Journal of Physics: Conference Series **283** (2011) 012041 doi:10.1088/1742-6596/283/1/012041

The Problem of Time in Quantum Cosmology: A Decoherent Histories View[1]

Theodosios Christodoulakis

University of Athens, Physics Department, Nuclear & Particle Physics Section
Panepistimiopolis 157-71, Ilissia, Athens, Greece

E-mail: tchris@cc.uoa.gr

Petros Wallden

[1] University of Athens, Physics Department, Nuclear & Particle Physics Section,
Panepistimiopolis 157-71, Ilissia Athens, Greece
[2] Technological Educational Institute of Chalkida, Department of Technological Applications, Psahna-34400, Greece

E-mail: petros.wallden@gmail.com, pwallden@phys.uoa.gr

Abstract.
The problem of time in quantum gravity arises due to the diffeomorphisms invariance of the theory and appears via the Hamiltonian constraint, in the canonical quantizations. There is a need for a description where one can ask some timeless questions that still encode some sense of temporality. The decoherent histories approach to quantum theory, already at the kinematical level admits an internal time. Several alternative proposals for resolving the problem of time via the decoherent histories, exist, and in this contribution we focus on one particular and examine how it manifests itself at some simple cosmological models.

1. The problem of time in quantum gravity

Quantum gravity is the synthesis of general relativity with quantum theory. In order to achieve this, several different approaches exist. While in this contribution we will focus on canonical approaches to QG, the problem of times exist in all approaches, manifesting itself in a different manner. The reason for this, is that it stems from the conceptual (and technical) different position that *time* has in quantum theory and general relativity[2]. In the former, time is a fixed external parameter as in Newtonian physics, while in the latter, time is on equal footing with space as manifested by the diffeomorphism invariance of the theory.

In quantum theory, time is not an observable, in the strict formal sense, as no self-adjoint operator can be constructed that corresponds to time. It solely, appears in the Schrodinger

[1] Based on talk given by P. Wallden at the NEB 14 conference
[2] Details on the nature and subtleties of the problem of time can be found in Refs. [1, 2]

equation as a parameter. In particular, any clock constructed from matter, does run backwards (even slightly) with respect to the abstract Newtonian time.

Time in general relativity is simply another coordinate, being in the same footing with the space dimensions. It is in general only locally defined. It is thus difficult to reconcile this with time in quantum theory that is based in the more-or-less Newtonian time.

Technically the problem arises due to the existence of diffeomorphisms invariance. The invariance of the theory appears as a gauge symmetry/constraint. Whenever such a constraint exist, the physical degrees of freedom do not coincide with the ones that appear at the kinematical level. For example in electromagnetism, the vector field A_μ, is not the true degree of freedom, but the equivalence classes of vector fields related by gauge transformations $A_\mu \equiv A_\mu + \partial_\mu \phi$. This is the case, because no physical property can depend on the arbitrary choice of gauge. When one is faced with the task to quantize a theory admitting such symmetries, has two options. (a) To impose the constraint classically, find the classical degrees of freedom, corresponding to the equivalent classes of gauge related configurations, and then quantize the resulting system with the usual prescription or (b) quantize the unconstrained system and impose the constraint on the quantum level (Dirac quantization). In particular if there is a classical constraint $\phi_i(x,p) = 0$ in quantum theory we require the operator version of this constraint (after suitable factor ordering choice) to annihilate any physical state

$$\hat{\phi}_i|\psi >= 0 \tag{1}$$

and furthermore that any observable $\hat{A}$ commutes with the constraints.

$$[\hat{A}, \hat{\phi}_i] = 0 \tag{2}$$

Therefore, while we start with a kinematical Hilbert space $\mathcal{H}_{kin}$ (unconstrained), after imposing the operator constraint Eq.(1) we end up with a subspace composed by the solutions of the constraint and this is the *physical* Hilbert space $\mathcal{H}_{phys}$.

Returning to general relativity, we see that the gauge symmetry in our case, is the invariance of the theory under diffeomorphisms Diff($\mathcal{M}$). In the canonical analysis of GR, we break space and time in 3+1, by selecting a foliation. The constraint then breaks into two parts:

(i) *Momentum constraint.* The theory is invariant under general diffeomorphisms in the 3-dimensional slice.

$$2iD_j \frac{\delta \Psi}{\delta h_{ij}} + H_i^{matter} \Psi = 0 \tag{3}$$

(ii) *Hamiltonian constraint.* The theory has as constraint, the Hamiltonian of the system, which reads

$$\hat{H} = -G_{ijkl} \frac{\delta^2}{\delta h_{ij} \delta h_{kl}} + h^{1/2} \left[-{}^3R(h) + 2\Lambda \right] + H^{matter} \tag{4}$$

This implies that the physical states solutions should obey the Wheeler-DeWitt equation

$$\hat{H}\Psi = 0 \tag{5}$$

and any observable should commute with the Hamiltonian/constraint

$$[\hat{H}, \hat{A}(t)] = 0 \tag{6}$$

However, the Hamiltonian, as we know from standard QT, corresponds to the generator of time translations. The time evolution of an observable $\hat{A}$ is given by

$$i\hbar\frac{d\hat{A}(t)}{dt} = [\hat{H}, \hat{A}(t)] \tag{7}$$

From Eqs. (6) and (7) we come to the profound conclusion that any observable should be independent of time. This is a general property of any theory that has weakly vanishing Hamiltonian[3]. These theories are called reparametrization invariant, since they are invariant under time reparametrizations.

There is a need therefore to construct a theory in which time plays no fundamental role. In such "timeless" theories, time should be an emergent property that appears due to coarse-grainning of the relative field configurations. In other words, all physical questions, should be translated to questions about the relative configurations of the universe and its material content. In this direction a lot of progress has been made, mainly in two approaches, the evolving constants (e.g. [3]) and the decoherent histories (e.g. [4, 5, 6, 7]). Here we will examine the latter.

2. The decoherent histories approach to quantum theory

It is an alternative formulation of quantum theory, specially designed to deal with closed systems (see for example [8]). The central objects in this approach, are histories of the system, and not one-moment-of-time propositions, and Hilbert space appears as a secondary object rather than as the starting point. The central mathematical objective of the approach is to assign probabilities to histories of a closed system. It can therefore easily deal with time-extended questions (as for example tunelling time, arrival time etc). It is an attempt to put time and space on equal footing, since we no longer deal with single time propositions (where time is obviously singled out)[4]. These considerations leads us to the conclusion that this formulation of quantum theory would be well suited to deal with the aforementioned problem of time.

If one wanted to calculate the probability, in standard "Copenhagen" QT, for a system being found in a sequence of positions in different times (a history), one would get:

$$P(\alpha_{t_1} \text{ at } t_1 \text{ and } \alpha_{t_2} \text{ at } t_2 \cdots \alpha_{t_n} \text{ at } t_n; \rho(t_0)) = \mathrm{Tr}(\alpha_{t_n}(t_n)\cdots\alpha_{t_1}(t_1)\rho(t_0)\alpha_{t_1}(t_1)\cdots\alpha_{t_n}(t_n)) \tag{8}$$

The $\alpha_{t_k}(t_k)$ are time evolved (Heisenberg picture) projection operators, and $\rho(t_0)$ is the initial state. We will call Eq.(8) *candidate* probability. However this is NOT the actual probability for this history to occur, for a given closed system. It assumes the existence of observers (and measuring apparatuses) at every moment t_k. If one attempted to use Eq.(8) as probability, he would fail, because the additivity of disjoint regions of the sample space, is not satisfied. The failure is caused due to interference. A well known example of the above, is the double slit, where the sum of the candidate probability (amplitude square in this trivial example) for crossing slit 1, and that of crossing slit 2, is not the candidate probability of crossing *any* slit. There are certain cases though, that we *can* assign the candidate probability to a closed

[3] The problem is more vital, when the Hamiltonian constraint is quadratic in all momenta, because in that case one cannot use any variable to "deparametrize" in order to recover time evolution.

[4] Other reasons to consider decoherent histories, involve emergence of classicality, decoherence etc, but these are not of interest for the topic of this contribution.

system. In particular, the decoherent histories approach, tells us when this is possible. The class operator is defined as

$$C_{\underline{\alpha}} = \alpha_{t_n}(t_n) \cdots \alpha_{t_1}(t_1) \tag{9}$$

The interference is measured by the decoherence functional, the key mathematical object in this approach. In the example of non-relativistic quantum mechanics we consider here, as a motivation, it corresponds to

$$\mathcal{D}(\underline{\alpha}, \underline{\alpha}') = \mathrm{Tr}(C_{\underline{\alpha}} \rho C^{\dagger}_{\underline{\alpha}'}) \tag{10}$$

Where $\underline{\alpha}$ is one history and $\underline{\alpha}'$ another, the decoherence functional simply measures the interference these two have[5]. We are interested in a set of histories $\{\underline{\alpha}_i\}$ that are *disjoint* (every pair of histories are orthogonal at at least one moment-of-time) and *exhaustive* (the set of histories span the full history space). Such a set is called *complete*. We can assign the probability given by Eq. (8) provided that we have a complete set of histories $\{\alpha_i\}$ that decoheres pairwise, i.e.

$$\mathcal{D}(\underline{\alpha}_i, \underline{\alpha}_j) = 0, \text{ for all } i \neq j \tag{11}$$

The probability we get for each history α_i to occur, is the candidate probability, that is equal to the diagonal parts of the decoherence functional

$$p(\underline{\alpha}_i) = \mathcal{D}(\underline{\alpha}_i, \underline{\alpha}_i) \tag{12}$$

One important issue is that there exist more than one complete set of histories (and possibly incompatible the one with the other) for a given quantum system. There is some interpretational ambiguity as to what is actually realized, or if there is a contextual reality (see for example Ref. [11, 12]).

One can generalize, all of the above, for the case of relativistic QT or even for quantum gravity. The important ingredient, one needs to consider in each of these cases, is (a) Construction of suitable Class Operators, that correspond to physical questions. (b) The use of the proper inner product, in order to be able to define probabilities. (c) The decoherence condition. Only sets of questions that decohere (obey Eq. (11)) can be used to get probabilities. The above steps will be taken below, for the case of timeless theories (with vanishing Hamiltonian). Particular (simple) quantum cosmological examples will conclude this contribution.

3. Decoherent histories and the problem of time

The first step is to consider classical (rather than quantum) timeless theories. Physical question that one can ask, which do not depend on time, could involve the full history/trajectory, of a classical object. For example:

Does a (classical) full trajectory cross a given region Δ of the configuration space?

This is indeed reparametrization invariant. However things get more complicated when we consider a quantum system. To deal with histories/trajectories, we immediately need to use the decoherent histories approach, and having the reparametrization invariance, adds further requirements, that would impose this invariance. In particular we need

[5] Further details can be found in e.g. Ref. [9] and its relation with the quantum measure in Ref. [10].

(i) An initial state obeying the constraint: $\hat{H}|\psi\rangle = 0$

(ii) Class operator (to be defined later): $[\hat{C}_\alpha, H] = 0$

(iii) We have to use the *induced* (or Rieffel) inner product, rather than the standard "Schrodinger" inner-product. It is an inner product defined on solutions of the constraint. The reader is referred to Ref. [13] for further details.

(iv) The decoherence condition needs to be satisfied in order to assign actual probabilities.

We need to find a class operator, that commutes with the Hamiltonian, gives (semi-classically) sensible results and corresponds to the question: "*What is the probability that the system crosses a region Δ of the configuration space, with no reference in time*". Here we will be following Ref. [5]. Since the classical reparametrization invariant object is the full trajectory, in our considerations we have the unphysical parameter time, running from $-\infty$ to $+\infty$. It is also clear, that the class operator for crossing the region Δ at any time, would be related to the Class operator of always remaining in the complement region $\bar{\Delta}$. The projections are related in this way $P_{\bar{\Delta}} = \mathbf{1} - P_\Delta$ and for notational simplicity we shall use $\bar{P}$ instead. The class operator for always remaining at $\bar{\Delta}$ is[6]

$$C_{\bar{\Delta}} = \prod_{t=-\infty}^{t=+\infty} \bar{P}(t) \tag{13}$$

One can see easily that this class operator commutes with the Hamiltonian,

$$[C_{\bar{\Delta}}, H] = 0 \tag{14}$$

while the crossing at some time class operator is given by

$$C_\Delta = \mathbf{1} - C_{\bar{\Delta}} \tag{15}$$

In particular one may rewrite the class operators with respect to the restricted propagator $g_r(t'', t')$ (see below for definitions and properties)

$$C_{\bar{\Delta}} = \lim_{t''\to\infty, t'\to-\infty} \exp(-iHt'') g_r(t'', t') \exp(iHt') \tag{16}$$

From this expression, one can see the resemblance with the arrival time problem in standard non-relativistic quantum mechanics (with different Hamiltonian though) as is examined in Refs. [6, 14, 15, 16][7].

The restricted propagator, seen above, is the normal quantum mechanical propagator, restricted to paths being only in a particular region ($\bar{\Delta}$ in our case). The path integral definition is

$$\begin{aligned} g_r(x, t \mid x_0, t_0) &= \int_{\bar{\Delta}} \mathcal{D}x \exp(iS[x(t)]) \\ &= \langle x | g_r(t, t_0) | x_0 \rangle \end{aligned} \tag{17}$$

[6] Some care is needed to define rigorously the continuous product of operators and taking the limits to infinity, but it can be done.

[7] Note that for periodic Hamiltonians (bounded systems), the above analysis changes slightly (see details in Ref. [5]).

One can generalize the above definition, using operators and taking the limit $\delta t \to 0$, $n \to \infty$ while keeping $\delta t \times n = (t - t_0)$:

$$\begin{aligned} g_r(t,t_0) &= \lim_{\delta t \to 0} \bar{P} e^{-iH(t_n - t_{n-1})} \bar{P} \cdots \bar{P} e^{-iH(t_1 - t_0)} \bar{P} \\ &= \bar{P} \exp\left(-i(t-t_0)\bar{P}H\bar{P}\right) \bar{P} \end{aligned} \tag{18}$$

The restricted operator has the important property (Ref. [6])

$$g_r^\dagger(t,t_0) g_r(t,t_0) = \bar{P} \tag{19}$$

Using the latter, we can compute the candidate probability, provided that we have an initial state satisfying the constraint (solution of the Wheeler DeWitt equation). The probability for not crossing is

$$p_{\bar{\Delta}} = \langle\psi|C^\dagger_{\bar{\Delta}} C_{\bar{\Delta}}|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle \tag{20}$$

And provided we have decoherence the probability for crossing becomes

$$p_\Delta = 1 - p_{\bar{\Delta}} = \langle\psi|P|\psi\rangle \tag{21}$$

Finally the decoherence condition is (requiring that the decoherence functional between the crossing and not-crossing class operators vanishes)

$$\lim_{t\to\infty, t_0 \to -\infty} e^{iE(t-t_0)} \langle\psi|g_r(t,t_0)|\psi\rangle = \langle\psi|\bar{P}|\psi\rangle \tag{22}$$

which becomes the very simple requirement, that *the wavefunction needs to vanish on the boundary of the region we consider*, i.e.:

$$\langle x|\psi\rangle = 0, \quad \forall x \in \partial\bar{\Delta} \tag{23}$$

At first sight, this seems to be very restrictive, since given an initial wavefunction (of a simple timeless system), there will be very few (if any) regions Δ that would satisfy the requirement. However, it is not too surprising, since with no external observer, and few or no matter degrees of freedom, one would not expect to have decoherence and be able to speak sharply about properties of the system. In realistic set-ups many non-trivial questions would exist. Below, we will see some very simple quantum cosmological models, that give non-trivial questions, and we will mention a possible comparison with the evolving constants approach to the problem of time.

4. Simple quantum cosmological examples

We will examine here the case of a Friedmann-Lemaitre-Robertson-Walker (FLRW) cosmology, i.e. a homogeneous and isotropic cosmology. In particular two specific examples will be considered. A $k = -1$ universe, with zero cosmological constant, (a) first empty and then (b) with a scalar field on it with potential $V(\Phi) = -\frac{\exp(2\Phi)}{2}$.

4.1. Empty FLRW

We have a universe with metric (in spherical polar coordinates)

$$(ds)^2 = -N^2(dt)^2 + \frac{\alpha^2}{1+r^2}(dr)^2 + r^2\alpha^2(d\theta)^2 + r^2\alpha^2(\sin\theta)^2(d\phi)^2 \tag{24}$$

Where $N(t)$ is the lapse function and $\alpha(t)$ is the scale factor. Following the usual prescription for quantization, we find the canonical Hamiltonian to be

$$H_c = N(\frac{\pi_\alpha^2}{4\alpha} - \alpha) \tag{25}$$

Where π_α is the conjugate momentum to the scale factor α. To get the Wheeler-DeWitt equation, we choose the operator ordering such that the kinetic part corresponds to the covariant Laplacian. We get the following equation

$$\Psi''(\alpha)/4\alpha - \Psi'(\alpha)/8\alpha^2 + \alpha\Psi(\alpha) = 0 \tag{26}$$

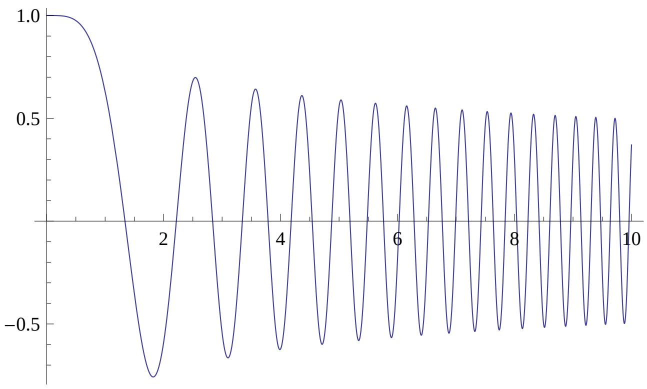
The general solutions are of the form

$$\Psi(\alpha) = C_1\alpha^{3/4}J_{-3/8}(\alpha^2) + C_2\alpha^{3/4}J_{3/8}(\alpha^2) \tag{27}$$

where the J's are Bessel functions of the first kind and C_1, C_2 are constants. We choose a particular solution as initial state of our universe

$$\Psi(\alpha) \propto \alpha^{3/4}J_{-\frac{3}{8}}(\alpha^2) \tag{28}$$

That looks like



Following the prescription we analyzed in the previous sections, we know that the only questions that we can ask are those that satisfy the decoherence condition. They involve the zeros of the Bessel function. We could ask for example, which is the probability for never crossing the region $\alpha > 6$, in other words, "What is the probability that our universe (with initial condition of Eq.(28)) never reaches a scale factor α greater than 6". The probability is

$$p_c = \frac{\int_0^6 |\Psi(\alpha)|^2 d\alpha}{\int_0^\infty |\Psi(\alpha)|^2 d\alpha} \simeq 0.27 \tag{29}$$

If one wished to check, he would find that the probability for the universe to become at any time greater than $\alpha = 6$ turns out to be $p_{nc} \simeq 0.73$. This is expected, since here we have decoherence and the candidate probabilities are actual probabilities summing up to 1.

4.2. FLRW with an exponential scalar field

We now consider an FLRW universe with $k = -1$, and with an exponential scalar field of the form $V(\Phi) = -\frac{\exp(2\Phi)}{2}$. The canonical Hamiltonian is

$$H_c = N(-\frac{\pi_\alpha^2}{4\alpha} + \frac{\pi_\Phi^2}{4\alpha^3} + \alpha - \frac{\alpha^3}{2}\exp(2\Phi)) \tag{30}$$

which after the appropriate operator ordering gives the Wheeler-DeWitt equation

$$(2\alpha - \alpha^3 e^{2\Phi})\Psi(\alpha,\Phi) - \frac{1}{2\alpha}\frac{\partial^2\Psi(\alpha,\Phi)}{\partial\alpha^2} - \frac{1}{2\alpha^2}\frac{\partial\Psi(\alpha,\Phi)}{\partial\alpha} + \frac{1}{2\alpha^3}\frac{\partial^2\Psi(\alpha,\Phi)}{\partial\Phi^2} = 0 \tag{31}$$

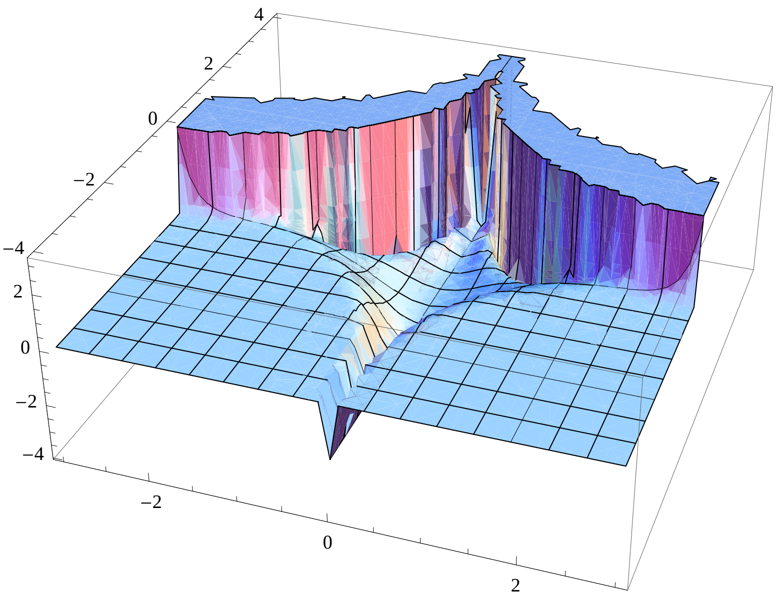
The general solution of Eq.(31) is of the form

$$\Psi(\alpha,\Phi) = C_2 \exp\frac{\alpha^2 e^{-2\Phi}(-4C_1^2 - 4e^{4\Phi} + 6e^{6\Phi}\alpha^2)}{8C_1} \tag{32}$$

For our analysis we take one particular solution

$$\Psi(\alpha,\Phi) = \exp\frac{\alpha^2 e^{-2\Phi}(-4^2 - 4e^{4\Phi} + 6e^{6\Phi}\alpha^2)}{8} - \exp\frac{\alpha^2 e^{-2\Phi}(-4\cdot 1.5^2 - 4e^{4\Phi} + 6e^{6\Phi}\alpha^2)}{12} \tag{33}$$

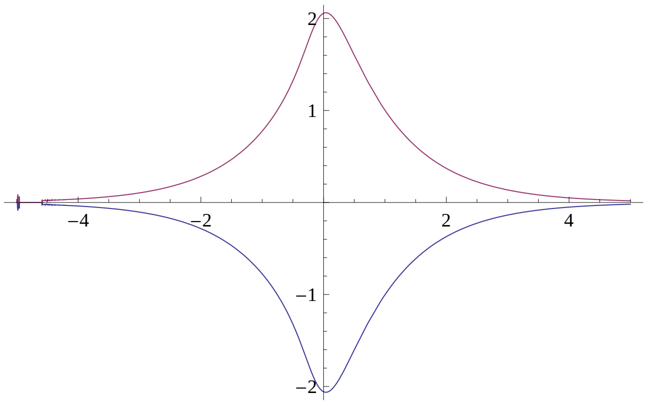
Note that this solution, is not normalizable in the normal inner product, and we need to use the induced inner product (an inner product on solutions).The solution looks pictorially



According to our scheme, we can ask a question, provided the wavefunction vanishes on the boundary of the region in question. In this example it vanishes at the following region[8].

(34)

[8] The closed form of this region is complicated



Since our solution vanishes at the above boundary, we have decoherence and we can assign the crossing probability, which however turns out to be zero $p_c = 0$. This happens even though the integral $|\Psi(\alpha,\Phi)|^2$ at the region is non-zero, because in the induced inner product it results to zero probability.

Other solutions (involving superpositions) or systems with more degrees of freedom (e.g. Bianchi Cosmologies, different matter content etc), will result to questions with non-trivial answers.

Here we should make a quick final note, on the relation of these results to similar questions in relational observables approach. One would classically evade the time dependence, by asking relational questions such as " which is the value(s) of α when $\Phi = 15$ or when $\Phi > 15$". This corresponds to an observable that projects at the range of Φ in question and since there is not explicit time dependence, classically it is reparametrization invariant. However, turning the above question to a quantum operator, it fails to commute with the Hamiltonian, and thus, is not a Dirac observable. A careful consideration, leads to the conclusion that the operator fails to commute with the Hamiltonian, due to some extra terms defined *only* on the boundary of the region considered. So while formally it is not an observable, if one was able to restrict attention to a single solution (the one realized for example) that vanishes on this boundary, then the operator would effectively commute with the Hamiltonian and be a well-defined observable. While this is not possible in the standard canonical quantization, it is allowed in the decoherent histories quantization, since the initial state is given and remains fixed in this approach. This is the advantage of using this approach, having however the major disadvantage that such questions can only give an answer, when we have decoherence. Interestingly the two conditions (commuting observable and decoherence) are both satisfied when the wavefunction vanishes at the boundary of the region considered.

5. Summary and Conclusions

We introduced the problem of time and the decoherent histories approach, in order to make a decoherent histories analysis of timeless quantum theories. We reviewed the class operator that results from this analysis and commutes with the Hamiltonian given in Ref. [5]. They gave a general enough set of physical questions of the type "Which is the probability that the system crosses a region in configuration space with no reference in time". However, following Ref. [6] we got a very restrictive but simple decoherence condition, namely "The initial state has to vanish on the boundary of the region considered". To see how this works in quantum cosmology, we took a couple of simple cosmological models (FLRW empty and with an exponential scalar field) and worked out the probabilities for crossing some regions that obeyed the decoherence condition we had obtained. Finally, some comments were made on how this result may compare with relational observables.

Acknowledgments
The authors thank Jonathan J. Halliwell for discussions in earlier stages of the work. PW was supported by I.K.Y. (State Scholarship Foundation).

References

[1] K. Kuchař, The problem of time in canonical quantization. in *Conceptual Problems of Quantum Gravity*, edited by A. Ashtekar and J. Stachel (Birkhäuser, Boston, 1991), K. Kuchař, in *Proceedings of the 4th Canadian Conference on General Relativity and Relativistic Astrophysics*, edited by G. Kunstatter, D. Vincent and J. Williams (World Scientific, Singapore, 1992).
[2] C.J. Isham, in *Integrable Systems, Quantum Groups, and Quantum Field Theories*, edited by L. A. Ibort and M. A. Rodrguez (Kluwer Academic Publishers, London, 1993).
[3] P. Hajicek, J. Math. Phys. **36**, 4612 (1995); M. Montesinos, C. Rovelli and T. Thiemann, Phys. Rev. D **60**, 044009 (1999); M. Montesinos, Gen. Relativ. Gravit. **33**, 1 (2001); R. Gambini and R.A. Porto, Phys. Rev. D **63**, 105014 (2001).
[4] K. Savvidou, J. Math. Phys. **40**, 5657 (1999); K. Savvidou, gr-qc/9912076; N. Savvidou, Braz. J. Phys. **35**, 307 (2005); C. Anastopoulos and N. Savvidou, Class. Quant. Grav. **22**, 1841 (2005); J.J. Halliwell and J. Thorwart, Phys. Rev. D **65**, 104009 (2002); J. Halliwell, in *The Future of Theoretical Physics and Cosmology*, edited by G.W. Gibbons, E.P.S. Shellard and S.J. Rankin (Cambridge University Press, 2003).
[5] J.J. Halliwell and P. Wallden, Phys. Rev. D **73**, 024011 (2006).
[6] P. Wallden, Int. J. Theor. Phys. **47**, 1512 (2008).
[7] J.J. Halliwell, Phys. Rev. D **80**, 124032 (2009).
[8] R.B. Griffiths, J. Stat. Phys., **36** 219, (1984); R. Omnès. J. Stat. Phys. **53**, 893 (1988); M. Gell-Mann and J. Hartle, in *Complexity, Entropy and the Physics of Information, SFI Studies in the Science of Complexity, Vol. VIII*, edited by W. Zurek, (Addison-Wesley, Reading, 1990); M. Gell-Mann and J. Hartle. Phys. Rev. D **47**, 3345 (1993).
[9] J.J. Halliwell, Annals N. Y. Acad. Sci. **755**, 726 (1995).
[10] R. D. Sorkin, Mod. Phys. Lett. A **9**, 3119 (1994); R. D. Sorkin, *Quantum Classical Correspondence: Proceedings of the 4th Drexel Symposium on Quantum Nonintegrability*, Philadelphia, September 8-11, 1994, pages 229-251, (International Press, Cambridge Mass. 1997) D.H. Feng and B-L Hu, editors. [arXiv:gr-qc/9507057].
[11] F. Dowker and A. Kent, Phys. Rev. Lett. **75**, 3038 (1995); F. Dowker and A. Kent, J. Statist. Phys. **82**, 1575 (1996).
[12] C.J. Isham, Int. J. Theor. Phys. **36**, 785 (1997).
[13] A. Ashtekar, J. Lewandowski, D. Marolf, J. Mourao and T. Thiemann, J. Math. Phys. **36**, 6456 (1995); A. Higuchi, Class. Quant. Grav. **8**, 1983 (1991). D. Giulini and D. Marolf, Class. Quant. Grav. **16**, 2489 (1999); Class. Quant. Grav. **16**, 2479 (1999). F. Embacher, Hadronic J. **21**, 337 (1998); N. Landsmann, J. Geom. Phys. **15**, 285 (1995); D. Marolf, gr-qc/0011112.
[14] J.J. Halliwell and E. Zafiris, Phys. Rev. D **57**, 3351 (1998).
[15] J. J. Halliwell. and J. M. Yearsley, Phys. Rev. **A 79** 062101 (2009) .
[16] P. Wallden, J. Phys.: Conf. Ser. **67**, 012043 (2007).