $x_i = 0$. We can then introduce rescaled coordinates $\tilde{x}$ and $\tilde{\psi}$ given by

$$\begin{aligned} \tilde{x} &= \sqrt{t}x & \tilde{\psi} &= \sqrt{t}\psi \\ \mathrm{d}\tilde{x} &= \sqrt{t}\mathrm{d}x & \mathrm{d}\tilde{\psi} &= \frac{\mathrm{d}\psi}{\sqrt{t}} \ . \end{aligned} \tag{8.9}$$

Using these new coordinates the integral in (8.7) becomes

$$Z(0) = \lim_{t\to\infty} \int_{\Pi T\mathcal{M}} \alpha\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right) \mathrm{e}^{-t\delta W\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right)} \mathrm{d}^n\tilde{x}\,\mathrm{d}^n\tilde{\psi} \ . \tag{8.10}$$

As the next step, we expand $t\delta W\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right)$ in the $t \to \infty$ limit,

$$\begin{aligned} &t\,\delta W\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right) = \\ &= [\partial_\mu V^\sigma(0)]\, g_{\sigma\rho}\, [\partial_\nu V^\rho(0)]\, \tilde{x}^\mu\tilde{x}^\nu + [\partial_\mu V^\rho(0) g_{\rho\nu}]\, \tilde{\psi}^\mu\tilde{\psi}^\nu + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) = \\ &= H_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu + S_{\mu\nu}\tilde{\psi}^\mu\tilde{\psi}^\nu + \mathcal{O}\left(\frac{1}{\sqrt{t}}\right) \ . \end{aligned} \tag{8.11}$$

Here it can be noted that terms proportional to t and $t^{1/2}$ vanish due to $V(x_i = 0) = 0$. We also introduced a constant symmetric matrix $H_{\mu\nu}$ and constant antisymmetric matrix $S_{\mu\nu}$, as their explicit forms will not be important, and we used the expansion of $V^\mu(x)$ around $x = 0$,

$$V^\mu(x) \simeq V^\mu(0) + \partial_\nu V^\mu(0) x^\nu + \dots \; . \tag{8.12}$$

We then recall the transformations in (8.2), which in the limit $t \to \infty$ takes the form

$$\begin{aligned} \delta \tilde{x}^\mu &= \tilde{\psi}^\mu \\ \delta \tilde{\psi}^\mu &= \sqrt{t} V^\mu(x) \simeq \partial_\nu V^\mu(0) \tilde{x}^\nu \; . \end{aligned} \tag{8.13}$$

Using these transformations and keeping only terms surviving the $t \to \infty$ limit in (8.11), the condition $\delta^2 W(x, \psi) = 0$ implies that

$$t\delta^2 W\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right) = 2\big(H_{\mu\nu} - S_{\mu\sigma}\partial_\nu V^\sigma(0)\big)\tilde{\psi}^\mu \tilde{x}^\nu = 0 \; , \tag{8.14}$$

using that $H_{\mu\nu}$ and $S_{\mu\nu}$ are constant symmetric and antisymmetric matrices respectively. In other words we require the two matrices to satisfy

$$H_{\mu\nu} = S_{\mu\sigma}\partial_\nu V^\sigma(0) \; . \tag{8.15}$$

The integral in (8.10) can then be evaluated to

$$\begin{aligned} Z(0) =& \alpha(0,0) \int_{\Pi T\mathcal{M}} \mathrm{e}^{-H_{\mu\nu}\tilde{x}^\mu \tilde{x}^\nu} \mathrm{d}^n \tilde{x} \int_{\Pi T\mathcal{M}} \mathrm{e}^{-S_{\mu\nu}\tilde{\psi}^\mu \tilde{\psi}^\nu} \mathrm{d}^n \tilde{\psi} = \\ =& \frac{\alpha(0,0)(2\pi)^n \mathrm{Pf}(S)}{\sqrt{\det(H)}} \; , \end{aligned} \tag{8.16}$$

recalling that $x^\mu, \tilde{x}^\mu$ are even coordinates and $\psi^\mu, \tilde{\psi}^\mu$ are odd. We also used that forms higher than the zero-form will have additional factors of $1/\sqrt{t}$ and so will not contribute as $t \to \infty$. Here $\det(M)$ is the determinant and $\mathrm{Pf}(M)$ is the Pfaffian of a matrix M. Using that the two are related by $\mathrm{Pf}(M) = \sqrt{\det(M)}$ provided that M is anti-symmetric together with the relation between S and H in (8.15), we find

$$Z(0) = \frac{(2\pi)^n \alpha(0,0)}{\sqrt{\det(\partial_\mu V^\nu(0))}} \; . \tag{8.17}$$

Generalising this result to the case of multiple isolated critical points at generic positions $\{x_i\}$ we obtain

$$Z(0) = \sum_i \frac{(2\pi)^n \alpha_0(x_i)}{\sqrt{\det(\partial_\mu V^\nu(x_i))}} \; , \tag{8.18}$$

thus recovering (8.1).

So far, it is not obvious to see how the localisation formula in (8.1) applies to supersymmetric gauge theories. In order to make this connection clearer, one needs to make some adjustments in the above reasoning. To begin with, in the gauge theory setting the objects of interest are observables of the theory. It could be for instance partition functions or correlation functions of gauge invariant operators which are δ-closed or also referred to as *protected* operators. Let us for clarity consider partition functions. Expressing a generic partition function through its path integral, it can be given as

$$Z(0) = \int \mathcal{D}\Phi \, \mathrm{e}^{-S[\Phi]} \ , \tag{8.19}$$

for an action $S[\Phi]$ where Φ encode all the fields. Here it should be noted that the first difference to (8.1) is that we have an infinite dimensional integral as we are integrating over all possible field configurations. Secondly, the transformations in (8.2) will be given by the supersymmetry transformations of the theory in consideration. In (8.19) we introduced the notation $Z(0)$ such that we, similarly to before, can introduce a deformation parametrised by t as

$$Z(t) = \int \mathcal{D}\Phi \, \mathrm{e}^{-S[\Phi] - t\delta V} \ , \tag{8.20}$$

for some deformation V again being δ-exact, $\delta^2 V = 0$. By the same reasoning as before, $Z(t)$ is independent of t and we are free to evaluate (8.19) at $t \to \infty$. Another difference to before, is that we choose that

$$V = \sum_{\Psi} (\delta\Psi)^{\dagger} \, \Psi \tag{8.21}$$

summing over fermionic fields Ψ. As the bosonic contribution to the deformation of the action $(\delta V)_{\mathrm{bos}}$ given by

$$(\delta V)_{\mathrm{bos}} = \sum_{\Psi} |\delta\Psi|^2 \tag{8.22}$$

is positive semi-definite, the integral is dominated by the region where $(\delta V)_{\mathrm{bos}} = 0$, i.e. where $\delta\Psi = 0$. Similarly to what was done around (8.12), we rescale and then expand the fields Φ around the fixed points, such that

$$\Phi \simeq \Phi_0 + \frac{1}{\sqrt{t}}\Phi' + \ldots \ . \tag{8.23}$$

We then find that in the limit $t \to \infty$ the partition function becomes

$$Z(0) = \int \mathcal{D}\Phi_0 \, \mathrm{e}^{-S[\Phi_0]} \, \mathrm{sdet}(\delta^2) \ . \tag{8.24}$$

Here we introduced the superdeterminant sdet which is taken over the appropriate space of fields. This generalises the usual determinant where now bosonic fields appear in the denominator and fermionic fields in the numerator. In other words this generalises the combination of the pfaffian and the (square root of the) determinant in (8.16). Thus the contributions to the partition function is the *classical contribution* given by the exponent and the *1-loop contribution* given by the superdeterminant, which are names we will refer to later. However, evaluation of the 1-loop contribution is typically not straightforward and methods to evaluate the superdeterminant include determining the spectrum of the fluctuations under δ^2 and also making use of so-called index theorems.

The localisation construction in the case of evaluating the path integral in the infinite dimensional setting was first done in [86], in which supersymmetric quantum mechanics was considered. More recently, the above described localisation procedure was applied in a supersymmetric gauge theory set-up in the pioneering work [3]. There it was used in order to exactly evaluate partition functions and expectation values of Wilson loops on S^4 for an $\mathcal{N}=2$ gauge theory where 8 supercharges were preserved. The transformations similar to (8.2), then originated from the supersymmetry and BRST transformations. The final result was finite dimensional integrals given by specific matrix models.

Following the work in [3], the applications of the supersymmetric localisation method has been extended to include various other supersymmetric theories, dimensions and backgrounds. We now review a few results which are crucial in the search for applications of our deformed matrix models.

8.2 Summary of localisation results

Let us now summarise the relevant localisation results which we will employ in the applications which follow later in this part. We consider three particular cases. Firstly, we consider three dimensional $\mathcal{N}=2$ gauge theories on $D^2 \times_q S^1$ and secondly on S^3_b. The third case is the four dimensional $\mathcal{N}=1$ gauge theories, where we focus on $S^3 \times S^1$.

8.2.1 $D^2 \times_q S^1$

Let us start with the 3d $\mathcal{N}=2$ gauge theory on $D^2 \times_q S^1$. The gauge theory we consider is a Yang-Mills Chern-Simons (YM-CS) theory with a single unitary gauge group $U(N)$. The geometry is a D^2 fibration over S^1. The parameter q appears as the rotation of the D^2 fiber when going around the base. In addition to the $\mathcal{N}=2$ vector multiplet, we allow for an adjoint chiral multiplet of mass t together with N_f fundamental antichiral multiplets of respective masses $\{u_k\}$ with $k=1,\dots,N_f$. Besides, we introduce a Fayet-Illiopoulos (FI) contribution parametrised by $\kappa_1 \in \mathbb{C}$. To then determine the $D^2 \times_q S^1$ partition function, one needs to establish the 1-loop contributions for each of these multiplets. This was done in [27, 87] where the latter reference used the method of localisation. The results are summarised in Table 8.1.

Multiplet	1-loop contribution
Vector	$\prod_{\alpha\in\Delta}(w_\alpha;q)_\infty$
Chiral in irrep $\mathcal{R}$ (Neumann)	$\prod_{\rho\in\mathcal{R}}(w_\rho m;q)_\infty^{-1}$
Chiral in irrep $\mathcal{R}$ (Dirichlet)	$\prod_{\rho\in\mathcal{R}}(qw_\rho^{-1}m^{-1};q)_\infty$

Table 8.1. *1-loop determinants of vector and chiral multiplets.*

Starting with the vector multiplet, the contribution is $\prod_{\alpha\in\Delta}(w_\alpha;q)_\infty$ with $\alpha \in \Delta$ being a root of the gauge Lie algebra excluding the zero root. Next, a chiral multiplet contributes with $\prod_{\rho\in\mathcal{R}}(w_\rho m;q)_\infty^{-1}$ for Neumann boundary conditions and $\prod_{\rho\in\mathcal{R}}(qw_\rho^{-1}m^{-1};q)_\infty$ for Dirichlet boundary conditions. Here ρ is a weight of the irreducible representation $\mathcal{R}$, whereas m is a global $U(1)$ fugacity. Then, upon collecting the above contributions [27, 87] showed that the partition function for the above gauge theory on $D^2 \times_q S^1$ with $U(N)$ gauge group is

$$Z^{N_f}_{D^2\times_q S^1} = \oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\lambda_i} Z^{\mathrm{cl}}_{D^2\times_q S^1}(\underline{\lambda}) Z^{1-\mathrm{loop}}_{D^2\times_q S^1}(\underline{\lambda}) \ . \tag{8.25}$$

Here we introduced the background $D^2 \times_q S^1$ and the number of fundamental anti-chiral fields N_f as labels on the partition function for clarity. The contour $\mathcal{C}$ is defined as N copies of the unit circle. The two contributions to the integrand $Z^{\mathrm{cl}}_{D^2\times_q S^1}(\underline{\lambda})$ and $Z^{1-\mathrm{loop}}_{D^2\times_q S^1}(\underline{\lambda})$ are then the classical and the 1-loop contributions respectively, given by

$$Z^{\mathrm{cl}}_{D^2\times_q S^1}(\underline{\lambda}) = \prod_{i=1}^{N} \lambda_i^{\kappa_1} \ , \tag{8.26}$$

$$Z^{1-\mathrm{loop}}_{D^2\times_q S^1}(\underline{\lambda}) = \prod_{1\leq k\neq l\leq N} \frac{(\lambda_k/\lambda_l;q)_\infty}{(t\lambda_k/\lambda_l;q)_\infty} \prod_{j=1}^{N}\prod_{k=1}^{N_f} (q\lambda_j u_k;q)_\infty \ . \tag{8.27}$$

The 1-loop determinants can be found from Table 8.1, as we have vector, adjoint chiral and fundamental anti-chiral multiplets contributing.

Let us close the discussion of partition functions on $D^2 \times_q S^1$ by the following remark. This partition function has sometimes been referred to as the *half-index* [88] or the 3d *holomorphic block* [27], expressed as

$$\mathcal{B}^{\mathrm{3d}}_\gamma = \oint_\gamma \prod_{i=1}^{N} \mathrm{d}w_i \Upsilon^{\mathrm{3d}}(\underline{w}) \ , \tag{8.28}$$

for a 3d integrand $\Upsilon^{3d}(\underline{w})$, a building block which we will return to shortly.

8.2.2 S_b^3

We now move on to consider 3d $\mathcal{N}=2$ theories on the squashed three sphere S_b^3. The 3d geometry is given by

$$\omega_1|z_1|^2+\omega_2|z_2|^2=1\ , \tag{8.29}$$

with $z_1,z_2\in\mathbb{C}$ and where $\omega_1,\omega_2\in\mathbb{R}$ are the squashing parameters. The squashing is often encoded in the real parameter b, where $b^2=\omega_2/\omega_1$ [64,89–91]. Although it may seem natural to have real squashing parameters from a geometrical point of view, the derivations which follow does allow for arbitrary complex values for the parameters ω_1,ω_2. Thus we take $\omega_1,\omega_2\in\mathbb{C}$.

Regarding the gauge theory picture, we consider – similarly to the $D^2\times_q S^1$ case – a single unitary gauge group $U(N)$ and an $\mathcal{N}=2$ vector multiplet, an adjoint chiral multiplet of mass M_a and N_f fundamental anti-chiral multiplets of masses $\{m_k\}$ with $k=1,\ldots,N_f$. We again allow for a non-trivial FI parameter κ_1 and this time also a CS term with parameter κ_2, as will be elaborated on in Section 9.2.

Then, in order to establish the 1-loop contributions of these gauge and matter multiplets, one can use supersymmetric localisation to determine the contributions as given in Table 8.2 (using the results of Paper III) [90,92]. In particular, it can be noted that the contributions are given in terms of double sine functions, defined in (5.9).

Multiplet	1-loop contribution
Vector	$\prod_{\alpha\in\Delta}S_2\left(\alpha(\underline{X})\vert\underline{\omega}\right)$
Chiral in irrep $\mathcal{R}$	$\prod_{w\in\mathcal{R}}S_2\left(w(\underline{X})+M_\mathcal{R}\vert\underline{\omega}\right)^{-1}$
Anti-chiral in irrep $\bar{\mathcal{R}}$	$\prod_{w\in\mathcal{R}}S_2\left(-w(\underline{X})+M_{\bar{\mathcal{R}}}\vert\underline{\omega}\right)^{-1}$

Table 8.2. *The 1-loop determinants of vector, chiral and anti-chiral multiplets.*

In Table 8.2, $w(X)\in\mathcal{R}$ is the weight of the representation $\mathcal{R}$ whereas $\alpha(\underline{X})\in\Delta$ are the roots of the algebra, excluding the zero root. $M_\mathcal{R}$ and $M_{\bar{\mathcal{R}}}$ are the masses of the chiral and anti-chiral fields respectively. Here we introduced the integration variables appearing in the partition function $\underline{X}=\{X_1,\ldots,X_N\}$, which take values in the Cartan of the gauge group.

We now specialise to the case of a $U(N)$ gauge group. In this case the roots are differences of fundamental weights w_i enabling us to write

$$\alpha_{ij}(\underline{X})=w_i(\underline{X})-w_j(\underline{X})=X_i-X_j\ , \tag{8.30}$$

for imaginary numbers X_i. As mentioned above, we consider a YM-CS theory, thus allowing for a CS term given by

$$\prod_{i=1}^{N} \mathrm{e}^{-\frac{\pi \mathrm{i}\kappa_2}{\omega_1\omega_2}X_i^2} \tag{8.31}$$

for a CS level $\kappa_2 \in \mathbb{Z}$. Furthermore, as there is a $U(1)$ in the center of the gauge group we also have an FI term

$$\prod_{i=1}^{N} \mathrm{e}^{\frac{2\pi \mathrm{i}\kappa_1}{\omega_1\omega_2}X_i} \tag{8.32}$$

parametrised by $\kappa_1 \in \mathbb{C}$. Then, the S_b^3 partition function as shown in [90, 92] is given by (and given in [89] for S^3)

$$Z_{S_b^3}^{N_f} = \int_{(\mathrm{i}\mathbb{R})^N} \prod_{i=1}^{N} \mathrm{d}X_i \, Z_{S_b^3}^{\mathrm{cl}}(\underline{X}) \, Z_{S_b^3}^{1-\mathrm{loop}}(\underline{X}) \ . \tag{8.33}$$

The integration variables $X_i \in \mathrm{i}\mathbb{R}$ are Coulomb branch variables and $Z_{S_b^3}^{\mathrm{cl}}(\underline{X})$ is the classical contribution given by

$$Z_{S_b^3}^{\mathrm{cl}}(\underline{X}) = \prod_{i=1}^{N} \exp\left(-\frac{\pi\mathrm{i}\kappa_2}{\omega_1\omega_2}X_i^2 + \frac{2\pi\mathrm{i}\kappa_1}{\omega_1\omega_2}X_i\right) \ . \tag{8.34}$$

Then $Z_{S_b^3}^{1-\mathrm{loop}}(\underline{X})$ is the product of 1-loop determinants

$$Z_{S_b^3}^{1-\mathrm{loop}}(\underline{X}) = \prod_{1\le k\neq j\le N} \frac{S_2(X_k - X_j|\underline{\omega})}{S_2(X_k - X_j + M_{\mathrm{a}}|\underline{\omega})} \prod_{k=1}^{N_f}\prod_{i=1}^{N} S_2\left(-X_i - m_k|\underline{\omega}\right)^{-1} \ , \tag{8.35}$$

given in terms of the double sine function $S_2(z|\underline{\omega})$ defined in (5.9).

It can be noted that topologically S_b^3 can be obtained by gluing two solid tori, i.e. two copies of $D^2 \times_q S^1$. One can then give each half of the sphere a label $\alpha = 1, 2$ and imagine each such half to have a corresponding $D^2 \times_{q_\alpha} S^1$ theory with its own modular parameter q_α given by

$$q_1 = \mathrm{e}^{2\pi\mathrm{i}\epsilon} \ , \qquad\qquad q_2 = \mathrm{e}^{-2\pi\mathrm{i} g\cdot\epsilon} \ , \tag{8.36}$$

for the gluing element $g \in SL(2, \mathbb{Z})$. Upon identifying $\epsilon = \omega/\omega_1$ we can write the two modular parameters as

$$q_1 = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_2}{\omega_1}} \ , \qquad q_2 = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_1}{\omega_2}} \ . \tag{8.37}$$

This seemingly simple geometric picture has also been found to hold at the level of partition functions, as shown in [27–29]. There, the S_b^3 partition function is found to take a factorised form into holomorphic blocks, in other words the $D^2 \times_q S^1$ partition functions introduced earlier. Recalling (8.28), the compact space partition functions then decompose as

$$Z^{\mathrm{3d}} = \sum_{\underline{\ell}} \int \prod_{i=1}^{N} \mathrm{d}x_i \Upsilon^{\mathrm{3d}} \left(\underline{w}(\underline{x}, \underline{\ell})\right)_1 \; \Upsilon^{\mathrm{3d}} \left(\underline{w}(\underline{x}, \underline{\ell})\right)_2 \tag{8.38}$$

for continuous variables $\{x_1, \ldots, x_N\}$ and discrete variables $\{\ell_1 \ldots, \ell_N\}$ parametrising the localisation locus. This can also decompose further to

$$Z^{\mathrm{3d}} = \sum_{\{c\}} \left(\mathcal{B}_c^{\mathrm{3d}}\right)_1 \left(\mathcal{B}_c^{\mathrm{3d}}\right)_2 \, , \tag{8.39}$$

summing over supersymmetric massive vacua $\{c\}$ and using the holomorphic blocks $\mathcal{B}_c^{\mathrm{3d}}$ in (8.28).

Finally, we remark that the object of interest from the gauge theory point of view namely expectation values of Wilson loops, can be conveniently obtained from expectation values of Schur polynomials using the above 3d partition functions. We will elaborate on this statement in Section 9.1.6.

8.2.3 $S^3 \times S^1$

We now consider the final case of interest, namely the 4d $\mathcal{N} = 1$ gauge theory on compact spaces of the form $\mathcal{M}_3 \times S^1$. For concreteness, we consider $S^3 \times S^1$. The goal now is to not only evaluate partition functions, but rather to be able to exactly compute more sophisticated observables. In the 3d case we are interested in expectation values of Wilson loops. In 4d, Wilson loops cannot be generically defined. Therefore we will instead consider the defect generating function, which can be thought to encode expectation values of BPS surface operators on $T^2 \subset \mathcal{M}^3 \times S^1$. This will be explored in Chapter 10.

To begin with, the partition function for $S^3 \times S^1$ is given by

$$Z_{S^3 \times S^1} = \mathrm{e}^{-\mathrm{i}\pi \mathcal{P}_3(\ln(\underline{\zeta}))} \oint_{T^{|G|}} \prod_{j=1}^{N} \frac{\mathrm{d}z_j}{2\pi \mathrm{i} z_j} \Delta_{S^3 \times S^1}(\underline{z}) \, . \tag{8.40}$$

Here we integrate over the maximal torus $T^{|G|}$ and the cubic polynomial $\mathcal{P}_3(\ln(\underline{\zeta}))$ encodes various gauge, mixed-gauge and global anomalies. Next, $\Delta_{S^3 \times S^1}(\underline{z})$ is given by one of the following measures. It is either the $\mathcal{N} = 1$ vector and chiral multiplets,

$$\Delta_{1\,\mathrm{vec}}(\underline{z}) = \prod_{\alpha \neq 0} \frac{1}{\Gamma(\boldsymbol{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})} \, , \quad \Delta_{\mathrm{chi}}(\underline{z}, \underline{\zeta}) = \prod_{I} \prod_{\rho, \phi \in \mathcal{R}_I} \Gamma((\mathfrak{p}\mathfrak{q})^{R_I/2} \boldsymbol{z}^{\rho} \boldsymbol{\zeta}^{\phi}; \mathfrak{p}, \mathfrak{q}) \, . \tag{8.41}$$

Here α are the roots of the Lie algebra, the product over I is over all the chiral multiplets of the theory with R_I being the R-charge, ρ, ϕ are weights of the representation $\mathcal{R}_I$ of the gauge and flavour groups and $\boldsymbol{z}$ and $\boldsymbol{\zeta}$ are the gauge and global holonomies in the Cartan tori. The parameters $\mathfrak{p}$ and $\mathfrak{q}$ are the moduli parameters which are related to the S^3 squashing parameters ω_1, ω_2 by $\mathfrak{p} = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_2}{\omega_3}}$ and $\mathfrak{q} = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_1}{\omega_3}}$, where ω_3 is the inverse S^1 radius. The measure can also be the the $\mathcal{N} = 2$ vector multiplet

$$\Delta_{2\,\mathrm{vec}}(\underline{z}) = \prod_{\alpha \neq 0} \frac{\Gamma(\hat{\mathfrak{t}}\boldsymbol{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})}{\Gamma(\boldsymbol{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})} \,, \tag{8.42}$$

with $\hat{\mathfrak{t}} = (\mathfrak{p}\mathfrak{q})^{\frac{R_{\mathrm{ad}}}{2}}$ where R_{ad} is the R-charge of the adjoint chiral, or the hypermultiplet

$$\Delta_{\mathrm{hyp}}(\underline{z}, \underline{\zeta}) = \prod_{I} \prod_{\rho,\phi \in \mathcal{R}_I} \frac{\Gamma((\mathfrak{p}\mathfrak{q})^{1/2}\hat{\mathfrak{t}}^{-1/2}\boldsymbol{z}^{\rho}\boldsymbol{\zeta}^{\phi}; \mathfrak{p}, \mathfrak{q})}{\Gamma((\mathfrak{p}\mathfrak{q})^{1/2}\hat{\mathfrak{t}}^{1/2}\boldsymbol{z}^{\rho}\boldsymbol{\zeta}^{\phi}; \mathfrak{p}, \mathfrak{q})} \,. \tag{8.43}$$

Finally we can have the $\mathcal{N} = 4$ vector multiplet contribution

$$\Delta_{4\,\mathrm{vec}}(\underline{z}) = \prod_{\alpha \neq 0} \frac{\Gamma(\hat{\mathfrak{t}}\boldsymbol{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})}{\Gamma(\boldsymbol{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})} \frac{\Gamma(\mathfrak{m}\boldsymbol{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})}{\Gamma(\hat{\mathfrak{t}}\mathfrak{m}\boldsymbol{z}^{\alpha}; \mathfrak{p}, \mathfrak{q})} \,, \tag{8.44}$$

for the parameter $\mathfrak{m}$ which can be associated with the $\mathcal{N} = 2^*$ deformation.

The next idea we wish to introduce is the decomposition of the compact space integrand on $\mathcal{M}_3 \times S^1$ into two *half-space* integrands, i.e. integrands on $D^2 \times T^2$ [28]. These two half-space integrands are then glued by an element $g \in SL(2, \mathbb{Z})$, as can be illustrated in Figure 8.1. More specifically,

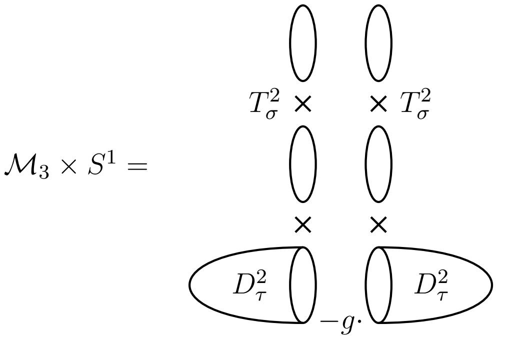

Figure 8.1. Illustration of the decomposition of $\mathcal{M}_3 \times S^1$ into two copies of $D^2 \times T^2$ glued by the element $g \in SL(2, \mathbb{Z})$ (as given in Paper I).

the compact space integrand $\Delta_{\mathcal{M}_3 \times S^1}(\underline{z}, \underline{\ell}, \underline{\zeta}, \underline{h})$, where $\Delta_{S^3 \times S^1}(\underline{z})$ is one example, can be given as

$$\Delta_{\mathcal{M}_3 \times S^1}(\underline{z}, \underline{\ell}, \underline{\zeta}, \underline{h}) = \mathrm{e}^{-\mathrm{i}\pi \mathcal{P}_3(\underline{X}, \Xi)}\, \Upsilon(\underline{w}; \tau, \sigma)^{(-g)}\, \Upsilon(\underline{w}; \tau, \sigma) \tag{8.45}$$

using the half-space integrands $\Upsilon(\underline{w};\tau,\sigma)$ together with the cubic polynomial $\mathcal{P}_3(\underline{X},\underline{\Xi})$. Here, we used the disk equivariant parameter τ and the torus modular parameter σ. The parameters $\boldsymbol{z} = \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{X}}$ and $\boldsymbol{\zeta} = \mathrm{e}^{2\pi\mathrm{i}\boldsymbol{\Xi}}$ give the gauge and background holonomies along S^1 or T^2, whereas ℓ and h describe gauge and background holonomies along non-contractible circle S^1 or fluxes through S^2. The half-space integrands are then composed of 1-loop determinants for the vector multiplet or the chiral multiplet with Neumann (N) or Dirichlet (D) boundary conditions, given by

$$\Upsilon_{\mathrm{vec}}(\underline{w}) = \prod_{\alpha\neq 0} \frac{1}{\Gamma(\boldsymbol{w}^{\alpha};q_\tau,q_\sigma)} \ , \tag{8.46}$$

$$\Upsilon^{\mathrm{N}}_{\mathrm{chi}}(\underline{w}) = \prod_{I} \prod_{\rho,\phi\in\mathcal{R}_I} \Gamma(\boldsymbol{w}^{\rho}\boldsymbol{\xi}^{\phi};q_\tau,q_\sigma) \ , \tag{8.47}$$

and

$$\Upsilon^{\mathrm{D}}_{\mathrm{chi}}(\underline{w}) = \prod_{I} \prod_{\rho,\phi\in\mathcal{R}_I} \frac{1}{\Gamma(q_\tau\boldsymbol{w}^{-\rho}\boldsymbol{\xi}^{-\phi};q_\tau,q_\sigma)} \ . \tag{8.48}$$

Here we introduced the two parameters

$$q_\tau = \mathrm{e}^{2\pi\mathrm{i}\tau} \ , \qquad q_\sigma = \mathrm{e}^{2\pi\mathrm{i}\sigma} \ , \tag{8.49}$$

and $\boldsymbol{w}$, $\boldsymbol{\xi}$ which are elements of the maximal torus of the gauge and global symmetry groups. As a final remark about the decomposition in (8.45), the action of the operation ${}^{(-g)}$ is to involute the half-space variables using the $g \in SL(2,\mathbb{Z})$ element. In the case of $S^3\times S^1$ the gluing element is

$$g = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \ . \tag{8.50}$$

We now make two simplifying assumptions. The first is that we assume the gauge and mixed-gauge anomalies to vanish. Then the polynomial $\mathcal{P}_3(\underline{X},\underline{\Xi})$ is a constant, and the decomposition in (8.45) simplifies such that the integrand completely factorises according to

$$\Delta_{\mathcal{M}_3\times S^1}(\underline{z},\underline{\ell},\underline{\zeta},\underline{h}) \propto \Upsilon(\underline{w};\tau,\sigma)^{(-g)}\ \Upsilon(\underline{w};\tau,\sigma) \ . \tag{8.51}$$

The second assumption is that q_τ and q_σ are parametrised as

$$q_\tau = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega}{\omega_1}} \ , \qquad q_\sigma = \mathrm{e}^{-2\pi\mathrm{i}\frac{\omega_3}{\omega_1}} \tag{8.52}$$

with $\omega = \omega_1 + \omega_2$. The ${}^{(-g)}$ action then results in interchanging $\omega_1 \leftrightarrow \omega_2$.

Let us now make a final comment about the half-space integrands $\Upsilon(\underline{w};\tau,\sigma)$ appearing in (8.51). Assuming that for the theory on $D^2\times T^2$ there are only isolated massive vacua, the decomposition can be taken a step further. The whole $S^3\times S^1$ partition function then factorises into

$$Z_{S^3\times S^1} \propto \sum_{\gamma\in\mathrm{vacua}} \mathcal{B}_\gamma(\underline{\xi};\tau,\sigma)^{(-g)}\ \mathcal{B}_\gamma(\underline{\xi};\tau,\sigma) \ , \tag{8.53}$$

using the 4d *holomorphic* or *anti-holomorphic blocks* [28, 31–33]

$$\mathcal{B}_\gamma(\underline{\xi};\tau,\sigma) = \oint_\gamma \prod_{a=1}^{|\Gamma_0|} \prod_{j=1}^{N_a} \frac{\mathrm{d}w_{j,a}}{2\pi\mathrm{i}w_{j,a}} \, \Upsilon(\underline{w};\tau,\sigma) \ . \tag{8.54}$$

$\mathcal{B}_\gamma(\underline{\xi};\tau,\sigma)$ is then the integral representation of the half-space, or $D^2 \times T^2$, partition function. We will return to this factorisation in Section 10.3.1.

9. 3d gauge theories and the q, t-deformation

In this chapter we discuss an application of the q, t-deformed matrix models introduced in Chapter 6. The applications we have in mind are three dimensional $\mathcal{N} = 2$ supersymmetric gauge theories and more specifically placed on the backgrounds $D^2 \times_q S^1$ and the squashed three sphere S_b^3, introduced in Section 8.2. The partition functions for such gauge theories can sometimes be interpreted as q-deformed matrix models and we will now make this interpretation more precise.

9.1 $D^2 \times_q S^1$

9.1.1 Details of the theory

Let us now introduce the details of the gauge theory we are considering. To begin with we consider the 3d $\mathcal{N} = 2$ YM-CS theory on $D^2 \times_q S^1$ with a single unitary gauge group $U(N)$, as described in Section 8.2.1. We then have an $\mathcal{N} = 2$ vector multiplet, an adjoint chiral multiplet of mass t and N_f fundamental anti-chiral multiplets of masses $\{u_k\}$. Lastly, we introduce an FI contribution parametrised by $\kappa_1 \in \mathbb{C}$. Generalising the result for the partition function given in (8.25) by introducing our familiar time variables $\{t_s\}$, we then obtain the generating function

$$\begin{aligned} Z^{N_f}_{D^2 \times_q S^1}(t) = \oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\lambda_i} \prod_{i=1}^{N} \lambda_i^{\kappa_1} \prod_{1 \le k \neq l \le N} \frac{(\lambda_k/\lambda_l; q)_\infty}{(t\lambda_k/\lambda_l; q)_\infty} \times \\ \times \prod_{j=1}^{N} \prod_{k=1}^{N_f} (q\lambda_j u_k; q)_\infty \, \mathrm{e}^{\sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} . \end{aligned} \tag{9.1}$$

This can be interpreted as a q-deformed matrix model upon recognising

$$\Delta_{q,t}(\underline{\lambda}) = \prod_{1 \le k \neq l \le N} \frac{(\lambda_k/\lambda_l; q)_\infty}{(t\lambda_k/\lambda_l; q)_\infty} \tag{9.2}$$

as the q-deformed Vandermonde determinant in (6.16). The remaining parts of the integral (9.1) is then viewed as a q-deformed version of the classical potential $V(\lambda)$ appearing in (2.21). This can be seen from rewriting the contribution from the fundamental multiplets in (8.27) as

$$\prod_{j=1}^{N} \prod_{k=1}^{N_f} (q\lambda_j u_k; q)_\infty = \exp\left(- \sum_{s=1}^{\infty} \frac{p_s(u)}{s(q^{-s} - 1)} \sum_{i=1}^{N} \lambda_i^s \right) , \tag{9.3}$$

using the power sum variables for the masses u_k, $p_s(u)$,

$$p_s(u) = \sum_{k=1}^{N_f} u_k^s \ . \tag{9.4}$$

Similarly to the classical potential in (2.12), (9.3) can be obtained by

$$t_s \mapsto t_s - \frac{p_s(u)}{s(q^{-s}-1)} \ , \tag{9.5}$$

hence the interpretation of (9.3) as a q-deformed potential.

9.1.2 The q-Virasoro constraints

The generating function in (9.1) satisfies q-Virasoro constraints, derived by the vanishing of the q-variation of the integrand as outlined in Section 6.3. The main difference compared to there is that now we allow for a generic number of anti-chiral fundamentals N_f together with an additional deformation parametrised by r. Following the derivation in Section 6.3, we obtain the equivalent of (6.35) for the $D^2 \times_q S^1$ model,

$$\text{(LHS)} = \text{(RHS)} \tag{9.6}$$

with

$$\begin{aligned}\text{(LHS)} = &\oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\lambda_i} \sum_{i=1}^{N} \left[\sum_{n\in\mathbb{Z}} (z\lambda_i)^n G_i(\underline{\lambda}) \right] \times \\ &\times Z^{\text{cl}}_{D^2\times_q S^1}(\underline{\lambda}) Z^{1-\text{loop}}_{D^2\times_q S^1}(\underline{\lambda}) \mathrm{e}^{\sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s}\end{aligned} \tag{9.7}$$

and

$$\begin{aligned}\text{(RHS)} = &\oint_{\mathcal{C}} \prod_{i=1}^{N} \frac{\mathrm{d}\lambda_i}{\lambda_i} \sum_{i=1}^{N} \hat{M}_{q^{-1},i} \times \\ &\times \left[\sum_{n\in\mathbb{Z}} (z\lambda_i)^n G_i(\underline{\lambda}) Z^{\text{cl}}_{D^2\times_q S^1}(\underline{\lambda}) Z^{1-\text{loop}}_{D^2\times_q S^1}(\underline{\lambda}) \mathrm{e}^{\sum_{s=1}^{\infty} t_s \sum_{i=1}^{N} \lambda_i^s} \right] \ .\end{aligned} \tag{9.8}$$

Upon evaluation we find for the (LHS)

$$\begin{aligned}\text{(LHS)} = &\frac{1}{1-t} \left\langle \exp\left(\sum_{s=1}^{\infty} z^{-s} \frac{(1-t^s)}{s} \sum_{i=1}^{N} \lambda_i^{-s} \right) \right\rangle_t^{N_f} + \\ &- \frac{t^N}{1-t} \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^{-s})}{s} \frac{\partial}{\partial t_s} \right) Z^{N_f}_{D^2\times_q S^1}(t) \ ,\end{aligned} \tag{9.9}$$

where we generalised the expectation value notation in (6.26) to also include a superscript N_f. Then, for the (RHS) we find

$$\begin{aligned}
\text{(RHS)} =& \frac{q^{-\kappa_1}}{1-t} \exp\left(\sum_{s=1}^{\infty} z^{-s}(1-q^s)\left(t_s + \frac{p_s(u)}{s(1-q^{-s})}\right)\right) \times \\
& \times \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^s)}{sq^s}\frac{\partial}{\partial t_s}\right) Z^{N_f}_{D^2\times_q S^1}(t) + \\
& - \frac{q^{-\kappa_1}t^N}{1-t} \exp\left(\sum_{s=1}^{\infty} z^{-s}(1-q^s)\left(t_s + \frac{p_s(u)}{s(1-q^{-s})}\right)\right) \times \\
& \times \left\langle \exp\left(\sum_{s=1}^{\infty} z^{-s}\frac{(1-t^{-s})}{sq^{-s}}\sum_{i=1}^{N}\lambda_i^{-s}\right)\right\rangle_t^{N_f}
\end{aligned} \tag{9.10}$$

using the power sum variables $p_s(u)$ in (9.4) together with the relation between powers of the integration variables and the time variables in (6.40). As before, we then consider the constraints at each order z^n with $n \geq -1$ separately. However, at $n = -1$ we find the contribution

$$\frac{1-q^{1-\kappa_1}t^{N-1}}{1-t} z^{-1} \left\langle \sum_{i=1}^{N} \lambda_i^{-1} \right\rangle_t^{N_f} , \tag{9.11}$$

which we cannot rewrite as a differential operator in the times acting on the generating function. We therefore require this term to vanish, just as in (6.61). Recalling $t = q^{\beta}$, the above vanishes upon imposing the *balancing condition*

$$\kappa_1 = \beta(N-1) + 1 . \tag{9.12}$$

It can be noted that this is in accordance with the choice of FI parameter used in [26]. In order to generalise this balancing condition, we introduce the *balancing parameter* ν defined as[1]

$$\nu = \kappa_1 - \beta(N-1) - 1 , \qquad r = q^{\nu} , \tag{9.13}$$

where we also introduced the parameter r for convenience. Thus, when the balancing condition is satisfied we have $\nu = 0$. Allowing for a non-trivial

[1]The parameter ν here should not be confused with the classical determinant insertion labelled by the same symbol.

balancing parameter, the q-Virasoro constraints take the form

$$\begin{aligned}
&t^N \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^{-s})}{s}\frac{\partial}{\partial t_s}\right) Z^{N_f}_{D^2\times_q S^1}(t) + \\
&+ r^{-1}q^{-1}t^{1-N}\exp\left(\sum_{s=1}^{\infty} z^{-s}(1-q^s)\left(t_s + \frac{p_s(u)}{s(1-q^{-s})}\right)\right)\times \\
&\qquad \times \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t^s)}{sq^s}\frac{\partial}{\partial t_s}\right) Z^{N_f}_{D^2\times_q S^1}(t) = \\
&= \left\langle \exp\left(\sum_{s=1}^{\infty} z^{-s}\frac{(1-t^s)}{s}\sum_{i=1}^{N}\lambda_i^{-s}\right)\right\rangle_t^{N_f} + \\
&+ r^{-1}q^{-1}t\exp\left(\sum_{s=1}^{\infty} z^{-s}(1-q^s)\left(t_s + \frac{p_s(u)}{s(1-q^{-s})}\right)\right)\times \\
&\qquad \times \left\langle \exp\left(\sum_{s=1}^{\infty} z^{-s}\frac{(1-t^{-s})}{sq^{-s}}\sum_{i=1}^{N}\lambda_i^{-s}\right)\right\rangle_t^{N_f} .
\end{aligned} \tag{9.14}$$

In order to be able to expand the above constraints in powers of z, we rewrite the shift in times (i.e. the contributions from the fundamental multiplets) using coefficients $A_k = A_k(u)$ defined via

$$\exp\left(-\sum_{s=1}^{\infty}\left(\frac{q}{z}\right)^s \frac{p_s(u)}{s}\right) = \prod_{k=1}^{N_f}\left(1-\frac{q}{z}u_k\right) = \sum_{k=0}^{N_f} A_k \left(\frac{q}{z}\right)^k . \tag{9.15}$$

As can be seen from the Cauchy identity specialised to antisymmetric Schur polynomials in (2.38), the A_k are nothing but antisymmetric Schur polynomials in the power sum variables for the fundamental masses u_k,

$$A_k = (-1)^k \mathsf{Schur}_{\{\underbrace{1,\ldots,1}_{k}\}}(p_s = p_s(u)) . \tag{9.16}$$

It should be noted that the summation on the right hand side in (9.15) truncates to N_f. This is due to that antisymmetric Schur polynomials of a degree higher than the number of variables are identically vanishing.

To then solve the q-Virasoro constraints in (9.14), we follow the procedure outlined in Section 6.4. In particular, we assume the expansion of the generating function in terms of the time variables

$$Z^{N_f}_{D^2\times_q S^1}(t) = \sum_{\rho}\frac{1}{|\mathrm{Aut}(\rho)|}c_\rho(u)\prod_{\mu\in\rho} t_\mu , \tag{9.17}$$

summing over all integer partitions ρ. Here $c_\rho(u)$ are the correlators, which similarly to the classical case depend on the fundamental masses $u = \{u_1, \ldots, u_{N_f}\}$. Let us now explore the solutions for the first few values of N_f, which are obtained via a recursion similarly to that in Section 6.4.

9.1.3 $N_f = 1$

Specialising to the case of $N_f = 1$, such that we only have the coefficient $A_1 = -u_1$, we find the correlators

$$\begin{aligned}
c_{\{1\}}(u_1) =& \frac{(t^N-1)\left(qrt^N-t\right)}{A_1(t-1)t} c_\emptyset(u_1) \\
c_{\{1,1\}}(u_1) =& \frac{(t^N-1)\left(qrt^N-t\right)}{A_1^2(t-1)^2t^3} \times \\
& \times \Big(qrt^{2N}(q(t-1)+1) - t^{N+1}(qr+t) + t^2\Big) c_\emptyset(u_1) \\
c_{\{2\}}(u_1) =& \frac{(t^N-1)\left(qrt^N-t\right)}{A_1^2t^3\left(t^2-1\right)} \times \\
& \times \Big(qrt^{2N}(qt+q+1) - t^{N+1}(qr+t) - t^2\Big) c_\emptyset(u_1) \ .
\end{aligned} \tag{9.18}$$

Here it can be noted that just as in the classical case in (3.70), all correlators of degree 2 and higher are proportional to $c_{\{1\}}$. Besides, the correlators are rational functions of the parameters q, t, r and $A_1 = -u_1$.

Another result found in the case of $N_f = 1$ is one related to superintegrability as introduced around (2.52). As the formulas in the β-deformed classical case involved Jack polynomials, it is natural to expect that the q-deformed case would involve the Macdonald polynomials. For the $N_f = 1$ theory on $D^2 \times_q S^1$ it can be observed that

$$\begin{aligned}
\langle \mathsf{Macdonald}_\rho(p_k) \rangle^{N_f=1} =& \frac{\mathsf{Macdonald}_\rho(p_k = \hat{\pi}_k^{(N)})}{\mathsf{Macdonald}_\rho(p_k = \delta^*_{k,1})} \times \\
& \times \mathsf{Macdonald}_\rho(p_k = \pi_k^{(N)}) c_\emptyset(u_1) \ .
\end{aligned} \tag{9.19}$$

Here we have defined the combinations of parameters

$$\begin{aligned}
\pi_k^{(N)} &= \frac{t^{\frac{k}{2}N} - t^{-\frac{k}{2}N}}{t^{\frac{k}{2}} - t^{-\frac{k}{2}}} \ , \\
\hat{\pi}_k^{(N)} &= r^{\frac{k}{2}} q^{\frac{k}{2}} t^{k(N-1)} \frac{r^{\frac{k}{2}} q^{\frac{k}{2}} t^{\frac{k}{2}(N-1)} - r^{-\frac{k}{2}} q^{-\frac{k}{2}} t^{-\frac{k}{2}(N-1)}}{t^{\frac{k}{2}} - t^{-\frac{k}{2}}}
\end{aligned} \tag{9.20}$$

together with

$$\delta^*_{k,1} = \frac{u_1^k}{t^{-\frac{k}{2}} - t^{\frac{k}{2}}} \ . \tag{9.21}$$

For consistency purposes, one can explore the semi-classical limit $t = q^\beta$, $r = q^\nu$, $u_1 = -(1-q^{-1})a_1$ and $q \to 1$. In this limit, we find

$$\pi_k^{(N)} \to N \ , \tag{9.22}$$

$$\hat{\pi}_k^{(N)} \to \beta^{-1}(\nu + \beta(N-1)+1) \ , \tag{9.23}$$

and

$$\delta^*_{k,1} = (-1)^{k+1} q^{-\frac{k}{2}} \frac{(q^{\frac{1}{2}} - q^{-\frac{1}{2}})^k}{t^{\frac{k}{2}} - t^{-\frac{k}{2}}} a_1^k \to \beta^{-1} a_1 \delta_{k,1} \ . \tag{9.24}$$

Since Macdonald polynomials degenerate to Jack polynomials the formula (9.19) coincides with that for Jack polynomials in (2.53). Finally, in the limit $\beta = 1$ we obtain the average of Schur polynomials given in (2.54).

9.1.4 $N_f = 2$

One can then perform a similar analysis for the case $N_f = 2$ as for the $N_f = 1$ case above. The first correlators are given by

$$\begin{aligned}
c_{\{1\}}(u_1,u_2) &= -\frac{A_1\left(t^N-1\right)}{A_2(t-1)} c_\emptyset(u_1,u_2) \\
c_{\{1,1\}}(u_1,u_2) &= \frac{\left(t^N-1\right)\left(A_2(q-1)(t-1)t^N + A_1^2 t\left(t^N-1\right)\right)}{A_2^2(t-1)^2 t} c_\emptyset(u_1,u_2) \\
c_{\{2\}}(u_1,u_2) &= \frac{\left(t^N-1\right)}{A_2^2 t\left(t^2-1\right)} \Big(A_2\left((q+1)t^N + (q-1)t^{N+1} - 2t\right) + \\
&\quad + A_1^2 t(t^N+1)\Big) c_\emptyset(u_1,u_2) \ ,
\end{aligned} \tag{9.25}$$

where we again can note that the correlators are rational functions of the parameters q, t, $A_1 = -(u_1+u_2)$ and $A_2 = u_1 u_2$. In the case of $N_f = 2$ we find the superintegrability result,

$$\begin{aligned}
\left\langle \mathsf{Macdonald}_\rho\left(p_k\right)\right\rangle^{N_f=2} =& \frac{\mathsf{Macdonald}_\rho\left(p_k = (-1)^k t^{\frac{k}{2}N} \frac{\left(u_1^k + u_2^k\right)}{1-t^k}\right)}{\mathsf{Macdonald}_\rho\left(p_k = (-1)^k t^{\frac{k}{2}} \frac{(u_1u_2)^k}{1-t^k}\right)} \times \\
& \times \mathsf{Macdonald}_\rho\left(p_k = \pi_k^{(N)}\right) c_\emptyset(u_1,u_2) \ ,
\end{aligned} \tag{9.26}$$

with $\pi_k^{(N)}$ as in (9.20). To then see the semi-classical limit, we again let $t = q^\beta$, but now we need to establish u_1 and u_2 as functions of a_1 and a_2. To do so, we require that in the semi-classical limit $q \to 1$, or $q = \mathrm{e}^\hbar$ with $\hbar \to 0$, u_1 and u_2 scale non-trivially with $\hbar$ at the same time as the shift in times in (9.5) is well-defined. One such parametrisation is

$$\begin{aligned}
u_1 &= \frac{(1-q)a_1 + \sqrt{(1-q)\left(a_1^2(q-1) + 2a_2(q+1)\right)}}{2q} \\
u_2 &= \frac{(1-q)a_1 - \sqrt{(1-q)\left(a_1^2(q-1) + 2a_2(q+1)\right)}}{2q} \ ,
\end{aligned} \tag{9.27}$$

(up to a permutation $u_1 \leftrightarrow u_2$) which implies that A_1 and A_2 behave as

$$A_k = a_k \hbar + \mathcal{O}(\hbar^2) \ , \qquad k = 1, 2 \ . \tag{9.28}$$

With this parametrisation, the arguments of the Macdonald polynomials in (9.26) become in the semi-classical limit

$$(-1)^k t^{\frac{k}{2}N} \frac{(u_1^k + u_2^k)}{1 - t^k} \to (-1)^k \beta^{-1}(a_1 \delta_{k,1} + a_2 \delta_{k,2}) \tag{9.29}$$

and

$$(-1)^k t^{\frac{k}{2}} \frac{(u_1 u_2)^k}{1 - t^k} \to \beta^{-1} a_2 \delta_{k,1} \ , \tag{9.30}$$

such that we again recover the β-deformed classical result for Jack polynomials in (2.55) and when $\beta = 1$ the result in (2.56) for Schur polynomials.

9.1.5 Comments about $N_f \geq 3$

As was discovered in paper IV, the cases $N_f \geq 3$ cannot be determined uniquely using the methods outlined in the sections above for $N_f = 1$ and $N_f = 2$. This resembles the classical case of $\mathsf{p} = 3$ mentioned in Section 3.2.2 in which we find that the correlators can only be determined up to dependence on the coupling a_1. What happens for $N_f \geq 3$ is that one needs to supply more initial conditions in order to solve the constraint equations, which originates from the kernel of the q-Virasoro constraints being infinite dimensional. Although it appears that the recursion relation still can be used to evaluate correlators in this case, the dependence on the mass parameters $\{u_k\}$ becomes more involved and the semi-classical behaviour is also not straightforward.

9.1.6 Final remarks about $D^2 \times_q S^1$

Upon comparing the above gauge theory construction in terms of a q-deformed matrix model to the classical β-deformed matrix model in (2.21) with potential (2.16), one can find the matching between the parameters of the theories as given in Table 9.1. The first matching between the β-deformation introduced in (2.21) and the adjoint mass t in (8.27), is through identifying the adjoint mass as the parameter $t = q^\beta$. Next, the matching between the degree of the classical potential p in (2.16) and the number of fundamental anti-chiral multiplets N_f in (8.27), originates from the interpretation of the fundamental multiplets as a q-deformed potential as discussed around (9.3). Also the third matching between the classical coupling constants a_k and the masses of the anti-chiral fundamental multiplets u_k comes from this q-deformed potential picture, where the a_k

can be non-trivially related to the u_k. Finally, the determinant insertion ν in the classical picture can be related to the effective FI parameter via the balancing parameter r in the quantum model.

Classical β-deformed matrix model		Quantum matrix model	
β-deformation	β	t	Adjoint mass
Degree of potential	p	N_f	Number of fundamentals
Coupling constants	a_k	u_k	Fundamental masses
Determinant insertion	ν	r	Balancing parameter

Table 9.1. *Matching the parameters of the classical β-deformed matrix model and the quantum matrix model describing the gauge theory on $D^2 \times_q S^1$.*

To conclude, we comment on the physical meaning of the correlators found for the above gauge theory on $D^2 \times_q S^1$. From the gauge theory perspective, the interesting objects to compute are gauge invariant quantities such as Wilson loops. On the q-deformed matrix model side, averages of Wilson loops correspond to averages of Schur polynomials. One can therefore in the $D^2 \times_q S^1$ gauge theory compute normalised expectation values for Wilson loops along S^1 in representation ρ via

$$\langle \mathrm{WL}_\rho \rangle = \frac{1}{c_\emptyset} \left\langle \mathsf{Schur}_\rho \left(p_k = \sum_{i=1}^{N} \lambda_i^k \right) \right\rangle^{N_f} . \tag{9.31}$$

As an example, we can consider the q,t-Gaussian model with the particular choices of $N_f = 2$ and masses $u_1 = q(1-q)^{1/2}$ and $u_2 = -q(1-q)^{1/2}$. We then find the normalised expectation values for Wilson loops

$$\begin{aligned}
\left\langle \mathrm{WL}_{\{1,1\}} \right\rangle = \Bigg(& t^{2N} \left(\frac{-3q-1}{2(q-1)(t-1)} + \frac{3q+5}{2(q-1)t} - \frac{2}{(q-1)(t+1)} \right) + \\
& + t^{N} \left(\frac{-3q-5}{2(q-1)t} + \frac{3q+5}{2(q-1)(t-1)} \right) + \\
& - \frac{2}{(q-1)(t-1)} + \frac{2}{(q-1)(t+1)} \Bigg) ,
\end{aligned}$$

$$\begin{aligned}
\left\langle \mathrm{WL}_{\{2\}} \right\rangle = \Bigg(& t^{2N} \left(\frac{-5q-3}{2(q-1)t} + \frac{5q-1}{2(q-1)(t-1)} + \frac{2}{(q-1)(t+1)} \right) + \\
& + t^{N} \left(\frac{-5q-3}{2(q-1)(t-1)} + \frac{5q+3}{2(q-1)t} \right) + \\
& + \frac{2}{(q-1)(t-1)} - \frac{2}{(q-1)(t+1)} \Bigg) .
\end{aligned} \tag{9.32}$$

9.2 S_b^3

9.2.1 Details of the theory

Let us move on to the 3d $\mathcal{N}=2$ YM-CS theory, now instead living on the squashed 3-sphere S_b^3. Similarly to the $D^2 \times_q S^1$ case, we consider the $\mathcal{N}=2$ vector multiplet, an adjoint chiral multiplet of mass M_{a}, N_f fundamental anti-chiral multiplets of masses $\{m_k\}$ for $k=1,\dots,N_f$ together with an FI contribution parametrised by κ_1. We also allow for a non-vanishing CS term parametrised by κ_2. This condition arises as a technical, rather than physical, requirement from writing the q-Virasoro constraints as a differential operator in the time variables acting on the generating function. More specifically, we require

$$N_f = 2\kappa_2 \ . \tag{9.33}$$

Then, using the partition function as shown in [90, 92] and as written in (8.33), upon introducing dependence on the time variables we find that the generating function for S_b^3 is given by

$$Z_{S_b^3}^{N_f}(t) = \int_{(\mathrm{i}\mathbb{R})^N} \prod_{i=1}^{N} \mathrm{d}X_i \, Z_{S_b^3}^{\mathrm{cl}}(\underline{X}) \, Z_{S_b^3}^{1-\mathrm{loop}}(\underline{X}) \prod_{\alpha=1,2} \exp\left(\sum_{s=1}^{\infty} t_{s,\alpha} \sum_{i=1}^{N} \lambda_{i,\alpha}^s \right) \ . \tag{9.34}$$

Here we introduce the following exponentiated variables to ease notation,

$$\begin{aligned} q_\alpha &= \mathrm{e}^{\frac{2\pi\mathrm{i}\omega}{\omega_\alpha}} \qquad & u_{k,\alpha} &= \mathrm{e}^{\frac{2\pi\mathrm{i}m_k}{\omega_\alpha}} \\ t_\alpha &= \mathrm{e}^{\frac{2\pi\mathrm{i}M_{\mathrm{a}}}{\omega_\alpha}} \qquad & \lambda_{i,\alpha} &= \mathrm{e}^{\frac{2\pi\mathrm{i}X_i}{\omega_\alpha}} \ , \end{aligned} \tag{9.35}$$

recalling the labels $\alpha = 1, 2$ for the holomorphic block, $k = 1,\dots,N_f$ for the fundamental anti-chirals and $i = 1,\dots,N$ for the integration variables. Here we also used $\omega = \omega_1 + \omega_2$. Finally, we have the complex parameter $\beta = M_{\mathrm{a}}/\omega$, so that $t_\alpha = q_\alpha^\beta$ is consistent with earlier discussions. We then recall from Section 8.2.2 that $Z_{S_b^3}^{\mathrm{cl}}(\underline{X})$ is the classical contribution given by

$$Z_{S_b^3}^{\mathrm{cl}}(\underline{X}) = \prod_{i=1}^{N} \exp\left(-\frac{\pi\mathrm{i}\kappa_2}{\omega_1\omega_2} X_i^2 + \frac{2\pi\mathrm{i}\kappa_1}{\omega_1\omega_2} X_i \right) \ , \tag{9.36}$$

and that $Z_{S_b^3}^{1-\mathrm{loop}}(\underline{X})$ is the product of 1-loop determinants given by

$$Z_{S_b^3}^{1-\mathrm{loop}}(\underline{X}) = \prod_{1\le k\ne j\le N} \frac{S_2(X_k - X_j|\underline{\omega})}{S_2(X_k - X_j + M_{\mathrm{a}}|\underline{\omega})} \prod_{k=1}^{N_f} \prod_{i=1}^{N} S_2\left(-X_i - m_k|\underline{\omega}\right)^{-1} \ . \tag{9.37}$$

The double sine function $S_2(z|\underline{\omega})$ is defined in (5.9). We now view

$$\prod_{1\le k\ne j\le N} \frac{S_2(X_k - X_j|\underline{\omega})}{S_2(X_k - X_j + M_{\mathrm{a}}|\underline{\omega})} \tag{9.38}$$

as the q-deformed Vandermonde, whereas the other contributions to (9.34) are considered a q-deformed version of the classical potential $V(\lambda)$.

9.2.2 Modular double construction revisited

Let us now recall the idea of the factorisation of the S_b^3 partition function in terms of $D^2 \times_q S^1$ partition functions as given in [29] and reviewed in Section 8.2.2. Extending this idea, it has been explored in [26] that the S_b^3 generating function in fact satisfies two independent copies of q-Virasoro constraints. Each copy then corresponds to the q-Virasoro constraints of the corresponding $D^2 \times_{q_\alpha} S^1$ theory and consequently one can then use the solution on $D^2 \times_q S^1$ to read off that on S_b^3. Referring to this factorisation property of the S_b^3 theory, the construction in [26] has been called the *modular double*. We will also refer to the two commuting copies of the q-Virasoro algebra as *chiral sectors*, similar to usual CFT terminology.

What this modular double construction implies more concretely, is that upon letting one set of the times to be zero in the S_b^3 generating function for instance $\{t_{s,2}\} = 0$, we recover the $D^2 \times_q S^1$ generating function,

$$Z_{S_b^3}^{N_f}(t)\Big|_{\{t_{s,2}\}=0} \simeq Z_{D^2\times_q S^1}^{N_f}(t) \ . \tag{9.39}$$

Here it should be noted that the equivalence holds at the level of the two objects satisfying the same set of q-Virasoro constraints, not at the level of explicit integral representation.

9.2.3 The q-Virasoro constraints

Let us now review the method to derive the q-Virasoro constraints in the case of S_b^3. This is very similar to the derivation in Section 6.3, with the difference being from the modular double construction that we have two sets of constraints for the generating function labelled by $\alpha = 1, 2$.

We begin with generalising the q-differential which then takes the form

$$\mathrm{d}_{q_\alpha^{-1},i,\alpha} = \hat{M}_{q_\alpha^{-1},i,\alpha} - 1 \tag{9.40}$$

using the q-shift operator $\hat{M}_{q,i}$ in (4.2), where we introduced a dependence on the additional parameter $\alpha = 1, 2$ labelling the two sets of constraints. $\hat{M}_{q_\alpha^{-1},i,\alpha}$ then shifts each set of variables separately, such that

$$\hat{M}_{q_\alpha^{-1},i,\alpha}\lambda_{j,\alpha'} = \begin{cases} q_\alpha^{-1}\lambda_{j,\alpha'} & \text{if } i = j \text{ and } \alpha = \alpha' \ , \\ \lambda_{j,\alpha'} & \text{otherwise} \ . \end{cases} \tag{9.41}$$

Alternatively, recalling the definitions of the exponentiated variables in (9.35), we find that when the q-shift acts on variables X_i we have

$$\begin{aligned}\hat{M}_{q_1^{-1},i,1} f(\underline{X}) =& f(\ldots, X_i - \omega_2, \ldots) \ , \\ \hat{M}_{q_2^{-1},i,2} f(\underline{X}) =& f(\ldots, X_i - \omega_1, \ldots) \ .\end{aligned} \tag{9.42}$$

To then generalise the insertion in (6.32) required to generate the constraints, this becomes

$$\sum_{i=1}^{N} \mathrm{d}_{q_\alpha^{-1},i,\alpha} \left[\sum_{n\in\mathbb{Z}} (z\lambda_{i,\alpha})^n \, G_{i,\alpha}(\underline{\lambda}) \ldots \right] \ , \tag{9.43}$$

with $\ldots$ denoting the integrand as before, and where now

$$G_{i,\alpha}(\underline{\lambda}) = \prod_{\substack{j=1 \\ j\neq i}}^{N} \frac{1 - t_\alpha \lambda_{i,\alpha}/\lambda_{j,\alpha}}{1 - \lambda_{i,\alpha}/\lambda_{j,\alpha}} \ . \tag{9.44}$$

We then proceed similarly to Section 6.3, to find the q-Virasoro constraints

$$\begin{aligned}
& t_\alpha^N \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t_\alpha^{-s})}{s} \frac{\partial}{\partial t_{s,\alpha}} \right) Z_{S_b^3}^{N_f}(t) + \\
& + r_\alpha^{-1} q_\alpha^{-1} t_\alpha^{1-N} \exp\left(\sum_{s=1}^{\infty} z^{-s} (1-q_\alpha^s) \left(t_{s,\alpha} + \frac{p_s(u)}{s(1-q_\alpha^{-s})} \right) \right) \times \\
& \qquad \times \exp\left(\sum_{s=1}^{\infty} z^s \frac{(1-t_\alpha^s)}{s q_\alpha^s} \frac{\partial}{\partial t_{s,\alpha}} \right) Z_{S_b^3}^{N_f}(t) = \\
& = \left\langle \exp\left(\sum_{s=1}^{\infty} z^{-s} \frac{(1-t_\alpha^s)}{s} \sum_{i=1}^{N} \lambda_{i,\alpha}^{-s} \right) \right\rangle_t^{N_f} + \\
& \qquad + r_\alpha^{-1} q_\alpha^{-1} t_\alpha \exp\left(\sum_{s=1}^{\infty} z^{-s} (1-q_\alpha^s) \left(t_{s,\alpha} + \frac{p_s(u)}{s(1-q_\alpha^{-s})} \right) \right) \times \\
& \qquad \times \left\langle \exp\left(\sum_{s=1}^{\infty} z^{-s} \frac{(1-t_\alpha^{-s})}{s q_\alpha^{-s}} \sum_{i=1}^{N} \lambda_{i,\alpha}^{-s} \right) \right\rangle_t^{N_f} \ ,
\end{aligned} \tag{9.45}$$

for $\alpha = 1, 2$. These constraints are identical to those on $D^2 \times_q S^1$ in (9.14) up to the index α. Similarly to (9.13), we also introduce the balancing condition for S_b^3 parametrised by ν and r_α,

$$\omega\nu = \kappa_1 - \omega - M_{\mathrm{a}}(N-1) + \frac{\omega}{2}\frac{N_f}{2} + \sum_{k=1}^{N_f} \frac{m_k}{2} \ , \qquad r_\alpha = \mathrm{e}^{\frac{2\pi \mathrm{i}\omega}{\omega_\alpha}\nu} = q_\alpha^\nu \ . \tag{9.46}$$

Again we find that for the $n = -1$ constraint we must require the balancing condition $\nu = 0$, in order to not have any contributions from expectation values of negative powers of $\lambda_{i,\alpha}$.

To then solve the above constraints, we assume as before that the generating function can be expanded in terms of the time variables as

$$Z^{N_f}_{S^3_b}(t) = \sum_\rho \sum_\sigma \frac{1}{|\mathrm{Aut}(\rho)|} \frac{1}{|\mathrm{Aut}(\sigma)|} c_{\rho;\sigma} \prod_{a\in\rho} t_{a,1} \prod_{b\in\sigma} t_{b,2} \ , \tag{9.47}$$

with the correlators $c_{\rho;\sigma}$ given by

$$c_{\rho;\sigma} = \left[\prod_{a\in\rho} \frac{\partial}{\partial t_{a,1}} \prod_{b\in\sigma} \frac{\partial}{\partial t_{b,2}} Z^{N_f}_{S^3_b}(t)\right]\Bigg|_{t=0} \ . \tag{9.48}$$

To see how the factorisation of the S^3_b generating function into two copies of $D^2 \times_q S^1$ appears, we review the logic outlined in paper III. From the q-Virasoro constraints in (9.45), we can obtain a recursion similar to that in (6.72). The starting point is the observation that this recursion treats the two copies separately. Then one assumes that correlators of a certain order d (in for instance the first set of times, i.e. $|\rho| = d$) factorises as

$$c_{\rho;\sigma} = \frac{c_{\rho;\emptyset} \cdot c_{\emptyset;\sigma}}{c_{\emptyset;\emptyset}} \ , \tag{9.49}$$

where $c_{\rho;\emptyset}$ is a correlation function of only the variables $\{\lambda_{i,1}\}$ and vice versa. If one then considers another correlator $c_{\rho';\sigma}$ of order $|\rho'| = d+1$, then all the correlators in the right hand side of the recursion equation (similar to that in (6.72)) are of order d as they are of one order lower. Then, from our assumption, all the correlators in the right hand side factorises as (9.49) since they are of order d, such that the dependence on the second copy can be extracted into an overall common factor of $c_{\emptyset;\sigma}$. After dividing by this factor, the left hand side of the recursion is therefore proportional to the ratio $c_{\rho';\sigma}/c_{\emptyset;\sigma}$. The right hand side can then be written entirely in terms of correlators $c_{\tilde{\rho};\emptyset}$ for some partition $\tilde{\rho}$ of order d or lower. This implies that the correlator on the left hand side of order $d+1$ is also forced to factorise according to the assumption in (9.49).

To complete the induction argument, we finally need that the first step of the recursion factorises, i.e. the empty correlator $c_{\emptyset;\emptyset}$. This correlator corresponds to an overall choice of normalisation, which we can choose as

$$c_{\emptyset;\emptyset} = 1 \ , \tag{9.50}$$

which therefore factorises. Thus by induction, it follows that all the correlators $c_{\rho;\sigma}$ factorises. Additionally, using the physical picture described around equation (9.31), the factorisation also holds for the expectation values of the Wilson loops. In other words,

$$\left\langle \mathrm{WL}^{(1)}_\rho \mathrm{WL}^{(2)}_\sigma \right\rangle = \left\langle \mathrm{WL}^{(1)}_\rho \right\rangle \left\langle \mathrm{WL}^{(2)}_\sigma \right\rangle \tag{9.51}$$

for any partitions ρ, σ, where the superscript α for a Wilson loop $\mathrm{WL}_\rho^{(\alpha)}$ labels the copy in the modular double construction $\alpha = 1, 2$.

Using the above logic we then find that the correlators factorises as desired according to (9.49), which in turns implies that also the generating function factorises as

$$Z_{S_b^3}^{N_f}(t) = Z_{S_b^3}^{N_f}(0) \left[\sum_\rho \frac{1}{|\mathrm{Aut}(\rho)|} \frac{c_{\rho;\emptyset}}{c_{\emptyset;\emptyset}} \prod_{a\in\rho} t_{a,1} \right] \left[\sum_\sigma \frac{1}{|\mathrm{Aut}(\sigma)|} \frac{c_{\emptyset;\sigma}}{c_{\emptyset;\emptyset}} \prod_{a\in\sigma} t_{a,2} \right] , \tag{9.52}$$

where we used the fact that the empty correlator is nothing but the partition function, $c_{\emptyset;\emptyset} = Z_{S_b^3}^{N_f}(0)$. Thus, each half of the generating function for S_b^3 satisfies its own set of q-Virasoro constraints and corresponds to one of the two constituent $D^2 \times_q S^1$ theories. The above factorisation of the generating function in (9.52), then gives a more precise meaning to the equivalence between the $D^2 \times_q S^1$ and S_b^3 generating functions in (9.39) motivated from the modular double construction. We therefore conclude that all the results for the S_b^3 theory can simply be obtained from those of the $D^2 \times_q S^1$ one.

10. 4d gauge theories and the q, t, q'-deformation

We now proceed to discuss an application of the second type of quantum matrix models, namely the q, t, q'-deformation as described in Chapter 7. The applications of this elliptic model that we would like to discuss are four dimensional supersymmetric gauge theories on compact spaces $\mathcal{M}_3 \times S^1$. In the 3d picture, it has been shown in [26] that there is a correspondence between the 3d $\mathcal{N} = 2$ unitary quiver gauge theory and the trigonometric $W_{q,t}(\Gamma)$ algebra for any quiver Γ of [67]. In paper I we therefore explored if this correspondence could be extended to 4d supersymmetric theories and the elliptic $W_{q,t;q'}$ algebra. Another motivation can be found from the property that partition functions of 4d supersymmetric gauge theories are related to important quantities like the superconformal indices in the particular case when $\mathcal{M}_3 = S^3$ [93, 94].

10.1 Details of the theory

Here we consider the 4d $\mathcal{N} = 1$ gauge theory on $\mathcal{M}_3 \times S^1$. More specifically, we provide the example of $S^3 \times S^1$ and we choose a 4d gauge group given by $G = \bigtimes_a U(N_a)$ together with the gauge and matter content as given in Section 8.2.3. Generalising the partition function in (8.40), the *defect generating function* for $S^3 \times S^1$ is given by

$$Z_{S^3\times S^1}(\{t^a_{0,\alpha}, t^a_\alpha\}) = \mathrm{e}^{\sum_{\alpha=1,2}\mathcal{N}_0(\{t^a_{0,\alpha}\})} \oint_{T^{|G|}} \Delta_{S^3\times S^1}(\underline{z}) \prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a} \frac{\mathrm{d}z_{j,a}}{2\pi\mathrm{i} z_{j,a}} \times \\ \times \exp\left(- \sum_{\alpha=1,2}\sum_{a=1}^{|\Gamma_0|}\sum_{j=1}^{N_a}\sum_{n\neq 0} \frac{t^a_{n,\alpha} z^n_{j,a}}{n(1-q^n_\alpha)} \right) \qquad (10.1)$$

encoding all possible observables which can be computed via localisation. $\Delta_{S^3\times S^1}(\underline{z})$ is then the measure as indicated in Section 8.2.3 and it should be noted that the time variables $\{t^a_{n,\alpha}\}$ are labelled by $\alpha = 1, 2$ mirroring the modular double structure or holomorphic blocks. We also recall the quiver construction in Section 7.1. The motivation behind studying this object is that it is precisely the object which we can recover from the $W_{q,t;q'}$ algebra side, as we will see in Section 10.2. As a final remark, similar to the 3d cases we can view $\Delta_{S^3\times S^1}(\underline{z})$ as giving rise to the elliptically deformed Vandermonde and a deformed version of the potential.

10.2 Elliptic modular double construction

We now move on to discuss how the idea of the modular double, as introduced by [26] and reviewed in Section 9.2 in the case of S_b^3, also can be applied to the case of 4d backgrounds. Similar to the 3d case, the idea mirrors the factorisation properties of 4d gauge theory observables [28, 31–33]. In short, one creates a modular double version of the elliptic $W_{q,t;q'}$ algebra (introduced in Chapter 7) by taking two commuting copies of the algebra and combining them into a bigger algebra upon imposing that the two are related by $SL(2,\mathbb{Z})$ transformations. We then denote the resulting algebra by $W^g_{q,t;q'}$ labelled by the element $g \in SL(2,\mathbb{Z})$.

Similarly to the modular double in the 3d case in Section 9.2.2, we use the index $\alpha = 1, 2$ to denote the two chiral sectors of the algebra. For each of the two commuting elliptic algebras $W_{q,t;q'}(\Gamma)_\alpha$ we have the screening current $\mathsf{S}^a(w)_\alpha$ defined by the relation

$$[T^a_{n,\alpha}, \mathsf{S}^b(w)_{\alpha'}] = \delta_{\alpha,\alpha'}\delta_{a,b}\left(\frac{\mathsf{O}^b_n(q_\alpha w)_\alpha - \mathsf{O}^b_n(w)_\alpha}{(q_\alpha - 1)(w)_\alpha}\right) , \tag{10.2}$$

generalising (7.15). We then parametrise the integration variables as

$$(w)_\alpha = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_\alpha}X} , \tag{10.3}$$

and combine the two screening currents into the operator $\mathcal{S}^a(X)$ given by

$$\mathcal{S}^a(X) = \mathsf{S}^a(w)_1 \otimes\ \mathsf{S}^a(w)_2 = \bigotimes_{\alpha=1,2}[\mathsf{S}^a(w)_\alpha]_-[\mathsf{S}^a(w)_\alpha]_+[\mathsf{S}^a(w)_\alpha]_0 \ . \tag{10.4}$$

Here, we let $[\ ]_{\pm,0}$ denote the positive, negative and the zero mode parts respectively. In general, $\mathcal{S}^a(X)$ does not commute with the generator $T^a_{n,\alpha}$ up to a total difference. For instance when $\alpha = 1$ we find

$$[T^a_{n,1}, \mathcal{S}^b(X)] = \delta_{a,b}\left(\frac{\mathsf{O}^b_n(q_1 w)_1 - \mathsf{O}^b_n(w)_1}{(q_1 - 1)(w)_1}\right) \otimes \mathsf{S}^b(w)_2 \ , \tag{10.5}$$

which cannot be written as a total difference and therefore $\mathcal{S}^a(X)$ is not suitable as a screening current for neither of the two $W_{q,t;q'}(\Gamma)_\alpha$ algebras. However, for the particular choice of deformation parameters given by

$$q_\alpha = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega}{\omega_\alpha}} \ , \quad q'_\alpha = \mathrm{e}^{-2\pi\mathrm{i}\frac{\omega_3}{\omega_\alpha}} \ , \quad t_\alpha = \mathrm{e}^{2\pi\mathrm{i}\beta\frac{\omega}{\omega_\alpha}} \ , \quad \mu_{e,\alpha} = \mathrm{e}^{2\pi\mathrm{i}\frac{M_e}{\omega_\alpha}} \ , \tag{10.6}$$

with $\omega = \omega_1 + \omega_2$ (where again we refer to Chapter 7 for details), together with the replacement of the zero mode part in (10.4) given by

$$[\mathsf{S}^a(w)_\alpha]_0 \longrightarrow \frac{\Theta((w)_\alpha\, q_\alpha^{-\mathsf{s}_{0,a,\alpha}}; q_\alpha)}{\Theta((w)_\alpha; q_\alpha)\Theta(q_\alpha^{-\mathsf{s}_{0,a,\alpha}}; q_\alpha)} \ , \tag{10.7}$$

we indeed find that $\mathcal{S}^a(X)$ is a good screening current.

To make a connection with the earlier introduced parameters τ and σ we can re-parametrise these as

$$\tau = \frac{\omega}{\omega_1} \,, \qquad \sigma = -\frac{\omega_3}{\omega_1} \,, \tag{10.8}$$

such that the $g \in SL(2,\mathbb{Z})$ action corresponds to interchanging $\omega_1 \leftrightarrow \omega_2$. Similarly, the action of g can be extended to the other deformation parameters of the two chiral sectors, as summarised in Table 10.1. Again we specialise to the case $S^3 \times S^1$ and for other backgrounds we refer to paper I.

$\alpha = 1$	$W^g_{q,t;q'}(\Gamma)$	$\alpha = 2$	$W^g_{q,t;q'}(\Gamma)$
q_1	$\mathrm{e}^{2\pi\mathrm{i}\tau}$	q_2	$\mathrm{e}^{-2\pi\mathrm{i}g\cdot\tau} = \mathrm{e}^{2\pi\mathrm{i}\frac{\tau}{\tau-1}}$
q'_1	$\mathrm{e}^{2\pi\mathrm{i}\sigma}$	q'_2	$\mathrm{e}^{-2\pi\mathrm{i}g\cdot\sigma} = \mathrm{e}^{2\pi\mathrm{i}\frac{\sigma}{\tau-1}}$
t_1	$\mathrm{e}^{2\pi\mathrm{i}\beta_1\tau}$	t_2	$\mathrm{e}^{-2\pi\mathrm{i}\beta_2 g\cdot\tau} = \mathrm{e}^{2\pi\mathrm{i}\beta_2\frac{\tau}{\tau-1}}$
$\mu_{e,1}$	$\mathrm{e}^{2\pi\mathrm{i}M_e}$	$\mu_{e,2}$	$\mathrm{e}^{-2\pi\mathrm{i}g\cdot M_e} = \mathrm{e}^{2\pi\mathrm{i}\frac{M_e}{\tau-1}}$
$(w)_1$	$\mathrm{e}^{2\pi\mathrm{i}X}\mathrm{e}^{2\pi\mathrm{i}\ell}$	$(w)_2$	$\mathrm{e}^{-2\pi\mathrm{i}g\cdot X}\mathrm{e}^{-2\pi\mathrm{i}g\cdot\ell} = \mathrm{e}^{2\pi\mathrm{i}\frac{X}{\tau-1}}\mathrm{e}^{2\pi\mathrm{i}(1-\ell)}$

Table 10.1. *The $g \in SL(2,\mathbb{Z})$ action on the parameters of the two chiral sectors of the $S^3 \times S^1$ theory.*

Proceeding similarly to Chapter 7, we can also construct a matrix model using the modular double elliptic $W^g_{q,t;q'}$ algebra. Using the operator $\mathcal{S}^a(X)$ introduced in (10.4) we consider the state

$$\mathsf{Z}|\underline{\alpha}\rangle = \int \prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a} \mathrm{d}X_{a,j}\mathcal{S}^a(X_{a,j})|\underline{\alpha}\rangle \,. \tag{10.9}$$

Comparing to (7.28) we then have the matrix model

$$\begin{aligned} Z_g(\{t^a_{0,\alpha}, t^a_\alpha\}) = \mathrm{e}^{\sum_{\alpha=1,2}\mathcal{N}_0(\{t^a_{0,\alpha}\})} \sum_{\underline{\ell}} \int \prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a} \mathrm{d}X_{a,j}\ \Delta_g(\underline{X},\underline{\ell})\times \\ \times \exp\left\{ \sum_{\alpha=1,2}\sum_{a=1}^{|\Gamma_0|}\sum_{j=1}^{N_a} \left(V^{(a)}_\alpha(X_{a,j},\ell_{a,j}) - \sum_{n>0} \frac{t^a_{n,\alpha}(w^n_{a,j})_\alpha}{n(1-q'^n_\alpha)} \right) \right\} \end{aligned} \tag{10.10}$$

for a potential

$$V^{(a)}_\alpha(X_{a,j},\ell_{a,j}) = \hat{\alpha}_a \ln(w_{a,j})_\alpha \,, \tag{10.11}$$

where we defined

$$\hat{\alpha}_a = \alpha_a + \beta\left(\sum_{b=1}^{|\Gamma_0|} C^{[0]}_{ab} N_b - (N_a - 1)\frac{C^{[0]}_{aa}}{2} - \sum_{a>b,b=1}^{|\Gamma_0|} C^{[0]}_{ab} N_b \right) \tag{10.12}$$

for later convenience. Finally, making use of the g-action on the deformation parameters as given in Table (10.1), we can identify

$$\Delta_g(\underline{X}, \underline{\ell}) = \Delta_{q,t,q'}(\underline{w})^{(-g)} \Delta_{q,t,q'}(\underline{w}) \tag{10.13}$$

with $\Delta_{q,t,q'}(\underline{w})$ as in (7.27). As a final remark, mirroring the discussion for the 3d modular double in Section 9.2.3, the matrix model above satisfies two sets of elliptic Virasoro constraints which are related by $g \in SL(2,\mathbb{Z})$.

10.3 The gauge theory and $W_{q,t;q'}$ algebra correspondence

In paper I we explored the following two correspondences between the elliptic $W_{q,t;q'}$ algebra and certain gauge theories. Firstly between the chiral elliptic $W_{q,t;q'}(\Gamma)$ matrix model and the generating function of a gauge theory on the half-space $D^2 \times T^2$. Secondly, we can match the modular double elliptic $W^g_{q,t;q'}(\Gamma)$ matrix model to the generating function of the gauge theory on $\mathcal{M}_3 \times S^1$. Furthermore, the $\mathcal{M}_3 \times S^1$ is created from gluing two $D^2 \times T^2$ using an element $g \in SL(2,\mathbb{Z})$, i.e. $\mathcal{M}_3 \times S^1 \simeq [D^2 \times T^2] \cup_g [D^2 \times T^2]$. We now explore the two cases in more detail.

10.3.1 Half-space/chiral matrix model

Let us start with the correspondence between the half-space gauge theory and the chiral $W_{q,t;q'}$ matrix model. We take for simplicity unitary gauge groups so we have an overall gauge group given by $G = \bigtimes_a U(N_a)$. We then consider an $\mathcal{N} = 2$ gauge theory content. As mentioned in Section 8.2.3, holomorphic blocks in four dimensions can be seen as gauge theory partition functions on $D^2 \times T^2$. In other words they are given by

$$\mathcal{B}_\gamma(\underline{\xi}; \tau, \sigma) = \oint_\gamma \prod_{a=1}^{|\Gamma_0|} \prod_{j=1}^{N_a} \frac{\mathrm{d}w_{j,a}}{2\pi \mathrm{i} w_{j,a}} \, \Upsilon(\underline{w}; \tau, \sigma) \ , \tag{10.14}$$

where the integration contours γ are along middle dimensional cycles $\gamma \subset (\mathbb{C}^\times)^{|G|}$. The essence of the correspondence is then given by the identification between the total integrand of the half-space gauge theory $\Upsilon(\underline{w}; \tau, \sigma)$ given in the holomorphic block above and the measure of the elliptic matrix model $\Delta_{q,t,q'}(\underline{w})$ in (7.27). In other words we identify

$$\Upsilon(\underline{w}; \tau, \sigma) = \Delta_{q,t,q'}(\underline{w}) \ . \tag{10.15}$$

In order for this correspondence to hold, we are required to identify the dictionary between the parameters as given in Table 10.2.

Gauge theory		$W_{q,t;q'}(\Gamma)$ matrix model	
Fibration moduli	q_τ	q	Algebra deformation
Fibration moduli	q_σ	q'	Algebra deformation
Adjoint mass	t_a	$t = q^\beta$	Algebra deformation
Half-space integration	$w_{a,j}$	$w_{a,j}$	M. M. integration
Bifundamental mass	ξ_e	μ_e	Cartan matrix deformation
Bifundamental mass	$\xi_e/\bar{\xi}_e$	$p = qt^{-1}$	Algebra deformation

Table 10.2. *Matching the parameters of the gauge theory and the chiral $W_{q,t;q'}(\Gamma)$ matrix model.*

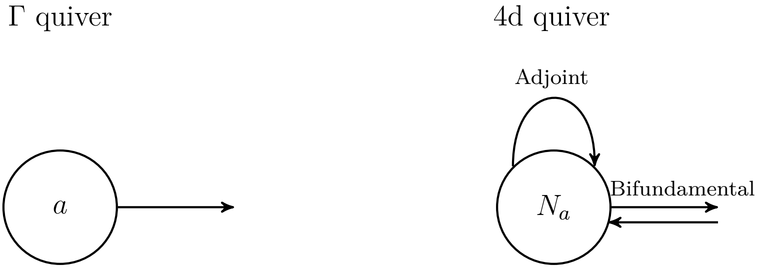

Figure 10.1. Illustration of a part of the quiver Γ (left) and its corresponding 4d $\mathcal{N} = 2$ gauge theory quiver (right) (following Paper I).

In addition to the above dictionary, one can also identify the number of screening currents N_a with the rank of the gauge group associated to the respective node $a \in \Gamma_0$. This can be illustrated as in Figure 10.1. Next, to each node $a \in \Gamma_0$ we associate one $U(N_a)$ vector multiplet and one adjoint chiral multiplet, i.e. an $\mathcal{N} = 2$ vector multiplet,

$$\Upsilon^{(a)}_{\text{vec}}(\underline{w_a}) = \prod_{1 \le j \neq k \le N_a} \frac{\Gamma(t_a w_{a,j}/w_{a,k}; q_\tau, q_\sigma)}{\Gamma(w_{a,j}/w_{a,k}; q_\tau, q_\sigma)} \,. \tag{10.16}$$

Then, for arrows $\Gamma_1 \ni e : a \to b$ or $e : b \to a$ between different nodes $(a < b)$, we associate a pair of bifundamental chiral multiplets, i.e. one $\mathcal{N} = 2$ bifundamental hypermultiplet,

$$\begin{aligned} \Upsilon^{(e:a\to b)}_{\text{bif}}(\underline{w_a}, \underline{w_b}) &= \prod_{j=1}^{N_a}\prod_{k=1}^{N_b} \frac{\Gamma(\xi_e w_{b,k}/w_{a,j}; q_\tau, q_\sigma)}{\Gamma(q_\tau \bar{\xi}_e w_{b,k}/w_{a,j}; q_\tau, q_\sigma)} \,, \\ \Upsilon^{(e:b\to a)}_{\text{bif}}(\underline{w_a}, \underline{w_b}) &= \prod_{j=1}^{N_a}\prod_{k=1}^{N_b} \frac{\Gamma(\bar{\xi}_e^{-1} w_{b,k}/w_{a,j}; q_\tau, q_\sigma)}{\Gamma(q_\tau \xi_e^{-1} w_{b,k}/w_{a,j}; q_\tau, q_\sigma)} \,. \end{aligned} \tag{10.17}$$

Next, for arrows starting and ending on the same node – i.e. loops – given by $\Gamma_1 \ni e : a \to a$, we associate a pair of adjoint chiral multiplets, i.e. one $\mathcal{N} = 2$ adjoint hypermultiplet,

$$\Upsilon^{(a)}_{\mathrm{ad}}(\underline{w_a}) = \prod_{1 \le j \ne k \le N_a} \frac{\Gamma(\xi_a w_{a,j}/w_{a,k}; q_\tau, q_\sigma)}{\Gamma(q_\tau \bar{\xi}_a w_{a,j}/w_{a,k}; q_\tau, q_\sigma)} \ . \tag{10.18}$$

Then, for each $U(1)$ factor one can have an FI contribution given by

$$\Upsilon^a_{\mathrm{FI}}(\underline{w_a}) = \prod_{j=1}^{N_a} w_{a,j}^{\kappa_a} \tag{10.19}$$

and fundamental/anti-fundamental chiral pairs of the form

$$\Upsilon^a_{\mathrm{f}}(\underline{w_a}) = \prod_{j=1}^{N_a} \prod_{f \ge 1} \frac{\Gamma(w_{a,j}/\bar{\xi}_{a,f}; q_\tau, q_\sigma)}{\Gamma(q_\tau w_{a,j}/\xi_{a,f}; q_\tau, q_\sigma)} \ . \tag{10.20}$$

We then have the correspondence between the gauge theory FI parameter κ_a and the parameter $\hat{\alpha}_a$ in (10.12) on the matrix model side.

As a final remark for the half-space discussion, we can include time variables to obtain the defect generating function on the half-space,

$$\begin{aligned} &\mathcal{B}_\gamma(\{t^a_0, t^a\}; \tau, \sigma) = \\ &= \mathrm{e}^{\mathcal{N}_0(\{t^a_0\})} \oint_\gamma \prod_{a=1}^{|\Gamma_0|} \prod_{j=1}^{N_a} \frac{\mathrm{d}w_{j,a}}{2\pi \mathrm{i} w_{j,a}} \Upsilon(\underline{w}; \tau, \sigma) \prod_{a=1}^{|\Gamma_0|} \exp\left(-\sum_{j=1}^{N_a} \sum_{n \ne 0} \frac{t^a_n w^n_{j,a}}{n(1-q^n_\sigma)} \right) \ . \end{aligned} \tag{10.21}$$

We can then identify the defect generating function from (10.21) with the matrix model in (7.28)

$$\mathcal{B}_\gamma(\{t^a_0, t^a\}; \tau, \sigma) \simeq \mathsf{Z}|\underline{\alpha}\rangle \simeq Z(\{t^a_0, t^a\}) \ . \tag{10.22}$$

10.3.2 Compact space/modular double matrix model

Moving on to the correspondence between the compact space $\mathcal{M}_3 \times S^1$ and the elliptic $W^g_{q,t;q'}(\Gamma)$ matrix model, we then have the identification between the gauge theory integrand $\Delta_{\ \mathcal{M}_3 \times S^1}(\underline{z}, \underline{\ell})$ and the matrix model measure $\Delta_g(\underline{X}, \underline{\ell})$ given by

$$\begin{aligned} \Delta_{\ \mathcal{M}_3 \times S^1}(\underline{z}, \underline{\ell}) &\propto \Upsilon(\underline{w}; \tau, \sigma)^{(-g)} \Upsilon(\underline{w}; \tau, \sigma) = \\ &= \Delta_{q,t,q'}(\underline{w})^{(-g)} \Delta_{q,t,q'}(\underline{w}) = \Delta_g(\underline{X}, \underline{\ell}) \ . \end{aligned} \tag{10.23}$$

Let us now review how the correspondence is achieved using the steps in the above relation for the case of $S^3 \times S^1$, and in particular understand how the factorisation $(\dots)^{(-g)}(\dots)$ with $g \in SL(2,\mathbb{Z})$ works. For other 4d backgrounds of the form $\mathcal{M}_3 \times S^1$ we refer to [28]. Let us consider the $S^3 \times S^1$ partition function for the gauge theory associated to the quiver Γ with gauge group $G = \bigtimes_a U(N_a)$

$$Z_{S^3\times S^1} = \oint_{T^{|G|}} \prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a} \frac{\mathrm{d}z_{a,j}}{2\pi\mathrm{i} z_{a,j}}\, \Delta_{S^3\times S^1}(\underline{z}) \ . \tag{10.24}$$

Here the integrand $\Delta_{S^3\times S^1}(\underline{z})$ is made up of the contributions

$$\Delta_{S^3\times S^1}(\underline{z}) = \Delta_{2\,\mathrm{vec}}(\underline{z})\Delta_{\mathrm{ad}}(\underline{z})\Delta_{\mathrm{bif}}(\underline{z})\Delta_{\mathrm{FI}}(\underline{z}) \tag{10.25}$$

which are respectively given by

$$\begin{aligned}
\Delta_{2\,\mathrm{vec}}(\underline{z}) &= \prod_{a=1}^{|\Gamma_0|} \prod_{1\le j\neq k\le N_a} \frac{\Gamma(\hat{\mathfrak{t}} z_{a,j}/z_{a,k};\mathfrak{p},\mathfrak{q})}{\Gamma(z_{a,j}/z_{a,k};\mathfrak{p},\mathfrak{q})} \ , \\
\Delta_{\mathrm{ad}}(\underline{z}) &= \prod_{a=1}^{|\Gamma_0|} \prod_{e:a\to a} \prod_{1\le j\neq k\le N_a} \frac{\Gamma(\mu_e z_{a,j}/z_{a,k};\mathfrak{p},\mathfrak{q})}{\Gamma(\mathfrak{p}\mathfrak{q}\bar{\mu}_e z_{a,j}/z_{a,k};\mathfrak{p},\mathfrak{q})} \ , \\
\Delta_{\mathrm{bif}}(\underline{z}) &= \prod_{1\le a<b\le|\Gamma_0|} \prod_{e:a\to b} \prod_{j=1}^{N_a}\prod_{k=1}^{N_b} \frac{\Gamma(\mu_e z_{b,k}/z_{a,j};\mathfrak{p},\mathfrak{q})}{\Gamma(\mathfrak{p}\mathfrak{q}\bar{\mu}_e z_{b,k}/z_{a,j};\mathfrak{p},\mathfrak{q})} \times \\
&\qquad \times \prod_{e:b\to a} \prod_{j=1}^{N_a}\prod_{k=1}^{N_b} \frac{\Gamma(\bar{\mu}_e^{-1} z_{b,k}/z_{a,j};\mathfrak{p},\mathfrak{q})}{\Gamma(\mathfrak{p}\mathfrak{q}\mu_e^{-1} z_{b,k}/z_{a,j};\mathfrak{p},\mathfrak{q})} \ , \\
\Delta_{\mathrm{FI}}(\underline{z}) &= \prod_{a=1}^{|\Gamma_0|}\prod_{j=1}^{N_a} z_{a,j}^{\kappa_a} \ .
\end{aligned} \tag{10.26}$$

Now, we aim for this integrand to be of the form $\Upsilon(\underline{w};\tau,\sigma)^{(-g)}\Upsilon(\underline{w};\tau,\sigma)$ in order to recover the desired modular double structure. Since we specialised to the case $S^3 \times S^1$ we have the parametrisation

$$\mathfrak{q} = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_1}{\omega_3}} \ , \quad \mathfrak{p} = \mathrm{e}^{2\pi\mathrm{i}\frac{\omega_2}{\omega_3}} \ , \tag{10.27}$$

and

$$z_{a,j} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_3}X_{a,j}} \ , \quad \hat{\mathfrak{t}} = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_3}\hat{T}} \ , \quad \mu_e = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_3}M_e} \ , \quad \bar{\mu}_e = \mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_3}\bar{M}_e} \ . \tag{10.28}$$

As indicated in Section 10.1, this precise form of the parametrisation enables the interpretation of the g-gluing as the exchange of parameters $\omega_1 \leftrightarrow \omega_2$. Using this, we employ the modular property of the elliptic Gamma function given in (5.7), which can be rewritten as

$$\begin{aligned}
\Gamma\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_3}X};\mathrm{e}^{2\pi\mathrm{i}\frac{\omega_1}{\omega_3}},\mathrm{e}^{2\pi\mathrm{i}\frac{\omega_2}{\omega_3}}\right) =& \mathrm{e}^{-\frac{\mathrm{i}\pi}{3}B_{33}(X|\underline{\omega})}\Gamma\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_1}X};\mathrm{e}^{2\pi\mathrm{i}\frac{\omega_2}{\omega_1}},\mathrm{e}^{-2\pi\mathrm{i}\frac{\omega_3}{\omega_1}}\right) \times \\
&\times \Gamma\left(\mathrm{e}^{\frac{2\pi\mathrm{i}}{\omega_2}X};\mathrm{e}^{2\pi\mathrm{i}\frac{\omega_1}{\omega_2}},\mathrm{e}^{-2\pi\mathrm{i}\frac{\omega_3}{\omega_2}}\right) \ .
\end{aligned} \tag{10.29}$$

This means that each Gamma function in the integrand $\Delta_{S^3\times S^1}(\underline{z})$ in (10.25) can separately be written as a product of two Gamma functions. Rewriting all of the contributions to $\Delta_{S^3\times S^1}(\underline{z})$ using (10.29), the entire integrand $\Delta_{S^3\times S^1}(\underline{z})$ can be cast as $(\dots)^{(-g)}(\dots)$. As a final step, we need to make either of the two assumptions

$$\sum_{j=1}^{N_a} X_{a,j} = 0 \qquad \text{or} \qquad C_{ab}^{[0]} = 0 \tag{10.30}$$

for $a \neq b$. The reason for these assumptions is to remove the cubic polynomial $\mathcal{P}_3$ given in (8.45). As it consists of mixed-gauge anomalies we require them to vanish for the theory to be sensible. Upon these assumptions, the correspondence takes the simple form

$$\Delta_{S^3\times S^1}(\underline{z}) = \Delta_g(\underline{X}) \ , \tag{10.31}$$

i.e. we precisely recover the modular double measure given in (10.13). We can then create a table of identifications similar to Table 10.2, where we match the parameters of the gauge theory and the modular double $W^g_{q,t;q'}(\Gamma)$ matrix model. This is shown in Table 10.3, where we recall the parametrisations in (10.27) and (10.28).

Gauge theory		$W^g_{q,t;q'}(\Gamma)$ matrix model	
Moduli	$\mathfrak{q}$	$\mathrm{e}^{2\pi\mathrm{i}\frac{\omega_1}{\omega_3}}$	S^3 squashing ω_1
Moduli	$\mathfrak{p}$	$\mathrm{e}^{2\pi\mathrm{i}\frac{\omega_2}{\omega_3}}$	S^3 squashing ω_2
Adjoint chiral mass	$\hat{T}$	$\beta\omega$	S^3 squashing ω
Compact space integration	$X_{a,j}$	$X_{a,j}$	M. M. integration
Adjoint chiral mass	M_e	M_e	Cartan matrix def.
Adjoint (anti-)chiral masses	$M_e - \bar{M}_e$	$\omega(1-\beta)$	S^3 squashing ω
FI parameter	κ_a/ω_3	$\hat{\alpha}_a\omega/\omega_1\omega_2$	M. M. potential

Table 10.3. *Matching between the parameters of the gauge theory and the modular double $W^g_{q,t;q'}(\Gamma)$ matrix model.*

10.4 Relations between classical and quantum models

So far we have considered the Hermitean matrix model, the β-deformed Hermitean matrix model, the 3d gauge theories on $D^2\times_q S^1$ and S^3_b given by q,t-deformed matrix models and now lastly the 4d gauge theories on

$D^2 \times T^2$ and $\mathcal{M}_3 \times S^1$ given by q, t, q'-deformed matrix models. Let us therefore conclude these chapters on applications of the quantum matrix models with a summary over the relations between these models and the classical ones. This is illustrated in Figure 10.2.

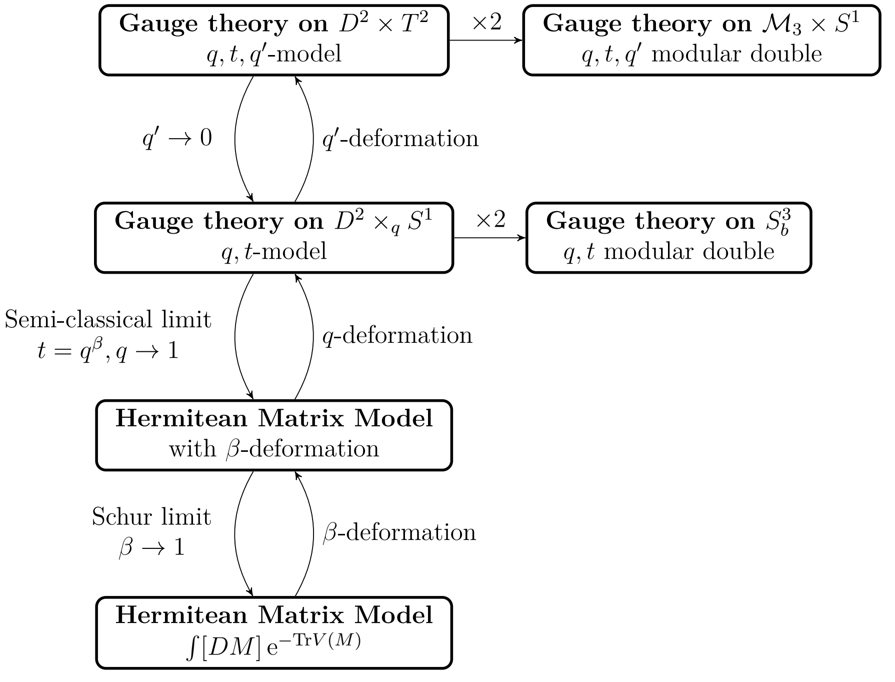

Figure 10.2. Relations between the classical and quantum matrix models.

11. Hirota equations versus Virasoro constraints

As a final application we explore the connection between the classical Virasoro constraints and the so-called Hirota equations. This connection arises when the generating function satisfying the Virasoro constraints also is a τ-function of an integrable hierarchy. Being a τ-function, the generating function satisfies a set of bilinear equations called the Hirota equations. However, for any solution of the Virasoro constraints, it is by no means guaranteed that the generating function satisfies the Hirota equations. One example of a class of models which do satisfy both the Virasoro constraints and the Hirota equations and therefore exist at the intersection of the two moduli spaces are the classical matrix models solutions. However, the exact relation between the two set of equations is not fully known, which was the motivation for paper V.

11.1 Reviewing the Hirota equations

Integrability and its relation to matrix models has been reviewed in for instance [35, 41]. Informally, integrability can be thought of as the property of dynamical systems that all the dynamical characteristics can be determined completely. Examples of systems which falls into this class are two dimensional CFT's and eigenvalue matrix models as introduced in Chapter 2. Having been originally expressed through non-linear dynamical equations, the bilinear Hirota equations was later introduced to describe some integrable systems, where the generating function satisfying these equations was then the τ-function as a function of time variables $\tau(t)$. One can from this recursively obtain a family of equations, where such a set of equations is usually referred to as an *integrable hierarchy*.

Let us now summarise how such Hirota equations are derived, while for a review we refer to [36, 95, 96]. We begin with introducing free fermionic operators ψ_s and ψ_s^* for $s \in \mathbb{Z}$ satisfying the anti-commutation relations

$$\{\psi_s, \psi_r\} = \{\psi_s^*, \psi_r^*\} = 0 \ , \qquad \{\psi_s, \psi_r^*\} = \delta_{s,r} \ , \tag{11.1}$$

and generating an infinite dimensional Clifford algebra. Collecting the operators into fermionic generating functions, $\psi(z)$ and $\psi^*(z)$, we have

$$\psi(z) = \sum_{s\in\mathbb{Z}} \psi_s z^{s-\frac{1}{2}}, \qquad \psi^*(z) = \sum_{s\in\mathbb{Z}} \psi_s^* z^{-s+\frac{1}{2}} \ . \tag{11.2}$$

Introducing a vacuum state $|0\rangle$ this satisfies

$$\begin{aligned} \psi_s|0\rangle &= 0\ , \qquad s<0\ , \\ \psi_s^*|0\rangle &= 0\ , \qquad s\geq 0\ , \end{aligned} \tag{11.3}$$

such that $|0\rangle$ is a "Dirac sea" with all negative mode states being empty and all positive mode states being occupied. Therefore, $\psi_{s<0}$ and $\psi^*_{s\geq 0}$ are annihilation operators and $\psi_{s\geq 0}$ and $\psi^*_{s<0}$ are creation operators with respect to the vacuum $|0\rangle$. One can then use the fermionic creation operators to create a charge k vacuum given by

$$|k\rangle = \begin{cases} \psi_{k-1}\dots\psi_1\psi_0\,|0\rangle\ , & k>0 \\ \psi_k^*\dots\psi_{-2}^*\psi_{-1}^*\,|0\rangle\ , & k<0\ . \end{cases} \tag{11.4}$$

From the fermionic operators we then construct Heisenberg oscillators α_s

$$\alpha_s = \sum_{r\in\mathbb{Z}} :\psi_r\psi^*_{r+s}:\ , \tag{11.5}$$

with $s\in\mathbb{Z}$, which satisfies the commutation relations

$$[\alpha_s,\alpha_r] = s\delta_{s+r,0}\ . \tag{11.6}$$

We also specify the fermionic charge operator α_0 given by

$$\alpha_0 = \sum_{r\in\mathbb{Z}} :\psi_r\psi^*_r:\ . \tag{11.7}$$

The charge k vacuum $|k\rangle$ then satisfies

$$\begin{aligned} \alpha_s|k\rangle &= 0\,, \qquad \text{if } s>k\ , \\ \alpha_0|k\rangle &= k|k\rangle\ . \end{aligned} \tag{11.8}$$

Collecting the Heisenberg oscillators into the current $\alpha(z)$, we have

$$\alpha(z) = \sum_{s\in\mathbb{Z}} \alpha_s z^{-s} = :\psi(z)\psi^*(z):\ . \tag{11.9}$$

Next, we introduce the bosonisation map Φ_N with $N>0$ defined by

$$\Phi_N = \langle N|\mathrm{e}^{\sum_{s=1}^{\infty} t_s\alpha_s}\ . \tag{11.10}$$

Upon bosonisation, we have the mapping of the Heisenberg oscillators to operators in the time variables $\{t_s\}$ given by

$$\begin{gathered} \Phi_N\alpha_s\Phi_N^{-1} = \frac{\partial}{\partial t_s}, \qquad \Phi_N\alpha_{-s}\Phi_N^{-1} = s\,t_s, \qquad \Phi_N\alpha_0\Phi_N^{-1} = N \\ \Phi_N\mathrm{i}q\Phi_N^{-1} = t_0 \qquad \Phi_N(|k\rangle) = \mathrm{e}^{kt_0}\ , \end{gathered} \tag{11.11}$$

where iq will appear shortly. Furthermore, under this map a vector $|\tau\rangle = g|N\rangle$ in the fermionic Fock space with $g \in GL(\infty, \mathbb{C})$ transforms as

$$|\tau\rangle \xrightarrow{\Phi_N} \tau(t) \ . \tag{11.12}$$

In other words, the τ-function $\tau(t)$ results from

$$\tau(t) = \Phi_N|\tau\rangle = \langle N|\mathrm{e}^{\sum_{s=1}^{\infty} t_s \alpha_s}|\tau\rangle \ . \tag{11.13}$$

As a next step, we introduce firstly operators $E_\pm(z)$

$$E_\pm(z) = \exp\left(\pm \sum_{s=1}^{\infty} \frac{z^{\mp s}\alpha_{\pm s}}{s}\right) \tag{11.14}$$

and secondly the unitary operator $Q = \mathrm{e}^{\mathrm{i}q}$ which satisfies

$$\begin{gathered} Q\psi_s = \psi_{s+1}Q, \qquad Q\psi_s^* = \psi_{s+1}^* Q, \qquad Q|k\rangle = |k+1\rangle, \\ [\alpha_0, Q] = Q, \qquad Q^\dagger = Q^{-1} \ . \end{gathered} \tag{11.15}$$

Using $E_\pm(z)$ and Q, we can express the fermionic generating currents as

$$\begin{aligned} \psi(z) &= z^{\alpha_0 - \frac{1}{2}} Q E_-(z)^{-1} E_+(z)^{-1} \ , \\ \psi^*(z) &= Q^{-1} z^{-\alpha_0 + \frac{1}{2}} E_-(z) E_+(z) \ . \end{aligned} \tag{11.16}$$

As the final part needed to obtain the Hirota equations, we introduce the Sato Grassmannian. It is defined as the $GL(\infty, \mathbb{C})$-orbit of the k-vacuum

$$\mathrm{Gr}_k(\mathbb{C}^\infty) \cong GL(\infty, \mathbb{C}) \cdot |k\rangle \tag{11.17}$$

as a submanifold of $\Lambda_k^\infty \mathbb{C}^\infty$. The Plücker relations on Sato Grassmannian then arise from the condition on an element $|\tau\rangle \in \Lambda_k^\infty \mathbb{C}^\infty$ to be in the orbit of the vacuum with charge k. This is true if and only if

$$\oint \frac{\mathrm{d}z}{z} \psi(z)|\tau\rangle \otimes \psi^*(z)|\tau\rangle = 0 \ , \tag{11.18}$$

which are the *Plücker relations*. Here, the integral over z is such that we extract the z^{-1} contribution in the Laurent series expansion of the integrand. Recalling the operator Q, this is invertible such that the above relation can be rewritten as

$$\oint \frac{\mathrm{d}z}{z} Q^{-1}\psi(z)|\tau\rangle \otimes Q\psi^*(z)|\tau\rangle = 0 \ . \tag{11.19}$$

Then, we use (11.16) to observe that

$$\begin{aligned} Q^{-1}\psi(z) &= z^{\alpha_0 - \frac{1}{2}} E_-(z)^{-1} E_+(z)^{-1} = \\ &= z^{\alpha_0 - \frac{1}{2}} \exp\left(\sum_{s=1}^{\infty} \frac{z^s \alpha_{-s}}{s}\right) \exp\left(-\sum_{s=1}^{\infty} \frac{z^{-s}\alpha_s}{s}\right) \end{aligned} \tag{11.20}$$

and

$$
\begin{aligned}
Q\psi^*(z) &= z^{-\alpha_0+\frac{1}{2}} E_-(z)E_+(z) = \\
&= z^{-\alpha_0+\frac{1}{2}} \exp\left(-\sum_{s=1}^{\infty} \frac{z^s \alpha_{-s}}{s}\right) \exp\left(\sum_{s=1}^{\infty} \frac{z^{-s}\alpha_s}{s}\right) .
\end{aligned} \tag{11.21}
$$

Using this, (11.19) can be rewritten as the bilinear *Hirota equations*

$$
\begin{aligned}
\oint \mathrm{d}z \exp\left(\sum_{s=1}^{\infty} \frac{z^s \alpha_{-s}}{s}\right) \exp\left(-\sum_{s=1}^{\infty} \frac{z^{-s}\alpha_s}{s}\right) |\tau\rangle \otimes \\
\otimes \exp\left(-\sum_{s=1}^{\infty} \frac{z^s \alpha_{-s}}{s}\right) \exp\left(\sum_{s=1}^{\infty} \frac{z^{-s}\alpha_s}{s}\right) |\tau\rangle = 0 \ .
\end{aligned} \tag{11.22}
$$

Using the mapping in (11.11), the Hirota equations take the form

$$
\oint \mathrm{d}z \exp\left(\sum_{s=1}^{\infty} z^s (t_s - \tilde{t}_s)\right) \exp\left(-\sum_{s=1}^{\infty} \frac{z^{-s}}{s}\left(\frac{\partial}{\partial t_s} - \frac{\partial}{\partial \tilde{t}_s}\right)\right) \tau(t)\tau(\tilde{t}) = 0 \ . \tag{11.23}
$$

Here it should be noted that there are two sets of time variables $\{t_s\}$ and $\{\tilde{t}_s\}$ due to the product of two vectors in (11.22). Upon a redefinition of the variables using the new time variables $\{u_s\}$ and $\{v_s\}$,

$$
u_s = \frac{t_s + \tilde{t}_s}{2} \,, \qquad\qquad v_s = \frac{t_s - \tilde{t}_s}{2} \tag{11.24}
$$

we can rewrite this as the integral identity

$$
\oint \mathrm{d}z \exp\left(2\sum_{s=1}^{\infty} z^s v_s\right) \exp\left(-\sum_{s=1}^{\infty} \frac{z^{-s}}{s} \frac{\partial}{\partial v_s}\right) \tau(u+v)\tau(u-v) = 0 \ , \tag{11.25}
$$

with $u \pm v$ denoting $u_s \pm v_s$ for $s \geq 1$. Again we view this integral as extracting the z^{-1} component of the Laurent series of the integrand and when this residue is vanishing then τ is a τ-function. The above is a compact way to express the relation at the level of generating functions $\tau(t)$. As we are now interested in studying the intersection of the solutions to the Virasoro constraints and the Hirota equations, the τ-functions in (11.25) can therefore be identified with the classical generating function $Z(t; a)$. Next, we wish to rewrite the above Hirota equations as constraints on the correlators c_ρ. To do so, we first use the Cauchy identity for symmetric Schur polynomials $\mathsf{Schur}_{\{m\}}(p_s)$ in (2.37) in order to rewrite the above identity as

$$\begin{aligned}
&\oint \frac{\mathrm{d}z}{2\pi \mathrm{i} z} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} z^{1+k-l} \mathsf{Schur}_{\{k\}}\left(p_s = 2 s v_s\right) \times \\
&\qquad \times \mathsf{Schur}_{\{l\}}\left(p_s = -\frac{\partial}{\partial v_s}\right) \tau(u+v)\tau(u-v) = \\
&= \sum_{k=0}^{\infty} \mathsf{Schur}_{\{k\}}\left(p_s = 2 s v_s\right) \times \\
&\qquad \times \mathsf{Schur}_{\{k+1\}}\left(p_s = -\frac{\partial}{\partial v_s}\right) \tau(u+v)\tau(u-v) = 0 \ .
\end{aligned} \tag{11.26}$$

The shifted τ-function can then be expanded as a power series in the v_s around the times u_s as

$$\tau(u \pm v) = \sum_{d=0}^{\infty} [\tau(u \pm v)]_{(d)} = \sum_{d=0}^{\infty} \left[\sum_{\lambda \vdash d} \frac{1}{|\mathrm{Aut}(\lambda)|} \tau_\lambda(u) \prod_{l \in \lambda} (\pm v_l) \right] \ , \tag{11.27}$$

where $[\tau(u \pm v)]_{(n)}$ is used to denote degree n in v_s and $\lambda \vdash d$ denotes that λ is an integer partition of d. Here we also use the notation $\tau_\lambda(u) = \prod_{a \in \lambda} \frac{\partial}{\partial u_a} \tau(u)$. Substituting this expansion in (11.26) and projecting onto the degree d part in v_s we obtain

$$\begin{aligned}
&\sum_{k=0}^{d} \mathsf{Schur}_{\{k\}}\left(p_s = 2 s v_s\right) \mathsf{Schur}_{\{k+1\}}\left(p_s = -\frac{\partial}{\partial v_s}\right) \times \\
&\qquad \times \sum_{l=0}^{d+1} [\tau(u+v)]_{(l)} [\tau(u-v)]_{(d+1-l)} = 0 \ .
\end{aligned} \tag{11.28}$$

Notice that for degree $0 \leq d < 3$ this equation is trivially satisfied, while for $d = 3$ we have the first non-trivial condition on the τ-function and its derivatives. Another remark we can make here is that for each degree in (11.28) there will be an independent equation. Consequently we have an infinite set of equations which the τ-function has to satisfy.

Finally, to make the connection to correlators as mentioned earlier, we set all times $u = 0$ in (11.28) and use the expansion of the generating function in terms of correlators in (2.48) which implies

$$\tau_\lambda(u)\big|_{u=0} = c_\lambda \ , \tag{11.29}$$

such that the Hirota equations then become bilinear constraints on the correlators. For instance in degree $d = 3$ we find

$$\begin{aligned}
&3 c_{\{1,1\}}^2 + 3 c_\emptyset c_{\{2,2\}} - 4 c_\emptyset c_{\{3,1\}} - 4 c_\emptyset c_{\{1,1,1\}} + \\
&\qquad + c_\emptyset c_{\{1,1,1,1\}} - 3 c_{\{2\}}^2 + 4 c_{\{1\}} c_{\{3\}} = 0 \ ,
\end{aligned} \tag{11.30}$$

and for degree $d = 4$

$$\begin{aligned}3c_{\{1,1\}}c_{\{2,1\}} + 2c_{\emptyset}c_{\{3,2\}} - 3c_{\{1\}}c_{\{2,1,1\}} + c_{\emptyset}c_{\{2,1,1,1\}} + 3c_{\{1\}}c_{\{4\}} + \\ - 3c_{\emptyset}c_{\{4,1\}} - c_{\{2\}}c_{\{1,1,1\}} - 2c_{\{2\}}c_{\{3\}} = 0 \ .\end{aligned} \tag{11.31}$$

In degree $d \geq 5$ there are several bilinear equations and more specifically the number of equations is equal to the number of partitions of size d. However, all such equations might not be independent. For instance at degree $d = 5$ there are 4 independent equations.

11.2 Relation between Virasoro and Hirota

As mentioned at the beginning of this chapter, the relation between the solutions of the Virasoro constraints and that of the Hirota equations is still not fully known and is the motivation behind paper V. More concretely, we wish to determine if the solutions to the classical Virasoro constraints for $\beta = 1$ given in (3.14) (i.e. the correlators) also satisfy the Hirota equations. As can be seen from the form of the Hirota equations in (11.28), this will be verified degree by degree.

Let us begin with reviewing why matrix models with $\beta = 1$ can provide a solution to the Hirota equations. The generating function for matrix models can be written as a determinant of an $N \times N$ symmetric matrix. This can be derived from what has been called *Andréief's integration formula* in [97], where for a review we refer to [98]. The restriction to $\beta = 1$ is then due to the fact that this derivation fails precisely when $\beta \neq 1$. We will therefore in this section assume that $\beta = 1$, in which case Andréief's integration formula takes the form

$$\tau_N(t; a) = \det_{1 \leq i,j \leq N} \left[\left(\frac{\partial}{\partial t_1} \right)^{i+j-2} \tau_1(t; a) \right] \ , \tag{11.32}$$

where we introduce a label N and a dependence on the couplings a of the τ-function. If the τ-function is of the form (11.32), together with satisfying

$$\frac{\partial}{\partial t_k} \tau_1(t; a) = \left(\frac{\partial}{\partial t_1} \right)^k \tau_1(t; a) \ , \tag{11.33}$$

then the τ-function is a solution to the Hirota equations [36, 99]. Thus we express the rank N matrix integral as a determinant of the rank 1 integral

$$\tau_1(t; a) = \int_\Gamma \mathrm{d}x \, \mathrm{e}^{-V(x) + \sum_{s=1}^{\infty} t_s x^s} \ , \tag{11.34}$$

for a contour Γ. We then recall the form of the potential in (2.16), which implies the additional constraint in (3.73). We can therefore trade derivatives with respect to t_1 with derivatives in the coupling constants a_1 in (11.32). Upon subsequently setting all time variables to zero, we find

$$\tau_N(0;a) = c_\emptyset(a) = \det_{1\le i,j\le N}\left[\left(-\frac{\partial}{\partial a_1}\right)^{i+j-2}\tau_1(0;a)\right] \tag{11.35}$$

where

$$\tau_1(0;a) = \int_\Gamma \mathrm{d}x\, \mathrm{e}^{-V(x)}\ . \tag{11.36}$$

Let us now summarise the cases of $\mathsf{p} = 1,2,3$ recalling the findings from Section 3.2.3, where the results are collected in Table 11.1. For $\mathsf{p} = 1$ we choose the contour Γ along the positive real line, whereas for $\mathsf{p} = 2$ it is along the entire real line. For both $\mathsf{p} = 1$ and $\mathsf{p} = 2$ it can be shown that upon insertion of the correlators and restricting to $\beta = 1$ that the Hirota equations are satisfied. Furthermore, it can also be checked that the empty correlators for $\mathsf{p} = 1$ and $\mathsf{p} = 2$ given in (3.80) and (3.83) respectively, can be recovered from the determinant representation in (11.35) up to constants $k_1^{N,\beta,\nu}$ and $k_2^{N,\beta}$ when $\beta = 1$.

However, in the case $\mathsf{p} = 3$ the Hirota equations are not immediately satisfied. To see this, we begin with recalling that correlators can only be determined up to correlators $c_{\{1,\ldots,1\}}(a_1,a_2,a_3)$ as shown in paper IV and reproduced in equation (3.72). Therefore, one needs to replace such correlators using the additional constraint (3.73),

$$c_{\underbrace{\{1,\ldots,1\}}_{k}}(a_1,a_2,a_3) = \left(-\frac{\partial}{\partial a_1}\right)^k c_\emptyset(a_1,a_2,a_3)\ . \tag{11.37}$$

The Hirota equations are then not directly satisfied, but one instead finds the constraint at degree 3

$$\begin{aligned} a_3^2\left(c_\emptyset\partial_{a_1}^4 c_\emptyset - 4\left(\partial_{a_1}^3 c_\emptyset\right)\left(\partial_{a_1} c_\emptyset\right) + 3\left(\partial_{a_1}^2 c_\emptyset\right)^2\right) + \\ + \left(a_2^2 - 4a_1a_3\right)\left(\left(\partial_{a_1} c_\emptyset\right)^2 - c_\emptyset\partial_{a_1}^2 c_\emptyset\right) - 2a_3 c_\emptyset\partial_{a_1} c_\emptyset + a_2 N c_\emptyset^2 = 0 \end{aligned} \tag{11.38}$$

where we let $c_\emptyset(a_1,a_2,a_3) = c_\emptyset$ and $\frac{\partial}{\partial a_1} = \partial_{a_1}$ to ease notation. To simplify this, we now apply the solution for $c_\emptyset(a_1,a_2,a_3)$ in (3.90) to find

$$2(zg'(z)^2 - 8g'''(z)g'(z) + 6g''(z)^2) + g(z)\left(g'(z) - 2zg''(z) + 4g''''(z)\right) = 0 \tag{11.39}$$

for the function g introduced in (3.89) and with z as in (3.95). The meaning of (11.39) is that for a generic choice of the function $g(z)$, the Hirota equations are not satisfied, even upon choosing $\beta = 1$. Specialising to the case $N = 1$ for $\mathsf{p} = 3$, we however find that for the solution in (3.98) the Hirota equations are satisfied. Another solution in the case of generic

N is the one given by matrix models, although identifying the explicit form of $g(z)$ by matching the integral and the generating function might not be straightforward. For instance in the case of $N=1$, we can make the change of variables $x = (a_3^{-1/3} x' - a_2/2a_3)$ to obtain

$$\begin{aligned} &\tau_1(0; a_1, a_2, a_3) = \\ &\quad = \int_\Gamma \exp\left(-\frac{1}{3}a_3 x^3 - \frac{1}{2}a_2 x^2 - a_1 x\right) \mathrm{d}x \\ &\quad = \exp\left(\frac{a_1 a_2}{2a_3} - \frac{a_2^3}{12 a_3^2}\right) a_3^{-1/3} \int_{\Gamma'} \exp\left(-\frac{x'^3}{3} + \left(\frac{a_2^2 - 4a_1 a_3}{4 a_3^{4/3}}\right) x'\right) \mathrm{d}x' \,. \end{aligned} \tag{11.40}$$

Upon comparing the above to (3.90) we can identify $g(z)$ as

$$g(z)|_{N=1} = \int_{\Gamma'} \exp\left(-\frac{x'^3}{3} + \frac{z}{2}x'\right) \mathrm{d}x' \,, \tag{11.41}$$

where for an appropriate contour Γ' we recover a linear combination of Airy functions. It can then be deduced from the Airy equation that the above solution for $\mathsf{p}=3$ and $N=1$ satisfies the Hirota equations, i.e. (11.39).

p	$c_\emptyset(a)$	Hirota equations
1	$k_1^{N,\beta,\nu} a_1^{-N(\nu+\beta(N-1)+1)}$	✓
2	$k_2^{N,\beta} a_2^{-\frac{1}{2}N(\beta(N-1)+1)} \exp\left(\frac{N a_1^2}{2a_2}\right)$	✓
3	$\exp\left(\frac{N a_1 a_2}{2a_3} - \frac{N a_2^3}{12 a_3^2}\right) a_3^{-\frac{1}{3}(1-\beta(N-1))N} g\left(\frac{a_2^2 - 4a_1 a_3}{2 a_3^{4/3}}\right)$	✗
	$N=1:$ $\exp\left(\frac{a_1 a_2}{2a_3} - \frac{a_2^3}{12 a_3^2}\right) a_3^{-1/3} \left[k_A \operatorname{Ai}\left(\frac{a_2^2 - 4a_1 a_3}{4 a_3^{4/3}}\right) + k_B \operatorname{Bi}\left(\frac{a_2^2 - 4a_1 a_3}{4 a_3^{4/3}}\right)\right]$	✓

Table 11.1. *Summary of the empty correlators for* $\mathsf{p} = 1, 2, 3$ *together with indication if this solves the Hirota equations or not.*

The q-analogue of the Hermitean matrix models, which can be given in terms of Jackson q-integrals, can also be written in determinant form using Andréief's formula. Consequently, such q-deformed models satisfies both the q-Virasoro constraints and the Hirota equations. These generalisations of the Hermitean matrix model therefore provide possible directions for further investigations.

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13. Svensk sammanfattning

Världen runt omkring oss beskrivs idag på kvantnivå med hjälp av ett ramverk som kallas Standardmodellen. I den delas samtliga universums partiklar upp i två klasser: bosoner och fermioner. Bosoner är de partiklar som förmedlar de fyra fundamentalkrafterna i universum. Med andra ord ger de upphov till de starka och svaga kärnkrafterna samt den elektromagnetiska kraften. Bosonen som ger upphov till gravitationskraften är dock ännu inte experimentellt bevisad. Fermioner är de partiklar som utgör all den materia som finns omkring oss. Det som skiljer de två klasserna åt är en egenskap som heter spinn, där bosoner har heltaligt spinn medan fermioner har halvtaligt spinn. Det finns dock problem med Standardmodellen. Ett sådant problem är något som kallas hierarkiproblemet. Det handlar om den stora skillnaden mellan storleken på den svaga kärnkraftens styrka och gravitationskraftens styrka. Ett annat problem handlar om föreningen av kopplingskonstanterna. Denna förening av de svaga och starka kärnkrafterna samt den elektromagnetiska kraften måste ske vid höga energinivåer för att en så kallad storförenad teori ska kunna existera. En lösning på dessa problem är *supersymmetri*. Detta är en symmetri mellan de två typer av partiklar som tidigare nämnts. Mer specifikt så är supersymmetri en teori där varje partikel har en motsvarande antipartikel, med samma egenskaper som partikeln förutom spinn. Varje boson har därför en supersymmetrisk partner som är en fermion och vice versa. Supersymmetri resulterar därför i en utökad version av Standardmodellen. Supersymmetrins existens har dock ännu inte bekräftats av experiment och nuvarande partikelacceleratorer, som exempelvis LHC vid CERN i Schweiz, behöver komma upp till högre energinivåer för att ha möjlighet att kunna upptäcka supersymmetri.

Standardmodellen är formulerad i ett matematiskt språk som kallas kvantfältteori och mer specifikt så beskrivs den genom gaugeteorier. Förutom att supersymmetri löser de tidigare nämnda problemen relaterade till hierarki och föreningen av kopplingskonstanter, så kan supersymmetri även underlätta uträkningar i gaugeteorier. I supersymmetriska gaugeteorier är man ofta intresserad av att beräkna olika typer av observabler. Under de senaste tio åren har sättet att beräkna sådana observabler på genomgått en metamorfos, i och med utvecklingen av en metod vid namn *supersymmetrisk lokalisering*. När denna lokaliseringsteknik används, resulterar den i att observabler reduceras till en enklare form och ibland till vad som kallas för matrismodeller. Dessa kan delas upp i två kategorier:

klassiska (eller odeformerade) samt deformerade matrismodeller. De deformerade matrismodellerna kan ses som en generalisering av de klassiska modellerna, där man i den enklaste typen av deformerade modeller introducerar en ny parameter, vanligtvis q, som ger upphov till deformationen. Båda dessa typer av matrismodeller har förekommit i olika lokaliseringsuträkningar. Klassiska matrismodeller kan ses som en lösning på en differentialekvation som kallas *Virasoro tvång* och det är dessa tvång samt dess q-deformerade motsvarighet som är centrala i våra utforskningar.

I artikel V studerar vi klassiska matrismodeller och undersöker dess Virasoro tvång. Vi undersöker även hur Virasoro tvång relaterar till ett annat område inom teoretisk fysik, nämligen integrabla system, med andra ord sådana system som kan lösas exakt. Vi undersöker sedan i artikel II hur man kan lösa q-deformerade Virasoro tvång och ger exempel på konkreta lösningar. Vi jämför sedan i artikel IV klassiska och deformerade modeller och inkluderar mer generella deformerade modeller än de studerade i artikel II. Slutligen, i artiklar I och III studerar vi olika typer av deformerade matrismodeller som har uppkommit som ett resultat av den tidigare nämnda lokaliseringstekniken.

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