The 5th International Workshop in

Group Analysis of Differential Equations and Integrable Systems

ΣΥΜΜΕΤΡΙΑ • СИМЕТРІЯ

V

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This book includes papers of participants of the Fifth International Workshop "Group Analysis of Differential Equations and Integrable Systems". The topics covered by the papers range from theoretical developments of group analysis of differential equations and the integrability theory to applications in a wide variety of disparate fields including fluid mechanics, classical mechanics, relativity, control theory, quantum mechanics, physiology and finance. The book may be useful for researchers and post graduate students who are interested in modern trends in symmetry analysis and its applications.

Editors: N.M. Ivanova, P.G.L. Leach, R.O. Popovych, C. Sophocleous and P.A. Damianou

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Preface

The Fifth International Workshop devoted to the Group Analysis of Differential Equations and Integrable Systems (GADEIS-V) was conducted at Protaras, Cyprus, during the period June 6–10, 2010. There were 50 participants from nineteen countries (Australia, Austria, Canada, Cyprus, Czech Republic, Egypt, Germany, Israel, Italy, People's Republic of China, Poland, Romania, Russia, South Africa, Spain, Switzerland, The Netherlands, Ukraine and United States of America) and thirty-three lectures were presented. The topics covered ranged from theoretical developments in group analysis of differential equations and the integrability theory to applications in a wide variety of disparate fields including fluid mechanics, classical mechanics, relativity, control theory, quantum mechanics, physiology and finance. Twenty papers are presented in this proceedings.

The Workshops are a joint initiative by the Department of Mathematics and Statistics, University of Cyprus, and the Department of Applied Research of the Institute of Mathematics, National Academy of Sciences, Ukraine. The Workshops evolved from close collaboration among Cypriot and Ukrainian scientists. The first three meetings were held at the Athalassa campus of the University of Cyprus (October 27, 2005, September 25–28, 2006, and October 4–5, 2007). The fourth (October 26–30, 2008) and fifth meetings were held at the Tetyk Hotel in the coastal resort of Protaras.

All of the papers in this volume have been reviewed by two independent referees. We express our appreciation of the care taken by the referees and thank them for making some suggestions for improvement to most of the papers. The importance of peer review in the maintenance of high standards of scholarship can never be overstated.

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Solution of Burgers' equation with time-dependent kinematic viscosity via Lie-group analysis

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Lie-group method is applied for determining symmetry reductions of a boundary value problem for the Burgers' equation with time-dependent kinematic viscosity. The resulting ordinary differential equations are solved numerically using shooting method coupled with Runge–Kutta scheme and the results are plotted.

1 Introduction

Burgers' equation [1] is a one dimensional model of the Navier–Stokes' hydrodynamical equations [2]. It used as a simplified model for turbulence. It describes a variety of nonlinear wave phenomena arising in the theory of wave propagation, acoustics, plasma physics, modeling of gas dynamics, traffic flow and other areas. Burgers' equation first introduced by Bateman [3] who derived it in a physical context and later treated by Burgers [1]. Exact and numerical solutions studies of Burgers' equation have received considerable attention of scientists. Cole [4] concluded that, the Burgers' equation can be transformed to the linear heat equation. Hopf [5] concluded the same result that Cole discovered. From which, it is known as Hopf–Cole transformation. About thirty-five distinct solutions for initial-value problem of Burgers' equation were surveyed by Benton and Platzman [6] in the infinite domain. Without taking into consideration the auxiliary conditions, Ames [7] studied how the Morgan–Michal method could be applied for determining the proper groups for Burgers' equation.

Peralta-Fabi and Plaschko [2] studied the stability and the bifurcation of the equilibrium solution of a controlled Burgers' equation. An integral term which represents a non-local behaviour has been added to the normal form of the equation describing flow through porous media. They found that, a supercritical bifurcation from the rest solution occurs when the viscosity reduced below a critical value. This critical value was calculated as a function of the porosity coefficient and the corresponding bifurcation solution is derived using perturbation forms up to fourth order.

Vorus [8] generated an exact solution to Burgers'nonlinear diffusion equation on a convective stream with sinusoidal excitation applied at the upstream boundary. He applied Hopf–Cole transformation in achieving the analytical solution after integrating the equation and its conditions to avoid nonlinearity in the transformed upstream boundary condition. He deduced a very simple limiting solution valid for high Reynolds number from the exact solution. This approximate solution was found to be amenable to an elegant geometrical interpretation.

Kingston and Sophocleous [9] classified all finite point transformations between generalized Burgers' equation. They found all the well-known invariant infinitesimal transformations and also the reciprocal point transformation that leave the Burgers' equation invariant, which is symmetry additional to the Lie point symmetries obtained from the classical approach. In their work, they did not take into consideration the invariance of the initial and boundary conditions.

Abd-el-Malek and El-Mansi [10] applied the one-parameter group of transformation to the Burgers' equation with unity kinematic viscosity associated with the initial and boundary conditions. Under this transformation, the given partial differential equation with the auxiliary conditions is reduced to an ordinary differential equation with the appropriate corresponding conditions. The obtained differential equation was solved analytically and the solution obtained in closed form.

Many other authors have used different numerical techniques to solve Burgers' equation such as the finite difference method, the finite element method and spectral methods, [11–21].

In this work, the Lie symmetry method is applied to the one-dimensional Burgers' equation with time-dependent kinematic viscosity, for determining symmetry reductions of the given partial differential equation, [22–31]. The resulting nonlinear ordinary differential equation is solved numerically using the shooting method coupled with Runge–Kutta scheme and the results are plotted. A particular case of our results is compared with those obtained by Abd-el-Malek and El-Mansi [10].

2 Mathematical formulation of the problem

For the one-dimensional velocity field $w(x,t)$, with time-dependent kinematic viscosity, the governing equation is given by

$$\frac{\partial w}{\partial t} + w\frac{\partial w}{\partial x} = f(t)\frac{\partial^2 w}{\partial x^2}, \quad x > 0,\ t > 0, \tag{1}$$

together with the initial and boundary conditions

$$\begin{aligned} &\text{(i)} \quad w(x,0) = 0, \\ &\text{(ii)} \quad w(0,t) = aq(t),\ t > 0,\ a \neq 0, \\ &\text{(iii)} \quad \lim_{x\to\infty} w(x,t) = 0. \end{aligned} \tag{2}$$

Lighthill [32] stated that the kinematic viscosity coefficient $f(t)$ of the diffusion term $\partial^2 w/\partial x^2$ is not normally a constant in applications, even approximately; the coefficient may actually be a function of the time. For bounded smooth time-dependent coefficient, i.e. $f(t)$ may be viewed as an approximation of a smoothly decaying diffusion.

A normalization of the boundary condition (2ii) is obtained by setting

$$w(x,t) = q(t)u(x,t), \tag{3}$$

where $u(x,t)$ is the normalize one-dimensional velocity field and $q(t)$ is an unknown function.

Substitution from (3) into (1) yields

$$q\frac{\partial u}{\partial t} + u\frac{dq}{dt} + q^2 u\frac{\partial u}{\partial x} - fq\frac{\partial^2 u}{\partial x^2} = 0, \quad x > 0,\ t > 0. \tag{4}$$

The initial and boundary conditions (2) will be

$$\begin{aligned} &\text{(i)} && u(x,0) = 0, \\ &\text{(ii)} && u(0,t) = a, \quad t > 0,\ a \neq 0, \\ &\text{(iii)} && \lim_{x\to\infty} u(x,t) = 0. \end{aligned} \tag{5}$$

3 Solution of the problem

At first, we derive the similarity solutions using Lie-group method under which (4) and the initial and boundary conditions (5) are invariant, and then we use these symmetries to determine the similarity variables.

3.1 Lie point symmetries

Consider the one-parameter Lie group of infinitesimal transformations in the space of $(x,t;u,q,f)$ given by

$$\begin{aligned} x^* &= x + \varepsilon X(x,t;u,q,f) + O(\varepsilon^2), \\ t^* &= t + \varepsilon T(x,t;u,q,f) + O(\varepsilon^2), \\ u^* &= u + \varepsilon U(x,t;u,q,f) + O(\varepsilon^2), \\ q^* &= q + \varepsilon Q(x,t;u,q,f) + O(\varepsilon^2), \\ f^* &= f + \varepsilon F(x,t;u,q,f) + O(\varepsilon^2), \end{aligned} \tag{6}$$

where ε is the group parameter. The partial differential equation (4) is said to admit a symmetry generated by the vector field

$$\Gamma \equiv X\frac{\partial}{\partial x} + T\frac{\partial}{\partial t} + U\frac{\partial}{\partial u} + Q\frac{\partial}{\partial q} + F\frac{\partial}{\partial f}, \tag{7}$$

if it is left invariant by the transformation $(x,t;u,q,f) \to (x^*,t^*;u^*,q^*,f^*)$. Assume $\Delta \equiv qu_t + uq_t + q^2uu_x - fqu_{xx}$, where subscripts denote partial derivatives. A vector field Γ given by (7) is a Lie symmetry vector field for (4) if

$$\Gamma^{[2]}\,(\Delta)|_{\Delta=0} = 0, \tag{8}$$

where

$$\Gamma^{[2]} = \Gamma + U^x\frac{\partial}{\partial u_x} + U^t\frac{\partial}{\partial u_t} + Q^t\frac{\partial}{\partial q_t} + U^{xx}\frac{\partial}{\partial u_{xx}}$$

is the essential part of the second prolongation of Γ.

To calculate the prolongation of the given transformation, we need to differentiate (6) with respect to each of the variables, x and t. To do this, we introduce the following total derivatives

$$\begin{aligned} D_x &= \partial_x + u_x\partial_u + u_{xx}\partial_{u_x} + u_{xt}\partial_{u_t} + \cdots, \\ D_t &= \partial_t + u_t\partial_u + q_t\partial_q + f_t\partial_f + u_{tt}\partial_{u_t} + h_{tt}\partial_{q_t} + u_{xt}\partial_{u_x} + \cdots. \end{aligned}$$

Equation (8) gives the following condition

$$\begin{aligned} &Uq_t + Uq^2u_x + Qu_t + 2Qquu_x - Qfu_{xx} - Fqu_{xx} \\ &\quad + q^2uU^x + qU^t + uQ^t - fqU^{xx} = 0, \end{aligned} \tag{9}$$

which should be satisfied in view of the equation (4). The components U^x, U^t, Q^t and U^{xx} can be determined from the following expressions

$$\begin{aligned} U^S &= D_SU - u_xD_SX - u_tD_ST, \\ Q^t &= D_tQ - q_tD_tT, \\ U^{JS} &= D_SU^J - u_{Jx}D_SX - u_{Jt}D_ST, \end{aligned}$$

where S stands for x and t and J stands for x. The substitution of these expressions into (9) leads to a cumbersome equation, then, equating to zero the coefficients of u_{xt}, u_xu_{xt}, u_tq_t, u_tf_t, u_xu_t, u_xq_t and u_xf_t, gives

$$T_x = T_u = T_q = T_f = X_u = X_q = X_f = 0. \tag{10}$$

The substitution of (10) into (9) removes many terms. Then, equating to zero the coefficients of ${u_x}^2$, q_t, u_x, u_t, f_t and the remaining terms, leads to the following system of determining equations:

$$\begin{aligned} &U_{uu} = 0, \\ &U - q^{-1}uQ - f^{-1}uF + qU_q + uQ_q - uT_t - uU_u + 2uX_x = 0, \\ &Uq^2 + Qqu - f^{-1}q^2uF - qX_t + fqX_{xx} + q^2uX_x - 2fqU_{xu} = 0, \\ &uQ_u + 2qX_x - qT_t - f^{-1}qF = 0, \\ &qU_f + uQ_f = 0, \\ &qU_t + q^2uU_x + uQ_t - fqU_{xx} = 0. \end{aligned}$$

We solve the system of determining equations, in view of the invariance of the initial and boundary conditions as well as using the facts that $q_x = 0$ and $f_x = 0$. So, the nonlinear equation (4) has the four-parameter Lie group of point symmetries generated by

$$\begin{aligned} &\Gamma_1 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + f\frac{\partial}{\partial f}, \quad \Gamma_2 = -x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} + q\frac{\partial}{\partial q}, \\ &\Gamma_3 = \frac{\partial}{\partial x}, \quad \Gamma_4 = \frac{\partial}{\partial t}. \end{aligned} \tag{11}$$

The one-parameter groups generated by Γ_1 and Γ_2 consist of scaling, whereas Γ_3 and Γ_4 generate translations.

The solutions $u = u(x,t)$, $q = q(t)$ and $f = f(t)$ are invariant under the symmetry (7) if $\Phi^u \equiv U - Tu_t - Xu_x = 0$, $\Phi^q \equiv Q - Tq_t = 0$, $\Phi^f \equiv F - Tf_t = 0$. These equations are called the invariant surface conditions.

Table 1 illustrates the solutions of the invariant surface conditions for some operators of the four-parameter Lie group of point symmetries.

Table 1. Solutions of the invariant surface conditions.

Generator	Characteristic $\Phi = (\Phi^u, \Phi^q, \Phi^f)$	Solution of the invariant surface conditions
Γ_1	$(-xu_x - tu_t, -tq_t, f - tf_t)$	$f(t) = t$
Γ_2	$(xu_x + 2tu_t, q + 2tq_t, 2tf_t)$	$f(t) = C$
Γ_3	$(-u_x, 0, 0)$	$u = u(t)$
Γ_4	$(-u_t, -q_t, -f_t)$	$u = u(x)$, $q = f = C$
$\Gamma_1 + \beta\Gamma_2$	$\Phi^u = (\beta-1)xu_x + (2\beta-1)tu_t$ $\Phi^q = \beta q + (2\beta-1)tq_t$ $\Phi^f = f + (2\beta-1)tf_t$	$u = u(\theta)$ $q(t) = K_1 t^{\frac{\beta}{1-2\beta}}$ $f(t) = K_2 t^{\frac{1}{1-2\beta}}$

As seen from Table 1, the solutions of the invariant surface conditions under Γ_1 and Γ_2 are $f(t) = t$ and $f(t) = C$, respectively. Practically, these solutions are not acceptable because for bounded smooth time-dependent coefficient, $f(t)$ may be viewed as an approximation of a smoothly decaying diffusion, as mentioned before.

The solution of the invariant surface conditions under Γ_3 is $u = u(t)$ which contradicts the boundary conditions.

The solutions of the invariant surface conditions under Γ_4 are $u = u(x)$ and $q = f = C$ which are the solutions of (4), even though they are not particularly interesting since they contradicts the initial condition.

On the other hand, the solutions of the invariant surface conditions under $\Gamma_1 + \beta\Gamma_2$ are

$$u \equiv u(\theta), \qquad q(t) = K_1 t^{\frac{\beta}{1-2\beta}}, \qquad f(t) = K_2 t^{\frac{1}{1-2\beta}}, \tag{12}$$

where $\theta = x\,t^{-\gamma}$ is the similarity variable, $\gamma = (1-\beta)/(1-2\beta)$, β, K_1 and K_2 are constants. As mentioned before, for bounded smooth time-dependent coefficient, i.e. $f(t)$ may be viewed as an approximation of a smoothly decaying diffusion, so, $1/(1-2\beta)$ should be negative. Also, from the similarity variable to avoid the contradiction in the initial and boundary conditions, $\gamma = (1-\beta)/(1-2\beta)$ should be positive. From which, we get $\beta > 1$.

Substitution from (12) into (4) yields

$$K_2 \frac{d^2u}{d\theta^2} + [\gamma\theta - K_1 u]\frac{du}{d\theta} - \frac{\beta}{1-2\beta}u = 0. \tag{13}$$

The initial and boundary conditions (5) will be

$$\begin{aligned} &\text{(i)} \quad u = a \quad \text{at} \quad \theta = 0, \\ &\text{(ii)} \quad u \to 0 \quad \text{as} \quad \theta \to \infty. \end{aligned} \tag{14}$$

3.2 Numerical solution

The ordinary differential equation (13) with the appropriate conditions (14) is solved numerically using the shooting method coupled with Runge–Kutta scheme for different values of K_1, K_2, β and a.

Assume $K_1 = K_2 = 1$, equation (13) will be

$$\frac{d^2u}{d\theta^2} + [\gamma\theta - u]\frac{du}{d\theta} - \frac{\beta}{1-2\beta}u = 0. \tag{15}$$

As $\beta \to \infty$, and assume $K_1 = K_2 = 1$, equation (12) will be

$$u \equiv u(\theta), \qquad q(t) = 1/\sqrt{t}, \qquad f(t) = 1, \tag{16}$$

where $\theta = x/\sqrt{t}$.

These solutions are the same as Abd-el-Malek and El-Mansi [10] obtained for the case of unity kinematic viscosity, where the solution shows the existence of shock waves.

Figures 1a and 1b illustrate the effect of the parameters β and a on the velocity field $w(x,t)$ at $K_1 = K_2 = 1$ and $t = 1$ with $a = 4$ and $\beta = 5$, respectively. As seen from these figures, the velocity $w(x,t)$ oscillates and it diminishes with increasing x. The decrease is more rapid near the infinity. Also, it is clear from the two figures the effect of the viscous term $f(t)w_{xx}$. It reduces the amplitude of the wave and prevents multivalued solutions from developing. Figures 1c and 1d illustrate the effect of the parameter β and a on the velocity field $w(x,t)$ at $K_1 =$ and $K_2 = 4$ and $t = 1$ with $a = 4$ and $\beta = 5$, respectively.

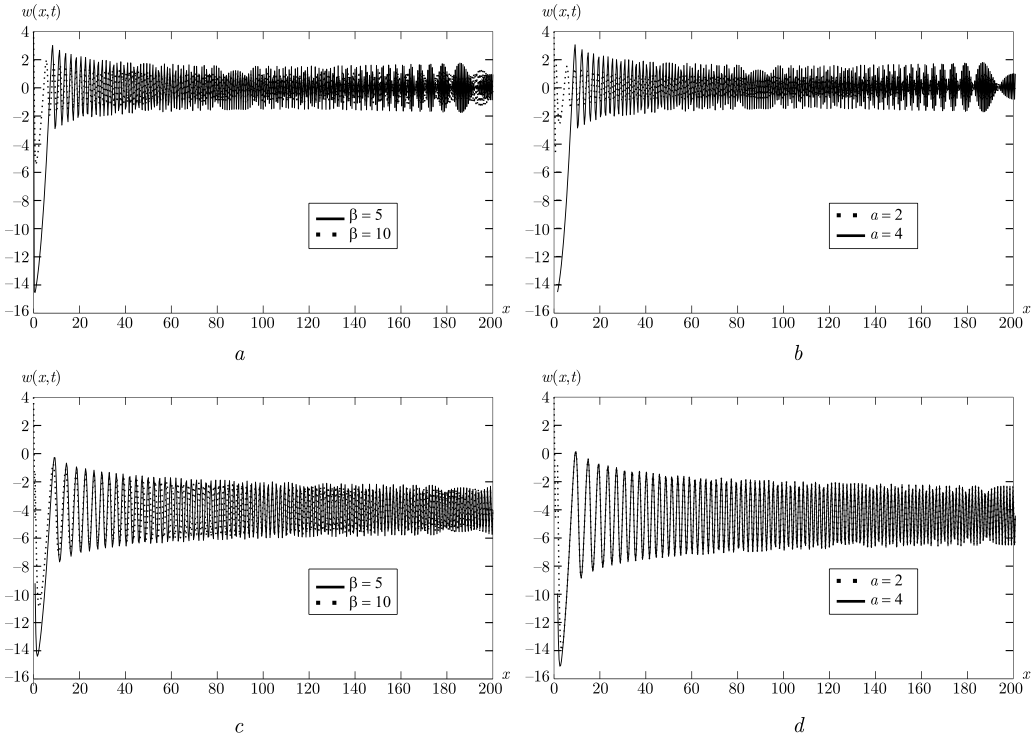

Figure 1. Velocity field profiles (a) with different values of β at $K_1 = K_2 = 1$ and $t = 1$ with $a = 4$; (b) with different values of a at $K_1 = K_2 = 1$ and $t = 1$ with $\beta = 5$; (c) with different values of β at $K_1 = 2$ and $K_2 = 4$ and $t = 1$ with $a = 4$; (d) with different values of a at $K_1 = 2$ and $K_2 = 4$ and $t = 1$ with $\beta = 5$.

4 Conclusion

We have used Lie-group method to obtain the similarity reductions of the Burgers' equation with time-dependent kinematic viscosity. By determining the transformation group under which the given partial differential equation and its initial and boundary conditions are invariant, we obtained the invariants and the symmetries of this equation. In turn, we used these invariants and symmetries to determine the similarity variables that reduced the number of independent variables. The resulting differential equation is solved numerically using shooting method coupled with Runge–Kutta scheme and the results are plotted. We have studied the effect of the parameters β and a on the velocity field $w(x,t)$. We found that, the velocity $w(x,t)$ oscillates and it diminishes with increasing x. The decrease is more rapid near the infinity. Also, the effect of the viscous term $f(t)w_{xx}$ reduces the amplitude of the wave and prevents multivalued solutions from developing. Particular case of our results is compared with those obtained by Abd-el-Malek and El-Mansi [10], it was found in complete agreement.

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[1] Burgers J.M., *A Mathematical Model Illustrating the Theory of Turbulence*, Academic Press, New York, 1948.

[2] Peralta-Fabi R. and Plaschko P., Bifurcation of solutions to the controlled Burgers equation, *Acta Mechanica*, 1993, V.96, 155–161.

[3] Bateman H., Some recent researches on the motion of the fluids, *Mon. Weather Rev.*, 1915, V.43, 163–170.

[4] Cole J.P., On a quasi-linear parabolic equation occurring in aerodynamics, *Quart. Appl. Math.*, 1951, V.9, 225–236.

[5] Hopf E., The partial differential equation $u_t + uu_x = \mu u_{xx}$, *Comm. Pure Appl. Math.*, 1950, V.3, 201–230.

[6] Benton E.R. and Platzman G.W., A table of solutions of the one dimensional Burgers' equation, *Quart. Appl. Math*, 1972, V.30, 195–212.

[7] Ames W.F., *Nonlinear Partial Differential Equations*, Academic Press, New York, 1972.

[8] Vorus W.S., The solution of Burgers' equation for sinusoidal excitation at the upstream boundary, *J. Eng. Math.*, 1989, V.23, 219–237.

[9] Kingston J.G. and Sophocleous C., On point transformations of a generalised Burgers equation, *Phys. Lett. A*, 1991, V.155, 15–19.

[10] Abd-el-Malek M.B. and El-Mani S.M.A., Group theoretic methods applied to Burgers' equation, *J. Comput. Appl. Math.*, 2000, V.115, 1–12.

[11] Fletcher C.A., A comparison of finite element and difference solutions of the one and two dimensional Burgers' equation, *J. Comput. Phys.*, 1983, V.51, 159–188.

[12] Kakuda K. and Tosaks N., The generalized boundary element approach to Burgers' equation, *Internat. J. Numer. Methods Eng.*, 1990, V.29, 245–261.

[13] Mittal R.C. and Singhal P., Numerical solution of Burgers' equation, *Comm. Numer. Methods Eng.*, 1993, V.9, 397–406.

[14] Mittal R.C. and Singhal P., Numerical solution of periodic Burgers' equation, *Indian J. Pure Appl. Math.*, 1996, V.27, 689–700.

[15] Kutluay S., Bahadir A.R., and Özdeş A., Numerical solution of one-dimensional Burgers equation: explicit and exact-explicit finite difference methods, *J. Comput. Appl. Math.*, 1999, V.103, 251–261.

[16] Öziş T., Aksan E.N. and Özdeş A., A finite element approach for solution of Burgers' equation, *Appl. Math. Comput.*, 2003, V. 139, 417–428.

[17] Kutluay S., Esen A. and Dağ İ., Numerical solutions of the Burgers' equation by the least-squares quadratic B-spline finite element method, *J. Comput. Appl. Math.*, 2004, V.167, 21–33.

[18] Aksan E.N. and Özdeş A., A numerical solution of Burgers' equation, *Appl. Math. Comput.*, 2004, V.156, 395–402.

[19] Aksan E.N., A numerical solution of Burgers' equation by finite element method constructed on the method of discretization in time, *Appl. Math. Comput.*, 2005, V.170, 895–904.

[20] Xie S., Heo S., Kim S., Woo G. and Yi S., Numerical solution of one-dimensional Burgers' equation using reproducing kernel function, *J. Comput. Appl. Math.*, 2008, V.214, 417–434.

[21] Saka B., and Dağ İ., A numerical study of the Burgers' equation, *J. Franklin Inst.*, 2008, V.345, 328–348.

[22] Hill J.M., *Solution of differential equations by means of one-parameter groups*, Pitman Publishing Co., 1982.

[23] Seshadri R. and Na T.Y., *Group invariance in engineering boundary value problems*, Springer-Verlag, New York, 1985.

[24] Ibragimov N.H., *Elementary Lie group analysis and ordinary differential equations*, Wiley, New York, 1999.

[25] Yi Z. and Fengxiang M., Lie symmetries of mechanical systems with unilateral holonomic constraints, *Chinese Sci. Bull.*, 2000, V.45, 1354–1358.

[26] Hydon P.E., *Symmetry methods for differential equations*, CUP, Cambridge, 2000.

[27] Burde G.I., Expanded Lie group transformations and similarity reductions of differential equations, *Proceedings of Fourth Conference Symmetry in Nonlinear Mathematical Physics (Kiev, Ukraine, 2001)*, 2002, 93–101.

[28] Olver P.J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[29] Boutros Y.Z., Abd-el-Malek M.B., Badran N.A. and Hassan H.S., Lie-group method for unsteady flows in a semi-infinite expanding or contracting pipe with injection or suction through a porous wall, *J. Comput. Appl. Math.*, 2006, V.197, 465–494.

[30] Boutros Y.Z., Abd-el-Malek M.B., Badran N.A. and Hassan H.S., Lie-group method of solution for steady two-dimensional boundary-layer stagnation-point flow towards a heated stretching sheet placed in a porous medium, *Meccanica*, 2007, V.41, 681–691.

[31] Boutros Y.Z., Abd-el-Malek M.B., Badran N.A. and Hassan H.S., Lie-group method solution for two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability, *Appl. Math. Model.*, 2007, V.31, 1092–1108.

[32] Lighthill M.J., Viscosity effects in sound waves of finite amplitude, in *Surveys in Mechanics*, Cambridge University Press, 1956, 250–351.

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Point symmetry group of the barotropic vorticity equation

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The complete point symmetry group of the barotropic vorticity equation on the β-plane is computed using the direct method supplemented with two different techniques. The first technique is based upon the preservation of any megaideal of the maximal Lie invariance algebra of a differential equation by the push-forwards of point symmetries of the same equation. The second technique involves a priori knowledge on normalization properties of a class of differential equations containing the equation under consideration. Both of these techniques are briefly outlined.

1 Introduction

It is well known that it is much easier to determine the continuous part of the complete point symmetry group of a differential equation than the entire group including discrete symmetries. The computation of continuous (Lie) symmetries is possible using infinitesimal techniques, which amounts to solving an overdetermined system of linear partial differential equations (referred to as *determining equations*) for coefficients of vector fields generating one-parameter Lie symmetry groups. Owing to the algorithmic nature of this problem, the automatic computation of Lie symmetries is already implemented in a number of symbolic calculation packages, see, e.g., papers [7, 9, 29] for detailed descriptions of certain packages and reviews [6, 10].

The relative simplicity of finding Lie symmetries of differential equations is also a primary reason why the overwhelming part of research on symmetries is devoted to symmetries of this kind. See, e.g., the textbooks [4,5,19–21] for general theory and numerous examples and additionally the works [1–3, 8, 18] for several applications of Lie methods in hydrodynamics and meteorology.

As is the case with continuous symmetries, also discrete symmetries are of practical relevance in a number of fields such as dynamical systems theory, quantum mechanics, crystallography and solid state physics. They can also be helpful in some issues related to Lie symmetries, e.g. allowing for a simplification of optimal lists of inequivalent subalgebras, and due to enabling the construction of new solutions of differential equations from known ones. It is not possible, in general, to determine the whole point symmetry group in terms of finite transformations by usage of infinitesimal techniques. On the other hand the direct computation of point symmetries based upon their definition boils down to solving a cumbersome nonlinear system of determining equations, which is difficult to be integrated. Similar determining equations also arise under calculations of equivalence groups and sets of admissible transformations of classes of differential equations by means of employing the direct method. In order to simplify the derivation of the determining equations, different special techniques have been developed involving, in particular, the implicit representation of unknown functions, the combined splitting with respect to old and new variables and the inverse expression of old derivative via new ones [23, 25, 28].

There exist two particular techniques that can be applied for *a priori* simplification of calculations concerning the point symmetry groups of differential equations.

The first technique was presented in [11] for equations the maximal Lie invariance algebras of whhich are finite dimensional. It is based on the fact that the push-forwards of point symmetries of a given system of differential equations to vector fields on the space of dependent and independent variables are automorphisms of the maximal Lie invariance algebra of the same system. This condition yields restrictions for those point transformations that can qualify as symmetries of the system of differential equations under consideration. We adapt this technique to the infinite-dimensional case using the notion of megaideals of Lie algebras, which are the most invariant algebraic structures.

The second technique involves available information on the set of admissible transformations of a class of differential equations [25], which contains the investigated equation.

In the present paper we demonstrate both of these techniques by computing the complete point symmetry group of the barotropic vorticity equation on the β-plane. This is one of the most classical models which are used in geophysical fluid dynamics. The techniques to be employed are briefly described in Section 2. The actual computations using the method based on the corresponding Lie invariance algebra and that involving a priori knowledge on admissible transformations of a class of generalized vorticity equations are presented in Section 3 and 4, respectively. A short summary concludes the paper.

2 Techniques of calculation of complete point symmetry groups

Both the techniques described in this section should be considered merely as tools for deriving preliminary restrictions on point symmetries. In either case calculations must finally be carried out within the framework of the direct approach.

2.1 Using megaideals of Lie invariance algebra

The most refined version of the technique involving Lie symmetries in the calculations of complete point symmetry groups was applied in [11]. It is outlined as follows: Given a system of differential equations $\mathcal{L}$ the maximal Lie invariance algebra of which, $\mathfrak{g}$, is n-dimensional with a basis $\{e_1,\dots,e_n\}$, $n<\infty$, one has to compute the entire automorphism group of $\mathfrak{g}$, $\mathrm{Aut}(\mathfrak{g})$. Supposing that $\mathcal{T}$ is a transformation from the complete point symmetry group G of $\mathcal{L}$, one has the condition $\mathcal{T}_*e_j=\sum_{i=1}^n e_ia_{ij}$ for $j=1,\dots,n$, where $\mathcal{T}_*$ denotes the push-forward of vector fields induced by $\mathcal{T}$ and (a_{ij}) is the matrix of an automorphism of $\mathfrak{g}$ in the chosen basis. This condition implies constraints on the transformation $\mathcal{T}$ which are then taken into account in further calculations with the direct method.

The method we propose here is different to those described in the previous paragraph. In fact it uses only the minimal information on the automorphism group $\mathrm{Aut}(\mathfrak{g})$ in the form of a set of megaideals of $\mathfrak{g}$. Due to this it is applicable also in the case for which the maximal Lie invariance algebra is infinite dimensional. The notion of megaideals was introduced in [24].

Definition 1. A *megaideal* $\mathfrak{i}$ is a vector subspace of $\mathfrak{g}$ that is invariant under any transformation from the automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$.

That is, we have $\mathfrak{T}\mathfrak{i}=\mathfrak{i}$ for a megaideal $\mathfrak{i}$ of $\mathfrak{g}$ whenever $\mathfrak{T}$ is a transformation from $\mathrm{Aut}(\mathfrak{g})$. Any megaideal of $\mathfrak{g}$ is an ideal and characteristic ideal of $\mathfrak{g}$. Both the improper subalgebras of $\mathfrak{g}$ (the zero subspace and $\mathfrak{g}$ itself) are megaideals of $\mathfrak{g}$. The following assertions are obvious.

Proposition 1. *If* $\mathfrak{i}_1$ *and* $\mathfrak{i}_2$ *are megaideals of* $\mathfrak{g}$*, then so are* $\mathfrak{i}_1+\mathfrak{i}_2$, $\mathfrak{i}_1\cap\mathfrak{i}_2$ *and* $[\mathfrak{i}_1,\mathfrak{i}_2]$*, i.e., sums, intersections and Lie products of megaideals are again megaideals.*

Proposition 2. *If* $\mathfrak{i}_2$ *is a megaideal of* $\mathfrak{i}_1$ *and* $\mathfrak{i}_1$ *is a megaideal of* $\mathfrak{g}$*, then* $\mathfrak{i}_2$ *is a megaideal of* $\mathfrak{g}$*, i.e., megaideals of megaideals are also megaideals.*

Corollary 1. *All elements of the derived, upper and lower central series of a Lie algebra are its megaideals. In particular the center and the derivative of a Lie algebra are its megaideals.*

Corollary 2. *The radical* $\mathfrak{r}$ *and nil-radical* $\mathfrak{n}$ *(i.e. the maximal solvable and nilpotent ideals, respectively) of* $\mathfrak{g}$ *as well as different Lie products, sums and intersections involving* $\mathfrak{g}$, $\mathfrak{r}$ *and* $\mathfrak{n}$ *(*$[\mathfrak{g},\mathfrak{r}]$, $[\mathfrak{r},\mathfrak{r}]$, $[\mathfrak{g},\mathfrak{n}]$, $[\mathfrak{r},\mathfrak{n}]$, $[\mathfrak{n},\mathfrak{n}]$ *etc.) are megaideals of* $\mathfrak{g}$.

Suppose that $\mathfrak{g}$ is finite dimensional and possesses a megaideal $\mathfrak{i}$ which, without loss of generality, can be assumed to be spanned by the first k basis elements, $\mathfrak{i}=\langle e_1,\dots,e_k\rangle$. Then the matrix (a_{ij}) of any automorphism of $\mathfrak{g}$ has block structure, namely, $a_{ij}=0$ for $i>k$. In other words in the finite-dimensional case we take into account only the block structure of automorphism matrices. This is reasonable as the entire automorphism group $\mathrm{Aut}(\mathfrak{g})$ (which should be computed within the method from [11]) may be much wider than the group of automorphisms of $\mathfrak{g}$ induced by elements of the point symmetry group G of $\mathcal{L}$. Moreover it seems difficult to find the entire group $\mathrm{Aut}(\mathfrak{g})$ if the algebra $\mathfrak{g}$ is infinite dimensional. At the same time, in view of the above assertions, it is easy to determine a set of megaideals for any Lie algebra.

2.2 Direct method and admissible transformations

The initial point of the second technique is to consider a given pth-order system $\mathcal{L}^0$ of l differential equations for m unknown functions $u=(u^1,\dots,u^m)$ of n independent variables $x=(x_1,\dots,x_n)$ as an element of a class $\mathcal{L}|_\mathcal{S}$ of similar systems $\mathcal{L}_\theta$: $L(x,u_{(p)},\theta(x,u_{(p)}))=0$ parameterized by a tuple of pth-order differential functions (arbitrary elements) $\theta=(\theta^1(x,u_{(p)}),\dots,\theta^k(x,u_{(p)}))$. Here $u_{(p)}$ denotes the set of all the derivatives of u with respect to x of order not greater than p, including u as the derivatives of order zero. The class $\mathcal{L}|_\mathcal{S}$ is determined by two objects: the tuple $L=(L^1,\dots,L^l)$ of l fixed functions depending upon x, $u_{(p)}$ and θ and θ running through the set $\mathcal{S}$. Within the framework of symmetry analysis of differential equations, the set $\mathcal{S}$ is defined as the set of solutions of an auxiliary system consisting of a subsystem $S(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))=0$ of differential equations with respect to θ and a no vanish condition $\Sigma(x,u_{(p)},\theta_{(q)}(x,u_{(p)}))\neq 0$ with another differential function Σ of θ. In the auxiliary system x and $u_{(p)}$ play the role of independent variables and $\theta_{(q)}$ stands for the set of all the partial derivatives of θ of order not greater than q with respect to the variables x and $u_{(p)}$. In view of the purpose of our consideration we should have that $\mathcal{L}^0=\mathcal{L}_{\theta_0}$ for some $\theta_0\in\mathcal{S}$.

Following [25], for $\theta,\tilde\theta\in\mathcal{S}$ we denote by $\mathrm{T}(\theta,\tilde\theta)$ the set of point transformations which map the system $\mathcal{L}_\theta$ to the system $\mathcal{L}_{\tilde\theta}$. The maximal point symmetry group G_θ of the system $\mathcal{L}_\theta$ coincides with $\mathrm{T}(\theta,\theta)$.

Definition 2. $\mathrm{T}(\mathcal{L}|_\mathcal{S})=\{(\theta,\tilde\theta,\varphi)\mid \theta,\tilde\theta\in\mathcal{S},\ \varphi\in\mathrm{T}(\theta,\tilde\theta)\}$ is called the *set of admissible transformations in* $\mathcal{L}|_\mathcal{S}$.

Sets of admissible transformations were first systematically described by Kingston and Sophocleous for a class of generalized Burgers equations [14] and Winternitz and Gazeau for a class of variable coefficient Korteweg–de Vries equations [30], in terms of *form-preserving* [14–16] and *allowed* [30] transformations, respectively. The notion of admissible transformations can be considered as a formalization of their approaches.

Any point symmetry transformation of an equation $\mathcal{L}_\theta$ from the class $\mathcal{L}|_\mathcal{S}$ generates an admissible transformation in this class. Therefore it obviously satisfies all restrictions which hold for admissible transformations [15]. For example, it is known for a long time that for any point (and even contact) transformation connecting a pair of $(1+1)$-dimensional evolution equations its component corresponding to t depends only upon t, cf. [17]. The equations in the pair can also coincide. As a result the same restriction should be satisfied by any point or contact symmetry transformation of every $(1+1)$-dimensional evolution equation.

The simplest description of admissible transformations is obtained for normalized classes of differential equations. Roughly speaking a class of (systems of) differential equations is called *normalized* if any admissible transformation in this class is induced by a transformation from its equivalence group. Different kinds of normalization can be defined depending upon what kind of equivalence group (point, contact, usual, generalized, extended etc.) is considered. Thus the *usual equivalence group* $G^{\sim}$ of the class $\mathcal{L}|_{\mathcal{S}}$ consists of those point transformations in the space of variables and arbitrary elements, which are projectable onto the variable space and preserve the whole class $\mathcal{L}|_{\mathcal{S}}$. The class $\mathcal{L}|_{\mathcal{S}}$ is called normalized in the usual sense if the set $\mathrm{T}(\mathcal{L}|_{\mathcal{S}})$ is generated by the usual equivalence group $G^{\sim}$. As a consequence all generalizations of the equivalence group within the framework of point transformations are trivial for this class. See [25] for precise definitions and further explanations. If the class $\mathcal{L}|_{\mathcal{S}}$ is normalized in a certain sense with respect to point transformations, the point symmetry group G_{θ_0} of any equation $\mathcal{L}_{\theta_0}$ from this class is contained in the projection of the corresponding equivalence group of $\mathcal{L}|_{\mathcal{S}}$ to the space of independent and dependent variables (taken for the value $\theta = \theta_0$ in the case when the generalized equivalence group is considered).

As a rule calculations of certain common restrictions on admissible transformations of the entire normalized class or its normalized subclasses or point symmetry transformations of a single equation from this class have the same level of complexity. For example, in order to derive the restriction that the transformation component corresponding to t depends only upon t, we should perform approximately the same operations, independently of considering the whole class of $(1+1)$-dimensional evolution equations, on any well-defined subclass from this class or any single evolution equation. This is why it is worthwhile firstly to construct nested series of normalized classes of differential equations by starting from a quite general, obviously normalized class, imposing on each step additional auxiliary conditions on the arbitrary elements and then studying the complete point symmetries of a single equation from the narrowest class of the constructed series.

In the way outlined above we have already investigated hierarchies of normalized classes of generalized nonlinear Schrödinger equations [25], (1+1)-dimensional linear evolution equations [26], (1+1)-dimensional third-order evolution equations including variable-coefficient Korteweg–de Vries and modified Korteweg–de Vries equations [27] and generalized vorticity equations arising in the study of local parameterization schemes for the barotropic vorticity equation [23].

If an equation does not belong to a class the admissible transformations of which have been studied earlier, one can try to map this equation using a point transformation to an equation from a class for which constraints on its admissible transformations are known a priori. Then one can either map the known constraints on admissible transformations back and then complete the calculations of point symmetries of the initial equation using the direct method or calculate the point symmetry group of the mapped equation using the direct method and then map this group back. The example on the application of this trick to the barotropic vorticity equation in presented in Section 4.

3 Calculations based on Lie invariance algebra of the barotropic vorticity equation

The barotropic vorticity equation on the β-plane is

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x = 0, \tag{1}$$

where $\psi = \psi(t, x, y)$ is the stream function and $\zeta := \psi_{xx} + \psi_{yy}$ is the relative vorticity, which is the vertical component of the vorticity vector. The barotropic vorticity equation in the formulation (1) is valid in situations in which the two-dimensional wind field can be regarded as almost nondivergent and the motion in North–South direction is confined to a relatively small region. It is then convenient to use a local Cartesian coordinate system. In such a coordinate system the effect of the sphericity of the Earth is conveniently taken into account by approximating the normal component of the vorticity due to the rotation of the Earth, $2\Omega\sin\varphi$, by its linear Taylor series expansion, where Ω is the angular rotation of the Earth and φ is the geographic latitude. This linear approximation at some reference latitude φ_0 is given by $2\Omega\sin\varphi_0 + \beta y$, where $\beta = 2\Omega\cos\varphi/a$ and a is the radius of the Earth. This is the traditional β-plane approximation, see [22] for further details. Then the taking of the vertical component of the curl of the two-dimensional ideal Euler equations and use of the β-plane approximation lead to Eq. (1).

It is straightforward to determine the maximal Lie invariance algebra $\mathfrak{g}$ of Eq. (1) using infinitesimal techniques:

$$\mathfrak{g} = \langle \mathcal{D}, \partial_t, \partial_y, \mathcal{X}(f), \mathcal{Z}(g)\rangle,$$

where $\mathcal{D} = t\partial_t - x\partial_x - y\partial_y - 3\psi\partial_\psi$, $\mathcal{X}(f) = f(t)\partial_x - f_t(t)y\partial_\psi$ and $\mathcal{Z}(g) = g(t)\partial_\psi$, and f and g run through the space of smooth functions of t. (In fact the precise interpretation of $\mathfrak{g}$ as a Lie algebra strongly depends on what space of smooth functions is chosen for f and g, cf. Note A.1 in [8, p. 178].) This result was first obtained in [13] and is now easily accessible in the handbook [12, p. 223]. See also [2] for related discussions and the exhaustive study of the classical Lie reductions of Eq. (1).

The nonzero commutation relations of the algebra $\mathfrak{g}$ in the above basis are

$$\begin{aligned}
&[\partial_t, \mathcal{D}] = \partial_t, \quad [\partial_y, \mathcal{D}] = -\partial_y,\\
&[\mathcal{D}, \mathcal{X}(f)] = \mathcal{X}(tf_t + f), \quad [\mathcal{D}, \mathcal{Z}(g)] = \mathcal{Z}(tg_t + 3g),\\
&[\partial_t, \mathcal{X}(f)] = \mathcal{X}(f_t), \quad [\partial_t, \mathcal{Z}(g)] = \mathcal{Z}(g_t), \quad [\partial_y, \mathcal{X}(f)] = -\mathcal{Z}(f_t).
\end{aligned}$$

It is easy to see from the commutation relations that the Lie algebra $\mathfrak{g}$ is solvable since

$$\begin{aligned}
&\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}] = \langle \partial_t, \partial_y, \mathcal{X}(f), \mathcal{Z}(g)\rangle,\\
&\mathfrak{g}'' = [\mathfrak{g}', \mathfrak{g}'] = \langle \mathcal{X}(f), \mathcal{Z}(g)\rangle,\\
&\mathfrak{g}''' = [\mathfrak{g}'', \mathfrak{g}''] = 0.
\end{aligned}$$

Therefore the radical $\mathfrak{r}$ of $\mathfrak{g}$ coincides with the entire algebra $\mathfrak{g}$. The nil-radical of $\mathfrak{g}$ is the ideal

$$\mathfrak{n} = \langle \partial_y, \mathcal{X}(f), \mathcal{Z}(g) \rangle.$$

Indeed this ideal is a nilpotent subalgebra of $\mathfrak{g}$ since

$$\mathfrak{n}^{(2)} = \mathfrak{n}' = [\mathfrak{n}, \mathfrak{n}] = \langle \mathcal{Z}(g) \rangle, \quad \mathfrak{n}^{(3)} = [\mathfrak{n}, \mathfrak{n}'] = 0.$$

It can be extended to a larger ideal of $\mathfrak{g}$ only with two sets of elements, $\{\partial_t\}$ and $\{\mathcal{D}, \partial_t\}$. Both resulting ideals are not nilpotent. In other words $\mathfrak{n}$ is the maximal nilpotent ideal.

Continuous point symmetries of Eq. (1) are determined from the elements of $\mathfrak{g}$ by integration of the associated Cauchy problems. It is obvious that Eq. (1) also possesses two discrete symmetries, $(t,x,y,\psi) \mapsto (-t,-x,y,\psi)$ and $(t,x,y,\psi) \mapsto (t,x,-y,-\psi)$, which are independent up to their composition and their compositions with continuous symmetries. The proof that the above symmetries generate the entire point symmetry group was, however, outstanding.

Theorem 1. *The complete point symmetry group of the barotropic vorticity equation on the β-plane* (1) *is formed by the transformations*

$$\mathcal{T}\colon \quad \tilde t = T_1 t + T_0, \quad \tilde x = \frac{1}{T_1}x + f(t), \quad \tilde y = \frac{\varepsilon}{T_1}y + Y_0,$$
$$\tilde\psi = \frac{\varepsilon}{(T_1)^3}\psi - \frac{\varepsilon}{(T_1)^2}f_t(t)y + g(t),$$

where $T_1 \neq 0$, $\varepsilon = \pm 1$ and f and g are arbitrary functions of t.

Proof. The discrete symmetries of the barotropic vorticity equation on the β-plane are computed as described in Section 2.1. The general form of a point transformation of the vorticity equation is:

$$\mathcal{T}\colon \quad (\tilde t, \tilde x, \tilde y, \tilde\psi) = (T, X, Y, \Psi),$$

where T, X, Y and Ψ are regarded as functions of t, x, y and ψ, the joint Jacobian of which does not vanish. To obtain the constrained form of $\mathcal{T}$ we use the above four proper nested megaideals of $\mathfrak{g}$, namely $\mathfrak{n}'$, $\mathfrak{g}''$, $\mathfrak{n}$ and $\mathfrak{g}'$, and $\mathfrak{g}$ itself. Recall once more that the transformation $\mathcal{T}$ must satisfy the conditions $\mathcal{T}_*\mathfrak{n}' = \mathfrak{n}'$, $\mathcal{T}_*\mathfrak{g}'' = \mathfrak{g}''$, $\mathcal{T}_*\mathfrak{n} = \mathfrak{n}$, $\mathcal{T}_*\mathfrak{g}' = \mathfrak{g}'$ and $\mathcal{T}_*\mathfrak{g} = \mathfrak{g}$ in order to qualify as a point symmetry of the vorticity equation, where $\mathcal{T}_*$ denotes the push-forward of $\mathcal{T}$ to vector fields. In other words we have

$$\mathcal{T}_*\mathcal{Z}(g) = g(T_\psi\partial_{\tilde t} + X_\psi\partial_{\tilde x} + Y_\psi\partial_{\tilde y} + \Psi_\psi\partial_{\tilde\psi}) = \tilde{\mathcal{Z}}(\tilde g^g), \tag{2}$$

$$\mathcal{T}_*\mathcal{X}(f) = \tilde{\mathcal{X}}(\tilde f^f) + \tilde{\mathcal{Z}}(\tilde g^f), \tag{3}$$

$$\mathcal{T}_*\partial_t = T_t\partial_{\tilde t} + X_t\partial_{\tilde x} + Y_t\partial_{\tilde y} + \Psi_t\partial_{\tilde\psi} = a_1\partial_{\tilde t} + a_2\partial_{\tilde y} + \tilde{\mathcal{X}}(\tilde f) + \tilde{\mathcal{Z}}(\tilde g), \tag{4}$$

$$\mathcal{T}_*\partial_y = T_y\partial_{\tilde t} + X_y\partial_{\tilde x} + Y_y\partial_{\tilde y} + \Psi_y\partial_{\tilde\psi} = b_1\partial_{\tilde y} + \tilde{\mathcal{X}}(\tilde f^y) + \tilde{\mathcal{Z}}(\tilde g^y), \tag{5}$$

$$\mathcal{T}_*\mathcal{D} = c_1\tilde{\mathcal{D}} + c_2\partial_{\tilde t} + c_3\partial_{\tilde y} + \tilde{\mathcal{X}}(\tilde f^D) + \tilde{\mathcal{Z}}(\tilde g^D), \tag{6}$$

where all $\tilde{f}$'s and $\tilde{g}$'s are smooth functions of $\tilde{t}$ which are determined, as the constant parameters a_1, a_2, b_1, c_1, c_2 and c_3, by $\mathcal{T}_*$ and the operator from the corresponding left-hand side.

We derive constraints on $\mathcal{T}_*$, consequently equating coefficients of vector fields in conditions (2)–(6) and taking into account constraints obtained on previous steps. Thus Eq. (2) immediately implies $T_\psi = X_\psi = Y_\psi = 0$ (hence $\Psi_\psi \neq 0$) and $g\Psi_\psi = \tilde{g}^g$. Evaluation of the last equation for $g = 1$ and $g = t$ and combination of the results give $t = \tilde{g}^t(T)/\tilde{g}^1(T)$, where $\tilde{g}^t = \tilde{g}^g|_{g=t}$ and $\tilde{g}^1 = \tilde{g}^g|_{g=1}$. As the derivative with respect to T in the right hand side of this equality does not vanish, the condition $T = T(t)$ must hold. This implies that Ψ_ψ depends only upon t.

As then $\mathcal{T}_*\mathcal{X}(f) = fX_x\partial_{\tilde{x}} + fY_x\partial_{\tilde{y}} + (f\Psi_x - f_ty\Psi_\psi)\partial_{\tilde{\psi}}$, it follows from Eq. (3) that $Y_x = 0$ and

$$fX_x = \tilde{f}^f, \quad f\Psi_x - f_ty\Psi_\psi = -\tilde{f}^f_{\tilde{t}}Y + \tilde{g}^f.$$

Evaluating the first of the displayed equalities for $f = 1$, we derive that $X_x = \tilde{f}^1(T) =: X_1(t)$. Therefore $\tilde{f}^f(T) = f(t)X_1(t)$. The second equality then becomes

$$f\Psi_x - f_ty\Psi_\psi = -\frac{(fX_1)_t}{T_t}Y + \tilde{g}^f.$$

The setting of $f = 1$ and $f = t$ in the last equality and combination of the resulting equalities yield $y\Psi_\psi = (T_t)^{-1}X_1Y + t\tilde{g}^1 - \tilde{g}^t$, where $\tilde{g}^t = \tilde{g}^f|_{f=t}$ and $\tilde{g}^1 = \tilde{g}^f|_{f=1}$. As $X_1 \neq 0$, this equation implies that $Y = Y_1(t)y + Y_0(t)$.

After analyzing Eq. (4) we find $T_t = \text{const}$, $Y_t = \text{const}$, which leads to $Y_1 = \text{const}$, $X_t = \tilde{f}(T)$ and thus $X_{tx} = 0$, i.e., $X_1 = \text{const}$. Finally Eq. (4) also implies $\Psi_t = -\tilde{f}_{\tilde{t}}Y + \tilde{g}$. In a similar manner, after we take into account the restrictions already derived so far, collection of coefficients in Eq. (5) gives the constraint $X_y = \tilde{f}^y =: X_2 = \text{const}$ since $X_{yt} = 0$. Moreover $\Psi_y = \tilde{g}^y$ as $\tilde{f}^y_{\tilde{t}} = 0$.

The final restrictions on $\mathcal{T}$ based on the preservation of $\mathfrak{g}$ are derivable from Eq. (6), where

$$\begin{aligned}\mathcal{T}_*\mathcal{D} = {} & tT_t\partial_{\tilde{t}} + (tX_t - xX_x - yX_y)\partial_{\tilde{x}} + (tY_t - yY_y)\partial_{\tilde{y}} \\ & + (t\Psi_t - x\Psi_x - y\Psi_y - 3\psi\Psi_\psi)\partial_{\tilde{\psi}}.\end{aligned}$$

Collecting the coefficients of $\partial_{\tilde{t}}$ and $\partial_{\tilde{y}}$, we obtain that $c_1 = 1$ and $Y_t = 0$. Similarly, equation of the coefficients of $\partial_{\tilde{\psi}}$ and the further splitting with respect to x implie that $\Psi_x = 0$.

The results obtained so far lead to the following constrained form of the general point symmetry transformation of the vorticity equation (1)

$$\begin{aligned}& T = T_1t + T_0, \quad X = X_1x + X_2y + f(t), \quad Y = Y_1y + Y_0, \\ & \Psi = \Psi_1\psi + \Psi_2(t)y + \Psi_4(t),\end{aligned} \tag{7}$$

where T_0, T_1, X_1, X_2, Y_0, Y_1 and Ψ_1 are arbitrary constants, $T_1X_1Y_1\Psi_1 \neq 0$, and $f(t)$, $\Psi_2(t)$ and $\Psi_4(t)$ are arbitrary time-dependent functions. The form (7) takes into account all constraints on point symmetries of (1), which follow from the preservation of the maximal Lie invariance algebra $\mathfrak{g}$ by the associated push-forward of vector fields.

Now the direct method should be applied. We carry out a transformation of the form (7) in the vorticity equation. For this aim we calculate the transformation rules for the partial derivative operators:

$$\partial_{\tilde t} = \frac{1}{T_1}\left(\partial_t - \frac{f_t}{X_1}\partial_x\right), \quad \partial_{\tilde x} = \frac{1}{X_1}\partial_x, \quad \partial_{\tilde y} = \frac{1}{Y_1}\left(\partial_y - \frac{X_2}{X_1}\partial_x\right).$$

Further restrictions on $\mathcal{T}$ can be imposed upon noting that the term ψ_{txy} can only arise in the expression for $\tilde\psi_{\tilde t\tilde y\tilde y}$, which is

$$\tilde\psi_{\tilde t\tilde y\tilde y} = -\frac{2\Psi_1}{T_1Y_1}\frac{X_2}{X_1}\psi_{txy} + \cdots .$$

This obviously implies that $X_2 = 0$. In a similar fashion the expression for $\tilde\zeta_{\tilde t}$ is

$$\tilde\zeta_{\tilde t} = \frac{\Psi_1}{T_1}\left(\frac{1}{(X_1)^2}\zeta_t + \left(\frac{1}{(Y_1)^2} - \frac{1}{(X_1)^2}\right)\psi_{yyt}\right) + \cdots ,$$

upon using $\psi_{xxt} = \zeta_t - \psi_{yyt}$. Hence $(X_1)^2 = (Y_1)^2$ as there are no other terms with ψ_{yyt} in the invariance condition. After taking into account these two more restrictions on $\mathcal{T}$, it is straightforward to expand the transformed version of the vorticity equation. This yields

$$\frac{\Psi_1}{T_1(X_1)^2}\zeta_t - \frac{f_t\Psi_1}{T_1(X_1)^3}\zeta_x + \frac{(\Psi_1)^2}{(X_1)^3Y_1}\psi_x\zeta_y - \left(\frac{\Psi_1}{Y_1}\psi_y + \frac{\Psi_2}{Y_1}\right)\frac{\Psi_1}{(X_1)^3}\zeta_x$$
$$+\beta\frac{\Psi_1}{X_1}\psi_x = \frac{\Psi_1}{T_1(X_1)^2}\left(\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x + \beta\psi_x\right).$$

The invariance condition is fulfilled provided that the constraints

$$\Psi_2 = -\frac{Y_1}{T_1}f_t, \quad X_1 = T_1(X_1)^2, \quad \frac{(\Psi_1)^2}{(X_1)^3Y_1} = \frac{\Psi_1}{T_1(X_1)^2}$$

hold. This completes the proof of the theorem. ∎

Corollary 3. *The barotropic vorticity equation on the β-plane possesses only two independent discrete point symmetries, which are given by*

$$\Gamma_1\colon (t,x,y,\psi)\mapsto(-t,-x,y,\psi), \quad \Gamma_2\colon (t,x,y,\psi)\mapsto(t,x,-y,-\psi).$$

They generate the group of discrete symmetry transformations of the barotropic vorticity equation on the β-plane, which is isomorphic to $\mathbb{Z}^2\times\mathbb{Z}^2$, where $\mathbb{Z}^2$ denotes the cyclic group of two elements.

4 Direct method and admissible transformations of classes of generalized vorticity equations

The construction of the complete point symmetry group G of the barotropic vorticity equation (1) by means of using only the direct method involves cumbersome and sophisticated calculations. As Eq. (1) is a third-order PDE in three independent variables, the system of determining equations for transformations from G is an overdetermined nonlinear system of PDEs in four independent variables, which should be solved by taking into account the nonsingularity condition of the point transformations. This is an extremely challenging task. Fortunately a hierarchy of normalized classes of generalized vorticity equations was recently constructed [23] that allows us to simplify strongly the whole investigation. Eq. (1) belongs only to the narrowest class of this hierarchy, which is quite wide and consists of equations of the general form

$$\zeta_t = F(t, x, y, \psi, \psi_x, \psi_y, \zeta, \zeta_x, \zeta_y, \zeta_{xx}, \zeta_{xy}, \zeta_{yy}), \quad \zeta := \psi_{xx} + \psi_{yy}, \tag{8}$$

where $(F_{\zeta_x}, F_{\zeta_y}, F_{\zeta_{xx}}, F_{\zeta_{xy}}, F_{\zeta_{yy}}) \neq (0,0,0,0,0)$. The equivalence group $G_1^\sim$ of this class is formed by the transformations

$$\tilde t = T(t), \quad \tilde x = Z^1(t,x,y), \quad \tilde y = Z^2(t,x,y), \quad \tilde\psi = \Upsilon(t)\psi + \Phi(t,x,y),$$
$$\tilde F = \frac{1}{T_t}\left(\frac{\Upsilon}{L}F + \left(\frac{\Upsilon}{L}\right)_0\zeta + \left(\frac{\Phi_{ii}}{L}\right)_0 - \frac{Z^i_tZ^i_j}{L}\left(\frac{\Upsilon}{L}\zeta_j + \left(\frac{\Upsilon}{L}\right)_j\zeta + \left(\frac{\Phi_{ii}}{L}\right)_j\right)\right),$$

where T, Z^i, Υ and Φ are arbitrary smooth functions of their arguments, satisfying the conditions $Z^1_kZ^2_k = 0$, $Z^1_kZ^1_k = Z^2_kZ^2_k := L$ and $T_t\Upsilon L \neq 0$. The subscripts 1 and 2 denote differentiation with respect to x and y, respectively, the indices i and j run through the set $\{1,2\}$ and the summation over repeated indices is understood. As Eq. (1) is an element of the class (8) and this class is normalized, the point symmetry group G of Eq. (1) is contained in the projection $\hat G_1^\sim$ of the equivalence group $G_1^\sim$ of the class (8) to the variable space (t,x,y,ψ). At the same time the group G is much narrower than the group $\hat G_1^\sim$, and in order to single out G from $\hat G_1^\sim$ we should still derive and solve a quite cumbersome system of additional constraints. Instead of this we use the trick described in the end of Section 2.2, namely, by the transformation

$$\check\psi = \psi + \frac{\beta}{6}y^3, \tag{9}$$

which identically acts on the independent variables and which is prolonged to the vorticity according to the formula $\check\zeta = \zeta + \beta y$, we map Eq. (1) to the equation

$$\check\zeta_t + \check\psi_x\check\zeta_y - \check\psi_y\check\zeta_x = -\frac{\beta}{2}y^2\check\zeta_x. \tag{10}$$

Eq. (10) belongs to the subclass of class (8) that is singled out by the constraints $F_\psi = 0$, $F_\zeta = 0$, $F_{\psi_x} = -\zeta_y$ and $F_{\psi_y} = \zeta_x$, i.e., the class consisting of the equations of the form

$$\zeta_t + \psi_x\zeta_y - \psi_y\zeta_x = H(t, x, y, \zeta_x, \zeta_y, \zeta_{xx}, \zeta_{xy}, \zeta_{yy}), \quad \zeta := \psi_{xx} + \psi_{yy}, \tag{11}$$

where H is an arbitrary smooth function of its arguments, which is assumed as an arbitrary element instead of $F = H - \psi_x\zeta_y + \psi_y\zeta_x$. The class (11) also is a member of the above hierarchy of normalized classes. Its equivalence group $G_2^\sim$ is much narrower than $G_1^\sim$ and is formed by the transformations

$$\begin{aligned}
&\tilde t = \tau, \quad \tilde x = \lambda(x\mathfrak{c} - y\mathfrak{s}) + \gamma^1, \quad \varepsilon\tilde y = \lambda(x\mathfrak{s} + y\mathfrak{c}) + \gamma^2,\\
&\tilde\psi = \varepsilon\frac{\lambda}{\tau_t}\left(\lambda\psi + \frac{\lambda}{2}\theta_t(x^2{+}y^2) - \gamma^1_t(x\mathfrak{s}{+}y\mathfrak{c}) + \gamma^2_t(x\mathfrak{c}{-}y\mathfrak{s})\right) + \delta + \frac{\sigma}{2}(x^2{+}y^2),\\
&\tilde H = \frac{\varepsilon}{\tau_t{}^2}\left(H - \frac{\lambda_t}{\lambda}(x\zeta_x + y\zeta_y) + 2\theta_{tt}\right) - \frac{\delta_y{+}\sigma y}{\tau_t\lambda^2}\zeta_x + \frac{\delta_x{+}\sigma x}{\tau_t\lambda^2}\zeta_y + \frac{2}{\tau_t}\left(\frac{\sigma}{\lambda^2}\right)_t,
\end{aligned}$$

where $\varepsilon = \pm1$, $\mathfrak{c} = \cos\theta$, $\mathfrak{s} = \sin\theta$; τ, λ, θ, γ^i and σ are arbitrary smooth functions of t satisfying the conditions $\lambda > 0$, $\tau_{tt} = 0$ and $\tau_t \neq 0$ and $\delta = \delta(t, x, y)$ runs through the set of solutions of the Laplace equation $\delta_{xx} + \delta_{yy} = 0$.

In order to derive the additional constraints that are satisfied by the group parameters of transformations from the point symmetry group G_2 of Eq. (10), we substitute the values $H = -\beta y^2\zeta_x/2$ and $\tilde H = -\beta\tilde y^2\tilde\zeta_{\tilde x}/2$ as well as expressions for the transformed variables and derivatives via the initial ones into the transformation component for H and then make all possible splitting in the obtained equality. As a result we derive the additional constraints

$$\theta = \gamma^2_t = 0, \quad \lambda = \frac{1}{\tau_t}, \quad \sigma = \frac{\varepsilon\beta\gamma^2}{2\tau_t{}^2}, \quad \delta_x = -\sigma x, \quad \delta_y = \sigma y + \frac{\varepsilon\beta(\gamma^2)^2}{2\tau_t}.$$

After projecting transformations from $G_2^\sim$ on the variable space (t, x, y, ψ), constraining the group parameters using the above conditions and taking the adjoint action of the inverse of the transformation (9), we obtain, up to redenoting, the transformations from Theorem 1.

5 Conclusion

In this paper we have computed the complete point symmetry group of the barotropic vorticity equation on the β-plane. It is obvious that both of the techniques presented in this paper are applicable to general systems of differential equations.

Despite of the apparent simplicity of the techniques employed above, there are several features that should be discussed properly. In particular the relation between discrete symmetries of a differential equation and discrete automorphisms of the corresponding maximal Lie invariance algebra is neither injective nor surjective. This is why it can be misleading to restrict the consideration to discrete automorphism when trying to finding discrete symmetries. This and related issues will be investigated and discussed more thoroughly in a forthcoming work.

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[1] Andreev V.K., Kaptsov O.V., Pukhnachov V.V. and Rodionov A.A., *Applications of Group-theoretical Methods in Hydrodynamics*, Kluwer, Dordrecht, 1998.

[2] Bihlo A. and Popovych R.O., Lie symmetries and exact solutions of the barotropic vorticity equation, *J. Math. Phys.*, 2009, V.50, 123102, 12 pp., arXiv:0902.4099.

[3] Bihlo A. and Popovych R.O., Symmetry analysis of barotropic potential vorticity equation, *Comm. Theor. Phys.*, 2009, V.52, 697–700, arXiv:0811.3008.

[4] Bluman G.W., Cheviakov A.F. and Anco S.C., *Application of Symmetry Methods to Partial Differential Equations*, Springer, New York, 2010.

[5] Bluman G.W. and Kumei S., *Symmetries and Differential Equations*, Springer, New York, 1989.

[6] Butcher J., Carminati J. and Vu K.T., A comparative study of some computer algebra packages which determine the Lie point symmetries of differential equations, *Comput. Phys. Comm.* 2003, V.155, 92–114.

[7] Carminati J. and Vu K., Symbolic computation and differential equations: Lie symmetries, *J. Symb. Comput.*, 2000, V.29, 95–116.

[8] Fushchych W.I. and Popovych R.O., Symmetry reduction and exact solutions of the Navier–Stokes equations, *J. Nonlinear Math. Phys.*, 1994, V.1, 75–113, 156–188, arXiv:math-ph/0207016.

[9] Head A.K., LIE, a PC program for Lie analysis of differential equations, *Comput. Phys. Comm.*, 1993, V.77, 241–248. (See also http://www.cmst.csiro.au/LIE/LIE.htm.)

[10] Hereman W., Review of symbolic software for Lie symmetry analysis. Algorithms and software for symbolic analysis of nonlinear systems, *Math. Comput. Modelling*, 1997, V.25, no. 8–9, 115–132.

[11] Hydon P.E., How to construct the discrete symmetries of partial differential equations, *Eur. J. Appl. Math.*, 2000, V.11, 515–527.

[12] Ibragimov N.H., Aksenov A.V., Baikov V.A., Chugunov V.A., Gazizov R.K. and Meshkov A.G., *CRC Handbook of Lie Group Analysis of Differential Equations. Vol. 2. Applications in Engineering and Physical Sciences*, edited by N. H. Ibragimov, CRC Press, Boca Raton, 1995.

[13] Katkov V.L., A class of exact solutions of the equation for the forecast of the geopotential, *Izv. Acad. Sci. USSR Atmospher. Ocean. Phys.*, 1965, V.1, 630–631.

[14] Kingston J.G. and Sophocleous C., On point transformations of a generalised Burgers equation, *Phys. Lett. A*, 1991, V.155, 15–19.

[15] Kingston J.G. and Sophocleous C., On form-preserving point transformations of partial differential equations, *J. Phys. A: Math. Gen.*, 1998, V.31, 1597–1619.

[16] Kingston J.G. and Sophocleous C., Symmetries and form-preserving transformations of one-dimensional wave equations with dissipation, *Int. J. Non-Lin. Mech.*, 2001, V.36, 987–997.

[17] Magadeev B.A., On group classification of nonlinear evolution equations, *Algebra i Analiz*, 1993, V.5, 141–156 (in Russian); English translation in *St. Petersburg Math. J.*, 1994, V.5, 345–359.

[18] Meleshko S.V., A particular class of partially invariant solutions of the Navier–Stokes equations, *Nonlin. Dyn.*, 2004, V.36, 47–68.

[19] Meleshko S.V., *Methods for Constructing Exact Solutions of Partial Differential Equations*, Mathematical and Analytical Techniques with Applications to Engineering, Springer, New York, 2005.

[20] Olver P.J., *Application of Lie Groups to Differential Equations*, Springer, New York, 2000.

[21] Ovsiannikov L.V., *Group Analysis of Differential Equations*, Acad. Press, San Diego, 1982.

[22] Pedlosky J., *Geophysical Fluid Dynamics*, Springer, New York, 1987.

[23] Popovych R.O. and Bihlo A., Symmetry preserving parameterization schemes, 2010, arXiv:1010.3010, 36 pp.

[24] Popovych R.O., Boyko V.M., Nesterenko M.O. and Lutfullin M.W., Realizations of real low-dimensional Lie algebras, arXiv:math-ph/0301029v7, 2005, 39 pp. (extended and revised version of the paper *J. Phys. A: Math. Gen.*, 2003, V.36, 7337–7360).

[25] Popovych R.O., Kunzinger M. and Eshraghi H., Admissible transformations and normalized classes of nonlinear Schrödinger equations, *Acta Appl. Math.*, 2010, V.109, 315–359, arXiv:math-ph/0611061.

[26] Popovych R.O., Kunzinger M. and Ivanova N.M., Conservation laws and potential symmetries of linear parabolic equations, *Acta Appl. Math.*, 2008, V.100, 113–185, arXiv:0706.0443.

[27] Popovych R.O. and Vaneeva O.O., More common errors in finding exact solutions of nonlinear differential equations. I, *Commun. Nonlinear Sci. Numer. Simul.*, 2010, V.15, 3887–3899, arXiv:0911.1848.

[28] Prokhorova M., The structure of the category of parabolic equations, 2005, arXiv: math.AP/0512094, 24 pp.

[29] Rocha Filho T.M. and Figueiredo A., [SADE] A Maple package for the symmetry analysis of differential equations, 2010, arXiv:1004.3339, 35 pp.

[30] Winternitz P. and Gazeau J.P., Allowed transformations and symmetry classes of variable coefficient Korteweg-de Vries equations, *Phys. Lett. A*, 1992, V.167, 246–250.

Simplest potential conservation laws of linear evolution equations

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Every simplest potential conservation law of any $(1+1)$-dimensional linear evolution equation of even order proves induced by a local conservation law of the same equation. This claim is true also for linear simplest potential conservation laws of $(1+1)$-dimensional linear evolution equations of odd order, which are related to linear potential systems. We also derive an effective criterion for checking whether a quadratic conservation law of a simplest linear potential system is a purely potential conservation law of a $(1+1)$-dimensional linear evolution equation of odd order.

1 Introduction

The notion of potential conservation laws arises as a natural generalization of the notion of local conservation laws of differential equations. We call a *potential conservation law* any local conservation law of a potential system constructed from a given system $\mathcal{S}$ of differential equations via introducing potentials by using local conservation laws of $\mathcal{S}$ [13]. This term first appeared in [4]. Potential conservation laws may be trivial in the sense that they induced by local conservation laws of the initial system [8,13]. The idea of iterative introduction of potentials by using local conservation laws of a potential system obtained on the previous step was first suggested in the famous paper [17] and later formalized in the form of the notion of *universal Abelian covering* of differential equations [6, 11, 16]. Although potential conservation laws of differential equations are interesting and important objects for study within the framework of symmetry analysis, nontrivial and complete results on such conservation laws were obtained only for a few classes of differential equations. See related reviews and references in [3, 8, 13].

As a generalization of similar results from [13] for the linear heat equation, it was proved in [14] that all potential conservation laws of $(1 + 1)$-dimensional linear second-order evolution equations are trivial. Local conservation laws of these equations are well understood. More precisely their spaces of the local conservation laws consist of linear conservation laws the characteristics of which depend only upon independent variables and are solutions of the corresponding adjoint equations. This finally solves the problem on potential conservation laws for these equations.

In the present paper we extend results from [14] onto simplest potential conservation laws (i.e. those which involve only single potentials) of $(1+1)$-dimensional linear second-order evolution equations to the case of an arbitrary order. The extension to equations of even order is quite direct and exhaustive. In the case of odd order we restrict our consideration with simplest potential systems constructed by linear conservation laws.

Consider an arbitrary linear evolution equation of order n in the two independent variables t and x and the dependent variable u,

$$u_t = \sum_{i=0}^{n} A^i u_i, \tag{1}$$

where $A^i = A^i(t,x)$ are arbitrary smooth functions, $A^n \neq 0$, $n \in \{2,3,4,\dots\}$, $u_t = \partial u/\partial t$, $u_i \equiv \partial^i u/\partial x^i$, $i = 1,\dots,n$, $u_0 \equiv u$ (i.e., the function u is assumed to be its zeroth-order derivative). We also employ, depending upon convenience or necessity, the following notation: $u_x = u_1$, $u_{xx} = u_2$ and $u_{xxx} = u_3$. D_t and D_x are the operators of total differentiation with respect to the variables t and x, respectively, and Div denotes the total divergence, $\mathrm{Div}\,\mathcal{V} = D_tF + D_xG$ for the tuple $\mathcal{V} = (F,G)$ of differential functions F and G. See [12,14] for other related notions.

Our paper is organized as follows: In the next section we briefly overview results of [15] on local conservation laws of equations from class (1) and extend the simplest potential frame, constructed in [14] for $(1+1)$-dimensional linear evolution equations of order two, to the case of an arbitrary order. The same extension of dual Darboux transformations is carried out in Section 3. Simplest potential conservation laws of $(1+1)$-dimensional linear evolution equations of even and odd order are studied in Sections 4 and 5, respectively. In the conclusion we discuss the results obtained and related problems which are still open.

2 Local conservation laws and simplest potential frame

It is well known that any linear partial differential equation $\mathcal{L}$ admits cosymmetries which are functions of independent variables only and are solutions of the corresponding adjoint equation $\mathcal{L}^\dagger$, and every solution of $\mathcal{L}^\dagger$ is a cosymmetry. Moreover any such cosymmetry is a characteristic of a conservation law of $\mathcal{L}$ which contains a conserved vector linear in the dependent variable and its derivatives, cf. [5]. Following [12, Section 5.3] we call the conservation laws of this kind *linear*.

It turns out that for any linear $(1+1)$-dimensional evolution equation of even order its space of conservation laws is exhausted by linear ones and therefore is isomorphic to the solution space of the corresponding adjoint equation [15].

In other words any cosymmetry of (1) does not depend upon derivatives of u, and a function $\alpha = \alpha(t,x)$ is a cosymmetry of (1) if and only if it is a solution of the adjoint equation

$$\alpha_t + \sum_{i=0}^{n} (-1)^i (A^i\alpha)_i = 0. \tag{2}$$

Any such cosymmetry is a characteristic of a linear conservation law for (1) with the canonical conserved vector $\mathcal{V} = (F, G)$, where

$$F = \alpha u, \quad G = \sum_{i=0}^{n-1} \sigma^i u_i \tag{3}$$

and the coefficients $\sigma^i = \sigma^i(t,x)$ are found recursively from the relations

$$\sigma^{n-1} = -\alpha A^n, \quad \sigma^i = -\alpha A^{i+1} - \sigma^{i+1}_x, \quad i = n-2, \dots, 0. \tag{4}$$

For any linear $(1+1)$-dimensional evolution equation of odd order the space of its conservation laws is spanned by linear and quadratic conservation laws [15]. There exist both such equations possessing infinite series of quadratic conservation laws of arbitrarily high orders and ones having no quadratic conservation laws. For all the formulas and claims obtained for equations of even order to be correct in the case of odd order it is necessary to restrict the consideration with linear local conservation laws, associated (linear) potential systems and their linear conservation laws.

Following the presentation of Section 7 of [14] we investigate certain objects related to simplest potential systems of (1), i.e. potential systems associated with single local conservation laws [13]. The theory of Darboux transformations for linear evolution equations [10] is strongly employed for this. A detailed study of the simplest potential systems is necessary for understanding the general case since such systems are components of more general potential systems.

Introducing the *potential* v by the nontrivial canonical conserved vector (3) associated with the characteristic $\alpha = \alpha(t,x) \neq 0$, we obtain the potential system

$$v_x = \alpha u, \quad v_t = -\sum_{i=0}^{n-1} \sigma^i u_i. \tag{5}$$

The initial equation (1) for u is a differential consequence of system (5). Another differential consequence of (5) is the equation

$$v_t = -\sum_{i=0}^{n-1} \sigma^i \left(\frac{v_x}{\alpha}\right)_i \tag{6}$$

on the potential dependent variable v, which is called the *potential equation* associated with the linear evolution equation (1) and the characteristic α. There is a one-to-one correspondence between solutions of the potential system and the potential equation due to the projection $(u,v)\to v$ on the one hand and due to the formula $u=v_x/\alpha$ on the other, cf. [14]. The correspondence between solutions of the initial equation and the potential system is one-to-one only up to a constant summand.

It is convenient to use another dependent variable $w=\psi v$ instead of v in our further considerations, where we introduce the notation $\psi=1/\alpha$. The function w is called the *modified potential* associated with the characteristic $\alpha=1/\psi$. After being written in terms of w and ψ instead of v and α, the potential system (5) and the potential equation (6) take the form

$$w_x-\frac{\psi_x}{\psi}w=u,\quad w_t-\frac{\psi_t}{\psi}w=-\psi\sum_{i=0}^{n-1}\sigma^i u_i \tag{7}$$

and

$$w_t=-\psi\sum_{i=0}^{n-1}\sigma^i\left(w_x-\frac{\psi_x}{\psi}w\right)_i+\frac{\psi_t}{\psi}w=:\sum_{i=0}^{n}B^iw_i, \tag{8}$$

respectively. Here

$$B^i=-\psi\sigma^{i-1}+\psi\sum_{j=i}^{n-1}\binom{i}{j}\sigma^i\left(\frac{\psi_x}{\psi}\right)_{j-i},\qquad i=n,n-1,\dots,1,$$
$$B^0=\frac{\psi_t}{\psi}+\psi\sum_{j=0}^{n-1}\sigma^i\left(\frac{\psi_x}{\psi}\right)_j.$$

In particular $B^n=A^n$ and $B^{n-1}=A^{n-1}-A^n_x$. System (7) and equation (8) are called the *modified potential system* and the *modified potential equation* associated with the characteristic α. These representations of the potential system and potential equation are more suitable for the study within the framework of symmetry analysis.

As the function $v=1$ obviously is a solution of (6) and therefore the function $w=\psi$ is a solution of (8), the first equation of (7) in fact represents the Darboux transformation [7, 10] of (8) to (1).

3 Dual Darboux transformation

The remarkable fact that Darboux covariance holds for $(1+1)$-dimensional linear evolution equations of arbitrary order was first established in [9] (see also [10, p. 17]). In contrast to the previous section for the coherent presentation we assume below that the initial object of the consideration is the equation (8),

$$w_t=\sum_{i=0}^{n}B^iw_i, \tag{9}$$

which is interpreted as an arbitrary representative of the class of linear evolution equations.

Denote by $\mathrm{DT}[\varphi]$ the Darboux transformation constructed with a nonzero solution φ of (9), i.e.,

$$\mathrm{DT}[\varphi](w) = w_x - \frac{\varphi_x}{\varphi} w.$$

The Darboux transformation possesses the useful property of duality. We formulate this in the same way as in [14], which is slightly different from [10, Section 2.4].

Lemma 1. *Let w^0 be a fixed nonzero solution of (9) and let the Darboux transformation $\mathrm{DT}[w^0]$ map (9) to equation (1). Then $\alpha^0 = 1/w^0$ is a solution of the equation (2) adjoint to equation (1) and $\mathrm{DT}[\alpha^0]$ maps (2) to the equation adjoint to (9), i.e.,*

$$\begin{array}{ccc} u_t = \sum\limits_{i=0}^{n} A^i u_i & \xleftarrow{\mathrm{DT}[w^0]} & w_t = \sum\limits_{i=0}^{n} B^i w_i, \\ & \Updownarrow & \\ \alpha_t + \sum\limits_{i=0}^{n} (-1)^i (A^i \alpha)_i = 0 & \xrightarrow{\mathrm{DT}[\alpha^0]} & \beta_t + \sum\limits_{i=0}^{n} (-1)^i (B^i \beta)_i = 0. \end{array}$$

Remark 1. Similarly to [14] the Darboux transformation $\mathrm{DT}[\alpha^0]$ is called the *dual* to the Darboux transformation $\mathrm{DT}[w^0]$. Since the twice adjoint equation coincides with the initial equation, the twice dual Darboux transformation is nothing but the initial Darboux transformation. This implies that 'then' in Lemma 1 can be replaced by 'if and only if'.

Remark 2. The Darboux transformation $\mathrm{DT}[w^0]$ from Lemma 1 is a linear mapping of the solution space of (9) to the solution space of (1). The kernel of this mapping coincides with the linear span $\langle w^0 \rangle$. Its image is the whole solution space of (1). Indeed for any solution u of (1) we can find a solution w of (9), mapped to u, by integrating system (7) with respect to w. By the Frobenius theorem system (7) is compatible in view of equation (1). Therefore $\mathrm{DT}[w^0]$ generates a one-to-one linear mapping between the solution space of (9), factorized by $\langle w^0 \rangle$, and the solution space of (1).

In the case of even order n Lemma 1 jointly with Remark 2 can be reformulated in terms of characteristics of conservation laws. Denote equations (1) and (9), where $n \in 2\mathbb{N}$, by $\mathcal{L}$ and $\widehat{\mathcal{L}}$ for convenience.

Lemma 2. *If w^0 is a nonzero solution of $\widehat{\mathcal{L}}$ and $\mathrm{DT}[w^0](\widehat{\mathcal{L}}) = \mathcal{L}$, then $\alpha^0 = 1/w^0$ is a characteristic of $\mathcal{L}$ and the Darboux transformation $\mathrm{DT}[\alpha^0]$ maps the characteristic space of $\mathcal{L}$ onto the characteristic space of $\widehat{\mathcal{L}}$.*

Proposition 1. *Let* w^0 *be a fixed nonzero solution of* (9) *and let the Darboux transformation* $\mathrm{DT}[w^0]$ *map* (9) *to equation* (1). *Then the operator associated with* $\mathrm{DT}[w^0]$ *is a splitting operator for the pair of operators associated with equations* (1) *and* (9), *i.e.,*

$$\left(\partial_t - \sum_{i=0}^{n} A^i \partial_x^i\right)\mathrm{DT}[w^0] = \mathrm{DT}[w^0]\left(\partial_t - \sum_{i=0}^{n} B^i \partial_x^i\right).$$

4 Simplest potential conservation laws: even order

If a potential system is constructed by introducing a potential v with a single local conservation law of (1), each of its local conservation laws is a *simplest potential conservation law* of (1), cf. [13]. We say that a simplest potential conservation law $\bar{\mathcal{F}}$ of (1) is *induced* by a local conservation law $\mathcal{F}$ of (1) if $\bar{\mathcal{F}}$ contains a conserved vector which is the pullback of a conserved vector from $\mathcal{F}$ with respect to the projection

$$\varpi\colon J^\infty(t,x\,|\,u,v) \to J^\infty(t,x\,|\,u),$$

where $J^\infty(t,x\,|\,u,v)$ (resp. $J^\infty(t,x\,|\,u)$) denotes the jet space with the independent variables t and x and the dependent variables u and v (resp. the dependent variable u). In view of Proposition 2 from [8] this is equivalent to that the conservation law $\bar{\mathcal{F}}$ contains a conserved vectors depending upon t, x and derivatives of u.

Theorem 1. *Every simplest potential conservation law of any* $(1{+}1)$*-dimensional linear evolution equation of even order is induced by a local conservation law of the same equation.*

Proof. Potentials v and $\tilde{v}$ introduced with equivalent conserved vectors are connected by the transformation $\tilde{v} = v + f[u]$, where $f[u]$ is a function of t, x and derivatives of u. This transformation preserves the property of inducing simplest potential conservation laws by local ones. Therefore exhaustively to investigate simplest potential conservation laws of equations from the class (1) with even n it is sufficient to study local conservation laws of potential systems of the form (5) associated with canonical conserved vectors of the form (3).

We fix an equation from the class (1) and its characteristic α and consider the corresponding potential system (5). As the usual potential v is connected with the modified potential w via a point transformation, we can investigate conservation laws of the modified potential system (7) instead of (5). Up to equivalence of conserved vectors on the solution set of (7), we can exclude derivatives of u from any conserved vector of (7). In other words each local conservation law $\bar{\mathcal{F}}$ of the modified potential system (7) contains a conserved vector depending solely on t, x and derivatives of w and therefore is induced by a local conservation law of the modified potential equation (8).

As equation (8) also is a $(1+1)$-dimensional linear evolution equation of even order as the initial equation, its space of conservation laws is exhausted by linear conservation laws, cf. the discussion in the beginning of Section 2. An arbitrary characteristic β of (8) depends only upon t and x and satisfies the equation adjoint to (8). In view of Lemma 2 there exists a characteristic $\tilde\alpha$ of (1) such that $\beta = \mathrm{DT}[\alpha]\tilde\alpha$.

The conserved vectors $\mathcal V^1$ and $\mathcal V^2$ of the modified potential system (7), which are the pullbacks of the canonical conserved vectors of the initial equation (1) and the modified potential equation (8), associated with the characteristic $\tilde\alpha$ and β, respectively, are equivalent. Indeed the sum of the densities of $\mathcal V^2$ and $\mathcal V^1$ is

$$\beta w + \tilde\alpha u = \Big(\tilde\alpha_x - \frac{\alpha_x}{\alpha}\tilde\alpha\Big)w + \tilde\alpha\Big(w_x + \frac{\alpha_x}{\alpha}w\Big) = D_x(\tilde\alpha w).$$

Denote by $\mathcal V^0$ the trivial conserved vector $(-D_x(\tilde\alpha w), D_t(\tilde\alpha w))$. The conserved vector $\mathcal V^0+\mathcal V^1+\mathcal V^2$ of the system (7) has zero density and therefore is a trivial conserved vector. (In fact the conserved vector $\mathcal V^0+\mathcal V^1+\mathcal V^2$ equals zero.) This means that the conserved vectors $-\mathcal V^1$ and $\mathcal V^2$ are equivalent.

In summary we prove that any simplest potential conservation law of equation (1) contains a conserved vector which is the pullback of a local conserved vector of (1). ■

Remark 3. The explicit construction of a local conserved vector which is equivalent to $\mathcal V^2$ in the end of the proof can be replaced by arguments based on the criterion of induction of potential conservation laws by local ones, cf. Proposition 8 from [8]. Indeed the canonical conserved vector of the modified potential equation (8), which is the trivial projection of $\mathcal V^2$, is associated with the characteristic $\beta = \beta(t,x)$. This is why

$$\begin{aligned}
\mathrm{Div}\,\mathcal V^2 &= \beta\Big(w_t - \sum_{i=0}^{n} B^i w_i\Big)\\
&= \beta\Big(w_t - \frac{\psi_t}{\psi}w + \psi\sum_{i=0}^{n-1}\sigma^i u_i + \psi\sum_{i=0}^{n-1}\sigma^i D_x^i\Big(w_x - \frac{\psi_x}{\psi}w - u\Big)\Big)\\
&= \psi\beta\Big(v_t + \sum_{i=0}^{n-1}\sigma^i u_i\Big) + \psi\Big(\sum_{i=0}^{n-1}(-D_x)^i(\psi\sigma^i\beta)\Big)(v_x - \alpha u) + D_x\Phi
\end{aligned}$$

for some differential function Φ of u and v for which the precise expression is not essential. (It is a linear function in derivatives of u and v with coefficients depending upon t and x.) In other words the conserved vector $\mathcal V^2$ belongs to the conservation law $\bar{\mathcal F}$ of the potential system (5) with the characteristic having the components

$$\psi\beta \quad\text{and}\quad \psi\sum_{i=0}^{n-1}(-D_x)^i(\psi\sigma^i\beta).$$

This characteristic is completely reduced since it does not depend upon derivatives of u and v. In particular it does not depend upon v. In view of Proposition 8 from [8], the associated local conservation law $\bar{\mathcal{F}}$ of the potential system (5) is induced by a local conservation law of the initial equation (1).

5 Simplest potential conservation laws: odd order

Theorem 1 cannot be directly extended to $(1+1)$-dimensional linear evolution equations of odd order since such equations may possess, additionally to linear, quadratic conservation laws. At the same time it is easy to see that a similar statement can be proved for the case of odd order after restricting to the completely linear case.

Theorem 2. *Every linear simplest potential conservation law of any* $(1+1)$*-dimensional linear evolution equation of odd order, which is related to a linear potential system, is induced by a local conservation law of the same equation.*

A $(1+1)$-dimensional linear evolution equation $\mathcal{L}$ of odd order may additionally possesses two kinds of simplest potential conservation laws:

- conservation laws of potential systems constructed with conserved vectors of $\mathcal{L}$ which nontrivially contain terms quadratic in derivatives of u and
- quadratic conservation laws of simplest linear potential systems.

Potential conservation laws of the first kind are difficult for investigation. Thus, in contrast to the linear case, related potential systems usually have no analogues of potential equations. It seems that an only possibility to study conservation laws of these systems is the direct application of general methods discussed, e.g., in [1–3, 6, 13, 18].

Here we consider only potential conservation laws of the second kind. There exists a simple criterion to check whether a potential conservation law of this kind is induced by a local conservation law of the initial equation (1).

Theorem 3. *Let* $\alpha=\alpha(t,x)$ *be a nonzero characteristic of a* $(1+1)$*-dimensional linear evolution equation of odd order* (1) *and*

$$\gamma=\Gamma w, \quad \Gamma=\sum_{k=0}^{r} g^k(t,x)D_x^k, \quad g^r\neq 0,$$

be a characteristic of the corresponding modified potential equation (8)*. Then the conservation law of potential system* (5)*, which is associated with* γ*, is induced by a local conservation law of* (1) *if and only if the solution* $\psi=1/\alpha$ *of* (8) *belongs to the kernel of operator* Γ*, i.e.,* $\Gamma\psi=0$*.*

Proof. Denote by $\mathcal{V}$ the conserved vector of the potential system (5), which is obtained by the pullback of the canonical conserved vector of the modified potential equation (8), associated with the characteristic γ, and the consequent transformation $v = \alpha w$. Analogously to Remark 3 we have

$$\begin{aligned}
\operatorname{Div}\mathcal{V} &= \gamma\Bigg(w_t - \sum_{i=0}^{n} B^i w_i\Bigg) \\
&= (\Gamma w)\Bigg(w_t - \frac{\psi_t}{\psi}w + \psi\sum_{i=0}^{n-1}\sigma^i u_i + \psi\sum_{i=0}^{n-1}\sigma^i D_x^i\Bigg(w_x - \frac{\psi_x}{\psi}w - u\Bigg)\Bigg) \\
&= \psi\Gamma(\psi v)\Bigg(v_t + \sum_{i=0}^{n-1}\sigma^i u_i\Bigg) + \psi\Bigg(\sum_{i=0}^{n-1}(-D_x)^i\big(\psi\sigma^i\Gamma(\psi v)\big)\Bigg)(v_x - \alpha u) \\
&\quad + D_x\Phi
\end{aligned}$$

for some differential function Φ of u and v the precise expression for which again is not essential. (It is a quadratic function in derivatives of u and v with coefficients depending on t and x.) This means that the conserved vector $\mathcal{V}$ belongs to the conservation law of the potential system (5) with the characteristic λ having the components

$$\psi\Gamma(\psi v) \quad \text{and} \quad \psi\Bigg(\sum_{i=0}^{n-1}(-D_x)^i\big(\psi\sigma^i\Gamma(\psi v)\big)\Bigg).$$

For the characteristic λ to be completely reduced we have to exclude the derivatives v_k, $k = 1, \dots, r + n - 1$, from it using the differential consequences of the equation $v_x = \alpha u$. The reduced form of λ depends upon the potential v if and only if $\Gamma\psi = 0$. Therefore the statement to be proved follows from the criterion of induction of potential conservation laws by local conservation laws formulated in [8] as Proposition 8. ■

Example 1. We construct an example starting from the corresponding modified potential equation with the known space of quadratic conservation laws. Consider the "linear Korteweg–de Vries equation"

$$w_t = w_{xxx} \tag{10}$$

which is identical with its adjoint. It was proved in [15] that the space of its quadratic conservation laws is spanned by the conservation laws with the characteristics $\Gamma_{ml}w$, where $\Gamma_{ml} = D_x^m(3tD_x^2 + x)^l D_x^m$ and $l, m = 0, 1, 2, \dots$.

As the solution ψ of the modified potential equation we choose the function $w = x$. The Darboux transformation $\mathrm{DT}[x]$ maps equation (10) to the "initial" equation

$$u_t = u_{xxx} - \frac{3}{x^2}u_x + \frac{3}{x^3}u \tag{11}$$

which also is identical with its adjoint. The solution α of the equation (11), dual to ψ, is $u = 1/\psi = 1/x$. The potential system constructed by the conservation law of (11) with the characteristic $\alpha = 1/x$ has the form

$$v_x = \frac{u}{x}, \quad v_t = \frac{u_{xx}}{x} + \frac{u_x}{x^2} - \frac{u}{x^3}.$$

We have that $\Gamma_{ml}\psi = 0$ if and only if $m \geqslant 2$. Therefore a complete set of independent simplest purely potential conservation laws of the equation (11), which are obtained via introducing the potential with the characteristic $\alpha = 1/x$, is exhausted by the quadratic conservation laws constructed by the pullback of the conservations laws of the corresponding modified potential equation (10), having the characteristics $\Gamma_{ml}w$, where $l = 0, 1, 2, \ldots$ and $m = 0, 1$.

6 Conclusions

In this paper we have studied simplest potential conservation laws of (1+1)-dimensional linear evolution equations, which are constructed via introducing single potentials using linear conservation laws. Such conservation laws are quite trivial in the case of equations of even order: All simplest potential conservation laws of any $(1 + 1)$-dimensional linear evolution equation of even order are induced by local conservation laws of the same equation and its space of local conservation laws is exhausted by linear ones. Similar results concerning equations of odd order are more complicated. Although all simplest linear potential conservation laws of these equations are induced by local ones, it is not the case for quadratic conservation laws. We derive an effective criterion which allows us to check easily whether a quadratic conservation law of a simplest modified potential equation leads to a purely potential conservation law for the initial equation. It is true if and only if any characteristic of this conservation law does not vanish for the value $w = 1/\alpha$ of the modified potential w, where α is the characteristic of the linear conservation law of the initial equation used for introducing the potential.

A preliminary analysis shows that all the results obtained in this paper for simplest conservation laws could be easily extended to the case of an arbitrary number of potentials introduced with linear conservation laws. The first step of this investigation should be the construction of the whole linear potential frame for the class of $(1+1)$-dimensional linear evolution equations of an arbitrary fixed order as this was done for order two in [14]. It is obvious that the linear potential frame coincides with the entire potential frame if the order of the equation is even.

The consideration of nonlinear potential systems constructed for equations of odd order with quadratic conservation laws calls for the development of new tools which are different from those used for linear potential systems.

Another possible direction for further investigations close to the subject of this paper is the description of potential symmetries of $(1+1)$-dimensional linear evolution equation of arbitrary order, at least those associated with linear potential systems. It seems to us that the approach which was developed in [14] for the case of equations of order two and is based upon the construction of the extended potential frame and the reduction of the consideration to the study of single modified potential equations also has to work for an arbitrary order.

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[1] Anco S.C. and Bluman G., Direct construction method for conservation laws of partial differential equations. I. Examples of conservation law classifications, *Eur. J. Appl. Math.*, 2002, V.13, 545–566, arXiv:math-ph/0108023.

[2] Anco S.C. and Bluman G., Direct construction method for conservation laws of partial differential equations. II. General treatment, *Eur. J. Appl. Math.*, 2002, V.13, 567–585, arXiv:math-ph/0108024.

[3] Bluman G.W., Cheviakov A.F. and Anco S.C., *Applications of Symmetry Methods to Partial Differential Equations*, Springer, New York, 2010.

[4] Bluman G. and Doran-Wu P., The use of factors to discover potential systems or linearizations. Geometric and algebraic structures in differential equations, *Acta Appl. Math.*, 1995, V.41, 21–43.

[5] Bluman G. and Shtelen V., Nonlocal transformations of Kolmogorov equations into the backward heat equation, *J. Math. Anal. Appl.*, 2004, V.291, 419–437.

[6] Bocharov A.V., Chetverikov V.N., Duzhin S.V., Khor'kova N.G., Krasil'shchik I.S., Samokhin A.V., Torkhov Yu.N., Verbovetsky A.M. and Vinogradov A.M., *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, American Mathematical Society, Providence, RI, 1999.

[7] Crum M.M., Associated Sturm–Liouville systems, *Quart. J. Math. Oxford Ser. (2)*, 1955, V.6, 121–127.

[8] Kunzinger M. and Popovych R.O., Potential conservation laws, *J. Math. Phys.*, 2008, V.49, 103506, 34 pp., arXiv:0803.1156.

[9] Matveev V.B., Darboux transformation and explicit solutions of the Kadomtcev–Petviaschvily equation, depending on functional parameters, *Lett. Math. Phys.*, 1979, V.3, 213–216.

[10] Matveev V.B. and Salle M.A., *Darboux Transformations and Solitons*, Springer-Verlag, Berlin, 1991.

[11] Marvan M., Reducibility of zero curvature representations with application to recursion operators, *Acta Appl. Math.*, 2004, V.83, 39–68, arXiv:nlin.SI/0306006.

[12] Olver P.J., *Application of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1993.

[13] Popovych R.O. and Ivanova N.M., Hierarchy of conservation laws of diffusion-convection equations, *J. Math. Phys.*, 2005, V.46, 043502, 22 pp., arXiv:math-ph/0407008.

[14] Popovych R.O., Kunzinger M. and Ivanova N.M., Conservation laws and potential symmetries of linear parabolic equations, *Acta. Appl. Math.*, 2008, V.100, 113–185, arXiv:0706.0443.

[15] Popovych R.O. and Sergyeyev A., Conservation laws and normal forms of evolution equations, *Phys. Lett. A*, 2010, V.374, 2210–2217, arXiv:1003.1648.

[16] Sergyeyev A., On recursion operators and nonlocal symmetries of evolution equations, in *Proceedings of the Seminar on Differential Geometry*, Silesian University in Opava, Opava, 2000, 159–173, arXiv:nlin.SI/0012011.

[17] Wahlquist H.D. and Estabrook F.B., Prolongation structures of nonlinear evolution equations, *J. Math. Phys.*, 1975, V.16, 1–7.

[18] Wolf T., A comparison of four approaches to the calculation of conservation laws, *Eur. J. Appl. Math.*, 2002, V.13, Part 5, 129–152, arXiv:cs.SC/0301027.

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Extended symmetry analysis of a “nonconservative Fokker–Planck equation”

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We show that all results of Yaşar and Özer [*Comput. Math. Appl.* **59** (2010), 3203–3210] on symmetries and conservation laws of a “nonconservative Fokker–Planck equation” can be easily derived from results existing in the literature by means of using the fact that this equation is reduced to the linear heat equation by a simple point transformation. Moreover nonclassical symmetries and local and potential conservation laws of the equation under consideration are exhaustively described in the same way as well as infinite series of potential symmetry algebras of arbitrary potential orders are constructed.

The investigation of Lie symmetries of two-dimensional second-order linear partial differential equations was one of the first problems considered within the group analysis of differential equations. The complete group classification of such equations was carried out by Lie [11] himself and essentially later revisited by Ovsiannikov [15]. This classification includes, as a special case, the group classification of linear (1+1)-dimensional homogeneous second-order evolution equations of the general form

$$u_t = a(t,x)u_{xx} + b(t,x)u_x + c(t,x)u, \tag{1}$$

where a, b and c are arbitrary smooth functions of t and x, $a \neq 0$. On the other hand the study of point equivalences between linear evolution equations also have a long history. It was started with Kolmogorov's paper [9], where the problem of the description of Kolmogorov equations that are reduced to the linear heat equation by certain point transformations was posed. This problem was exhaustively solved by Cherkasov [2]. Symmetry criteria on equivalence of equations within class (1) obviously follow from the above Lie–Ovsiannikov group classification. They were also discussed by a number of authors [1,22,23]. See additionally the review in [19] and references therein.

A constructive approach to investigation of point equivalences of equations from the class (1) was suggested by Ibragimov [4]. This approach is based upon the computation of differential invariants and semi-invariants of the associated equivalence group $G^{\sim}$. In fact in [4] only the semi-invariants of the subgroup of $G^{\sim}$, consisting of linear transformations in the dependent variable, were obtained. An efficient criterion on point equivalence of equations from class (1) to the linear heat equation in terms of semi-invariants of the entire equivalence group $G^{\sim}$, depending upon the coefficients of equation (1), was proposed in [8]. This criterion was reformulated in a more compact form in [12]. Therein criteria of point reducibility to Lie’s other canonical forms were found within Ibragimov’s approach. It is necessary to note that earlier the equivalence problem for equations from class (1) was exhaustively investigated by Morozov [13] within the framework of the method of moving frames although the expressions constructed for related differential invariants and semi-invariants are quite cumbersome.

An exhaustive review of the previous results on group analysis and the equivalence problem within the class (1) as well as the complete description of conservation laws and potential symmetries of equations from this class can be found in the paper by Popovych, Kunzinger and Ivanova [19].

In a recent paper [24] Lie symmetries, conservation laws, potential symmetries and solutions were considered for the “nonconservative Fokker–Planck equation”

$$u_t = u_{xx} + xu_x \tag{2}$$

which is a representative of the class (1). After the sign of t is changed, this equation becomes a Kolmogorov backward equation. Although equation (2) is quite interesting from the mathematical point of view and appears in numerous applications, its investigation in the framework of symmetry analysis is unnecessary due to the fact that it is reduced to the linear heat equation

$$w_\tau = w_{yy} \tag{3}$$

by the simple point transformation

$$\tau = \frac{1}{2}e^{2t}, \quad y = e^t x, \quad w = u. \tag{4}$$

The inverse of (4),

$$t = \frac{1}{2}\ln(2\tau), \quad x = \frac{y}{\sqrt{2\tau}}, \quad u = w, \tag{5}$$

is well defined only for $\tau > 0$.

Because of the transformation (4) the results of [24] as well as more general ones can be easy derived from well-known results on the linear heat equation (3) existing in the literature for a long time, namely, using the transformation (4), one can obtain the maximal Lie invariance algebra of the equation (2), the complete description of its local and potential conservation laws and its potential symmetries of any potential order as well as nonclassical symmetries and wide families of its exact solutions. (A similar example on usage of point equivalence of a Fokker–Planck equation to the linear heat equation was presented in [19, p. 156].)

The maximal Lie invariance algebra of the linear heat equation (3) is

$$\mathfrak{g} = \langle \partial_\tau,\ \partial_y,\ 2\tau\partial_\tau + y\partial_y,\ 2\tau\partial_y - yw\partial_w,\\ 4\tau^2\partial_\tau + 4\tau y\partial_y - (y^2 + 2\tau)w\partial_w,\ w\partial_w,\ f\partial_w\rangle,$$

where the function $f = f(\tau, y)$ is an arbitrary solution of (3) [11, 14, 15]. The maximal Lie invariance algebra $\tilde{\mathfrak{g}}$ of the equation (2) was calculated in Section 3 of [24] by the classical infinitesimal method. In contrast to this we directly obtain $\tilde{\mathfrak{g}}$ from $\mathfrak{g}$ using the pushforward of vector fields associated with (5):

$$\tilde{\mathfrak{g}} = \langle \partial_t,\ e^{-t}\partial_x,\ e^{-2t}\partial_t - e^{-2t}x\partial_x + e^{-2t}u\partial_u,\ e^t\partial_x - e^t xu\partial_u,\\ e^{2t}\partial_t + e^{2t}x\partial_x - e^{2t}x^2u\partial_u,\ u\partial_u,\ f\partial_u\rangle,$$

where the function $f = f(t, x)$ runs through the solution set of (2).

In emulation of Ibragimov's approach [5], in Section 4 of [24] conservation laws of the joint system of the initial equation (2) and its adjoint

$$\alpha_t + \alpha_{xx} - (x\alpha)_x = 0 \tag{6}$$

were constructed using Lie symmetries of the system, namely essential Lie symmetries of (2) were prolonged to the additional dependent variable α. The prolonged symmetries are Lie symmetries of the joint system of (2) and (6) and variational symmetries of the corresponding Lagrangian $\alpha(u_t - u_{xx} - xu_x)$. Therefore conserved vectors of the system are obtained from the prolonged symmetries in view of the Noether theorem on connections between symmetries and conservation laws.

All conservation laws of the joint system of the linear heat equation (3) and its adjoint (the linear backward heat equation),

$$\tilde{\alpha}_\tau = -\tilde{\alpha}_{yy}, \tag{7}$$

obtainable with this approach, were calculated in [5, 6]. The point transformation (4) prolonged to the dependent variable α according to the formula $\tilde{\alpha} = e^{-t}\alpha$ (cf. Section 5 of [19] and in particular equation (26)) maps the joint system of (2) and (6) to the joint system of (3) and (7). Therefore the result of Section 4 of [24] is a direct consequence of the result for linear heat equation from [5, 6].

The complete description of local and potential conservation laws of (2) can be simply obtained as a particular case of the same results of [19] on the entire class of linear (1+1)-dimensional second-order evolution equations (1). According to [19, Lemma 3] every local conservation law of any equation from class (1) is of the first order and moreover it possesses a conserved vector with density depending at most upon t, x and u and flux depending at most on t, x, u and u_x. As was proven in [19, Theorem 4], each local conservation law of an arbitrary equation from the class (1) contains a conserved vector of the canonical form

$$\big(\alpha u,\ -\alpha a u_x + ((\alpha a)_x - \alpha b)u\big),$$

where the characteristic $\alpha = \alpha(t,x)$ runs through the solution set of the adjoint equation

$$\alpha_t + (a\alpha)_{xx} - (b\alpha)_x + c\alpha = 0. \qquad (8)$$

In other words the space of local conservation laws of equation (1) is isomorphic to the solution space of the adjoint equation (8). An analogous statement is true for an arbitrary $(1+1)$-dimensional linear evolution equation of even order [20].

As a result the equation (2) possesses, up to the equivalence of conserved vectors [19, Definition 3], only conserved vectors of the canonical form

$$\big(\alpha u,\ -\alpha u_x + (\alpha_x - \alpha x)u\big),$$

where the characteristic $\alpha = \alpha(t,x)$ is an arbitrary solution of the adjoint equation (6). Moreover it follows from Theorem 5 of [19] that any potential conservation law of equation (2) is induced by a local conservation law of this equation.

Potential symmetries of (2) also can be constructed from known potential symmetries of the linear heat equation (3).

It was proven in [19, Theorem 7] that the linear heat equation (3) admits, up to the equivalence generated by its point symmetry group, only two simplest potential systems the Lie symmetries of which are not induced by Lie symmetries of (3) and therefore are nontrivial potential symmetries of (3). (A potential system is called simplest if it involves a single potential.) These potential systems are associated with the characteristics $\alpha^1 = 1$ and $\alpha^2 = y$. The inverse

$$t = \frac{1}{2}\ln(2\tau), \quad x = \frac{y}{\sqrt{2\tau}}, \quad u = w, \quad \alpha = e^t\tilde\alpha$$

of the transformation (4) prolonged to the dependent variable α maps the characteristics α^1 and α^2 of equation (3) to the characteristics $\tilde\alpha^1 = e^t$ and $\tilde\alpha^2 = e^{2t}x$ of equation (2). The potential systems for equation (2), associated with the characteristics $\tilde\alpha^1$ and $\tilde\alpha^2$, respectively are

$$\hat v_x = e^t u, \quad \hat v_t = e^t u_x + e^t xu \qquad (9)$$

and

$$\check v_x = e^{2t}xu, \quad \check v_t = e^{2t}xu_x + e^{2t}(x^2-1)u. \qquad (10)$$

Their maximal Lie invariance algebras are

$$\begin{aligned}\tilde{\mathfrak p}_1 = \langle\ & e^{-2t}\partial_t - e^{-2t}x\partial_x,\ e^{-t}\partial_x,\ \partial_t - u\partial_u,\ e^t\partial_x - e^t(xu + x\hat v)\partial_u - \hat v\partial_{\hat v},\\ & e^{2t}\partial_t + e^{2t}x\partial_x - e^{2t}(x^2u + 3u + 2xe^{-t}\hat v)\partial_u - e^{2t}(x^2+1)\hat v\partial_{\hat v},\\ & u\partial_u + \hat v\partial_{\hat v},\ e^{-t}g_x\partial_u + g\partial_{\hat v}\ \rangle,\end{aligned}$$

$$\begin{aligned}\tilde{\mathfrak p}_2 = \langle\ & \partial_t - u\partial_u,\ e^{-2t}\partial_t - e^{-2t}x\partial_x,\\ & e^{2t}\partial_t + e^{2t}x\partial_x - e^{2t}(x^2u + 3u + 2e^{-2t}\check v)\partial_u - e^{2t}(x^2-1)\check v\partial_{\check v},\\ & u\partial_u + \check v\partial_{\check v},\ e^t x^{-1}h_x\partial_u + h\partial_{\check v}\ \rangle,\end{aligned}$$

where the functions $g = g(t,x)$ and $h = h(t,x)$ run through the solution set of the associated potential equations, $\hat{v}_t - \hat{v}_{xx} - x\hat{v}_x = 0$ and $\check{v}_t - \check{v}_{xx} + (2x^{-1} - x)\check{v}_x = 0$, respectively. The maximal Lie invariance algebras of these potential equations are projections of $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$ to the spaces $(t,x,\hat{v})$ and $(t,x,\check{v})$. Instead of using the infinitesimal Lie method, the algebras $\tilde{\mathfrak{p}}_1$ and $\tilde{\mathfrak{p}}_2$ can be obtained from the potential symmetry algebras $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of the linear heat equation (see [19, pp. 155–156]) by the transformation (5) trivially prolonged to the corresponding potentials.

Moreover the linear heat equation admits an infinite series $\{\mathfrak{g}_p,\, p \in \mathbb{N}\}$ of potential symmetry algebras [19, Proposition 12], namely, for any $p \in \mathbb{N}$ the algebra $\mathfrak{g}_p$ is of strictly pth potential order (i.e., it involves exactly p independent potentials) and is associated with p-tuples of characteristics which are linearly independent polynomial solutions of lowest order of the backward heat equation. Each of the algebras $\mathfrak{g}_p$ is isomorphic to the maximal invariance algebra $\mathfrak{g}$ of the linear heat equation. The linear heat equation possesses also other infinite series of nontrivial potential symmetry algebras. Due to the change of variables (5), trivially prolonged to the corresponding potentials, similar results hold true for the equation (2).

In [24] the classical Lie algorithm was used for finding the maximal Lie invariance algebra of the potential system of the equation (2), which is associated with the characteristic $e^{-x^2/2}$. In fact this system is equivalent, with respect to the Lie symmetry group of (2), to the simpler potential system (9) associated with the characteristic $\tilde{\alpha}^1 = e^t$. To reduce the characteristic $e^{-x^2/2}$ to the characteristic $\tilde{\alpha}^1$ it is necessary to apply a point symmetry transformation which is the composition of a projective transformation and a shift with respect to t.

Nonclassical symmetries of the linear heat equation (3) were exhaustively investigated in [3]. Roughly speaking, it was proved that in both the singular and regular cases (when the coefficient of ∂_t in a nonclassical symmetry operator vanishes or does not, respectively) the corresponding determining equations are reduced by nonpoint transformations to the initial equations. Later these no-go results were extended to all equations from the class (1) [17, 18, 25].

Continuing the list of common errors in finding exact solutions of differential equations made up by Kudryashov [10], Popovych and Vaneeva [21] indicated one more common error of such a kind: Solutions are often constructed with no relation to equivalence of differential equations with respect to point (resp. contact, resp. potential etc.) transformations. A number of multiparametric families of exact solutions of the linear heat equation (3) are well known for a long time and are presented widely in the literature. See e.g. the review [7], the textbook [14, Examples 3.3 and 3.17], the handbook [16] and the website EqWorld `http://eqworld.ipmnet.ru/`. A simple solution of (2) was constructed in Section 5 of [24] which is similar with respect to the point transformation (4) to the obvious solution $w = c_1 y$ of the linear heat equation (3).

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[1] Bluman G.W., On the transformation of diffusion processes into the Wiener process, *SIAM J. Appl. Math.*, 1980, V.39, 238–247.

[2] Cherkasov I.D., On the transformation of the diffusion process to a Wiener process, *Theory Probab. Appl.*, 1957, V.2, 373–377.

[3] Fushchych W.I., Shtelen W.M., Serov M.I. and Popovych R.O., Q-conditional symmetry of the linear heat equation, *Proc. Acad. Sci. Ukraine*, 1992, no. 12, 28–33.

[4] Ibragimov N.H., Laplace type invariants for parabolic equations, *Nonlinear Dynam.*, 2002, V.28, 125–133.

[5] Ibragimov N.H., A new conservation theorem, *J. Math. Anal. Appl.*, 2007, V.333, 311–328.

[6] Ibragimov N.H. and Kolsrud T., Lagrangian approach to evolution equations: symmetries and conservation laws, *Nonlinear Dynam.*, 2004, V.36, 29–40.

[7] Ivanova N.M., Exact solutions of diffusion-convection equations, *Dynam. PDEs*, 2008, V.5, 139–171.

[8] Johnpillai I.K. and Mahomed F.M., Singular invariant equation for the $(1 + 1)$ Fokker–Planck equation, *J. Phys. A: Math. Gen.*, 2001, V.34, 11033–11051.

[9] Kolmogorov A.N., On analytic methods in probability, *Uspehi Mat. Nauk*, 1938, V.5, 5–41.

[10] Kudryashov N.A., Seven common errors in finding exact solutions of nonlinear differential equations, *Commun. Nonlinear Sci. Numer. Simul.*, 2009, V.14, 3507–3529.

[11] Lie S., Über die Integration durch bestimmte Integrale von einer Klasse linear partieller Differentialgleichung, *Arch. Math.*, 1881, V.6, no. 3, 328–368 (translation by N.H. Ibragimov: Lie S., On integration of a class of linear partial differential equations by means of definite integrals, *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 2, 1994, 473–508).

[12] Mahomed F., Complete invariant characterization of scalar linear $(1 + 1)$ parabolic equations, *J. Nonlinear Math. Phys.*, 2008, V.15, 112–123.

[13] Morozov O.I., Contact equivalence problem for linear parabolic equations, arXiv: math-ph/0304045.

[14] Olver P.J., *Application of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1993.

[15] Ovsiannikov L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[16] Polyanin A.D., *Handbook of Linear Partial Differential Equations for Engineers and Scientists*, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[17] Popovych R.O., No-go theorem on reduction operators of linear second-order parabolic equations, in *Symmetry and Integrability of Equations of Mathematical Physics*, *Collection of Works of Institute of Mathematics, Kyiv*, 2006, V.3, no. 2, 231–238.

[18] Popovych R.O., Reduction operators of linear second-order parabolic equations, *J. Phys. A: Math. Theor.*, 2008, V.41, 185202, 31 pp.

[19] Popovych R.O., Kunzinger M. and Ivanova N.M., Conservation laws and potential symmetries of linear parabolic equations, *Acta. Appl. Math.*, 2008, V.100, 113–185.

[20] Popovych R.O. and Sergyeyev A., Conservation laws and normal forms of evolution equations, *Phys. Lett. A*, 2010, V.374, 2210–2217.

[21] Popovych R.O. and Vaneeva O.O., More common errors in finding exact solutions of nonlinear differential equations. I, *Commun. Nonlinear Sci. Numer. Simul.*, 2010, V.15, 3887–3899.

[22] Shtelen W.M. and Stogny V.I., Symmetry properties of one- and two-dimensional Fokker–Planck equations, *J. Phys. A: Math. Gen.*, 1989, V.22, L539–L543.

[23] Spichak S.V. and Stognii V.I., Symmetry classification and exact solutions of the one-dimensional Fokker–Planck equation with arbitrary coefficients of drift and diffusion, *J. Phys. A: Math. Gen.*, 1999, V.32, 8341–8353.

[24] Yaşar E. and Özer T., Invariant solutions and conservation laws to nonconservative FP equation, *Comput. Math. Appl.*, 2010, V.59, 3203–3210.

[25] Zhdanov R.Z. and Lahno V.I., Conditional symmetry of a porous medium equation, *Phys. D*, 1998, V.122, 178–186.

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Λ-symmetries of dynamical systems, Hamiltonian and Lagrangian equations

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After a brief survey of the definition and the properties of Λ-symmetries in the general context of dynamical systems we introduce the notion of "Λ-constant of motion" for Hamiltonian equations. If the Hamiltonian problem is derived from a Λ-invariant Lagrangian, it is shown how the Lagrangian Λ-invariance can be transferred into the Hamiltonian context and shown that the Hamiltonian equations turn out to be Λ-symmetric. Finally the "partial" (Lagrangian) reduction of the Euler–Lagrange equations is compared with the reduction obtained for the corresponding Hamiltonian equations.

1 Introduction (λ-symmetries)

I briefly recall for the reader's convenience the basic definition of λ-symmetry (with lower case λ), originally introduced by C. Muriel and J.L. Romero in 2001 [1,2].

Consider the simplest case of a single ordinary differential equation (ODE) $\Delta(t, u(t), \dot{u}, \ddot{u}, \dots) = 0$ for the unknown function $u = u(t)$ (I denote by t the independent variable, with the only exception being in Section 4, because the applications I am going to propose concern the case of Dynamical Systems (DS), where the independent variable is precisely the time t, and $\dot{u} = \mathrm{d}u/\mathrm{d}t$ etc.). Given a vector field

$$X = \varphi(u,t)\frac{\partial}{\partial u} + \tau(u,t)\frac{\partial}{\partial t}$$

the idea is to *modify* suitably its prolongation rules. The first λ-prolongation $X_\lambda^{(1)}$ is defined by

$$X_\lambda^{(1)} = X^{(1)} + \lambda(\varphi - \tau\dot{u})\frac{\partial}{\partial \dot{u}}, \tag{1}$$

where $\lambda = \lambda(u, \dot{u}, t)$ is a C^∞ function, and $X^{(1)}$ is the standard first prolongation. Other modifications have to be introduced for higher prolongations, but in the present paper I need only just the first one.

An nth-order ODE $\Delta = 0$ is said to be λ-invariant under X if $X_\lambda^{(n)}\Delta\Big|_{\Delta=0} = 0$, where $X_\lambda^{(n)}$ is the appropriate λ-prolongation of X.

It should be emphasized that λ-symmetries are not properly symmetries because they do not transform in general solutions of a λ-invariant equation into solutions. Nevertheless they share with standard Lie point-symmetries some important properties, namely: if an equation is λ-invariant, then

- the order of the equation can be lowered by one,
- invariant solutions can be found (note that conditional symmetries do the same, but λ-symmetries are clearly *not* conditional symmetries),
- convenient new ("symmetry adapted") variables can be suggested.

In the context of DS, which is the main object of this paper, the first two properties are not effective, the third one is instead one of my starting points.

Before considering the role of λ-symmetries in DS, we recall that many applications and extensions of this notion have been proposed in these past 10 years: these include extensions to systems of ODE's, to PDE's, applications to variational principles and Noether-type theorems, the analysis of their connections with nonlocal symmetries, with symmetries of exponential type, with hidden or "lost" symmetries, with potential, telescopic symmetries as well. Other investigations concern their deep geometrical interpretation with the introduction of a suitable notion of deformed Lie differential operators, the study of their dynamical effects in terms of changes of reference frames and so forth. Only the papers more directly involved with the argument considered in this paper are quoted; for a fairly complete list of references see e.g. [3–5]. A very recent application concerns discrete difference equations [6].

2 Λ-symmetries for DS

I am going to consider the case of dynamical systems, i.e. systems of first-order ODE's

$$\dot u_a \ = \ f_a(u,t) \qquad (a=1,\dots,m)$$

for the $m>1$ unknowns $u_a \ = \ u_a(t)$.

I start with a trivial (but significant) case: if the DS admits a rotation symmetry, then it is completely natural to introduce new variables, the radius r and the angle θ, and the DS immediately takes a simplified form, as is well known. However, in general symmetries of DS may be very singular and/or difficult to detect. An example can be useful: the DS

$$\dot u_1 \ = \ u_1u_2, \quad \dot u_2 \ = \ -u_1^2$$

admits the (not very useful or illuminating) symmetry generated by (with $r^2 = u_1^2+u_2^2$)

$$X \ = \ \left(\frac{2u_1}{r^2} - \frac{u_1u_2}{r^3}\log\frac{u_2-r}{u_2+r}\right)\frac{\partial}{\partial u_1} + \left(\frac{2u_2}{r^2} - \frac{u_1^2}{r^3}\log\frac{u_2-r}{u_2+r}\right)\frac{\partial}{\partial u_2} \ .$$

In this example the rotation (with a commonly accepted abuse of language the same symbol X denotes both the symmetry and its Lie generator) $X = u_2 \frac{\partial}{\partial u_1} - u_1 \frac{\partial}{\partial u_2}$ is a λ-*symmetry* (its precise definition is given just below) and *not* a symmetry in the "standard" sense; *nevertheless*, still introducing the variables as above, i.e. r and θ, the DS takes the very simple form

$$\dot{r} = 0, \qquad \dot{\theta} = -r \cos\theta.$$

This is just a first, simple, example of the possible role of λ-symmetries in the context of DS.

2.1 Λ-symmetries of general DS

The natural way to extend the definition (1) of the first λ-prolongation of the vector field

$$X = \varphi_a(u,t) \frac{\partial}{\partial u_a} + \tau(u,t) \frac{\partial}{\partial t} = \varphi \cdot \nabla_u + \tau \partial_t$$

to the case of $m > 1$ variables u_a is the following (sum over repeated indices)

$$X^{(1)}_\Lambda = X^{(1)} + \Lambda_{ab}(\varphi_b - \tau \dot{u}_b) \cdot \nabla_{\dot{u}_a},$$

where now $\Lambda = \Lambda(t, u_a, \dot{u}_a)$ is an $(m \times m)$ matrix; accordingly I denote by the upper case Λ these symmetries in this context.

To simplify we assume from now that $\tau = 0$ (or use evolutionary vector field; it is not restrictive).

Then the given DS is Λ-invariant under X (or X is a Λ-symmetry for the DS), i.e., $X^{(1)}_\Lambda(\dot{u} - f)|_{\dot{u}=f} = 0$ if and only if

$$[\, f, \varphi\,]_a + \partial_t \varphi_a = -(\Lambda\varphi)_a \quad (a = 1, \ldots, m),$$

where $[\, f, \varphi\,]_a \equiv f_b \nabla_{u_b} \varphi_a - \varphi_b \nabla_{u_b} f_a$.

Given X, we now introduce the following new $m+1$ "canonical" (or *symmetry-adapted*) variables (*notice that they are independent of* Λ): precisely, $m-1$ variables $w_j = w_j(u)$ which, together with the time t, are X- invariant:

$$X\, w_j = X\, t = 0 \qquad (j = 1, \ldots, m-1)$$

and the coordinate z, "rectifying" the action of X, i.e. $X = \partial/\partial z$. Writing the given DS in these new variables, we obtain a "reduced" form of the DS, as stated by the following theorem [7–9].

Theorem 1. *Let X be a Λ-symmetry for a given DS; once the DS is written in terms of the new variables w_j, z, t, i.e.*

$$\dot{w}_j = W_j(w, z, t), \quad \dot{z} = Z(w, z, t),$$

the dependence on z of the r.h.s., W_j, Z, is controlled by the formulas

$$\frac{\partial W_j}{\partial z} = \frac{\partial w_j}{\partial u_a}(\Lambda\varphi)_a \equiv M_j, \qquad \frac{\partial Z}{\partial z} = \frac{\partial z}{\partial u_a}(\Lambda\varphi)_a \equiv M_m.$$

One has:

- *If $\Lambda = 0$, then $M_j = M_m = 0$ and W_j and Z are independent of z.*
- *If $\Lambda = \lambda I$, then only Z depends upon z.*
- *Otherwise a "partial" reduction is obtained: If some $M_k = 0$, then W_k is independent of z. In terms of the new variables the Λ-prolongation becomes*

$$X_\Lambda^{(1)} = \frac{\partial}{\partial z} + M_j\frac{\partial}{\partial \dot{w}_j} + M_m\frac{\partial}{\partial \dot{z}}.$$

The first case ($\Lambda = 0$) clearly means that X is an *exact*, or standard, Lie point-symmetry [7]; the second has been considered in detail by Muriel and Romero [8] (notice that actually it would be enough to require $\Lambda\varphi = \lambda\varphi$); the last case has been treated in [9]. Several situations can be met depending upon the number of vanishing M_j (e.g., one may obtain triangular or similar DS).

2.2 Hamiltonian DS

I now consider the special case in which the DS is an *Hamiltonian* DS. Obvious changes in the notation can be introduced: the m variables $u = u_a(t)$ are replaced by the $m = 2n$ variables, $q_\alpha(t)$ and $p_\alpha(t)$ ($\alpha = 1, \ldots, n$), and the DS is now the system of the Hamiltonian equations of motion for the given Hamiltonian $H = H(q, p, t)$:

$$\dot{u} = J\nabla H \equiv F(u,t)\,, \qquad \nabla \equiv \nabla_u \equiv (\nabla_q, \nabla_p),$$

where J is the standard symplectic matrix

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

A vector field X can be written accordingly (with $a = 1, \ldots, 2n\,;\ \alpha = 1, \ldots, n$) as

$$X = \varphi_\alpha(u,t)\frac{\partial}{\partial q_\alpha} + \psi_\alpha(u,t)\frac{\partial}{\partial p_\alpha} \equiv \mathbf{\Phi}\cdot\nabla_u, \quad \mathbf{\Phi} \equiv (\varphi_\alpha, \psi_\alpha)$$

and all the above discussion clearly holds if X is a Λ-symmetry for an Hamiltonian DS. Clearly here Λ is a $(2n \times 2n)$ matrix. However, Hamiltonian problems possess certainly a *richer* structure with respect to general DS and this deserves to be exploited; a first instance is clearly provided by the notion of conservation rules with its related topics.

I distinguish two cases:

(*i*) X admits a *generating function* $G(u,t)$ (then X is often called an "Hamiltonian symmetry"):

$$\mathbf{\Phi} = J\nabla G, \quad \text{i.e.} \quad \varphi = \nabla_p G \ , \ \psi = -\nabla_q G. \tag{2}$$

This implies $\nabla D_t G = 0$, where D_t is the total derivative, i.e., G is a constant of motion, $D_t G = 0$, possibly apart from an additional time-dependent term as is well known.

(*ii*) X does not admit a generating function: also in this case, defining

$$S(u,t) \equiv \nabla \cdot \mathbf{\Phi}, \quad \text{one has that} \quad D_t S = 0 \tag{3}$$

and therefore, if $S \neq \text{const}$, then S is a first integral (the examples known to me of first integrals of this form are rather tricky, being usually obtained multiplying symmetries by first integrals, but they "in principle" exist and their presence is important for the following discussion, see Subsection 3.4).

Direct calculations can show the following:

Theorem 2. *If the Hamiltonian equations of motion admit a* Λ*-symmetry* X *with a matrix* Λ*, then in case* (*i*) $\nabla(D_t G) = J\,\Lambda\,\mathbf{\Phi} = J\,\Lambda\,J\,\nabla\,G$ *and in case* (*ii*) $D_t S = -\nabla(\Lambda\,\mathbf{\Phi})$.

When this happens, G (resp. S) is called a "Λ-*constant of motion*".

If $\Lambda = 0$, i.e., when X is a "standard" (or "exact") symmetry, the above equations become clearly the usual conservation rules; Λ-symmetries can then be viewed as "perturbations" of the exact symmetries. More explicitly the equations in Theorem 2 state the precise "deviation" from the conservation of G (resp. of S) due to the fact that the invariance under X is "broken" by the presence of a nonzero matrix Λ.

As a special case for case (*i*) the following Corollary may be of interest:

Corollary 1. *Under mild assumptions* ($\Lambda\,\mathbf{\Phi} = \lambda\,\mathbf{\Phi}$, $\lambda = \lambda(G)$) *the* Λ*-constant of motion* G *satisfies a "completely separated equation" involving only* $G(t)$*:*

$$\dot{G} = \gamma(t, G).$$

This equation expresses how much the conservation of $G(t)$ is "violated" along the time evolution. If Λ is in some sense "small", then G is "almost" conserved.

3 When a Λ-symmetry of the Hamiltonian equations is inherited by a Λ-invariant Lagrangian

3.1 Λ-invariant Lagrangians, Noether theorem and Λ-conservation rules

I consider (for simplicity) only first-order Lagrangians:

$$\mathcal{L} = \mathcal{L}(q_\alpha, \dot{q}_\alpha, t) \quad (\alpha = 1, \ldots, n).$$

Such a Lagrangian is said to be $\Lambda^{(\mathcal{L})}$-invariant [10, 11] under

$$X^{(\mathcal{L})} = \varphi_\alpha(q,t)\frac{\partial}{\partial q_\alpha} = \varphi\cdot\nabla_q$$

if there is an $(n\times n)$ matrix $\Lambda^{(\mathcal{L})} = \Lambda^{(\mathcal{L})}(q,\dot q,t)$ such that $\big(X_\Lambda^{(\mathcal{L})}\big)^{(1)}(\mathcal{L}) = 0$, where $\big(X_\Lambda^{(\mathcal{L})}\big)^{(1)}$ is the first $\Lambda^{(\mathcal{L})}$-prolongation of $X^{(\mathcal{L})}$ (the notation is rather heavy in order to distinguish carefully the Lagrangian case from the Hamiltonian one which is considered in the next subsection). We then have [11]

Theorem 3. *If the Lagrangian $\mathcal{L}$ is $\Lambda^{(\mathcal{L})}$-invariant under $X^{(\mathcal{L})}$, then, putting $\mathcal{P}_{\alpha\beta} = \varphi_\alpha p_\beta$ with $p_\beta = \frac{\partial\mathcal{L}}{\partial\dot q_\beta}$, one has*

$$D_t\mathbf{P} = -\Lambda^{(\mathcal{L})}_{\alpha\beta}\varphi_\beta\frac{\partial\mathcal{L}}{\partial\dot q_\alpha} = -\big(\Lambda^{(\mathcal{L})}\varphi\big)_\alpha p_\alpha,$$

where $\mathbf{P} = \mathtt{Tr}(\mathcal{P}) = \varphi_\alpha p_\alpha$. If one introduces a "deformed derivative" $\widehat{D}_t$

$$(\widehat{D}_t)_{\alpha\beta} \equiv D_t\delta_{\alpha\beta} + \Lambda^{(\mathcal{L})}_{\alpha\beta}, \quad \textit{then} \quad \mathtt{Tr}(\widehat{D}_t\mathcal{P}) = 0.$$

This result can be called the *"Noether $\Lambda^{(\mathcal{L})}$-conservation rule"*. Indeed, if $\Lambda^{(\mathcal{L})}=0$, the standard Noether theorem is recovered.

In the special case $\Lambda^{(\mathcal{L})}\varphi = \lambda\varphi$ the above result becomes

$$\widehat{D}_t\mathbf{P} = 0, \quad \text{where} \quad \widehat{D}_t = D_t + \lambda.$$

Theorem 3 can be extended [11] to divergence symmetries and to generalized symmetries as well. Also higher-order Lagrangians can be included. The $\Lambda^{(\mathcal{L})}$-conservation rule has the same form, but $\mathcal{P}_{\alpha\beta}$ is different: for instance for second-order Lagrangians one has

$$\mathcal{P}_{\alpha\beta} = \varphi_\alpha\frac{\partial\mathcal{L}}{\partial\dot q_\beta} + \big((\widehat{D}_t)_{\alpha\gamma}\varphi_\gamma\big)\frac{\partial\mathcal{L}}{\partial\ddot q_\beta} - \varphi_\alpha D_t\frac{\partial\mathcal{L}}{\partial\ddot q_\beta}\ .$$

3.2 From Lagrangians to Hamiltonians

Assume that one has a Lagrangian which is $\Lambda^{(\mathcal{L})}$-invariant under a vector field

$$X^{(\mathcal{L})} = \varphi_\alpha\frac{\partial}{\partial q_\alpha}$$

and introduces the corresponding Hamiltonian H with its Hamiltonian equations of motion. The natural question is whether the $\Lambda^{(\mathcal{L})}$-symmetry $X^{(\mathcal{L})}$ of the Lagrangian is transferred to some $\Lambda^{(H)}$-symmetry $X^{(H)}$ of the Hamiltonian equations of motion. Two problems then arise: i) to extend the vector field $X^{(\mathcal{L})}$ to a suitable vector field $X^{(H)}$ and ii) to extend the $(n\times n)$ matrix $\Lambda^{(\mathcal{L})}$ to a suitable $(2n\times 2n)$ matrix $\Lambda^{(H)}$.

Firstly the vector field $X^{(H)}$ is expected to have the form

$$X \equiv X^{(H)} = \varphi_\alpha \frac{\partial}{\partial q_\alpha} + \psi_\alpha \frac{\partial}{\partial p_\alpha}, \tag{4}$$

where the coefficient functions ψ must be determined. This can be done observing that the variables p are related to $\dot q$ (and then the first $\Lambda^{(\mathcal{L})}$-prolongation of $X^{(\mathcal{L})}$ is needed, where the "effect" of $\Lambda^{(\mathcal{L})}$ is present). One finds, after some explicit calculations, that

$$\psi_\alpha = \frac{\partial}{\partial \dot q_\alpha}\Big(D_t \mathbf{P} + \Lambda^{(\mathcal{L})}_{\beta\gamma}\varphi_\gamma \frac{\partial \mathcal{L}}{\partial \dot q_\beta}\Big) - \frac{\partial \Lambda^{(\mathcal{L})}_{\beta\gamma}}{\partial \dot q_\alpha}\varphi_\gamma \frac{\partial \mathcal{L}}{\partial \dot q_\beta} - p_\beta \frac{\partial \varphi_\beta}{\partial q_\alpha}. \tag{5}$$

The term in parenthesis vanishes if the Lagrangian is $\Lambda^{(\mathcal{L})}$-invariant thanks to Theorem 3. In addition, if $\Lambda^{(\mathcal{L})}$ does not depend upon $\dot q$ (as happens in most cases, otherwise a separate treatment is needed, see Subsection 3.4), then we are left with

$$\psi_\alpha = -p_\beta \frac{\partial \varphi_\beta}{\partial q_\alpha}. \tag{6}$$

This implies that X admits a generating function, which is just $G = \varphi_\alpha p_\alpha \equiv \mathbf{P}$ in the notation introduced in Theorem 3.

Secondly one now introduces the $(2n \times 2n)$ matrix

$$\Lambda \equiv \Lambda^{(H)} = \begin{pmatrix} \Lambda^{(\mathcal{L})} & 0 \\ -\dfrac{\partial \Lambda^{(\mathcal{L})}}{\partial q_\alpha} p_\gamma & \Lambda^{(2)} \end{pmatrix},$$

where $\Lambda^{(2)}$ must satisfy (Λ is not uniquely defined, as is well known)

$$\Lambda^{(2)}_{\alpha\beta} \frac{\partial \varphi_\gamma}{\partial q_\beta} = \Lambda^{(\mathcal{L})}_{\gamma\beta} \frac{\partial \varphi_\beta}{\partial q_\alpha}.$$

It is well known that Euler–Lagrange equations coming from a $\Lambda^{(\mathcal{L})}$-invariant Lagrangian do *not exhibit* Λ*-symmetry in general.* In contrast with this it is not difficult to verify explicitly that the Hamiltonian equations of motion turn out to be $\Lambda^{(H)}$-symmetric under the vector field $X^{(H)}$ obtained according to the above prescription.

In conclusion I have shown the following

Theorem 4. *If $\mathcal{L}$ is a Λ-invariant Lagrangian under a vector field $X^{(\mathcal{L})}$ with a matrix $\Lambda^{(\mathcal{L})}$ (not depending on $\dot q$), one can extend $X^{(\mathcal{L})}$ to a vector field $X \equiv X^{(H)}$ and the $(n \times n)$ matrix $\Lambda^{(\mathcal{L})}$ to a $(2n \times 2n)$ matrix $\Lambda \equiv \Lambda^{(H)}$ in such a way that the resulting Hamiltonian equations of motion are Λ-symmetric under X; in addition $G = \varphi_\alpha p_\alpha$ is a Λ-constant of motion.*

Example 1. The Lagrangian (with $n = 2$)

$$\mathcal{L} = \frac{1}{2}\left(\frac{\dot{q}_1}{q_1} - q_1\right)^2 + \frac{1}{2}(\dot{q}_1 - q_1\dot{q}_2)^2 \exp(-2q_2) + q_1 \exp(-q_2)$$

is $\Lambda^{(\mathcal{L})}$-invariant under

$$X^{(\mathcal{L})} = q_1 \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} \quad \text{with} \quad \Lambda^{(\mathcal{L})} = \text{diag}\,(q_1, q_1).$$

It is easy to write the Hamiltonian equations of motion and to check that they are indeed Λ-symmetric under

$$X = q_1 \frac{\partial}{\partial q_1} + \frac{\partial}{\partial q_2} - p_1 \frac{\partial}{\partial p_1}$$

with

$$\Lambda = \Lambda^{(H)} = \begin{pmatrix} q_1 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 \\ -p_1 & -p_2 & q_1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

X-invariant coordinates are $w_1 = q_1 \exp(-q_2)$, $w_2 = q_1 p_1$ and $w_3 = p_2$. As is expected, the generating function $G = w_2 + w_3$ satisfies the Λ-conservation rule

$$\nabla_u D_t G = J\Lambda\mathbf{\Phi} \quad \text{or} \quad D_t G = -q_1 G.$$

A special, but rather common, case is described by the following:

Corollary 2. *If $\Lambda^{(\mathcal{L})}\varphi = c\,\varphi$, where c is a constant, then also $\Lambda\mathbf{\Phi} = c\,\mathbf{\Phi}$ and the "most complete" reduction of the Hamiltonian equations of motion is obtained:*

$$\dot{G} = \gamma(G, t), \quad \dot{w}_j = W_j(w, G, t) \quad \textit{and} \quad \dot{z} = Z(w, G, z, t).$$

3.3 Reduction of the Euler–Lagrange equations versus the Hamiltonian equations

In this section I want to compare the reduction procedure which is provided by the presence of a Λ-symmetry of a Lagrangian (i.e. the reduction of Euler–Lagrange equations) with the analogous reduction of the Hamiltonian equations of motion.

I start by recalling that any vector field $X = \varphi_\alpha \partial/\partial q_\alpha$ admits n (0th-order) invariants (as already said, see Subsection 2.1)

$$w_j = w_j(q, t) \quad (j = 1, \ldots, n-1) \quad \text{and the time } t$$

and n first-order differential invariants $\eta_\alpha = \eta_\alpha(q, t, \dot{q})$ under the first prolongation $X^{(1)}$

$$X^{(1)}\eta_\alpha = 0 \quad (\alpha = 1, \ldots, n).$$

Both if $X^{(1)}$ is standard *and* if it is a Λ prolongation (*under the condition* $\Lambda\,\varphi = \lambda\,\varphi$), it is well known that $\dot{w}_j$ are $n-1$ first-order differential invariants (note that this is an "algebraic" property and is not related to dynamics). If one now chooses another independent first-order differential invariant $\zeta = \zeta(q,t,\dot{q})$, then one has that any first-order $\Lambda^{(\mathcal{L})}$-invariant Lagrangian is a function of the above $2n$ invariants t, w_j, $\dot{w}_j$ and ζ. When one writes the Lagrangian in terms of these variables, the Euler–Lagrange equation for ζ is then simply

$$\frac{\partial \mathcal{L}}{\partial \zeta} \;=\; 0.$$

This first-order equation provides in general a "partial" reduction, i.e., it produces only *particular solutions*, even when the Euler–Lagrange equations for the other variables are considered [3, 10] (notice that this is true both for exactly invariant and for $\Lambda^{(\mathcal{L})}$-invariant Lagrangians).

I want to emphasize that, if one introduces Λ-symmetric Hamiltonian equations of motion along the lines stated in Theorem 4, a "better" reduction is obtained and no solution is lost. The following example clarifies this point.

Example 2. The Lagrangian ($n = 2$)

$$\mathcal{L} \;=\; \frac{1}{2}\left(\frac{\dot{q}_1}{q_1} - \log q_1\right)^2 + \frac{1}{2}\left(\frac{\dot{q}_1}{q_1} + \frac{\dot{q}_2}{q_2}\right)^2 \qquad (q_1 > 0)$$

is $\Lambda^{(\mathcal{L})}$-invariant under

$$X^{(\mathcal{L})} \;=\; q_1\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial q_2}$$

with $\Lambda^{(\mathcal{L})} = \mathrm{diag}\ (1,1)$. With $w = q_1 q_2$, $\dot{w} = \dot{q}_1 q_2 + q_1\dot{q}_2$, $\zeta = \frac{\dot{q}_1}{q_1} - \log q_1$ the Lagrangian becomes

$$\mathcal{L} \;=\; \frac{1}{2}\zeta^2 + \frac{1}{2}\frac{\dot{w}^2}{w^2}$$

and the Euler–Lagrange equation for ζ is

$$\partial\mathcal{L}/\partial\zeta = \zeta = 0 \quad \text{or} \quad \dot{q}_1 \;=\; q_1 \log q_1$$

with the particular solution $q_1(t) \;=\; \exp(c\,e^t)$. The corresponding Hamiltonian equations of motion are Λ-symmetric under

$$X \;=\; q_1\frac{\partial}{\partial q_1} - q_2\frac{\partial}{\partial q_2} - p_1\frac{\partial}{\partial p_1} + p_2\frac{\partial}{\partial p_2}$$

with $\Lambda = \mathrm{diag}\ (1,1,1,1)$. Invariants under this X are

$$w_1 \;=\; q_1 q_2, \quad w_2 \;=\; q_1 p_1, \quad w_3 \;=\; q_2 p_2$$

and X is generated by $G = w_2 - w_3$. A "complete" reduction is obtained. With $z = \log q_1$ we get

$$\dot w_1 \,=\, w_1 w_3, \quad \dot w_2 \,=\, w_3 - w_2, \quad \dot G \,=\, -G, \quad \dot z \,=\, z + w_2 - w_3.$$

The above "partial" (Lagrangian) solution $\zeta = 0$ corresponds to

$$\dot z \,=\, z, \quad w_2 = w_3 = c = \text{const}, \quad \dot w_1 = c w_1.$$

From the Hamiltonian equations, instead, one has e.g. $q_1(t) = \exp(ce^t) + c_1 \exp(-t)$ etc. The reader can easily complete the calculations.

3.4 When $\Lambda^{(\mathcal{L})}$ depends upon $\dot q$

If Λ depends also on $\dot q$ (see equations (4) and (5)), the calculations performed in Subsection 3.2 cannot be repeated, the coefficient functions ψ_α cannot be expressed in the simple form (6) and the vector field X does not admit a generating function G. In this case one can resort to the other quantity S, introduced in (3), which provides a Λ-constant of motion. An example can completely illustrate this situation.

Example 3. $(n = 1)$ The Lagrangian

$$\mathcal{L} \,=\, \frac{1}{2}\Big(\frac{\dot q}{q} + 1\Big)^2 \exp(-2q)$$

is $\Lambda^{(\mathcal{L})}$-invariant under $X^{(\mathcal{L})} \,=\, q\frac{\partial}{\partial q}$ with $\Lambda^{(\mathcal{L})} \,=\, q + \dot q$. One finds $\psi = -qp - p$ and the resulting vector field

$$X \,=\, q\frac{\partial}{\partial q} - (qp + p)\frac{\partial}{\partial p}$$

does *not* admit a generating function. Nevertheless the Hamiltonian equations of motion are Λ-symmetric under X with

$$\Lambda \,=\, \begin{pmatrix} q + \dot q & 0 \\ -p & q + \dot q \end{pmatrix}.$$

Here $S \,=\, -q$ satisfies $D_t S = -\nabla(\Lambda\Phi)$ and is a Λ-constant of motion.

4 A digression: general Λ-invariant Lagrangians

The Λ-invariance of a Lagrangian $\mathcal{L} \,=\, \mathcal{L}(q, \dot q, t)$ considered in Subsection 3.1 is a special case of a much more general situation. Instead of n time-dependent quantities $q_\alpha(t)$, consider now n "fields"

$$u_\alpha(x_i) \quad (\alpha = 1, \ldots, n\,;\; i = 1, \ldots, s)$$

depending upon $s > 1$ real variables x_i. Now the Euler–Lagrange equations become a system of PDEs and the notion of μ-symmetry [12, 13] replaces that of λ-symmetry (or Λ-symmetry if $n > 1$).

In this case there are $s > 1$ matrices Λ_i $(n \times n)$, which must satisfy the compatibility condition

$$D_i\Lambda_j - D_j\Lambda_i + [\Lambda_i, \Lambda_j] = 0 \quad (D_i \equiv D_{x_i}). \tag{7}$$

This can be rewritten putting $\widehat{D}_i = D_i\,\delta + \Lambda_i$ (or in explicit form $(\widehat{D}_i)_{\alpha\beta} = D_i\delta_{\alpha\beta} + (\Lambda_i)_{\alpha\beta}$ with a notation extending the one introduced in Theorem 3) as $[\widehat{D}_i, \widehat{D}_j] = 0$. Then one has [12, 13]:

Theorem 5. *Given $s > 1$ matrices Λ_i satisfying* (7)*, there exists (locally) an $(n \times n)$ nonsingular matrix Γ such that $\Lambda_i = \Gamma^{-1}(D_i\Gamma)$. If a Lagrangian $\mathcal{L}$ is Λ-invariant under a vector field $X = \varphi_\alpha \partial/\partial u_\alpha$, then there is a matrix-valued vector $\mathcal{P}_i \equiv (\mathcal{P}_i)_{\alpha\beta}$ which is Λ-conserved. This Λ-conservation law holds in the form (with the symbol* `Tr` *defined to be $\sum_{\alpha=1}^{n}$)*

$$\mathtt{Tr}\left[\Gamma^{-1}D_i\big(\Gamma\,\mathcal{P}_i\big)\right] = 0$$

or in the equivalent forms

$$D_i\mathbf{P}_i = -(\Lambda_i)_{\alpha\beta}(\mathcal{P}_i)_{\beta\alpha} = -\mathtt{Tr}(\Lambda_i\mathcal{P}_i), \quad \textit{where} \quad \mathbf{P}_i = (\mathcal{P}_i)_{\alpha\alpha} = \mathtt{Tr}\,\mathcal{P}_i,$$
$$\mathtt{Tr}(\widehat{D}_i\,\mathcal{P}_i) = 0.$$

For first-order Lagrangians one has

$$(\mathcal{P}_i)_{\alpha\beta} = \varphi_\alpha \frac{\partial L}{\partial u_{\beta,i}}, \quad \text{where} \quad u_{\beta,i} = \frac{\partial u_\beta}{\partial x_i}.$$

and for second-order Lagrangians

$$(\mathcal{P}_i)_{\alpha\beta} = \varphi_\beta \frac{\partial \mathcal{L}}{\partial u_{\alpha,i}} + \big((\widehat{D}_j)_{\beta\gamma}\varphi_\gamma\big)\frac{\partial \mathcal{L}}{\partial u_{\alpha,ij}} - \varphi_\beta D_j \frac{\partial \mathcal{L}}{\partial u_{\alpha,ij}}.$$

Example 4. Let $n = s = 2$. For ease of notation we write x, y instead of x_1, x_2 and $u = u(x, y)$, $v = v(x, y)$ instead of u_1, u_2. Consider the vector field

$$X = u\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \tag{8}$$

and the two matrices

$$\Lambda_1 = \begin{pmatrix} 0 & 0 \\ u_x & 0 \end{pmatrix}, \quad \Lambda_2 = \begin{pmatrix} 0 & 0 \\ u_y & 0 \end{pmatrix} \quad \text{and then} \quad \Gamma = \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}.$$

It is easy to check that the Lagrangian

$$\mathcal{L} = \frac{1}{2}\Big(u_x^2 + u_y^2\Big) - \frac{1}{u}\big(u_xv_x + u_yv_y\big) + u^2\exp(-2v)$$

is Λ-invariant (or better, in this context, μ-invariant) but not invariant under the above vector field X. The μ-conservation law $\mathtt{Tr}(\widehat{D}_i \mathcal{P}_i) = 0$ takes here the form

$$D_i \mathbf{P}^i \equiv D_x \left(u u_x - v_x - \frac{u_x}{u} \right) + D_y \left(u u_y - v_y - \frac{u_y}{u} \right) \ = \ u_x^2 + u_y^2 \, .$$

In agreement with Theorem 5, the r.h.s. of this expression is precisely equal to

$$-\mathtt{Tr}(\Lambda_i \mathcal{P}_i) = -(\Lambda_i \varphi)_\alpha \frac{\partial \mathcal{L}}{\partial u_{\alpha,i}}.$$

Notice that in this case the quantity $u_x^2 + u_y^2$ is just the "symmetry-breaking term", i.e. the term which prevents the above Lagrangian from being exactly symmetric under the vector field (8).

It should be remarked that μ-symmetries are actually strictly related to *standard* symmetries, or – more precisely – are *locally gauge-equivalent* to them (see for details [4, 11, 14]).

Given indeed the vector field $X = \varphi_\alpha \partial / \partial u_\alpha$ and the s matrices Λ_i, I denote by

$$X_\Lambda^{(\infty)} \ = \ \sum_J \Psi_\alpha^{(J)} \frac{\partial}{\partial u_{\alpha,J}}$$

the infinite Λ-prolongation of X, where the sum is over all multi-indices J as usual and $\Psi_\alpha^{(0)} = \varphi_\alpha$. Introducing now the other vector field $\widetilde{X}$

$$\widetilde{X} \equiv \widetilde{\phi}_\alpha \frac{\partial}{\partial u_\alpha} \quad \text{with} \quad \widetilde{\varphi}_\alpha \equiv (\Gamma \, \varphi)_\alpha ,$$

where Γ is assigned in Theorem 5, and denoting by

$$\widetilde{X}^{(\infty)} \ = \ \sum_J \widetilde{\varphi}_\alpha^{(J)} \frac{\partial}{\partial u_{\alpha,J}}$$

the *standard* prolongation of $\widetilde{X}$, one has [11,13] that the coefficient functions $\Psi_\alpha^{(J)}$ of the Λ prolongation of X are connected to the coefficient functions $\widetilde{\varphi}_\alpha^{(J)}$ of the standard prolongation of $\widetilde{X}$ by the relation

$$\Psi_\alpha^{(J)} \ = \ \Gamma^{-1} \widetilde{\varphi}_\alpha^{(J)}.$$

In the particularly simple case $n = 1$ (i.e. a single "field" $u(x_i)$) the $s > 1$ matrices Λ_i and the matrix Γ as well become (scalar) functions λ_i and γ; in this case, if a Lagrangian is μ-invariant under the vector field X, then it is also invariant under the *standard* symmetry $\widetilde{X} = \gamma X$. In addition the μ conservation law can be also expressed as a standard conservation rule

$$D_i \widetilde{\mathbf{P}}^i \ = \ 0$$

where $\widetilde{\mathbf{P}}^i = \gamma \, \varphi_\alpha \partial \mathcal{L} / \partial u_{\alpha,i}$ is the "current density vector" determined by the vector field $\widetilde{X} = \gamma X$.

Example 5. Let now $n = 1$, $s = 2$, and let me introduce for convenience as independent variables the polar coordinates r, θ. I am considering a single "field" $u = u(r, \theta)$ and the rotation vector field $X = \partial/\partial\theta$. The Lagrangian

$$\mathcal{L} = \frac{1}{2} r^2 \exp(-\epsilon\theta) u_r^2 + \frac{1}{2} \exp(\epsilon\theta) u_\theta^2$$

is clearly not invariant under rotational symmetry (if $\epsilon \neq 0$), but is μ-invariant with $\lambda_1 = 0$, $\lambda_2 = \epsilon$. The above Lagrangian is the Lagrangian of a perturbed Laplace equation, indeed the Euler–Lagrange equation is the PDE

$$r^2 u_{rr} + 2r u_r + \exp(2\epsilon\theta)(u_{\theta\theta} + \epsilon\, u_\theta) = 0.$$

It is easy to check that the current density vector

$$\mathbf{P} \equiv \Big(- r^2 \exp(-\epsilon\theta) u_r u_\theta,\ \frac{1}{2} r^2 \exp(-\epsilon\theta) u_r^2 - \frac{1}{2} \exp(\epsilon\theta) u_\theta^2 \Big)$$

satisfies the μ-conservation law

$$D_i \mathbf{P}_i = -\epsilon \mathbf{P}_2.$$

According to the above remark on the (local) equivalence of the μ-symmetry X to the standard symmetry $\widetilde{X} = \gamma X = \exp(\epsilon\theta)\, \partial/\partial\theta$ also the (standard) conservation law $D_i \widetilde{\mathbf{P}}^i = 0$ holds with

$$\widetilde{\mathbf{P}} \equiv \Big(- r^2 u_r u_\theta,\ \frac{1}{2} r^2 u_r^2 - \frac{1}{2} \exp(2\epsilon\theta) u_\theta^2 \Big).$$

5 Conclusions

I have shown that the notion of λ-symmetry and the related procedures for studying differential equations can be conveniently extended to the case of dynamical systems.

The use and the interpretation of this notion becomes particularly relevant when the DS is an Hamiltonian system and even more if the symmetry is inherited by an invariant Lagrangian: in this context indeed it is possible to introduce in a natural way and to draw a comparison between the notions of Λ-constant of motion and of Noether Λ-conservation rule. Similarly the symmetry properties of Euler–Lagrange equations and of the Hamiltonian ones can be compared, and some reduction techniques for the equations can be conveniently introduced.

Finally I have shown that the Λ-invariance of the Lagrangians in the context of the DS is a special case of a more general and richer situation, where several independent variables are present and a Λ-conservation rule of very general form is true.

Another interesting problem is the nontrivial relationship between λ (or Λ or μ) symmetries with the standard ones. An aspect of this problem has been mentioned in the above section of this paper. In different situations this may involve the introduction of nonlocal symmetries and other concepts in differential geometry, as briefly indicated in the Introduction, which clearly go beyond the scope of the present contribution.

[1] Muriel C. and Romero J.L., New method of reduction for ordinary differential equations, *IMA J. Appl. Math.*, 2001, V.66, 111–125.

[2] Muriel C. and Romero J.L., C^∞-symmetries and nonsolvable symmetry algebras, *IMA J. Appl. Math.*, 2001, V.66, 477–498.

[3] Cicogna G., Symmetries of Hamiltonian equations and Λ-constants of motion, *J. Nonlinear Math. Phys.*, 2009, V.16, 43–60.

[4] Gaeta G., Twisted symmetries of differential equations, *J. Nonlinear Math. Phys.*, 2009, V.16, 107–136.

[5] Gaeta G. and Cicogna G., Twisted symmetries and integrable systems, *Int. J. Geom. Meth. Math. Phys.*, 2009, V.6, 1306–1321.

[6] Levi D. and Rodrìguez M.A., λ-symmetries for discrete equations, *J. Phys. A: Math. Theor.*, 2010, V.43, 292001, 9 pp.

[7] Olver P.J., *Application of Lie Groups to Differential Equations*, Springer, Berlin, 1986.

[8] Muriel C. and Romero J.L., C^∞-symmetries and integrability of ordinary differential equations, *Proceedings of the First Colloquium on Lie Theory and Applications*, Editors: Bajo I. and Sanmartin E., Publicacións da Universidade de Vigo, Vigo, 2002.

[9] Cicogna G., Reduction of systems of first-order differential equations via Λ-symmetries, *Phys. Lett. A*, 2008, V.372, 3672–3677.

[10] Muriel C., Romero J.L. and Olver P.J., Variational C^∞-symmetries and Euler–Lagrange equations, *J. Differential Equations*, 2006, V.222, 164–184.

[11] Cicogna G. and Gaeta G., Noether theorem for μ-symmetries, *J. Phys. A: Math. Theor.*, 2007, V.40, 11899–11921.

[12] Gaeta G. and Morando P., On the geometry of lambda-symmetries and PDEs reduction, *J. Phys. A: Math. Gen.*, 2004, V.37, 6955–6975.

[13] Cicogna G., Gaeta G. and Morando P., On the relation between standard and μ-symmetries for PDEs, *J. Phys. A: Math. Gen.*, 2004, V.37, 9467–9486.

[14] Gaeta G., A gauge-theoretic description of μ-prolongations, and μ-symmetries of differential equations, *J. Geom. Phys.*, 2009, V.59, 519–539.

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Poisson brackets with prescribed Casimirs. I

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We consider the problem of constructing Poisson brackets with prescribed Casimir functions. The problem is completely solved in the case of an almost symplectic manifold.

1 Introduction

Let f be an arbitrary smooth function on $\mathbb{R}^3$ and define

$$\{x, y\} = \frac{\partial f}{\partial z}, \quad \{x, z\} = -\frac{\partial f}{\partial y} \quad \text{and} \quad \{y, z\} = \frac{\partial f}{\partial x}. \tag{1}$$

Then this bracket is Poisson and it has f as Casimir function.

More generally let $f_1, f_2, \ldots, f_k$ be functionally independent smooth functions on $\mathbb{R}^{k+2}$ and Ω a nonvanishing $(k+2)$-smooth form on $\mathbb{R}^{k+2}$. Then the formula

$$\{g, h\}\Omega = dg \wedge dh \wedge df_1 \wedge \cdots \wedge df_k \tag{2}$$

defines a Poisson bracket on $\mathbb{R}^{k+2}$ with $f_1, \ldots, f_k$ as Casimir invariants. The Jacobian Poisson structure (2) (the bracket $\{g, h\}$ is equal, up to a coefficient function, with the usual Jacobian determinant of $(g, h, f_1, \ldots, f_k)$) appeared in [2] in 1989 where it was attributed to H. Flaschka and T. Ratiu. The first explicit proof appeared in [9]. The first application of formula (2) is in [2,3] in conjunction with transverse Poisson structures to subregular elements in $gl(n, \mathbb{C})$. It was shown that the transverse Poisson structure to the subregular orbit which is usually computed using Dirac's constraint formula can be calculated using the Jacobian Poisson structure (2). This fact was extended to any simple Lie algebra in [5]. In the same paper it is also proved that with a suitable change of coordinates the transverse Poisson structure is reduced to a three-dimensional structure of the form (1).

The purpose of this paper is to generalize formula (2) in the case of $\mathbb{R}^{2n}$ with given $(2n-2k)$ functionally independent functions $f_1, f_2, \ldots, f_{2n-2k}$. Actually we prove the result in the more general setting of an almost symplectic manifold (M, ω_0) with a volume form Ω which is a power of ω_0. We assume that the matrix $\big(\{f_i, f_j\}_0\big)$ of the brackets of $f_1, \ldots, f_{2n-2k}$ with respect to Λ_0 (the bivector field on M corresponding to ω_0) is invertible on an open and dense subset $\mathcal{U}$ of M. Then the $(2n-2)$-form

$$\Phi = -\frac{1}{f}\Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}, \tag{3}$$

where f satisfies $f^2 = \det\big(\{f_i, f_j\}_0\big)$, σ is a 2-form on M satisfying some special requirements (see Theorem 1) and $g = i_{\Lambda_0}\sigma$, corresponds to a Poisson tensor field Λ on M with orbits of dimension at most $2k$ for which $f_1, \dots, f_{2n-2k}$ are Casimir functions. Precisely $\Lambda = \Lambda_0^{\#}(\sigma)$ and the associated bracket of Λ on $C^\infty(M)$ is given for any $h_1, h_2 \in C^\infty(M)$ by

$$\{h_1, h_2\}\Omega = -\frac{1}{f} dh_1 \wedge dh_2 \wedge \Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}. \quad (4)$$

Conversely, if Λ is a Poisson tensor on (M, ω_0) of rank at most $2k$ on an open and dense subset $\mathcal{U}$ of M, then there are $2n - 2k$ functionally independent smooth functions $f_1, \dots, f_{2n-2k}$ on $\mathcal{U}$ and a suitable 2-form σ on M such that $\Psi_\Lambda = -i_\Lambda \Omega$ and $\{\cdot, \cdot\}$ are given, respectively, by (3) and (4).

The proof of this result is given in Section 3. Section 2 consists of preliminaries and fixing the notation. Finally in Section 4 we give two applications. One on Dirac brackets and one which gives as a byproduct a new method to obtain the Volterra lattice from the Toda lattice.

2 Preliminaries

We start by fixing our notation and by recalling the most important notions and formulas needed in this paper. Let M be a real smooth m-dimensional manifold, TM and T^*M its tangent and cotangent bundles and $C^\infty(M)$ the space of smooth functions on M. For each $p \in \mathbb{Z}$ we denote by $\mathcal{V}^p(M)$ and $\Omega^p(M)$ the spaces of smooth sections, respectively, of $\bigwedge^p TM$ and $\bigwedge^p T^*M$. By convention we set $\mathcal{V}^p(M) = \Omega^p(M) = \{0\}$, for $p < 0$, $\mathcal{V}^0(M) = \Omega^0(M) = C^\infty(M)$ and, taking into account the skew-symmetry, we have $\mathcal{V}^p(M) = \Omega^p(M) = \{0\}$, for $p > m$. Finally we set $\mathcal{V}(M) = \oplus_{p\in\mathbb{Z}}\mathcal{V}^p(M)$ and $\Omega(M) = \oplus_{p\in\mathbb{Z}}\Omega^p(M)$.

2.1 From multivector fields to differential forms and back

There is a natural *pairing* between the elements of $\Omega(M)$ and $\mathcal{V}(M)$ that is the $C^\infty(M)$-bilinear map $\langle\cdot,\cdot\rangle\colon \Omega(M) \times \mathcal{V}(M) \to C^\infty(M)$, $(\eta, P) \mapsto \langle \eta, P\rangle$, defined as follows: For any $\eta \in \Omega^q(M)$ and $P \in \mathcal{V}^p(M)$ with $p \neq q$, $\langle \eta, P\rangle = 0$; for any $f, g \in \Omega^0(M)$, $\langle f, g\rangle = fg$; while, if $\eta = \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_p \in \Omega^p(M)$ is a decomposable p-form ($\eta_i \in \Omega^1(M)$) and $P = X_1 \wedge X_2 \wedge \cdots \wedge X_p$ is a decomposable p-vector field ($X_i \in \mathcal{V}^1(M)$),

$$\langle \eta, P\rangle = \langle \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_p, X_1 \wedge X_2 \wedge \cdots \wedge X_p\rangle = \det(\langle \eta_i, X_j\rangle).$$

The above definition is extended to the nondecomposable forms and multivector fields by bilinearity in a unique way. Precisely, for any $\eta \in \Omega^p(M)$ and $X_1, \dots, X_p \in \mathcal{V}^1(M)$,

$$\langle \eta, X_1 \wedge X_2 \wedge \cdots \wedge X_p\rangle = \eta(X_1, X_2, \dots, X_p).$$

Similarly, for $P \in \mathcal{V}^p(M)$ and $\eta_1, \eta_2, \ldots, \eta_p \in \Omega^1(M)$,

$$\langle \eta_1 \wedge \eta_2 \wedge \cdots \wedge \eta_p, P\rangle = P(\eta_1, \eta_2, \ldots, \eta_p).$$

For the *interior product* $i_P\colon \Omega(M) \to \Omega(M)$ *of differential forms by a p-vector field P*, viewed as a $C^\infty(M)$-linear endomorphism of $\Omega(M)$ of degree $-p$, we adopt the following convention. If $P = X \in \mathcal{V}^1(P)$ and η is a q-form, $i_X\eta$ is the element of $\Omega^{q-1}(M)$ defined for any $X_1, \ldots, X_{q-1} \in \mathcal{V}^1(M)$ by

$$(i_X\eta)(X_1, \ldots, X_{q-1}) = \eta(X, X_1, \ldots, X_{q-1}).$$

If $P = X_1 \wedge X_2 \wedge \cdots \wedge X_p$ is a decomposable p-vector field, we put

$$i_P\eta = i_{X_1\wedge X_2\wedge\cdots\wedge X_p}\eta = i_{X_1}i_{X_2}\cdots i_{X_p}\eta.$$

More generally, recalling that each $P \in \mathcal{V}^p(M)$ can be locally written as the sum of decomposable p-vector fields, we define as $i_P\eta$, with $\eta \in \Omega^q(M)$ and $q \geq p$, the unique element of $\Omega^{q-p}(M)$ such that for any $Q \in \mathcal{V}^{q-p}(M)$

$$\langle i_P\eta, Q\rangle = (-1)^{(p-1)p/2}\langle \eta, P \wedge Q\rangle. \tag{5}$$

Similarly we define the *interior product* $j_\eta\colon \mathcal{V}(M) \to \mathcal{V}(M)$ *of multivector fields by a q-form* η. If $\eta = \alpha \in \Omega^1(M)$ and $P \in \mathcal{V}^p(M)$, then $j_\alpha P$ is the unique $(p-1)$-vector field on M given, for any $\alpha_1, \ldots, \alpha_{p-1}$, by

$$(j_\alpha P)(\alpha_1, \ldots, \alpha_{p-1}) = P(\alpha_1, \ldots, \alpha_{p-1}, \alpha).$$

Moreover, if $\eta = \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_q$ is a decomposable q-form, we set

$$j_\eta P = j_{\alpha_1\wedge\alpha_2\wedge\cdots\wedge\alpha_q}P = j_{\alpha_1}j_{\alpha_2}\cdots j_{\alpha_q}P.$$

Hence, using the fact that any $\eta \in \Omega^q(M)$ can be locally written as the sum of decomposable q-forms, we define j_η to be the $C^\infty(M)$-linear endomorphism of $\mathcal{V}(M)$ of degree $-q$ which associates with each $P \in \mathcal{V}^p(M)$ $(p \geq q)$ the unique $(p-q)$-vector field $j_\eta P$ defined for any $\zeta \in \Omega^{p-q}(M)$ by

$$\langle \zeta, j_\eta P\rangle = \langle \zeta \wedge \eta, P\rangle.$$

If the degrees of η and P are equal, i.e., $q = p$, the interior products $j_\eta P$ and $i_P\eta$ are up to sign equal:

$$j_\eta P = (-1)^{(p-1)p/2} i_P\eta = \langle \eta, P\rangle.$$

The *Schouten bracket* $[\cdot,\cdot]\colon \mathcal{V}(M) \times \mathcal{V}(M) \to \mathcal{V}(M)$, which is a natural extension of the usual Lie bracket of vector fields on the space $\mathcal{V}(M)$ [7, 11], is related with the operator i with the following useful formula due to Koszul [11]. For any $P \in \mathcal{V}^p(M)$ and $Q \in \mathcal{V}^q(M)$

$$i_{[P,Q]} = [[i_P, d], i_Q], \tag{6}$$

where the brackets on the right hand side of (6) denote the graded commutator of graded endomorphisms of $\Omega(M)$, i.e., for any two endomorphisms E_1 and E_2 of $\Omega(M)$ of degrees e_1 and e_2, respectively, $[E_1, E_2] = E_1 \circ E_2 - (-1)^{e_1 e_2} E_2 \circ E_1$. Hence we have

$$\begin{aligned} i_{[P,Q]} &= i_P \circ d \circ i_Q - (-1)^p\, d \circ i_P \circ i_Q \\ &\quad - (-1)^{(p-1)q}\, i_Q \circ i_P \circ d + (-1)^{(p-1)q-p}\, i_Q \circ d \circ i_P. \end{aligned} \tag{7}$$

Furthermore for a given smooth *volume form* Ω on M, i.e., Ω is a nowhere vanishing element of $\Omega^m(M)$, the interior product of p-vector fields on M, $p = 0, 1, \dots, m$, with Ω yields a $C^\infty(M)$-linear isomorphism Ψ of $\mathcal{V}(M)$ onto $\Omega(M)$ such that for each degree p, $0 \le p \le m$,

$$\begin{aligned} \Psi\colon \mathcal{V}^p(M) &\to \Omega^{m-p}(M), \\ P &\mapsto \Psi(P) = \Psi_P = (-1)^{(p-1)p/2} i_P \Omega. \end{aligned}$$

Its inverse maps $\Psi^{-1}\colon \Omega^{m-p}(M) \to \mathcal{V}^p(M)$ is defined for any $\eta \in \Omega^{m-p}(M)$ by $\Psi^{-1}(\eta) = j_\eta \tilde{\Omega}$, where $\tilde{\Omega}$ denotes the dual m-vector field of Ω, i.e., $\langle \Omega, \tilde{\Omega} \rangle = 1$. By composing Ψ with the exterior derivative d on $\Omega(M)$ and Ψ^{-1} we obtain the operator $D = -\Psi^{-1} \circ d \circ \Psi$ which was introduced by Koszul [11]. It is of degree -1 and of square 0 and it generates the Schouten bracket. For any $P \in \mathcal{V}^p(M)$ and $Q \in \mathcal{V}(M)$

$$[P, Q] = (-1)^p \big(D(P \wedge Q) - D(P) \wedge Q - (-1)^p P \wedge D(Q) \big). \tag{8}$$

2.2 Poisson manifolds

We recall the notion of *Poisson manifold* and some of its properties the proofs of which may be found, for example, in the books [7, 13, 15].

A *Poisson structure* on a smooth manifold M is a Lie algebraic structure on $C^\infty(M)$ the bracket of which $\{\,,\,\}\colon C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfies Leibniz' rule:

$$\{f, gh\} = \{f, g\}h + g\{f, h\}, \quad \forall\, f, g, h \in C^\infty(M).$$

In [14] A. Lichnérowicz remarks that $\{\,,\,\}$ gives rise to a contravariant antisymmetric tensor field Λ on M of order 2 such that $\Lambda(df, dg) = \{f, g\}$ for $f, g \in C^\infty(M)$. Conversely each such bivector field Λ on M gives rise to a bilinear and antisymmetric bracket $\{\,,\,\}$ on $C^\infty(M)$, $\{f, g\} = \Lambda(df, dg)$, $f, g \in C^\infty(M)$, that satisfies the Jacobi identity, i.e., for any $f, g, h \in C^\infty(M)$, $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$, if and only if $[\Lambda, \Lambda] = 0$, where $[\,,\,]$ denotes the Schouten bracket on $\mathcal{V}(M)$. In this case Λ is called a *Poisson tensor* and the manifold (M, Λ) *Poisson manifold*.

As was proved in [9], it is a consequence of expression (2.1) of $[\cdot, \cdot]$ that an element $\Lambda \in \mathcal{V}^2(M)$ defines a Poisson structure on M if and only if

$$2 i_\Lambda d\Psi_\Lambda + d\Psi_{\Lambda \wedge \Lambda} = 0.$$

Equivalently, using formula (8) of $[\cdot,\cdot]$ and the fact that, for any $P \in \mathcal{V}^p(M)$, $\Psi^{-1} \circ i_P = (-1)^{(p-1)p/2} P \wedge \Psi^{-1}$, the last condition can be written as

$$2\Lambda \wedge D(\Lambda) = D(\Lambda \wedge \Lambda). \tag{9}$$

Given a Poisson tensor Λ on M, we can associate to it a natural homomorphism $\Lambda^{\#}\colon \Omega^1(M) \to \mathcal{V}^1(M)$, which maps each element α of $\Omega^1(M)$ to a unique vector field $\Lambda^{\#}(\alpha)$ such that for any $\beta \in \Omega^1(M)$

$$\langle \alpha \wedge \beta, \Lambda\rangle = \langle \beta, \Lambda^{\#}(\alpha)\rangle = \Lambda(\alpha,\beta).$$

If $\alpha = df$, $f \in C^\infty(M)$, the vector field $\Lambda^{\#}(df)$ is called the *hamiltonian vector field of* f and it is denoted by X_f. The image $\mathrm{Im}\Lambda^{\#}$ of $\Lambda^{\#}$ is a completely integrable distribution on M and defines the *symplectic foliation* of (M,Λ) the space of first integrals of which is the space of *Casimir functions of* Λ, i.e. the space of the functions $f \in C^\infty(M)$ that are solutions of $\Lambda^{\#}(df) = 0$.

Moreover $\Lambda^{\#}$ can be extended to a homomorphism, also denoted by $\Lambda^{\#}$, from $\Omega^p(M)$ to $\mathcal{V}^p(M)$, $p \in \mathbb{N}$, by setting, for any $f \in C^\infty(M)$, $\Lambda^{\#}(f) = f$ and, for any $\zeta \in \Omega^p(M)$ and $\alpha_1,\ldots,\alpha_p \in \Omega^1(M)$,

$$\Lambda^{\#}(\zeta)(\alpha_1,\ldots,\alpha_p) = (-1)^p \zeta(\Lambda^{\#}(\alpha_1),\ldots,\Lambda^{\#}(\alpha_p)). \tag{10}$$

Thus $\Lambda^{\#}(\zeta\wedge\eta) = \Lambda^{\#}(\zeta)\wedge\Lambda^{\#}(\eta)$ for all $\eta \in \Omega(M)$. When $\Omega(M)$ is equipped with the Koszul bracket $\{\!\!\{\cdot,\cdot\}\!\!\}$ defined for any $\zeta \in \Omega^p(M)$ and $\eta \in \Omega(M)$ by

$$\{\!\!\{\zeta,\eta\}\!\!\} = (-1)^p\big(\Delta(\zeta\wedge\eta) - \Delta(\zeta)\wedge\eta - (-1)^p\zeta\wedge\Delta(\eta)\big), \tag{11}$$

where $\Delta = i_\Lambda \circ d - d \circ i_\Lambda$, and $\mathcal{V}(M)$ with the Schouten bracket $[\cdot,\cdot]$, $\Lambda^{\#}$ becomes a graded Lie algebraic homomorphism. Explicitly

$$\Lambda^{\#}(\{\!\!\{\zeta,\eta\}\!\!\}) = [\Lambda^{\#}(\zeta),\Lambda^{\#}(\eta)]. \tag{12}$$

Example 1. Any symplectic manifold (M,ω_0), where ω_0 is a nondegenerate closed smooth 2-form on M, is equipped with a Poisson structure Λ_0 defined by ω_0 as follows. The tensor field Λ_0 is the image of ω_0 by the extension of the isomorphism $\Lambda_0^{\#}\colon \Omega^1(M) \to \mathcal{V}^1(M)$, inverse of $\omega_0^{\flat}\colon \mathcal{V}^1(M) \to \Omega^1(M)$, $X \mapsto \omega_0^{\flat}(X) = -\omega_0(X,\cdot)$, to $\Omega^p(M)$, $p \in \mathbb{N}$, given by (10).

2.3 Decomposition theorem for exterior differential forms

In this subsection we begin by reviewing some important results concerning the decomposition theorem for exterior differential forms on an almost symplectic manifold (M,ω_0) of dimension $2n$, i.e., ω_0 is a nondegenerate smooth 2-form on M a complete study of which is given in [13] and [12].

Let (M,ω_0) be an almost symplectic manifold, Λ_0 the bivector field on M associated with ω_0 (see, Example 1), $\Omega = \omega_0^n/n!$ the corresponding volume form on M and $\tilde{\Omega} = \Lambda_0^n/n!$ the dual $2n$-vector field of Ω. We define an isomorphism $*\colon \Omega^p(M) \to \Omega^{2n-p}(M)$ by setting

$$* \varphi = (-1)^{(p-1)p/2} \, i_{\Lambda_0^\#(\varphi)} \frac{\omega_0^n}{n!}$$

for any $\varphi \in \Omega^p(M)$.

Remark 1. In order to be in agreement with the convention of sign adopted in (5) for the interior product, we make a sign convention for $*$ different from the one given in [13].

The $(2n-p)$-form $*\varphi$ is called the *adjoint of φ relative to ω_0*. The isomorphism $*$ has the following properties:

i) $** = \mathrm{Id}$.

ii) For any $\varphi \in \Omega^p(M)$ and $\psi \in \Omega^q(M)$

$$\begin{aligned} * (\varphi \wedge \psi) &= (-1)^{(p+q-1)(p+q)/2} \, i_{\Lambda_0^\#(\varphi) \wedge \Lambda_0^\#(\psi)} \frac{\omega_0^n}{n!} \\ &= (-1)^{(p-1)p/2} \, i_{\Lambda_0^\#(\varphi)} (* \psi) = (-1)^{pq + (q-1)q/2} i_{\Lambda_0^\#(\psi)} (* \varphi). \end{aligned} \tag{13}$$

iii) For any $k \leq n$

$$* \frac{\omega_0^k}{k!} = \frac{\omega_0^{n-k}}{(n-k)!}.$$

Definition 1. A smooth form $\psi \in \Omega(M)$ such that $i_{\Lambda_0} \psi = 0$ everywhere on M is said to be *effective*. A smooth form φ on M is said to be *simple* if it can be written as $\varphi = \psi \wedge (\omega_0^k/k!)$, where ψ is effective.

Proposition 1. *The adjoint of an effective differential form ψ of degree $p \leq n$ is*

$$* \psi = (-1)^{p(p+1)/2} \, \psi \wedge \frac{\omega_0^{n-p}}{(n-p)!}.$$

The adjoint $*\varphi$ of a smooth $(p+2k)$-simple form $\varphi = \psi \wedge \dfrac{\omega_0^k}{k!}$ is

$$* \varphi = (-1)^{p(p+1)/2} \, \psi \wedge \frac{\omega_0^{n-p-k}}{(n-p-k)!}. \tag{14}$$

Lepage's Decomposition Theorem 2. *Every differential form $\varphi \in \Omega(M)$ of degree $p \leq n$ may be uniquely decomposed as the sum*

$$\varphi = \psi_p + \psi_{p-2} \wedge \omega_0 + \cdots + \psi_{p-2q} \wedge \frac{\omega_0^q}{q!}$$

with $q \leq [p/2]$ *(*$[p/2]$ *being the largest integer less than or equal to* $p/2$*), where for* $s = 0, \ldots, q$ *the differential forms* ψ_{p-2s} *are effective and may be calculated from* φ *by means of iteration of the operator* i_{Λ_0}*. Then its adjoint* $*\,\varphi$ *may be uniquely written as the sum*

$$*\,\varphi = (-1)^{p(p+1)/2}\Bigg(\psi_p - \psi_{p-2} \wedge \frac{\omega_0}{n-p+1} + \cdots + (-1)^q \frac{(n-p)!}{(n-p+q)!}\psi_{p-2q} \wedge \omega_0^q\Bigg) \wedge \frac{\omega_0^{n-p}}{(n-p)!}.$$

We continue by indicating the relation which links $*$ with Ψ and its effect on Poisson structures. Since $\Lambda_0^{\#} : \Omega^p(M) \to \mathcal{V}^p(M)$, $p \in \mathbb{N}$, defined by (10) is an isomorphism, for any smooth p-vector field P on M there exists an unique p-form $\sigma_p \in \Omega^p(M)$ such that $P = \Lambda_0^{\#}(\sigma_p)$. So it is clear that

$$\Psi_P = *\,\sigma_p. \tag{15}$$

In particular a bivector field Λ on (M, ω_0) can be viewed as the image $\Lambda_0^{\#}(\sigma)$ of a 2-form σ on M by the isomorphism $\Lambda_0^{\#}$. We want to establish the condition on σ under which $\Lambda = \Lambda_0^{\#}(\sigma)$ is a Poisson tensor. For this reason we consider the *codifferenrial operator* $\delta = *\,d\,*$ introduced in [12], which is of degree -1 and satisfies the relation $\delta^2 = 0$, and we prove

Lemma 1. *For any differential form* ζ *on* (M, ω_0) *of degree* $p \leq n$

$$\Psi^{-1}(\zeta) = \Lambda_0^{\#}(*\,\zeta). \tag{16}$$

Proof. Let η be a smooth $(2n-p)$-form on M. We have

$$\begin{aligned}
\langle \eta,\, \Psi^{-1}(\zeta)\rangle &= \left\langle \eta,\, j_\zeta \frac{\Lambda_0^n}{n!}\right\rangle = \left\langle \eta \wedge \zeta,\, \frac{\Lambda_0^n}{n!}\right\rangle = (-1)^{2n}\left\langle \frac{\omega_0^n}{n!},\, \Lambda_0^{\#}(\eta) \wedge \Lambda_0^{\#}(\zeta)\right\rangle \\
&= (-1)^{p(2n-p)}\left\langle \frac{\omega_0^n}{n!},\, \Lambda_0^{\#}(\zeta) \wedge \Lambda_0^{\#}(\eta)\right\rangle \\
&= (-1)^{p(2n-p)}(-1)^{(p-1)p/2}\left\langle i_{\Lambda_0^{\#}(\zeta)}\frac{\omega_0^n}{n!},\, \Lambda_0^{\#}(\eta)\right\rangle \\
&= (-1)^{p(2n-p)}(-1)^{(p-1)p/2}(-1)^{2n-p}\left\langle \eta,\, \Lambda_0^{\#}\left(i_{\Lambda_0^{\#}(\zeta)}\frac{\omega_0^n}{n!}\right)\right\rangle \\
&= (-1)^{(p-1)p/2}\left\langle \eta,\, \Lambda_0^{\#}\left(i_{\Lambda_0^{\#}(\zeta)}\frac{\omega_0^n}{n!}\right)\right\rangle = \langle \eta,\, \Lambda_0^{\#}(*\,\zeta)\rangle,
\end{aligned}$$

whence (16) follows. (We remark that the number $p(2n-p) + (2n-p) = (2n-p)(p+1)$ is even for any $p \in \mathbb{N}$.) ∎

Theorem 1. *Under the above notations* $\Lambda = \Lambda_0^{\#}(\sigma)$ *defines a Poisson structure on* (M, ω_0) *if and only if*

$$2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma). \tag{17}$$

Proof. We have seen that Λ is a Poisson tensor if and only if (9) holds, but in our case $\Lambda = \Lambda_0^{\#}(\sigma)$ so that $\Lambda \wedge \Lambda = \Lambda_0^{\#}(\sigma \wedge \sigma)$ and $\Lambda_0^{\#}$ is an isomorphism. Therefore

$$
\begin{aligned}
2\Lambda \wedge D(\Lambda) = D(\Lambda \wedge \Lambda) \quad &\Leftrightarrow \quad -2\Lambda \wedge \big((\Psi^{-1} \circ d \circ \Psi)(\Lambda)\big) = -(\Psi^{-1} \circ d \circ \Psi)(\Lambda \wedge \Lambda) \\
&\overset{(15)}{\Leftrightarrow} \quad 2\Lambda_0^{\#}(\sigma) \wedge \big(\Psi^{-1}(d * \sigma)\big) = \Psi^{-1}\big(d * (\sigma \wedge \sigma)\big) \\
&\overset{(16)}{\Leftrightarrow} \quad 2\Lambda_0^{\#}(\sigma) \wedge \Lambda_0^{\#}(* d * (\sigma)) = \Lambda_0^{\#}(* d * (\sigma \wedge \sigma)) \\
&\Leftrightarrow \quad \Lambda_0^{\#}(2\sigma \wedge \delta\sigma) = \Lambda_0^{\#}\big(\delta(\sigma \wedge \sigma)\big) \\
&\Leftrightarrow \quad 2\sigma \wedge \delta(\sigma) = \delta(\sigma \wedge \sigma)
\end{aligned}
$$

and we are done. ■

Remark 2. Brylinski [1] observed that, when the manifold is symplectic, i.e., $d\omega_0 = 0$, δ is equal, up to sign, to $\Delta = i_{\Lambda_0} \circ d - d \circ i_{\Lambda_0}$. Then in this framework (17) is equivalent to $\{\!\{\sigma, \sigma\}\!\}_0 = 0$ ($\{\!\{\cdot, \cdot\}\!\}_0$ being the Koszul bracket (11) associated to Λ_0), which means that σ is a complementary 2-form on (M, Λ_0) in the sense of I. Vaisman [16].

3 Poisson structures with prescribed Casimirs on almost symplectic manifolds

Let (M, ω_0) be a $2n$-dimensional almost symplectic manifold, Λ_0 the bivector field associated with ω_0 and $\Omega = \omega_0^n/n!$ the corresponding volume form on M. We consider $2n - 2k$ smooth functions, $f_1, f_2, \ldots, f_{2n-2k}$, on (M, ω_0) such that the matrix $\big(\{f_i, f_j\}_0\big)$ of their brackets with respect to Λ_0 is invertible on an open and dense subset $\mathcal{U}$ of M. This last assumption assures that $f_1, \ldots, f_{2n-2k}$ are functionally independent on $\mathcal{U}$. We want to describe the Poisson structures Λ on (M, ω_0) with symplectic leaves of dimension at most $2k$ which have as Casimirs the functions $f_1, f_2, \ldots, f_{2n-2k}$.

We denote by X_{f_i} the hamiltonian vector field of f_i, $i = 1, \ldots, 2n - 2k$, with respect to Λ_0, i.e., $X_{f_i} = \Lambda_0^{\#}(df_i)$, by $D = \langle X_{f_1}, X_{f_2}, \ldots, X_{f_{2n-2k}}\rangle$ the distribution on M generated by $X_{f_1}, \ldots, X_{f_{2n-2k}}$ and by D° its annihilator. The invertibility assumption of the matrix $\big(\{f_i, f_j\}_0\big)$ on $\mathcal{U}$ implies that the function

$$
f = \left\langle df_1 \wedge \cdots \wedge df_{2n-2k},\ \frac{\Lambda_0^{n-k}}{(n-k)!} \right\rangle = \left\langle \frac{\omega_0^{n-k}}{(n-k)!},\ X_{f_1} \wedge \cdots \wedge X_{f_{2n-2k}} \right\rangle, \tag{18}
$$

which satisfies $f^2 = \det\big(\{f_i, f_j\}_0\big)$, never vanishes on $\mathcal{U}$ and that at every point $x \in \mathcal{U}$ $D_x = D \cap T_xM$ is a symplectic subspace of T_xM. Thus $T_xM = D_x \oplus \mathrm{orth}_{\omega_0} D_x = D_x \oplus \Lambda_{0_x}^{\#}(D_x^\circ)$, where $D_x^\circ = D^\circ \cap T_x^*M$, and $T_x^*M = D_x^\circ \oplus (\Lambda_{0_x}^{\#}(D_x^\circ))^\circ = D_x^\circ \oplus \langle df_1, \ldots, df_{2n-2k}\rangle_x$. Finally we denote by σ the smooth 2-form on M which corresponds via the isomorphism $\Lambda_0^{\#}$ to an element Λ of $\mathcal{V}^2(M)$.

Proposition 2. *A bivector field Λ on (M,ω_0) of rank at most $2k$ on M has as unique Casimirs the functions $f_1,\dots,f_{2n-2k}$ if and only if its corresponding 2-form σ is a smooth section of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{U}$.*

Proof. Effectively for any f_i, $i=1,\dots,2n-2k$

$$\Lambda(df_i,\cdot)=0 \Leftrightarrow \Lambda_0^{\#}(\sigma)(df_i,\cdot)=0 \Leftrightarrow \sigma(X_{f_i},\Lambda_0^{\#}(\cdot))=0. \tag{19}$$

Thus $f_1,\dots,f_{2n-2k}$ are the unique Casimir functions of Λ on $\mathcal{U}$ if and only if the vector fields $X_{f_1},\dots,X_{f_{2n-2k}}$ with functionally independent hamiltonians on $\mathcal{U}$ generate $\ker\sigma$, i.e., for any $x\in\mathcal{U}$ $D_x=\ker\sigma_x^\flat$. The last relation means that σ is a section of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{U}$. ■

With the notation above we can formulate the following theorem:

Theorem 2. *Let $f_1,\dots,f_{2n-2k}$ be smooth functions on an almost symplectic manifold (M,ω_0) with volume form $\Omega=\frac{\omega_0^n}{n!}$ such that the Poisson matrix $(\{f_i,f_j\}_0)$ with respect to Λ_0 is invertible on an open and dense subset $\mathcal{U}$ of M and σ a section of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{U}$ that satisfies (17). Then the $(2n-2)$-form*

$$\Phi=-\frac{1}{f}\Big(\sigma+\frac{g}{k-1}\omega_0\Big)\wedge\frac{\omega_0^{k-2}}{(k-2)!}\wedge df_1\wedge\cdots\wedge df_{2n-2k}, \tag{20}$$

where f is given by (18) and $g=i_{\Lambda_0}\sigma$, corresponds to a Poisson tensor field Λ with orbits of dimension at most $2k$ for which $f_1,\dots,f_{2n-2k}$ are Casimirs. Precisely $\Lambda=\Lambda_0^{\#}(\sigma)$ and the associated bracket of Λ on $C^\infty(M)$ is given for any $h_1,h_2\in C^\infty(M)$ by

$$\{h_1,h_2\}\Omega=-\frac{1}{f}dh_1\wedge dh_2\wedge\Big(\sigma+\frac{g}{k-1}\omega_0\Big)\wedge\frac{\omega_0^{k-2}}{(k-2)!}\wedge df_1\wedge\cdots\wedge df_{2n-2k}. \tag{21}$$

Conversely, if Λ is a Poisson tensor on (M,ω_0) of rank $2k$ on an open and dense subset $\mathcal{U}$ of M, then there are $2n-2k$ functionally independent smooth functions $f_1,\dots,f_{2n-2k}$ on $\mathcal{U}$ and a section σ of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{U}$ satisfying (17), such that Ψ_Λ and $\{\cdot,\cdot\}$ are given, respectively, by (20) and (21).

Proof. We denote by $\tilde\Omega=\Lambda_0^n/n!$ the dual $2n$-vector field of Ω on M and we set $\Lambda=j_\Phi\tilde\Omega$. For any f_i, $i=1,\dots,2n-2k$, we have

$$\Lambda^{\#}(df_i)=-j_{df_i}\Lambda=-j_{df_i}j_\Phi\tilde\Omega=-j_{df_i\wedge\Phi}\tilde\Omega=-j_0\tilde\Omega=0,$$

which means that $f_1,\dots,f_{2n-2k}$ are Casimir functions of Λ. We see below that $\Lambda=\Lambda_0^{\#}(\sigma)$. Thus Λ defines a Poisson structure on M having the required properties. We calculate the adjoint form $*\,\Phi$ of Φ:

$$*\,\Phi=-\frac{1}{f}*\bigg(\Big(\sigma+\frac{g}{k-1}\omega_0\Big)\wedge\frac{\omega_0^{k-2}}{(k-2)!}\wedge df_1\wedge\cdots\wedge df_{2n-2k}\bigg)\overset{(13)}{=}$$
$$-(-1)^{(2n-2k-1)(2n-2k)/2}\,\frac{1}{f}i_{X_{f_1}\wedge\cdots\wedge X_{f_{2n-2k}}}\bigg[*\bigg(\Big(\sigma+\frac{g}{k-1}\omega_0\Big)\wedge\frac{\omega_0^{k-2}}{(k-2)!}\bigg)\bigg].$$

From Lepage's decomposition theorem σ can be written as $\sigma = \psi_2 + \psi_0\omega_0$, where ψ_2 is an effective 2-form on M and $\psi_0 = i_{\Lambda_0}\sigma/(i_{\Lambda_0}\omega_0) = -g/n$. (It is easy to check that $i_{\Lambda_0}\omega_0 = -\langle\omega_0, \Lambda_0\rangle = -\mathrm{Tr}(\omega_0^\flat \circ \Lambda_0^\#)/2 = -\mathrm{Tr}(I_{2n})/2 = -n$.) Hence

$$\Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} = \Big(\psi_2 - \frac{g}{n}\omega_0 + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!}$$
$$= \psi_2 \wedge \frac{\omega_0^{k-2}}{(k-2)!} + \frac{n-k+1}{n} g \frac{\omega_0^{k-1}}{(k-1)!}$$

and

$$*\Big(\Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!}\Big) = *\Big(\psi_2 \wedge \frac{\omega_0^{k-2}}{(k-2)!}\Big) + \frac{n-k+1}{n} g\Big(*\frac{\omega_0^{k-1}}{(k-1)!}\Big)$$
$$\overset{(14)}{=} -\psi_2 \wedge \frac{\omega_0^{n-(k-2)-2}}{(n-(k-2)-2)!} + \frac{n-k+1}{n} g \frac{\omega_0^{n-(k-1)}}{(n-(k-1))!}$$
$$= -\Big(\psi_2 - \frac{g}{n}\omega_0\Big) \wedge \frac{\omega_0^{n-k}}{(n-k)!} = -\sigma \wedge \frac{\omega_0^{n-k}}{(n-k)!}. \tag{22}$$

Consequently

$$*\,\Phi = -(-1)^{(2n-2k-1)(2n-2k)/2}\,\frac{1}{f} i_{X_{f_1}\wedge\cdots\wedge X_{f_{2n-2k}}}\Big[-\sigma \wedge \frac{\omega_0^{n-k}}{(n-k)!}\Big]$$
$$\overset{(5)(19)}{=} \frac{1}{f}\Big\langle \frac{\omega_0^{n-k}}{(n-k)!}, X_{f_1} \wedge \cdots \wedge X_{f_{2n-2k}}\Big\rangle \sigma = \frac{1}{f} f\,\sigma = \sigma. \tag{23}$$

Applying (16) to the above relation, we obtain $\Lambda_0^\#(\sigma) = \Lambda_0^\#(*\,\Phi) = \Psi^{-1}(\Phi) = j_\Phi\tilde{\Omega} = \Lambda$. Thus according to Theorem 1 Λ defines a Poisson structure on M with orbits of dimension at most $2k$ for which $f_1, \ldots, f_{2n-2k}$ are Casimir functions. Obviously the associated bracket of Λ on $C^\infty(M)$ is given by (21). For any $h_1, h_2 \in C^\infty(M)$

$$\{h_1, h_2\} = j_{dh_1\wedge dh_2}\Lambda = j_{dh_1\wedge dh_2} j_\Phi \tilde{\Omega} = j_{dh_1\wedge dh_2\wedge\Phi}\tilde{\Omega} \quad \Leftrightarrow$$
$$\{h_1, h_2\}\Omega = -\frac{1}{f} dh_1 \wedge dh_2 \wedge \Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}.$$

Conversely, if Λ is a Poisson tensor on (M, ω_0) with symplectic leaves of dimension at most $2k$, then in a neighborhood U of a nonsingular point we can construct a system of local coordinates $(z_1, \ldots, z_{2k}, f_1, \ldots, f_{2n-2k})$ in which the symplectic leaves of Λ are defined by $f_l = \text{const}$, $l = 1, \ldots, 2n-2k$ and the matrix of Λ_0 has the form

$$\begin{pmatrix} A_{2k} & 0 \\ 0 & B_{2n-2k} \end{pmatrix}, \quad \text{where} \quad A_{2k} = \big(\{z_i, z_j\}_0\big) \quad \text{and} \quad B_{2n-2k} = \big(\{f_l, f_m\}_0\big).$$

Also, according to Proposition 2 and to Theorem 1, there is a section σ of $\bigwedge^2 D^\circ$ of maximal rank almost everywhere on M that satisfies (17) such that $\Lambda = \Lambda_0^\#(\sigma)$.

We prove that the $(2n-2)$-form $\Psi_\Lambda = -i_{\Lambda_0^\#(\sigma)}\Omega = *\,\sigma$ can be written as in (20). Since ω_0 is nondegenerate, $f^2 = \big(\langle df_1 \wedge \cdots \wedge df_{2n-2k},\ \Lambda_0^n/n!\rangle\big)^2 = \det B_{2n-2k} \neq 0$ on U. Thus Ω can be written on U as

$$\Omega = \frac{1}{f}\frac{\omega_0^k}{k!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}$$

and

$$\Psi_\Lambda = -i_\Lambda \Omega = -\frac{1}{f}\left(i_\Lambda \frac{\omega_0^k}{k!}\right) \wedge df_1 \wedge \cdots \wedge df_{2n-2k}. \tag{24}$$

We now proceed to calculate the $(2k-2)$-form $-i_\Lambda \frac{\omega_0^k}{k!}$. Note that $\frac{\omega_0^k}{k!} = *\,\frac{\omega_0^{n-k}}{(n-k)!}$. So from (13) we get that

$$-i_\Lambda \frac{\omega_0^k}{k!} = *\left(\sigma \wedge \frac{\omega_0^{n-k}}{(n-k)!}\right). \tag{25}$$

Taking again the calculation of (22) in the inverse direction, we have

$$\begin{aligned} *\left(\sigma \wedge \frac{\omega_0^{n-k}}{(n-k)!}\right) &= -**\left(\left(\sigma + \frac{g}{k-1}\omega_0\right) \wedge \frac{\omega_0^{k-2}}{(k-2)!}\right) \\ &= -\left(\sigma + \frac{g}{k-1}\omega_0\right) \wedge \frac{\omega_0^{k-2}}{(k-2)!}. \end{aligned} \tag{26}$$

Therefore by replacing (26) in (25) and the obtained relation in (24) we prove that Ψ_Λ results in the expression (20). Then it is clear that $\{\cdot,\cdot\}$ is given by (21). ∎

4 Some examples

4.1 Dirac brackets

Let (M, ω_0) be a symplectic manifold of dimension $2n$, Λ_0 its associated Poisson structure and $f_1, \ldots, f_{2n-2k}$ smooth functions on M the differentials of which are linearly independent at each point in the submanifold M_0 of M defined by the equations $f_1(x) = 0, \ldots, f_{2n-2k}(x) = 0$. We assume that the matrix $\big(\{f_i, f_j\}_0\big)$ is invertible on an open neighborhood $\mathcal{W}$ of M_0 in M and we denote by c_{ij} the coefficients of its inverse matrix which are smooth functions on $\mathcal{W}$ such that $\sum_{j=1}^{2n-2k}\{f_i, f_j\}_0 c_{jk} = \delta_{ik}$. We consider on $\mathcal{W}$ the 2-form

$$\sigma = \omega_0 + \sum_{i<j} c_{ij}\, df_i \wedge df_j. \tag{27}$$

We prove that it is a section of $\bigwedge^2 D^\circ$ of maximal rank on $\mathcal{W}$ which verifies (17). As in Section 3, D denotes the subbundle of TM generated by the Hamiltonian vector fields X_{f_i} of f_i, $i = 1, \dots, 2n-2k$, with respect to Λ_0 and D° its annihilator. For any X_{f_l}, $l = 1, \dots, 2n-2k$, we have

$$\sigma(X_{f_l}, \cdot) = \omega_0(X_{f_l}, \cdot) + \sum_{i<j} c_{ij} \langle df_i, X_{f_l} \rangle df_j \;=\; -df_l + \sum_{i<j} c_{ij} \{f_l, f_i\}_0 df_j$$
$$= -df_l + \sum_j \delta_{lj} df_j \;=\; -df_l + df_l \;=\; 0,$$

which means that σ is a section of $\bigwedge^2 D^\circ \to \mathcal{W}$. The assumption that $\big(\{f_i, f_j\}_0\big)$ is invertible assures us that D is a symplectic subbundle of $T_{\mathcal{W}}M$. So for any $x \in \mathcal{W}$ $T_x^*M = D_x^\circ \oplus \langle df_1, \dots, df_{2n-2k}\rangle_x$ and $\bigwedge^2 T_x^*M = \bigwedge^2 D_x^\circ + \bigwedge^2 \langle df_1, \dots, df_{2n-2k}\rangle_x + D_x^\circ \wedge \langle df_1, \dots, df_{2n-2k}\rangle_x$. However, ω_0 is a nondegenerate section of $\bigwedge^2 T^*M$ and the part $\sum_{i<j} c_{ij} df_i \wedge df_j$ of σ is a smooth section of $\bigwedge^2 \langle df_1, \dots, df_{2n-2k}\rangle$ of maximal rank on $\mathcal{W}$ because $\det(c_{ij}) \neq 0$ on $\mathcal{W}$. Thus σ is of maximal rank on $\mathcal{W}$. Also we have

$$g = i_{\Lambda_0}\sigma = -\Big\langle \omega_0 + \sum_{i<j} c_{ij} df_i \wedge df_j, \Lambda_0 \Big\rangle = -n - \sum_{i<j} c_{ij} \{f_i, f_j\}_0$$
$$= -n + (n-k) = -k, \quad \text{and}$$
$$*\,\sigma \overset{(23)(20)}{=} -\frac{1}{f}\Big(\sigma + \frac{g}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}$$
$$= -\frac{1}{f}\Big(\omega_0 + \sum_{i<j} c_{ij} df_i \wedge df_j - \frac{k}{k-1}\omega_0\Big) \wedge \frac{\omega_0^{k-2}}{(k-2)!} \wedge df_1 \wedge \ldots \wedge df_{2n-2k}$$
$$= \frac{1}{f}\frac{\omega_0^{k-1}}{(k-1)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}. \tag{28}$$

Consequently

$$\delta\sigma = (*\,d\,*)\sigma \overset{(28)}{=} *\Big(-\frac{df}{f} \wedge (*\,\sigma)\Big) \overset{(13)}{=} -\frac{1}{f} i_{X_f}\sigma \quad \text{and}$$
$$2\sigma \wedge \delta(\sigma) = -\frac{2}{f}\sigma \wedge (i_{X_f}\sigma) = -\frac{1}{f} i_{X_f}(\sigma \wedge \sigma). \tag{29}$$

On the other hand

$$*\,(\sigma \wedge \sigma) \overset{(13)}{=} -i_{\Lambda_0^\#(\sigma)}(*\,\sigma) \overset{(28)}{=} -\frac{1}{f}\Big(i_{\Lambda_0^\#(\sigma)}\frac{\omega_0^{k-1}}{(k-1)!}\Big) \wedge df_1 \wedge \cdots \wedge df_{2n-2k}$$
$$\overset{(25)}{=} \frac{1}{f}\Big[*\Big(\sigma \wedge \frac{\omega_0^{n-k+1}}{(n-k+1)!}\Big)\Big] \wedge df_1 \wedge \cdots \wedge df_{2n-2k}$$
$$\overset{(22)(27)}{=} -\frac{1}{f}\Big(\omega_0 + \sum_{i<j} c_{ij} df_i \wedge df_j - \frac{k}{k-2}\omega_0\Big) \wedge \frac{\omega_0^{k-3}}{(k-3)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}$$
$$= \frac{2}{f} \wedge \frac{\omega_0^{k-3}}{(k-3)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k} \quad \text{and} \tag{30}$$
$$\delta(\sigma \wedge \sigma) = *\,d\,*\,(\sigma \wedge \sigma) \overset{(30)}{=} *\Big(-\frac{df}{f} \wedge *\,(\sigma \wedge \sigma)\Big) \overset{(13)}{=} -\frac{1}{f} i_{X_f}(\sigma \wedge \sigma). \tag{31}$$

From (29) and (31) we conclude that σ satisfies (17). Thus according to Theorem 2 the bivector field

$$\Lambda = \Lambda_0^{\#}(\sigma) = \Lambda_0 + \sum_{i<j} c_{ij} X_{f_i} \wedge X_{f_j}$$

defines a Poisson structure on $\mathcal{W}$ the corresponding bracket of which $\{\cdot,\cdot\}$ on $C^\infty(\mathcal{W},\mathbb{R})$ is given for any $h_1, h_2 \in C^\infty(\mathcal{W},\mathbb{R})$ by

$$\{h_1,h_2\}\Omega = \frac{1}{f} dh_1 \wedge dh_2 \wedge \frac{\omega_0^{k-1}}{(k-1)!} \wedge df_1 \wedge \cdots \wedge df_{2n-2k}. \tag{32}$$

In the above expression of Λ we recognize the Poisson structure defined by Dirac [6] on an open neighborhood $\mathcal{W}$ of the constrained submanifold M_0 of M and in (32) the expression of Dirac bracket given in [10].

4.2 Periodic Toda and Volterra lattices

In this paragraph we study the linear Poisson structure Λ_T associated to the periodic Toda lattice of n particles. This Poisson structure has two well-known Casimir functions. Following Theorem 2 we construct another Poisson structure having the same Casimir invariants with Λ_T. It turns out that this structure decomposes as a direct sum of two Poisson tensors one of which (involving only the a variables in Flaschka's coordinates) is the quadratic Poisson bracket of the Volterra lattice (also known as the KM-system). It agrees with the general philosophy (see [4]) that one obtains the Volterra lattice from the Toda lattice by restricting to the a variables.

The periodic Toda lattice of n particles ($n \geq 2$) is the system of ordinary differential equations on $\mathbb{R}^{2n}$ which in Flaschka's [8] coordinate system ($a_1, \ldots, a_n, b_1, \ldots, b_n$) takes the form

$$\dot{a}_i = a_i(b_{i+1} - b_i) \text{ and } \dot{b}_i = 2(a_i^2 - a_{i-1}^2) \ (i \in \mathbb{Z} \text{ and } (a_{i+n}, b_{i+n}) = (a_i, b_i)).$$

This system is hamiltonian with respect to the nonstandard Lie–Poisson structure

$$\Lambda_T = \sum_{i=1}^{n} a_i \frac{\partial}{\partial a_i} \wedge \Big(\frac{\partial}{\partial b_i} - \frac{\partial}{\partial b_{i+1}}\Big)$$

on $\mathbb{R}^{2n}$ and it has as hamiltonian the function $H = \sum_{i=1}^n (a_i^2 + b_i^2/2)$. The structure Λ_T is of rank $2n-2$ on

$$\mathcal{U} = \Big\{ (a_1, \ldots, a_n, b_1, \ldots, b_n) \in \mathbb{R}^{2n} \ \Big| \ \sum_{i=1}^{n} a_1 \cdots a_{i-1} a_{i+1} \cdots a_n \neq 0 \Big\}$$

and it admits two Casimir functions: $C_1 = b_1 + b_2 + \cdots + b_n$ and $C_2 = a_1 a_2 \ldots a_n$.

We consider on $\mathbb{R}^{2n}$ the standard symplectic form $\omega_0 = \sum_{i=1}^n da_i \wedge db_i$, its associated Poisson tensor $\Lambda_0 = \sum_{i=1}^n \frac{\partial}{\partial a_i} \wedge \frac{\partial}{\partial b_i}$ and the corresponding volume element $\Omega = \frac{\omega_0^n}{n!} = da_1 \wedge db_1 \wedge \cdots \wedge da_n \wedge db_n$. The hamiltonian vector fields of C_1 and C_2 with respect to Λ_0 are

$$X_{C_1} = -\sum_{i=1}^{n} \frac{\partial}{\partial a_i} \quad \text{and} \quad X_{C_2} = \sum_{i=1}^{n} a_1 \dots a_{i-1} a_{i+1} \dots a_n \frac{\partial}{\partial b_i}.$$

So $D = \langle X_{C_1}, X_{C_2} \rangle$ and

$$D^\circ = \Bigg\{ \sum_{i=1}^{n} (\alpha_i da_i + \beta_i db_i) \in \Omega^1(\mathbb{R}^{2n}) \;\Big|\; \sum_{i=1}^{n} \alpha_i = 0 \quad \text{and}$$
$$\sum_{i=1}^{n} a_1 \cdots a_{i-1} \beta_i a_{i+1} \cdots a_n = 0 \Bigg\}.$$

The family of 1-forms $(\sigma_1, \dots, \sigma_{n-1}, \sigma'_1, \dots, \sigma'_{n-1})$,

$$\sigma_j = da_j - da_{j+1} \quad \text{and} \quad \sigma'_j = a_j db_j - a_{j+1} db_{j+1}, \quad j = 1, \dots, n-1,$$

provides a basis of $D^\circ_{(a,b)}$ at every point $(a, b) \in \mathcal{U}$. The section of maximal rank σ_T of $\bigwedge^2 D^\circ \to \mathcal{U}$, which corresponds to Λ_T via the isomorphism $\Lambda_0^\#$ and satisfies (17), is written in this basis as

$$\sigma_T = \sum_{j=1}^{n-1} \sigma_j \wedge \Bigg(\sum_{l=j}^{n-1} \sigma'_l \Bigg).$$

We now consider on $\mathbb{R}^{2n}$ the 2-form

$$\begin{aligned}
\sigma &= \sum_{j=1}^{n-2} \sigma_j \wedge \Bigg(\sum_{l=j+1}^{n-1} \sigma_l \Bigg) + \sum_{j=1}^{n-2} \sigma'_j \wedge \Bigg(\sum_{l=j+1}^{n-1} \sigma'_l \Bigg) \\
&= \sum_{j=1}^{n-2} \Big[(da_j - da_{j+1}) \wedge (da_{j+1} - da_n) \\
&\qquad + (a_j db_j - a_{j+1} db_{j+1}) \wedge (a_{j+1} db_{j+1} - a_n db_n) \Big] \\
&= \sum_{j=1}^{n} \Big(da_j \wedge da_{j+1} + a_j a_{j+1} db_j \wedge db_{j+1} \Big).
\end{aligned}$$

It is a section of $\bigwedge^2 D^\circ$ the rank of which depends upon the parity of n; if n is odd, its rank is $2n-2$ on $\mathcal{U}$, while, if n is even, its rank is $2n-4$ almost everywhere on $\mathbb{R}^{2n}$. Also after a long computation we can confirm that it satisfies (17). Thus its image via $\Lambda_0^\#$, i.e. the bivector field

$$\Lambda = \sum_{j=1}^{n} \Big(a_j a_{j+1} \frac{\partial}{\partial a_j} \wedge \frac{\partial}{\partial a_{j+1}} + \frac{\partial}{\partial b_j} \wedge \frac{\partial}{\partial b_{j+1}} \Big),$$

defines a Poisson structure on $\mathbb{R}^{2n}$ with symplectic leaves of dimension at most $2n-2$, when n is odd, that has C_1 and C_2 as Casimir functions. (When n is even, Λ has two more Casimir functions.) We remark that $(\mathbb{R}^{2n},\Lambda)$ can be viewed as the product of Poisson manifolds $(\mathbb{R}^n,\Lambda_V)\times(\mathbb{R}^n,\Lambda')$, where

$$\Lambda_V=\sum_{j=1}^n a_ja_{j+1}\frac{\partial}{\partial a_j}\wedge\frac{\partial}{\partial a_{j+1}}\quad\text{and}\quad\Lambda'=\sum_{j=1}^n\frac{\partial}{\partial b_j}\wedge\frac{\partial}{\partial b_{j+1}}.$$

The Poisson tensor Λ_V is the quadratic bracket of the periodic Volterra lattice on $\mathbb{R}^n$ and, when n is odd, it has C_2 as unique Casimir function.

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[1] Brylinski J.-L., A differential complex for Poisson manifolds, *J. Differential Geom.*, 1988, V.28, 93–114.

[2] Damianou P.A., Nonlinear Poisson brackets, Ph.D. Dissertation, University of Arizona, 1989.

[3] Damianou P.A., Transverse Poisson structures of coadjoint orbits, *Bulletin des Sciences Mathématiques*, 1996, V.120, 195–214.

[4] Damianou P.A. and Fernandes R.L., From the Toda lattice to the Volterra lattice and back, *Rep. Math. Phys.*, 2002, V.50, 361–378.

[5] Damianou P.A., Sabourin H. and Vanhaecke P., Transverse Poisson structures to adjoint orbits in semi-simple Lie algebras, *Pacific J. Math.*, 2007, V.232, 111–139.

[6] Dirac P.A.M., *Lectures in Quantum Mechanics*, Yeshida University, 1964.

[7] Dufour J.-P. and Zung N.T., *Poisson Structures and Their Normal Forms*, Progress in Mathematics, V.242, Birkhauser Verlag, Basel, 2005.

[8] Flaschka H., The Toda lattice. II. Existence of integrals, *Phys. Rev. B*, 1974, V.9, 1924–1925.

[9] Grabowski J., Marmo G. and Perelomov A.M., Poisson structures: Towards a classification, *Modern Phys. Lett. A*, 1993, V.8, 1719–1733.

[10] Grabowski J. and Marmo G., Generalized n-Poisson brackets on a symplectic manifold, *Modern Phys. Lett. A*, 1998, V.13, 3185–3192.

[11] Koszul J.-L., Crochet de Schouten–Nijenhuis et cohomologie, in *Élie Cartan et les Mathématiques d'aujourd'hui, Astérisque, Numéro Hors Série*, 1985, 257–271.

[12] Libermann P., Sur le problème d'équivalence de certaines structures infinitésimales régulières, *Ann. Mat. Pura Appl.*, 1954, V.36, 27–120.

[13] Libermann P. and Marle Ch.-M., *Symplectic Geometry and Analytical Mechanics, Mathematics and its Applications*, V.35, D. Reidel Publishing Co., Dordrecht, 1987.

[14] Lichnérowicz A., Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geom.*, 1977, V.12, 253–300.

[15] Vaisman I., *Lectures on the Geometry of Poisson Manifolds*, Progress in Mathematics, V.118, Birkhäuser Verlag, Basel, 1994.

[16] Vaisman I., Complementary 2-forms of Poisson structures, *Compositio Math.*, 1996, V.101, 55–75.

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Height-2 Toda systems

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We announce some results which involve some new, evidently integrable systems of Toda type. More specifically we construct a large family of Hamiltonian systems which interpolate between the classical Kostant–Toda lattice and the full Kostant–Toda lattice and we discuss their integrability. There is one such system for every nilpotent ideal $\mathcal{I}$ in a Borel subalgebra $\mathfrak{b}_+$ of an arbitrary simple Lie algebra $\mathfrak{g}$. The classical Kostant–Toda lattice corresponds to the case of $\mathcal{I} = [\mathfrak{n}_+, \mathfrak{n}_+]$, where $\mathfrak{n}_+$ is the unipotent ideal of $\mathfrak{b}_+$, while the full Kostant–Toda lattice corresponds to $\mathcal{I} = \{0\}$. We mainly focus on the case of $\mathfrak{g}$ being of type A, B or C with $\mathcal{I} = [[\mathfrak{n}_+, \mathfrak{n}_+], \mathfrak{n}_+]$ which we call the height-2 Toda lattice. Complete proofs of the announced results will appear in a future publication.

1 Introduction

The classical Toda lattice is the mechanical system with Hamiltonian function

$$H(q_1, \dots, q_N, p_1, \dots, p_N) = \sum_{i=1}^{N} \frac{1}{2} p_i^2 + \sum_{i=1}^{N-1} e^{q_i - q_{i+1}}.$$

It describes a system of N particles on a line connected by exponential springs. The differential equations which govern this lattice can be transformed via a change of variables due to Flaschka [9] to a Lax equation $\dot{L} = [L_+, L]$, where L is the Jacobi matrix

$$L = \begin{pmatrix} b_1 & a_1 & 0 & \cdots & \cdots & 0 \\ a_1 & b_2 & a_2 & \cdots & & \vdots \\ 0 & a_2 & b_3 & \ddots & & \\ \vdots & & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & a_{N-1} \\ 0 & \cdots & & \cdots & a_{N-1} & b_N \end{pmatrix}, \qquad (1)$$

and L_+ is the skew-symmetric part of L in the Lie algebra decomposition lower triangular plus skew-symmetric. Lax equations define isospectral deformations; though the entries of L vary over time, the eigenvalues of L remain constant. It follows that the functions $H_i = \frac{1}{i}\operatorname{Tr} L^i$ are constants of motion. Moreover they are in involution with respect to a Poisson structure associated to the above Lie algebra decomposition.

There is a generalization due to Deift, Li, Nanda and Tomei [5] who showed that the system remains integrable when L is replaced by a full symmetric $N \times N$ matrix. The resulting system is called the full symmetric Toda lattice. The functions $H_i := \frac{1}{i}\operatorname{Tr} L^i$ are still in involution, but they are not enough to ensure integrability. It was shown in [5] that there are additional integrals, called chop integrals, which are rational functions of the entries of L. They are constructed as follows. For $k = 0, \dots, [\frac{(N-1)}{2}]$, denote by $(L - \lambda \operatorname{Id}_N)_k$ the result of removing the first k rows and the last k columns from $L - \lambda \operatorname{Id}_N$ and let

$$\det(L - \lambda \operatorname{Id}_N)_k = E_{0k}\lambda^{N-2k} + \cdots + E_{N-2k,k}. \tag{2}$$

Set

$$\frac{\det\,(L - \lambda \operatorname{Id}_N)_k}{E_{0k}} = \lambda^{N-2k} + I_{1k}\lambda^{N-2k-1} + \cdots + I_{N-2k,k}. \tag{3}$$

The functions I_{rk}, where $r = 1, \dots, N-2k$ and $k = 0, \dots, [\frac{N-1}{2}]$, are independent constants of motion, they are in involution and sufficient to account for the integrability of the full Toda lattice.

1.1 Bogoyavlensky–Toda

The classical Toda lattice was generalized in another direction. One can define a Toda-type system for each simple Lie algebra. The finite, nonperiodic Toda lattice corresponds to a root system of type A_ℓ. This generalization is due to Bogoyavlensky [3]. These systems were studied extensively in [10] in which the solution of the system was connected intimately with the representation theory of simple Lie groups. See also Olshanetsky–Perelomov [11] and Adler–van Moerbeke [1]. We call these systems the Bogoyavlensky–Toda lattices. They can be described as follows.

Let $\mathfrak{g}$ be any simple Lie algebra equipped with its Killing form $\langle \cdot \,|\, \cdot \rangle$. One chooses a Cartan subalgebra, $\mathfrak{h}$ of $\mathfrak{g}$, and a basis Π of simple roots for the root system Δ of $\mathfrak{h}$ in $\mathfrak{g}$. The corresponding set of positive roots is denoted by Δ^+. To each positive root α one can associate a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of vectors in $\mathfrak{g}$ which generate a Lie subalgebra isomorphic to $\mathrm{sl}_2(\mathbf{C})$. The set $(X_\alpha, X_{-\alpha})_{\alpha \in \Delta^+} \cup (H_\alpha)_{\alpha \in \Pi}$ is basis of $\mathfrak{g}$ and is called a root basis. To these data one associates the Lax equation $\dot{L} = [L_+, L]$, where L and L_+ are defined as follows:

$$L = \sum_{i=1}^{\ell} b_i H_{\alpha_i} + \sum_{i=1}^{\ell} a_i (X_{\alpha_i} + X_{-\alpha_i}), \quad L_+ = \sum_{i=1}^{\ell} a_i (X_{\alpha_i} - X_{-\alpha_i}).$$

The affine space M of all elements L of $\mathfrak{g}$ of the above form is the phase space of the Bogoyavlensky–Toda lattice associated to $\mathfrak{g}$. The functions which yield the integrability of the system are the Ad-invariant functions on $\mathfrak{g}$which are restricted to M.

1.2 Kostant form

Let D be the diagonal $N \times N$ matrix with entries $d_i := \prod_{j=1}^{i-1} a_j$. In [10] Kostant conjugates the matrix L, given by (1), by the matrix D to obtain a matrix of the form

$$X = \begin{pmatrix} b_1 & 1 & 0 & \cdots & \cdots & 0 \\ c_1 & b_2 & 1 & \ddots & & \vdots \\ 0 & c_2 & b_3 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & & 0 \\ \vdots & & & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & 0 & c_{N-1} & b_N \end{pmatrix}. \tag{4}$$

The Lax equation takes the form $\dot{X} = [X_+, X]$, where X_+ is the strictly lower triangular part of X, according to the Lie algebra decomposition strictly lower plus upper triangular. This form is convenient in applying Lie theoretic techniques to describe the system. Note that the diagonal elements correspond to the Cartan subalgebra while the subdiagonal elements correspond to the set Π of simple roots. The full Kostant–Toda lattice is obtained by replacing Π with Δ^+ in the sense that one fills the lower triangular part of X in (4) with additional variables. It leads on the affine space of all such matrices to the Lax equation

$$\dot{X} = [X_+, X], \tag{5}$$

where X_+ is again the projection to the strictly lower part of X.

1.3 Adapted sets in a root system

Generalizing the above procedure we can introduce the Lax pair (L_Φ, B_Φ), where Φ is any subset of Δ^+ containing Π. Thus we have

$$L_\Phi = \sum_{\alpha\in\Pi} b_\alpha H_\alpha + \sum_{\alpha\in\Phi} a_\alpha(X_\alpha + X_{-\alpha}), \quad B_\Phi = \sum_{\alpha\in\Phi} a_\alpha(X_\alpha - X_{-\alpha}).$$

In order to have consistency in the Lax equation, since the Lax matrix is symmetric, the bracket $[B_\Phi, L_\Phi]$ should give an element of the form $\sum_{\alpha\in\Phi} c_\alpha H_\alpha + \sum_{\alpha\in\Phi} d_\alpha(X_\alpha + X_{-\alpha})$. In this case we say that Φ is adapted. A straightforward computation leads to the following result:

Proposition 1. *The set* Φ *is adapted if and only if it satisfies the following property:*

$$\forall \alpha, \beta \in \Phi, \quad \alpha - \beta \text{ or } \beta - \alpha \in \Phi \cup \{0\}.$$

Recall that $\alpha - \beta = 0$ means that $\alpha - \beta$ is not a root.

Thus for each Φ which is adapted we obtain a corresponding Hamiltonian system and the problem is to study this system and determine whether it is integrable. We conjecture that in fact it is integrable. We prove this claim for a particular class of such systems. Note that the special case $\Phi = \Pi$ corresponds to the classical Toda lattice while the case $\Phi = \Delta^+$ corresponds to the full symmetric Toda of [5].

Example 1. We consider a Lie algebra of type B_2. The set of positive roots $\Delta^+ = \{\alpha, \beta, \alpha + \beta, \beta + 2\alpha\}$ which corresponds to the full symmetric Toda lattice with Lax matrix

$$L = \begin{pmatrix} b_1 & a_1 & a_3 & a_4 & 0 \\ a_1 & b_2 & a_2 & 0 & -a_4 \\ a_3 & a_2 & 0 & -a_2 & -a_3 \\ a_4 & 0 & -a_2 & -b_2 & -a_1 \\ 0 & -a_4 & -a_3 & -a_1 & -b_1 \end{pmatrix}.$$

This system is completely integrable with integrals $h_2 = \frac{1}{2}\mathrm{Tr}L^2$ which is the Hamiltonian, $h_4 = \frac{1}{2}\mathrm{Tr}L^4$ and a rational integral which is obtained by the method of chopping as in [5].

Taking $\Phi = \{\alpha, \beta, \alpha + \beta\}$ we obtain another integrable system with Lax matrix

$$L = \begin{pmatrix} b_1 & a_1 & a_3 & 0 & 0 \\ a_1 & b_2 & a_2 & 0 & 0 \\ a_3 & a_2 & 0 & -a_2 & -a_3 \\ 0 & 0 & -a_2 & -b_2 & -a_1 \\ 0 & 0 & -a_3 & -a_1 & -b_1 \end{pmatrix}.$$

The matrix L_+ is defined as above, i.e. the skew-symmetric part of L. Again there is rational integral given by $I_{11} = (a_1 a_2 - a_3 b_2)/a_3$. Defining the Poisson bracket by $\{a_1, a_2\} = a_3$, $\{a_i, b_i\} = -a_i$, $i = 1, 2$, and $\{a_1, b_2\} = a_1$ we verify easily that h_2 plays the role of the Hamiltonian and I_{11} is a Casimir. The set $\{h_2, h_4, I_{11}\}$ is an independent set of functions in involution.

2 Intermediate Toda lattices

We have defined some Hamiltonian systems associated to a subset Φ consisting of positive roots (which we call adapted). The associated matrix is symmetric. As in the case of classical and full Toda there is also an analogous system defined by a Lax matrix which is lower triangular (the Kostant–Toda lattices). In this paper we restrict our attention to this version of the systems. In this section we show that these Hamiltonian systems are associated to a nilpotent ideal of a Borel subalgebra of a semisimple Lie algebra $\mathfrak{g}$. Since for particular (extreme) choices of the ideal one finds the classical Kostant–Toda lattice or the full Kostant–Toda lattice associated to $\mathfrak{g}$, we call these Hamiltonian systems *intermediate Toda lattices.*

2.1 The phase space $M_{\mathcal{I}}$

Throughout this section $\mathfrak{g}$ is an arbitrary complex semisimple Lie algebra, the rank of which we denote by ℓ. We fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a basis $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ of the root system Δ of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The choice of Π amounts to the choice of a Borel subalgebra $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$. It also leads to a Borel subalgebra $\mathfrak{b}_- = \mathfrak{h} \oplus \mathfrak{n}_-$ corresponding to the negative roots. We fix an element ε in $\mathfrak{n}_+$, satisfying $\langle \varepsilon \mid [\mathfrak{n}_-, \mathfrak{n}_-] \rangle = 0$, where $\langle \cdot \mid \cdot \rangle$ stands for the Killing form of $\mathfrak{g}$. One usually picks for ε a principal nilpotent element of $\mathfrak{n}_+$. For example, for $\mathfrak{g} = \mathrm{sl}_N(\mathbf{C})$, viewed as the Lie algebra of traceless $N \times N$ matrices, one can take for $\mathfrak{h}$ and for $\mathfrak{b}_+$ the subalgebras of diagonal, respectively upper triangular, matrices and for ε one can choose

$$\varepsilon := \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & 1 \\ 0 & \dots & \dots & \dots & 0 \end{pmatrix}.$$

Let $\mathcal{I}$ be a nilpotent ideal of $\mathfrak{b}_+$. The quotient map $\mathfrak{b}_+ \to \mathfrak{b}_+/\mathcal{I}$ is denoted by $P_{\mathcal{I}}$. Using the isomorphism $\mathfrak{b}_+^* \simeq \mathfrak{b}_-$ induced by the Killing form, we can think of the orthogonal $\mathcal{I}^\perp$ of $\mathcal{I}$ in $\mathfrak{b}_+^*$ as a vector subspace of $\mathfrak{b}_-$. We consider the affine space $M_{\mathcal{I}} := \varepsilon + \mathcal{I}^\perp$. Explicitly

$$M_{\mathcal{I}} = \{X + \varepsilon \mid X \in \mathfrak{b}_- \text{ and } \langle X \mid \mathcal{I} \rangle = 0\}.$$

When $\mathcal{I} = \{0\}$, $M_{\mathcal{I}} = \mathfrak{b}_- + \varepsilon$, which is the phase space of the full Kostant–Toda lattice. On the other extreme, taking $\mathcal{I} = [\mathfrak{n}_+ , \mathfrak{n}_+]$ the manifold $M_{\mathcal{I}}$ is the phase space of the classical Kostant–Toda lattice. We therefore call $M_{\mathcal{I}}$ the intermediate Kostant–Toda phase space. Notice that, if $\mathcal{I} \subset \mathcal{J}$, then $M_{\mathcal{J}} \subset M_{\mathcal{I}}$.

2.2 Hamiltonian structure

We show that $M_{\mathcal{I}}$ has a natural Poisson structure. To do this we prove that $M_{\mathcal{I}}$ is a Poisson submanifold of $\mathfrak{g}$ equipped with a Poisson structure $\{\cdot\,,\cdot\}$ the construction of which[1] we firstly recall. We use the theory of R-matrices (see [2, Chapter 4.4]

[1]See the appendix of [6] for an alternative construction using symplectic reduction to the cotangent bundle $T^*\mathbf{G}$, where $\mathbf{G}$ is any Lie group integrating $\mathfrak{g}$.

for the general theory of R-matrices). Write $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ where $\mathfrak{g}_+ := \mathfrak{b}_+$ and $\mathfrak{g}_- := \mathfrak{n}_-$. For $X \in \mathfrak{g}$ its projection in $\mathfrak{g}_\pm$ is denoted by $X_\pm$. The endomorphism $R : \mathfrak{g} \to \mathfrak{g}$, defined for all $X \in \mathfrak{g}$ by $R(X) := X_+ - X_-$, is an R-matrix which means that the bracket on $\mathfrak{g}$, defined by

$$[X, Y]_R := \frac{1}{2}([R(X), Y] + [X, R(Y)]) = [X_+, Y_+] - [X_-, Y_-]$$

for all $X, Y \in \mathfrak{g}$, is a (new) Lie bracket on $\mathfrak{g}$. The Lie–Poisson bracket on $\mathfrak{g}$, which corresponds to $[\cdot, \cdot]_R$ and which we denote simply by $\{\cdot\,, \cdot\}$ (since it is the only Poisson bracket on $\mathfrak{g}$ which we use), is given by

$$\{F, G\}(X) = \langle X \mid [(\nabla_X F)_+, (\nabla_X G)_+]\rangle - \langle X \mid [(\nabla_X F)_-, (\nabla_X G)_-]\rangle \tag{6}$$

for every pair of functions, F and G, on $\mathfrak{g}$ and for all $X \in \mathfrak{g}$. In this formula the gradient $\nabla_X F$ of F at X is the element of $\mathfrak{g}$ defined by

$$\langle \nabla_X F \mid Y\rangle = \langle \mathrm{d}_X F, Y\rangle = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} F(X + tY). \tag{7}$$

Proposition 2. *Let $\mathcal{I}$ be a nilpotent ideal of $\mathfrak{b}_+$.*

(1) The affine space $M_\mathcal{I}$ is a Poisson submanifold of $(\mathfrak{g}, \{\cdot\,, \cdot\})$;

(2) Equipped with the induced Poisson structure $M_\mathcal{I}$ is isomorphic to $(\mathfrak{b}_+/\mathcal{I})^*$, which is equipped with the canonical Lie–Poisson bracket;

(3) A function F on $M_\mathcal{I}$ is a Casimir function if and only if $(\nabla_X \tilde{F})_+ \in \mathcal{I}$ for all $X \in M_\mathcal{I}$, where $\tilde{F}$ is an arbitrary extension of F to $\mathfrak{g}$.

For a function H on $M_\mathcal{I}$ we denote its Hamiltonian vector field by $\mathcal{X}_H$; our sign convention is that $\mathcal{X}_H := \{\cdot\,, H\}$ so that $\mathcal{X}_H[F] = \{F, H\}$ for all $F \in \mathcal{F}(M)$. The Hamiltonian of the intermediate Kostant–Toda lattice is the polynomial function on $M_\mathcal{I}$ given by

$$H := \frac{1}{2}\langle X \mid X\rangle \tag{8}$$

so that the vector field of the intermediate Kostant–Toda lattice is given by the Lax equation (on $M_\mathcal{I}$)

$$\dot{X} = [X_+, X]\,. \tag{9}$$

2.3 Height k Kostant–Toda lattices

In the sequel of this paper we mainly study the case for which $\mathcal{I}$ is an ideal of height 2, a notion which we introduce in this paragraph. We firstly give some information on the nilpotent ideals of $\mathfrak{b}_+$ (see [4]). If $\mathcal{I}$ is a nilpotent ideal of $\mathfrak{b}_+$, then $\mathcal{I}$ is contained in $\mathfrak{n}_+$. For example $\mathfrak{n}_+$ itself is a nilpotent ideal of $\mathfrak{b}_+$.

For $\alpha \in \Delta^+$ let $X_{\mathfrak{a}}$ denote an arbitrary root vector corresponding to α, i.e., $[H, X_\alpha] = \langle \alpha, H \rangle X_\alpha$ for all $H \in \mathfrak{h}$. Consider a subset, Φ, of Δ^+ which has the property that, if $\alpha \in \Phi$, then every root of the form $\alpha + \beta$ with $\beta \in \Delta^+$ belongs to Φ; we call such a set Φ an admissible set of roots. For such α and β the Jacobi identity implies that $[X_\alpha, X_\beta]$ is a multiple of $X_{\alpha+\beta}$. It follows that the (vector space) span of $\{X_\alpha \mid \alpha \in \Phi\}$ is a nilpotent ideal of $\mathfrak{b}_+$. Most importantly every nilpotent ideal of $\mathfrak{b}_+$ is of this form for a certain admissible set of roots Φ. Thus the nilpotent ideals of a given Borel subalgebra $\mathfrak{b}_+$ of $\mathfrak{g}$ are parametrized by the family of all subsets Φ of Π^+, which have the property that, if $\alpha \in \Phi$, then every root of the form $\alpha + \beta$ with $\beta \in \Delta^+$ belongs to Φ.

Every positive root $\alpha \in \Delta^+$ can be written as a linear combination of the simple roots, $\alpha = \sum_{i=1}^{\ell} n_i \alpha_i$, where all n_i are nonnegative integers. The integer $ht(\alpha) := \sum_{i=1}^{\ell} n_i$ is called the *height* of α. For $k \in \mathbf{N}$, let Φ_k denote the set of all roots of height larger than k. It is clear that Φ_k is an admissible set of roots. We denote the corresponding ideal of $\mathfrak{b}_+$ by $\mathcal{I}_k$ and we call it a height k ideal. An alternative description of $\mathcal{I}_k$ is as $\mathrm{ad}^k_{\mathfrak{n}_+} \mathfrak{n}_+$. For $k = 1$, $\mathcal{I}_1 = [\mathfrak{n}_+, \mathfrak{n}_+]$ is the ideal which leads to the classical Toda lattice. We consider in the sequel mainly $\mathcal{I}_2 = [\mathfrak{n}_+, [\mathfrak{n}_+, \mathfrak{n}_+]]$ and the corresponding affine space $M_{\mathcal{I}_2}$.

Example 2. Consider a Lie algebra of type C_4. Take $\Phi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1 + \alpha_2, \alpha_2 + \alpha_3, \alpha_3 + \alpha_4\}$. It gives rise to a height 2 Toda system.

The Lax matrix is

$$L = \begin{pmatrix} a_1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & 1 & 0 & 0 & 0 & 0 & 0 \\ c_1 & b_2 & a_3 & 1 & 0 & 0 & 0 & 0 \\ 0 & c_2 & b_3 & a_4 & 1 & 0 & 0 & 0 \\ 0 & 0 & c_3 & b_4 & -a_4 & -1 & 0 & 0 \\ 0 & 0 & 0 & c_3 & -b_3 & -a_3 & -1 & 0 \\ 0 & 0 & 0 & 0 & -c_2 & -b_2 & -a_2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -c_1 & -b_1 & -a_1 \end{pmatrix}.$$

The function

$$a_1 - a_2 + a_3 - a_4 + \frac{2b_1b_2c_3 + b_1c_2b_4 + b_3b_4c_1}{c_1c_3}$$

is a Casimir. We need five functions to establish integrability. Since $\det(L - \lambda I)$ is an even polynomial of the form $\lambda^8 + \sum_{i=0}^{3} f_i \lambda^{2i}$, we obtain four polynomial integrals f_0, f_1, f_2, f_3. Using an one-chop we obtain a characteristic polynomial of the form $A\lambda^2 + B$. The function $f_4 = B/A$ is the fifth integral.

3 Computation of the rank

In this section we compute the index of the Lie algebra $\mathfrak{b}_+/\mathcal{I}_2$ when $\mathfrak{b}_+$ is a Borel subalgebra of a simple Lie algebra of type A_ℓ, B_ℓ or C_ℓ. It yields the rank of the corresponding intermediate Kostant–Toda phase space (see Subsection 2.3). We firstly recall a few basic facts about stable linear forms, the index of a Lie algebra and the relation to the rank of the corresponding Lie–Poisson structure.

3.1 Stable linear forms

Let $\mathfrak{a}$ be any complex algebraic Lie algebra and $\mathfrak{a}^*$ its dual vector space. The *stabilizer* of a linear form $\varphi \in \mathfrak{a}^*$ is given by

$$\mathfrak{a}^\varphi := \{x \in \mathfrak{a} \mid \mathrm{ad}_x^* \varphi = 0\} = \{x \in \mathfrak{a} \mid \forall y \in \mathfrak{a},\ \langle \varphi, [x, y]\rangle = 0\}.$$

The integer $\min\{\dim \mathfrak{a}^\varphi \mid \varphi \in \mathfrak{a}^*\}$ is called the *index* of $\mathfrak{a}$ and is denoted by $\mathrm{Ind}(\mathfrak{a})$. Since the symplectic leaves of the canonical Lie–Poisson structure on $\mathfrak{a}^*$ are the coadjoint orbits, the codimension of the symplectic leaf through φ is the dimension of $\mathfrak{a}^\varphi$. It follows that the index of $\mathfrak{a}$ is the codimension of a symplectic leaf of maximal dimension, i.e., the rank of the canonical Lie–Poisson structure on $\mathfrak{a}^*$ is given by $\dim \mathfrak{a} - \mathrm{Ind}(\mathfrak{a})$; notice that, since the latter rank is always even, the index of $\mathfrak{a}$ and the dimension of $\mathfrak{a}$ have the same parity. A linear form $\varphi \in \mathfrak{a}^*$ is said to be *regular* if $\dim \mathfrak{a}^\varphi = \mathrm{Ind}(\mathfrak{a})$. Thus we can use regular linear forms to compute the index of $\mathfrak{a}$ and hence the rank of the canonical Lie–Poisson structure on $\mathfrak{a}^*$.

We use the following proposition to compute the index of $\mathfrak{b}_+/\mathcal{I}_2$.

Proposition 3. *Let $\mathfrak{a}$ be a subalgebra of a semisimple complex Lie algebra $\mathfrak{g}$. Suppose that φ is a linear form on $\mathfrak{a}$ such that $\mathfrak{a}^\varphi$ is a commutative Lie algebra composed of semisimple elements. Then φ is regular so that the index of $\mathfrak{a}$ is given by* $\dim \mathfrak{a}^\varphi$.

Proof. A linear form $\varphi \in \mathfrak{a}^*$ is said to be *stable* if there exists a neighborhood U of φ in $\mathfrak{a}^*$ such that for every $\psi \in U$ the stabilizer $\mathfrak{a}^\psi$ is conjugate to $\mathfrak{a}^\varphi$ with respect to the adjoint group of $\mathfrak{a}$. According to [8] every stable linear form is regular. According to [7] and [8, Theorem 1.7, Corollary 1.8] φ is stable if and only if $[\mathfrak{a}, \mathfrak{a}^\varphi] \cap \mathfrak{a}^\varphi = \{0\}$. The latter equality holds when $\mathfrak{a}^\varphi$ is a commutative Lie algebra composed of semisimple elements (see [8, Lemma 2.6]). Thus φ is stable, hence regular. ■

3.2 Computation of the index

In this paragraph we compute the index of $\mathfrak{b}/\mathcal{I}$ under the following assumption on (the root system of) $\mathfrak{g}$:

(*H*) *The roots of height* 2 *of* $\mathfrak{g}$ *are given by* $\{\alpha_k + \alpha_{k+1} \mid 1 \le k \le \ell - 1\}$.

For classical Lie algebras the basis Π can be ordered such that this assumption occurs when $\mathfrak{g}$ is of type A_ℓ, B_ℓ or C_ℓ. Let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha\in\Delta^+}(\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha})$ be the decomposition of $\mathfrak{g}$ according to the adjoint action of $\mathfrak{h}$. To each positive root α there corresponds a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of elements of $\mathfrak{g}$, where $X_\alpha \in \mathfrak{g}_\alpha, X_{-\alpha} \in \mathfrak{g}_{-\alpha}, H_\alpha \in \mathfrak{h}$ and $(X_\alpha, X_{-\alpha}, H_\alpha)$ generates a subalgebra isomorphic to $\mathrm{sl}_2(\mathbf{C})$. We recall shortly how such a triple can be constructed. Let h_α be the unique element in $\mathfrak{h}$ such that $\langle \alpha, H\rangle = \langle h_\alpha \mid H\rangle$ for all $H \in \mathfrak{h}$. Define a scalar product on the real vector-space $\mathfrak{h}^*_{\mathbf{R}}$ by

$$\langle \alpha \mid \beta\rangle := \langle h_\alpha h_\beta\rangle = \langle \beta, h_\alpha\rangle = \langle \alpha, h_\beta\rangle$$

for all α and $\beta \in \Delta$. We set $H_\alpha := \dfrac{2}{\langle \alpha \mid \alpha\rangle} h_\alpha$. It is clear that $\langle \alpha, H_\alpha\rangle = 2$. Choose $X_\alpha \in \mathfrak{g}_\alpha, X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that

$$\langle X_\alpha \mid X_{-\alpha}\rangle = \frac{2}{\langle \alpha \mid \alpha\rangle}.$$

Then $(X_\alpha, X_{-\alpha}, H_\alpha)$ is the required triple. Moreover

$$\left[X_{\pm\alpha_k}, X_{\mp\alpha_k\mp\alpha_{k+1}}\right] = \epsilon_k^{\pm} X_{\mp\alpha_{k+1}}, \qquad \left[X_{\pm\alpha_{k+1}}, X_{\mp\alpha_k\mp\alpha_{k+1}}\right] = \eta_k^{\pm} X_{\mp\alpha_k},$$

where each of the integers $\epsilon_k^{\pm}$ and $\eta_k^{\pm}$ is equal to 1 or to -1 depending upon $\mathfrak{g}$.

For all $\alpha, \beta \in \Pi$ let

$$C_{\alpha\beta} := \langle \beta, H_\alpha\rangle = 2\frac{\langle \alpha \mid \beta\rangle}{\langle \alpha \mid \alpha\rangle}.$$

The $\ell \times \ell$-matrix $C := (C_{ij}, 1 \le i, j \le \ell)$, where $C_{ij} := C_{\alpha_i\alpha_j}$, is invertible. It is called the *Cartan matrix* of $\mathfrak{g}$.

Proposition 4. *Consider the linear form φ on $\mathfrak{b}_+$ defined for $Z \in \mathfrak{b}_+$ by $\langle \varphi, Z\rangle := \langle X \mid Z\rangle$, where X is defined by*

$$X := \delta_\ell X_{-\alpha_\ell} + \sum_{i=1}^{\ell-1} X_{-\alpha_i-\alpha_{i+1}} \tag{10}$$

with $\delta_\ell := 1$ if ℓ is odd and $\delta_\ell := 0$ otherwise. Denote by $\bar\varphi$ the induced linear form on $\mathfrak{b}_+/\mathcal{I}_2$.

(1) $\bar\varphi$ is a regular linear form on $\mathfrak{b}_+/\mathcal{I}_2$;

(2) $\dim(\mathfrak{b}_+/\mathcal{I}_2)^{\bar\varphi} = 1 - \delta_\ell$;

(3) The index of $\mathfrak{b}_+/\mathcal{I}_2$ is 1 if the rank ℓ of $\mathfrak{g}$ is even and is 0 otherwise.

4 Integrability

We now get to the integrability of the intermediate Kostant–Toda lattice on $M_{\mathcal{I}_2} \subset \mathfrak{g}$ for any semisimple Lie algebra $\mathfrak{g}$ of type A_ℓ, B_ℓ or C_ℓ. Recall that this means that the Hamiltonian is part of a family of s independent functions in involution, where s is related to the dimension and the rank of the Poisson manifold $M_{\mathcal{I}_2}$ by the formula $\dim M_{\mathcal{I}_2} = \frac{1}{2}\mathrm{Rk}\, M_{\mathcal{I}_2} + s$. Since $\dim M_{\mathcal{I}_2} = 3\ell - 1$ and since the corank of $M_{\mathcal{I}_2}$ is 1 if ℓ is even and 0 otherwise (see item (3) in Proposition 4), we need $s = [3\ell/2]$ such functions. According to the Adler–Kostant–Symes Theorem the ℓ basic Ad-invariant polynomials provide already ℓ independent functions in involution. Thus one needs $[\ell/2]$ additional ones. As we see, they can be constructed by restricting certain chop-type integrals, except for the case of C_ℓ for which another integral (Casimir) is needed. We firstly recall from [5] the construction of the chop integrals on $M := \varepsilon + \mathfrak{b}_-$ in the case that $\mathfrak{g} = \mathrm{sl}_N(\mathbf{C})$ and explain why they are in involution. Since $M_{\mathcal{I}_2}$ is a Poisson submanifold of M, their restrictions to $M_{\mathcal{I}_2}$ are still in involution (but they may become trivial or dependent).

We consider $\mathfrak{g} = \mathrm{sl}_N(\mathbf{C})$ with the standard choice of $\mathfrak{h}$ and Π (see Subsection 2.1). Let k be an integer, $0 \le k \le [\frac{N-1}{2}]$. For any matrix X we denote by X_k the matrix obtained by removing the first k rows and last k columns from X. We denote by $\mathbf{G}_k$ the subgroup of $\mathbf{GL}_N(\mathbf{C})$ consisting of all $N \times N$ invertible matrices of the form

$$g = \begin{pmatrix} \Delta & A & B \\ 0 & D & C \\ 0 & 0 & \Delta' \end{pmatrix}, \tag{11}$$

where Δ and Δ' are arbitrary upper triangular matrices of size $k \times k$ and A, B, C and D are arbitrary[2]. The Lie algebra of $\mathbf{G}_k$ is denoted by $\mathfrak{g}_k$. A first, fundamental and nontrivial observation, due to [5], is that for all $g \in \mathbf{G}_k$, decomposed as in (11),

$$\det \left(gXg^{-1}\right)_k = \frac{\det \Delta'}{\det \Delta} \det X_k. \tag{12}$$

This leads to (rational) $\mathbf{G}_k$-invariant functions on $\mathfrak{g}$ (and hence on M) which are constructed as follows. For $X \in \mathfrak{g}$ and for an arbitrary scalar l consider the so-called k-chop polynomial of X defined by $Q_k(X,\lambda) := \det(X - \lambda\,\mathrm{Id}_N)_k$. In view of (12) the coefficients of Q_k (as a polynomial in l) define polynomial functions on $\mathfrak{g}$, which transform under the action of $g \in \mathbf{G}_k$ with the same factor $\det \Delta' / \det \Delta$. We write

$$Q_k(X,\lambda) = \sum_{i=0}^{N-2k} E_{i,k}(X)\lambda^{N-2k-i}.$$

Each of the rational functions $E_{i,k}/E_{j,k}$ is $\mathbf{G}_k$-invariant. By restriction to M this yields $\mathbf{G}_k$-invariant elements of $\mathcal{F}(M)$. They are called k-chop integrals because they are integrals (constants of motion) for the full Kostant–Toda lattice. Notice that the constants of motion $H_i := \frac{1}{i}\,\mathrm{Tr}\, X^i$ are 0-chop integrals and that the Toda Hamiltonian is expressible in terms of them as $H = (H_1^2 - 2H_2)/2$.

We show that all chop integrals are in involution. To do this we let F be a k-chop integral and let $\tilde{F}$ denote its extension to a $\mathbf{G}_k$-invariant rational function

[2] With the understanding that, since X is supposed invertible, Δ, Δ' and D are invertible.

on $\mathfrak{g}$. Similarly let G be a l-chop integral with $\mathbf{G}_\ell$-invariant extension $\tilde{G}$. We may suppose that $k \leq \ell$. Infinitesimally the fact that $\tilde{F}$ is $\mathbf{G}_k$ invariant yields that

$$\left\langle X \left[\nabla_X \tilde{F}, Y\right]\right\rangle = 0 \tag{13}$$

for all $X \in \mathfrak{g}$ and for all $Y \in \mathfrak{g}_k$. Since $\mathfrak{b}_+ \subset \mathfrak{g}_k$, it follows that

$$\left\langle X \;\middle|\; \left[(\nabla_X \tilde{F})_+, \nabla_X \tilde{G}\right]\right\rangle = 0 = \left\langle X \;\middle|\; \left[\nabla_X \tilde{F}, (\nabla_X \tilde{G})_+\right]\right\rangle$$

so that (6) can be rewritten for $X \in M$ as

$$\{F, G\}(X) = -\left\langle X \mid \left[\nabla_X \tilde{F}, \nabla_X \tilde{G}\right]\right\rangle. \tag{14}$$

We claim that $\nabla_X \tilde{G} \in \mathfrak{g}_\ell$. This follows from the construction of the function $\tilde{G} \in \mathcal{F}(\mathfrak{g})$: the rational function $\tilde{G}(X)$ depends only upon X_ℓ, the ℓ-chop of X, while, if an element Z of $\mathfrak{g}$ satisfies $\langle \mathfrak{g}_\ell \mid Z\rangle = 0$, then Z_ℓ is the zero matrix. Thus $\nabla_X \tilde{G} \in \mathfrak{g}_\ell \subset \mathfrak{g}_k$ so that (13) implies that the right hand side of (14) is zero for all $X \in M$. It follows that F and G have zero Poisson bracket.

Notice that in the case of the height 2 intermediate Kostant–Toda lattice all k-chops with $k > 1$ vanish and that only a few 1-chops survive. In what follows we consider separately the cases of A_ℓ, B_ℓ and C_ℓ.

4.1 The case of A_ℓ

We firstly consider $\mathfrak{g} = \mathrm{sl}_{\ell+1}(\mathbf{C})$, the Lie algebra of traceless matrices of size $N = \ell + 1$, and take for $\mathfrak{h}$, Π and ε the standard choices as before. A general element of $\mathcal{M}_{\mathcal{I}_2}$ is then of the form

$$X = \begin{pmatrix} a_1 & 1 & 0 & \dots & \dots & 0 \\ b_1 & a_2 & 1 & \ddots & & \vdots \\ c_1 & b_2 & a_3 & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & c_{\ell-1} & b_\ell & a_{\ell+1} \end{pmatrix}$$

with $\sum_{i=1}^{\ell+1} a_i = 0$. The 1-chop matrix of X is given by

$$(X - \lambda\, \mathrm{Id}_{\ell+1})_1 = \begin{pmatrix} b_1 & a_2^\lambda & 1 & 0 & \dots & 0 \\ c_1 & b_2 & a_3^\lambda & 1 & \ddots & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & 0 \\ \vdots & \ddots & c_3 & b_4 & \ddots & 1 \\ \vdots & & \ddots & \ddots & \ddots & a_\ell^\lambda \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_\ell \end{pmatrix},$$

where a_i^λ is a shorthand for $a_i - \lambda$. We also use the matrix $X(\lambda,\alpha)$, defined by

$$X(\lambda,\alpha) = \begin{pmatrix} b_1 & a_2^\lambda & \alpha_{13} & \dots & \dots & \alpha_{1\ell} \\ c_1 & b_2 & a_3^\lambda & \alpha_{24} & & \vdots \\ 0 & c_2 & b_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & c_3 & b_4 & \ddots & \alpha_{\ell-2,\ell} \\ \vdots & & \ddots & \ddots & \ddots & a_\ell^\lambda \\ 0 & \dots & \dots & 0 & c_{\ell-1} & b_\ell \end{pmatrix}.$$

Proposition 5. *The polynomials* $\det(X - \lambda\,\mathrm{Id}_{\ell+1})_1$ *and* $\det X(\lambda,\alpha)$ *have degree* $d := [\frac{\ell}{2}]$ *in* λ.

4.2 The case of B_ℓ

A Lie algebra of type B_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell+1$, satisfying $XJ + JX^t = 0$, where J is the matrix of size $2\ell+1$, all of whose entries are zero except for the entries on the antidiagonal, which are all equal to one. Clearly X satisfies $XJ + JX^t = 0$ if and only if X is skew-symmetric with respect to its antidiagonal. It follows for such X that $\det(X - \lambda\,\mathrm{Id}_{\ell+1}) = (-1)^N \det(X + \lambda\,\mathrm{Id}_{\ell+1})$ so that the characteristic polynomial is an odd polynomial in λ. The 1-chop matrix X_1 satisfies the same relation $X_1 J + JX_1^t = 0$ so that its determinant is an even polynomial in λ. As a Cartan subalgebra of $\mathfrak{g}$ one can take the diagonal matrices in $\mathfrak{g}$ and one can take as a basis for Δ^+ the matrices $E_{i,i+1} - E_{2\ell-i,2\ell-i+1}$ for $i = 1,\dots,\ell$. If one finally chooses ε to be the matrix $\sum_{i=1}^{\ell}(E_{i,i+1} - E_{2\ell-i,2\ell-i+1})$, then the height 2 phase space is given by all matrices of the form

$$\begin{pmatrix} a_1 & 1 & & & & & & & \\ b_1 & \ddots & \ddots & & & & & & \\ c_1 & \ddots & \ddots & 1 & & & & & \\ & \ddots & b_{n-1} & a_n & 1 & & & & \\ & & c_{n-1} & b_n & 0 & -1 & & & \\ & & & 0 & -b_n & -a_n & \ddots & & \\ & & & & -c_{n-1} & -b_{n-1} & \ddots & \ddots & \\ & & & & & \ddots & \ddots & & -1 \\ & & & & & -c_1 & -b_1 & -a_1 & \end{pmatrix}.$$

In this case $N = 2\ell+1$, the 1-chop polynomial is even and so the 1-chop polynomial is degree ℓ when ℓ is even and of degree $\ell-1$ when ℓ is odd. This yields $\frac{\ell}{2}$ integrals when ℓ is even and $\frac{\ell-1}{2}$ when ℓ is odd. Therefore the number of integrals is correct in each case.

4.3 The case of C_ℓ

A Lie algebra of type C_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell$, satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

$$J = \begin{pmatrix} 0 & I_l \\ -I_l & 0 \end{pmatrix}.$$

It follows for such X that $\det(X - \lambda\,\mathrm{Id}_{\ell+1}) = (-1)^{2l}\det(X + \lambda\,\mathrm{Id}_{\ell+1})$ so that the characteristic polynomial is an even polynomial in λ. The 1-chop matrix X_1 satisfies the same relation $X_1 J + JX_1^t = 0$ so that its determinant is an even polynomial in λ. As a Cartan subalgebra of $\mathfrak{g}$ one can take the diagonal matrices in $\mathfrak{g}$ and one can take as a basis for Δ^+ the matrices $E_{i,i+1} - E_{2\ell-1-i,2\ell-i}$ for $i = 1, \ldots, \ell$. The height 2 phase space for C_ℓ is given by all matrices of the form

$$\begin{pmatrix}
a_1 & 1 & & & & & & & \\
b_1 & a_2 & \ddots & & & & & & \\
c_1 & b_2 & \ddots & 1 & & & & & \\
 & \ddots & \ddots & a_n & 1 & & & & \\
 & & c_{n-1} & b_n & -a_n & -1 & & & \\
 & & & c_{n-1} & -b_{n-1} & \ddots & \ddots & & \\
 & & & & -c_{n-2} & \ddots & \ddots & \ddots & \\
 & & & & & \ddots & \ddots & & -1 \\
 & & & & & -c_1 & -b_1 & -a_1 &
\end{pmatrix}.$$

In this case, $N = 2\ell$, the 1-chop polynomial is even so that we get $\frac{l}{2} - 1$ integrals from the 1-chop when l is even and $\frac{l-2}{2}$ integrals when l is odd. Therefore the odd case gives the correct number of integrals. For the even case there exists a Casimir function which does not arise from the method of chopping and we describe it as follows:

The Casimir f has the form $f = A + B/C$, where

$$A = \sum_{i=1}^{\ell-1}(a_i - a_{i+1}), \quad B = \sum_{i,j} d_{ij} m_{ij}, \quad \text{and} \quad C = \prod_{i=1}^{\ell-1} c_{2i-1}.$$

The term m_{ij} in B is determined as follows: We associate the variables b_1, b_2, $\ldots$, b_l to the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_l$ and the variables c_1, c_2, $\ldots$, c_{l-1} to the height 2 roots $\alpha_1 + \alpha_2$, $\alpha_2 + \alpha_3$, $\ldots$, $\alpha_{l-1} + \alpha_l$.

Take simple roots α_i and α_j (with corresponding variables b_i, b_j) such that i is odd and j is even. The remaining variables correspond to the height two roots $\alpha_k + \alpha_{k+1}$, where $k \neq i, i-1$, $k \neq j, j-1$. The term m_{ij} is a product of b_i, b_j and $\frac{l-1}{2}$ c variables.

The coefficient d_{ij} is 2 if m_{ij} includes the term c_{l-1} (corresponding to the root $\alpha_{l-1}+\alpha_l$) and is equal to 1 otherwise.

Example 3. $l = 6$.

$$f = a_1 - a_2 + a_3 - a_4 + a_5 - a_6 + \frac{b_5b_6c_1c_3 + 2b_1b_4c_2c_5 + b_3b_6c_1c_4 + 2b_1b_2c_3c_5 + 2b_3b_4c_1c_5 + b_1b_6c_2c_4}{c_1c_3c_5}.$$

4.4 The case of D_ℓ

We conclude with some comments on the case of D_ℓ. A Lie algebra of type D_ℓ can be realized as the Lie algebra $\mathfrak{g}$ of all square matrices of size $N = 2\ell$ satisfying $XJ + JX^t = 0$, where J is the matrix of size 2ℓ, given by

$$J = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}.$$

As in the case of C_ℓ the characteristic polynomial is an even polynomial. On the other hand the 1-chop polynomial is odd so that the degree of this polynomial is $\ell-1$ when ℓ is even. However, when ℓ is odd the degree of the 1-chop polynomial is again ℓ. This gives $\frac{\ell}{2}-1$ integrals when ℓ is even and $\frac{\ell-1}{2}$ integrals when ℓ is odd. In the even case we need an additional function, i.e. a Casimir, but at this point we do not have an explicit formula. There is no stable form in this case, but we can produce a form which gives a lower bound for the rank and this lower bound is good enough, once we have the Casimir.

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[1] Adler M. and van Moerbeke P., Completely integrable systems, Euclidean Lie algebras, and curves, *Adv. Math.*, 1980, V.38, 267–317.

[2] Adler M., van Moerbeke P., Vanhaecke P., *Algebraic Integrability, Painlevé Geometry and Lie Algebras* (Ergebnisse der Mathematik und ihrer grenzgebiete), V.47, Springer-Verlag, Berlin Heidelberg, 2004.

[3] Bogoyavlensky O.I., On perturbations of the periodic Toda lattice, *Comm. Math. Phys.*, 1976, V.51, 201–209.

[4] Cellini P. and Papi P., Ad-nilpotent ideals of a Borel subalgebra, *J. Algebra*, 2000, V.225, 130–141.

[5] Deift P.A., Li L.C., Nanda T. and Tomei C., The Toda Lattice on a generic orbit is integrable *Comm. Pure Appl. Math.*, 1986, V.39, 183–232.

[6] Ercolani N.M., Flaschka H. and Singer S., The geometry of the full Kostant–Toda lattice, *Colloque Verdier, Progress in Mathematics Series*, Birkhäuser Verlag, 1994, 181–225.

[7] Kosmann Y. and Sternberg S., Conjugaison des sous-algèbres d'isotropie, *C.R.A.S, Paris Ser. A*, 1974, V.279, 777–779.

[8] Tauvel P. and Yu R., Indice et formes linéaires stables dans les algèbres de Lie, *J. Algebra*, 2004, V.273, 507–516.

[9] Flaschka H., The Toda lattice I. Existence of integrals, *Phys. Rev. B*, 1974, V.9, 1924–1925.

[10] Kostant B., The solution to a generalized Toda lattice and representation theory, *Adv. Math.*, 1979, V.34, 195–338.

[11] Olshanetsky M.A. and Perelomov A.M., Explicit solutions of classical generalized Toda models, *Invent. Math.*, 1979, V.54, 261–269.

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On nonclassical symmetries of generalized Huxley equations

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We search for nonclassical symmetries of a class of generalized Huxley equations of the form $u_t = u_{xx} + k(x)u^2(1-u)$. We completely classify the functions $k(x)$ such that this class admits nonclassical symmetries of the form $Q = \partial_t + \xi\partial_x + \eta\partial_u$.

1 Introduction

We consider reaction-diffusion equation of the form

$$u_t = u_{xx} + k(x)u^2(1-u), \tag{1}$$

where $k(x) \neq 0$. This equation models many phenomena that occur in different areas of mathematical physics and biology. In particular, it can be used to describe the spread of a recessive advantageous allele through a population in which there are only two possible alleles at the locus in question. Equation (1) is interesting also in the area of nerve axon potentials [18]. Case $k = \text{const}$ is the known Huxley equation. For more details about application see [3, 4] and references therein.

Group analysis of differential equations provides us with systematic methods for deducing exact solutions of nonlinear general partial differential equations. One of these methods, called by the authors "non-classical", was introduced by Bluman and Cole [2]. A precise and rigorous definition of nonclassical invariance was firstly formulated in [10] as "a generalization of the Lie definition of invariance" (see also [21]). Later operators satisfying the nonclassical invariance criterion were also called, by different authors, nonclassical symmetries, conditional symmetries, Q-conditional symmetries and reduction operators [8, 9, 13]. The necessary definitions, including ones of equivalence of reduction operators, and relevant statements on this subject are collected in [17, 19].

Initially Bradshaw-Hajek *et al.* [3, 4] studied the class (1) from the symmetry point of view. More precisely, they found some cases of equations (1) admitting Lie and/or nonclassical symmetries. Complete classification of Lie symmetries of class (1) is performed in [12]. Conditional symmetries of Huxley and Burgers–Haxley equations having nontrivial intersection with class (1) are investigated in [1, 5–7, 11]. The present paper is a step towards to the complete classification of nonclassical symmetries of the class (1).

Reduction operators of equations (1) have the general form $Q = \tau\partial_t + \xi\partial_x + \eta\partial_u$, where τ, ξ and η are functions of t, x and u, and $(\tau, \xi) \neq (0, 0)$. In order to derive such operators one needs to consider two cases:

1. $\tau \neq 0$. Without loss of generality $\tau = 1$.
2. $\tau = 0$, $\xi \neq 0$. Without loss of generality $\xi = 1$.

Here we present the complete classification for the case 1 and we give the first steps for the case 2.

2 Equivalence transformations and Lie symmetries

As classification of nonclassical symmetries is impossible without detailed knowledge of Lie invariance properties, we review [12] the equivalence group and results of the group classification of class (1). The complete equivalence group $G^\sim$ of class (1) contains only scaling and translation transformations of independent variables t and x. More precisely it consists of transformations

$$\tilde{t} = \varepsilon_1^2 t + \varepsilon_2, \quad \tilde{x} = \varepsilon_1 x + \varepsilon_3, \quad \tilde{u} = u, \quad \tilde{k} = \varepsilon_1^{-2} k,$$

where ε_i, $i = 1, 2, 3$ are arbitrary constants, $\varepsilon_1 \neq 0$.

Theorem 1. *There exists three $G^\sim$-inequivalent cases of equations from class* (1) *admitting nontrivial Lie invariance algebras (the values of k are given together with the corresponding maximal Lie invariance algebras, $c = \mathrm{const}$) (see [12]):*

$$\begin{aligned} &1: \quad \forall k, \quad \langle \partial_t \rangle; \\ &2: \quad k = c, \quad \langle \partial_t,\, \partial_x \rangle; \\ &3: \quad k = cx^{-2}, \quad \langle \partial_t,\, 2t\partial_t + x\partial_x \rangle. \end{aligned}$$

In the following section we search for nonclassical symmetries which are not equivalent to the above Lie symmetries.

3 Nonclassical symmetries

Firstly, we recall the definition of nonclassical symmetry (or conditional symmetry, or reduction operator). Reduction operators (nonclassical symmetries, Q-conditional symmetries) of a differential equation $\mathcal{L}$ of form $L(t, x, u_{(r)}) = 0$ have the general form

$$Q = \tau\partial_t + \xi\partial_x + \eta\partial_u,$$

where τ, ξ and η are functions of t, x and u, and $(\tau,\xi)\neq(0,0)$. Here $u_{(r)}$ denotes the set of all the derivatives of the function u with respect to t and x of order not greater than r, including u as the derivative of order zero.

The first-order differential function $Q[u] := \eta(t,x,u)-\tau(t,x,u)u_t-\xi(t,x,u)u_x$ is called the *characteristic* of the operator Q. The characteristic PDE $Q[u]=0$ is called also the *invariant surface condition.* Denote the manifold defined by the set of all the differential consequences of the characteristic equation $Q[u]=0$ in the jet space $J^{(r)}$ by $\mathcal{Q}^{(r)}$.

Definition 1. The differential equation $\mathcal{L}$ of form $L(t,x,u_{(r)})=0$ is called *conditionally (nonclassicaly) invariant* with respect to the operator Q if the relation $Q_{(r)}L(t,x,u_{(r)})\big|_{\mathcal{L}\cap\mathcal{Q}^{(r)}}=0$ holds, which is called the *conditional invariance criterion.* Then Q is called an operator of *conditional symmetry* (or Q-conditional symmetry, nonclassical symmetry, reduction operator etc) of the equation $\mathcal{L}$.

In Definition 1 the symbol $Q_{(r)}$ stands for the standard r-th prolongation of the operator Q [14,15].

The classical (Lie) symmetries are, in fact, partial cases of nonclassical symmetries. Therefore, below we solve the problem on finding only pure nonclassical symmetries which are not equivalent to classical ones. Moreover, our approach is based on application of the notion of equivalence of nonclassical symmetries with respect to a transformation group (see, e.g., [17]). For more details, necessary definitions and properties of nonclassical symmetries we refer the reader to [16,17,19,21].

Since (1) is an evolution equation, there exist two principally different cases of finding Q: 1. $\tau\neq0$ and 2. $\tau=0$.

In the present paper, we consider the case with $\tau\neq0$. Here without loss of generality we can assume that $\tau=1$. The results are summarized in the following theorem.

Theorem 2. *All possible cases of equations* (1) *admitting nonclassical symmetries with* $\tau=1$ *are exhausted by the following ones:*

1. $k=c\tan^2 x$: $\quad Q=\partial_t-\cot x\partial_x$,
2. $k=c\tanh^2 x$: $\quad Q=\partial_t-\coth x\partial_x$,
3. $k=c\coth^2 x$: $\quad Q=\partial_t-\tanh x\partial_x$,
4. $k=cx^2$: $\quad Q=\partial_t-\frac{1}{x}\partial_x$,
5. $k=\frac{c^2}{2}$ $(c>0)$: $\quad Q=\partial_t\pm\frac{c}{2}(3u-1)\partial_x-\frac{3c^2}{4}u^2(u-1)\partial_u$
6. $k=2x^{-2}$: $\quad Q=\partial_t+\frac{3}{x}(u-1)\partial_x-\frac{3}{x^2}u(u-1)^2\partial_u$,

where c is an arbitrary constant.

Proof. We search for operators of nonclassical (Q-conditional) symmetry (reduction operators) in form $Q = \partial_t + \xi(t,x,u)\partial_x + \eta(t,x,u)\partial_u$. Then the system of determining equations for the coefficients of operator Q has the form

$$\begin{aligned}
&\xi_{uu} = 0,\\
&2\xi\xi_u - 2\xi_{xu} + \eta_{uu} = 0,\\
&2\xi\xi_x - 2\eta\xi_u - 3k\xi_u u^3 + 3k\xi_u u^2 + 2\eta_{xu} - \xi_{xx} + \xi_t = 0,\\
&-k\eta_u u^2(1-u) + 2k\xi_x u^2(1-u) + \eta_{xx} - 2\xi_x\eta\\
&= \eta_t - k_x\xi u^2(1-u) - 2k\eta u + 3k\eta u^2.
\end{aligned}$$

From the first equation we obtain immediately that

$$\xi = \phi(t,x)u + \psi(t,x).$$

Substituting it to the second equation we derive

$$\eta = -\frac{1}{3}\phi^2 u^3 - \phi\psi u^2 + \phi_x u^2 + A(t,x)u + B(t,x).$$

Then, splitting the rest of determining equations with respect to different powers of u implies the following system of equations for coefficients ϕ, ψ, A and B.

$$\begin{aligned}
&\frac{2}{3}\phi^3 - 3k\phi = 0, \quad -4\phi\phi_x + 2\phi^2\psi + 3k\phi = 0,\\
&-2\phi_x\psi + \phi_t - 2\phi\psi_x - 2\phi A + 3\phi_{xx} = 0,\\
&2\psi\psi_x - 2\phi B + 2A_x - \psi_{xx} + \psi_t = 0,\\
&-\frac{1}{3}\phi^2\phi_x + \frac{1}{3}k\phi^2 + k\phi\psi = 3k\phi_x + k_x\phi,\\
&\frac{2}{3}\phi^2\psi_x - \frac{2}{3}\phi\phi_{xx} - \frac{8}{3}\phi_x^2 - 2k\psi_x - 2kA + 2\phi\phi_x\psi\\
&= -\frac{2}{3}\phi\phi_t - 2k\phi_x - k_x\phi + k_x\psi,\\
&-2\phi_x A + \phi_{xxx} + 2\phi\psi\psi_x + kA - \phi_{xx}\psi - 4\phi_x\psi_x - \phi\psi_{xx} + 2k\psi_x\\
&= \phi_{tx} - \phi_t\psi - \phi\psi_t - k_x\psi + 3kB,\\
&A_{xx} - 2\psi_x A = A_t + 2\phi_x B - 2kB,\\
&-2\psi_x B + B_{xx} = B_t.
\end{aligned}\tag{2}$$

Now from the first equation of (2) it is obvious that either (i) $\phi = 0$ or (ii) $\phi_t = 0$, $k = \frac{2}{9}\phi^2$. Consider separately these two possibilities.

Case (i). $\phi(t,x) = 0$. System (2) is read now like

$$\begin{aligned}
&2A_x - \psi_{xx} + 2\psi\psi_x + \psi_t = 0,\\
&-2k\psi_x - 2kA - \psi k_x = 0,\\
&2k\psi_x + kA + k_x\psi - 3kB = 0,\\
&A_{xx} - 2A\psi_x + 2kB - A_t = 0,\\
&B_{xx} - 2\psi_x B - B_t = 0.
\end{aligned}$$

From the second and third equations we deduce that $A = -3B$. After substituting this to the previous system we obtain

$$
\begin{aligned}
&2\psi\psi_x - 6B_x - \psi_{xx} + \psi_t = 0,\\
&k_x\psi + 2k\psi_x - 6kB = 0,\\
&6\psi_x B - 3B_{xx} + 3B_t + 2kB = 0,\\
&B_{xx} - 2B\psi_x - B_t = 0.
\end{aligned}
$$

It follows from the last two equations that $B = 0$. Then the rest of the determining equations is read like

$$2\psi\psi_x - \psi_{xx} + \psi_t = 0, \quad k_x\psi + 2k\psi_x = 0.$$

General solution of this system is

$$
\begin{aligned}
&k = c, \quad \psi = \text{const},\\
&k = \frac{c}{(ax+b)^2}, \quad \psi = \frac{ax+b}{2ta+m}, \quad \text{and}\\
&k = \frac{c}{\psi^2}, \quad \psi' = \psi^2 + a.
\end{aligned}
$$

Nonclassical symmetry operator obtained from the first two branches of the solution of the above system are equivalent to the usual Lie symmetry. The third branch (up to equivalence transformations of scaling and translations of x) gives cases 1–4 of the theorem.

Case (ii). $\phi_t = 0$, $k = \frac{2}{9}\phi^2$. Substituting this to system (2) we obtain easily that $\psi_t = A_t = B_t = 0$. Then, the rest of the system (2) has the form

$$
\begin{aligned}
&-4\phi_x + 2\phi\psi + \frac{2}{3}\phi^2 = 0,\\
&-2\phi_x\psi - 2\psi_x\phi - 2\phi A + 3\phi_{xx} = 0,\\
&2\psi\psi_x - 2\phi B + 2A_x - \psi_{xx} = 0,\\
&\frac{2}{9}\phi^2\psi_x - \frac{2}{3}\phi\phi_{xx} - \frac{8}{3}\phi_x^2 - \frac{4}{9}\phi^2 A + \frac{14}{9}\phi\psi\phi_x = -\frac{8}{9}\phi^2\phi_x,\\
&\phi_{xxx} + \frac{4}{9}\phi^2\psi_x - \frac{2}{3}\phi^2 B + \frac{2}{9}\phi^2 A - \phi\psi_{xx} - \phi_{xx}\psi - 4\phi_x\psi_x - 2\phi_x A + 2\phi\psi\psi_x\\
&\qquad = -\frac{4}{9}\phi\phi_x\psi - \phi\psi_t,\\
&A_{xx} - 2\psi_x A = -\frac{4}{9}\phi^2 B + 2\phi_x B + A_t,\\
&B_{xx} = 2\psi_x B + B_t.
\end{aligned}
$$

It follows then that

$$B = \frac{9(2\psi_x A - A_{xx})}{2(2\phi^2 - 9\phi_x)}, \quad \psi = \frac{6\phi_x - \phi^2}{3\phi}, \quad A = \frac{4\phi\phi_x - 3\phi_{xx}}{6\phi}.$$

Substituting this values to the above system we obtain a system of 4 differential equations for one function ϕ only that has first order differential consequence of form $3\phi_x^2 + \phi^2\phi_x = 0$. It is not difficult to show that general solution of this constraint $\phi = \frac{3}{x+c}$, $\phi = c$ satisfies the whole system for ϕ. These two values of ϕ (taken up to equivalence transformations) give respectively cases 6 and 5 of the theorem. ■

Note 1. Cases 1, 2 and 4 with $c > 0$ were known in [3, 4], constant coefficient case 5 with $c = 2$ can be found in, e.g., [11], while cases 1, 2 and 4 with $c < 0$, 3 and 6 are new.

4 Final remarks

Here we have partially completed an open problem of classification of nonclassical symmetries for the class (1). In particular, we have presented all forms of (1) that admit nonclassical symmetry operators of the form $Q = \partial_t + \xi(x,t,u)\partial_x + \eta(t,x,u)\partial_u$. The problem can be completed by determining all nonclassical symmetry operator of the form

$$Q = \partial_x + \eta(t,x,u)\partial_u.$$

Any nonclassical symmetry operator of the above form satisfies the following equation

$$\eta_t - \eta_{xx} - 2\eta\eta_{xu} - \eta^2\eta_{uu} + (k\eta_u - k_x)u^2(1-u) - 2k\eta u + 3k\eta u^2 = 0. \qquad (3)$$

The corresponding invariant surface condition is $u_x = \eta$. Eliminating u_x and u_{xx}, equation (1) reads

$$u_t = \eta\eta_u + \eta_x + k(x)u^2(1-u). \qquad (4)$$

Using a solution of (3), we can find u by integrating invariance surface condition and then substituting in (4) to derive a solution of (1).

Now, as was shown in [16, 20] for the more general case of $(1+n)$-dimensional evolution equations, integration of equation (3) is, in some sense, equivalent to integration of the initial equation (1). Therefore it is *impossible* to integrate the equation completely (such a situation for the evolution equation is often called the "no-go case"). Since it contains larger number of unknown variables, it is possible to construct certain partial solutions. Thus, for example, we have succeeded to find all ($G^{\sim}$-inequivalent) partial solutions of equation (3) of the form

$$\eta(x,t,u) = \sum_{p=-m}^{n} \phi_p(x,t)u^p,$$

where m and n are positive integers and $\phi_p(x,t)$ are unknown functions. However, all the explicit forms of $k(x)$ found, which are expressed in terms of elementary functions, that admit new nonclassical symmetries lead to similarity solutions obtainable either by Lie symmetries or the nonclassical symmetries with $\tau = 1$. Further forms of $k(x)$ that admit nonclassical symmetries exist, for example, in terms of Bessel functions or though certain nonlinear ordinary differential equations which need to be solved.

More detailed investigation of conditional symmetries, especially for the no-go case, and construction of associate similarity solutions of equations from class (1) will be the subject of a forthcoming paper.

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[1] Arrigo D.J., Hill J.M. and Broadbridge P., Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source, *IMA J. Appl. Math.*, 1994, V.52, 1–24.

[2] Bluman G.W. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, 1025–1042.

[3] Bradshaw-Hajek B., Reaction-diffusion equations for population genetics, PhD thesis, School of Mathematics and Applied Statistics, University of Wollongong, 2004, http://ro.uow.edu.au/theses/201 .

[4] Bradshaw-Hajek B.H., Edwards M.P., Broadbridge P. and Williams G.H., Nonclassical symmetry solutions for reaction-diffusion equations with explicit spatial dependence, *Nonlin. Analysis*, 2007, V.67, 2541–2552.

[5] Clarkson P.A. and Mansfield E.L., Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Phys. D*, 1994, V.70, 250–288.

[6] Estevez P.G., Non-classical symmetries and the singular manifold method: the Burgers and the Burgers–Huxley equations, *J. Phys. A: Math. Gen.*, 1994, V.27, 2113–2127.

[7] Estevez P.G., Gordoa P.R., Nonclassical symmetries and the singular manifold method: theory and six examples, *Stud. Appl. Math.*, 1995, V.95, 73–113.

[8] Fushchich W.I. and Serov N.I., Conditional invariance and exact solutions of a nonlinear acoustics equation, *Dokl. Akad. Nauk Ukrain. SSR, Ser. A*, 1988, no. 10, 27–31 (in Russian)

[9] Fushchich W.I., Shtelen W.M., Serov M.I. and Popovych R.O., Q-conditional symmetry of the linear heat equation, *Proc. Acad. Sci. Ukraine*, 1992, no. 12, 28–33.

[10] Fushchich W.I. and Tsyfra I.M., On a reduction and solutions of the nonlinear wave equations with broken symmetry, *J. Phys. A: Math. Gen.*, 1987, V.20, L45–L48.

[11] Hydon, P.E., *Symmetry Methods for Differential Equations*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2000.

[12] Ivanova N.M., On Lie symmetries of a class of reaction-diffusion equations, *Proceedings of 4th Int. Workshop "Group Analysis of Differential Equations and Integrable Systems" (Protaras, Cyprus, 2008)*, 2009, 84–86.

[13] Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A: Math. Gen.*, 1989, V.22, 2915–2924.

[14] Olver P., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[15] Ovsiannikov L.V., Group analysis of differential equations, New York, Academic Press, 1982.

[16] Popovych R.O., On a class of Q-conditional symmetries and solutions of evolution equations, in Symmetry and Analytic Methods in Mathematical Physics, *Proceedings of Institute of Mathematics*, Kyiv, 1998, V.19, 194–199 (in Ukrainian).

[17] Popovych R.O., Vaneeva O.O. and Ivanova N.M., Potential nonclassical symmetries and solutions of fast diffusion equation, *Phys. Lett. A*, 2007, V.362, 166–173, arXiv: math-ph/0506067.

[18] Scott A.C., The electrophysics of a nerve fiber, *Rev. Modern Phys.*, 1975, 487–533.

[19] Vaneeva O.O., Popovych R.O. and Sophocleous C., Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, *Acta Appl. Math.*, 2009, V.106, 1–46, arXiv:0708.3457.

[20] Zhdanov R.Z. and Lahno V.I., Conditional symmetry of a porous medium equation, *Phys. D*, 1998, V.122, 178–186.

[21] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, 101–123, arXiv:math-ph/0207023.

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Involutive distributions of operator-valued evolutionary vector fields and their affine geometry

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We generalize the notion of a Lie algebroid over an infinite jet bundle by replacing the variational anchor with an N-tuple of differential operators the images of which in the Lie algebra of evolutionary vector fields of the jet space are subject to collective commutation closure. The linear space of such operators becomes an algebra with bidifferential structural constants, of which we study the canonical structure. In particular we show that these constants incorporate bidifferential analogues of Christoffel symbols.

1 Introduction

Lie algebroids [21] are an important and convenient construction that appear, e.g., in classical Poisson dynamics [2] or the theory of quantum Poisson manifolds [1, 22]. Essentially Lie algebroids extend the tangent bundle TM over a smooth manifold M, retaining the information about the $C^\infty(M)$-module structure for its sections. In the paper [10] we defined Lie algebroids over the infinite jet spaces for mappings between smooth manifolds (e.g., from strings to space-time); the classical definition [21] is recovered by shrinking the source manifold to a point. A special case of Lie algebroids over spaces of finite jets for sections of the tangent bundle was firstly considered in [15]. Within the variational setup the anchors become linear matrix differential operators that map sections which belong to horizontal modules [13] to the generating sections φ of evolutionary derivations ∂_φ on the jet space; by assumption the images of such anchors are closed under commutation in the Lie algebra of evolutionary vector fields. The two main examples of variational anchors are the recursions with involutive images [8] and the Hamiltonian operators (see [12, 13, 19] and [8]) the domains of which consist of variational vectors and covectors, respectively.

In [8] we studied the *linear compatibility* of variational anchors, meaning that N operators with a common domain span an N-dimensional linear space $\mathcal{A}$ such that each point $A_{\boldsymbol{\lambda}} \in \mathcal{A}$ is itself an anchor with involutive image. For example Poisson-compatible Hamiltonian operators are linearly compatible and *vice versa* (Hamiltonian operators are *Poisson-compatible* if their linear combinations remain

Hamiltonian). The linear compatibility[1] allows us to reduce the case of many operators $A_1, \ldots, A_N$ to one operator $A_{\boldsymbol{\lambda}} = \sum \lambda_i \cdot A_i$ with the same properties.

In this paper we introduce a different notion of compatibility for the N operators. Strictly speaking we consider the class of structures which is wider than the set of Lie algebroids over jet spaces, namely, we relax the assumption that each operator alone is a variational anchor, but instead we deal with N-tuples of total differential operators $A_1, \ldots, A_N$ the images of which are subject to the collective commutation closure: $\left[\sum_{i=1}^N \operatorname{im} A_i, \sum_{j=1}^N \operatorname{im} A_j\right] \subseteq \sum_{k=1}^N \operatorname{im} A_k$. This involutivity condition converts the linear space of operators to an algebra with bidifferential structural constants $\mathbf{c}_{ij}^k$, see (6) below. The Magri scheme [16] for the restriction of compatible Hamiltonian operators to the hierarchy of Hamiltonians yields an example of such an overlapping for $N = 2$ with $\mathbf{c}_{ij}^k \equiv 0$.

We study the standard decomposition of the structural constants $\mathbf{c}_{ij}^k$, which is similar to the previously known case (1) for $N = 1$ ([7, 8, 10]). From the bidifferential constants $\mathbf{c}_{ij}^k$ we extract the components Γ_{ij}^k that act by total differential operators on both arguments at once. Our main result, Theorem 3, states that under a change of coordinates in the domain the symbols Γ_{ij}^k are transformed by a proper analogue (11) of the classical rule $\Gamma \mapsto g\,\Gamma g^{-1} + \mathrm{d}g\, g^{-1}$ for the connection 1-forms Γ and reparametrizations g. We note that the bidifferential symbols Γ_{ij}^k are symmetric in their lower indices if the common domain of the N operators A_i consists of the variational covectors and hence its elements acquire their own odd grading.[2]

This note is organized as follows. In Section 2 we introduce operators with collective closure under commutation. For consistency we recall here the cohomological formulation [11] of the Magri scheme which gives us an example. In Section 3 we study the properties of the bidifferential constants that appear in such algebras of operators. The analogues of Christoffel symbols emerge here; as an example we calculate them for the symmetry algebra of the Liouville equation.

2 Compatible differential operators

We begin with some notation; for a more detailed exposition of the geometry of integrable systems we refer to [19] and [4, 12, 14, 17]. In the sequel the ground field is the field $\mathbb{R}$ of real numbers and all mappings are C^∞-smooth.

Let $\pi\colon E^{m+n} \xrightarrow[M^m]{} B^n$ be a vector bundle over an orientable n-dimensional manifold B^n and, similarly, let $\xi\colon N^{d+n} \longrightarrow B^n$ be another vector[3] bundle

[1] When the set of admissible linear combinations $\{\boldsymbol{\lambda}\} \subsetneq \mathbb{R}^N$ has punctures near which the homomorphisms $A_{\boldsymbol{\lambda}}$ exhibit a nontrivial analytic behaviour, this concept reappears in the theory of continuous contractions of Lie algebras (see [18] and references therein).

[2] Throughout this paper we deal with a purely commutative setup, refraining from the treatment of supermanifolds. However, we emphasize that on a supermanifold the two notions of *parity* and *grading* (or *weight*) may be totally uncorrelated, see [22].

[3] For this paper the established term '*vector* bundle' is particularly unfortunate because in our

over B^n. Consider the bundle $\pi_\infty\colon J^\infty(\pi) \to B^n$ of infinite jets of sections for the bundle π and take the pull-back $\pi_\infty^*(\xi)\colon N^{d+n} \times_{B^n} J^\infty(\pi) \to J^\infty(\pi)$ of the bundle ξ along π_∞. By definition the $C^\infty(J^\infty(\pi))$-module of sections $\Gamma\bigl(\pi_\infty^*(\xi)\bigr) = \Gamma(\xi) \otimes_{C^\infty(B^n)} C^\infty(J^\infty(\pi))$ is called *horizontal*, see [13] for further details.

For example let $\xi := \pi$. Then the variational vectors $\varphi \in \Gamma\bigl(\pi_\infty^*(\pi)\bigr)$ are the generating sections of evolutionary derivations ∂_φ on $J^\infty(\pi)$. For convenience we use the shorthand notation $\varkappa(\pi) \equiv \Gamma\bigl(\pi_\infty^*(\pi)\bigr)$ and $\Gamma\Omega\bigl(\xi_\pi\bigr) \equiv \Gamma\bigl(\pi_\infty^*(\xi)\bigr)$ in the general setup.

We consider firstly the case $N = 1$ for which there is only one total differential operator, $A\colon \Gamma\Omega(\xi_\pi) \to \varkappa(\pi)$, with involutive image

$$[\operatorname{im} A, \operatorname{im} A] \subseteq \operatorname{im} A. \tag{1}$$

The operator A transfers the bracket in the Lie algebra $\mathfrak{g}(\pi) = \bigl(\varkappa(\pi), [\,,\,]\bigr)$ to the Lie algebraic structure $[\,,\,]_A$ on the quotient of its domain by the kernel. The standard decomposition of this bracket is [8, 10]

$$[\boldsymbol{p}, \boldsymbol{q}]_A = \partial_{A(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{A(\boldsymbol{q})}(\boldsymbol{p}) + \{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A, \qquad \boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi). \tag{2}$$

The *linear compatibility* of operators (4), which means that their arbitrary linear combinations $A_{\boldsymbol{\lambda}} = \sum_i \lambda_i \cdot A_i$ satisfy (1), reduces the case of $N \geq 2$ operators to the previous case with $N = 1$ as follows.

Theorem 1 ([8]). *The bracket $\{\{\,,\,\}\}_{A_{\boldsymbol{\lambda}}}$ induced by the combination $A_{\boldsymbol{\lambda}} = \sum_i \lambda_i \cdot A_i$ on the domain of the linearly compatible normal*[4] *operators A_i is*

$$\{\{\,,\,\}\}_{\sum\limits_{i=1}^{N} \lambda_i A_i} = \sum_{i=1}^{N} \lambda_i \cdot \{\{\,,\,\}\}_{A_i}.$$

The pairwise linear compatibility implies the collective linear compatibility of $A_1, \ldots, A_N$.

Proof. Consider the commutator $\bigl[\sum_i \lambda_i A_i(\boldsymbol{p}), \sum_j \lambda_j A_j(\boldsymbol{q})\bigr]$, here $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$. On one hand it is equal to

$$\begin{aligned} &= \sum_{i \neq j} \lambda_i \lambda_j \bigl[A_i(\boldsymbol{p}), A_j(\boldsymbol{q})\bigr] \\ &+ \sum_i \lambda_i^2 A_i\bigl(\partial_{A_i(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{A_i(\boldsymbol{q})}(\boldsymbol{p}) + \{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{A_i}\bigr). \end{aligned} \tag{3}$$

main Example 1 the sections of such a bundle are variational *co*vectors and obey a nonvectorial transformation law.

[4]By definition, a total differential operator A is *normal* if $A \circ \nabla = 0$ implies $\nabla = 0$; in other words it may be that $\ker A \neq 0$, but the kernel does not have any functional freedom for its elements, see [7].

On the other hand the linear compatibility of A_i implies

$$= A_{\boldsymbol{\lambda}}\bigl(\partial_{A_{\boldsymbol{\lambda}}(\boldsymbol{p})}(\boldsymbol{q})\bigr) - A_{\boldsymbol{\lambda}}\bigl(\partial_{A_{\boldsymbol{\lambda}}(\boldsymbol{q})}(\boldsymbol{p})\bigr) + A_{\boldsymbol{\lambda}}\bigl(\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_{\boldsymbol{\lambda}}}\bigr).$$

The entire commutator is quadratically homogeneous in $\boldsymbol{\lambda}$, whence the bracket $\{\{\,,\,\}\}_{A_{\boldsymbol{\lambda}}}$ is linear in $\boldsymbol{\lambda}$. From (3) we see that the individual brackets $\{\{\,,\,\}\}_{A_i}$ are contained in it. Therefore

$$\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_{\boldsymbol{\lambda}}} = \sum_\ell \lambda_\ell \cdot \{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_\ell} + \sum_\ell \lambda_\ell \cdot \gamma_\ell(\boldsymbol{p},\boldsymbol{q}),$$

where $\gamma_\ell \colon \Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)$.

We claim that all summands $\gamma_\ell(\cdot,\cdot)$, which do not depend upon $\boldsymbol{\lambda}$ at all, vanish. Indeed, assume the converse. Let there be $\ell \in [1,\ldots,N]$ such that $\gamma_\ell(\boldsymbol{p},\boldsymbol{q}) \neq 0$; without loss of generality suppose that $\ell = 1$. Then set $\boldsymbol{\lambda} = (1,0,\ldots,0)$, whence

$$\Bigl[\sum_i \lambda_i A_i(\boldsymbol{p}), \sum_j \lambda_j A_j(\boldsymbol{q})\Bigr] = \Bigl[(\lambda_1 A_1)(\boldsymbol{p}), (\lambda_1 A_1)(\boldsymbol{q})\Bigr] = (\lambda_1 A_1)\bigl(\lambda_1\gamma_1(\boldsymbol{p},\boldsymbol{q})\bigr)$$
$$+ (\lambda_1 A_1)\Bigl(\partial_{(\lambda_1 A_1)(\boldsymbol{p})}(\boldsymbol{q}) - \partial_{(\lambda_1 A_1)(\boldsymbol{q})}(\boldsymbol{p}) + \lambda_1\{\{\boldsymbol{p},\boldsymbol{q}\}\}_{A_1}\Bigr) = \lambda_1 A_1\bigl(\lambda_1[\boldsymbol{p},\boldsymbol{q}]_{A_1}\bigr).$$

Consequently, $\gamma_\ell(\boldsymbol{p},\boldsymbol{q}) \in \ker A_\ell$ for all $\boldsymbol{p}$ and $\boldsymbol{q}$. Now we use the assumption that each operator A_ℓ is normal. This implies that $\gamma_\ell = 0$ for all ℓ which concludes the proof. ■

Now we let $N > 1$ and consider N-tuples of linear total differential operators

$$A_1,\ldots,A_N \colon \Gamma\Omega\bigl(\xi_\pi\bigr) \longrightarrow \varkappa(\pi), \tag{4}$$

the images of which in the Lie algebra $\mathfrak{g}(\pi)$ of evolutionary vector fields on $J^\infty(\pi)$ are subject to collective closure of commutators.

Definition 1. We say that $N \geq 2$ total differential operators (4) are *strongly compatible* if the sum of their images is closed under commutation in the Lie algebra $\mathfrak{g}(\pi) = \bigl(\varkappa(\pi), [\,,\,]\bigr)$ of evolutionary vector fields,

$$\Bigl[\sum_i \operatorname{im} A_i, \sum_j \operatorname{im} A_j\Bigr] \subseteq \sum_k \operatorname{im} A_k, \qquad 1 \leq i,j,k \leq N. \tag{5}$$

The involutivity (5) gives rise to the bidifferential operators

$$\mathbf{c}_{ij}^k \colon \Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)$$

through

$$\bigl[A_i(\boldsymbol{p}), A_j(\boldsymbol{q})\bigr] = \sum_k A_k\bigl(\mathbf{c}_{ij}^k(\boldsymbol{p},\boldsymbol{q})\bigr), \qquad \boldsymbol{p},\boldsymbol{q} \in \Gamma\Omega(\xi_\pi). \tag{6}$$

The structural constants $\mathbf{c}_{ij}^k$ absorb the bidifferential action on $\boldsymbol{p}$, $\boldsymbol{q}$ under commutation in the images of the operators.

Remark 1. If $N = 1$ and there is a unique operator $A\colon \Gamma\Omega(\xi_\pi) \to \varkappa(\pi)$ satisfying (1), then we recover the definition of the variational anchor in the Lie algebroid over the infinite jet space $J^\infty(\pi)$, see [10]. By construction, $\mathbf{c}_{11}^1 \equiv [\,,\,]_{A_1}$ if $N = 1$. However, for $N > 1$ we obtain a wider class of structures because we do not assume that the image of each operator A_i alone is involutive. Therefore it may well occur that $\mathbf{c}_{ii}^k \neq 0$ for some $k \neq i$.

The Magri scheme [16] for the restriction of two compatible Hamiltonian operators A_1, A_2 onto the commutative hierarchy of the descendants $\mathcal{H}_i$ of the Casimirs $\mathcal{H}_0$ for A_1 gives us an example of (5) with $N = 2$ and $\mathbf{c}_{ij}^k \equiv 0$. We consider it in more detail; from now we standardly identify the Hamiltonian operators A with the variational Poisson bivectors $\boldsymbol{A}$, see [13]. We recall that the variational Schouten bracket $[\![\,,\,]\!]$ of such bivectors satisfies the Jacobi identity

$$[\![[\![\boldsymbol{A}_1,\boldsymbol{A}_2]\!],\boldsymbol{A}_3]\!] + [\![[\![\boldsymbol{A}_2,\boldsymbol{A}_3]\!],\boldsymbol{A}_1]\!] + [\![[\![\boldsymbol{A}_3,\boldsymbol{A}_1]\!],\boldsymbol{A}_2]\!] = 0. \tag{7}$$

Hence the defining property $[\![\boldsymbol{A},\boldsymbol{A}]\!] = 0$ for a Poisson bivector $\boldsymbol{A}$ implies that $\mathrm{d}_A = [\![\boldsymbol{A},\cdot\,]\!]$ is a differential, giving rise to the Poisson cohomology H_A^k. Obviously the Casimirs $\mathcal{H}_0$ such that $[\![\boldsymbol{A},\mathcal{H}_0]\!] = 0$ for a Poisson bivector $\boldsymbol{A}$ constitute the group H_A^0.

Theorem 2 ([11,16]). *Suppose* $[\![\boldsymbol{A}_1,\boldsymbol{A}_2]\!] = 0$, $\mathcal{H}_0 \in H_{A_1}^0$ *is a Casimir of* $\boldsymbol{A}_1$ *and the first Poisson cohomology w.r.t.* $\mathrm{d}_{A_1} = [\![\boldsymbol{A}_1,\cdot\,]\!]$ *vanishes. Then for any* $k > 0$ *there is a Hamiltonian* $\mathcal{H}_k$ *such that*

$$[\![\boldsymbol{A}_2,\mathcal{H}_{k-1}]\!] = [\![\boldsymbol{A}_1,\mathcal{H}_k]\!]. \tag{8}$$

Put $\varphi_k := A_1\bigl(\delta/\delta u(\mathcal{H}_k)\bigr)$ *such that* $\partial_{\varphi_k} = [\![\boldsymbol{A}_1,\mathcal{H}_k]\!]$. *The Hamiltonians* $\mathcal{H}_i$, $i \geq 0$, *pairwise Poisson commute w.r.t. either* A_1 *or* A_2, *the densities of* $\mathcal{H}_i$ *are conserved on any equation* $u_{t_k} = \varphi_k$ *and the evolutionary derivations* ∂_{φ_k} *pairwise commute for all* $k \geq 0$.

Standard proof of existence. The main homological equality (8) is established by induction on k. Starting with a Casimir $\mathcal{H}_0$ we obtain

$$0 = [\![\boldsymbol{A}_2, 0]\!] = [\![\boldsymbol{A}_2,[\![\boldsymbol{A}_1,\mathcal{H}_0]\!]]\!] = -[\![\boldsymbol{A}_1,[\![\boldsymbol{A}_2,\mathcal{H}_0]\!]]\!] \mod [\![\boldsymbol{A}_1,\boldsymbol{A}_2]\!] = 0,$$

using the Jacobi identity (7). The first Poisson cohomology $H_{A_1}^1 = 0$ is trivial by an assumption of the theorem. Hence the closed element $[\![\boldsymbol{A}_2,\mathcal{H}_0]\!]$ in the kernel of $[\![\boldsymbol{A}_1,\cdot]\!]$ is exact: $[\![\boldsymbol{A}_2,\mathcal{H}_0]\!] = [\![\boldsymbol{A}_1,\mathcal{H}_1]\!]$ for some $\mathcal{H}_1$. For $k \geq 1$ we have

$$[\![\boldsymbol{A}_1,[\![\boldsymbol{A}_2,\mathcal{H}_k]\!]]\!] = -[\![\boldsymbol{A}_2,[\![\boldsymbol{A}_1,\mathcal{H}_k]\!]]\!] = -[\![\boldsymbol{A}_2,[\![\boldsymbol{A}_2,\mathcal{H}_{k-1}]\!]]\!] = 0$$

using (7) and by $[\![\boldsymbol{A}_2,\boldsymbol{A}_2]\!] = 0$. Consequently by $H_{A_1}^1 = 0$ we have that $[\![\boldsymbol{A}_2,\mathcal{H}_k]\!] = [\![\boldsymbol{A}_1,\mathcal{H}_{k+1}]\!]$, and we thus proceed infinitely. ■

We see now that the inductive step — the existence of the $(k+1)$th Hamiltonian functional in involution — is possible if and only if H_0 is a Casimir,[5] and therefore the operators A_1 and A_2 are restricted onto the linear subspace which is spanned in the space of variational covectors by the Euler derivatives of the descendants of $\mathcal{H}_0$, i.e. of the Hamiltonians of the hierarchy. We note that the image under A_2 of a generic section from the domain of operators A_1 and A_2 cannot be resolved w.r.t. A_1 by (8). For example the first and second Hamiltonian structures for the KdV equation, which equal, respectively, $A_1 = \mathrm{d}/\mathrm{d}x$ and $A_2 = -\frac{1}{2}\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 2u\frac{\mathrm{d}}{\mathrm{d}x} + u_x$, are not strongly compatible unless they are restricted onto some subspaces of their arguments. On the linear subspace of descendants of the Casimir $\int u\,\mathrm{d}x$ we have $\operatorname{im} A_2 \subset \operatorname{im} A_1$ and, since the image of the Hamiltonian operator A_1 is involutive, we conclude that $[\operatorname{im} A_1, \operatorname{im} A_2] \subset \operatorname{im} A_1$.

On the other hand the strong compatibility of the restrictions of Poisson-compatible operators A_1 and A_2 onto the hierarchy is valid since their images are commutative Lie algebras. Regarding the converse statement as a potential generator of multidimensional completely integrable systems we formulate the open problem: Is the strong compatibility of Poisson-compatible Hamiltonian operators achieved *only* for their restrictions onto the hierarchies of Hamiltonians in involution so that the bidifferential constants $\mathbf{c}_{ij}^k$ necessarily vanish? If so, this would have a remarkable similarity with the technique of the Bethe ansatz, one component of which is the extension of a commutative algebra of Hamiltonian operators on a Hilbert space to a bigger noncommutative algebra.

3 Bidifferential Christoffel symbols

Similarly to (2), we extract the total bidifferential parts of the structural constants $\mathbf{c}_{ij}^k$ in (6) and obtain

$$\mathbf{c}_{ij}^k = \partial_{A_i(\boldsymbol{p})}(\boldsymbol{q}) \cdot \delta_j^k - \partial_{A_j(\boldsymbol{q})}(\boldsymbol{p}) \cdot \delta_i^k + \Gamma_{ij}^k(\boldsymbol{p}, \boldsymbol{q}), \quad \boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi), \tag{9}$$

where $\Gamma_{ij}^k \in \mathcal{C}\mathrm{Diff}\big(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)\big)$ and δ_i^k, δ_j^k are the Kronecker delta symbols. By definition the three indices in Γ_{ij}^k match the respective operators A_i, A_j, A_k in (6). (The total number of the indices is much greater than three; moreover the proper upper or lower location of the omitted indices depends upon the (co)vector nature of the domain $\Gamma\Omega(\xi_\pi)$.) Obviously the convention

$$\Gamma_{11}^1 = \{\{\,,\,\}\}_{A_1}$$

holds if $N = 1$. At the same time for fixed i, j, k the symbol Γ_{ij}^k remains a (class of) matrix differential operator in each of its two arguments $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$.

[5] The Magri scheme starts from any two Hamiltonians $\mathcal{H}_{k-1}$, $\mathcal{H}_k$ that satisfy (8), but we operate with maximal subspaces of the space of functionals such that the sequence $\{\mathcal{H}_k\}$ cannot be extended with $k < 0$.

The symbol Γ_{ij}^{k} represents a class of bidifferential operators because they are not uniquely defined. Indeed they are gauged by the conditions

$$\sum_{k=1}^{N} A_k\Bigl(\partial_{A_i(\boldsymbol{p})}(\boldsymbol{q})\delta_j^k - \partial_{A_j(\boldsymbol{q})}(\boldsymbol{p})\delta_i^k + \Gamma_{ij}^k(\boldsymbol{p},\boldsymbol{q})\Bigr) = 0, \quad \boldsymbol{p},\boldsymbol{q} \in \Gamma\Omega(\xi_\pi). \tag{10}$$

We let the r.h.s. of (10) be zero if the sum $\sum_\ell \operatorname{im} A_\ell$ of the images is indecomposable, which mean that no nontrivial sections commute with all the others: $\bigl[A_k(\boldsymbol{p}), \sum_{\ell=1}^N \operatorname{im} A_\ell\bigr] = 0$ implies that $\boldsymbol{p} \in \ker A_k$. For this it is sufficient that the sum of the images of A_ℓ in $\mathfrak{g}(\pi)$ be semisimple and the Whitehead lemma holds for it [5]. Otherwise the right-hand side of (10) belongs to the linear subspace of such nontrivial sections.

Example 1 (see [9, 10]). Consider the Liouville equation $\mathcal{E}_{\text{Liou}} = \{u_{xy} = \exp(2u)\}$. The differential generators of its conservation laws are $w = u_x^2 - u_{xx} \in \ker \frac{\mathrm{d}}{\mathrm{d}y}\big|_{\mathcal{E}_{\text{Liou}}}$ and $\bar{w} = u_y^2 - u_{yy} \in \ker \frac{\mathrm{d}}{\mathrm{d}x}\big|_{\mathcal{E}_{\text{Liou}}}$. The operators[6] $\Box = u_x + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}$ and $\overline{\Box} = u_y + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}y}$ determine higher symmetries $\varphi, \bar{\varphi}$ of $\mathcal{E}_{\text{Liou}}$ by the formulas

$$\varphi = \Box\bigl(p(x,[w])\bigr), \quad \bar{\varphi} = \overline{\Box}\bigl(\overline{p}(y,[\bar{w}])\bigr)$$

for any variational covectors p, $\overline{p}$. The images of $\Box$ and $\overline{\Box}$ are closed w.r.t. the commutation; for instance the bracket (2) for $\Box$ contains $\{\{p,q\}\}_\Box = \frac{\mathrm{d}}{\mathrm{d}x}(p)\cdot q - p\cdot\frac{\mathrm{d}}{\mathrm{d}x}(q)$, and similarly for $\overline{\Box}$. The two summands in the symmetry algebra $\operatorname{sym}\mathcal{E}_{\text{Liou}} \simeq \operatorname{im}\Box \oplus \operatorname{im}\overline{\Box}$ commute between each other, $[\operatorname{im}\Box, \operatorname{im}\overline{\Box}] \doteq 0$ on $\mathcal{E}_{\text{Liou}}$. The operators $\Box$, $\overline{\Box}$ generate the bidifferential symbols

$$\Gamma_{\Box\Box}^{\Box} = \{\{\,,\,\}\}_\Box = \tfrac{\mathrm{d}}{\mathrm{d}x}\otimes\mathbf{1} - \mathbf{1}\otimes\tfrac{\mathrm{d}}{\mathrm{d}x}, \quad \Gamma_{\overline{\Box}\overline{\Box}}^{\overline{\Box}} = \{\{\,,\,\}\}_{\overline{\Box}} = \tfrac{\mathrm{d}}{\mathrm{d}y}\otimes\mathbf{1} - \mathbf{1}\otimes\tfrac{\mathrm{d}}{\mathrm{d}y},$$
$$\Gamma_{\Box\overline{\Box}}^{\Box} = \tfrac{\mathrm{d}}{\mathrm{d}y}\otimes\mathbf{1}, \quad \Gamma_{\Box\overline{\Box}}^{\overline{\Box}} = -\mathbf{1}\otimes\tfrac{\mathrm{d}}{\mathrm{d}x}, \quad \Gamma_{\overline{\Box}\Box}^{\Box} = -\mathbf{1}\otimes\tfrac{\mathrm{d}}{\mathrm{d}y}, \quad \Gamma_{\overline{\Box}\Box}^{\overline{\Box}} = \tfrac{\mathrm{d}}{\mathrm{d}x}\otimes\mathbf{1},$$

where the notation is obvious. We note that $\Gamma_{\Box\overline{\Box}}^{\Box}(p,q) \doteq \Gamma_{\Box\overline{\Box}}^{\overline{\Box}}(p,q) \doteq \Gamma_{\overline{\Box}\Box}^{\Box}(q,p) \doteq \Gamma_{\overline{\Box}\Box}^{\overline{\Box}}(q,p) \doteq 0$ on $\mathcal{E}_{\text{Liou}}$ for any $p(x,[w])$ and $q(y,[\bar{w}])$.

The matrix operators $\Box$, $\overline{\Box}$ are well defined [7] for each 2D Toda chain $\mathcal{E}_{\text{Toda}}$ associated with a semisimple complex Lie algebra. They exhibit the same properties as above.

Remark 2. The operators $\Box$, $\overline{\Box}$ yield the involutive distributions of evolutionary vector fields that are tangent to the *integral manifolds*, the 2D Toda differential equations. Generally there is no Frobenius theorem for such distributions. Still, if the integral manifold exists and is an infinite prolongation of a differential equation $\mathcal{E} \subset J^\infty(\pi)$, then by construction this equation admits infinitely many symmetries of the form $\varphi = A_i(\boldsymbol{p})$ with free functional parameters $\boldsymbol{p} \in \Gamma\Omega(\xi_\pi)$. This property is close but not equivalent to the definition of systems of Liouville type (see [7, 9] and references therein).

[6] We denote the operators by $\Box$ and $\overline{\Box}$ following the notation of [7, 9], see also references therein.

The method by which we introduced the symbols Γ^k_{ij} suggests that, under reparametrizations g in the domain of the operators (4), they obey a proper analogue of the standard rule $\Gamma \mapsto g\,\Gamma\,g^{-1} + \mathrm{d}g\cdot g^{-1}$ for the connection 1-forms Γ. This is indeed so.

Theorem 3 (Transformations of Γ^k_{ij}). *Let g be a reparametrization $\boldsymbol{p} \mapsto \tilde{\boldsymbol{p}} = g\boldsymbol{p}$, $\boldsymbol{q} \mapsto \tilde{\boldsymbol{q}} = g\boldsymbol{q}$ of sections $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$ in the domains*[7] *of strongly compatible operators (4). In this notation the operators $A_1, \ldots, A_N$ are transformed by the formula $A_i \mapsto \tilde{A}_i = A_i \circ g^{-1}\big|_{w=w[\tilde{w}]}$. Then the bidifferential symbols $\Gamma^k_{ij} \in \mathcal{C}\mathit{Diff}\big(\Gamma\Omega(\xi_\pi) \times \Gamma\Omega(\xi_\pi) \to \Gamma\Omega(\xi_\pi)\big)$ are transformed according to the rule*

$$\Gamma^k_{ij}(\boldsymbol{p},\boldsymbol{q}) \mapsto \tilde{\Gamma}^{\tilde{k}}_{\tilde{\imath}\tilde{\jmath}}(\tilde{\boldsymbol{p}},\tilde{\boldsymbol{q}}) = \big(g \circ \Gamma^{\tilde{k}}_{\tilde{\imath}\tilde{\jmath}}\big)\big(g^{-1}\tilde{\boldsymbol{p}}, g^{-1}\tilde{\boldsymbol{q}}\big) + \delta^{\tilde{k}}_{\tilde{\imath}} \cdot \partial_{\tilde{A}_{\tilde{\jmath}}(\tilde{\boldsymbol{q}})}(g)\big(g^{-1}\tilde{\boldsymbol{p}}\big) - \delta^{\tilde{k}}_{\tilde{\jmath}} \cdot \partial_{\tilde{A}_{\tilde{\imath}}(\tilde{\boldsymbol{p}})}(g)\big(g^{-1}\tilde{\boldsymbol{q}}\big). \tag{11}$$

Proof. Denote $A = A_i$ and $B = A_j$; without loss of generality we assume $i = 1$ and $j = 2$. We calculate the commutators of vector fields in the images of A and B using two systems of coordinates in the domain. We equate the commutators straighforwardly because the fibre coordinates in the images of the operators are not touched at all. So we have originally

$$\big[A(\boldsymbol{p}), B(\boldsymbol{q})\big] = B\big(\partial_{A(\boldsymbol{p})}(\boldsymbol{q})\big) - A\big(\partial_{B(\boldsymbol{q})}(\boldsymbol{p})\big) + A\big(\Gamma^A_{AB}(\boldsymbol{p},\boldsymbol{q})\big) + B\big(\Gamma^B_{AB}(\boldsymbol{p},\boldsymbol{q})\big) + \sum_{k=3}^{N} A_k\big(\Gamma^k_{AB}(\boldsymbol{p},\boldsymbol{q})\big).$$

On the other hand we substitute $\tilde{\boldsymbol{p}} = g\boldsymbol{p}$ and $\tilde{\boldsymbol{q}} = g\boldsymbol{q}$ into $\big[\tilde{A}(\tilde{\boldsymbol{p}}), \tilde{B}(\tilde{\boldsymbol{q}})\big]$ whence by the Leibnitz rule we obtain

$$\begin{aligned}\big[\tilde{A}(\tilde{\boldsymbol{p}}), \tilde{B}(\tilde{\boldsymbol{q}})\big] &= \tilde{B}\big(\partial_{\tilde{A}(\tilde{\boldsymbol{p}})}(g)(\boldsymbol{q})\big) + \big(\tilde{B}\circ g\big)\big(\partial_{\tilde{A}(\tilde{\boldsymbol{p}})}(\boldsymbol{q})\big) \\ &- \tilde{A}\big(\partial_{\tilde{B}(\tilde{\boldsymbol{q}})}(g)(\boldsymbol{p})\big) - \big(\tilde{A}\circ g\big)\big(\partial_{\tilde{B}(\tilde{\boldsymbol{q}})}(\boldsymbol{p})\big) \\ &+ \big(A\circ g^{-1}\big)\big(\Gamma^{\tilde{A}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\big) + \big(B\circ g^{-1}\big)\big(\Gamma^{\tilde{B}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\big) \\ &+ \sum_{\tilde{k}=3}^{N} \big(A_{\tilde{k}}\circ g^{-1}\big)\big(\Gamma^{\tilde{k}}_{\tilde{A}\tilde{B}}(g\boldsymbol{p}, g\boldsymbol{q})\big).\end{aligned}$$

Therefore

$$\begin{aligned}\Gamma^A_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \big(g^{-1}\circ\Gamma^{\tilde{A}}_{\tilde{A}\tilde{B}}\big)(g\boldsymbol{p}, g\boldsymbol{q}) - \big(g^{-1}\circ\partial_{B(\boldsymbol{q})}(g)\big)(\boldsymbol{p}), \\ \Gamma^B_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \big(g^{-1}\circ\Gamma^{\tilde{B}}_{\tilde{A}\tilde{B}}\big)(g\boldsymbol{p}, g\boldsymbol{q}) + \big(g^{-1}\circ\partial_{A(\boldsymbol{p})}(g)\big)(\boldsymbol{q}), \\ \Gamma^k_{AB}(\boldsymbol{p},\boldsymbol{q}) &= \big(g^{-1}\circ\Gamma^{k}_{\tilde{A}\tilde{B}}\big)(g\boldsymbol{p}, g\boldsymbol{q}) \qquad \text{for } k \geq 3.\end{aligned}$$

[7] Under an invertible change $\tilde{w} = \tilde{w}[w]$ of fibre coordinates (see Example 1) the variational covectors are transformed by the inverse of the adjoint linearization $g = \big[\big(\ell^{(w)}_{\tilde{w}}\big)^\dagger\big]^{-1}$ whereas for variational vectors, $g = \ell^{(w)}_{\tilde{w}}$ is the linearization.

Acting by g upon these equalities and expressing $\boldsymbol{p} = g^{-1}\tilde{\boldsymbol{p}}$, $\boldsymbol{q} = g^{-1}\tilde{\boldsymbol{q}}$ we obtain (11) and conclude the proof. ■

Remark 3. Within the Hamiltonian formalism it is very productive to postulate that the arguments of Hamiltonian operators, the variational covectors, are *odd*,[8] see [22] and [13]. Indeed in this particular situation they can be conveniently identified with Cartan 1-forms times the pull-back of the volume form $\mathrm{dvol}\,(B^n)$ for the base of the jet bundle. We preserve this *grading* for such domains of operators (when $N = 1$, we referred to such operators in [10] as variational anchors of *second kind*). If moreover π and ξ are superbundles with Grassmann-valued sections, then the operators become bigraded [22]. Their proper grading is -1 because their images in $\mathfrak{g}(\pi)$ have grading zero, but the $\mathbb{Z}_2$-*parity*, if any, can be arbitrary.

Corollary 1. *For strongly compatible operators the domain $\Gamma\Omega(\xi_\pi)$ of which consists of variational covectors, the grading of the arguments equals 1. Therefore for any $i, j, k \in [1, \ldots, N]$ and for any $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$ we have that*

$$\Gamma^k_{ij}(\boldsymbol{p}, \boldsymbol{q}) = -\Gamma^k_{ji}(\boldsymbol{q}, \boldsymbol{p}) = (-1)^{|\boldsymbol{p}|_{gr}\cdot|\boldsymbol{q}|_{gr}} \cdot \Gamma^k_{ji}(\boldsymbol{q}, \boldsymbol{p}) \tag{12}$$

due to the skew-symmetry of the commutators in (5). Hence the symbols Γ^k_{ij} are symmetric in this case.

Proposition 1. *If two normal operators A_i and A_j are simultaneously linear and strongly compatible, then their 'individual' brackets Γ^i_{ii} and Γ^j_{jj} are*

$$\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{A_i} = \Gamma^j_{ij}(\boldsymbol{p}, \boldsymbol{q}) + \Gamma^j_{ji}(\boldsymbol{p}, \boldsymbol{q}) \quad \text{and} \quad \{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{A_j} = \Gamma^i_{ij}(\boldsymbol{p}, \boldsymbol{q}) + \Gamma^i_{ji}(\boldsymbol{p}, \boldsymbol{q})$$

for any $\boldsymbol{p}, \boldsymbol{q} \in \Gamma\Omega(\xi_\pi)$.

Proof. For brevity denote $A = A_i$, $B = A_j$ and consider the linear combination $\mu A + \nu B$; by assumption its image is closed under commutation. By Theorem 1 we have

$$\begin{aligned}&\big(\mu A + \nu B\big)\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_{\mu A + \nu B}\big)\\ &= \mu^2 A\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big) + \mu\nu\cdot A\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_B\big) + \mu\nu\cdot B\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big) + \nu^2 B\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big).\end{aligned}$$

On the other hand

$$\begin{aligned}&\Big[\big(\mu A + \nu B\big)(\boldsymbol{p}), \big(\mu A + \nu B\big)(\boldsymbol{q})\Big]\\ &= \mu^2 \big[A(\boldsymbol{p}), A(\boldsymbol{q})\big] + \mu\nu\big[A(\boldsymbol{p}), B(\boldsymbol{q})\big] - \mu\nu\big[A(\boldsymbol{q}), B(\boldsymbol{p})\big] + \nu^2\big[B(\boldsymbol{p}), B(\boldsymbol{q})\big].\end{aligned}$$

Taking into account (9) and equating the coefficients of $\mu\nu$ we obtain

$$\begin{aligned}&A\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_B\big) + B\big(\{\{\boldsymbol{p}, \boldsymbol{q}\}\}_A\big)\\ &= A\big(\Gamma^A_{AB}(\boldsymbol{p}, \boldsymbol{q})\big) + B\big(\Gamma^B_{AB}(\boldsymbol{p}, \boldsymbol{q})\big) - A\big(\Gamma^A_{AB}(\boldsymbol{q}, \boldsymbol{p})\big) - B\big(\Gamma^B_{AB}(\boldsymbol{q}, \boldsymbol{p})\big).\end{aligned}$$

Using the formulas $\Gamma^A_{AB}(\boldsymbol{q}, \boldsymbol{p}) = -\Gamma^A_{BA}(\boldsymbol{p}, \boldsymbol{q})$ and $\Gamma^B_{AB}(\boldsymbol{q}, \boldsymbol{p}) = -\Gamma^B_{BA}(\boldsymbol{p}, \boldsymbol{q})$, see (12), we isolate the arguments of the operators and obtain the assertion. ■

[8]Here we assume for simplicity that all fibre coordinates in π are permutable.

Conclusion

For every $\Bbbk$-vector space V the space of endomorphisms $\mathrm{End}_{\Bbbk}(V)$ is a monoid with respect to the composition $\circ$. In this context one can study relations between recursion operators. For instance the structural relations for recursion operators of the Krichever–Novikov equations are described by hyperelliptic curves, see [3]. Likewise we have the relation $R_1 \circ R_2 - R_2 \circ R_1 = R_1^2$ between two recursions for the dispersionless 3-component Boussinesq system, see [6]. Simultaneously the space of endomorphisms carries the structure of a Lie algebra which is given by the formula $[R_i, R_j] = R_i \circ R_j - R_j \circ R_i$ for every $R_i, R_j \in \mathrm{End}_{\Bbbk}(V)$.

In this paper we proceed further and consider the class of structures on the linear spaces of total differential operators that generally do not in principle admit any associative composition. (The bracket of recursion operators that appears through (6) is different from the Richardson–Nijenhuis bracket [12], although we use similar geometric techniques.) The classification problem for such algebras of operators is completely open.

Discussion

We performed all the reasonings for local differential operators in a purely commutative setup; all the structures were defined on the empty jet spaces. A rigorous extension of these objects to $\mathbb{Z}_2$-graded nonlocal operators on differential equations is a separate problem for future research. In addition the use of difference operators subject to (5) can be a fruitful idea *au début* for the discretization of integrable systems with free functional parameters in their symmetries (e.g., Toda-like difference systems [20]).

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[1] Alexandrov M., Schwarz A., Zaboronsky O. and Kontsevich M., The geometry of the master equation and topological quantum field theory, *Int. J. Modern Phys. A*, 1997, V.12, 1405–1429.

[2] Crainic M. and Fernandes R.L., Integrability of Poisson brackets, *J. Differential Geom.*, 2004, V.66, 71–137.

[3] Demskoi D.K. and Sokolov V.V., On recursion operators for elliptic models, *Nonlinearity*, 2008, V.21, 1253–1264.

[4] Dubrovin B.A., Geometry of 2D topological field theories, in *Integrable Systems and Quantum Groups (Montecatini Terme, 1993)*, Lect. Notes in Math., V.1620, Springer, Berlin, 1996, 120–348.

[5] Fuks D.B., *Cohomology of Infinite-Dimensional Lie Algebras*, Contemp. Sov. Math., Consultants Bureau, New York, 1986.

[6] Kersten P., Krasil'shchik I. and Verbovetsky A., A geometric study of the dispersionless Boussinesq type equation, *Acta Appl. Math.*, 2006, V.90, 143–178.

[7] Kiselev A.V. and van de Leur J.W., Symmetry algebras of Lagrangian Liouville-type systems, *Theor. Math. Phys.*, 2010, V.162, 149–162, arXiv:0902.3624.

[8] Kiselev A.V. and van de Leur J.W., A family of second Lie algebra structures for symmetries of dispersionless Boussinesq system, *J. Phys. A*: *Math. Theor.*, 2009, V.42, 404011, arXiv:0903.1214.

[9] Kiselev A.V. and van de Leur J.W., A geometric derivation of KdV-type hierarchies from root systems, 2009, *Proc. 4th Int. Workshop "Group Analysis of Differential Equations and Integrable Systems" (Protaras, Cyprus, 2008)*, 2009, 87–106, arXiv:0901.4866.

[10] Kiselev A.V. and van de Leur J.W., Variational Lie algebroids and homological evolutionary vector fields, *Theor. Math. Phys.*, 2011, in press, arXiv:1006.4227.

[11] Krasil'shchik I.S., Schouten bracket and canonical algebras, in *Global analysis — studies and applications.* III, Lecture Notes in Math., V.1334 (Yu. G. Borisovich and Yu. E. Gliklikh, eds.), Springer, Berlin, 1988, 79–110.

[12] Krasil'shchik I. and Verbovetsky A., *Homological Methods in Equations of Mathematical Physics*, Open Education and Sciences, Opava, 1998, arXiv:math/9808130.

[13] Krasil'shchik J. and Verbovetsky A., Geometry of jet spaces and integrable systems, 2010, arXiv:1002.0077.

[14] Krasil'shchik I.S. and Vinogradov A.M., eds. *Symmetries and Conservation Laws for Differential Equations of Mathematical Physics*, AMS, Providence, RI, 1999.

[15] Kumpera A. and Spencer D., *Lie Equations. I: General Theory*, Annals of Math. Stud., V.73, Princeton Univ. Press, Princeton, NJ, 1972.

[16] Magri F., A simple model of the integrable equation, *J. Math. Phys.*, 1978, V.19, 1156–1162.

[17] Manin Yu.I., Algebraic aspects of nonlinear differential equations, *Current problems in mathematics*, 1978, V.11, AN SSSR, VINITI, Moscow, 5–152 (in Russian).

[18] Nesterenko M. and Popovych R., Contractions of low-dimensional Lie algebras, *J. Math. Phys.*, 2006, V.47, 123515.

[19] Olver P.J., *Applications of Lie Groups to Differential Equations*, Grad. Texts in Math., V.107 (2nd ed.), Springer-Verlag, New York, 1993.

[20] Suris Yu.B., *The problem of integrable discretization: Hamiltonian approach*, Progr. in Math., V.219, Birkhäuser Verlag, Basel, 2003.

[21] Vaintrob A.Yu., Lie algebroids and homological vector fields, *Russ. Math. Surv.*, 1997, V.52, 428–429.

[22] Voronov T., Graded manifolds and Drinfeld doubles for Lie bialgebroids, *Proc. Conf. Quantization, Poisson Brackets, and Beyond* (Voronov T., ed.), *Contemp. Math.*, 2002, V.315, AMS, Providence, RI, 131–168.

On realizations of Lie algebras of Poincaré groups and new Poincaré-invariant equations

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We study realizations of the Poincaré groups $P(1,1)$ and $P(1,2)$ acting as local transformation groups in the space of one dependent variable and two and three independent variables, respectively. Realizations of the Lie algebras $p(1,1)$, $\tilde{p}(1,1)$, $c(1,1)$ and $p(1,2)$ by vector fields are classified. Using the classification results, we construct sets of second-order scalar Poincaré-invariant differential equations in the two- and three-dimensional space-time.

1 Introduction

In this paper we consider the problem of construction of partial differential equations of certain order admitting a given group G as an invariance group, which is one of the most important problems of classical group analysis of differential equations. It is well known that the complete solution of this problem requires the classification of realizations of the group G as a group of point transformations and the construction of a complete set of functionally independent differential invariants of certain order for each of the realizations found. Then any G-invariant equation is equivalent to the condition of the vanishing of a function of invariants of certain realization. The above problem seems algorithmically solvable for realizations in spaces of low dimension.

In theoretical and mathematical physics an important role is played by the Euclid, Poincaré, Galilei groups and their natural generalizations that are invariance groups of a number of model equations including the d'Alambert, Euclid, heat, Schrödinger, Dirac, Maxwell equations etc. Differential invariants of these groups were widely investigated in the literature. In particular Fushchych and Yehorchenko [1–3] obtained the exhaustive set of functionally independent second-order differential invariants for known scalar realizations (representations) of the Euclid, Poincaré and Galilei algebras by linear differential first-order operators. In an investigation of wave and evolution equations in two-dimensional space-time that are invariant with respect to the Galilei and Poincaré algebras Rideau and Winternitz [4, 5] preliminarily described realizations of these algebras by vector fields in the space of three variables. As a result new realizations were obtained that made it possible, after finding the corresponding differential invariants, to construct new Galilei- and Poincaré-invariant equations. Special attention was paid to nonlinear realizations of important Lie algebras in papers by Fushchych, Zhdanov, Yehorchenko, Boyko, Tsyfra and others [6–11], in which new realizations of Lie algebras of the Poincaré groups $P(1,2)$, $P(1,3)$ and Galilei groups $G(1,2)$, $G(1,3)$ were constructed.

The main purpose of this paper is to obtain a complete list of realizations of Lie algebras of the Poincaré groups $P(1,1)$ and $P(1,2)$ by vector fields and to describe the corresponding invariant partial differential equations.

2 Realizations of Lie algebras of groups $\boldsymbol{P(1,1)}$, $\boldsymbol{\tilde{P}(1,1)}$ and $\boldsymbol{C(1,1)}$ and invariant equations

In this section we study realizations of the Poincaré algebra $p(1,1)$, which is Lie algebra of the Poincaré group $P(1,1)$, and its natural generalizations (the extended Poincaré algebra $\tilde{p}(1,1)$ and the conformal algebra $c(1,1)$) in the space $V = X \times U$ of two independent and one dependent variables. Here X is the two-dimensional Minkowski space with coordinates t and x and U is the space of the dependent variable $u = u(t,x)$. Vector fields on the realization space V have the form

$$v = \tau\partial_t + \xi\partial_x + \eta\partial_u, \tag{1}$$

where $\tau = \tau(t,x,u)$, $\xi = \xi(t,x,u)$ and $\eta = \eta(t,x,u)$ are arbitrary smooth functions in a domain of the space V, $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$ and $\partial_u = \partial/\partial u$.

Realizations of the above algebras with operators of the form (1) are Lie invariance algebras of a number of known two-dimensional partial differential equations of relativistic physics (for example, the Klein–Gordon, Liouville, sin-d'Alambert and eikonal equations).

We say that the operators P_μ, K, D and C_μ of the form (1) realize *a representation of the conformal algebra* $c(1,1)$ if they are linearly independent and satisfy the following commutation relations:

$$\begin{aligned}
&[P_0,K] = P_1,\ [P_1,K] = P_0,\ [P_\mu,D] = P_\mu,\ [C_0,K] = C_1,\\
&[C_1,K] = C_0,\ [C_\mu,D] = -C_\mu,\ [P_\mu,C_\nu] = 2(g_{\mu\nu}D - \varepsilon_{\mu\nu}K),\\
&[K,D] = [P_0,P_1] = [C_0,C_1] = 0.
\end{aligned} \tag{2}$$

Here $g_{00} = -g_{11} = 1$, $g_{01} = g_{10} = 0$, $\varepsilon_{01} = -\varepsilon_{10} = 1$, $\varepsilon_{00} = \varepsilon_{11} = 0$, the subscripts μ and ν run from 0 to 1 and $[v_1,v_2] \equiv v_1v_2 - v_2v_1$ denotes the commutator (Lie bracket) of vector fields. The subalgebra of the algebra $c(1,1)$ with the basic operators P_0, P_1 and K is the Poincaré algebra $p(1,1)$ and the subalgebra spanned by the operators P_0, P_1, K and D is the extended Poincaré algebra $\tilde{p}(1,1)$.

It is obvious that commutation relations (2) are not changed under the push-forward of the basic operators of a realization by any nondegenerate *point transformation*

$$t \to \bar{t} = f(t,x,u), \quad x \to \bar{x} = g(t,x,u), \quad u \to \bar{u} = h(t,x,u), \tag{3}$$

where f, g and h are arbitrary smooth functions with vanishing Jacobian. Such transformations form a group (the diffeomorphism group) that generates a natural equivalence relation on the set of all possible realizations of the algebra $c(1,1)$. Realizations of the conformal algebra are called *equivalent* if their respective basic operators can be simultaneously transformed to each other by a change of variables of the form (3). Realizations of the algebra $c(1,1)$ will be described up to this equivalence.

A list of inequivalent realizations of the algebras $p(1,1)$, $\tilde{p}(1,1)$ and $c(1,1)$ were obtained in [4] under the assumption that the operators P_0 and P_1 are reduced to the form

$$P_0 = \partial_t, \quad P_1 = \partial_x \tag{4}$$

by a point transformation. The list comprises the following realizations:

1. **Inequivalent realizations of the algebra $p(1,1)$:**

$$\begin{aligned}
&p^1(1,1)\colon\ P_0 = \partial_t,\ P_1 = \partial_x,\ K = x\partial_t + t\partial_x;\\
&p^2(1,1)\colon\ P_0 = \partial_t,\ P_1 = \partial_x,\ K = x\partial_t + t\partial_x + u\partial_u.
\end{aligned} \tag{5}$$

2. **Inequivalent realizations of the algebra $\tilde{p}(1,1)$:**

$$\begin{aligned}
&\tilde{p}^1(1,1)\colon\ p^1(1,1),\ D = t\partial_t + x\partial_x;\\
&\tilde{p}^2(1,1)\colon\ p^1(1,1),\ D = t\partial_t + x\partial_x + u\partial_u;\\
&\tilde{p}^3(1,1)\colon\ p^2(1,1),\ D = (t + au + bu^{-1})\partial_t + (x + au - bu^{-1})\partial_x + \lambda u\partial_u,
\end{aligned} \tag{6}$$

where $(a,b) = (1,0)$ if $\lambda = 1$, $(a,b) = (0,1)$ if $\lambda = -1$ and $(a,b) = (0,0)$ otherwise.

3. **Inequivalent realizations of the algebra $c(1,1)$:**

$$\begin{aligned}
c^1(1,1)\colon\ &\tilde{p}^1(1,1),\ C_0 = (t^2 + x^2)\partial_t + 2tx\partial_x,\\
&C_1 = -(t^2 + x^2)\partial_x - 2tx\partial_t;\\
c^2(1,1)\colon\ &\tilde{p}^2(1,1),\ C_0 = (t^2 + x^2 + au^2)\partial_t + 2tx\partial_x + 2tu\partial_u,\\
&C_1 = -(t^2 + x^2 + au^2)\partial_x - 2tx\partial_t - 2xu\partial_u,\ a \in \{0, 1, -1\};\\
c^3(1,1)\colon\ &\tilde{p}^3(1,1), \lambda \in \mathbb{R}, a = b = 0,\\
&C_0 = (t^2 + x^2 + cu^2 + du^{-2})\partial_t + (2tx + cu^2 - du^{-2})\partial_x +\\
&+ (2u(x + \lambda t) + eu^2 + k)\partial_u,\ C_1 = -[K, C_0].
\end{aligned} \tag{7}$$

In the last realization, $c^3(1,1)$,

$$\begin{aligned}
&c = d = e = k = 0 \quad \text{if} \quad \lambda \in \mathbb{R}\backslash\{-1, 1\};\\
&d = k = 0,\ c = \pm 1,\ e \in \mathbb{R} \quad \text{or} \quad c = 0,\ e = 0, \pm 1 \quad \text{if} \quad \lambda = 1;\\
&c = e = 0,\ d = \pm 1,\ k \in \mathbb{R} \quad \text{or} \quad d = 0,\ k = 0, 1 \quad \text{if} \quad \lambda = -1.
\end{aligned}$$

In general the basic elements P_0 and P_1 are not always reduced to the form (4). As a result inequivalent realizations of the algebras $p(1,1)$ and $\tilde{p}(1,1)$ are not exhausted by realizations (5) and (6), respectively. Below we show that there are exactly one more inequivalent realization of the algebra $p(1,1)$ and two more inequivalent realizations of the algebra $\tilde{p}(1,1)$.

Lemma 1. *Let P_0 and P_1 be linearly independent operators of the form* (1)*. Then there exists a point transformation* (3) *that reduces these operators to either the form* (4) *or the following form:*

$$P_0 = \partial_t, \quad P_1 = x\partial_t. \tag{8}$$

Proof. Denote by M the matrix formed by coefficients of the operators P_0 and P_1.

Case 1. $\operatorname{rank} M = 2$. It is well-known that then the commuting operators P_0 and P_1 can be reduced by change of variables (3) to the form (4).

Case 2. $\operatorname{rank} M = 1$. Using a point transformation, we reduce the operator P_0 to the form $P_0 = \partial_t$. Then we have $P_1 = \tau(x,u)\partial_t$, where $(\tau_x, \tau_u) \neq (0,0)$, in view of $\operatorname{rank} M = 1$ and $[P_0, P_1] = 0$. Hence the transformation

$$\bar{t} = t, \quad \bar{x} = \tau(x,u), \quad \bar{u} = h(x,u), \quad \frac{\partial(\tau,h)}{\partial(x,u)} \neq 0,$$

reduces the operators P_0 and P_1 to the form (8). ∎

Theorem 1. *Inequivalent realizations of the algebra $p(1,1)$ are exhausted by* (5) *and the realization*

$$p^3(1,1)\colon\ P_0 = \partial_t,\ P_1 = x\partial_t,\ K = xt\partial_t + (x^2-1)\partial_x. \tag{9}$$

Proof. Inequivalent realizations of the two-dimensional Abelian algebra with the basic operators P_0 and P_1 are exhausted by realizations (4) and (8). The case of realization (4) was analyzed in [4]. Let the operators P_0 and P_1 have the form (8). We take the operator K having the general form (1). The commutation relations $[P_0, K] = P_1$ and $[P_1, K] = P_0$ imply that

$$K = [tx + \tau(x,u)]\partial_t + (x^2-1)\partial_x + \eta(x,u)\partial_u.$$

Using the change of variables $\bar{t} = t + f(x,u)$, $\bar{x} = x$, $\bar{u} = h(x,u)$, where the functions f and h are solutions of the system

$$Kf = xf - \tau, \quad Kh = 0, \quad h_u \neq 0,$$

we reduce the operators P_0, P_1 and K to the form (9). ∎

Theorem 2. *Inequivalent realizations of the algebra $\tilde{p}(1,1)$ are exhausted by* (6) *and the realizations*

$$\begin{aligned} &\tilde{p}^4(1,1)\colon\ p^3(1,1),\ D = t\partial_t, \\ &\tilde{p}^5(1,1)\colon\ p^3(1,1),\ D = t\partial_t + u\partial_u. \end{aligned} \tag{10}$$

Theorem 3. *Inequivalent realizations of the algebra $c(1,1)$ are exhausted by* (7).

The proofs of Theorems 2 and 3 are similar to the proof of Theorem 1. Note only that the realizations $\tilde{p}^4(1,1)$ and $\tilde{p}^5(1,1)$ do not admit extensions by vector fields of the form (1) into realizations of the conformal algebra.

The general form of invariant equations corresponding to the above realizations can be found by a standard procedure within the framework of the classical Lie approach. Let vector fields v_a, $a = 1,\dots,p$, form a basis of the Lie algebra of a local transformation group G which acts in the space V. In the case under consideration V is the space of the variables t, x and u and vector fields v_a have the form (1). A second-order partial differential equation

$$\Phi(t,x,u,u_t,u_x,u_{tt},u_{xx},u_{tx}) = 0 \tag{11}$$

is invariant with respect to the group G if the function Φ satisfies the system

$$\mathrm{pr}^{(2)}v_a\Phi = 0, \quad a = 1,\dots,p. \tag{12}$$

Here $\mathrm{pr}^{(2)}v_a$ denotes the second prolongation of the operator v_a, $a = 1,\dots,p$. When we solve system (12), we obtain a complete set of functionally independent second-order differential invariants

$$J_k = J_k(x,t,u,u_\mu,u_{\mu\nu}), \quad (\mu,\nu) = (t,x), \quad k = 1,\dots,s.$$

Then all G-invariant equations from the class (11) have the form

$$F(J_1,\dots,J_s) = 0. \tag{13}$$

In other words the description of equations invariant with respect to the group G is reduced to construction of a basis of differential invariants of this group.

$P(1,1)$-, $\tilde{P}(1,1)$-, and $C(1,1)$-invariant equations which possess realizations (5)–(7) were studied in [4]. To complete the description of equations of the general form (11), which are invariant with respect to the groups $P(1,1)$ and $\tilde{P}(1,1)$, it remains to consider realizations (9) and (10). The function Φ depends upon eight arguments and the general orbits of the second prolongations of these realizations are of dimensions three and four, respectively. Therefore bases of second-order differential invariants of realization (9) and realizations (10) consist of five and four invariants, respectively.

The results of the calculation of these bases are given below.

1. Elementary invariants of the algebra $p^3(1,1)$:

$$\begin{aligned}
&I_1 = u, \quad I_2 = u_t^2(x^2-1), \quad I_3 = u_{tt}(x^2-1),\\
&I_4 = (x^2-1)^2(u_xu_{tt} - u_tu_{tx}) - x(x^2-1)u_t^2,\\
&I_5 = (x^2-1)^3(u_{tt}u_{xx} - u_{tx}^2) + 2x(x^2-1)^2(u_xu_{tt} - u_tu_{tx}) -\\
&\qquad - x^2(x^2-1)u_t^2.
\end{aligned} \tag{14}$$

2. Elementary invariants of the algebra $\tilde{p}^4(1,1)$*:*

$$\Sigma_1 = I_1, \quad \Sigma_2 = I_2^{-1} I_3, \quad \Sigma_3 = I_2^{-2} I_4, \quad \Sigma_4 = I_2^{-5} I_5. \tag{15}$$

Here and in what follows $I_1, \ldots, I_5$ have the form (14).

3. Elementary invariants of the algebra $\tilde{p}^5(1,1)$*:*

$$\Sigma_1 = I_1 I_3, \ \Sigma_2 = I_2, \ \Sigma_3 = I_4, \ \Sigma_4 = I_5. \tag{16}$$

As a result we obtain new classes $P(1,1)$- and $\tilde{P}(1,1)$-invariant equations of the form (11):

$$\Phi(I_1, I_2, \ldots, I_5) = 0 \tag{17}$$

and

$$\Phi(\Sigma_1, \ldots, \Sigma_4) = 0, \tag{18}$$

where $\Sigma_1, \ldots, \Sigma_4$ are presented in (15) or (16).

Note that classes of equations (17) and (18) include equations which are natural generalizations of the well-known Monge–Amperé equation.

3 Realizations of Lie algebras of the group $P(1,2)$

Consider the space $V = X \times U$, where X is the three-dimensional Minkowski space with coordinates x_0, x_1 and x_2 and U is the space of the real dependent variable $u = u(x_0, x_1, x_2)$. Vector fields on the space V have the general form

$$v = \xi^\mu(x,u)\partial_{x_\mu} + \eta(x,u)\partial_u, \tag{19}$$

where ξ^μ and η are real smooth functions defined in an open domain of the space V. The Greek indices run from 0 to 2 and we use the summation convention for repeated indices. Additional or other constraints on indices are indicated explicitly.

We say that the operators P_μ and $J_{\mu\nu} = -J_{\mu\nu}$ of the form (19) form a basis of a realization of the Lie algebra $p(1,2)$ of the Poincaré group $P(1,2)$ if they are linearly independent and satisfy the commutation relations

$$\begin{aligned} &[P_\mu, J_{\alpha\beta}] = g_{\mu\alpha} P_\beta - g_{\mu\beta} P_\alpha, \quad [P_\mu, P_\nu] = 0, \\ &[J_{\mu\nu}, J_{\alpha\beta}] = g_{\mu\beta} J_{\nu\alpha} + g_{\nu\alpha} J_{\mu\beta} - g_{\mu\alpha} J_{\nu\beta} - g_{\nu\beta} J_{\mu\alpha}, \end{aligned} \tag{20}$$

where

$$g_{\mu\nu} = \begin{cases} 1, & \mu = \nu = 0, \\ 0, & \mu \neq \nu, \\ -1, & \mu = \nu \in \{1,2\}. \end{cases}$$

We study realizations of the algebra $p(1,2)$ in the class of vector fields (19) up to equivalence which is defined by the action of the local diffeomorphism group

$$x_\mu \to \bar{x_\mu} = f^\mu(x,u),\ u \to \bar{u} = g(x,u), \tag{21}$$

where f^μ and g are arbitrary smooth functions on the space V with nonvanishing Jacobian.

Commutation relations (20) imply that $p(1,2) = o(1,2) \Subset T$, where $o(1,2) = \langle J_{\mu\nu}\rangle$ and $T = \langle P_\mu\rangle$ is a commutative ideal. This is why we begin the study of realizations of the algebra $p(1,2)$ with the consideration of possible realizations of the translation operators P_μ.

Lemma 2. *There exist transformations* (21) *that reduce the operators* P_μ *to one of the following triples of operators:*

$$\begin{aligned}
&(a)\ P_0 = \partial_{x_0},\ P_1 = \partial_{x_1},\ P_2 = \partial_{x_2},\\
&(b)\ P_0 = \partial_{x_0},\ P_1 = \partial_{x_1},\ P_2 = x_2\partial_{x_0} + u\partial_{x_1},\\
&(c)\ P_0 = \partial_{x_0},\ P_1 = \partial_{x_1},\ P_2 = h(x_2)\partial_{x_0} + \varphi(x_2)\partial_{x_1},\\
&(d)\ P_0 = \partial_{x_0},\ P_1 = x_1\partial_{x_0},\ P_2 = \partial_{x_2},\\
&(e)\ P_0 = \partial_{x_0},\ P_1 = x_1\partial_{x_0},\ P_2 = \psi(x_1)\partial_{x_0},\\
&(g)\ P_0 = \partial_{x_0},\ P_1 = x_1\partial_{x_0},\ P_2 = x_2\partial_{x_0},
\end{aligned} \tag{22}$$

where φ, ψ *and* h *are arbitrary smooth functions of their arguments, and, in view of the linear independence of the operators* P_μ, $\psi_{x_1x_1} \neq 0$ *and* $(h_{x_2}, \varphi_{x_2}) \neq (0,0)$.

As the proof of Lemma 2 is cumbersome but similar to the proof of Lemma 1, we do not present it here.

Further the realizations (22) of the ideal T should be extended to realizations of the algebra $p(1,2)$ with operators $J_{\mu\nu}$ of the form (19). As the realizations (22) are inequivalent to each other, the realizations of the algebra $p(1,2)$ corresponding to different realizations from (22) also are inequivalent.

Note that the problem of expansion of the ideal T to the algebra $p(1,2)$ for the first triple of operators (22) was solved in [9,12] and other papers of the same authors. It was shown that the operators $J_{\mu\nu}$ have one of the following forms:

$$J_{\mu\nu} = g_{\mu\gamma}x_\gamma\partial_{x_\nu} - g_{\nu\gamma}x_\gamma\partial_{x_\mu} \tag{23}$$

or

$$\begin{aligned}
&J_{01} = x_0\partial_{x_1} + x_1\partial_{x_0} + \sin u\partial_u,\\
&J_{02} = x_0\partial_{x_2} + x_2\partial_{x_0} + \cos u\partial_u,\\
&J_{12} = -x_1\partial_{x_2} + x_2\partial_{x_1} + \sin u\partial_u.
\end{aligned} \tag{24}$$

When we made the expansion of the ideal T to the algebra $p(1,2)$ for remaining realizations (22), we have obtained the number of new realizations of the algebra $p(1,2)$. A complete result of classification of inequivalent realizations of the algebra $p(1,2)$ in the class of Lie vector fields (19) is represented in the following theorem that we give without proof.

Theorem 4. *Inequivalent realizations of the algebra* $p(1,2)$ *are exhausted by realizations* (22) *(a)*, (23)*;* (22) *(a)*, (24)*; and by the following realizations*

1. P_μ *of the form* (22) *(b)*,

$$J_{01} = x_0\partial_{x_1} + x_1\partial_{x_0} + u\partial_{x_2} + x_2\partial_u,$$
$$J_{02} = x_0x_2\partial_{x_0} + x_0u\partial_{x_1} + (x_2^2-1)\partial_{x_2} + x_2u\partial_{x_1}, \qquad (25)$$
$$J_{12} = -x_1x_2\partial_{x_0} - ux_1\partial_{x_1} - ux_2\partial_{x_2} - (1+u^2)\partial_u.$$

2. P_μ *of the form* (22) *(c), where* $h(x_2) = x_2$,

$$J_{01} = x_0\partial_{x_1} + x_1\partial_{x_0} + \varphi\partial_{x_2},$$
$$J_{02} = x_0x_2\partial_{x_0} + x_0\varphi\partial_{x_1} + \varphi^2\partial_{x_2} + a\partial_{x_0} + b\partial_{x_1} + q\partial_u, \qquad (26)$$
$$J_{12} = -x_1x_2\partial_{x_0} - \varphi x_1\partial_{x_1} - \varphi x_2\partial_{x_2} + \alpha\partial_{x_0} + \beta\partial_{x_1} + p\partial_u,$$

where $\varphi = \pm\sqrt{x_2^2-1}$, $|x_2| > 1$, *and for the functions* a, b, α, β, p *and* q *we have one of the following cases:*

1) $\alpha = \beta = \mathrm{const}, \quad a = b = \epsilon e^{-2u}, \quad q = -x_2, \quad p = \varphi, \quad \epsilon = 0, 1;$

2) $\alpha = \beta = \lambda_1\left[\dfrac{u}{1-u^2} + \dfrac{1}{2}\ln\left|\dfrac{1+u}{1-u}\right|\right] + \lambda_2, \quad a = b = \dfrac{\lambda_1}{1-u^2},$

$q = \varphi - x_2u, \quad p = \varphi u - x_2, \quad \lambda_1, \lambda_2 = \mathrm{const};$

3) $a = -\beta = \lambda x_2\varphi, \quad b = \lambda x_2^2, \quad \alpha = -\lambda\varphi^2, \quad p = q = 0, \quad \lambda = \mathrm{const};$

4) $a = -\beta = x_2\varphi u, \quad b = x_2^2u, \quad \alpha = -\varphi^2u, \quad p = q = 0.$

3. P_μ *of the form* (22) *(e)*,

$$J_{01} = (x_0x_1 + B\psi)\partial_{x_0} - \psi^2\partial_{x_1} + (Cx_1 + D\psi)\partial_{x_2} + A\psi\partial_u,$$
$$J_{02} = (x_0\psi - x_1B)\partial_{x_0} + x_1\psi\partial_{x_1} + (C\psi - x_1D)\partial_{x_2} - Ax_1\partial_u, \qquad (27)$$
$$J_{12} = \psi\partial_{x_1},$$

where $\psi = \pm\sqrt{1-x_1^2}$, $|x_1| < 1$, *and the parameters* A, B, C *and* D *take one of the following values:*

1) $A = B = C = D = 0;$

2) $A = \sqrt{|x_2|}g(u), \quad B = x_2, \quad C = 2x_2, \quad D = x_2\sqrt{|x_2|}f(u);$

3) $A = x_2f(u), \quad B = 0, \quad C = x_2, \quad D = x_2^2g(u),$

with f *and* g *being arbitrary smooth functions of* u.

4. P_μ *of the form* (22) *(g)*,

$$J_{01} = x_0x_1\partial_{x_0} + (x_2^2-1)\partial_{x_1} + x_1x_2\partial_{x_2} + x_2\theta\partial_{x_0} + x_2\rho\partial_u,$$
$$J_{02} = x_0x_2\partial_{x_0} + x_1x_2\partial_{x_1} + (x_2^2-1)\partial_{x_2} - x_1\theta\partial_{x_0} - x_1\rho\partial_u, \qquad (28)$$
$$J_{12} = x_2\partial_{x_1} - x_1\partial_{x_2},$$

where either $\theta = f(u)(1-\omega^{-1})$, $\rho = 0$ *with* f *being an arbitrary function of* u *or* $\theta = 0$, $\rho = \omega^{-1}\sqrt{|\omega-1|}$ *with* $\omega = x_1^2 + x_2^2$.

Note that the realization (22) (d) of the ideal T does not admit an extension to a realization of the algebra $p(1,2)$.

4 Discussion of results and summary

As it follows from the results of the present paper, the problem of classification of inequivalent realizations of Poincaré algebras $p(1,1)$ and $p(1,2)$ in the class of vector fields in the space of a low dimension is completely constructive. The lists of realizations found here and in the papers mentioned give the complete solution of this problem in spaces of three and four variables, respectively. By the way the partition of variables into dependent and independent variables is quite conditional. Here we consider a single variable as a dependent one in order to describe later the general form of scalar equations which admit the realizations obtained as Lie invariance algebras. These realizations can be also used for the description of Poincaré-invariant systems of differential equations in spaces of low dimensions.

We also completely solve of the problem of description of Poincaré-invariant scalar second-order partial equations in the two-dimensional space-time. The problem of description of Poincaré-invariant equations in a three-dimensional space-time should be additionally studied. Currently there exist only partial solutions of this problem. Thus for the realizations (22) (a), (23) and (22) (a), (24) this problem was solved in the works of W. Fushchych and I. Yehorchenko [1–3,9]. We have succeeded to obtain other four of seven differential invariants for the last realization from Theorem 4, where $\theta = \rho = 0$. These are the invariants

$$I_1 = u, \quad I_2 = u_{x_0}^2 u_{x_0x_0}, \quad I_3 = (\Sigma_1 - 1)^2 u_{x_0}^2,$$
$$I_4 = \Sigma_1^{-1}\left[(1-\Sigma_1)^3(2\Sigma_1\Sigma_2 u_{x_0}^2 + (1-\Sigma_1)\Sigma_2^2 + \Sigma_3^2) + (1-\Sigma_1)^2\Sigma_1^2 u_{x_0}^4\right],$$

where

$$\Sigma_1 = x_1^2 + x_2^2, \quad \Sigma_2 = x_1(u_{x_1}u_{x_0x_0} - u_{x_0}u_{x_0x_1}) + x_2(u_{x_2}u_{x_0x_0} - u_{x_0}u_{x_0x_2}),$$
$$\Sigma_3 = x_2(u_{x_1}u_{x_0x_0} - u_{x_0}u_{x_0x_1}) - x_1(u_{x_2}u_{x_0x_0} - u_{x_0}u_{x_0x_2}).$$

The remaining cases are not considered up to now. Also note that the list of realizations found for the algebra $p(1,2)$ facilitates solution of the problem on classification of inequivalent realizations of the algebra $c(1,2)$.

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[1] Fushchich V.I. and Egorchenko I.A., Differential invariants of the Galilei algebra, *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, 1989, no. 4, 19–24 (in Russian).

[2] Fushchich V.I. and Egorchenko I.A., Differential invariants of the Poincaré algebra, *Dokl. Akad. Nauk Ukrain. SSR Ser. A*, 1989, no. 5, 46–53 (in Russian).

[3] Fushchich V.I. and Egorchenko I.A., Second-order differential invariants of the rotation group O(n) and of its extensions: $E(n)$, $P(1,n)$, $G(1,n)$, *Acta Appl. Math.*, 1992, V.28, 69–92.

[4] Rideau G. and Winternitz P., Nonlinear equations invariant under the Poincare, similitude, and conformal groups in two-dimensional space-time, *J. Math. Phys.*, 1990, V.31, 1095–1105.

[5] Rideau G. and Winternitz P., Evolution equations invariant under two-dimensional space-time Schrodinger group, *J. Math. Phys.*, 1993, V.34, 558–570.

[6] Fushchych W.I. and Lahno V.I., On new nonlinear equations invariant under Poincaré group in two-dimentional space-times, *Proc. Acad. Sci. Ukraine*, 1996, no. 11, 60–65 (in Ukrainian).

[7] Fushchych W.I. and Lahno V.I., Linear and nonlinear representation of the Galilei groups in a two-dimensional space-time, *Ukrainian Math. J.*, 1998, V.50, 470–480.

[8] Fushchych W.I., Tsyfra I.M. and Boyko V.M., Nonlinear representations for Poincaré and Galilei algebras and nonlinear equations for electromagnetic fields, *J. Nonlinear Math. Phys.*, 1994, V.1, 210–221.

[9] Yehorchenko I., Nonlinear representation of the Poincaré algebra and invariant equations, in *Symmetry Analysis of Equations of Mathematical Physics*, Kyiv, Inst. of Math., 1992, 62–66.

[10] Zhdanov R.Z. and Fushchych W.I., On new representations of Galilei groups, *J. Nonlinear Math. Phys.*, 1997, V.4, 426–435.

[11] Zhdanov R.Z., Lahno V.I. and Fushchych W.I., On covariant realizations of the Euclid group, *Comm. Math. Phys.*, 2000, V.212, 535–556.

[12] Fushchych W.I., Zhdanov R.Z. and Lahno V.I., On linear and non-linear representations of the generalized Poincaré groups in the class of Lie vector fields, *J. Nonlinear Math. Phys.*, 1994, V.1, 295–308.

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Lie symmetries and certain equations of Financial Mathematics

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We explore the application of symmetry in the sense of Lie's Theory to the algebraic resolution of evolution partial differential equations for problems arising in Financial Mathematics and demonstrate that many problems are susceptible to an algorithmic treatment. In particular we show how certain equations which have been solved by *ad hoc* methods are easily solved using the algebraic approach. Some equations considered are the Cox–Ingersoll–Ross equation with time-dependent parameters and the Heston Problem of Stochastic Volatility.

1 Introduction

Although the application of symmetry to the resolution of differential equations – be they ordinary or partial – is well established in the traditionally exact sciences, there are some fields of scientific investigation newly entering into the realm of mathematical exactitude in which such application is largely absent, nay, even rejected. Such a troglodytic approach is difficult to understand when one considers the advantages of an algorithmic approach to the determination of solutions of differential equations. Given that it is almost 140 years since Lie developed his theory of examining differential equations for their symmetries to make clear the route to solution, one can only be surprised that there exists those who still believe that the only route to take is that of the stage magician armed with tophat and an incredible supply of rabbits to be drawn from it.

Financial transactions in the markets of the World have become increasingly complex over the last four decades. There was a time when matters such as options, insurance and reinsurance were not so prominent in the scheme of financial affairs. This is no longer the case. As has happened with so many other areas of human endeavour, Finance in its broader interpretation has become increasingly mathematical, as opposed to its intrinsically arithmetical nature, and now the topic of Financial Mathematics is a discipline unto itself.

One of the measures of the extent of the quantification of a discipline is the degree to which its processes are modelled by differential equations[1]. When differential equations are introduced into a discipline, it should be automatic that effective methods of the treatment of differential equations should also be introduced. It is the purpose of this paper to aid that process in the case of the application of symmetry to the solution of evolution partial differential equations which arise in Financial Mathematics.

We begin our discussion with a brief resumé of the classical heat equation in terms of its origin and of its properties, well-known, in terms of Lie point symmetries. We then compare this with the standard form of modelling of a financial process so that the emergence of evolution partial differential equations as an important aspect of Financial Mathematics becomes obvious. In subsequent sections we illustrate several applications of Lie theory in the solution of such differential equations both in the linear and the nonlinear situations.

The heat equation for a uniform one-dimensional medium is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

in which $u(t,\, x)$ represents the temperature of the medium at time, t, and position, x. Although the original derivation of (1) was from considerations of the continuum due to Laplace and in terms of Fourier's Law, the development of Statistical Physics as a natural evolution from the observation of the behaviour of particles of very small mass by Brown provided a probabilistic basis for the equation. A model for the observed apparently random motions of these particles was provided by the concept of stochastic processes. This led to some very useful results, the Lemma of Itô [18], in terms of the relationship between stochastic processes and evolution partial differential equations. The Theorems of Fokker–Planck [15,26] and Feynman–Kac [14,19] are important in that under appropriate conditions they guarantee uniqueness of the solution of the equation subject to some conditions which are not terribly onerous.

Equation (1) has the Lie point symmetries[2]

$$\begin{aligned} &\Gamma_1 = \partial_x, \quad \Gamma_2 = 2t\partial_x - xu\partial_u, \quad \Gamma_3 = u\partial_u, \quad \Gamma_4 = \partial_t, \\ &\Gamma_5 = 2t\partial_t + x\partial_x, \quad \Gamma_6 = 4t^2\partial_t + 4tx\partial_x - \left(2t + x^2\right) u\partial_u, \quad \Gamma_7 = f(t,\, x)\partial_u, \end{aligned}$$

where $f(t,\, x)$ is any solution[3] of (1). In the Mubarakzyanov Classification Scheme [22–25] the algebra is $\{A_{3,8} \oplus_s A_{3,1}\} \oplus_s \infty A_1$ which in a more common parlance is written as $\{\mathrm{sl}(2, R) \oplus_s W_3\} \oplus_s \infty A_1$, where W_3 is the three-element Heisenberg–Weyl algebra more familiar from considerations of the simple harmonic oscillator

[1] We are well aware that other approaches to mathematical modelling are to be found and their worth is not to be gainsayed!

[2] Calculated using the Mathematica add-on, Sym [7–9]. One should note that the calculation is not original and the symmetries can be found listed in such texts as Bluman and Kumei [3].

[3] One must emphasise the anyness of the solution. There is no need to take into consideration initial or boundary conditions.

in Quantum Mechanics. In terms of point symmetries this algebra is the maximal algebra for a $(1+1)$ evolution partial differential equation.

Normally the heat equation, (1), is solved with some requirement such as an initial condition or some boundary conditions. In terms of finance one would be looking for a solution, $u(t,\,x)$, which would lead to the dependent variable, $u(t,\,x)$, having a particular value, U, at some time in the future, say T.

Essentially the problem reduces to finding a symmetry or symmetries of (1) compatible with the dual requirements

$$t = T \quad \text{and} \quad u(T,\,x) = U \quad \forall\, x. \tag{2}$$

To determine the symmetry we take the linear combination[4]

$$\Gamma = \sum_{i=1}^{i=6} \alpha_i \Gamma_i. \tag{3}$$

We apply the symmetry in (3) to the dual conditions in (2) to obtain a system of equations,

$$\begin{aligned} 0 &= \alpha_4 + 2T\alpha_5 + 4T^2\alpha_6 \quad \text{and} \\ 0 &= -xU\alpha_2 + U\alpha_3 - \left(2T + x^2\right) U\alpha_6. \end{aligned} \tag{4}$$

Since x is a free variable, (4) separates into

$$0 = U\alpha_3 - 2TU\alpha_6, \quad 0 = -U\alpha_2 \quad \text{and} \quad 0 = -U\alpha_6$$

from which it is evident that $\alpha_6 = 0$, $\alpha_2 = 0$, $\alpha_3 = 0$, $\alpha_4 + 2T\alpha_5 = 0$ and α_1 is arbitrary so that there are two symmetries of (1) compatible with the requirements and they are given by

$$\begin{aligned} \Sigma_1 &= \partial_x \quad \text{and} \\ \Sigma_2 &= 2(t-T)\partial_t + x\partial_x. \end{aligned} \tag{5}$$

The obvious symmetry for the reduction of (1) is (5) and it follows immediately that the solution is $u(t,\,x) = U$, which perhaps may not be regarded as terribly interesting.

On the other hand, if one takes $T = 0$ and sets $u(0,\,x) = U(x)$, the application of (3) to the dual conditions gives

$$0 = \alpha_4 \quad \text{and} \quad U'(x)\alpha_1 + U'(x)x\alpha_5 = -xU(x)\alpha_2 + U(x)\alpha_3 - x^2U(x)\alpha_6.$$

From the latter we obtain

$$\frac{U'(x)}{U(x)} = -\frac{x\alpha_2 - \alpha_3 + x^2\alpha_6}{\alpha_1 + x\alpha_5}$$

[4]Note that we do not include the infinite-dimensional subalgebra contained in Γ_7. As the coefficient function is a solution of the differential equation, (1), it cannot play a role in the discriminatory procedure being undertaken.

which for the purposes of this example we simplify by taking $\alpha_2 = \alpha_1$, $\alpha_6 = \alpha_5$ and $\alpha_3 = 0$. Then

$$\frac{U'(x)}{U(x)} = -x \quad \Longrightarrow \quad U(x) = K\exp[-\tfrac{1}{2}x^2] \tag{6}$$

and the corresponding symmetries are

$$\Sigma_{\alpha_1} = (1+2t)\partial_x - xu\partial_u \quad \text{and}$$
$$\Sigma_{\alpha_5} = 2t(1+2t)\partial_t + (1+4t)x\partial_x - \left(2t + x^2\right)u\partial_u.$$

Since $[\Sigma_{\alpha_1}, \Sigma_{\alpha_5}]_{LB} = \Sigma_{\alpha_1}$, the preferred route for reduction is via Σ_{α_1}. The invariants are t and $u\exp[x^2/(2(1+2t))]$. The reduction and reduced equation are

$$u(t, x) = \exp\left[-\frac{x^2}{2(1+2t)}\right]g(t) \quad \text{and} \quad \frac{\dot{g}(t)}{g(t)} = -\frac{1}{1+2t}$$

so that the solution corresponding to the initial distribution (6) is

$$u(t, x) = \frac{K}{\sqrt{1+2t}}\exp\left[-\frac{x^2}{2(1+2t)}\right].$$

We have dealt in some length with these two elementary solutions of the classical heat equation, (1), to provide simple demonstrations of the methods of solution for some of the equations which arise in Financial Mathematics. In the first example we tailored the solution to fit a precise terminal condition. In the second example we tailored[5] the terminal condition so that it would admit some symmetry which would then permit reduction and solution.

In the sections below we examine the following problems. The first is a generalisation of the Cox–Ingersoll–Ross Equation [6], which models the zero-coupon bond-pricing problem, to the case in which the parameters are explicitly time-dependent. The second is the equation describing a model of stochastic volatility developed by Heston [17] in which a second varible is introduced to allow for the assumed stochastic nature of the volatility. We conclude with an adaptation of the Black–Scholes model [4] to allow for a market which is illiquid due to the dominance of a single trader. These examples give some idea of the uses of symmetry in the resolution of the evolution partial differential equations which arise in the field of Financial Mathematics.

2 The Cox–Ingersoll–Ross Equation with a difference

The Cox–Ingersoll–Ross Equation [6][6]

$$u_t + \tfrac{1}{2}\sigma^2 x u_{xx} - (\kappa - \lambda x)\,u_x - xu = 0$$

[5]Note that the degree of 'tailoring' was designed to produce a simple calculation. It should be quite evident that a far more complicated function would be obtained if the restrictions on the parameters, α_i, were relaxed.

[6]Studies of similar equations are to be found in [5, 10, 16, 27].

is an example of an equation which has a number of symmetries which varies depending upon the existence or lack of existence of a specific relationship between the parameters. The equation is rather typical of equations to be found in the practical applications of the theory underlying Financial Mathematics. In the development of the partial differential equations from the underlying stochastic equations all sorts of possible dependencies can be carried since they are not really relevant to the probabilistic aspects being considered. However, when it comes to the solution of the resultant partial differential equations, functions quickly seem to develop a nature of constancy rarely to be found in the real world. We recall that the parameters have the meaning of variance in the case of σ, an underlying trend rate in the case of κ and a measure of reversion to the mean in the case of λ.

Here we examine the case in which all of the parameters can depend upon time. Admittedly this is not a complete case since one could also imagine a scenario in which the parameters were affected by the underlying price/cost of the commodity under consideration. Nevertheless the allowance for a temporal variation in the parameters is a move in the direction of accepting reality. We examine the equation

$$u_t + \tfrac{1}{2}\sigma(t)^2 x u_{xx} - (\kappa(t) - \lambda(t)x)\, u_x - xu = 0 \tag{7}$$

for its Lie point symmetries using Sym in interactive mode. The consequent calculations are notable more for the complexity of the expressions rather than the complexity of the actual calculations and so there is no real point to repeat them here. What we do find is a symmetry which depends upon three functions, $a(t)$, $b(t)$ and $g(t)$, of a fairly familiar form, *ie*

$$a(t)\partial_t + (b(t) + xf_1(a,\, \sigma))\partial_x + \left(x^2 f_2(a,\, \sigma,\, \lambda) + xf_3(a,\, \sigma,\, \lambda,\, \kappa) + g\right) u\partial_u,$$

in which we omit the infinite-dimensional subalgebra of solution symmetries which are a consequence of the linearity of (7).

The three functions to be determined, $a(t)$, $b(t)$ and $g(t)$, to establish the coefficient functions of the symmetries of (7) solutions of three linear ordinary differential equations. For the function $a(t)$ the equation is of the third order and has the typical structure for the time-dependent function in an $\mathrm{sl}(2, R)$ subalgebra, namely

$$\dddot{a}(t) + 2p(t)\dot{a}(t) + \dot{p}(t)a(t) = 0,$$

where $p(t)$ depends upon $s(t)$ and $\lambda(t)$. The equation for $b(t)$ is of the second order and somewhat more complicated, but is consistent with one part being a two-dimensional subalgebra and the other being related to the coefficient function $a(t)$. The same story applies to $g(t)$ which is determined as the solution of a first-order equation with the nonhomogeneous terms depending upon the other coefficient functions.

The interesting point about the time-dependent version of the Cox–Ingersoll–Ross Equation is that it admits the maximal number of Lie point symmetries.

Consequently one would expect to be able to solve any problem with a terminal condition. This is an interesting development for it broadens the class of models for which one can add least construct the structure of an analytic solution. It must be admitted that a viewing of the third-, second- and first-order ordinary differential equations to be satisfied by $a(t)$, $b(t)$ and $g(t)$ does not imbue one with an expectation to be able to determine a solution in closed form. Nevertheless this lack of a solution in closed form may not present a serious impediment to progress. It may happen that one must resort to numerical procedures to go from the second-last line to the last line [20]. That one must eventually resort to numerical procedures is more or less a fact of life since very few equations can be solved in a form which obviates the necessity for numerics. The critical point is at which level one must make these numerical computations. As a general observation one may state quite comfortably that the further into the resolution of a given problem that one can defer the implementation of numerical procedures the more effective the modelling undertaken.

In the case of the Cox–Ingersoll–Ross Equation with time-dependent parameters we have an explicit expression for the structure of the symmetries. The very fact that we know the precise dependence of the symmetries upon x and u makes it possible to construct the form of the invariants for the reduction of the equation to an ordinary differential equation. It is this ability to be able to reduce the problem which makes the application of symmetry even in this somewhat nebulous form advantageous.

3 Stochastic Volatility: the Heston Model

The Heston Model of Stochastic Volatility [17] leads to the (1+2) evolution partial differential equation

$$\begin{aligned}&\frac{\partial V}{\partial t}+\tfrac{1}{2}S^2y\frac{\partial^2 V}{\partial S^2}+\varepsilon rSy\frac{\partial^2 V}{\partial S\partial y}+\tfrac{1}{2}\varepsilon^2y\frac{\partial^2 V}{\partial y^2}+(r-D)S\frac{\partial V}{\partial S}\\&+(\omega-y\theta-\Lambda)\frac{\partial V}{\partial y}-rV=0,\end{aligned}\tag{8}$$

where V is the valuation of a volatility-dependent instrument, S the value of the underlying asset, D the yield on the asset, r is the interest rate and the coefficient of u_y is the real-world drift term less the market price of risk. The terminal conditions are

$$t=T\quad\text{and}\quad V(T,\,S,\,y)=\mathrm{Max}\{S-K,0\},$$

where K is the cut-off value of the asset. In the case that the coefficient of V_y was taken to have the form $\kappa\theta-(\kappa+\lambda)y$ Heston [17] has provided a solution in closed form. The exponent on y need not be so simple. In a more general model the exponent of y in the mixed derivative is $\gamma+\frac{1}{2}$ and of the second derivative with respect to y it is 2γ, where γ is the exponent of the variance in the stochastic differential equation for the variance. In the case that $\gamma = \frac{1}{2}$ the two exponents become simpler and more tractible numbers. The solution for general values of γ proves to be elusive [29].

The only Lie point symmetries of (8), apart from the infinite class of solution symmetries, are the obvious ones of

$$\Gamma_1 = V\partial_V, \quad \Gamma_2 = S\partial_S \quad \text{and} \quad \Gamma_3 = \partial_t.$$

The first is a consequence of the linearity of (8) and the third of the autonomy of the equation. The second symmetry is a consequence of the equation being autonomous in the variable $\log S$. We make a change of variables from the original equation. The change of variables is given by

$$\tau = T - t, \quad x = \log S + (r - D)\tau \quad y = y, \quad V = u\mathrm{e}^{-r\tau}. \tag{9}$$

Equation (8) is now

$$2\frac{\partial u}{\partial \tau} = y\left\{\frac{\partial^2 u}{\partial x^2} + 2\varepsilon r\frac{\partial^2 u}{\partial x \partial y} + \varepsilon^2\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} - 2(\kappa + \lambda)\frac{\partial u}{\partial y}\right\} + 2\kappa\theta\frac{\partial u}{\partial y}. \tag{10}$$

We take the Fourier transform of (10) with respect to x to obtain the $(1 + 1)$ evolution equation

$$2\frac{\partial \tilde{u}}{\partial \tau} + y\left\{\omega^2\tilde{u} - 2i\omega\varepsilon r\frac{\partial \tilde{u}}{\partial y} - \varepsilon^2\frac{\partial^2 \tilde{u}}{\partial y^2} + 2(\kappa + \lambda)\frac{\partial \tilde{u}}{\partial y} - i\omega\tilde{u}\right\} - 2\kappa\theta\frac{\partial \tilde{u}}{\partial y} = 0. \tag{11}$$

Note that we have replaced the coefficient of V_y in (8) with the expression given by Heston [17], namely $\kappa\theta - (\kappa + \lambda)y$. The given terminal condition becomes

$$\tilde{u}(0, \omega, y) = \frac{1}{\sqrt{2\pi}}\frac{K^{i\omega+1}}{i\omega - \omega^2}, \tag{12}$$

which is evaluated at $\tau = 0$ and this is significant when one considers the change of variables given in (9).

An analysis of the Lie point symmetries of (11) gives the result that the symmetry (apart from the generic solution symmetry) has the general form

$$\begin{aligned}\Gamma = {} & a(\tau)\partial_\tau + \left(\sqrt{y}b(\tau) + ya'(\tau)\right)\partial_y + \left\{C_0 + \frac{\kappa\theta}{\varepsilon^2}\left((\kappa + \lambda - i\varepsilon r\omega)a(\tau)\right.\right. \\ & \left. - a'(\tau)\right) + \frac{1}{4\varepsilon^2\sqrt{y}}\left(\varepsilon^2 - 4\kappa\theta\right)b(\tau) + \frac{1}{\varepsilon^2}\left[(\kappa + \lambda - i\varepsilon r\omega)b(\tau) - 2b'(\tau)\right]\sqrt{y} \\ & \left. + \frac{1}{\varepsilon^2}\left[(\kappa + \lambda - i\varepsilon r\omega)a'(\tau) - a''(\tau)\right]y\right\}u\partial_u,\end{aligned}$$

where

$$\begin{aligned} a(\tau) &= A_0 + A_1\exp[P\tau] + A_2\exp[-P\tau], \\ b(\tau) &= B_1\exp[P\tau/2] + B_2\exp[-P\tau/2] \end{aligned}$$

and the upper case letters are constants of integration with

$$P^2 = \left[(\kappa+\lambda-i\varepsilon r\omega)^2 - \varepsilon^2\left(i\omega-\omega^2\right)\right].$$

There is also the constraint that, if $(4\kappa\theta-3\varepsilon^2)(4\kappa\theta-\varepsilon^2)\neq 0$, it follows that $b(\tau)$ must be identically zero.

In the case that the constraint be satisfied (11) has the maximal number of Lie point symmetries for an $(1+1)$ evolution equation and the algebra is $\{\mathrm{sl}(2,R)\oplus W_3\}\oplus_\infty A_1$. Otherwise the algebra is $\{\mathrm{sl}(2,R)\oplus A_1\}\oplus_\infty A_1$, *ie*, the Weyl–Heisenberg subalgebra is reduced to the single homogeneity symmetry.

When the general symmetry, Γ, with the functional dependence in $a(\tau)$ and $b(\tau)$ substituted is applied to the terminal conditions, (12), one finds that there is compatibility provided $b(\tau)=0$ and the relations

$$A_0 = -m, \quad A_1 = \frac{m+P}{2} \quad \text{and} \quad A_2 = \frac{m-P}{2}$$

hold, where $m=\kappa+\lambda-i\varepsilon r\omega$ and the common multiplier of the symmetry, C_0, is given the value $\kappa\theta P^2/\varepsilon^2$. Thus we have

$$a(\tau) = P\sinh P\tau - m(1-\cosh P\tau). \tag{13}$$

For general values of the parameters, when the expression for the symmetry compatible with the terminal conditions is simplified, it is

$$\begin{aligned}\Gamma_c &= a(\tau)\partial_t + ya'(\tau)\partial_y \\ &+\left[C_0+\frac{\kappa\theta m a(\tau)}{\varepsilon^2}-\frac{\kappa\theta a'(\tau)}{\varepsilon^2}+\frac{y}{\varepsilon^2}\left(ma'(\tau)-a''(\tau)\right)\right]u\partial_u\end{aligned} \tag{14}$$

with $a(\tau)$ and C_0 as given above.

The invariants of Γ_c are

$$q=\frac{y}{a(\tau)} \quad \text{and}$$

$$z = u\exp\left[-\frac{1}{\varepsilon^2}\left(\kappa\theta m\tau + mqa(\tau) - \kappa\theta\log a(\tau) - qa'(\tau)\right) + j(\tau)\right], \tag{15}$$

where $j(\tau)=\int C_0/a(\tau)\mathrm{d}t$. When we make the substitution

$$\begin{aligned}\tilde{u}(\tau,\,\omega,\,y) &= \exp\left[\frac{1}{\varepsilon^2}\left(\kappa\theta m\tau + mqa(\tau) - \kappa\theta\log a(\tau)\right.\right.\\ &\left.\left.-qa'(\tau)\right) + j(\tau)\right]Q(y/a(\tau)),\end{aligned} \tag{16}$$

the resulting ordinary differential equation is simply

$$qQ'' + \frac{2\kappa\theta}{\varepsilon^2}Q' - \frac{P^2}{\varepsilon^4}\left(qP^2+2\kappa\theta\right)Q = 0$$

which has the solution

$$Q(p) = \exp\left[\frac{P^2 q}{\varepsilon^2}\right]\left\{C_1 - C_2\left(\frac{2P^2}{\varepsilon^2}\right)^{2\kappa\theta/\varepsilon^2-1}\Gamma\left[1-\frac{2\kappa\theta}{\varepsilon^2}, \frac{2P^2 q}{\varepsilon^2}\right]\right\}, \tag{17}$$

where C_1 and C_2 are the constants of integration and $\Gamma[\cdot, \cdot]$ is the incomplete gamma function. Since the terminal condition is evaluated at $\tau = 0$, the term involving the incomplete gamma function makes no contribution to the required solution as its argument becomes infinite and so we set $C_2 = 0$.

From (13), (16) and (17) it follows that

$$\begin{aligned}\tilde{u}(\tau, \omega, y) = {} & C_1 \exp\left\{\frac{1}{\varepsilon^2}\left[\frac{P^2 y}{P\sinh(P\tau) - m(1-\cosh(P\tau))}\right.\right.\\ & + \kappa\theta m\tau + \kappa\theta\left[\log(\sinh(P\tau/2)) - \log\left(P\cosh(P\tau/2) + m\sinh(P\tau/2)\right)\right]\\ & - \kappa\theta\log\left(P\sinh(P\tau) - m(1-\cosh(P\tau))\right) + my\\ & \left.\left. - \frac{Py\left(P\cosh(P\tau) + m\sinh(P\tau)\right)}{P\sinh(P\tau) - m(1-\cosh(P\tau))}\right]\right\}.\end{aligned} \tag{18}$$

When one takes into account the terminal condition, (12), on the Fourier transform, it follows that

$$C_1 = \frac{\left(2P^2\right)^{\kappa\theta/\varepsilon^2}}{\sqrt{2\pi}}\frac{K^{i\omega+1}}{i\omega - \omega^2}$$

from (12). After some simplification one finds that

$$\begin{aligned}\tilde{u}(\tau, \omega, y) = {} & \frac{1}{\sqrt{2\pi}}\left(\frac{P}{P\cosh(P\tau/2) + m\sinh(P\tau/2)}\right)^{2\kappa\theta/\varepsilon^2}\frac{K^{i\omega+1}}{(i\omega-\omega^2)}\\ & \times\exp\left[\frac{m\kappa\theta\tau}{\varepsilon^2} + \frac{y\left(i\omega-\omega^2\right)\sinh(P\tau/2)}{P\cosh(P\tau/2) + m\sinh(P\tau/2)}\right].\end{aligned} \tag{19}$$

The solution to (8) follows from the evaluation of the inverse Fourier transform

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp[-i\omega x]\tilde{u}(\tau, \omega, y)\mathrm{d}\omega$$

with $\tilde{u}(\tau, \omega, y)$ as given in (19) and inverting the transformation (9).

4 Black–Scholes in an illiquid market

The Black–Scholes Equation

$$2u_t + \sigma^2 x^2 u_{xx} + bxu_x - ru = 0$$

was constructed under certain assumptions regarding the nature of the market. A departure from those assumptions can be found in the case of what is known as an illiquid market which has been modelled in [11–13,28]. One cause can be found in the existence of a single trader dominating the market for which the equation governing the model is [4]

$$2u_t\left(1-\rho x\lambda(x)u_{xx}\right)^2+\sigma^2(t)x^2u_{xx}=0, \tag{20}$$

where ρ is a measure of the influence of the large-scale trader and $\lambda(x)$ is intended to produce the desired payoff, but both must necessarily be estimated from the realities of the market. One notes that the variance includes a dependence upon time. In terms of the analysis of the equation this is quite spurious since it may be removed by means of a rescaling of time. Consequently we treat it as a constant.

Equation (20) has been treated in terms of symmetry by Yang *et al* [30] in that they constructed the optimal systems for the equation. Here we consider (20) in terms of a problem with a standard terminal condition and with the option of a terminal condition dependent upon the stock price. Yang *et al* present the Lie point symmetries of (20) as[7]

$$\Gamma_1=\partial_t,\quad \Gamma_2=\partial_u,\quad \Gamma_3=x\partial_u,\quad \Gamma_4=x\partial_x+(1-k)u\partial_u$$

provided $\lambda(x)=\omega x^k$, where ω is a constant. If this be not the case, Γ_4 is absent. The algebra has a structure which does depend upon the value of the parameter k. It is $A_1\oplus A_{3,5}^a$ in the Mubarakzyanov Classification Scheme for general values of k. For the specific values $k=0$ and $k=1$ the algebra is $2A_1\oplus A_2$ and, if $k=\frac{1}{2}$, the algebra is $A_1\oplus A_{3,4}$, where the three-dimensional subalgebra is commonly known as $E(1,1)$. Note that in all cases the algebra is the direct sum of a one-dimensional subalgebra (Γ_1) and a three-dimensional subalgebra.

We can contemplate (20) as the differential equation for a problem with a terminal condition of the nature $u(T,\,S)=U(x)$, *ie* at some time T in the future the dependent variable is required to take a specific value depending upon the price of the underlying asset. We take a linear combination, $\sum_{i=1}^4\alpha_i\Gamma_i$, of the symmetries above and apply them to the conditions $t=T$ and $u(T,\,S)=U(x)$. It is immediately obvious that $\alpha_1=0$. The remaining coefficients are related according to

$$\alpha_2+\alpha_3x+\alpha_4(1-k)U(x)=\alpha_4xU'(x)$$

from which it is evident that the terminal function, $U(x)$, is compatible with the symmetry provided

$$U(x)=U_0x^{1-k}+\frac{1}{\alpha_4}\left(\frac{\alpha_3x}{k}-\frac{\alpha_2}{1-k}\right),$$

ie in addition to the parameter, k, of the model there are three constants in the expression for $U(x)$ which may be chosen at will.

[7]We have verified the correctness of the calculation using our own methods. Yang *et al.* cite Bordag [4] as their source of these symmetries.

The corresponding symmetry is

$$\Gamma = x\partial_x + \left(\frac{\alpha_2}{\alpha_4} + \frac{\alpha_3}{\alpha_4}x + (1-k)u\right)\partial_u.$$

The characteristics are t and

$$\frac{u}{x^{1-k}} - \frac{1}{\alpha_4}\left\{\frac{\alpha_3}{k}x^k - \frac{\alpha_2}{(1-k)}x^{k-1}\right\}$$

so that the reduction to an ordinary differential equation is given by

$$u(t,\,x) = \frac{1}{\alpha_4}\left(\frac{\alpha_3}{k}x - \frac{\alpha_2}{1-k}\right) + x^{1-k}q(t).$$

The reduced equation is

$$\dot{q}(t) = \frac{k\sigma^2(1-k)q(t)}{2\left(1+k\rho\omega(1-k)q(t)\right)^2}$$

which, not surprisingly, cannot be solved in closed form for $q(t)$. The solution in implicit form is

$$t - t_0 = \frac{2}{\sigma^2 k(1-k)}\log q(t) + \frac{4\rho\omega}{\sigma^2}q(t) + \frac{\rho^2\omega^2}{\sigma^2}k(1-k)q^2(t).$$

We may, probably without loss of generality, take $t_0 = T$. This means that $q(T) = U_0$ which is required to satisfy the equation

$$0 = \frac{2}{\sigma^2 k(1-k)}\log U_0 + \frac{4\rho\omega}{\sigma^2}U_0 + \frac{\rho^2\omega^2}{\sigma^2}k(1-k)U_0.$$

If one assumes that the parameters ρ, ω and σ are beyond the control even of a dominant trader, there is a necessary relationship between k and U_0 which may or may not be good news for the market.

5 Conclusion

We have considered some examples chosen from the many models which have been developed in the area now known as Financial Mathematics. Although one may trace the development of the field back many years, it is generally agreed that the real thrust in development is to be found in the papers of Merton [21] and Black and Scholes [1, 2] of about forty years ago. The original context was in the pricing of options as interpreted in the narrow sense of the stock market. Indeed Merton observes that 'since options are specialised and relatively unimportant financial securities, the amount of time and space devoted to the development of a pricing theory might be questioned.' At about the same time Black and Scholes had observed that their results could be extended to many other situations and, in a sense, that virtually every financial instrument could be regarded in terms of an option. It reminds one of the story about how a man views a doughnut! Given the explosion of the field in recent decades it would appear that Black and Scholes had a greater understanding of the practical implications of their models.

Traditionally – if tradition be already established within forty years – the solvers of the evolution partial differential equations which emerge at the end of what can be a very long and complicated process of modelling have used *ad hoc* methods to solve the equations. Experience counts. What has been observed is that many of these equations are rich in symmetry. This in itself is not likely to be *a priori* expected although one could argue from hindsight that the underlying processes are very similar in nature to those physical processes which lead through their modelling to the classical heat equation and its variations.

The employment of the methods of symmetry analysis has become standard in some fields. In others, we mention Financial Mathematics, Epidemiology and Ecology in particular, there seems to be not only a marked reluctance to employ these methods but even to reject them when they are employed. Since so many of the equations which arise in Financial Mathematics are rich in symmetry, one finds that this attitude rather strange.

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[1] Black F. and Scholes M., The valuation of option contracts and a test of marketing efficiency, *J. Financ.*, 1972, V.27, 399–417.

[2] Black F. and Scholes M., The pricing of options and corporate liabilities, *J. Polit. Econ.*, 1973, V.81, 637–659.

[3] Bluman G.W. and Kumei S., *Symmetries and Differential Equations*, Applied Mathematical Sciences, V.81, Springer-Verlag, New York, 1989.

[4] Bordag L.A., Symmetry reductions of a nonlinear option pricing model, 2006, arXiv: math/0604207.

[5] Chan K., Karolyi A., Longstaff F. and Sanders A., An empirical comparison of alternate models of the short-term interest rate, *J. Finance*, 1992, V.47, 1209–1227.

[6] Cox J.C., Ingersoll J.E. and Ross S.A., An intertemporal general equilibrium model of asset prices, *Econometrica*, 1985, V.53, 363–384.

[7] Dimas S. and Tsoubelis D., SYM: A new symmetry-finding package for Mathematica, *Proceedings of Tenth International Conference in Modern Group Analysis (Larnaca, Cyprus, 2004)*, 2005, 64–70.

[8] Dimas S. and Tsoubelis D., A new Mathematica-based program for solving overdetermined systems of PDEs, *8th International Mathematica Symposium (Avignon, France)*, 2006.

[9] Dimas S., Andriopoulos K., Tsoubelis D. and Leach P.G.L., Complete specification of some partial differential equations that arise in Financial Mathematics, *J. Nonlinear Math. Phys.*, 2009, V.16, s01, 73–92.

[10] Dothan U.L., On the term structure of interest rates, *J. Financ. Econ.*, 1978, V.6, 59–69.

[11] Frey R., *Perfect Option Replication for a Large Trader*, thesis, ETH, Zürich, 1996.

[12] Frey R. and Stremme A., Market volatility and feedback effects from dynamic hedging, *Math. Finance*, 1997, V.7, 351–374.

[13] Frey R., *Market illiquidity as a source of model risk in dynamic hedging, model risk*, Risk Publication, London, 2000, 125–136.

[14] Feynman R.J., Space-time approach to nonrelativistic quantum mechanics, *Rev. Modern Physics*, 1948, V.20, 367–387.

[15] Fokker A.D., Die mittlere Energie rotierender elektrischer Dipole im Strahlungsfeld, *Annalen der Physik*, 1914, V.43, 810–820.

[16] Goard J., New solutions to the bond-pricing equation via Lie's classical method, *Math. Comput. Modelling*, 2000, V.32, 299–313.

[17] Heston S.L., A closed-form solution for options with stochastic volatility and applications to bond and currency options, *Rev. Fin. Studies*, 1993, V.6, 327–343.

[18] Itô K., Stochastic Integral, *Proc. Imp. Acad. Tokyo*, 1944, V.20, 519–524.

[19] Kac M., On the distribution of certain Wiener functionals, *Trans. Amer. Math. Soc.*, 1941, V.65, 1–13.

[20] Leach P.G.L., Berry's phase and wave functions for time-dependent Hamiltonian systems, *J. Phys. A: Math. Gen.*, 1990, V 23, 2695–2699.

[21] Merton R.C., Theory of rational option pricing, *Bell J. Econom. and Management Sci.*, 1973, V 4, 141–183.

[22] Morozov V.V., Classification of six-dimensional nilpotent Lie algebras, *Izv. Vyssh. Uchebn. Zavend. Matematika*, 1958, no. 4 (5), 161–171.

[23] Mubarakzyanov G.M., On solvable Lie algebras, *Izv. Vyssh. Uchebn. Zavend. Matematika*, 1963, no. 1 (32), 114–123.

[24] Mubarakzyanov G.M., Classification of real structures of Lie algebras of fifth order, *Izv. Vyssh. Uchebn. Zavend. Matematika*, 1963, no. 3 (34), 99–106.

[25] Mubarakzyanov G.M., Classification of solvable Lie algebras of sixth order with a non-nilpotent basis element, *Izv. Vyssh. Uchebn. Zavend. Matematika*, 1963, no. 4 (35), 104–116.

[26] Planck M., Über einer Satz der statistischen Dynamik und seine Erwerterung in Quanten Theorie, *Sitzungbereich Koenigliche Preussische Akademie der Wissenschaften physik-mathematik Klasse*, 1917, V.24, 324–341.

[27] Pooe C.A., Mahomed F.M. and Wafo Soh C., Fundamental solutions for zero-coupon bond-pricing models, *Nonlinear Dynam.*, 2004, V.36, 69–76.

[28] Rodrigo M.R. and Mamon R.S., An alternative approach to solving the Black–Scholes equation with time-varying parameters, *Appl. Math. Lett.*, 2006, V.19, 398–402.

[29] Sophocleous C., O'Hara J.G. and Leach P.G.L., Stochastic volatility: A critical feature of the Heston model, to appear.

[30] Yang X.-L., Zhang S.-L. and Qu C.-Z., Symmetry-breaking for Black–Scholes Equations, *Commun. Theor. Phys.*, 2007, V.47, 995–1000.

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Orbit functions of SU(n) and Chebyshev polynomials

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Orbit functions of a simple Lie group/algebra L consist of exponential functions summed over the Weyl group of L. They are labeled by the highest weights of irreducible finite-dimensional representations of L. They are of three types: C-, S- and E-functions. Orbit functions of the Lie algebras A_n or, equivalently, of the Lie group, SU($n+1$), are considered. Firstly orbit functions in two different bases – one orthonormal, the other given by the simple roots of SU($n+1$) – are written using the isomorphism of the permutation group of $n+1$ elements and the Weyl group of SU($n+1$). Secondly it is demonstrated that there is a one-to-one correspondence between classical Chebyshev polynomials of the first and second kinds and C- and S-functions of the simple Lie group SU(2). It is then shown that the well-known orbit functions of SU($n+1$) are straightforward generalizations of Chebyshev polynomials to n variables. Properties of the orbit functions provide a wealth of properties of the polynomials. Finally multivariate exponential functions are considered, and their connection with orbit functions of SU($n+1$) is established.

1 Introduction

The history of the Chebyshev polynomials dates back over a century. Their properties and applications have been considered in many papers. Studies of polynomials in more than one variable were undertaken by several authors, e.g., [2–4, 13, 17, 18, 25, 26]. Of these none follow the path we have laid down here.

In this paper we demonstrate that the classical Chebyshev polynomials in one variable are naturally associated with the action of the Weyl group of SU(2) or equivalently with the action of the Weyl group $W(A_1)$ of the simple Lie algebra of type A_1. The association is so simple that it has been ignored so far. However, by making $W(A_1)$ the cornerstone of our rederivation of Chebyshev polynomials, we have gained insight into the structure of the theory of polynomials. In particular the generalization of Chebyshev polynomials to any number of variables was a straightforward task. The polynomials of [14] correspond to our case in spite of a different approach and terminology. The $2D$ generalizations of Chebyshev polynomials of [13] coincide with our polynomials of A_2 even if no Lie group connection is mentioned there. n-dimensional generalizations of Chebyshev polynomials were also constructed in [15]. Further it was shown in [21] that our polynomials obtained from orbit functions are special cases of the Macdonald polynomials [17].

We proceed in three steps. In Section 2 we exploit the isomorphism of the group of permutations of $n+1$ elements S and the Weyl group of SU$(n+1)$ or, equivalently of A_n, and define the orbit functions of A_n. This opens the possibility to write the orbit functions in two rather different bases, the orthonormal basis and the basis determined by the simple roots of A_n, which considerably alters the appearance of the orbit functions. In the paper we use the nonorthogonal basis because of its direct generalization to simple Lie algebras of types other than A_n.

In Section 3 we consider classical Chebyshev polynomials of the first and second kinds and compare them with the C- and S-orbit functions of A_1. We show that polynomials of the first kind are in one-to-one correspondence with C-functions. Polynomials of the second kind coincide with the appropriate S-function divided by the unique lowest nontrivial S-function. We point out that polynomials of the second kind can be identified as irreducible characters of finite-dimensional representations of SU(2). Useful properties of Chebyshev polynomials can undoubtedly be traced to that identification because the fundamental object of representation theory of semisimple Lie groups/algebras is character. In principle all one needs to know about an irreducible finite-dimensional representation can be deduced from its character. An important aspect of this conclusion is that characters are known and uniformly described for all simple Lie groups/algebras.

In Section 4 we provide details of the recursive procedure from which the classical form of Chebyshev polynomials in n variables can be found. Thus there are n generic recursion relations for A_n, having at least $n+2$ terms, and at most $\binom{n+1}{[(n+1)/2]}+1$ terms. Irreducible polynomials are divided into $n+1$ exclusive classes with the property that monomials within one irreducible polynomial belong to the same congruence class[1]. This follows directly from the recognition of the presence and properties of the underlying Lie algebra. In Section 4.2 the simple substitution $z=e^{2\pi ix}$, $x\in\mathbb{R}^n$, is used in orbit functions to form Laurent analogs of Chebyshev polynomials in n variables in their nontrigonometric form.

In Section 5 we present the orbit functions of A_n disguised as polynomials built from multivariate exponential functions of the symmetric group. In Section 2 such a possibility is described in terms of related bases, one orthonormal (symmetric group) and the other nonorthogonal (simple roots of A_n and their dual ω-basis). Both forms of the same polynomials appear rather different, but may prove useful in different situations.

The last section contains a few comments and some questions related to the subject of this paper that we find intriguing.

[1]It is well known that each Chebyshev polynomial has only even or odd power monomials. This is caused by two congruence classes of A_1.

2 Preliminaries

This section is intended to fix notation and terminology. We also briefly recall some facts about S_{n+1} and A_n, dwelling particularly on various bases in $\mathbb{R}^{n+1}$ and $\mathbb{R}^n$. In Section 2.3 we identify elementary reflections that generate the A_n Weyl group W with the permutation of two adjacent objects in an ordered set of $n+1$ objects. Finally we present some standard definitions and properties of orbit functions.

2.1 Permutation group S_{n+1}

The group S_{n+1} of order $(n+1)!$ transforms the ordered number set $[l_1, \dots, l_n, l_{n+1}]$ by permuting the numbers.

We introduce an orthonormal basis in the real Euclidean space $\mathbb{R}^{n+1}$,

$$e_i \in \mathbb{R}^{n+1}, \quad \langle e_i, e_j \rangle = \delta_{ij}, \quad 1 \le i, j \le n+1, \tag{1}$$

and use the l_k's as the coordinates of a point μ in the e-basis: $\mu = \sum_{k=1}^{n+1} l_k e_k$, $l_k \in \mathbb{R}$.

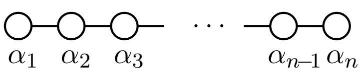
The group S_{n+1} permutes the coordinates l_k of μ thereby generating other points from it. The set of all distinct points, obtained by application of S_{n+1} to μ, is called the orbit of S_{n+1}. We denote an orbit by W_λ, where λ is a unique point of the orbit, such that $l_1 \ge l_2 \ge \dots \ge l_n \ge l_{n+1}$. If there is no pair of equal l_k's in λ, the orbit W_λ consists of $(n+1)!$ points.

Below we only consider points μ from the n-dimensional subspace $\mathcal{H} \subset \mathbb{R}^{n+1}$ defined by the equation

$$\sum_{k=1}^{n+1} l_k = 0. \tag{2}$$

2.2 Lie algebra A_n

We recall basic properties of the simple Lie algebra A_n of the compact Lie group SU$(n+1)$. Consider the general value $(1 \le n < \infty)$ of the rank. The Coxeter–Dynkin diagram, Cartan matrix $\mathfrak{C}$ and inverse Cartan matrix $\mathfrak{C}^{-1}$ of A_n are, respectively,



$$\mathfrak{C} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$\mathfrak{C}^{-1} = \frac{1}{n+1} \begin{pmatrix} 1\cdot n & 1\cdot(n-1) & 1\cdot(n-2) & 1\cdot(n-3) & \dots & 1\cdot 3 & 1\cdot 2 & 1\cdot 1 \\ 1\cdot(n-1) & 2\cdot(n-1) & 2\cdot(n-2) & 2\cdot(n-3) & \dots & 2\cdot 3 & 2\cdot 2 & 2\cdot 1 \\ 1\cdot(n-2) & 2\cdot(n-2) & 3\cdot(n-2) & 3\cdot(n-3) & \dots & 3\cdot 3 & 3\cdot 2 & 3\cdot 1 \\ 1\cdot(n-3) & 2\cdot(n-3) & 3\cdot(n-3) & 4\cdot(n-3) & \dots & 4\cdot 3 & 4\cdot 2 & 4\cdot 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 1\cdot 3 & 2\cdot 3 & 3\cdot 3 & 4\cdot 3 & \dots & (n-2)\cdot 3 & (n-2)\cdot 2 & (n-2)\cdot 1 \\ 1\cdot 2 & 2\cdot 2 & 3\cdot 2 & 4\cdot 2 & \dots & (n-2)\cdot 2 & (n-1)\cdot 2 & (n-1)\cdot 1 \\ 1\cdot 1 & 2\cdot 1 & 3\cdot 1 & 4\cdot 1 & \dots & (n-2)\cdot 1 & (n-1)\cdot 1 & n\cdot 1 \end{pmatrix}.$$

The simple roots α_i, $1 \le i \le n$ of A_n form a basis (α-basis) of a real Euclidean space $\mathbb{R}^n$. We choose them in $\mathcal{H}$:

$$\alpha_i = e_i - e_{i+1}, \quad i = 1, \ldots, n.$$

This choice fixes the lengths and relative angles of the simple roots. Their length is equal to $\sqrt{2}$ with relative angles between α_k and α_{k+1} $(1 \le k \le n-1)$ equal to $\frac{2\pi}{3}$ and $\frac{\pi}{2}$ for any other pair.

In addition to e- and α-bases we introduce the ω-basis as the $\mathbb{Z}$-dual basis to the simple roots α_i:

$$\langle \alpha_i, \omega_j \rangle = \delta_{ij}, \quad 1 \le i, j \le n.$$

It is also a basis in the subspace $\mathcal{H} \subset \mathbb{R}^{n+1}$ (see (2)). The bases α and ω are related by the Cartan matrix

$$\alpha = \mathfrak{C}\omega, \quad \omega = \mathfrak{C}^{-1}\alpha.$$

Throughout the paper we use $\lambda \in \mathcal{H}$. Here we fix the notation for its coordinates relative to the e- and ω-bases:

$$\lambda = \sum_{j=1}^{n+1} l_j e_j =: (l_1, \ldots, l_{n+1})_e = \sum_{i=1}^{n} \lambda_i \omega_i =: (\lambda_1, \ldots, \lambda_n)_\omega, \quad \sum_{i=1}^{n+1} l_i = 0.$$

Consider a point $\lambda \in \mathcal{H}$ with coordinates l_j and λ_i in the e- and ω-bases, respectively. Using $\alpha = \mathfrak{C}\omega$, i.e., $\omega_i = \sum_{k=1}^{n} (\mathfrak{C}^{-1})_{ik} \alpha_k$, we obtain the relations between λ_i and l_j:

$$l_1 = \sum_{k=1}^{n} \lambda_k \mathfrak{C}^{-1}_{k1}, \quad l_{n+1} = -\sum_{k=1}^{n} \lambda_k \mathfrak{C}^{-1}_{kn},$$
$$l_j = \lambda_1(\mathfrak{C}^{-1}_{1\,j} - \mathfrak{C}^{-1}_{1\,j-1}) + \lambda_2(\mathfrak{C}^{-1}_{2\,j} - \mathfrak{C}^{-1}_{2\,j-1}) + \cdots + \lambda_n(\mathfrak{C}^{-1}_{n\,j} - \mathfrak{C}^{-1}_{n\,j-1}), \quad j=2, \ldots, n,$$

or explicitly

$$\lambda_i = l_i - l_{i+1}, \quad i = 1, 2, \ldots, n. \tag{3}$$

The inverse formulas are much more complicated being

$$l = A\lambda, \tag{4}$$

where $l = (l_1, \ldots, 1_{n+1})_e$, $\lambda = (\lambda_1, \ldots, \lambda_n)_\omega$ and A is the $(n+1) \times n$ matrix

$$A = \frac{1}{n+1} \begin{pmatrix} n & n-1 & n-2 & \cdots & 2 & 1 \\ -1 & n-1 & n-2 & \cdots & 2 & 1 \\ -1 & -2 & n-2 & \cdots & 2 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -1 & -2 & -3 & \cdots & -(n-1) & 1 \\ -1 & -2 & -3 & \cdots & -(n-1) & -n \end{pmatrix}.$$

2.3 The Weyl group of A_n

The Weyl group $W(A_n)$ of order $(n+1)!$ acts in $\mathcal{H}$ by permuting coordinates in the e-basis, i.e. as the group S_{n+1}. Indeed let r_i, $1 \le i \le n$ be the generating elements of $W(A_n)$, i.e. reflections with respect to the hyperplanes perpendicular to α_i and passing through the origin. Let $x = \sum_{k=1}^{n+1} x_k e_k = (x_1, x_2, \dots, x_{n+1})_e$ and $\langle \cdot, \cdot \rangle$ denote the inner product. We then have the reflection by r_i:

$$\begin{aligned} r_i x &= x - \tfrac{2}{\langle \alpha_i, \alpha_i \rangle} \langle x, \alpha_i \rangle \alpha_i = (x_1, x_2, \dots, x_{n+1})_e - (x_i - x_{i+1})(e_i - e_{i+1}) \\ &= (x_1, \dots, x_{i-1}, x_{i+1}, x_i, x_{i+2}, \dots, x_{n+1})_e. \end{aligned} \tag{5}$$

Such transpositions generate the full permutation group S_{n+1}. Thus $W(A_n)$ is isomorphic to S_{n+1} and the points of the orbit $W_\lambda(\mathrm{S}_{n+1})$ and $W_\lambda(A_n)$ coincide.

2.4 Definitions of orbit functions

The notion of an orbit function in n variables depends essentially on the underlying semisimple Lie group G of rank n. In our case $G = \mathrm{SU}(n+1)$ (equivalently, Lie algebra A_n). Let the basis of the simple roots be denoted by α and the basis of fundamental weights by ω.

The *weight lattice* P is formed by all integer linear combinations of the ω-basis, i.e.,

$$P = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2 + \cdots + \mathbb{Z}\omega_n.$$

In the weight lattice, P, we define the *cone of dominant weights* P^+ and its subset of strictly dominant weights P^{++} as

$$P \supset P^+ = \mathbb{Z}^{\ge 0}\omega_1 + \cdots + \mathbb{Z}^{\ge 0}\omega_n \supset P^{++} = \mathbb{Z}^{>0}\omega_1 + \cdots + \mathbb{Z}^{>0}\omega_n.$$

Hereinafter $W^e \subset W$ denotes the *even subgroup* of the Weyl group formed by an even number of reflections that generate W. W_λ and W_λ^e are the corresponding group orbits of a point $\lambda \in \mathbb{R}^n$.

We also introduce the notion of fundamental region $F(G) \subset \mathbb{R}^n$. For A_n the *fundamental region* F is the convex hull of the vertices $\{0, \omega_1, \omega_2, \dots, \omega_n\}$.

Definition 1. The C orbit function $C_\lambda(x)$, $\lambda \in P^+$ is defined as

$$C_\lambda(x) := \sum_{\mu \in W_\lambda(G)} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n. \tag{6}$$

Definition 2. The S orbit function $S_\lambda(x)$, $\lambda \in P^{++}$ is defined as

$$S_\lambda(x) := \sum_{\mu \in W_\lambda(G)} (-1)^{p(\mu)} e^{2\pi i \langle \mu, x \rangle}, \quad x \in \mathbb{R}^n, \tag{7}$$

where $p(\mu)$ is the number of reflections necessary to obtain μ from λ. Of course the same μ can be obtained by different successions of reflections, but all routes from λ to μ have a length of the same parity and thus the salient detail given by $p(\mu)$ in the context of an S-function is meaningful and unchanging.

Definition 3. We define E orbit function $E_\lambda(x)$, $\lambda \in P^e$, as

$$E_\lambda(x) := \sum_{\mu\in W^e_\lambda(G)} e^{2\pi i\langle\mu,x\rangle}, \quad x \in \mathbb{R}^n, \tag{8}$$

where $P^e := P^+ \cup r_i P^+$ and r_i is a reflection from W.

If we always suppose that $\lambda, \mu \in P$ are given in the ω-basis and $x \in \mathbb{R}^n$ is given in the α basis, namely $\lambda = \sum\limits_{j=1}^n \lambda_j\omega_j$, $\mu = \sum\limits_{j=1}^n \mu_j\omega_j$, $\lambda_j, \mu_j \in \mathbb{Z}$ and $x = \sum\limits_{j=1}^n x_j\alpha_j$, $x_j \in \mathbb{R}$, then the orbit functions of A_n have the following forms

$$C_\lambda(x) = \sum_{\mu\in W_\lambda} e^{2\pi i \sum\limits_{j=1}^n \mu_j x_j} = \sum_{\mu\in W_\lambda} \prod_{j=1}^n e^{2\pi i\mu_j x_j}, \tag{9}$$

$$S_\lambda(x) = \sum_{\mu\in W_\lambda} (-1)^{p(\mu)} e^{2\pi i \sum\limits_{j=1}^n \mu_j x_j} = \sum_{\mu\in W_\lambda} (-1)^{p(\mu)} \prod_{j=1}^n e^{2\pi i\mu_j x_j}, \tag{10}$$

$$E_\lambda(x) = \sum_{\mu\in W^e_\lambda} e^{2\pi i \sum\limits_{j=1}^n \mu_j x_j} = \sum_{\mu\in W^e_\lambda} \prod_{j=1}^n e^{2\pi i\mu_j x_j}. \tag{11}$$

2.5 Some properties of orbit functions

For S functions the number of summands is always equal to the size of the Weyl group. Note that in the 1-dimensional case C-, S- and E-functions are respectively a cosine, a sine and an exponential function up to the constant.

All three families of orbit functions are based on semisimple Lie algebras. The number of variables coincides with the rank of the Lie algebra. In general C-, S- and E- functions are finite sums of exponential functions. Therefore they are continuous and have continuous derivatives of all orders in $\mathbb{R}^n$.

The S-functions are antisymmetric with respect to the $(n-1)$-dimensional boundary of F. Hence they are zero on the boundary of F. The C-functions are symmetric with respect to the $(n-1)$-dimensional boundary of F. Their normal derivative at the boundary is equal to zero (because the normal derivative of a C-function is an S-function).

For simple Lie algebras of any type the functions $C_\lambda(x)$, $E_\lambda(x)$ and $S_\lambda(x)$ are eigenfunctions of the appropriate Laplace operator. The Laplace operator has the same eigenvalues on every exponential function summand of an orbit function with eigenvalue $-4\pi\langle\lambda, \lambda\rangle$.

2.5.1 Orthogonality

For any two complex square-integrable functions $\phi(x)$ and $\psi(x)$ defined on the fundamental region, F, we define a continuous scalar product as

$$\langle \phi(x), \psi(x) \rangle := \int\limits_F \phi(x)\overline{\psi(x)}\mathrm{d}x. \tag{12}$$

Here integration is performed with respect to the Euclidean measure, the bar means complex conjugation and $x \in F$, where F is the fundamental region of either W or W^e (note that the fundamental region of W^e is $F^e = F \cup r_i F$, where $r_i \in W$).

Any pair of orbit functions from the same family is orthogonal on the corresponding fundamental region with respect to the scalar product (12), namely

$$\langle C_\lambda(x), C_{\lambda'}(x) \rangle = |W_\lambda| \cdot |F| \cdot \delta_{\lambda\lambda'}, \tag{13}$$
$$\langle S_\lambda(x), S_{\lambda'}(x) \rangle = |W| \cdot |F| \cdot \delta_{\lambda\lambda'}, \tag{14}$$
$$\langle E_\lambda(x), E_{\lambda'}(x) \rangle = |W^e_\lambda| \cdot |F^e| \cdot \delta_{\lambda\lambda'}, \tag{15}$$

where $\delta_{\lambda\lambda'}$ is the Kronecker delta, $|W|$ is the order of the Weyl group, $|W_\lambda|$ and $|W^e_\lambda|$ are the sizes of the Weyl group orbits (the number of distinct points in the orbit) and $|F|$ and $|F^e|$ are volumes of fundamental regions. The volume $|F|$ was calculated in [6].

Proof. Proof of the relations (13), (14) and (15) follows from the orthogonality of the usual exponential functions and from the fact that a given weight $\mu \in P$ belongs to precisely one orbit function. ■

The families of C-, S- and E-functions are complete on the fundamental domain. The completeness of these systems follows from the completeness of the system of exponential functions, i.e., there does not exist a function $\phi(x)$ such that $\langle \phi(x), \phi(x) \rangle > 0$ and at the same time $\langle \phi(x), \psi(x) \rangle = 0$ for all functions $\psi(x)$ from the same system.

2.5.2 Orbit functions of A_n acting in $\mathbb{R}^{n+1}$

Relations (4) allow us to rewrite variables λ and x in an orbit function in the e-basis. Therefore we can obtain the C-, S- and E- functions acting in $\mathbb{R}^{n+1}$,

$$C_\lambda(x) = \sum_{s \in \mathrm{S}_{n+1}} e^{2\pi i (s(\lambda), x)}, \tag{16}$$

$$S_\lambda(x) = \sum_{s \in \mathrm{S}_{n+1}} (\operatorname{sgn} s) e^{2\pi i (s(\lambda), x)}, \tag{17}$$

$$E_\lambda(x) = \sum_{s \in \mathrm{Alt}_{n+1}} e^{2\pi i (s(\lambda), x)}, \tag{18}$$

where $(\cdot,\cdot)$ is a scalar product in $\mathbb{R}^{n+1}$, $\operatorname{sgn} s$ is the permutation sign and Alt_{n+1} is the alternating group acting on an $(n+1)$-tuple of numbers. Note that variables x and λ are in the hyperplane $\mathcal{H}$. For C- and E-functions it is essential that λ is a generic point (i.e., does not have zero coordinates in ω-basis).

When one uses the identity $\langle \lambda, r_i x\rangle = \langle r_i\lambda, x\rangle$ for the reflection r_i, $i=1,\dots,n$, it can be verified that

$$C_\lambda(r_i x) = C_{r_i\lambda}(x) = C_\lambda(x) \quad \text{and} \quad S_{r_i\lambda}(x) = S_\lambda(r_i x) = -S_\lambda(x). \tag{19}$$

Note that it is easy to see for generic points that $E_\lambda(x) = \frac{1}{2}\big(C_\lambda(x) + S_\lambda(x)\big)$ and from the relations (19) we obtain

$$E_{r_i\lambda}(x) = E_\lambda(r_i x) = \tfrac{1}{2}\left(C_\lambda(x) - S_\lambda(x)\right) = E_\lambda(x). \tag{20}$$

A number of other properties of orbit functions are presented in [8, 9, 11].

3 Orbit functions and Chebyshev polynomials

We recall known properties of Chebyshev polynomials [24] in order to be able subsequently to make an unambiguous comparison between them and the appropriate orbit functions.

3.1 Classical Chebyshev polynomials

Chebyshev polynomials are orthogonal polynomials which are usually defined recursively. One distinguishes between Chebyshev polynomials of the first kind T_n,

$$T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1}(x) = 2xT_n - T_{n-1}, \tag{21}$$

$$\text{hence} \quad T_2(x) = 2x^2 - 1, \quad T_3(x) = 4x^3 - 3x, \quad \dots, \tag{22}$$

and Chebyshev polynomials of the second kind U_n,

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad U_{n+1}(x) = 2xU_n - U_{n-1}, \tag{23}$$

$$\text{in particular} \quad U_2(x) = 4x^2 - 1, \quad U_3(x) = 8x^3 - 4x, \quad \dots. \tag{24}$$

The polynomials T_n and U_n are of degree n in the variable x. All terms in a polynomial have the parity of n. The coefficient of the leading term of T_n is 2^{n-1} and 2^n for U_n, $n = 1, 2, 3, \dots$.

The roots of the Chebyshev polynomials of the first kind are widely used as nodes for polynomial interpolation in approximation theory. The Chebyshev polynomials are a special case of Jacobi polynomials. They are orthogonal with the following weight functions:

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} T_n(x)T_m(x)\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ \pi, & n = m = 0, \\ \frac{\pi}{2}, & n = m \neq 0, \end{cases} \tag{25}$$

$$\int_{-1}^{1} \sqrt{1-x^2}\, U_n(x)U_m(x)\mathrm{d}x = \begin{cases} 0, & n \neq m, \\ \frac{\pi}{2}, & n = m. \end{cases} \tag{26}$$

There are other useful relations between Chebyshev polynomials of the first and second kinds.

$$\frac{\mathrm{d}}{\mathrm{d}x} T_n(x) = nU_{n-1}(x), \quad n = 1, 2, 3, \dots, \tag{27}$$

$$T_n(x) = \frac{1}{2}(U_n(x) - U_{n-2}(x)), \quad n = 2, 3, \dots, \tag{28}$$

$$T_{n+1}(x) = xT_n(x) - (1 - x^2)U_{n-1}, \quad n = 1, 2, 3, \dots, \tag{29}$$

$$T_n(x) = U_n(x) - xU_{n-1}, \quad n = 1, 2, 3, \dots . \tag{30}$$

3.1.1 Trigonometric form of Chebyshev polynomials

When one introduces the trigonometric variable $x = \cos y$, polynomials of the first kind become

$$T_n(x) = T_n(\cos y) = \cos(ny), \quad n = 0, 1, 2, \dots, \tag{31}$$

and polynomials of the second kind are written as

$$U_n(x) = U_n(\cos y) = \frac{\sin((n+1)y)}{\sin y}, \quad n = 0, 1, 2, \dots . \tag{32}$$

The first few lowest polynomials are

$$\begin{aligned}
T_0(x) &= T_0(\cos y) = \cos(0y) = 1, \\
T_1(x) &= T_1(\cos y) = \cos(y) = x, \\
T_2(x) &= T_2(\cos y) = \cos(2y) = \cos^2 y - \sin^2 y = 2cos^2 y - 1 = 2x^2 - 1; \\
U_0(x) &= U_0(\cos y) = \frac{\sin y}{\sin y} = 1, \\
U_1(x) &= U_1(\cos y) = \frac{\sin(2y)}{\sin y} = 2\cos y = 2x, \\
U_2(x) &= U_2(\cos y) = \frac{\sin(3y)}{\sin y} = \frac{\sin(2y)\cos y + \sin y \cos(2y)}{\sin y} = \\
&= 4\cos^2 y - 1 = 4x^2 - 1.
\end{aligned}$$

3.2 Orbit functions of A_1 and Chebyshev polynomials

We consider the orbit functions of one variable. There is only one simple Lie algebra of rank 1, namely A_1. Our aim is to build the recursion relations in a way that generalizes to higher rank groups, unlike the standard relations of the classical theory presented above.

3.2.1 Orbit functions of A_1 and trigonometric form of T_n and U_n

The orbit of $\lambda = m\omega_1$ has two points for $m \neq 0$, namely $W_\lambda = \{(m), (-m)\}$. The orbit of $\lambda = 0$ has just one point, $W_0 = \{0\}$.

One-dimensional orbit functions have the form (see (9), (10), (11))

$$C_\lambda(x) = e^{2\pi imx}+e^{-2\pi imx} = 2\cos(2\pi mx) = 2\cos(my), \tag{33}$$
$$\text{where} \quad y = 2\pi x,\ m \in \mathbb{Z}^{\geqslant 0};$$
$$S_\lambda(x) = e^{2\pi imx}-e^{-2\pi imx} = 2i\sin(2\pi mx) = 2i\sin(my) \tag{34}$$
$$\text{for} \quad y = 2\pi x,\ m \in \mathbb{Z}^{>0};$$
$$E_\lambda(x) = e^{2\pi imx} = y^m, \quad \text{where} \quad y = e^{2\pi ix},\ m \in \mathbb{Z}. \tag{35}$$

From (33) and (31) it directly follows that polynomials generated from C_m functions of A_1 are doubled Chebyshev polynomials T_m of the first kind for $m = 0, 1, 2, \dots$.

Analogously from (34) and (32) it follows that polynomials $\frac{S_{m+1}}{S_1}$ are Chebyshev polynomials U_m of the second kind for $m = 0, 1, 2, \dots$.

The polynomials generated from E_m functions of A_1 form a standard monomial sequence y^m, $m = 0, 1, 2 \dots$, which is the basis for the vector space of polynomials.

C- and S-orbit functions are orthogonal on the interval $F = [0, 1]$ (see (13) and (14)) which implies the orthogonality of the corresponding polynomials.

3.2.2 Orbit functions of A_1 and their polynomial form

In this subsection we start a derivation of the A_1 polynomials in a way which emphasizes the role of the Lie algebra and, more importantly, in a way that directly generalizes to simple Lie algebras of any rank n and any type resulting in polynomials of n variables and of a new type for each algebra. In the present case of A_1 this leads us to a different normalization of the polynomials and their trigonometric variables than is common for classical Chebyshev polynomials. No new polynomials emerge than those equivalent to Chebyshev polynomials of the first and second kinds. Insight is nevertheless gained into the structure of the problem, which, to us, turned out to be of considerable importance. We are inclined to consider the Chebyshev polynomials in the form derived here as the canonical polynomials.

The underlying Lie algebra A_1 is often denoted $sl(2,\mathbb{C})$ or $su(2)$. In fact this case is so simple that the presence of the Lie algebras has never been acknowledged.

Orbit functions of A_1 are of two types (33) and (34); in particular $C_0(x) = 2$ and $S_0(x) = 0$ for all x.

The simplest substitution of variables to transform the orbit functions into polynomials is $y = e^{2\pi ix}$. Monomials in such a polynomial are y^m and y^{-m}. Instead we introduce new ('trigonometric') variables X and Y as follows:

$$X := C_1(x) = e^{2\pi ix} + e^{-2\pi ix} = 2\cos(2\pi x), \tag{36}$$

$$Y := S_1(x) = e^{2\pi ix} - e^{-2\pi ix} = 2i\sin(2\pi x). \tag{37}$$

We can now start to construct polynomials recursively in the degrees of X and Y by calculating the products of the appropriate orbit functions. Omitting the dependence on x from the symbols we have

$$\begin{array}{lcl} X^2 = C_2 + 2 & \Longrightarrow & C_2 = X^2 - 2, \\ XC_2 = C_3 + X & \Longrightarrow & C_3 = X^3 - 3X, \\ XC_m = C_{m+1} + C_{m-1} & \Longrightarrow & C_{m+1} = XC_m - C_{m-1}, \quad m \geq 3. \end{array} \tag{38}$$

Therefore we obtain the following recursive polynomial form of the C-functions

$$C_0 = 2, \quad C_1 = X, \quad C_2 = X^2 - 2, \quad C_3 = X^3 - 3X, \ \ldots\ . \tag{39}$$

After the substitution $z = \frac{1}{2}X$ we have

$$C_0 = 2\cdot 1, \quad C_1 = 2z, \quad C_2 = 2(2z^2 - 1), \quad C_3 = 2(4z^3 - 3z), \ \ldots\ .$$

Hence we conclude that $C_m = 2T_m$ for $m = 0, 1, \ldots$.

Remark 1. In our opinion the normalization of orbit functions is also more 'natural' for the Chebyshev polynomials. For example the equality $C_2^2 = C_4 + 2$ does not hold for T_2 and T_4.

Remark 2. Each C_m also can be written as a polynomial of degree m in X, Y and S_{m-1}. It suffices to consider the products YS_m. For example $C_2 = Y^2 + 2$, $C_3 = YS_2 + X$ etc. Equating the polynomials obtained in such a way with the corresponding polynomials from (38), we obtain a trigonometric identity for each m. Above we find two ways to write C_2, one from the product X^2 and one from Y^2. Equating the two we get

$$X^2 - Y^2 = 4 \quad \Longleftrightarrow \quad \sin^2(2\pi x) + \cos^2(2\pi x) = 1$$

because Y is defined in (37) to be purely imaginary.

Just as the polynomials representing C_m were obtained above, it is possible to to find polynomial expressions for S_m for all m.

Fundamental relations between the S- and C-orbit functions follow from the properties of the character $\chi_m(x)$ of the irreducible representation of A_1 of dimension $m + 1$.

The character can be written in two ways: as in the Weyl character formula and also as the sum of appropriate C-functions. Explicitly we have the A_1 character:

$$\chi_m(x)=\frac{S_{m+1}(x)}{S_1(x)}=C_m(x)+C_{m-2}(x)+\cdots+\begin{cases} C_2(x)+1 & \text{for } m \text{ even}, \\ C_3(x)+C_1(x) & \text{for } m \text{ odd}. \end{cases}$$

We write down a few characters

$$\chi_0 = \frac{S_1(x)}{S_1(x)} = 1, \quad \chi_1 = \frac{S_2(x)}{S_1(x)} = C_1 = X,$$
$$\chi_2 = \frac{S_3(x)}{S_1(x)} = C_2 + C_0 = X^2 - 1, \quad \chi_3 = \frac{S_4(x)}{S_1(x)} = C_3 + C_1 = X^3 - 2X,$$
$$\chi_4 = \frac{S_5(x)}{S_1(x)} = C_4 + C_2 + C_0 = X^4 - 3X^2 + 1, \quad \dots .$$

Again the substitution $z = \frac{1}{2}X$ transforms these polynomials into the Chebyshev polynomials of the second kind $\frac{S_{m+1}}{S_1} = U_m$, $m = 0, 1, \dots$. Indeed

$$\frac{S_1(x)}{S_1(x)} = 1, \quad \frac{S_2(x)}{S_1(x)} = 2z, \quad \frac{S_3(x)}{S_1(x)} = 4z^2-1, \quad \frac{S_4(x)}{S_1(x)} = 8z^3-4z, \quad \dots .$$

Remark 3. Note that in the character formula we used $C_0 = 1$, while above (see (11) and (39)) we used $C_0 = 2$. It is just a question of normalization of orbit functions. For some applications/calculations it is convenient to scale orbit functions of nongeneric points on the factor equal to the order of the stabilizer of that point in the Weyl group $W(A_1)$.

4 Orbit functions of A_n and their polynomials

This section proposes two approaches to construct orthogonal polynomials of n variables based on orbit functions. The first comes from the decomposition of Weyl orbit products into sums of orbits. Its result is the analog of the trigonometric form of the Chebyshev polynomials. The second approach is the exponential substitution in [8].

4.1 Recursive construction

Since the C- and S- functions are defined for A_n of any rank $n = 1, 2, 3, \dots$, it is natural to take C-functions and the ratio of S-functions as multidimensional generalizations of Chebyshev polynomials of the first and second kinds respectively

$$T_\lambda(x) := C_\lambda(x), \quad x \in \mathbb{R}^n,$$
$$U_\lambda(x) := \frac{S_{\lambda+\rho}(x)}{S_\rho(x)}, \quad \rho = \omega_1 + \omega_2 + \cdots + \omega_n = (1, 1, \dots, 1)_\omega, \quad x \in \mathbb{R}^n,$$

where λ is one of the dominant weights of A_n.

The functions T_λ and U_λ can be constructed as polynomials using the recursive scheme proposed in Section 3.2.2. In the n-dimensional case we start from C-orbit functions of the fundamental weights,

$$X_1 := C_{\omega_1}(x), \quad X_2 := C_{\omega_2}(x), \quad \dots, \quad X_n := C_{\omega_n}(x), \quad x \in \mathbb{R}^n.$$

By decomposing the products $X_j(x)C_\lambda(x)$, $j = 1, 2\dots$, into the sum of orbit functions, we build polynomials for any C- and S-function.

The generic recursion relations are found as the decomposition of the products $C_{\omega_j}C_{(a_1,a_2,\dots,a_n)}$ with 'sufficiently large' $a_1, a_2, \dots, a_n$. Such a recursion relation has $\binom{n+1}{j} + 1$ terms, where $\binom{n+1}{j}$ is the size of the Weyl orbit of ω_j.

An efficient way to find the decompositions is to work with products of Weyl group orbits rather than with orbit functions. Their decomposition has been studied and many examples have been described in [5]. It is useful to be aware of the congruence class of each product because all of the orbits in its decomposition necessarily belong to that class. The *congruence number* # of an orbit λ of A_n, which is also the congruence number of the orbit functions C_λ and S_λ, specifies the class. It is calculated as follows:

$$\#(C_{(a_1,a_2,\dots,a_n)}(x)) = \#(S_{(a_1,a_2,\dots,a_n)}(x)) = \sum_{k=1}^{n} k a_k \mod (n+1). \tag{40}$$

In particular each X_j, where $j = 1, 2, \dots, n$, is in its own congruence class. During the multiplication congruence numbers add $\mathrm{mod}(n + 1)$.

Recursions and polynomials in two and three variables originating from orbit functions of the simple Lie algebras A_2, C_2, G_2, A_3, B_3, and C_3 are obtained in [21] and [22].

4.2 Exponential substitution

There is another approach to multivariate orthogonal polynomials which is also based on orbit functions. Such polynomials can be constructed by the continuous and invertible change of variables,

$$y_j = e^{2\pi i x_j}, \quad x_j \in \mathbb{R}, \quad j = 1, 2, \dots, n. \tag{41}$$

Consider an A_n orbit function $C_\lambda(x)$, $S_\lambda(x)$ or $E_\lambda(x)$, in the case that λ is given in the ω-basis and x is given in the α-basis. Each of these functions consists of summands $\prod_{j=1}^{n} e^{2\pi i \mu_j x_j}$, where $\mu_j \in \mathbb{Z}$ are coordinates of an orbit point μ. Then the summand is transformed by (41) into a monomial of the form $\prod_{j=1}^{n} y_j^{\mu_j}$.

It turns out that that C- and S-polynomials formed by these monomials are the special cases of the Macdonald symmetric polynomials, see [21] for details.

Polynomials of two variables obtained from the orbit functions by the substitution (41) are already described in the literature [13], where they are derived from very different considerations. The detailed comparison is made in the following example.

Example 1. Consider the A_2 Weyl orbits of the lower weights $(0,m)_\omega$, $(m,0)_\omega$ and the orbit of the generic point $(m_1,m_2)_\omega$, $m,m_1,m_2 \in \mathbb{Z}^{>0}$

$$\begin{aligned}
&W_{(0,m)}(A_2) = \{(0,m),\ (-m,0),\ (m,-m)\},\\
&W_{(m,0)}(A_2) = \{(m,0),\ (-m,m),\ (0,-m)\},\\
&W_{(m_1,m_2)}(A_2) = \{(m_1,m_2)^+,\ (-m_1,m_1{+}m_2)^-,\ (m_1{+}m_2,-m_2)^-,\\
&\qquad\qquad (-m_2,-m_1)^-,\ (-m_1{-}m_2,m_1)^+,\ (m_2,-m_1{-}m_2)^+\}.
\end{aligned}$$

Suppose $x=(x_1,x_2)$ is given in the α-basis. Then the orbit functions assume the forms

$$\begin{aligned}
&C_{(0,0)} = 1, \quad C_{(0,m)}(x) = e^{-2\pi i m x_1} + e^{2\pi i m(x_1-x_2)} + e^{2\pi i m x_2},\\
&C_{(m_1,m_2)}(x) = e^{2\pi i m_1 x_1} e^{2\pi i m_2 x_2} + e^{-2\pi i m_1 x_1} e^{2\pi i (m_1+m_2) x_2} +\\
&\qquad e^{2\pi i (m_1+m_2) x_1} e^{-2\pi i m_2 x_2} + e^{-2\pi i m_2 x_1} e^{-2\pi i m_1 x_2} +\\
&\qquad e^{-2\pi i (m_1+m_2) x_1} e^{2\pi i m_1 x_2} + e^{2\pi i m_2 x_1} e^{-2\pi i (m_1+m_2) x_2},
\end{aligned} \tag{42}$$

$$\begin{aligned}
&S_{(m_1,m_2)}(x) = e^{2\pi i m_1 x_1} e^{2\pi i m_2 x_2} - e^{-2\pi i m_1 x_1} e^{2\pi i (m_1+m_2) x_2} -\\
&\qquad e^{2\pi i (m_1+m_2) x_1} e^{-2\pi i m_2 x_2} - e^{-2\pi i m_2 x_1} e^{-2\pi i m_1 x_2} +\\
&\qquad e^{-2\pi i (m_1+m_2) x_1} e^{2\pi i m_1 x_2} + e^{2\pi i m_2 x_1} e^{-2\pi i (m_1+m_2) x_2}.
\end{aligned}$$

The polynomials e^+ and e^- given in (2.6) of [13, III] coincide with those in (42) whenever the correspondence $\sigma = 2\pi x_1$, $\tau = 2\pi x_2$ is made. So both the orbit functions' polynomials of A_2 and $e^\pm$ are orthogonal on the interior of Steiner's hypocycloid and the regular tessellation of the plane by equilateral triangles considered in [13] is the standard tiling of the weight lattice of A_2. The fundamental region R of [13] coincides with the fundamental region $F(A_2)$ in our notations. The corresponding isometry group is the affine Weyl group of A_2.

Furthermore, continuing the comparison with [13], we point out that orbit functions are eigenfunctions of the Laplace operator written in the appropriate basis, e.g. in ω-basis (the corresponding eigenvalues bring $-4\pi^2\langle\lambda,\lambda\rangle$, where λ is the representative from the dominant Weyl chamber, which labels the orbit function). This property holds not only for the Lie algebra A_n and not only for the Laplace operator but also for all simple Lie algebras and for the differential operators built from the elementary symmetric polynomials, see [8, 9].

An independent approach to the polynomials in two variables is proposed in [26] and the generalization of classical Chebyshev polynomials to the case of several variables is also presented in [4].

5 Multivariate exponential functions

In this section we consider one more class of special functions which, as it will be shown, are closely related to orbit functions of A_n. Such a relation allows us to view orbit functions in the orthonormal basis and to represent them in the form of determinants and permanents. At the same time we obtain the straightforward procedure to construct polynomials from multivariate exponential functions.

Definition 4 ([12]). For a fixed point, $\lambda = (l_1, l_2, \ldots, l_{n+1})_e$, such that $l_1 \geq l_2 \geq \cdots \geq l_{n+1}$, $\sum\limits_{k=1}^{n+1} l_k = 0$, the symmetric multivariate exponential function D^+_λ of $x = (x_1, x_2, \ldots, x_{n+1})_e$ is defined as follows

$$D^+_\lambda(x) := \det{}^+ \begin{pmatrix} e^{2\pi i l_1 x_1} & e^{2\pi i l_1 x_2} & \ldots & e^{2\pi i l_1 x_{n+1}} \\ e^{2\pi i l_2 x_1} & e^{2\pi i l_2 x_2} & \ldots & e^{2\pi i l_2 x_{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i l_{n+1} x_1} & e^{2\pi i l_{n+1} x_2} & \ldots & e^{2\pi i l_{n+1} x_{n+1}} \end{pmatrix}.$$

Here $\det^+$ is calculated as a conventional determinant, except that all of its monomial terms are taken with positive sign. It is also called *permanent* [16] or *antideterminant*.

It was shown in [12] that it suffices to consider $D^+_\lambda(x)$ on the hyperplane $x \in \mathcal{H}$ (see (2)). Furthermore, due to the following property of the permanent

$$\det{}^+(a_{ij})_{i,j=1}^m = \sum_{s \in \mathrm{S}_m} a_{1,s(1)} a_{2,s(2)} \cdots a_{m,s(m)} = \sum_{s \in \mathrm{S}_m} a_{s(1),1} a_{s(2),2} \cdots a_{s(m),m},$$

we have

$$D^+_\lambda(x) = \sum_{s \in \mathrm{S}_{n+1}} e^{2\pi i l_1 x_{s(1)}} \cdots e^{2\pi i l_m x_{s(n+1)}} = \sum_{s \in \mathrm{S}_{n+1}} e^{2\pi i(\lambda, s(x))} = \sum_{s \in \mathrm{S}_{n+1}} e^{2\pi i(s(\lambda), x)}.$$

Proposition 1. *For all $\lambda, x \in \mathcal{H} \subset \mathbb{R}^{n+1}$ we have the following connection between the symmetric multivariate exponential functions in $n+1$ variables and C orbit functions of A_n: $D^+_\lambda(x) = kC_\lambda(x)$, where $k = \frac{|W|}{|W_\lambda|}$, $|W|$ and $|W_\lambda|$ are sizes of the Weyl group and Weyl orbit respectively. In particular for generic points $k = 1$.*

Proof. The proof follows from the definitions of the functions C and D^+ (Definitions 1 and 4, respectively) and properties of orbit functions formulated in Section 2.5.2. ■

Definition 5 ([12]). For a fixed point $\lambda = (l_1, l_2, \ldots, l_{n+1})_e$ such that $l_1 \geq l_2 \geq \cdots \geq l_{n+1}$ and $\sum\limits_{k=1}^{n+1} l_k = 0$ the antisymmetric multivariate exponential function D^-_λ

of $x=(x_1, x_2, \dots, x_{n+1})_e \in \mathcal{H}$ is defined as follows

$$D^-_\lambda(x) := \det \begin{pmatrix} e^{2\pi i l_1 x_1} & e^{2\pi i l_1 x_2} & \dots & e^{2\pi i l_1 x_{n+1}} \\ e^{2\pi i l_2 x_1} & e^{2\pi i l_2 x_2} & \dots & e^{2\pi i l_2 x_{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i l_{n+1} x_1} & e^{2\pi i l_{n+1} x_2} & \dots & e^{2\pi i l_{n+1} x_{n+1}} \end{pmatrix} =$$

$$= \sum_{s \in \mathrm{S}_{n+1}} (\operatorname{sgn} s) e^{2\pi i (s(\lambda), x)}, \quad \text{where “sgn” is the permutation sign.}$$

Proposition 2. *For all generic points $\lambda \in \mathcal{H} \subset \mathbb{R}^{n+1}$ we have the connection $D^-_\lambda(x) = S_\lambda(x)$.*

The antisymmetric multivariate exponential functions, D^- and S orbit functions, equal zero for nongeneric points.

Proof. The proof directly follows from the definitions of functions S and D^- (Definitions 2 and 5, respectively) and properties of S functions formulated in Section 2.5.2. ∎

Definition 6 ([7]). The alternating multivariate exponential function, $D^{\mathrm{Alt}}_\lambda(x)$ for $x = (x_1, \dots, x_{n+1})_e$, $\lambda = (l_1, \dots, l_{n+1})_e$, is defined as the function

$$D^{\mathrm{Alt}}_\lambda(x) := \operatorname{sdet} \begin{pmatrix} e^{2\pi i l_1 x_1} & e^{2\pi i l_1 x_2} & \dots & e^{2\pi i l_1 x_{n+1}} \\ e^{2\pi i l_2 x_1} & e^{2\pi i l_2 x_2} & \dots & e^{2\pi i l_2 x_{n+1}} \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i l_{n+1} x_1} & e^{2\pi i l_{n+1} x_2} & \dots & e^{2\pi i l_{n+1} x_{n+1}} \end{pmatrix},$$

where Alt_{n+1} is the alternating group (even subgroup of S_{n+1}) and

$$\operatorname{sdet}\left(e^{2\pi i l_j x_k}\right)_{j,k=1}^{n+1} := \sum_{w \in \mathrm{Alt}_{n+1}} e^{2\pi i l_1 x_{w(1)}} \dots e^{2\pi i l_{n+1} x_{w(n+1)}} = \sum_{w \in \mathrm{Alt}_{n+1}} e^{2\pi i (\lambda, w(x))}.$$

Here (λ, x) denotes the scalar product in the $(n+1)$-dimensional Euclidean space.

Note that Alt_m consists of even substitutions of S_m and is usually denoted as A_m; here we change the notation in order to avoid confusion with the notation for the simple Lie algebra A_n.

It was shown in [7] that it is sufficient to consider the function $D^{\mathrm{Alt}}_\lambda(x)$ on the hyperplane $\mathcal{H}$: $x_1 + x_2 + \dots + x_{n+1} = 0$ for λ such that $l_1 \geq l_2 \geq l_3 \geq \dots \geq l_{n+1}$.

Alternating multivariate exponential functions are obviously connected with symmetric and antisymmetric multivariate exponential functions. This connection is the same as that of the cosine and sine with the exponential function of one variable $D^{\mathrm{Alt}}_\lambda(x) = \frac{1}{2}(D^+_\lambda(x) + D^-_\lambda(x))$.

Proposition 3. *For all generic points $\lambda \in \mathcal{H} \subset \mathbb{R}^{n+1}$, the following relation between the alternate multivariate exponential functions D^{Alt} and E-orbit functions of A_n holds true: $D^{\mathrm{Alt}}_\lambda(x) = E_\lambda(x)$.*

For nongeneric points λ, we have $E_\lambda(x) = C_\lambda(x)$ and therefore $E_\lambda(x) = kD^+_\lambda(x)$, where $k = \frac{|W|}{|W_\lambda|}$.

Proof. The proof directly follows from Definitions 3 and 6, from the relation $E = \frac{1}{2}(C+S)$ and from the properties of orbit functions formulated in Section 2.5.2. ■

6 Concluding remarks

1. Consequences of the identification of W-invariant orbit functions of compact simple Lie groups and multivariable Chebyshev polynomials merit further exploitation. Some of the properties of orbit functions translate readily into properties of Chebyshev polynomials of many variables. However, there are other properties the discovery of which from the theory of polynomials is difficult to imagine. As an example, consider the decomposition of the Chebyshev polynomial of the second kind into the sum of Chebyshev polynomials of the first kind. In one variable it is a familiar problem that can be solved by elementary means. For two and more variables the problem turns out to be equivalent to a more general question about representations of simple Lie groups. In general the coefficients of that sum are the dominant weight multiplicities. Again simple specific cases can be computed, but a sophisticated algorithm is required to deal with it in general [19]. In order to provide a solution for such a problem extensive tables have been prepared [1] (see also references therein).

2. Our approach to the derivation of multidimensional orthogonal polynomials hinges on the knowledge of appropriate recursion relations. The basic mathematical property underlying the existence of the recursion relation is the complete decomposability of products of the orbit functions. Numerous examples of the decompositions of products of orbit functions, involving also other Lie groups than SU(n), were shown elsewhere [8, 9]. An equivalent problem is the decomposition of products of Weyl group orbits [5].

3. Possibility to discretize the polynomials is a consequence of the known discretization of orbit functions. For orbit functions it is a simpler problem in that it is carried out in the real Euclidean space $\mathbb{R}^n$. In principle it carries over to the polynomials, but variables of the polynomials happen to be on the maximal torus of the underlying Lie group. Only in the case of A_1 are the variables are (the imaginary unit multiplying the S-functions can be normalized away). For A_n with $n > 1$ the functions are complex valued. Practical aspects of discretization deserve to be thoroughly investigated. Several important results are already obtained in [6] and [20].

4. For simplicity of formulation we insisted throughout this paper that the underlying Lie group be simple. The extension to compact semisimple Lie group and their Lie algebras is straightforward. Orbit functions are products of orbit functions of simple constituents and different types of orbit functions can be mixed.

5. Polynomials formed from E-functions by the same substitution of variables should be equally interesting once $n > 1$. We know of no analogs of such polynomials in the standard theory of polynomials with more than one variable. Intuitively they would be formed as 'halves' of Chebyshev polynomials although their domain of orthogonality is twice as large as that of Chebyshev polynomials [11].

6. Notions of multivariate trigonometric functions [10] lead us to the idea of new, yet to be defined, classes of W-orbit functions based on trigonometric sine and cosine functions. From another point of view it is possible to introduce new orbit functions in the similar "exponential" manner, but the signs of summands should be chosen in such a way that linear combination of new orbit functions produce multivariate trigonometric functions.

7. Analogs of orbit functions of Weyl groups can be introduced also for the finite Coxeter groups that are not Weyl groups of a simple Lie algebra. Many of the properties of orbit functions extend to these cases. Only their orthogonality, continuous or discrete, has not been shown so far.

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[1] Bremner M.R., Moody R.V. and Patera J., *Tables of Dominant Weight Multiplicities for Representations of Simple Lie Algebras*, Marcel Dekker, New York, 1985.

[2] Dunkl Ch. and Xu Yu., *Orthogonal Polynomials of Several Variables*, Cambridge University Press, New York, 2008.

[3] Dunn K.B. and Lidl R., Multidimensional generalizations of the Chebyshev polynomials. I, II, *Proc. Japan Acad. Ser. A Math. Sci.*, 1980, V.56, no. 4, 154–159, 160–165.

[4] Eier R. and Lidl R., A class of orthogonal polynomiasl in k variables, *Math. Ann.*, 1982, V.260, 93–99.

[5] Háková L., Larouche M. and Patera J., The rings of n-dimensional polytopes, *J. Phys. A: Math. Theor.*, 2008, V.41, 495202.

[6] Hrivnák J. and Patera J., On discretization of tori of compact simple Lie groups, *J. Phys. A: Math. Theor.*, 2009, V.42, 385208, arXiv:0905.2395.

[7] Klimyk A. and Patera J., Alternating group and multivariate exponential functions, *AMS-CRM Proceedings and Lectures Notes Series "Groups and Symmetries: From Neolithic Scots to John McKay"*, 2009, V.47, eds. J. Harnad and P. Winternitz, American Mathematical Society, Providence, RI, 233–246, arXiv:0907.0601.

[8] Klimyk A. and Patera J., Orbit functions, *SIGMA*, 2006, V.2, 006, 60 pp., arXiv: math-ph/0601037.

[9] Klimyk A. and Patera J., Antisymmetric orbit functions, *SIGMA*, 2007, V.3, 023, 83 pp., arXiv:math-ph/0702040.

[10] Klimyk A. and Patera J., (Anti)symmetric multidimensional trigonometric functions and the corresponding Fourier transforms, *J. Math. Phys.*, 2007, V.48, 093504, arXiv:0705.4186.

[11] Klimyk A. and Patera J., E-orbit functions, *SIGMA*, 2008, V.4, 002, 57 pp., arXiv: 0801.0822.

[12] Klimyk A. and Patera J., (Anti)symmetric multidimensional exponential functions and the corresponding Fourier transforms, *J. Phys. A: Math. Theor.*, 2007, V.40, 10473–10489, arXiv:0705.3572.

[13] Koornwinder T.H., Orthogonal polynomials in two variables which are eigenfunctions of two algebraically independent partial differential operators. I–IV, *Indag. Math.*, 1974, V.36, 48–66, 357–381.

[14] Li H. and Xu Yu., Discrete Fourier analysis on fundamental domain of A_d lattice and on simplex in d-variables, *J. Fourier Anal. Appl.*, 2009, V.16, 383–433, arXiv:0809.1079.

[15] Lidl R. and Wells Ch., Chebyshev polynomials in several variables, *J. Reine Angew. Math.*, 1972, V.255, 104–111.

[16] Minc H., *Permanents*, Addison-Wesley, New York, 1978.

[17] Macdonald I.G., Orthogonal polynomials associated with root systems, *Sém. Lothar. Combin.*, 2000/01, V.45, Art. B45a, 40 pp.

[18] Mason J.C. and Handscomb D.C., *Chebyshev Polynomials*, Chapman&Hall/CRC, Boca Raton, FL, 2003.

[19] Moody R.V. and Patera J., Fast recursion formula for weight multiplicities, *Bull. Amer. Math. Soc.*, 1982, V.7, 237–242.

[20] Moody R.V. and Patera J., Cubature formulae for orthogonal polynomials in terms of elements of finite order of compact simple Lie groups, arXiv:1005.2773.

[21] Nesterenko M., Patera J. and Tereszkiewicz A., Orthogonal polynomials of compact simple Lie groups, arXiv:1001.3683v2.

[22] Nesterenko M., Patera J., Szajewska M. and Tereszkiewicz A., Orthogonal polynomials of compact simple Lie groups: Branching rules for polynomials, arXiv:1007.4431v1, 2010.

[23] Patera J., Compact simple Lie groups and theirs C-, S-, and E-transforms, *SIGMA*, 2005, V.1, 025, 6 pp., arXiv:math-ph/0512029.

[24] Rivlin T.J., *The Chebyshef Polynomials*, Wiley, New York, 1974.

[25] Suetin P.K., *Classical Orthogonal Polynomials*, Nauka, Moscow, 1979 (in Russian).

[26] Suetin P.K., *Orthogonal Polynomials in Two Variables*, Gordon and Breach, Amsterdam, 1999.

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Group analysis of equations of axion electrodynamics

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Group classification of systems of nonlinear equations which model a generalized axion electrodynamics is carried out. The related conservation laws are discussed. Using the Inönü–Wigner contraction the nonrelativistic limit of equations of the standard axion electrodynamics is found. An extended class of closed-form solutions for the electromagnetic and axionic fields is presented. It is shown that in spite of the manifest relativistic invariance of the theory such solutions can describe propagation with velocities faster than the velocity of light.

1 Introduction

Group analysis (GA) of differential equations is a nice and fundamental field including a number of internal problems, but maybe its main value consists in the powerful tools presented for construction of solutions of complicated nonlinear problems. Sometimes the GA looks as the only hope to develop the complicated physical (chemical, biological, ...) problem and it is not a pure accident that the classical book of Petrov "New Methods in the General Theory of Relativity" [1] includes the analysis of the low-dimensional Lie algebras and their applications to construction of solutions of partial differential equations (PDEs).

In this paper we present some results obtained with application of the Lie theory to the complicated physical model called axion electrodynamics. We start with physical motivations for this research.

Axions are hypotetical particles belonging to the main candidates to constitute dark matter, see, e.g. [2] and references therein. Additional arguments to investigate axionic theories appeared in solid states physics. It happens that the interaction terms of axionic type appear in the theoretical description of crystalline solids called topological insulators [3]. In addition the axionic hypothesis makes it possible to resolve a fundamental problem of quantum chromodynamics connected with its prediction of the CP symmetry violation in interquark interactions which never was observed experimentally [4–6]. Thus it is interesting to make group analysis of axionic theories which are requested in three fundamental branches of physics.

We present two more motivations which are very inspiring for us. Recently we have found a new solvable model for neutral Dirac fermions [7] and indicate existence of other new integrable models for such particles, but these models involve an external electromagnetic (EM) field which does not satisfy the Maxwell equations with physically reasonable currents. However, these fields satisfy the equations of axion electrodynamics.

Some time ago we described the finite-dimensional indecomposable vector representations of the homogeneous Galilei group and constructed Lagrangians which admit the corresponding symmetries [8–10]. Axion electrodynamics appears precisely to be the relativistic counterpart of some of our models.

In addition axion electrodynamics is a nice and rather complicated mathematical model which certainly needs good group-theoretical grounds. In the present paper we are trying to create such grounds and also to find an extended class of closed-form solutions for the equations of axion electrodynamics.

2 Equations of axion electrodynamics

We start with the Lagrangian of axion electrodynamics:

$$L = \frac{1}{2}p_\mu p^\mu - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{8M}\theta F_{\mu\nu}\tilde{F}^{\mu\nu} + \varphi(\theta). \tag{1}$$

Here $F_{\mu\nu}$ is the vector-potential of the electromagnetic field, $\tilde{F}_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, $p_\mu = \partial_\mu\theta$, θ is the potential of the pseudoscalar axion field, $\varphi(\theta)$ is a function of θ which usually is supposed to be linear and M is a dimensionless constant.

Setting in (1) $\theta = 0$ we obtain the Lagrangian for the Maxwell field. We see that the axion theory is a nonlinear generalization of Maxwell electrodynamics, which includes an additional (pseudo)scalar axion field, θ. The interaction Lagrangian

$$L_{int} = \frac{1}{8M}\theta F_{\mu\nu}\tilde{F}^{\mu\nu}$$

is a (1+3)-dimensional version of the Shern–Simon topological term [11].

We write the Euler–Lagrange equations corresponding to Lagrangian (1):

$$\begin{aligned}
&\nabla\cdot\mathbf{E} = \frac{1}{M}\mathbf{p}\cdot\mathbf{B}, \quad \partial_0\mathbf{E} - \nabla\times\mathbf{B} = \frac{1}{M}(p_0\mathbf{B} + \mathbf{p}\times\mathbf{E}),\\
&\nabla\cdot\mathbf{B} = 0, \quad \partial_0\mathbf{B} + \nabla\times\mathbf{E} = 0, \qquad (2)\\
&\Box\theta = -\frac{1}{M}\mathbf{E}\cdot\mathbf{B} + F, \qquad (3)
\end{aligned}$$

where

$$\mathbf{B} = \{B^1, B^2, B^3\}, \quad \mathbf{E} = \{E^1, E^2, E^3\}, \quad E^a = F^{0a}, \quad B^a = \tilde{F}^{0a},$$
$$F = \frac{\partial\varphi}{\partial\theta}, \quad \Box = \partial_0^2 - \nabla^2, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad p_0 = \frac{\partial\theta}{\partial x_0}, \quad \mathbf{p} = \nabla\theta.$$

In the l.h.s. of (2) we recognize the standard Maxwellian terms. However, in contrast to the usual electrodynamics there are nonlinear interaction terms in the r.h.s. of (2) and (3).

We consider also the system

$$\nabla\cdot\mathbf{E}=\frac{1}{M}\mathbf{p}\cdot\mathbf{E},\quad \partial_0\mathbf{E}-\nabla\times\mathbf{B}=\frac{1}{M}(p_0\mathbf{E}-\mathbf{p}\times\mathbf{B}),$$
$$\nabla\cdot\mathbf{B}=0,\quad \partial_0\mathbf{B}+\nabla\times\mathbf{E}=0,\quad \Box\theta=\frac{1}{M}(\mathbf{E}^2-\mathbf{B}^2)+F \qquad (4)$$

which models generalized axion electrodynamics with a *scalar* axionic field (we recall that in system (2)–(3) this field was pseudoscalar). Only equations (2)–(4) with an arbitrary (in general nonlinear) function $F(\theta)$ is the main subject of the group classification.

3 Group classification of equations (2)–(3)

Equation (3) includes the free element, $F(\theta)$, so we can expect that symmetries of system (2)–(3) depend upon the explicit form of F.

In accordance to the classical Lie algorithm (refer, e.g., to [12]) to find symmetries of system (2)–(3) w.r.t. continuous groups of transformations $\mathbf{B}\to\mathbf{B}'$, $\mathbf{E}\to\mathbf{E}'$, $\theta\to\theta'$, $x_\mu\to x'_\mu$ we consider the infinitesimal operator

$$Q=\xi^\mu\partial_\mu+\eta^j\partial_{B^j}+\zeta^j\partial_{E^j}+\sigma\partial_\theta, \qquad (5)$$

and its prolongation

$$Q_{(2)}=Q+\eta_i{}^j\frac{\partial}{\partial B_i^j}+\zeta_i{}^j\frac{\partial}{\partial E_i^j}+\sigma_i\partial_{\theta_i}+\eta_{ik}{}^j\frac{\partial}{\partial B_{ik}^j}+\zeta_{ik}{}^j\frac{\partial}{\partial E_{ik}^j}+\sigma_{ik}\partial_{\theta_{ik}}. \qquad (6)$$

The invariance condition for system (2), (3) has the form: $Q_{(2)}\mathcal{F}|_{\mathcal{F}=0}=0$, where $\mathcal{F}$ is the manifold defined by this system. Integrating the system of determining equations obtained (see [13] for details) we find that for F arbitrary the generator Q should be a linear combination of the following operators:

$$P_0=\partial_0,\quad P_a=\partial_a,$$
$$J_{ab}=x_a\partial_b-x_b\partial_a+B^a\partial_{B^b}-B^b\partial_{B^a}+E^a\partial_{E^b}-E^b\partial_{E^a}, \qquad (7)$$
$$J_{0a}=x_0\partial_a+x_a\partial_0+\varepsilon_{abc}\big(E^b\partial_{B^c}-B^b\partial_{E^c}\big),$$

where ε_{abc} is the unit antisymmetric tensor, $a,b,c=1,2,3$.

Operators (7) form a basis of the Lie algebra p(1,3) of the Poincaré group P(1,3) which is is the maximal continuous invariance group of system (2)–(3) with an *arbitrary* function $F(\theta)$.

This symmetry is more extensive provided function F has one of the following particular forms: $F=0$, $F=c$ or $F=b\exp(a\theta)$, where c, a and b are nonzero constants. The corresponding additional elements of the invariance algebra are

$$
\begin{aligned}
&P_4 = \partial_\theta && \text{if} \quad F(\theta) = c,\\
&X = aD - P_4 && \text{if} \quad F(\theta) = be^{a\theta},\\
&P_4 = \partial_\theta, \quad D = x_0\partial_0 + x_i\partial_i - B^i\partial_{B^i} - E^i\partial_{E^i} && \text{if} \quad F(\theta) = 0.
\end{aligned} \tag{8}
$$

Operator P_4 generates shifts of the dependent variable θ, D is the dilatation operator generating a consistent scaling of dependent and independent variables and X generates the simultaneous shift and scaling.

Finally the group classification of equations (4) gives the same results: this system is invariant w.r.t. the Poincaré group for arbitrary F. System (4) admits more extensive symmetry in the cases enumerated in (8).

4 Nonrelativistic limit

The definition of nonrelativistic limit is not a simple problem in general and in the case of theories of massless fields in particular. Such limit is not unique and well defined in general.

To find a nonrelativistic limit of equations (2) and (3) we use the Inönü–Wigner contraction [14]. Firstly we denote $1/M = \kappa$ and rewrite equations (3) with $F = 0$ in the equivalent form

$$\partial_0 p_0 - \nabla\cdot\mathbf{p} = -\kappa\mathbf{E}\cdot\mathbf{B}, \tag{9}$$

$$\partial_0\mathbf{p} - \nabla p_0 = 0, \tag{10}$$

$$\nabla\times\mathbf{p} = 0. \tag{11}$$

Equations (9)–(11) are equivalent to equation (3) together with the definitions $\partial_0\theta = p_0$ and $\nabla\theta = \mathbf{p}$.

Prolonging the basis elements (7) to the first derivatives of θ we obtain generators of the Poincaré group for system (2), (9)–(11):

$$
\begin{aligned}
&\hat{P}_0 = \partial_0, \quad \hat{P}_a = \partial_a,\\
&\hat{J}_{ab} = x_a\partial_b - x_b\partial_a + B^a\partial_{B^b} - B^b\partial_{B^a} + E^a\partial_{E^b} - E^b\partial_{E^a} + p^a\partial_{p^b} - p^b\partial_{p^a},\\
&J_{0a} = x_0\partial_a + x_a\partial_0 + \varepsilon_{abc}\big(E^b\partial_{B^c} - B^b\partial_{E^c}\big) + p^0\partial_{p^a} - p^a\partial_{p^0}.
\end{aligned} \tag{12}
$$

The Inönü–Wigner contraction consists of transformation to a new basis $J_{ab} \to J_{ab}$, $J_{0a} \to \varepsilon J_{0a}$, where ε is a small parameter associated with the inverse speed of light. In addition the dependent and independent variables in (12) undergo the invertible transformations $E^a \to E'^a$, $B^a \to B'^a$, $p^\mu \to p'^\mu$, where the primed quantities are functions of the initial ones and of ε, and $x^\mu \to x'^\mu$, where $x'^\mu = \varphi^\mu(x^0, x^1, x^2, x^3, \varepsilon)$. For representation (12) this transformation looks as [13]:

$$
\begin{aligned}
&x'_0 = t = \varepsilon x_0, \quad x'_a = x_a,\\
&\mathbf{p}' = \frac{\varepsilon}{2}(\mathbf{E} + \mathbf{p}), \quad \mathbf{E}' = \varepsilon^{-1}(\mathbf{p} - \mathbf{E}), \quad \mathbf{B}' = \mathbf{B}, \quad p'_0 = p_0
\end{aligned} \tag{13}
$$

and so $\partial_0 = \varepsilon\partial_t$ and $\partial_{x_a} = \partial_{x'_a}$. As a result the system (2), (9)–(10) is reduced to the following form [13]:

$$\begin{aligned}
&\partial_t p'_0 - \nabla\cdot\mathbf{E}' + \kappa\mathbf{B}'\cdot\mathbf{E}' = 0, \quad \partial_t\mathbf{p}' + \nabla\times\mathbf{B}' + \kappa(p'_0\mathbf{B}' + \mathbf{p}'\times\mathbf{E}') = 0,\\
&\nabla\cdot\mathbf{p}' + \kappa\mathbf{p}'\cdot\mathbf{B}' = 0, \quad \nabla\cdot\mathbf{B}' = 0,\\
&\partial_t\mathbf{B}' + \nabla\times\mathbf{E}' = 0, \quad \partial_t\mathbf{p}' - \nabla p'_0 = 0, \quad \nabla\times\mathbf{p}' = 0
\end{aligned} \tag{14}$$

and $p'^0 = \partial_t\theta'$, $\mathbf{p}' = \nabla\theta'$.

Equations (14) present the nonrelativistic limit of system (2), (9)–(10). These equations coincide with the Galilei invariant system for an indecomposable ten-component field obtained earlier [10].

5 Closed-form solutions

5.1 Three-dimensional subalgebras of p(1,3)

To generate solutions of system (2)–(3) we can exploit its invariance w.r.t. the Poincaré group. To do this we need the three-dimensional subalgebras of algebra p(1,3) which give rise to reductions of this system to ordinary differential equations.

The subalgebras of algebra p(1,3) defined up to the group of internal automorphisms has been found for the first time by the Belorussian mathematician Bel'ko [15]. We use a more advanced classification of these subalgebras proposed by Patera, Winternitz and Zassenhaus [16] who proved that there exist 30 nonequivalent three-dimensional subalgebras $A_1, A_2, \ldots, A_{30}$ of the algebra p(1,3) which we present in the following formulae by specifying their basis elements:

$$\begin{aligned}
&A_1\colon\ \langle P_0, P_1, P_2\rangle; \quad A_2\colon\ \langle P_1, P_2, P_3\rangle; \quad A_3\colon\ \langle P_0 - P_3, P_1, P_2\rangle;\\
&A_4\colon\ \langle J_{03}, P_1, P_2\rangle; \quad A_5\colon\ \langle J_{03}, P_0 - P_3, P_1\rangle; \quad A_6\colon\ \langle J_{03} + \alpha P_2, P_0, P_3\rangle;\\
&A_7\colon\ \langle J_{03} + \alpha P_2, P_0 - P_3, P_1\rangle; \quad A_8\colon\ \langle J_{12}, P_0, P_3\rangle;\\
&A_9\colon\ \langle J_{12} + \alpha P_0, P_1, P_2\rangle; \quad A_{10}\colon\ \langle J_{12} + \alpha P_3, P_1, P_2\rangle;\\
&A_{11}\colon\ \langle J_{12} - P_0 + P_3, P_1, P_2\rangle; \quad A_{12}\colon\ \langle G_1, P_0 - P_3, P_2\rangle;\\
&A_{13}\colon\ \langle G_1, P_0 - P_3, P_1 + \alpha P_2\rangle; \quad A_{14}\colon\ \langle G_1 + P_2, P_0 - P_3, P_1\rangle;\\
&A_{15}\colon\ \langle G_1 - P_0, P_0 - P_3, P_2\rangle; \quad A_{16}\colon\ \langle G_1 + P_0, P_1 + \alpha P_2, P_0 - P_3\rangle;\\
&A_{17}\colon\ \langle J_{03} + \alpha J_{12}, P_0, P_3\rangle; \quad A_{18}\colon\ \langle \alpha J_{03} + J_{12}, P_1, P_2\rangle;\\
&A_{19}\colon\ \langle J_{12}, J_{03}, P_0 - P_3\rangle; \quad A_{20}\colon\ \langle G_1, G_2, P_0 - P_3\rangle;\\
&A_{21}\colon\ \langle G_1 + P_2, G_2 + \alpha P_1 + \beta P_2, P_0 - P_3\rangle;\\
&A_{22}\colon\ \langle G_1, G_2 + P_1 + \beta P_2, P_0 - P_3\rangle; \quad A_{23}\colon\ \langle G_1, G_2 + P_2, P_0 - P_3\rangle;\\
&A_{24}\colon\ \langle G_1, J_{03}, P_2\rangle; \quad A_{25}\colon\ \langle J_{03} + \alpha P_1 + \beta P_2, G_1, P_0 - P_3\rangle;\\
&A_{26}\colon\ \langle J_{12} - P_0 + P_3, G_1, G_2\rangle; \quad A_{27}\colon\ \langle J_{03} + \alpha J_{12}, G_1, G_2\rangle;\\
&A_{28}\colon\ \langle G_1, G_2, J_{12}\rangle; \quad A_{29}\colon\ \langle J_{01}, J_{02}, J_{12}\rangle; \quad A_{30}\colon\ \langle J_{12}, J_{23}, J_{31}\rangle.
\end{aligned} \tag{15}$$

Here P_μ and $J_{\mu\nu}$ are generators of the Poincaré group which in our case are given by relations (7), $G_1 = J_{01} - J_{13}$, $G_2 = J_{02} - J_{23}$, α and β are arbitrary parameters.

Using subalgebras (15) we can deduce solutions for system (2)–(3). Note that to make an effective reduction using the Lie algorithm we can use only such subalgebras the basis elements of which satisfy the transversality condition, i.e., their rank of representation coincides with the rank of representation restricted to space of independent variables [12]. This condition is satisfied by basis element of the algebras $A_1 - A_{27}$, but is not satisfied by A_{28}, A_{29}, A_{30} and A_6 with $\alpha = 0$. Nevertheless the latter symmetries also can be used to generate solutions in frames of the weak transversality approach discussed by Grundland, Tempesta and Winternitz [17].

The completed list of reductions can be found in [13]. We note that there appear the following types of reductions:

- Reductions to algebraic equations which are induced by algebras A_{11}, A_{12}, A_{21}, A_{22}, A_{23} and A_{26};
- Reductions to linear ODE induced by A_1, A_2, A_3, A_5, A_7, A_{15}, A_{16} and A_{25};
- Reductions to nonlinear ODE induced by A_6 ($\alpha \neq 0$), A_9 ($\alpha \neq 0$), A_{10}, A_{13}, A_{14}, A_{17} and A_{18};
- Reductions to PDE induced by A_6 ($\alpha = 0$), A_{28}, A_{29} and A_{30}.

Reductions to algebraic or linear equations make it possible to find closed-form solutions of equations (2) and (3) while a part of the reduced nonlinear equations is not integrable by quadratures. Nevertheless it is possible to find particular solutions for any type of the reductions.

In this paper we present solutions of equations (2) and (3) which can be obtained using reductions to algebraic equations and some particular solutions for the other reductions. The complete list of solutions obtained via reductions induced by three-dimensional subalgebras of p(1,3) can be found in preprint [13].

5.2 Plane-wave solutions

Firstly we note that by scaling the dependent variables we can reduce the constant M in (2)–(3) to unity. Thus without loss of generality we set

$$M = 1 \tag{16}$$

and search for solutions of system (2)–(3), (16). To obtain solutions for M arbitrary it will be sufficient simply to multiply by M all vectors $\mathbf{B}$, $\mathbf{E}$ and scalars θ presented in the following formulae.

We find solutions which are invariant w.r.t. the subalgebras A_1, A_2 and A_3. Basis elements of these subalgebras can be represented in the following unified form

$$A\colon\ \langle P_1, P_2, kP_0 + \varepsilon P_3\rangle, \tag{17}$$

where ε and k are parameters. Indeed for $\varepsilon = -k$, $\varepsilon^2 < k^2$ or $k^2 < \varepsilon^2$ algebra (17) is equivalent to A_3, A_1 or A_2, respectively.

To find the related invariant solutions it is necessary to change variables in (2)–(3) by invariants of the group the generators of which are given in (17). These invariants are E_a, B_a, θ $(a = 1, 2, 3)$ and the only independent variable $\omega = \varepsilon x_0 - kx_3$. Thus we search for solutions of (2)–(3) which are functions of ω. As a result we reduce equations (2) to the following system:

$$\begin{aligned} &\dot{B}_3 = 0, \quad \dot{E}_3 = \dot{\theta} B_3, \quad k\dot{E}_2 = -\varepsilon \dot{B}_1, \quad k\dot{E}_1 = \varepsilon \dot{B}_2, \\ &\varepsilon \dot{E}_1 - k\dot{B}_2 = \dot{\theta}(kE_2 + \varepsilon B_1), \quad k\dot{B}_1 + \varepsilon \dot{E}_2 = \dot{\theta}(\varepsilon B_2 - kE_1), \end{aligned} \tag{18}$$

where $\dot{B}_3 = \partial B_3/\partial\omega$.

The system (18) is easily integrated. For $\varepsilon^2 = k^2$ and $\varepsilon^2 \neq k^2$ we obtain

$$E_1 = \frac{\varepsilon}{k} B_2 = \varphi_1, \quad E_2 = -\frac{\varepsilon}{k} B_1 = \varphi_2, \quad E_3 = e\theta + b, \quad B_3 = e \tag{19}$$

and

$$B_1 = ke_1\theta - kb_1 + \varepsilon e_2, \quad B_2 = ke_2\theta - kb_2 - \varepsilon e_1, \quad B_3 = e_3, \tag{20}$$

$$E_1 = \varepsilon e_2\theta - \varepsilon b_2 - ke_1, \quad E_2 = -\varepsilon e_1\theta + \varepsilon b_1 - ke_2,$$

$$E_3 = e_3\theta - b_3(\varepsilon^2 - k^2) \tag{21}$$

correspondingly. Here φ_1 and φ_2 are arbitrary functions of ω while e, b, b_a and e_a $(a = 1, 2, 3)$ are constants of integration.

If (19) is valid, then it follows from (3) that the corresponding equation (3) is reduced to the form $e^2\theta = F(\theta) - be$, i.e., θ is proportional to $F(\theta) - be$ if $e \neq 0$. If both e and F equal to zero, then θ is an arbitrary function of ω.

For $\varepsilon^2 \neq k^2$ solutions of (18) have the following form:

$$\ddot{\theta} = -\left(e_1^2 + e_2^2 + \frac{e_3^2}{\varepsilon^2 - k^2}\right)\theta + c + \frac{F}{(\varepsilon^2 - k^2)}, \tag{22}$$

where $c = e_1b_1 + e_2b_2 + e_3b_3$.

If $F = 0$ or $F = -m^2\theta$, then (22) is reduced to the linear equation:

$$\ddot{\theta} = -a\theta + c, \tag{23}$$

where $a = e_1^2 + e_2^2 + (e_3^2 + m^2)/(\varepsilon^2 - k^2)$.

We present bounded solutions of equation (23) corresponding to $a > 0$ and $c = 0$:

$$\theta = a_\mu \cos\tilde{\omega} + b_\mu \sin\tilde{\omega}, \quad \omega = \tilde{\varepsilon} x_0 - \tilde{k} x_1, \tag{24}$$

where we denote $a = \mu^2$, $\tilde{\varepsilon} = \mu\varepsilon$, $\tilde{k} = \mu k$, a_μ and b_μ are arbitrary constants.

Equations (20), (24) give plane-wave solutions of system (2)–(3). The corresponding dispersive relations are

$$\tilde{\varepsilon}^2 = \tilde{k}^2 + \frac{e_3^2 + m^2}{1 - e_1^2 - e_2^2}. \tag{25}$$

Equation (22) can be solved in closed form for various functions $F(\theta)$. In particular for $F = 3\lambda\theta^2$ we obtain a solitary-wave solution:

$$\theta = \frac{2(\varepsilon^2 - k^2)}{\lambda}\tanh^2(kx_3 - \varepsilon x_0 + b), \tag{26}$$

where b is a constant of integration. The related parameters ε, k and c in (22) should satisfy the conditions

$$\varepsilon^2 = k^2 + \frac{e_3^2}{8(1 - e_1^2 - e_2^2)}, \quad c = \frac{e_3^2}{2(1 - e_1^2 - e_2^2)\lambda}. \tag{27}$$

For $F = \lambda\theta^3$ and $c = 0$ there exists a shock-wave solution

$$\theta = \sqrt{\frac{\varepsilon^2 - k^2}{2\lambda}}\tanh(\varepsilon x_0 - kx_3) \tag{28}$$

which has to be completed by the following relation:

$$\varepsilon^2 = k^2 + \frac{e_3^2}{2(1 - e_1^2 - e_2^2)}. \tag{29}$$

In an analogous (but as a rule much more complicated) way we can find solutions corresponding to other subalgebras (15). We present one more particular solution of equations (2)–(3) with $M = 1$ and $F = 0$, obtained with using of algebras A_9 and A_{10}:

$$\begin{aligned}
&E_2 = c_k\varepsilon\cos(\varepsilon x_0 + kx_1) + d_k\varepsilon\sin(\varepsilon x_0 + kx_1),\\
&E_3 = c_k\varepsilon\sin(\varepsilon x_0 + kx_1) - d_k\varepsilon\cos(\varepsilon x_0 + kx_1),\\
&B_2 = c_k k\sin(\varepsilon x_0 + kx_1) - d_k k\cos(\varepsilon x_0 + kx_1),\\
&B_3 = -c_k k\cos(\varepsilon x_0 + kx_1) - d_k k\sin(\varepsilon x_0 + kx_1),\\
&E_1 = e,\ B_1 = 0, \quad \theta = \alpha x_0 + \nu x_1 + c_3,
\end{aligned} \tag{30}$$

where e, c_k, d_k, ε, k, α and ν are arbitrary constants restricted by the constraint

$$\varepsilon^2 - k^2 = \nu\varepsilon - \alpha k. \tag{31}$$

If $\varepsilon = k$, then $\alpha = \nu$ and all solutions (30) depend upon the light cone variable $x_0 - x_1$. However, for $\varepsilon \neq k$ we have plane-wave solutions depending upon two different plane-wave variables, namely $\varepsilon x_0 + kx_1$ and $\alpha x_0 + \nu x_1$.

We note that for fixed parameters α and ν solutions (30) for E_a and B_a satisfy the superposition principle, i.e., a sum of solutions with different ε, k, c_k and d_k is also a solution of equations (2)–(3), (16) with $F = 0$. Thus it is possible to sum (integrate) solutions (30) for E_a and B_a over k treating c_k and d_k as functions of k.

5.3 Radial and planar solutions

Using invariants of the subalgebra A_{30} we can find the solution for equations (2)–(3), (16) with $F = -m^2\theta$ in the following form:

$$B_a = \frac{qx_a}{r^3}, \quad E_a = \frac{q\theta x_a}{r^3}, \quad \theta = c_1 \sin(mx_0)\mathrm{e}^{-\frac{q}{r}}, \tag{32}$$

where c_1 and q are arbitrary parameters. The components of magnetic field B_a are singular at $r = 0$ while E_a and θ are bounded for $0 \leq r \leq \infty$.

We present solutions which depend upon two spatial variables, but are rather similar to the three-dimensional Coulombic field. We denote $x = \sqrt{x_1^2 + x_2^2}$. Then functions

$$E_1 = -B_2 = \frac{x_1}{x^3},\ E_3 = 0,\ B_1 = E_2 = \frac{x_2}{x^3},\ B_3 = b,\ \theta = \arctan\left(\frac{x_2}{x_1}\right), \tag{33}$$

where b is a number, solve equations (2) and (3) with $M = 1$ and $F = 0$.

A pecularity of the planar solutions (33) is that the related electric field decreases with an increase of x as the field of a point charge in the three-dimensional space.

We write one more solution of equations (2), (3) with $F = 0$:

$$B_1 = \frac{x_1 x_3}{r^2 x}, \quad B_2 = \frac{x_2 x_3}{r^2 x}, \quad B_3 = -\frac{x}{r^2}, \quad \theta = \arctan\left(\frac{x}{x_3}\right), \tag{34}$$

$$E_a = \frac{x_a}{r^2}, \quad a = 1, 2, 3, \tag{35}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$, $x = \sqrt{x_1^2 + x_2^2}$. The electric field (35) is directed like the three-dimensional field of point charge, but its strength is proportional to $1/r$ instead of $1/r^2$.

We note that functions (34), (35) also solve equations (4) with $\kappa = 1$, $F = 0$. One more stationary solution for these equations can be written as

$$E_a = \frac{x_a}{r},\ B_a = b_a, \quad \theta = \ln r, \tag{36}$$

where b_a are constants satisfying the condition $b_1^2 + b_2^2 + b_3^2 = 1$.

Functions (36) solve equations (4) with $F = 0$ for $0 < r < \infty$.

6 Solutions with arbitrary functions

Solutions considered above include arbitrary parameters the number of which can be extended by application of Lorentz transformations. Here we present a class of solutions depending upon arbitrary functions. This class covers all reductions which can be obtained using subalgebras A_{12}–A_{14} and A_{20}–A_{23}.

We define

$$
\begin{aligned}
&B_1 = E_2 = \psi_1(x_1, x_2, \omega) - x_1\dot{\varphi}_1(\omega),\\
&B_2 = -E_1 = \psi_2(x_1, x_2, \omega) + x_1\big(\dot{\varphi}_2(\omega) - \varphi_1(\omega)\dot{\theta}(\omega)\big),\\
&B_3 = \varphi_1(\omega), \quad E_3 = \varphi_2(\omega),
\end{aligned}
\tag{37}
$$

where φ_1, φ_2 and ψ_1, ψ_2 are functions of $\omega = x_0 + x_3$ and x_1, x_2, ω, respectively, and

$$
\theta = -\frac{\varphi_1\varphi_2}{m^2} \quad \text{if} \quad m^2 \neq 0, \qquad \theta = \varphi_3(\omega), \quad \varphi_1\varphi_2 = 0 \quad \text{if} \quad m^2 = 0. \tag{38}
$$

Up to the restriction present in (38) functions φ_1, φ_2 and φ_3 are arbitrary while ψ_1 and ψ_2 should satisfy the Cauchy–Riemann condition with respect to the variables x_1 and x_2:

$$
\partial_1\psi_1 + \partial_2\psi_2 = 0, \quad \partial_1\psi_2 - \partial_2\psi_1 = 0. \tag{39}
$$

It is easily to verify that Ansatz (37)–(38) does satisfy equations (2)–(3). Thus any solution of the Cauchy–Riemann equations (39) depending upon the parameter ω and three arbitrary functions $\varphi_1(\omega)$, $\varphi_2(\omega)$, $\varphi_3(\omega)$ satisfying (38) give rise to solution (37) of system (2)–(3).

7 Discussion

Thus we present the results of the group classification of possible generalizations of axion electrodynamics the Lagrangian of which includes an arbitrary function of θ. In accordance with our analysis the Poincaré invariance is the maximal symmetry of the standard axion electrodynamics. In addition we find three cases when the theory admits more extensive symmetry, see equations (8). These results form certain group-theoretical grounds for constructing various axionic models.

The second goal of this paper was to find a correct nonrelativistic limit of equations of axion electrodynamics. As a result we prove that the limiting case of these equations is nothing but the Galilei-invariant system obtained earlier in [10].

At the third place we find families of closed-form solutions of equations of axion electrodynamics using invariants of three-parameter subgroups of the Poincaré group. Among them are solutions including sets of arbitrary parameters and arbitrary functions as well. In addition it is possible to generate more extensive families of solutions by applying inhomogeneous Lorentz transformations.

Except the particular examples (34)–(36) we did not discuss solutions for the system (4) which has the same symmetries as (2), (3). We note that reductions of these equations can be made in a very straightforward way. Indeed, making the gauge transformation $E_a \to \mathsf{e}^{\theta}E_a$ and $B_a \to \mathsf{e}^{\theta}B_a$, we can reduce these equations to a system including the Maxwell equation for the electromagnetic field in a vacuum and the following equation:

$$
\Box\theta = \kappa(\mathbf{B}^2 - \mathbf{E}^2)\mathsf{e}^{-2\theta} + F. \tag{40}
$$

Since reductions of the free Maxwell equations using three-dimensional subalgebras of p(1,3) have been done in paper [18], to find the related solutions for system (4) it is sufficient to solve equation (40) with $\mathbf{B}$ and $\mathbf{E}$ being the closed-form solutions found in [18].

Some of the solutions, especially those which include arbitrary functions or, like (30), satisfy the superposition principle, are good candidates to solve various initial- and boundary-value problems in axion electrodynamics. We apply the solutions found to demonstrate a specific property of the models discussed.

We consider in more detail plane-wave solutions found in Section 5.2, namely, the solutions given by equations (20), (24) and (30).

Solutions (24) describe oscillating waves moving along the third coordinate axis. Using the corresponding dispersion relation (25) we can find the corresponding group velocity V_g which is equal to the derivative of $\tilde{\varepsilon}$ w.r.t. $\tilde{k}$, i.e.,

$$V_g = \frac{1}{\sqrt{1+\delta}}, \quad \text{where} \quad \delta = \frac{e_3^2 + m^2}{(1 - e_1^2 - e_2^2)k^2}. \tag{41}$$

For fixed e_1 and e_2 the parameter δ is either positive or negative. In the latter case solutions (24) describe the waves which propagate faster than the velocity of light (remember that we use the Heaviside units in which the velocity of light is equal to 1). These solutions are smooth and bounded functions which correspond to positive definite and bounded energy density which is defined by the following relation [13]:

$$T^{00} = \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2 + p_0^2 + \mathbf{p}^2) + V(\theta) \tag{42}$$

(in our case $V(\theta) = \frac{m}{2}\theta^2$). Thus we can conclude that the tachyon modes are natural constituents of the axion electrodynamics.

Analogously considering solutions (30) we deal with the dispersion relations (31). The corresponding group velocity is given by relation (41), where

$$\delta = \frac{\nu^2 - \alpha^2}{(2k - \alpha)^2}.$$

Thus we again can conclude that the axion electrodynamics admits bounded and smooth solutions propagating faster thnn light. This conclusion is correct in the case for which the arbitrary function $F(\theta)$ is nonlinear, in particular, for $F(\theta) = 2\lambda\theta^2$, $\lambda < 0$. The corresponding acausal solutions can be chosen in the form (20), (26) with $e_1^2 + e_2^2 > 1$.

We believe that the closed-form solutions presented in Sections 5 and 6 can find various applications in axion electrodynamics. In particular solutions, which correspond to the algebras A_9, A_1, A_{17}, A_{18} and A_{28}, generate well-visible dynamical contributions to the axion mass. In addition, as was indicated in [7], the vectors of the electric and magnetic fields described by relations (33) give rise to an exactly solvable Dirac equation for a charged particle anomalously interacting with these fields. We plan to present elsewhere the detailed analysis of the solutions obtained.

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[1] Petrov A.Z., *New Methods in the General Theory of Relativity*, Nauka, Moscow, 1966 (in Russian).

[2] Rafflet G.G., Astrophisical methods to constrain axions and other novel particle phenomena, *Phys. Rep.*, 1990, V.198, 1–113.

[3] Qi X-L., Hughes T.L. and Zhang S-C., Topological field theory of time-reversal invariant insulators, *Phys. Rev. B*, 2008, V.78, 195424.

[4] Peccei R.D. and Quinn H., Constraints imposed by CP conservation in the presence of pseudoparticles, *Phys. Rev. Lett.*, 1977, V.16, 1791–1797.

[5] Weinberg S., A new light boson?, *Phys. Rev. Lett.*, 1977, V.40, 223–226.

[6] Wilczek F., Problem of strong P and T invariance in the presence of instantons, *Phys. Rev. Lett.*, 1978, V.40, 279–282.

[7] Ferraro E., Messina N. and Nikitin A.G., New exactly solvable relativistic models with anomalous interaction, *Phys. Rev. A*, 2010, V.81, 042108.

[8] de Montigny M., Niederle J. and Nikitin A.G., Galilei invariant theories: I. Constructions of indecomposable finite-dimensional representations of the homogeneous Galilei group: directly and via contractions, *J. Phys. A: Math. Gen.*, 2006, V.39, 9365–9385.

[9] Niederle J. and Nikitin A.G., Construction and classification of indecomposable finite-dimensional representations of the homogeneous Galilei group, *Czech. J. Phys.*, 2006, V.56, 1243–1250.

[10] Niederle J. and Nikitin A.G., Galilei-invariant equations for massive fields, *J. Phys. A: Math. Theor.*, 2009, V.42, 105207.

[11] Chern S.S. and Simons J., Characteristic forms and geometric invariants, *Ann. of Math. (2)*, 1974, V.99, 48–69.

[12] Olver P., *Application of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[13] Kuriksha O. and Nikitin A.G., Group analysis and exact solutions for equations of axion electrodynamics, arXiv:1002.0064.

[14] Inönü E. and Wigner E.P., On the contraction of groups and their representations, *Proc. Nat. Acad. Sci. U. S. A.*, 1953, V.39, 510–524.

[15] Bel'ko I.V., Subgroups of Lorentz–Poincaré group, *Izv. Akad. Nauk Bel. SSR*, 1971, V.1, 5–13 (in Russian).

[16] Patera J., Winternitz P. and Zassenhaus H., Continuous subgroups of the fundamental groups of physics, *J. Math. Phys.*, 1975, V.16, 1597–1614.

[17] Grundland M.A., Tempesta P. and Winternitz P., Weak transversality and partially invariant solutions, *J. Math. Phys.*, 2003, V.44, 2704–2731.

[18] Lahno H.O. and Smalij V.F., Subgroups of extended Poincaré group and new exact solutions of Maxwell equations, *Proc. Inst. Math. of NAS of Ukraine*, 2002, V.43, 162–166.

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A Ermakov–Ray–Reid reduction in 2+1-dimensional magnetogasdynamics

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A 2+1-dimensional system governing rotating homentropic magnetogasdynamics with a parabolic gas law is shown to admit an elliptic vortex ansatz determined by an eight-dimensional nonlinear dynamical system with underlying integrable Ermakov–Ray–Reid structure. A novel magnetogasdynamic analogue of the pulsrodon of shallow water f-plane theory is isolated thereby.

1 Introduction

In general, the nonlinear Lundquist magnetohydrodynamics (mhd) equations are analytically intractable [1]. However, under certain physically acceptable approximations, reductions have been made to canonical equations of soliton theory. Thus, in particular, recent work has established that the uniaxial propagation of magneto-acoustic waves in a cold plasma subject to a purely transverse magnetic field may be modelled by the integrable resonant nonlinear Schrödinger (NLS) equation [2]. In [3–5] an anholonomic geometric formalism, originally used in a magnetohydrodynamics context in [6] was adapted to obtain reduction of a steady spatial mhd system to an in integrable Pohlmeyer–Regge–Lund model subject to a volume-preserving constraint. Novel Bernoulli-type integrals of motion for certain planar mhd systems have also recently been shown to provide a means to construct exact solutions [7].

Integrable reductions via approximation such as in the original study of interaction processes in collisionless plasma by Zabusky and Kruskal [8] are well-known. The established methods of modern soliton theory such as the Inverse Scattering Transform and invariance under Bäcklund transformations are then available (see e.g. [9, 10]). In general, in the absence of approximation, Lie group methods may be applied in a systematic manner to isolate substitution principles and privileged symmetry reductions corresponding to restricted classes of exact solutions of the mhd equations [11–14].

In [15–17], Neukirch *et al.* introduced a novel solution procedure in which the nonlinear acceleration terms in the governing Lundquist momentum equations either vanish or, more generally, are assumed to be conservative. Here, an approach to a 2+1-dimensional mhd system is adopted which has its roots in classical work in hydrodynamics of Goldsbrough [18] wherein a class of exact elliptical vortex solutions were constructed in a study of tidal oscillations in an elliptical basin. This work, in turn, is related to that of Kirchoff [19] on vortex structures in the classical 2+1-dimensional Euler system. In the present work, a procedure developed in [20] in an oceanographic elliptic warm-core eddy context is adapted to analyse a 2+1-dimensional magnetohydrodynamic system of the type investigated in [17]. An elliptical vortex ansatz is introduced and reduction obtained thereby to an eight-dimensional nonlinear dynamical system. Time-modulated physical variables are introduced to reduce the system to a form amenable to exact solution, generally, in terms of an elliptic integral representation. In addition, a magnetogasdynamic analogue of the pulsrodon of [20] is isolated. Moreover, the mhd system is shown to have remarkable underlying Hamiltonian structure of Ermakov–Ray–Reid type (see e.g. [21–25]). Nonlinear coupled systems of the latter type have arisen extensively in optics in the description of elliptic Gaussian beam propagation via paraxial approximation [26–33]. They are distinguished by their admittance of a novel integral of motion, namely the Ray–Reid invariant. The Hamiltonian nature of the Ermakov–Ray–Reid system underlying the present 2+1-dimensional magnetogasdynamic system is established.

2 The magnetogasdynamic system

Here, we consider the rotating 2+1-dimensional homentropic magnetogasdynamic system

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho\,\mathbf{q}) = 0, \tag{1}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}\cdot\nabla\mathbf{q} - \frac{\mu}{\rho}\operatorname{curl}\mathbf{H}\times\mathbf{H} + f(\mathbf{k}\times\mathbf{q}) + \frac{1}{\rho}\nabla p = \mathbf{0}, \tag{2}$$

$$\operatorname{div}\mathbf{H} = 0, \tag{3}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \operatorname{curl}(\mathbf{q}\times\mathbf{H}) \tag{4}$$

with purely transverse magnetic field

$$\mathbf{H} = h\mathbf{k} \tag{5}$$

and where the parabolic pressure-density law $p = p_0 + \epsilon\rho^2$, $\epsilon > 0$ is adopted. The latter law has previously arisen in astrophysical contexts (see e.g. [34]). In the above, the magneto-gas density $\rho(\mathbf{x},t)$, pressure $p(\mathbf{x},t)$ and $\mathbf{H}(\mathbf{x},t)$ are all assumed to be dependent only on x, y and t. It is noted that the magnetic induction equation (3) holds identically for the representation (5).

Insertion of (5) into Faraday's law (4) produces

$$\frac{\partial h}{\partial t} + \operatorname{div}(h\mathbf{q}) = 0,$$

whence, in view of the continuity equation (1), we set $h = \lambda\rho$, $\lambda \in \mathbb{R}$. Accordingly, the system reduces to consideration of

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{q}) = 0, \quad \frac{\partial \mathbf{q}}{\partial t} + \mathbf{q}.\nabla\mathbf{q} + f(\mathbf{k} \times \mathbf{q}) + (2\epsilon + \mu\lambda^2)\nabla\rho = \mathbf{0}.$$

This system is in direct analogy with the shallow water f-plane system [20]. Here, an elliptic vortex type ansatz is introduced with

$$\mathbf{q} = \mathbf{L}(t)\mathbf{x} + \mathbf{M}(t), \tag{6}$$

$$\rho = \left[\mathbf{x}^T\mathbf{E}(t)\mathbf{x} + h_0(t)\right]/(2\epsilon + \mu\lambda^2), \quad 2\epsilon + \mu\lambda^2 \neq 0, \tag{7}$$

$$\mathbf{x} = \begin{pmatrix} x - q(t) \\ y - p(t) \end{pmatrix} \tag{8}$$

where

$$\mathbf{L} = \begin{pmatrix} u_1(t) & u_2(t) \\ v_1(t) & v_2(t) \end{pmatrix}, \quad \mathbf{M} = \begin{pmatrix} \dot{q}(t) \\ \dot{p}(t) \end{pmatrix}, \quad \mathbf{E} = \begin{pmatrix} a(t) & b(t) \\ b(t) & c(t) \end{pmatrix}. \tag{9}$$

Insertion of the ansatz (6)–(9) into the continuity equation yields

$$\begin{pmatrix} \dot{a} \\ \dot{b} \\ \dot{c} \end{pmatrix} + \begin{pmatrix} 3u_1 + v_2 & 2v_2 & 0 \\ u_2 & 2(u_1 + v_2) & v_1 \\ 0 & 2u_2 & u_1 + 3v_2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \mathbf{0}, \tag{10}$$

together with

$$\dot{h}_0 = -(u_1 + v_2)h_0. \tag{11}$$

Substitution into the momentum equation gives

$$\begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{v}_1 \\ \dot{v}_2 \end{pmatrix} + \begin{pmatrix} \mathbf{L}^T & -f\mathbf{I} \\ f\mathbf{I} & \mathbf{L}^T \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \end{pmatrix} + 2\begin{pmatrix} a \\ b \\ b \\ c \end{pmatrix} = \mathbf{0}, \tag{12}$$

augmented by the auxiliary equations

$$\ddot{p} + f\dot{q} = 0, \quad \ddot{q} + f\dot{p} = 0. \tag{13}$$

In the sequel, it proves convenient to proceed in terms of new variables

$$\begin{aligned} &G = u_1 + v_2,\ G_R = \frac{1}{2}(v_1 - u_2),\ G_S = \frac{1}{2}(v_1 + u_2),\ G_N = \frac{1}{2}(u_1 - v_2), \\ &B = a + c,\ B_S = b,\ B_N = \frac{1}{2}(a - c). \end{aligned} \tag{14}$$

Here, G and G_R represent, in turn, the divergence and spin of the velocity field, while G_S and G_N represent shear and normal deformation rates.

On use of the expressions (14), the system (10)–(11) together with (12) produces an eight-dimensional nonlinear dynamical system in $\{B,\ B_S,\ B_N,\ G,\ G_S,\ G_R,\ G_N,\ h_0\}$, namely:

$$\begin{aligned}
&\dot{h}_0 + h_0 G = 0, \quad \dot{B} + 2[BG + 2(B_N G_N + B_S G_S)] = 0,\\
&\dot{B}_S + 2B_S G + G_S B - 2B_N G_R = 0,\\
&\dot{B}_N + 2B_N G + G_N B + 2B_S G_R = 0,\\
&\dot{G} + G^2/2 + 2(G_N^2 + G_S^2 - G_R^2) - 2fG_R + 2B = 0,\\
&\dot{G}_R + GG_R + fG/2 = 0, \quad \dot{G}_N + GG_N - fG_S + 2B_N = 0,\\
&\dot{G}_S + GG_S + fG_N + 2B_S = 0.
\end{aligned} \tag{15}$$

If we now introduce Ω via

$$G = \frac{2\dot{\Omega}}{\Omega}, \tag{16}$$

then $(15)_6$ and $(15)_1$, yield, in turn,

$$G_R + \frac{1}{2}f = c_0\Omega^{-2}, \tag{17}$$

and $h_0 = c_{\mathrm{I}}\Omega^{-2}$, where c_0, c_{I} are arbitrary constants of integration.

New modulated variables are now introduced according to

$$\bar{B} = \Omega^4 B,\ \bar{B}_S = \Omega^4 B_S,\ \bar{B}_N = \Omega^4 B_N,\ \bar{G}_S = \Omega^2 G_S,\ \bar{G}_N = \Omega^2 G_N \tag{18}$$

whence the system (15) reduces to

$$\dot{\bar{B}} + 4(\bar{B}_N\bar{G}_N + \bar{B}_S\bar{G}_S)/\Omega^2 = 0, \tag{19}$$

$$\dot{\bar{B}}_S + (\bar{B}\bar{G}_S - 2c_0\bar{B}_N)/\Omega^2 + f\bar{B}_N = 0, \tag{20}$$

$$\dot{\bar{B}}_N + (\bar{B}\bar{G}_N + 2c_0\bar{B}_S)/\Omega^2 - f\bar{B}_S = 0, \tag{21}$$

$$\dot{\bar{G}}_N - f\bar{G}_S + 2\bar{B}_N/\Omega^2 = 0, \tag{22}$$

$$\dot{\bar{G}}_S + f\bar{G}_N + 2\bar{B}_S/\Omega^2 = 0, \tag{23}$$

together with

$$\Omega^3\ddot{\Omega} + \frac{f^2}{4}\Omega^4 - c_0^2 + \bar{G}_N^2 + \bar{G}_S^2 + \bar{B} = 0. \tag{24}$$

Combination of (20) and (21) with use of (19) produces the integral of motion

$$\bar{B}_S^2 + \bar{B}_N^2 - \frac{\bar{B}^2}{4} = c_{\mathrm{II}}, \tag{25}$$

while (22) and (23) together give a further integral of motion

$$\bar{G}_S^2 + \bar{G}_N^2 - \bar{B} = c_{\mathrm{III}}, \tag{26}$$

where c_{II}, c_{III} are constants of integration.

3 A parametrisation

The integrals of motion (25) and (26) may be conveniently parametrised, in turn, according to

$$\begin{aligned} \bar{B}_S &= \pm\sqrt{c_{\mathrm{II}} + \frac{1}{4}\bar{B}^2}\cos\phi(t), \quad \bar{B}_N = \pm\sqrt{c_{\mathrm{II}} + \frac{1}{4}\bar{B}^2}\sin\phi(t), \\ \bar{G}_S &= \pm\sqrt{c_{\mathrm{III}} + \bar{B}}\sin\theta(t), \quad \bar{G}_N = \pm\sqrt{c_{\mathrm{III}} + \bar{B}}\cos\theta(t). \end{aligned} \tag{27}$$

Here, signs in (27) will be adopted which are compatible with the subsequent construction of pulsrodon-type solutions whence, we set

$$\begin{aligned} \bar{B}_S &= -\sqrt{c_{\mathrm{II}} + \frac{1}{4}\bar{B}^2}\cos\phi(t), \quad \bar{B}_N = -\sqrt{c_{\mathrm{II}} + \frac{1}{4}\bar{B}^2}\sin\phi(t), \\ \bar{G}_S &= -\sqrt{c_{\mathrm{III}} + \bar{B}}\sin\theta(t), \quad \bar{G}_N = +\sqrt{c_{\mathrm{III}} + \bar{B}}\cos\theta(t). \end{aligned} \tag{28}$$

Substitution of the parametrisation (28) into (19) yields

$$\dot{\bar{B}} + \frac{4}{\Omega^2}\sqrt{c_{\mathrm{II}} + \bar{B}^2/4}\sqrt{c_{\mathrm{III}} + \bar{B}}\sin(\theta - \phi) = 0, \tag{29}$$

while conditions (20) and (21) reduce to the single requirement

$$\sqrt{c_{\mathrm{II}} + \bar{B}^2/4}\left[\dot{\phi} + \frac{2c_0}{\Omega^2} - f\right] - \frac{\bar{B}}{\Omega^2}\sqrt{c_{\mathrm{III}} + \bar{B}}\cos(\theta - \phi) = 0 \tag{30}$$

and similarly, conditions (22), (23) produce the single additional condition

$$\sqrt{c_{\mathrm{III}} + \bar{B}}\left[f - \dot{\theta}\right] - \frac{2}{\Omega^2}\sqrt{c_{\mathrm{II}} + \bar{B}^2/4}\cos(\theta - \phi) = 0. \tag{31}$$

Two relations which are key to the subsequent development and which may be established by appeal to system (15) are now recorded:

Theorem 1.

$$\dot{M} = -3GM, \quad \dot{Q} = -3GQ, \tag{32}$$

where

$$\begin{aligned} M &= a\left(u_2 - \frac{f}{2}\right) + b(v_2 - u_1) - c\left(v_1 + \frac{f}{2}\right), \\ Q &= -a(u_2^2 + v_2^2) + 2b(u_1u_2 + v_1v_2) - c(u_1^2 + v_1^2) + 4\triangle \end{aligned}$$

and $\triangle = ac - b^2$.

Corollary 1. *On use of* (16) *and* (14), *it is seen that*

$$M = c_{\mathrm{IV}}\Omega^{-6}, \quad Q = c_{\mathrm{V}}\Omega^{-6}, \tag{33}$$

where

$$M = 2(B_N G_S - B_S G_N) - B\left(\frac{f}{2} + G_R\right), \tag{34}$$

$$Q = -B(G_S^2 + G_N^2 + G_R^2 + \frac{1}{4}G^2) + 4G_R(B_N G_S - B_S G_N)$$
$$+2G(B_S G_S + B_N G_N) + 4\triangle,$$

and c_{IV}, c_{V} *are arbitrary constants of integration.*

In particular, the relations (17), (33) and (34) show that

$$c_0\bar{B} = -c_{\mathrm{IV}} + 2(\bar{B}_N\bar{G}_S - \bar{B}_S\bar{G}_N)$$

whence one obtains:

Corollary 2.

$$c_0\bar{B} = -c_{\mathrm{IV}} + 2\sqrt{(C_{\mathrm{II}} + \bar{B}^2/4)(c_{\mathrm{III}} + \bar{B})}\cos(\theta - \phi). \tag{35}$$

Elimination of $\cos(\theta - \phi)$ between (35) and (30), (31) in turn, yields

$$\dot{\phi} = f + \frac{1}{\Omega^2}\left[-2c_0 + \frac{\bar{B}}{2}\frac{(c_0\bar{B} + c_{\mathrm{IV}})}{(c_{\mathrm{II}} + \frac{\bar{B}^2}{4})}\right], \tag{36}$$

and

$$\dot{\theta} = f - \frac{1}{\Omega^2}\frac{(c_0\bar{B} + c_{\mathrm{IV}})}{(c_{\mathrm{III}} + \bar{B})}. \tag{37}$$

It remains to consider the nonlinear equation (24) for Ω, namely

$$\Omega^3\ddot{\Omega} + \frac{f^2}{4}\Omega^4 + c_{\mathrm{III}} + 2\bar{B} - c_0^2 = 0 \tag{38}$$

If $\bar{B} = \lambda + \mu\Omega^4$, then (38) reduces to the classical Steen–Ermakov equation [35,36] with explicit solution given via its well-known nonlinear superposition principle. In general, use of Theorem 1 shows that

$$\ddot{(\Omega^2\bar{B})} + f^2\Omega^2\bar{B} = -2(Q + fM)\Omega^6 = -2(c_{\mathrm{V}} + fc_{\mathrm{IV}})$$

whence,

$$\Omega^2\bar{B} = \begin{cases} c_{\mathrm{VI}}\cos ft + c_{\mathrm{VII}}\sin ft - 2(c_{\mathrm{V}} + fc_{\mathrm{IV}})/f^2, f \neq 0 \\ -c_{\mathrm{V}}t^2 + c_{\mathrm{VI}}t + c_{\mathrm{VII}}, f = 0. \end{cases} \tag{39}$$

On elimination of $\theta - \phi$ and Ω in (29) via the relations (35) and (39) it is seen that if $\bar{B} \neq \text{const}$, then $\bar{B}$ is given by the elliptic integral relation

$$\begin{aligned}
&\int_{c_{\mathrm{VIII}}}^{\bar{B}} \frac{d\bar{B}^*}{\bar{B}^* \sqrt{(\bar{B}^{*2} + 4c_{\mathrm{II}})(\bar{B}^* + c_{\mathrm{III}}) - (c_0 \bar{B}^* + c_{\mathrm{IV}})^2}} \\
&= -2 \int_0^t \frac{dt^*}{c_{\mathrm{VI}} \cos ft^* + c_{\mathrm{VII}} \sin ft^* - 2(c_{\mathrm{V}} + f c_{\mathrm{IV}})/f^2} \quad (f \neq 0) \\
&= -2 \int_0^t \frac{dt^*}{-c_{\mathrm{V}} t^{*2} + c_{\mathrm{VI}} t^* + c_{\mathrm{VII}}} \quad (f = 0)
\end{aligned} \tag{40}$$

where $\bar{B}|_{t=0} = c_{\mathrm{VIII}}$. With $\bar{B}$ obtained thereby, Ω is given via (39) and may be shown to be compatible with the nonlinear equation (38). The angles $\theta(t)$, $\phi(t)$ are determined by (36), (37) while the velocity components u_1, u_2, v_1, v_2 and the quantities a, b, c are given, in turn, by

$$\begin{aligned}
u_1 &= \frac{\dot{\Omega}}{\Omega} + \frac{1}{\Omega^2} \sqrt{c_{\mathrm{III}} + \bar{B}} \cos \theta(t), \ \ v_1 = -\frac{1}{\Omega^2} \sqrt{c_{\mathrm{III}} + \bar{B}} \sin \theta(t) + \frac{c_0}{\Omega^2} - \frac{f}{2}, \\
u_2 &= -\frac{1}{\Omega^2} \sqrt{c_{\mathrm{III}} + \bar{B}} \sin \theta(t) - \frac{c_0}{\Omega^2} + \frac{f}{2}, \ \ v_2 = \frac{\dot{\Omega}}{\Omega} - \frac{1}{\Omega^2} \sqrt{c_{\mathrm{III}} + \bar{B}} \cos \theta(t),
\end{aligned}$$

together with

$$\begin{aligned}
a &= \frac{1}{\Omega^4} \left[\frac{\bar{B}}{2} - \sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}} \sin \phi(t) \right], \ \ b = -\frac{1}{\Omega^4} \sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}} \cos \phi(t), \\
c &= \frac{1}{\Omega^4} \left[\frac{\bar{B}}{2} + \sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}} \sin \phi(t) \right], \ \ h_0 = \frac{c_{\mathrm{I}}}{\Omega^2}.
\end{aligned}$$

The above completes the solution of the nonlinear dynamical system (10)–(12) if $\bar{B} \neq \text{const}$.

4 The pulsrodon

If $\bar{B} = \text{const}$ then (38) shows that Ω is determined by the nonlinear oscillator equation

$$\left(\ddot{\Omega} + \frac{f^2}{4} \right) \Omega = \frac{c_0^2 - c_{\mathrm{III}} - 2\bar{B}}{\Omega^3}. \tag{41}$$

This is commonly known as the Ermakov equation although it originated in a paper of 1874 by Steen [35]. It arises in a wide range of areas of physical importance, most notably in quantum mechanics, optics and nonlinear elasticity (see e.g. [37–39]). It is characterised by its admittance of a well-known nonlinear superposition principle. Thus, the general solution of (41) is given by

$$\Omega = \sqrt{\lambda\Omega_1^2 + 2\mu\Omega_1\Omega_2 + \nu\Omega_2^2} \tag{42}$$

where Ω_1, Ω_2 are linearly independent solutions of $\ddot{\Omega}_L + f^2\Omega_L/4 = 0$, with unit Wronskian $W(\Omega_1, \Omega_2) = \Omega_1\dot{\Omega}_2 - \Omega_2\dot{\Omega}_1$ where the constants λ, μ, ν are constrained by the relation

$$\lambda\nu - \mu^2 = c_0^2 - c_{\mathrm{III}} - 2\bar{B}. \tag{43}$$

If we set

$$\Omega_1 = \frac{2}{f}\cos\frac{ft}{2}, \quad \Omega_2 = \frac{2}{f}\sin\frac{ft}{2} \quad (f \neq 0)$$

then the general solution of (41) is given by

$$\Omega = \sqrt{\zeta\cos(ft + \phi) + \eta} \tag{44}$$

where ζ, η are arbitrary constants subject to $\zeta < \eta$ and ϕ is a phase chosen as to accommodate the constraint (43).

In view of the condition $\bar{B} = \text{const}$, the relation (29) shows that, if $\bar{B}^2/4 + c_{\mathrm{II}} \neq 0$ and $\bar{B} + c_{\mathrm{III}} \neq 0$ then $\theta = \phi + n\pi$ whence (36), (37) yield

$$\dot{\theta} = \dot{\phi} = f - \frac{1}{\Omega^2}\frac{(c_0\bar{B} + c_{\mathrm{IV}})}{(c_{\mathrm{III}} + \bar{B})}$$

with consistency condition

$$\frac{1}{2}(c_0\bar{B} + c_{\mathrm{IV}})\left[\frac{\bar{B}}{2\left(c_{\mathrm{II}} + \frac{\bar{B}^2}{4}\right)} + \frac{1}{\bar{B} + c_{\mathrm{III}}}\right] = 1.$$

Here, we proceed with $\theta = \phi$ so that the relations (18) together with the parametrisation (28) yield

$$B_S = -\frac{1}{\Omega^4}\sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}}\cos\theta(t), \ B_N = -\frac{1}{\Omega^4}\sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}}\sin\theta(t),$$

$$G_S = -\frac{1}{\Omega^2}\sqrt{c_{\mathrm{III}} + \bar{B}}\sin\theta(t), \ G_N = +\frac{1}{\Omega^2}\sqrt{c_{\mathrm{III}} + \bar{B}}\cos\theta(t)$$

where

$$\theta(t) - \theta(0) = ft - \frac{c_0\bar{B} + c_{\mathrm{IV}}}{\bar{B} + c_{\mathrm{III}}}\int_0^t \frac{1}{\Omega^2(\tau)}d\tau$$

and Ω is given by (44). The velocity and density distributions are given by the relations (6)–(9) wherein

$$\begin{aligned} u_1 &= \frac{G}{2} + G_N = \frac{\dot{\Omega}}{\Omega} + \frac{1}{\Omega^2}\sqrt{c_{\mathrm{III}} + \bar{B}}\cos\theta(t), \\ v_1 &= G_S + G_R = -\frac{1}{\Omega^2}\sqrt{c_{\mathrm{III}} + \bar{B}}\sin\theta(t) + \frac{c_0}{\Omega^2} - \frac{f}{2}, \\ u_2 &= G_S - G_R = -\frac{1}{\Omega^2}\sqrt{c_{\mathrm{III}} + \bar{B}}\sin\theta(t) - \frac{c_0}{\Omega^2} + \frac{f}{2}, \\ v_2 &= \frac{G}{2} - G_N = \frac{\dot{\Omega}}{\Omega} - \frac{1}{\Omega^2}\sqrt{c_{\mathrm{III}} + \bar{B}}\cos\theta(t), \end{aligned} \tag{45}$$

together with

$$\begin{aligned} a &= \frac{1}{\Omega^4}\left[\frac{\bar{B}}{2} - \sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}}\sin\theta(t)\right], \quad b = -\frac{1}{\Omega^4}\sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}}\cos\theta(t), \\ c &= \frac{1}{\Omega^4}\left[\frac{\bar{B}}{2} + \sqrt{c_{\mathrm{II}} + \frac{\bar{B}^2}{4}}\sin\theta(t)\right], \quad h_0 = \frac{c_{\mathrm{I}}}{\Omega^2}, \end{aligned} \tag{46}$$

and p, q are given by the coupled system (13).

The magneto-gasdynamic vortex solutions presented above are analogous to the pulsrodons constructed in the context of a rotating elliptic-warm core hydrodynamic system in [20]. These pulsrodon solutions and their duals were later derived by Holm via an elegant Lagrangian formulation in [40]. Therein, the pulsrodon was shown to be orbitally Lyapunov stable to perturbations within the class of elliptical vortex solutions. Here, the pulsrodons may be shown to describe pulsating, rotating elliptic plasma cylinders bounded externally by a vacuum state (Rogers and Schief [41]). It is noted that magnetohydrostatic boundary value problems involving elliptic plasma cylinders bounded by an exterior vacuum state have been treated via the Grad–Shafranov equation (vide Biskamp [1]).

5 Ermakov–Ray–Reid structure

It turns out that the eight-dimensional nonlinear dynamical system (15) has remarkable underlying structure and may be reduced to consideration of a Ermakov–Ray–Reid system

$$\ddot{\alpha} + \omega^2(t)\alpha = \frac{1}{\alpha^2\beta}F(\beta/\alpha), \quad \ddot{\beta} + \omega^2(t)\beta = \frac{1}{\alpha\beta^2}G(\alpha\beta). \tag{47}$$

Such systems have their origin in work of Ermakov [36] and were introduced by Ray and Reid in [21,22]. The main theoretical interest in the system (47) resides in its admittance of a distinctive integral of motion, namely, the Ray–Reid invariant

$$I = \frac{1}{2}(\alpha\dot{\beta} - \beta\dot{\alpha})^2 + \int^{\beta/\alpha} F(z)dz + \int^{\alpha/\beta} G(w)dw$$

together with a concomitant nonlinear superposition principle. In the case of the Steen–Ermakov reduction (41) this adopts the form (42). This result was set down originally in the classical paper of Steen.

In the sequel, it proves convenient to proceed with $p(t) = q(t) = 0$ in the ansatz (6)–(9). However, the terms involving p, q, $\dot{p}$, $\dot{q}$ are readily re-introduced by use of a Lie group invariance of the original magnetogasdynamic system.

The semi-axes of the time-modulated ellipse

$$a(t)x^2 + 2b(t)xy + c(t)y^2 + h_0(t) = 0 \quad (ac - b^2 > 0) \tag{48}$$

are given by

$$\Phi = \sqrt{\frac{2h_0}{[\sqrt{(a-c)^2 + 4b^2} - (a+c)]}} = \sqrt{\frac{h_0}{(B_N^2 + B_S^2)^{1/2} - \dfrac{B}{2}}},$$

$$\Psi = \sqrt{\frac{2h_0}{[-\sqrt{(a-c)^2 + 4b^2} - (a+c)]}} = \sqrt{\frac{h_0}{-(B_N^2 + B_S^2)^{1/2} - \dfrac{B}{2}}}.$$

On use of (18) and (25), these relations yield

$$\Phi = \Omega\sqrt{c_{\rm I}}\Big/\sqrt{\left(c_{\rm II} + \frac{\bar{B}^2}{4}\right)^{1/2} - \frac{\bar{B}}{2}}, \tag{49}$$

$$\Psi = \Omega\sqrt{c_{\rm I}}\Big/\sqrt{-\left(c_{\rm II} + \frac{\bar{B}^2}{4}\right)^{1/2} - \frac{\bar{B}}{2}} \tag{50}$$

whence, the ratio of the semi-axes is given by

$$\Phi/\Psi = \frac{\sqrt{-c_{\rm II}}}{\left(c_{\rm II} + \dfrac{\bar{B}^2}{4}\right)^{1/2} - \dfrac{\bar{B}}{2}} = \frac{-1}{\sqrt{-c_{\rm II}}}\left[\left(c_{\rm II} + \frac{\bar{B}^2}{4}\right)^{1/2} + \frac{\bar{B}}{2}\right] > 0 \tag{51}$$

where it is required that $\bar{B} < 0$ and $\bar{B}^2/4 > -c_{\rm II} > 0$. Thus, $\bar{B} = \bar{B}(\Phi/\Psi)$ so that, if $\bar{B} \neq$ const then the ratio of the semi-axes the ellipse is determined by the elliptic integral relation (40). If $\bar{B} =$ const, corresponding to the pulsrodon case, then the ratio of the semi-axes of the ellipse (48) is likewise constant.

The relation (51) together with (29) and (35) produce the Ray–Reid type relation $\dot{\Psi}\Phi - \Psi\dot{\Phi} = Z(\Phi/\Psi)$, where

$$\begin{aligned} Z(\Phi/\Psi) &= \frac{2c_{\rm I}}{\sqrt{-c_{\rm II}}}\sqrt{\bar{B} + c_{\rm III}}\sin(\theta - \phi) \\ &= \frac{2c_{\rm I}}{\sqrt{-c_{\rm II}}}\sqrt{\frac{(\bar{B}^2 + 4c_{\rm II})(\bar{B} + c_{\rm III}) - (\bar{B} + c_{\rm IV})^2}{\bar{B}^2 + 4c_{\rm II}}} \end{aligned}$$

and $\bar{B} = -\sqrt{-c_{\rm II}}\,[\Psi/\Phi + \Phi/\Psi]$,with the requirement that $0 < |\bar{B}| < c_{\rm III}$.

It is now readily established that the semi-axes Φ, Ψ of the ellipse (48) are governed by the Ermakov–Ray–Reid system

$$\begin{aligned}
\ddot{\Phi} + \frac{1}{4}f^2\Phi &= \frac{1}{\Phi^2\Psi}\left[\frac{ZZ'}{[1+(\Psi/\Phi)^2]} - \left(\frac{\Psi}{\Phi}\right)\frac{(Z^2+\frac{k}{4})}{[1+(\Psi/\Phi)^2]^2}\right],\\
\ddot{\Psi} + \frac{1}{4}f^2\Psi &= \frac{1}{\Psi^2\Phi}\left[-\left(\frac{\Phi}{\Psi}\right)\frac{(Z^2+\frac{k}{4})}{[1+(\Phi/\Psi)^2]^2} - \frac{ZZ'}{[1+(\Psi/\Phi)^2]}\right],
\end{aligned} \tag{52}$$

where

$$k = (c_{\mathrm{I}}/c_{\mathrm{II}})^2\left[f^2(c_{\mathrm{VI}}^2 + c_{\mathrm{VII}}^2) - \frac{4}{f^2}(c_{\mathrm{V}} + fc_{\mathrm{IV}})^2\right] \quad (f \neq 0).$$

In addition, the system (52) is seen to be Hamiltonian with invariant

$$H = \frac{1}{2}(\dot{\Phi}^2 + \dot{\Psi}^2) - \frac{1}{2(\Phi^2+\Psi^2)}\left[Z^2 + \frac{1}{4}f^2(\Phi^2+\Psi^2)^2 + \frac{k}{4}\right]$$

and is readily integrated via the general procedure summarised below.

6 Hamiltonian Ermakov systems

The Ermakov–Ray–Reid system

$$\ddot{\alpha} + \Theta'(\alpha^2+\beta^2)\alpha = \frac{1}{\alpha^2\beta}F(\beta/\alpha), \quad \ddot{\beta} + \Theta'(\alpha^2+\beta^2)\beta = \frac{1}{\alpha\beta^2}G(\alpha/\beta) \tag{53}$$

is Hamiltonian with $\ddot{\alpha} = -\partial V/\partial\alpha$, $\ddot{\beta} = -\partial V/\partial\beta$ provided

$$\frac{\partial}{\partial\beta}\left[\frac{1}{\alpha^2\beta}F(\beta/\alpha)\right] = \frac{\partial}{\partial\alpha}\left[\frac{1}{\alpha\beta^2}G(\alpha/\beta)\right].$$

Accordingly, the system (53) necessarily adopts the form

$$\begin{aligned}
\ddot{\alpha} + \Theta'(\alpha^2+\beta^2)\alpha &= \frac{1}{\alpha^2\beta}\left[2\left(\frac{\beta}{\alpha}\right)J(\alpha/\beta) - \frac{dJ(\alpha/\beta)}{d(\alpha/\beta)}\right],\\
\ddot{\beta} + \Theta'(\alpha^2+\beta^2)\beta &= \frac{1}{\alpha\beta^2}\frac{dJ(\alpha/\beta)}{d(\alpha/\beta)}
\end{aligned}$$

so that $V = J(\alpha/\beta)/\alpha^2 + \Theta(\alpha^2+\beta^2)/2$ and the Hamiltonian is given by

$$H = \frac{1}{2}\left[\dot{\alpha}^2 + \dot{\beta}^2 + \Theta(\alpha^2+\beta^2)\right] + \frac{1}{\alpha^2}J(\alpha/\beta). \tag{54}$$

The Ray–Reid invariant becomes

$$I = \frac{1}{2}(\alpha\dot{\beta} - \dot{\alpha}\beta)^2 + \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right) J(\alpha/\beta) \tag{55}$$

and combination with (54) yields

$$(\alpha^2+\beta^2)H - I = \frac{1}{2}(\alpha\dot{\alpha} + \beta\dot{\beta})^2 + \frac{1}{2}(\alpha^2+\beta^2)\Theta(\alpha^2+\beta^2). \tag{56}$$

If we set $\Sigma = \alpha^2 + \beta^2$, then (56) shows that

$$\frac{1}{8}\dot{\Sigma}^2 + \left[\frac{1}{2}\Theta(\Sigma) - H\right]\Sigma + I = 0.$$

In particular, if $\Theta = \Sigma$ then $\Sigma = [H \pm \sqrt{H^2 - 2I}\sin 2(t - t_0)]$.

Introduction of the expression $\Lambda = 2\alpha\beta/(\alpha^2+\beta^2)$into the Ray–Reid invariant (55) yields

$$I = \frac{1}{8}\dot{\Lambda}^2 \frac{(\alpha^2+\beta^2)^4}{(\alpha^2-\beta^2)^2} + \left(\frac{\alpha^2+\beta^2}{\alpha^2}\right) J(\alpha/\beta),$$

while, introduction of T according to $dT/dt = 1/\Sigma$ shows that

$$\frac{d\Lambda}{dT} = \pm 2\sqrt{2(1-\Lambda^2)\left[I - \frac{2}{\Lambda}\frac{\beta}{\alpha}J(\alpha/\beta)\right]}$$

where β/α is given in terms of Λ via $\beta/\alpha = (1 \pm \sqrt{1-\Lambda^2})/\Lambda$. Thus, Λ is given by

$$\pm\frac{1}{2\sqrt{2}}\int\left[\frac{\Lambda}{(1-\Lambda^2)(\Lambda I - 2L(\Lambda))}\right]^{1/2} d\Lambda = T + T_0,$$

where $L(\Lambda) = (\beta/\alpha)J(\alpha/\beta)$. The original Ermakov variables α, β are given in terms of Σ and Λ by the relations

$$\alpha = \left[\sqrt{\Sigma(1+\Lambda)} \mp \sqrt{\Sigma(1-\Lambda)}\right]/2,$$

$$\beta = \left[\sqrt{\Sigma(1+\Lambda)} \pm \sqrt{\Sigma(1-\Lambda)}\right]/2.$$

For the Ermakov–Ray–Reid system (52), once Φ, Ψ have been determined then Ω and $\bar{B}$ are given by (49)–(50) and then θ, ϕ are obtained by integration of the relations (36), (37). The residual magnetogasdynamic variables are then readily constructed.

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[1] Biskamp D., *Nonlinear Magnetohydrodynamics*, Cambridge University Press, 1993.

[2] Lee J.H., Pashaev O.K., Rogers C. and Schief W.K., The resonant nonlinear Schrödinger equation in cold plasma physics. Application of Bäcklund–Darboux transformations and superposition principles, *J. Plasma Phys.*, 2007, V.73, 257–272.

[3] Schief W.K., Hidden integrability in ideal magnetohydrodynamics. The Pohlmeyer–Lund–Regge Model, *Phys. Plasmas*, 2003, V.10, 2677–2685.

[4] Schief W.K., Nested toroidal flux surfaces in magnetohydrostatics: generation via soliton theory, *J. Plasma Phys.*, 2003, V.69, 465–484.

[5] Rogers C. and Schief W.K., Novel integrable reductions in nonlinear continuum mechanics via geometric constraints, *J. Math. Phys.*, 2003, V.44, 3341–3369.

[6] Rogers C. and Kingston J.G., Non–dissipative magnetodydrodynamic flows with magnetic and velocity field lines orthogonal geodesics on a normal congruence, *SIAM J. Appl. Math*, 1974, V.26, 183–195.

[7] Rogers C. and Schief W.K., On a Bernoulli–integrable magnetohydrodynamic system, *Zeit. angew. Math. Phys.*, 2009, V.60, 608–620.

[8] Zabusky N.J. and Kruskal M.D., Interaction of solitons in a collisionless plasma and the recurrence of initial states, *Phys. Rev. Lett.*, 1965, V.15, 240–243.

[9] Ablowitz M.J. and Clarkson P.A., *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, London Mathematical Society Lecture Notes Series, 1992.

[10] Rogers C. and Schief W.K., *Bäcklund and Darboux Transformations. Geometry and Modern Applications in Soliton Theory*, Cambridge University Press, 2002.

[11] Power G. and Rogers C., Substitution principles in non–steady magnetogasdynamics, *Appl. Sci. Res.*, 1969, V.21, 176–184.

[12] Oliveri F., On substitution principles in ideal magneto-gasdynamics by means of Lie group analysis, *Nonlinear Dynam.*, 2005, V.42, 217–231.

[13] Galas F. and Richter E.W., Exact similarity solutions of ideal MHD equations for plane motions, *Phys. D*, 1991, V.50, 297–307.

[14] Fuchs J.C., Symmetry groups and similarily solutions of mhd equations, *J. Math. Phys.*, 1991, V.32, 1703–1708.

[15] Neukirch T., Quasi-equilibria: a special class of time-dependent solutions for two-dimensional magnetohydrodynamics, *Phys. Plasmas*, 1995, V.2, 4389–4399.

[16] Neukirch T. and Priest E.R., Generalization of a special class of time-dependent solutions of the two-dimensional magnetohydrodynamic equations to arbitrary pressure profiles, *Phys. Plasmas*, 2000, V.7, 3105–3107.

[17] Neukirch T. and Cheung D.L.G., A class of accelerated solutions of the two-dimensional ideal magnetohydrodynamic equations, *Proc. Roy. Soc. Lond. Ser. A*, 2001, V.457, 2547–2566.

[18] Goldsbrough G.R., The tidal oscillations in an elliptic basin of variable depth, *Proc. Roy. Soc. Lond. Ser. A*, 1930, V.130, 157–167.

[19] Kirchhoff G., *Vorlesungen über Mathematische Physik*, V.I, Leipzig: Teubner, 1876.

[20] Rogers C., Elliptic warm-core theory: the pulsrodon, *Phys. Lett. A*, 1989, V.138, 267–273.

[21] Ray J.R., Nonlinear superposition law for generalised Ermakov systems, *Phys. Lett. A*, 1980, V.78, 4–6.

[22] Reid J.L. and Ray J.R., Ermakov systems, nonlinear superposition and solution of nonlinear equations of motion, *J. Math. Phys.*, 1980, V.21, 1583–1587.

[23] Rogers C., Hoenselaers C. and Ray J.R., On (2+1)-dimensional Ermakov systems, *J. Phys. A: Math. Gen.*, 1993, V.26, 2623–2633.

[24] Rogers C. and Schief W.K., Multi-component Ermakov systems: structure and linearization, *J. Math. Anal. Appl.*, 1996, V.198, 194–220.

[25] Schief W.K., Rogers C. and Bassom A., Ermakov systems with arbitrary order and dimension. Structure and linearization, *J. Phys. A: Math Gen*, 1996, V.29, 903–911.

[26] Wagner W.G., Haus H.A. and Marburger J.H., Large-scale self-trapping of optical beams in the paraxial ray approximation, *Phys. Rev.*, 1968, V.175, 256–266.

[27] Giuliano C.R., Marburger J.H. and Yariv A., Enhancement of self-focussing threshold in sapphire with elliptical beams, *Appl. Phys. Lett.*, 1972, V.21, 58–60.

[28] Cornolti F., Lucchesi M. and Zambon B., Elliptic Gaussian beam self-focussing in nonlinear media, *Optics Comm.*, 1990, V.75, 129–135.

[29] Goncharenko A.M., Logvin Yu A., Samson A.M., Shapovalov P.S. and Turovets S.I., Ermakov Hamiltonian systems in nonlinear optics of elliptic Gaussian beams, *Phys. Lett. A*, 1991, V.160, 138–142.

[30] Goncharenko A.M., Logvin Yu A., Samson A.M. and Shapovalov P.S., Rotating elliptical gaussian beams in nonlinear media, *Optics Comm.*, 1991, V.81, 225–230.

[31] Goncharenko A.M., Kukuskin V.G., Logvin Yu A. and Samson A.M., Self-focussing of orthogonally polarised light beams in a nonlinear medium, *Optical and Quantum Electronics*, 1993, V.25, 97–104.

[32] Capobianco A.D., Constantini B., Angeles C. and Nalesso G.F., Two-dimensional Hamiltonian description of nonlinear optical waveguides, *Opt. Quant. Electron.*, 1994, V.26, 5335–5348.

[33] Rogers C., Malomed B., Chow K.W. and An H., Ermakov–Ray–Reid systems in nonlinear optics, *J. Phys. A: Math. Theor.*, 2010, V.43, 455214, 15 pp.

[34] Schäfer G., *Gravity-wave astrophysics*, Lecture Notes in Physics, 1992, V.410, 163–183.

[35] Steen A., Om Formen for Integralet af den lineaere Differentialligning af anden Orden, Overs. over d.k. Danske Vidensk. Selsk. Forh, 1874, 1–12.

[36] Ermakov V.P., Second-order differential equations. Conditions for complete integrability, *Univ. Izv. Kiev*, 1880, V.20, 1–25.

[37] Nassar A.B., Ermakov and non-Ermakov systems in quantum dissipative models, *J. Math. Phys.*, 1986, V.27, 755–758.

[38] Fernández-Guasti M., Jiménez J.L., Granados-Agustin F. and Cornejo-Rodriguez A., Amplitude and phase representation of monochromatic fields in physical optics, *J. Opt. Soc. Am. A*, 2003, V.20, 1629–1634.

[39] Shahinpoor M. and Nowinski J.L., Exact solutions to the problem of forced large amplitude radial oscillations of a thin hyperelastic tube, *Int. J. Nonlinear Mech.*, 1971, V.6, 193–208.

[40] Holm D.D., Elliptical vortices and integrable Hamiltonian dynamics of the rotating shallow–water equations, *J. Fluid Mech.*, 1991, V.227, 393–406.

[41] Rogers C. and Schief W.K., The pulsrodon in 2+1-dimensional magnetogasdynamics. Hamiltonian structure and integrability, preprint, 2010.

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Travelling-wave solutions for viscoelastic models

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We study a model proposed in some recent papers by Ruggieri and Valenti from the point of view of the theory of symmetry reduction of partial differential equations. The author, among the solutions admitted by such a class of viscoelastic models, obtains travelling waves.

1 Introduction

In some recent papers, motivated by a number of physical problems discussed in [1, 2], Ruggieri and Valenti have studied the group properties of

$$w_{tt} = f(w_x)w_{xx} + [\lambda(w_x)\, w_{tx}]_x, \tag{1}$$

where f and λ are smooth functions, $w(t,x)$ is the dependent variable and subscripts denote partial differentiation with respect to the independent variables t and x.

Ruggieri and Valenti in Ref. [3], after having observed that by setting $w_x = u$ and $w_t = v$ equation (1) can be written as a 2×2 system in conservative form

$$u_t - v_x = 0, \quad v_t - \left(\int^u f(s)\, ds + \lambda(u)\, v_x\right)_x = 0, \tag{2}$$

have proved that the group classifications of (1) and (2) are identical in the sense that *for any f and λ a point symmetry admitted by* (1) *induces a point symmetry admitted by* (2) *and vice versa.*

Moreover it is worthwhile noting that system (2) can be regarded as the potential system associated with the equation

$$u_{tt} = [f(u)u_x]_x + [\lambda(u)\, u_t]_{xx} \tag{3}$$

so that point symmetries of the potential system (2), if they exist, allow one to obtain nonlocal symmetries (potential symmetries [4]) of equation (3).

In a recent work [5] Ruggieri and Valenti studied the group properties of equation (3) and, comparing the classification of the equation (3) to that of the system (2), they stated that *the point symmetries of the system* (2) *do not induce any potential symmetries of the equation* (3) *but only point symmetries; conversely there are point symmetries of the equation* (3) *which do not induce point symmetries of the potential system* (2).

When $\lambda(u) \equiv 0$, equation (3) includes the nonlinear homogeneous vibrating-string equation $u_{tt} = [f(u)\, u_x]_x$ which was classified by Ames et al. [6] and gives rise to numerous publications on symmetry analysis of nonlinear wave phenomena (see [7] and references therein for a review).

When $\lambda(u) \equiv \lambda_0$ with λ_0 a positive constant, a complete symmetry classification can be found in the papers of Ruggieri and Valenti [8, 9]. Moreover, when $\lambda_0 = \varepsilon \ll 1$, a study performed by means of approximate symmetries can be found in Valenti [10, 11].

In [5] Ruggieri and Valenti also found travelling-wave solutions for equation (3) in the case of *ideally hard material*, the main feature of which is that the Lagrangian speed of sound increases monotonically without bound.

In this paper the author seeks travelling-wave solutions of the third-order partial differential equation (3) in other cases of physical interest.

2 Travelling-wave solutions

We consider an homogeneous viscoelastic bar of uniform cross-section and assume that the material is a nonlinear Kelvin solid. This model is described by a stress-strain relation of the following form [12]

$$\tau = \sigma(w_x) + \lambda(w_x)\, w_{xt},$$

where τ is the stress, x the position of a cross-section in the homogeneous rest configuration of the bar, $w(t, x)$ the desplacement at time t of the section from the rest position, $\sigma(w_x)$ is the elastic part of the stress while $\lambda(w_x)w_{tx}$ is the dissipative part.

In the absence of body forces the equation of linear momentum, $w_{tt} = \tau_x$, can be reduced to equation (3) after setting $w_x = u$ and introducing the function $f(u)$ such that

$$\sigma(u) = \int^u f(s)\, ds. \tag{4}$$

In this section we search for travelling-wave solutions of equation (3) for specific functional forms of f and λ of physical interest.

Travelling waves are very interesting from the point of view of applications. These types of waves do not change their shapes during propagation and are thus easy to detect. Of particular interest are three types of travelling waves: the solitary waves, which are localized travelling waves, asymptotically zero at large distances; the periodic waves; and the kink waves, which rise or descend from one asymptotic state to another.

In order to search for travelling-wave solutions for equation (3) we consider that the Principal Lie Algebra $\mathcal{L}_\mathcal{P}$ of (3) (see Ref. [5]) is two-dimensional and is spanned by the operators

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}$$

and we observe that the third-order partial differential equation (3) admits travelling-wave solutions for arbitrary f and λ.

Reducing the equation (3) by means of Principal Lie Algebra we obtain that the similarity variable, the similarity solution and the reduced ordinary differential equation (ODE) of (3) respectively are

$$z = x - c_2\, t, \quad u = \phi(z), \tag{5}$$

$$c_2^2\, \phi'' - \left(f\, \phi'\right)' + c_2\, \left(\lambda\, \phi'\right)'' = 0 \tag{6}$$

with f and λ arbitrary functions of ϕ.

We consider the following form for the tension function, $\sigma(u)$,

$$-\sigma(u) = 2\,\tilde{\gamma}^2 \rho_0 (-u)^{\frac{1}{2}}, \quad \tilde{\gamma} = \text{const},$$

which arises, as Bell has shown [13,14], in polycrystalline solids during a dynamic uniaxial compression.

When we make this assumption, take into account (4) and choose for the compatibility of the problem [5] the following expression for the function $\lambda = \lambda_0\,(-\phi)^{-\frac{1}{2}}$ with $\lambda_0 > 0$, the reduced equation (6) becomes

$$c_2^2\, \phi'' - \tilde{\gamma}^2 \rho_0 \left[(-\phi)^{-\frac{1}{2}}\ \phi'\right]' + c_2\, \lambda_0 \left[(-\phi)^{-\frac{1}{2}}\ \phi'\right]'' = 0. \tag{7}$$

A closed-form solution of (7) is

$$\phi = \left[\frac{2\,\tilde{\gamma}^2 \rho_0}{e^{\frac{-\,\tilde{\gamma}^2 \rho_0}{c_2\,\lambda_0}\,(z+k_1)} + c_2^2}\right]^2 \tag{8}$$

with k_1 an arbitrary constant of integration.

When we revert to the original variables (5) and take (8) into account, the solution can be written as

$$u = \left[\frac{2\tilde{\gamma}^2 \rho_0}{e^{\frac{-\,\tilde{\gamma}^2 \rho_0}{c_2\,\lambda_0}\,(x - c_2\, t + k_1)} + c_2^2}\right]^2. \tag{9}$$

Another solution of physical interest can be obtained when we consider the following form of the tension

$$\sigma(u) = -\sigma_0 \left(\frac{3T_0}{\rho V_0^2}\right)^3 \left(\frac{3T_0}{\rho V_0^2} + u\right)^{-3} + \sigma_0,$$

which models the *ideal soft material* the main feature of which is that the lagrangian speed of sound decreases monotonically to zero as u increases without bound [13,15].

In this case, after we take into account (4) and choose for the compatibility of the problem [5] the following expression for the function

$$\lambda = \lambda_0 \left(\phi + \frac{3T_0}{\rho V_0^2} \right)^{-4}$$

with $\lambda_0 > 0$, the reduced equation (6) becomes

$$c_2^2\,(\rho V_0^2)^3 \phi'' - (3T_0)^4 \left[\left(\phi + \frac{3T_0}{\rho V_0^2} \right)^{-4} \phi' \right]'$$
$$+c_2\,\lambda_0 (\rho V_0^2)^3 \left[\left(\phi + \frac{3T_0}{\rho V_0^2} \right)^{-4} \phi' \right]'' = 0. \tag{10}$$

A closed-form solution of (10) is

$$\phi = 3T_0 \left[(\rho V_0^2)^3 \left(e^{\frac{108\,T_0^4}{c_2\,\lambda_0\,\rho^3 V_0^6}\,(z+k)} - 3\,c_2^2 \right) \right]^{-\frac{1}{4}} - \frac{3T_0}{\rho V_0^2} \tag{11}$$

with k an arbitrary constant of integration.

When we revert to the original variables (5) and take (11) into account, the solution can be written as

$$u = 3T_0 \left[(\rho V_0^2)^3 \left(e^{\frac{108\,T_0^4}{c_2\,\lambda_0\,\rho^3 V_0^6}\,(x-c_2\,t+k)} - 3\,c_2^2 \right) \right]^{-\frac{1}{4}} - \frac{3T_0}{\rho V_0^2}. \tag{12}$$

The travelling-waves solutions (9)–(12) have the form of a *kink* and it is known that *kinks* may propagate in a viscoelastic medium (see [2] and bibliography therein).

It is worthy of note that by means of the relations $w_x = u$ and $w_t = v$, starting from the above solutions, we can construct solutions of the equation (1) and system (2). More precisely solutions (9)–(12) give rise to invariant solutions of (1) and (2). In fact it is a simple matter to ascertain that, if we set $\psi' = \phi$, expressions (8)–(11) satisfy the reduced equation (15) in Ref. [16] when $c_4 = 0$.

In fact, if we consider (9) from

$$\psi = \int^z \phi(s)ds,$$

we obtain the solution

$$\psi = 4\,\tilde{\gamma}^4 \rho_0^2 \left[\frac{\lambda_0 \log\left(e^{-\frac{\tilde{\gamma}^2 \rho_0\, z}{c_2 \lambda_0}} + c_2^2\right)}{\tilde{\gamma}^2 \rho_0 c_2^3} + \frac{z}{c_2^4} - \frac{\lambda_0}{c_2 \tilde{\gamma}^2 \rho_0 e^{-\frac{\tilde{\gamma}^2 \rho_0\, z}{c_2 \lambda_0}} + c_2^3 \tilde{\gamma}^2 \rho_0} \right] \tag{13}$$

for the reduced equation (15) in [16] and coming back to the original variables we can obtain the corresponding invariant solutions of both equation (1) and system (2).

In a similar way it is possible to proceed to the solution of (12). In fact from $\psi = \int^z \phi(s)ds$ we obtain the solution

$$\psi = \log\left[\frac{\sqrt{\mu - 3c_2^2} - 3^{\frac{1}{4}}\sqrt{2\,|c_2|}\,\left(\mu - 3c_2^2\right)^{\frac{1}{4}} + \sqrt{3}|c_2|}{\sqrt{\mu - 3c_2^2} + 3^{\frac{1}{4}}\sqrt{2\,|c_2|}\,\left(\mu - 3c_2^2\right)^{\frac{1}{4}} + \sqrt{3}|c_2|}\right]^{p/2}$$

$$+p\left\{\arctan\left[\frac{2\left(\mu - 3c_2^2\right)^{\frac{1}{4}}}{3^{\frac{1}{4}}\sqrt{2\,|c_2|}} + 1\right] + \arctan\left[\frac{2\left(\mu - 3c_2^2\right)^{\frac{1}{4}}}{3^{\frac{1}{4}}\sqrt{2\,|c_2|}} - 1\right]\right\},$$

where

$$\mu = \exp\left(\frac{108T_0^4\,(z+k)}{c_2\,\lambda_0(\rho V_0^2)^3}\right), \quad p = \frac{(\rho V_0^2)^{\frac{9}{4}} c_2\lambda_0}{3^{\frac{9}{4}}\,2\sqrt{2|c|}T_0^4},$$

of the reduced equation (15) in [16] and coming back to the original variables we can obtain the corresponding invariant solutions of both equation (1) and system (2).

3 Conclusions

In this paper we have derived travelling-wave solutions for the third-order partial differential equation (3) in the cases of ideal soft material and in polycrystalline solids during a dynamic uniaxial compression.

We have also observed that, by means of the relations $w_x = u$ and $w_t = v$, starting from these solutions of the equation (3) we can construct invariant solutions of the equation (1) and the system (2).

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[1] Rajagopal K.R. and Saccomandi G., Shear waves in some models of nonlinear viscoelasticity, *Q. J. Mech. Appl. Math.*, 2003, V.56, 311–326.

[2] Destrade M., Jordan P.M. and Saccomandi G., Compact travelling waves in viscoelastic solids, *Europhys. Lett.*, 2009, V.87, 48001.

[3] Ruggieri M. and Valenti A., Symmetry analysis of a viscoelastic model, *Proceedings of WASCOM07*, World Scientific, 2008, 514–519.

[4] Bluman G.W. and Kumei S., *Symmetries and Differential Equations*, Springer-Verlag, New York, 1989.

[5] Ruggieri M. and Valenti A., Exact solutions for a nonlinear model of dissipative media, 2010.

[6] Ames, W.F., Lohner, R.J. and Adams, E., Group properties of $u_{tt} = [f(u)u_x]_x$, *Int. J. Non-Linear Mech.*, 1981, V.16, 439–447.

[7] Ibragimov N.H., *CRC Handbook of Lie Group Analysis of Differential Equations*, CRC Press, Boca Raton, FL, 1994.

[8] Ruggieri M. and Valenti A., Group analysis of a nonlinear model describing dissipative media, *Proceedings of Tenth International Conference in Modern Group Analysis (Larnaca, Cyprus, 2004)*, 2005, 175–181.

[9] Ruggieri M. and Valenti A., Symmetries and reduction techniques for a dissipative model, *Proceedings of WASCOM05*, 2006, 481–492.

[10] Valenti A., Approximate symmetries for a model describing dissipative media, *Proceedings of Tenth International Conference in Modern Group Analysis (Larnaca, Cyprus, 2004)*, 2005, 236–243.

[11] Valenti A., Approximate symmetries of a viscoelastic model, *Proceedings of WASCOM07*, World Scientific, 2008, 582.

[12] MacCamy R.C., Existence, uniqueness and stability of $u_{tt} = \frac{\partial}{\partial x}[\sigma(u_x) + \lambda(u_x)u_{xt}]$, *Indiana Univ. Math. J.*, 1970, V.20, 231–238.

[13] Rogers C. and Shadwick W.F., *Bäcklund Transformations and Their Applications*, Accademic Press, New York, 1982.

[14] Bell J.F., *The Physics of Large Deformations in Crystalline Solids*, Springer-Verlag, Berlin, 1986.

[15] Cekirge H.M. and Varley E., Large amplitude waves in bounded media: reflection and transmission of large amplitude shockless at interface, *Phil. Trans. R. Soc. Lond.*, 1973, V.273, 261–313.

[16] Ruggieri M. and Valenti A., Symmetries and reduction techniques for dissipative models, *J. Math. Phys.*, 2009, V.50, 063506, 9 pp.

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On algebraic classification of Hermitian quasi-exactly solvable matrix Schrödinger operators on line

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We construct six multiparameter families of Hermitian quasi-exactly solvable matrix Schrödinger operators in one variable and five multiparameter families of exactly solvable Schrödinger operators. The method for finding quasi-exactly operators relies heavily upon a special representation of the Lie algebra $o(2,2) \cong sl(2) \oplus sl(2)$ the representation space of which contains an invariant finite-dimensional subspace. For finding exactly solvable model a special set of operators is used, which is an expansion of the matrix representation of the solvable four-dimensional algebra. Furthermore we select those models that have square integrable eigenfunctions on $\mathbb{R}$. These models are in direct analogy with the quasi-exactly solvable scalar Schrödinger operators obtained by Turbiner and Ushveridze.

1 Introduction

In papers [1, 2] we have extended the Turbiner–Shifman approach [3–5] to the construction of quasi-exactly solvable (QES) models on line for the case of matrix Hamiltonians. We recall that originally their method was applied to scalar one-dimensional stationary Schrödinger equations. Later it was extended to the case of multidimensional scalar stationary Schrödinger equations [5–8] (see also [9]).

The procedure of constructing a QES matrix (scalar) model is based upon the concept of a Lie-algebraic Hamiltonian. We call a second-order operator in one variable Lie-algebraic if the following requirements are satisfied:

- The Hamiltonian is a constant coefficient quadratic form of first-order operators, $Q_1, Q_2, \ldots, Q_n$, forming a Lie algebra g;
- The Lie algebra g has a finite-dimensional invariant subspace, $\mathcal{I}$, of the whole representation space.

Now, if a given Hamiltonian $H[x]$ is Lie-algebraic, then after being restricted to the space $\mathcal{I}$ it becomes a matrix operator $\mathcal{H}$ the eigenvalues and eigenvectors of whhich are computed in a purely algebraic way. This means that the Hamiltonian $H[x]$ is quasi-exactly solvable (for further details on scalar QES models see [9]).

We impose no *a priori* restrictions on the form of basis elements of the space $\mathcal{I}$, namely we fix the class to which the basis elements of the Lie algebra g should belong. Following [1, 2] we choose this class $\mathcal{L}$ as the set of matrix differential operators of the form

$$\mathcal{L} = \{Q\colon Q = a(x)\partial_x + A(x)\}\,.$$

Here $a(x)$ is a smooth real-valued function and $A(x)$ is an $N \times N$ matrix the entries of which are smooth complex-valued functions of x. Hereafter we denote d/dx as ∂_x.

Evidently $\mathcal{L}$ can be treated as an infinite-dimensional Lie algebra with a standard commutator as a Lie bracket. Given a subalgebra $\langle Q_1,\, Q_2,\, \ldots,\, Q_n\rangle$ of the algebra $\mathcal{L}$, the representation space of whhich contains a finite-dimensional invariant subspace, we can easily construct a QES matrix model. To this end we compose a bilinear combination of the operators $Q_1,\, Q_2,\, \ldots,\, Q_n$ (one of them may be the unit $N \times N$ matrix I) with constant complex coefficients α_{jk} and get

$$H[x] = \left(\sum_{j,k=1}^{n} \alpha_{jk} Q_j Q_k\right). \qquad (1)$$

As is well-known, a physically meaningful QES matrix Schrödinger operator has to be Hermitian. This requirement imposes restrictions on the choice of QES models which somehow were beyond considerations of our previous papers [1,2]. It should be noted that a problem of reducing QES scalar operator to an Hermitian form is fairly trivial and is solved straightforwardly by rearranging a dependent variable and making an appropriate gauge transformation of the wave-function. However, for the case of matrix QES first- or second-order operators the problem of transforming these to Hermitian Schrödinger forms becomes nontrivial and requires very involved calculations. It occurs that, in contrast to the scalar case, not every second-order matrix QES operator can be reduced to an Hermitian form. One of the principal aims of the present paper is to develop a systematic algebraic procedure for constructing QES Hermitian matrix Schrödinger operators

$$\hat{H}[x] = \partial_x^2 + V(x). \qquad (2)$$

This requires a slight modification of the algebraic procedure used in [2]. We consider as an algebra g the direct sum of two $sl(2)$ algebras which is equivalent to the algebra $o(2, 2)$. The necessary algebraic structures are introduced in Section 2. The next section is devoted to constructing in a regular way Hermitian QES matrix Schrödinger operators on line which is a core result of the paper. We give the list of QES models thus obtained in Section 4.

A stronger constraint imposed on the QES Schrödinger operators is that the basis elements of invariant space $\mathcal{I}$ must be square integrable on $\mathbb{R}$. A detailed study of this problem for the case of scalar QES Schrödinger operators has been carried out recently in [10]. Using the results mentioned above we have constructed in the present paper several classes of QES matrix Schrödinger operators (Hamiltonians) having finite-dimensional invariant spaces the basis elements of which are square integrable on $\mathbb{R}$. As examples we present below two such Hamiltonians without giving derivation details which are based on tedious calculations of Sections 2–4.

Model 1. $(\hat{H}[y]+E)\psi(y)=0$, where

$$\hat{H}[y]=\partial_y^2-\frac{y^6}{256}+\frac{4m-1}{16}y^2-\frac{1}{4}y^2\sigma_3-\sigma_1.$$

The invariant space $\mathcal{I}$ of this operator has the dimension $2m$ and is spanned by the vectors

$$\begin{aligned}\vec{f}_j &= \exp\left(-\frac{y^4}{64}\right)\left(\frac{y}{2}\right)^{2j}\vec{e}_1,\\ \vec{g}_k &= \exp\left(-\frac{y^4}{64}\right)\left(m\left(\frac{y}{2}\right)^{2k}\vec{e}_2-k\left(\frac{y}{2}\right)^{2k-2}\vec{e}_1\right),\end{aligned}$$

where $j=0,\dots,m-2,\ k=0,\dots,m,\ \vec{e}_1=(1,0)^T,\vec{e}_2=(0,1)^T$ and m is an arbitrary natural number.

It is not difficult to verify that the basis vectors of the invariant space $\mathcal{I}$ are square integrable on the interval $(-\infty,+\infty)$. One further remark is that there exists an analogous QES scalar Schrödinger operator the invariant space of which has square integrable basis vectors (see, for more details [3, 11]).

Model 2. $(\hat{H}[y]+E)\psi(y)=0$, where

$$\begin{aligned}\hat{H}[y]=\partial_y^2&-\frac{1}{4}-\frac{1}{4}\exp(-2y)+m\exp(-y)+\frac{1}{2}\exp(2y)\\ &+\left[m\frac{\sqrt{3}+1}{2}\sin(\sqrt{2}e^y)-\frac{\sqrt{6}}{2}\cos(\sqrt{2}e^y)-\exp(-y)\sin(\sqrt{2}e^y)\right]\sigma_1\\ &+\left[m\frac{\sqrt{3}+1}{2}\cos(\sqrt{2}e^y)+\frac{\sqrt{6}}{2}\sin(\sqrt{2}e^y)-\exp(-y)\cos(\sqrt{2}e^y)\right]\sigma_3.\end{aligned}$$

The invariant space $\mathcal{I}$ of this operator has dimension $2m$ and is spanned by the vectors

$$\begin{aligned}&\vec{f}_j=U^{-1}(y)\exp(-jy)\vec{e}_1,\\ &\vec{g}_k=U^{-1}(y)\left(m\exp(-ky)\vec{e}_2-k\exp(-(k-1)y)\vec{e}_1\right),\end{aligned}$$

where $j=0,\dots,m-2, k=0,\dots,m$, m is an arbitrary natural number and

$$\begin{aligned}U^{-1}(y)=&\frac{1}{2\sqrt{2}}\exp\left(-\frac{y}{2}\right)\exp\left(-\frac{1}{2}e^{-y}\right)\\ &\times(\sqrt{3}+\sqrt{2}-\sigma_3)\left[\cos(\sqrt{2}e^y)+\frac{i\sqrt{3}\sigma_2-\sigma_1}{\sqrt{2}}\sin(\sqrt{2}e^y)\right].\end{aligned}$$

The basis vectors of the invariant space $\mathcal{I}$ are square integrable. Indeed the functions $\vec{f}_j(y)$ and $\vec{g}_k(y)$ behave asymptotically as $\exp\{-(2j+1)y/2\}$ and $\exp\{-(2k+1)y/2\}$, correspondingly, with $y \to +\infty$. Furthermore they behave as $\exp\{-(2j+1)y/2\}\exp\{-e^{-y}/2\}$ and $\exp\{-(2k+1)y/2\}\exp\{-e^{-y}/2\}$, correspondingly, with $y \to -\infty$. This means that they vanish rapidly provided $y \to \pm\infty$.

2 Extension of the algebra $sl(2)$

Following [1,2] we consider the realization of the algebra $sl(2)$

$$
\begin{aligned}
sl(2) &= \langle Q_-,\ Q_0,\ Q_+\rangle \\
&= \Big\langle \partial_x,\ x\partial_x - \frac{m-1}{2} + S_0,\ x^2\partial_x - (m-1)x + 2S_0x + S_+\Big\rangle,
\end{aligned}
\tag{3}
$$

where $S_0 = \sigma_3/2$, $S_+ = (i\sigma_2 + \sigma_1)/2$, σ_k are the 2×2 Pauli matrices,

$$
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},
$$

and $m \geq 2$ is an arbitrary natural number. This representation gives rise to a family of QES models and furthermore algebra (3) has the following finite-dimensional invariant space

$$
\begin{aligned}
\mathcal{I}_{sl(2)} = \mathcal{I}_1 \oplus \mathcal{I}_2 = \langle \vec{e}_1, x\vec{e}_1, \ldots, x^{m-2}\vec{e}_1\rangle \oplus \\
\langle m\vec{e}_2, \ldots, mx^j\vec{e}_2 - jx^{j-1}\vec{e}_1, \ldots, mx^m\vec{e}_2 - mx^{m-1}\vec{e}_1\rangle.
\end{aligned}
\tag{4}
$$

Since the spaces $\mathcal{I}_1$, $\mathcal{I}_2$ are invariant with respect to an action of any of the operators (3), the above representation is reducible. A more serious problem is that it is not possible to construct a QES operator, that is equivalent to a Hermitian Schrödinger operator, by taking a bilinear combination (1) of operators (3) with coefficients being complex numbers. To overcome this difficulty we use the idea indicated in [2] and let the coefficients of the bilinear combination (1) be constant 2×2 matrices. To this end we introduce a wider Lie algebra and add to (3) the following three matrix operators:

$$
R_- = S_-, \quad R_0 = S_-x + S_0, \quad R_+ = S_-x^2 + 2S_0x + S_+, \tag{5}
$$

where $S_\pm = (i\sigma_2 \pm \sigma_1)/2$. It is straightforward to verify that the space (4) is invariant with respect to an action of a linear combination of the operators (5). Consider next the following set of operators:

$$
\langle T_\pm = Q_\pm - R_\pm,\ T_0 = Q_0 - R_0,\ R_\pm,\ R_0,\ I\rangle, \tag{6}
$$

where Q and R are operators (3) and (5), respectively, and I is a unit 2×2 matrix. By a direct computation we check that the operators $T_\pm$, T_0 as well as the operators $R_\pm$, R_0, fulfill the commutation relations of the algebra $sl(2)$. Furthermore any of the operators $T_\pm$, T_0 commutes with any of the operators $R_\pm$ and R_0. Consequently operators (6) form the Lie algebra

$$sl(2) \oplus sl(2) \oplus I \cong o(2,2) \oplus I.$$

In the sequel we denote this algebra as g. The Casimir operators of the Lie algebra g are multiples of the unit matrix

$$C_1 = T_0^2 - T_+T_- - T_0 = \left(\frac{m^2-1}{4}\right) I, \quad K_2 = R_0^2 - R_+R_- - R_0 = \frac{3}{4} I.$$

Using this fact it can be shown that the representation of g realized on the space $\mathcal{I}_{sl(2)}$ is irreducible.

One more remark is that the operators (6) satisfy the following relations:

$$\begin{aligned} &R_-^2 = 0, \quad R_0^2 = \frac{1}{4}, \quad R_+^2 = 0, \\ &\{R_-, R_0\} = 0, \quad \{R_+, R_0\} = 0, \quad \{R_-, R_+\} = -1, \\ &R_-R_0 = \frac{1}{2}R_-, \quad R_0R_+ = \frac{1}{2}R_+, \quad R_-R_+ = R_0 - \frac{1}{2}. \end{aligned} \tag{7}$$

Here $\{Q_1, Q_2\} = Q_1Q_2 + Q_2Q_1$. One of the consequences of this fact is that the algebra g may be considered as a superalgebra which shows an evident link to the results of the paper [12].

3 The general form of the Hermitian QES operator

Using the commutation relations of the Lie algebra g together with relations (7) one can show that any bilinear combination of the operators (6) is a linear combination of twenty-one (basis) quadratic forms of these operators. Then it is necessary to transform the bilinear combination (1) to the standard form (2). What is more it is essential that the corresponding transformation should be given by explicit formulae since we need to write explicitly the matrix potential $V(x)$ of the QES Schrödinger operator thus obtained and the basis functions of its invariant space.

The general form of QES model obtainable within the framework of our approach is as follows

$$H[x] = \xi(x)\partial_x^2 + B(x)\partial_x + C(x), \tag{8}$$

where $\xi(x)$ is some real-valued function and $B(x)$ and $C(x)$ are matrix functions of the dimension 2×2. Let $U(x)$ be an invertible 2×2 matrix-function satisfying the system of ordinary differential equations

$$U'(x) = \frac{1}{2\xi(x)} \left(\frac{\xi'(x)}{2} - B(x)\right) U(x), \tag{9}$$

and the function $f(x)$ be defined by the relation

$$f(x) = \pm \int \frac{dx}{\sqrt{\xi(x)}}. \tag{10}$$

Then the change of variables reducing (8) to the standard form (2) is

$$x \to y = f(x), \quad H[x] \to \hat{H}[y] = \hat{U}^{-1}(y) H[f^{-1}(y)] \hat{U}(y), \tag{11}$$

where f^{-1} stands for the inverse of f and $\hat{U}(y) = U(f^{-1}(y))$.

Performance of the transformation (11) yields the Schrödinger operator

$$\hat{H}[y] = \partial_y^2 + V(y) \tag{12}$$

with

$$\begin{aligned} V(y) = \Bigg\{ & U^{-1}(x) \left[-\frac{1}{4\xi} B^2(x) - \frac{1}{2} B'(x) + \frac{\xi'}{2\xi} B(x) + C(x) \right] U(x) + \\ & + \frac{\xi''}{4} - \frac{3\xi'^2}{16\xi} \Bigg\} \Bigg|_{x=f^{-1}(y)}. \end{aligned} \tag{13}$$

Hereinafter the notation $\{W(x)\}_{x=f^{-1}(y)}$ means that we should replace x with $f^{-1}(y)$ in the expression $W(x)$.

Furthermore, if we denote the basis elements of the invariant space (4) as $\vec{f}_1(x)$, ..., $\vec{f}_{2m}(x)$, then the invariant space of the operator $\hat{H}[y]$ takes the form

$$\hat{\mathcal{I}}_{sl(2)} = \left\langle \hat{U}^{-1}(y) \vec{f}_1(f^{-1}(y)), \ldots, \hat{U}^{-1}(y) \vec{f}_{2m}(f^{-1}(y)) \right\rangle. \tag{14}$$

In view of the remark made at the beginning of this section we are looking for such QES models that the transformation law (11) can be given explicitly. This means that we should be able to construct a solution of system (9) in an explicit form. To achieve this goal we select from the above-mentioned set of twenty-one linearly independent quadratic forms of operators (6) the twelve forms,

$$\begin{aligned} & A_0 = \partial_x^2, \; A_1 = x\partial_x^2, \quad A_2 = x^2\partial_x^2 + (m-1)\sigma_3, \\ & B_0 = \partial_x, \; B_1 = x\partial_x + \tfrac{\sigma_3}{2}, \quad B_2 = x^2\partial_x - (m-1)x + \sigma_3 x + \sigma_1, \\ & C_1 = \sigma_1\partial_x + \tfrac{m}{2}\sigma_3, \quad C_2 = i\sigma_2\partial_x + \tfrac{m}{2}\sigma_3, \quad C_3 = \sigma_3\partial_x, \\ & D_1 = x^3\partial_x^2 - 2\sigma_1 x\partial_x + (3m - m^2 - 3)x + (2m-3)x\sigma_3 + (4m-4)\sigma_1, \\ & D_2 = x^3\partial_x^2 - 2i\sigma_2 x\partial_x + (3m - m^2 - 3)x + (2m-3)x\sigma_3 + (4m-4)\sigma_1, \\ & D_3 = 2\sigma_3 x\partial_x + (1-2m)\sigma_3, \end{aligned} \tag{15}$$

the linear combinations of whhich have such a structure that system (9) can be integrated in closed form. However, in the present paper we study systematically the first nine quadratic forms from the above list and exclude the quadratic forms D_1, D_2 and D_3 from further considerations.

Thus the general form of the Hamiltonian to be considered in the sequel is

$$\begin{aligned} H[x] &= \textstyle\sum_{\mu=0}^{2}(\alpha_\mu A_\mu + \beta_\mu B_\mu) + \sum_{i=1}^{3}\gamma_i C_i = (\alpha_2 x^2 + \alpha_1 x + \alpha_0)\partial_x^2 \\ &+ (\beta_2 x^2 + \beta_1 x + \beta_0 + \gamma_1\sigma_1 + i\gamma_2\sigma_2 + \gamma_3\sigma_3)\partial_x + \beta_2\sigma_3 x \\ &- \beta_2(m-1)x + \beta_2\sigma_1 + \left[\alpha_2(m-1) + \tfrac{\beta_1}{2} + \tfrac{m}{2}(\gamma_1+\gamma_2)\right]\sigma_3. \end{aligned} \tag{16}$$

Here α_0, α_1, α_2 are arbitrary real constants and $\beta_0, \ldots, \gamma_3$ are arbitrary complex constants.

If we denote

$$\begin{aligned} &\tilde\gamma_1 = \gamma_1,\ \tilde\gamma_2 = i\gamma_2,\ \tilde\gamma_3 = \gamma_3,\ \delta = 2\alpha_2(m-1) + \beta_1 + m(\gamma_1+\gamma_2), \\ &\xi(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0,\ \eta(x) = \beta_2 x^2 + \beta_1 x + \beta_0, \end{aligned} \tag{17}$$

then the general solution of system (9) is

$$U(x) = \xi^{1/4}(x)\exp\left[-\frac{1}{2}\int\frac{\eta(x)}{\xi(x)}dx\right]\exp\left[-\frac{1}{2}\tilde\gamma_i\sigma_i\int\frac{1}{\xi(x)}dx\right]\Lambda, \tag{18}$$

where Λ is an arbitrary constant invertible 2×2 matrix. Performance of the transformation (11) with $U(x)$ being given by (18) reduces the QES operator (16) to the Schrödinger form (12), where

$$\begin{aligned} V(y) = \Bigg\{ &\frac{1}{4\xi}\Lambda^{-1}\{-\eta^2 + 2\xi'\eta - 2\xi\eta' - 4\beta_2(m-1)x\xi - \tilde\gamma_i^2 \\ &+ 2(\xi'-\eta)\tilde\gamma_i\sigma_i + 4\beta_2\xi U^{-1}(x)\sigma_1 U(x) + (4\beta_2 x + 2\delta)\xi \\ &\times U^{-1}(x)\sigma_3 U(x)\}\Lambda + \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16\xi}\Bigg\}\Bigg|_{x=f^{-1}(y)}. \end{aligned} \tag{19}$$

Here ξ and η are the functions of x defined in (17) and $f^{-1}(y)$ is the inverse of $f(x)$ which is given by (10).

The requirement of hermiticity of the Schrödinger operator (12) is equivalent to the requirement of hermiticity of the matrix $V(y)$. To select from the multi-parameter family of matrices (19) those which are Hermitian we make use of the following technical lemmas (we omit the proof of the first).

Lemma 1. *The matrices $z\sigma_a$, $w(\sigma_a \pm i\sigma_b)$, $a \neq b$, with $\{z, w\} \subset \mathbf{C}$, $z \notin \mathbb{R}$, $w \neq 0$ cannot be reduced to Hermitian matrices with the help of a transformation*

$$A \to A' = \Lambda^{-1}A\Lambda, \tag{20}$$

where Λ is an invertible constant 2×2 matrix.

Lemma 2. *Let $\vec a = (a_1, a_2, a_3)$, $\vec b = (b_1, b_2, b_3)$, $\vec c = (c_1, c_2, c_3)$ be complex vectors and $\vec\sigma$ be the vector the components of which are the Pauli matrices $(\sigma_1, \sigma_2, \sigma_3)$. Then the following assertions hold true.*

1. *A nonzero matrix $\vec{a}\vec{\sigma}$ is reduced to a Hermitian form with the help of a transformation* (20) *iff* $\vec{a}^2 > 0$ *(this inequality means, in particular, that* $\vec{a}^2 \in \mathbb{R}$*);*

2. *Nonzero matrices $\vec{a}\vec{\sigma}$, $\vec{b}\vec{\sigma}$ with $\vec{b} \neq \lambda\vec{a}$, $\lambda \in \mathbb{R}$, are reduced simultaneously to Hermitian forms with the help of a transformation* (20) *iff*
$$\vec{a}^2 > 0, \quad \vec{b}^2 > 0, \quad (\vec{a} \times \vec{b})^2 > 0;$$

3. *Matrices $\vec{a}\vec{\sigma}$, $\vec{b}\vec{\sigma}$, $\vec{c}\vec{\sigma}$ with $\vec{a} \neq \vec{0}$, $\vec{b} \neq \lambda\vec{a}$, $\vec{c} \neq \mu\vec{b}$, $\{\lambda, \mu\} \subset \mathbb{R}$ are reduced simultaneously to Hermitian forms with the help of a transformation* (20) *iff*
$$\vec{a}^2 > 0, \quad \vec{b}^2 > 0, \quad (\vec{a} \times \vec{b})^2 > 0,$$
$$\left\{\vec{a}\vec{c},\ \vec{b}\vec{c},\ (\vec{a} \times \vec{b})\vec{c}\right\} \subset \mathbb{R}.$$

Here we designate the scalar product of vectors $\vec{a}, \vec{b}$ as $\vec{a}\vec{b}$ and their vector product as $\vec{a} \times \vec{b}$.

Proof. We firstly prove the necessity of Assertion 1 of the lemma. Suppose that the nonzero matrix $\vec{a}\vec{\sigma}$ can be reduced to a Hermitian form. We prove that hence the inequality $\vec{a}^2 > 0$ follows.

Consider the matrices:
$$\Lambda_{ij}(a,b) = \begin{cases} 1 + \epsilon_{ijk}\dfrac{\sqrt{a^2+b^2} - b}{a} i\sigma_k, & a \neq 0, \\ 1, & a = 0, \end{cases} \tag{21}$$
where $(i,j,k) = \text{cycle}\,(1,2,3)$. It is not difficult to verify that these matrices are invertible provided
$$\sqrt{a^2+b^2} \neq 0. \tag{22}$$

Under the given condition, (22), the following relations hold
$$\sigma_l \to \Lambda_{ij}^{-1}(a,b)\ \sigma_l\ \Lambda_{ij}(a,b) = \begin{cases} \sigma_k, & l = k, \\ \dfrac{b\sigma_i + a\sigma_j}{\sqrt{a^2+b^2}}, & l = i, \\ \dfrac{-a\sigma_i + b\sigma_j}{\sqrt{a^2+b^2}}, & l = j. \end{cases} \tag{23}$$

As $\vec{a}$ is a nonzero vector, there exists at least one pair of the indices i, j such that $a_i^2 + a_j^2 \neq 0$. Applying the transformation (23) with $a = a_i$ and $b = a_j$ we get
$$\vec{a}\vec{\sigma} \to \vec{a}'\vec{\sigma} = \sqrt{a_i^2 + a_j^2}\,\sigma_j + a_k\sigma_k \tag{24}$$

(no summation over the indices i, j, k is carried out). As the direct check shows, the quantity $\vec{a}^2$ is invariant with respect to transformation (23), i.e. $\vec{a}^2 = \vec{a}'^2$.

If $\vec{a}^2 = 0$, then $a'^2_j + a'^2_k = 0$, or $a'_i = \pm i a'_k$. Hence by force of Lemma 1 it follows that the matrix (24) cannot be reduced to a Hermitian form. Consequently $\vec{a}^2 \neq 0$ and the relation $a'^2_j + a'^2_k \neq 0$ holds true. Applying transformation (23) with $a = \sqrt{a_i^2 + a_j^2}$ and $b = a_k$ we get

$$\vec{a}'\vec{\sigma} \to \sqrt{\vec{a}^2}\,\sigma_k. \tag{25}$$

Due to Lemma 1, if the number $\sqrt{\vec{a}^2}$ is complex, then the above matrix cannot be transformed to a Hermitian matrix. Consequently, the relation $\vec{a}^2 > 0$ holds true.

The sufficiency of Assertion 1 of the lemma follows from the fact that, given the condition $\vec{a}^2 > 0$, the matrix (25) is Hermitian.

Now we prove the necessity of Assertion 2 of the lemma. Firstly we note that due to Assertion 1, $\vec{a}^2 >$ and $\vec{b}^2 > 0$. Next, without loss of generality, we can again suppose that $a_i^2 + a_j^2 \neq 0$. The superposition of two transformations of the form (23) with $a = a_i$, $b = a_j$ and $a = \sqrt{a_i^2 + a_j^2}$, $b = a_k$ yields

$$\begin{aligned}\Lambda_{ij}(a_i, a_j)\Lambda_{jk}\left(\sqrt{a_i^2 + a_j^2}, a_k\right) = 1 + i\epsilon_{ijk}\frac{\sqrt{\vec{a}^2} - a_k}{\sqrt{a_i^2 + a_j^2}}\sigma_i \\ + i\epsilon_{ijk}\frac{\sqrt{a_i^2 + a_j^2} - a_j}{a_i}\sigma_k - i\epsilon_{ijk}\frac{\sqrt{a_i^2 + a_j^2} - a_j}{a_i}\frac{\sqrt{\vec{a}^2} - a_k}{\sqrt{a_i^2 + a_j^2}}\sigma_j\end{aligned} \tag{26}$$

(here the finite limit exists when $a_i \to 0$). When we use this formula and take into account (23), it follows that

$$\begin{aligned}&\vec{a}\vec{\sigma} \to \sqrt{\vec{a}^2}\sigma_k, \\ &\vec{b}\vec{\sigma} \to \vec{b}'\vec{\sigma} = \frac{b_i a_j - b_j a_i}{\sqrt{a_i^2 + a_j^2}}\sigma_i + \frac{a_k \vec{a}\vec{b} - b_k \vec{a}^2}{\sqrt{\vec{a}^2}\sqrt{a_i^2 + a_j^2}}\sigma_j + \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}}\sigma_k.\end{aligned} \tag{27}$$

We show that the necessary condition for the matrices $\sqrt{\vec{a}^2}\sigma_k$, $\vec{b}'\vec{\sigma}$ to be reducible to Hermitian forms simultaneously means $\vec{a}\vec{b} \in \mathbb{R}$. Indeed, as the matrices $\vec{b}'\vec{\sigma}, \sigma_k$ are simultaneously reduced to Hermitian forms, the matrix $\vec{b}'\vec{\sigma} + \lambda\sigma_k$ can be reduced to a Hermitian form with any real λ. Hence, in view of Assertion 1, we conclude that

$$b'^2_i + b'^2_j + (b'_k + \lambda)^2 > 0, \tag{28}$$

where λ is an arbitrary real number. The above equality may be valid only when $b'_k = \vec{a}\vec{b}/\sqrt{\vec{a}^2} \in \mathbb{R}$.

The choice $\lambda = -b'_k$ in (28) yields $b'^2_i + b'^2_j > 0$. Since $b'^2_i + b'^2_j = (\vec{a} \times \vec{b})^2$, we get the desired inequality $(\vec{a} \times \vec{b})^2 > 0$. The necessity is proved.

In order to prove the sufficiency of Assertion 2 we consider transformation (23) with

$$a = \frac{b_i a_j - b_j a_i}{\sqrt{a_i^2 + a_j^2}}, \qquad b = \frac{a_k \vec{a}\vec{b} - b_k \vec{a}^2}{\sqrt{\vec{a}^2}\sqrt{a_i^2 + a_j^2}}. \tag{29}$$

This transformation leaves the matrix $\sqrt{\vec{a}^2}\sigma_k$ invariant while the matrix $\vec{b}'\vec{\sigma}$ (27) transforms as

$$\vec{b}'\vec{\sigma} \to \vec{b}''\vec{\sigma} = \frac{\sqrt{(\vec{a} \times \vec{b})^2}}{\sqrt{\vec{a}^2}}\sigma_j + \frac{\vec{a}\vec{b}}{\sqrt{\vec{a}^2}}\sigma_k, \tag{30}$$

whence the sufficiency of Assertion 2 follows.

The proof of Assertion 3 of the lemma is similar to that of Assertion 2. The first three conditions are obtained with account of Assertion 2. A sequence of transformations (23) with a, b of the form (26), (29) transforms the matrix $\vec{c}\vec{\sigma}$ to

$$\vec{c}\vec{\sigma} \to \vec{c}\,''\vec{\sigma} = \frac{\epsilon_{ijk}\vec{a}(\vec{c} \times \vec{b})}{\sqrt{(\vec{c} \times \vec{b})^2}}\sigma_i + \frac{(\vec{a} \times \vec{b})(\vec{a} \times \vec{c})}{\sqrt{(\vec{c} \times \vec{b})^2}\sqrt{\vec{a}^2}}\sigma_j + \frac{\vec{a}\vec{c}}{\sqrt{\vec{a}^2}}\sigma_k. \tag{31}$$

Using the standard identities for the mixed vector products we establish that the coefficients by the matrices σ_i, σ_j, σ_k are real if and only if the relations

$$\left\{\vec{a}\vec{c},\ \vec{b}\vec{c},\ (\vec{a} \times \vec{b})\vec{c}\right\} \subset \mathbb{R}$$

hold true. ∎

Lemma 2 plays the crucial role when reducing the potentials (19) to Hermitian forms. This is done as follows. Firstly we reduce the QES operator to the Schrödinger form

$$\partial_y^2 + f(y)\vec{a}\vec{\sigma} + g(y)\vec{b}\vec{\sigma} + h(y)\vec{c}\vec{\sigma} + r(y).$$

Here f, g, h and r are some linearly independent real-valued functions and $\vec{a} = (a_1, a_2, a_3)$, $\vec{b} = (b_1, b_2, b_3)$ and $\vec{c} = (c_1, c_2, c_3)$ are complex constant vectors whose components depend on the parameters $\vec{\alpha}$, $\vec{\beta}$, $\vec{\gamma}$. Next, using Lemma 2, we obtain the conditions for the parameters $\vec{\alpha}$, $\vec{\beta}$ and $\vec{\gamma}$ that provide a simultaneous reducibility of the matrices $\vec{a}\vec{\sigma}$, $\vec{b}\vec{\sigma}$ and $\vec{c}\vec{\sigma}$ to Hermitian forms. Then, making use of formulae (21), (26) and (29), we find the form of the matrix Λ. Formulae (25), (30) and (31) yield explicit forms of the transformed matrices $\vec{a}\vec{\sigma}$, $\vec{b}\vec{\sigma}$, $\vec{c}\vec{\sigma}$ and, consequently, the Hermitian form of the matrix potential $V(y)$.

4 QES matrix models

Applying the algorithm mentioned at the end of the previous section we have obtained a complete description of QES matrix models (16) that can be reduced to Hermitian Schrödinger matrix operators. We give below the final results, namely, the restrictions on the choice of parameters and the explicit forms of the QES Hermitian Schrödinger operators and then consider in some detail a derivation of the corresponding formulae for one of the six inequivalent cases. In the formulae below we denote the disjunction of two statements A and B as $[A]\vee[B]$.

Case 1. $\tilde\gamma_1=\tilde\gamma_2=\tilde\gamma_3=0$ and

$$[\beta_0,\ \beta_1,\ \beta_2\in\mathbb{R}]\vee[\beta_2=0,\ \beta_1=2\alpha_2,\ \beta_0=\alpha_1+i\mu,\ \mu\in\mathbb{R}];$$

$$\begin{aligned}\hat H[y]=\partial_y^2+\Bigg\{&\frac{1}{4(\alpha_2x^2+\alpha_1x+\alpha_0)}\{-\beta_2^2x^4-[2\beta_1\beta_2+4\alpha_2\beta_2(m-1)]x^3+\\&+\left[2\alpha_2\beta_1-2\alpha_1\beta_2-\beta_1^2-2\beta_0\beta_2-4\alpha_1\beta_2(m-1)\right]x^2+\\&+[4\alpha_2\beta_0-2\beta_0\beta_1-4m\alpha_0\beta_2]x+2\alpha_1\beta_0-2\alpha_0\beta_1-\beta_0^2+\\&+4\beta_2(\alpha_2x^2+\alpha_1x+\alpha_0)\sigma_1+(4\beta_2x+2\delta)(\alpha_2x^2+\alpha_1x+\alpha_0)\sigma_3\}+\\&+\frac{\alpha_2}{2}-\frac{3(2\alpha_2x+\alpha_1)^2}{16(\alpha_2x^2+\alpha_1x+\alpha_0)}\Bigg\}\Bigg|_{x=f^{-1}(y)},\end{aligned}$$

$$\Lambda=1.$$

Case 2. $\beta_2,\delta=0$ and

$$2\alpha_2\beta_1-\beta_1^2\in\mathbb{R},\ 2\alpha_2\beta_0-\beta_0\beta_1\in\mathbb{R},\ 2\alpha_1\beta_0-2\beta_1\alpha_0-\beta_0^2-\tilde\gamma_i^2\in\mathbb{R},$$

$$\left[(2\alpha_2-\beta_1)^2\tilde\gamma_i^2>0\right]\vee[2\alpha_2-\beta_1=0],\ \left[(\alpha_1-\beta_0)^2\tilde\gamma_i^2>0\right]\vee[\alpha_1-\beta_0=0];$$

$$\begin{aligned}\hat H[y]=\partial_y^2+\Bigg\{&\frac{1}{4(\alpha_2x^2+\alpha_1x+\alpha_0)}\Bigg\{\beta_1(2\alpha_2-\beta_1)x^2+2\beta_0(2\alpha_2-\beta_1)x+\\&+2\alpha_1\beta_0-2\beta_1\alpha_0-\beta_0^2-\tilde\gamma_i^2+[2(2\alpha_2-\beta_1)x+2(\alpha_1-\beta_0)]\sqrt{\tilde\gamma_i^2}\sigma_3\Bigg\}+\\&+\frac{\alpha_2}{2}-\frac{3(2\alpha_2x+\alpha_1)^2}{16(\alpha_2x^2+\alpha_1x+\alpha_0)}\Bigg\}\Bigg|_{x=f^{-1}(y)},\end{aligned}$$

$$\Lambda=\Lambda_{12}(\tilde\gamma_1,\tilde\gamma_2)\Lambda_{23}(\sqrt{\tilde\gamma_1^2+\tilde\gamma_2^2},\tilde\gamma_3),\ \tilde\gamma_1^2+\tilde\gamma_2^2\neq0.$$

(If $\tilde\gamma_1^2+\tilde\gamma_2^2=0$, then one can choose another matrix Λ (27) with $\tilde\gamma_i^2+\tilde\gamma_j^2\neq0$.)

Case 3. $\alpha_2\neq0$, $\beta_2\neq0$ and

$$\left[\{\beta_2,\gamma_1\}\subset\mathbb{R},\ \gamma_3=0,\ \gamma_2=\sqrt{\gamma_1^2-2\alpha_2\gamma_1},\alpha_2\gamma_1<0,\right.$$

$$\left.\beta_1=2\alpha_2+\beta_2\frac{\alpha_1}{\alpha_2},\ \beta_0=\alpha_1+\beta_2\frac{\alpha_0}{\alpha_2}\right];$$

$$\hat{H}[y] = \partial_y^2 + \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \left\{ -\beta_2^2 x^4 - \right.\right.$$
$$- \left[2\beta_2^2 \frac{\alpha_1}{\alpha_2} + 4\alpha_2\beta_2 m \right] x^3 - \left[\frac{\beta_2^2}{\alpha_2^2}(\alpha_1^2 + 2\alpha_0\alpha_2) + 2\alpha_1\beta_2(1+2m) \right] x^2 -$$
$$- \left[\frac{2\alpha_1\beta_2(\alpha_1\alpha_2 + \alpha_0\beta_2)}{\alpha_2^2} + 4\alpha_0\beta_2 m \right] x + \alpha_1^2 - \beta_2^2 \frac{\alpha_0^2}{\alpha_2^2} - 4\beta_2 \frac{\alpha_0\alpha_1}{\alpha_2} -$$
$$- 4\alpha_0\alpha_2 - 2\alpha_2\gamma_1 + 4\beta_2 x(\alpha_2 x^2 + \alpha_1 x + \alpha_0) \times$$
$$\times \left[\sin\left(\theta(y)\sqrt{-2\alpha_2\gamma_1}\right)\sigma_1 + \cos\left(\theta(y)\sqrt{-2\alpha_2\gamma_1}\right)\sigma_3 \right] +$$
$$+ 2(\alpha_2 x^2 + \alpha_1 x + \alpha_0) \times$$
$$\times \left[\frac{\sin\left(\theta(y)\sqrt{-2\alpha_2\gamma_1}\right)}{\sqrt{-2\alpha_2\gamma_1}} \left(\delta\sqrt{-2\alpha_2\gamma_1}\sigma_1 - 2\beta_2\sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}\sigma_3 \right) + \right.$$
$$\left.\left.\left. + \cos\left(\theta(y)\sqrt{-2\alpha_2\gamma_1}\right) \left(\frac{2\beta_2\sqrt{\gamma_1^2 - 2\alpha_2\gamma_1}}{\sqrt{-2\alpha_2\gamma_1}}\sigma_1 + \delta\sigma_3 \right) \right] \right\} \right\} \Bigg|_{x=f^{-1}(y)},$$
$$\Lambda = 1 + \left(\sqrt{1 - \frac{2\alpha_2}{\gamma_1}} - \sqrt{\frac{-2\alpha_2}{\gamma_1}} \right) \sigma_3.$$

Case 4. $\alpha_2 \neq 0$, $\beta_2 = 0$.

Subcase 4.1. $\delta \neq 0$, γ_1, γ_2 do not vanish simultaneously and

$$\gamma_1^2 - \gamma_2^2 < 0, \quad \gamma_3 = i\mu, \quad \{\mu, \delta\} \subset \mathbb{R}, \quad i(\alpha_1 - \beta_0) \in \mathbb{R}, \quad \beta_1 = 2\alpha_2;$$
$$\hat{H}[y] = \partial_y^2 + \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} + \frac{1}{4\xi} \left\{ -\beta_0^2 + 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \right.\right.$$
$$- \tilde{\gamma}_i^2 + 2(\alpha_2 x^2 + \alpha_1 x + \alpha_0) \left[\delta\sqrt{\gamma_2^2 - \gamma_1^2}\sigma_1 \frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\right)}{\sqrt{-\tilde{\gamma}_i^2}} + \right.$$
$$\left. + \frac{-i\delta\gamma_3\sqrt{\gamma_2^2 - \gamma_1^2}\sigma_2 + \delta(\gamma_1^2 - \gamma_2^2)\sigma_3}{\tilde{\gamma}_i^2} \cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\right) \right] +$$
$$+ \frac{2\delta\alpha_1\gamma_3}{\tilde{\gamma}_i^2} x + \left[\frac{2\delta\alpha_2\gamma_3}{\tilde{\gamma}_i^2} x^2 + \frac{(2\alpha_1 - 2\beta_0)\tilde{\gamma}_i^2 + 2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2} \right] \times$$
$$\left.\left. \times \left(i\sqrt{\gamma_2^2 - \gamma_1^2}\sigma_2 + \gamma_3\sigma_3 \right) \right\} \right\} \Bigg|_{x=f^{-1}(y)},$$
$$\Lambda = \Lambda_{21}(i\gamma_1, \gamma_2).$$

Subcase 4.2. $\delta \neq 0$, $\gamma_1 = \gamma_2 = 0$, $\gamma_3 \neq 0$ and

$$\{\delta,\ \beta_1(2\alpha_2 - \beta_1),\ \beta_0(2\alpha_2 - \beta_1),\ -\beta_0^2 + 2\alpha_1\beta_0 - 2\alpha_0\beta_1, \gamma_3(2\alpha_2 - \beta_1),$$
$$\gamma_3(\alpha_1 - \beta_0)\} \subset \mathbb{R};$$

$$\hat{H}[y] = \partial_y^2 + \left\{ \frac{\alpha_2}{2} - \frac{3(2\alpha_2 x + \alpha_1)^2}{16(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_2 x^2 + \alpha_1 x + \alpha_0)} \times \right.$$
$$\times \{\beta_1(2\alpha_2 - \beta_1)x^2 + 2\beta_0(2\alpha_2 - \beta_1)x - \beta_0^2 + 2\alpha_1\beta_0 - 2\beta_1\alpha_0 -$$
$$- \gamma_3^2 + \left[2\delta\alpha_2 x^2 + 2x((2\alpha_2 - \beta_1)\gamma_3 + \delta\alpha_1) + 2(\alpha_1 - \beta_0)\gamma_3 + \right.$$
$$\left.\left. + 2\delta\alpha_0\right]\sigma_3\} \right\}\Bigg|_{x=f^{-1}(y)},$$

$\Lambda = 1.$

Case 5. $\alpha_2 = 0,\ \beta_2 \neq 0$ and

$$\alpha_1 \neq 0,\ \gamma_1^2 - \gamma_2^2 < 0,\ \tilde{\gamma}_i^2 < 0,\ \gamma_3 = \frac{\tilde{\gamma}_i^2}{2\alpha_1},$$
$$\{\beta_0,\ \beta_1,\ \beta_2,\ \gamma_2,\ \delta(\gamma_2^2 - \gamma_1^2) + 2\beta_2\gamma_1\gamma_3,\ i(2\alpha_0\beta_2\gamma_3 - \beta_1\tilde{\gamma}_i^2 +$$
$$+ 2\beta_2\alpha_1\gamma_1 + \delta\alpha_1\gamma_3),\ i((\alpha_1 - \beta_0)\tilde{\gamma}_i^2 + 2\beta_2\alpha_0\gamma_1 + \delta\alpha_0\gamma_3)\} \subset \mathbb{R};$$
$$\hat{H}[y] = \partial_y^2 + \left\{ -\frac{3\alpha_1^2}{16(\alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_1 x + \alpha_0)} \left\{ -\beta_2^2 x^4 - 2\beta_1\beta_2 x^3 + \right.\right.$$
$$+ \left[(2 - 4m)\alpha_1\beta_2 - \beta_1^2 - 2\beta_0\beta_2\right] x^2 - \left[2\beta_0\beta_1 + 4m\alpha_0\beta_2\right] x +$$
$$+ 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \beta_0^2 - \tilde{\gamma}_i^2 + 4x(\alpha_1 x + \alpha_0) \times$$
$$\times \left[\beta_2\sqrt{\gamma_2^2 - \gamma_1^2}\,\sigma_1 \frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\right)}{\sqrt{-\tilde{\gamma}_i^2}} + \right.$$
$$\left. + \frac{\beta_2\sqrt{(\gamma_1^2 - \gamma_2^2)\tilde{\gamma}_i^2}}{\tilde{\gamma}_i^2}\sigma_3 \cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\right) \right] + 2(\alpha_1 x + \alpha_0) \times$$
$$\times \left[\left(\frac{\delta(\gamma_2^2 - \gamma_1^2) + 2\beta_2\gamma_1\gamma_3}{\sqrt{\gamma_2^2 - \gamma_1^2}}\sigma_1 - \frac{2\beta_2\gamma_2\tilde{\gamma}_i^2}{\sqrt{(\gamma_1^2 - \gamma_2^2)\tilde{\gamma}_i^2}}\sigma_3 \right) \frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\right)}{\sqrt{-\tilde{\gamma}_i^2}} + \right.$$
$$\left. + \left(\frac{2\beta_2\gamma_2}{\sqrt{\gamma_2^2 - \gamma_1^2}}\sigma_1 + \frac{\delta(\gamma_1^2 - \gamma_2^2) - 2\beta_2\gamma_1\gamma_3}{\sqrt{(\gamma_1^2 - \gamma_2^2)\tilde{\gamma}_i^2}}\sigma_3 \right) \cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\right) \right] +$$
$$+ \left[x\frac{4\alpha_0\beta_2\gamma_3 - 2\beta_1\tilde{\gamma}_i^2 + 4\alpha_1\beta_2\gamma_1 + 2\delta\alpha_1\gamma_3}{\tilde{\gamma}_i^2} + \right.$$
$$\left.\left.\left. + \frac{(2\alpha_1 - 2\beta_0)\tilde{\gamma}_i^2 + 4\alpha_0\beta_2\gamma_1 + 2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2} \right] \left(-i\sqrt{-\tilde{\gamma}_i^2}\sigma_2 \right) \right\} \right\}\Bigg|_{x=f^{-1}(y)},$$
$$\Lambda = \Lambda_{21}(i\gamma_1, \gamma_2)\Lambda_{23}\left(-i\gamma_3\sqrt{\gamma_2^2 - \gamma_1^2}, \gamma_1^2 - \gamma_2^2 \right).$$

Case 6. $\alpha_2 = 0,\ \beta_2 = 0.$

Subcase 6.1. $\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ do not vanish simultaneously, $\delta \neq 0$ and

$$\tilde{\gamma}_i^2 < 0,\ \{\delta^2(\gamma_1^2 - \gamma_2^2) < 0,\ \beta_0,\ \beta_1\} \subset \mathbb{R},$$
$$\{i(-\beta_1\tilde{\gamma}_i^2 + \delta\alpha_1\gamma_3),\ i((\alpha_1 - \beta_0)\tilde{\gamma}_i^2 + \delta\alpha_0\gamma_3)\} \subset \mathbb{R};$$
$$\hat{H}[y] = \partial_y^2 + \Bigg\{-\frac{3\alpha_1^2}{16(\alpha_1 x + \alpha_0)} + \frac{1}{4(\alpha_1 x + \alpha_0)}\Bigg\{-\beta_1^2 x^2 - 2\beta_0\beta_1 x + 2\alpha_1\beta_0 -$$
$$- 2\alpha_0\beta_1 - \beta_0^2 - \tilde{\gamma}_i^2 + 2(\alpha_1 x + \alpha_0)\left[\delta\sqrt{\gamma_2^2 - \gamma_1^2}\sigma_1 \frac{\sin\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)}{\sqrt{-\tilde{\gamma}_i^2}} + \right.$$
$$\left. + \frac{\delta(\gamma_1^2 - \gamma_2^2)}{\sqrt{(\gamma_1^2 - \gamma_2^2)\tilde{\gamma}_i^2}}\sigma_3 \cos\left(\theta(y)\sqrt{-\tilde{\gamma}_i^2}\,\right)\right] + \left[x\frac{-2\beta_1\tilde{\gamma}_i^2 + 2\delta\alpha_1\gamma_3}{\tilde{\gamma}_i^2} + \right.$$
$$\left. + \frac{(2\alpha_1 - 2\beta_0)\tilde{\gamma}_i^2 + 2\delta\alpha_0\gamma_3}{\tilde{\gamma}_i^2}\right]\left(-i\sqrt{-\tilde{\gamma}_i^2}\sigma_2\right)\Bigg\}\Bigg\}\Bigg|_{x=f^{-1}(y)},$$
$$\Lambda = \Lambda_{21}(i\gamma_1, \gamma_2)\Lambda_{23}\left(-i\gamma_3\sqrt{\gamma_2^2 - \gamma_1^2}, \gamma_1^2 - \gamma_2^2\right).$$

Subcase 6.2.

$$\gamma_1 = \gamma_2 = 0,\ \gamma_3 \neq 0,\ \{\beta_1^2,\ \beta_0\beta_1\} \subset \mathbb{R},$$
$$\{-\beta_1\gamma_3 + \delta\alpha_1,\ (\alpha_1 - \beta_0)\gamma_3 + \delta\alpha_0,\ -\beta_0^2 + 2\alpha_1\beta_0 - 2\alpha_0\beta_1\} \subset \mathbb{R};$$
$$\hat{H}[y] = \partial_y^2 + \Bigg\{-\frac{3\alpha_1^2}{16(\alpha_1 x + \alpha_0)}$$
$$+ \frac{1}{4(\alpha_1 x + \alpha_0)}\{-\beta_1^2 x^2 - 2\beta_0\beta_1 x + 2\alpha_1\beta_0 - 2\alpha_0\beta_1 - \beta_0^2 - \gamma_3^2$$
$$+ 2(\alpha_1 x + \alpha_0)[2x\beta_1(\alpha_1 - \gamma_3) + 2(\alpha_1 - \beta_0)\gamma_3 + 2\beta_1\alpha_0]\sigma_3\}\Bigg\}\Bigg|_{x=f^{-1}(y)},$$
$$\Lambda = 1.$$

In the above formulae we denote the inverse of the function

$$y = f(x) \equiv \int \frac{dx}{\sqrt{\alpha_2 x^2 + \alpha_1 x + \alpha_0}},$$

as $f^{-1}(y)$, moreover the function $\theta = \theta(y)$ is defined as

$$\theta(y) = -\left\{\int \frac{dx}{\alpha_2 x^2 + \alpha_1 x + \alpha_0}\right\}\Bigg|_{x=f^{-1}(y)}$$

and $\tilde{\gamma}_i^2$ stands for $\tilde{\gamma}_1^2 + \tilde{\gamma}_2^2 + \tilde{\gamma}_3^2$.

The whole procedure of derivation of the above formulae is very cumbersome and here we omit it.

A further restriction narrowing the choice of QES matrix Hamiltonians is a requirement that the basis elements of the corresponding invariant space be square integrable on the interval $(-\infty, \infty)$. For example, if we put in Case I $\alpha_1 = 1$, $\beta_2 = -1$, $\beta_0 = 1/2$, the remaining coefficients being equal to zero, then we arrive at the model 1 from the list of QES Hamiltonians given in the Introduction. The second model given there is obtained in an analogous way.

5 Some conclusions

A principal aim of the paper is to give a systematic algebraic treatment of Hermitian QES Hamiltonians within the framework of the approach to constructing QES matrix models suggested in our papers [1, 2]. The whole procedure is based upon a specific representation of the algebra $o(2,2)$ given by formulae (3), (5) and (6). Making use of the fact that the representation space of the algebra (6) has a finite-dimensional invariant subspace (4) we have constructed in a systematic way six multiparameter families of Hermitian QES Hamiltonians on line. Due to computational reasons we do not present here a systematic description of Hermitian QES Hamiltonians with potentials depending upon elliptic functions.

The problem of constructing all Hermitian QES Hamiltonians of the form (16) having square integrable eigenfunctions is also beyond the scope of the present paper. We restricted our analysis of this problem to giving several examples of such Hamiltonians and postpone its further investigation for our future publications.

A very interesting problem is a comparison of the results of the present paper based on structure of representation space of the representation (3), (5), (6) of the Lie algebra $o(2,2)$ to those of the paper [12], where some superalgebras of matrix-differential operators come into play. The link to the results of [12] is provided by the fact that the Lie algebra $o(2,2)$ has a structure of a superalgebra. This is a consequence of the fact that operators (6) fulfill identities (7).

One more challenging problem is a utilization of the obtained results for integrating the multidimensional Pauli equation with the help of the method of separation of variables. As an intermediate problem to be solved within the framework of the method in question is a reduction of the Pauli equation to four second-order systems of ordinary differential equations with the help of an Ansatz of separation. The next step is studying whether the corresponding matrix-differential operators belong to one of the six classes of QES Hamiltonians constructed in Section 4.

Investigation of the problems enumerated above is in progress now and we hope to report the results obtained in one of our future publications.

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[1] Zhdanov R.Z., Quasi-exactly solvable matrix models, *Phys. Lett. B*, 1997, V.405, 253–256.

[2] Zhdanov R.Z., On algebraic classification of quasi-exactly solvable matrix models, *J. Phys. A: Math. Gen.*, 1997, V.30, 8761–8770.

[3] Turbiner A.V., Quasi-exactly-solvable problems and sl(2) algebra, *Comm. Math. Phys.*, 1988, V.118, 467–474.

[4] Shifman M.A., New findings in quantum mechanics (partial algebraization of the spectral problem), *Internat. J. Modern Phys. A*, 1989, V.4, 2897–2952.

[5] Shifman M.A. and Turbiner A.V., Quantal problems with partial algebraization of the spectrum, *Comm. Math. Phys.*, 1989, V.126, 347–365.

[6] González-López A., Kamran N. and Olver P., Quasi-exactly solvable Lie algebras of differential operators in two complex variables, *J. Phys. A: Math. Gen.*, 1991, V.24, 3995–4008.

[7] González-López A., Kamran N. and Olver P., New quasi-exactly solvable Hamiltonians in two dimensions, *Comm. Math. Phys.*, 1994, V.159, 503–537.

[8] González-López A., Kamran N. and Olver P., Quasi-exact solvability. Lie algebras, cohomology, and new applications to quantum mechanics, *Contemp. Math.*, 1994, V.160, 113–140.

[9] Ushveridze A.G., *Quasi-exactly Solvable Models in Quantum Mechanics*, Bristol: Institute of Physics Publishing, 1994, 465 pp.

[10] González-López A., Kamran N. and Olver P., Normalizability of one-dimensional quasi-exactly solvable Schrödinger operators, *Comm. Math. Phys.*, 1993, V.153, 117–146.

[11] Ushveridze A.G., Quasi-exact solvability in quantum mechanics of one-dimensional systems, *Phys. Lebedev Inst. Rep.*, 1988, no. 2, 50–53.

[12] Finkel F., González-López A. and Rodriguez M., Quasi-exactly solvable Lie superalgebras of differential operators, *J. Phys. A: Math. Gen.*, 1997, V.30, 6879–6892.

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Symmetries of equations for vibrational convection in binary mixture

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The differential equations describing vibrational convection in a binary mixture with the Soret effect are considered. The limit of high frequency and small amplitude of vibration allows the application of an averaging approach. The symmetry classification of averaged equations with respect to the physical parameters of the system is performed.

1 Introduction

It is well known that symmetry analysis provides a powerful tool for studying partial and ordinary differential equations [1]. This method is especially fruitful in application to the equations of physics and mechanics since many of them are derived on the basis of invariance principles.

This paper deals with the symmetry analysis of equations describing vibrational convection in a binary mixture. Vibrational convection refers to the specific flows that appear when a fluid with density gradient is subjected to external vibration [2]. In a binary mixture the density gradient can be induced by the gradients of temperature and concentration. The flow dynamics in mixtures is more complex than in one-component fluids due to an interplay between convection, heat conduction, diffusion and thermal diffusion (or the Soret effect). Note that convection induced by vibrations can appear in pure weightlessness. It provides a mechanism of heat and mass transfer in the absence of gravity and can be used to control and operate fluids in space. Experimental and theoretical study of vibrational convection in one-component fluid has been recently reported in [3,4].

The symmetries of equations for convection in binary mixture in the absence of vibration were investigated in several works. The symmetry classification of the governing equations was performed for linear dependence of density on temperature and concentration [5,6] as well as for the general case, where density is an arbitrary function of temperature, concentration, and pressure [7]. Based on these results closed-form solutions describing the flow of a binary mixture in plane and cylindrical layers were constructed [8,9].

In this paper we investigate the symmetries of equations describing vibrational convection in a binary mixture. The symmetry classification of the system with respect to the control parameters is performed.

2 Governing equations

Consider a binary mixture with the equation of state

$$\rho = \rho_0(1 - \beta_T T - \beta_C C),$$

where ρ_0 is the density of the mixture at mean values of the temperature T_0 and concentration C_0, T and C are the deviations from these mean values, β_T and β_C are the thermal and concentration expansion coefficients, respectively. It is assumed that C is the concentration of the lighter component so that $\beta_C > 0$.

The binary mixture is subjected to harmonic oscillations with angular frequency ω and displacement amplitude A in the direction of the unit vector $\mathbf{e} = (\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3)$. In what follows we consider the limit of high-frequency vibrations. It means that the period of vibration $\tau = 2\pi/\omega$ is much smaller than all characteristic hydrodynamic times:

$$\tau \ll \min(L^2/\nu, L^2/\chi, L^2/D),$$

where L is the characteristic scale and ν, χ and D are the kinematic viscosity, thermal diffusivity and diffusion coefficient, respectively. In this case the velocity, pressure, temperature and concentration fields can be represented as a sum of two parts: 'slow' averaged part (which is obtained by averaging the corresponding quantity over the period of vibration) and 'fast' oscillatory part (the difference between the corresponding quantity and its averaging). We assume that the amplitude of oscillation is sufficiently small so that the equations for the averaged fields can be written as [2]

$$\boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\rho_0^{-1}\nabla p + \nu\nabla^2\boldsymbol{u} - (\beta_T T + \beta_C C)\mathbf{g} + \\ + \frac{(A\omega)^2}{2}\big(((\beta_T T + \beta_C C)\mathbf{e} - \nabla\Phi)\cdot\nabla\big)\nabla\Phi, \tag{1}$$

$$T_t + \boldsymbol{u}\cdot\nabla T = \chi\nabla^2 T, \tag{2}$$

$$C_t + \boldsymbol{u}\cdot\nabla C = D\nabla^2 C + D_T\nabla^2 T, \tag{3}$$

$$\nabla\cdot\boldsymbol{u} = 0, \tag{4}$$

$$\nabla^2\Phi - (\beta_T\nabla T + \beta_C\nabla C)\cdot\mathbf{e} = 0, \tag{5}$$

while the oscillatory fields are described by the formulas

$$\boldsymbol{u}' = -A\omega\sin(\omega t)\boldsymbol{w}, \quad p' = -\rho_0 A\omega^2\cos(\omega t)\Phi,$$
$$T' = -A\cos(\omega t)\boldsymbol{w}\cdot\nabla T, \quad C' = -A\cos(\omega t)\boldsymbol{w}\cdot\nabla C.$$

The vector $\boldsymbol{w}$ and function Φ, which characterize the amplitudes of velocity and pressure oscillations, respectively, are related to the decomposition of the vector $(\beta_T T + \beta_C C)\mathbf{e}$ into an irrotational part, $\nabla\Phi$, and a solenoidal part, $\boldsymbol{w}$:

$$(\beta_T T + \beta_C C)\mathbf{e} = \boldsymbol{w} + \nabla\Phi, \tag{6}$$

$$\nabla\cdot\boldsymbol{w} = 0, \quad \boldsymbol{w}\cdot\boldsymbol{n}\big|_\Gamma = 0. \tag{7}$$

The vector $\boldsymbol{w}$ has zero normal projection to the boundary Γ of the domain of motion.

In system (1)–(5) $\boldsymbol{x} = (x_1, x_2, x_3)$ is the coordinate vector, $\boldsymbol{u} = (u_1, u_2, u_3)$ is the velocity vector, p is the difference between total and hydrostatic pressure, $\mathbf{g} = (\mathrm{g}_1, \mathrm{g}_2, \mathrm{g}_3)$ is the acceleration due to a constant external force and D_T is the thermal diffusion coefficient. The case $D_T < 0$ ($D_T > 0$) corresponds to positive (negative) Soret effect in which the lighter component is driven towards the higher (lower) temperature region.

3 Symmetry properties of the governing equations

The equations of motion (1)–(5) contain fifteen arbitrary constant parameters (including components of vectors $\mathbf{g}$ and $\mathbf{e}$). In this section we consider the symmetry classification of governing equations with respect to these parameters. It is supposed that $\mathbf{g}$, β_T, β_C and D_T can be zero (in this case the corresponding terms in the equations are omitted). At the same time we assume that $\beta_T^2 + \beta_C^2 \neq 0$. Otherwise it follows from (5)–(7) that $\Phi = 0$ and equations (1)–(5) are reduced to the model of gravitational convection in a binary mixture (the symmetry properties of this model were investigated in [5,6]). We also assume that $|\mathbf{e}| \neq 0$ and ρ_0, ν, χ, D, A and ω are positive.

We introduce the following notation. If $f(\boldsymbol{x}, t)$ is an arbitrary function, then its derivatives are denoted as follows:

$$\frac{\partial f}{\partial t} = f_t, \quad \frac{\partial f}{\partial x^i} = f_i, \quad \frac{\partial f}{\partial t \partial x^i} = f_{ti}, \quad \frac{\partial f}{\partial x^i \partial x^j} = f_{ij}, \quad i, j = 1, 2, 3.$$

To find the admissible Lie symmetry group, we calculate the corresponding Lie symmetry algebra of the generators of infinitesimal transformations. The admissible generator is sought in the form

$$X = \xi^t \frac{\partial}{\partial t} + \xi^i \frac{\partial}{\partial x^i} + \eta^i \frac{\partial}{\partial u^i} + \eta^p \frac{\partial}{\partial p} + \eta^T \frac{\partial}{\partial T} + \eta^C \frac{\partial}{\partial C} + \eta^\Phi \frac{\partial}{\partial \Phi}.$$

Its coordinates depend upon all dependent and independent variables (summation over $i = 1, 2, 3$ is assumed). To derive the determining equations we need to apply the prolongation generator $\underset{2}{X}$ to equations (1)–(5) and to make the transition to the manifold given by this system. However, the equations are not in involution, which makes it difficult to choose the external and internal variables. We supplement the system with its differential consequence

$$(\nabla \boldsymbol{u})^2 + 2(u_2^1 u_1^2 + u_3^1 u_1^3 + u_3^2 u_2^3) + \rho_0^{-1} \nabla^2 p + \nabla J \cdot \mathbf{g} + \frac{(A\omega)^2}{2} \Big[\Phi_{11}^2 + \Phi_{22}^2 + \\ + \Phi_{33}^2 + 2(\Phi_{12}^2 + \Phi_{13}^2 + \Phi_{23}^2) - \nabla J \cdot (\Upsilon \mathbf{e}) + (\nabla \Phi - J\mathbf{e}) \cdot (\Upsilon^* \mathbf{e}) \Big] = 0. \qquad (8)$$

Here $J = \beta_T T + \beta_C C$, while Υ and Υ^* are matrices of the second spatial derivatives with components $\{\Upsilon\}_{ij} = \Phi_{ij}$ and $\{\Upsilon^*\}_{ij} = J_{ij}$, $i, j = 1, 2, 3$. Relation (8) is obtained by differentiating equations (1) with respect x^1, x^2, x^3, respectively, and using (4) and (5). When making the transition to the manifold, we also take into account the differential consequences from (4):

$$u^1_{t1} + u^2_{t2} + u^3_{t3} = 0, \quad u^1_{1i} + u^2_{2i} + u^3_{3i} = 0, \quad i = 1, 2, 3. \tag{9}$$

Equations (1)–(5), (8) and (9) are in involution and it is easy to choose the external variables: u^1_{11}, u^2_{11}, u^3_{11}, p_{11}, T_{11}, C_{11}, Φ_{11}, u^1_1, u^3_{t3}, u^3_{13}, u^3_{23}, u^3_{33}. The determining equations are found by applying the prolongation generator $\underset{2}{X}$ to the system and substituting the expressions for external variables in the equations obtained. After a considerable amount of calculations the solution is written in the form

$$\begin{aligned}
&\xi^t = 2c_4 t + c_0,\\
&\xi^1 = c_4 x^1 + c_1 x^2 + c_2 x^3 + f^1(t), \quad \eta^1 = -c_4 u^1 + c_1 u^2 + c_2 u^3 + f^1_t(t),\\
&\xi^2 = -c_1 x^1 + c_4 x^2 + c_3 x^3 + f^2(t), \quad \eta^2 = -c_1 u^1 - c_4 u^2 + c_3 u^3 + f^2_t(t),\\
&\xi^3 = -c_2 x^1 - c_3 x^2 + c_4 x^3 + f^3(t), \quad \eta^3 = -c_2 u^1 - c_3 u^2 - c_4 u^3 + f^3_t(t),\\
&\eta^p = -\rho_0\big(f^1_{tt}(t)x^1 + f^2_{tt}(t)x^2 + f^3_{tt}(t)x^3 +\\
&\qquad + (\beta_T c_5 + \beta_C c_6)(\mathrm{g}_1 x^1 + \mathrm{g}_2 x^2 + \mathrm{g}_3 x^3)\big) - 2c_4 p + f^0(t),\\
&\eta^T = c_7 T + c_9 C + c_5, \qquad \eta^C = c_8 T + c_{10} C + c_6,\\
&\eta^\Phi = (\beta_T c_5 + \beta_C c_6)(\mathrm{e}_1 x^1 + \mathrm{e}_2 x^2 + \mathrm{e}_3 x^3) + \varphi(t).
\end{aligned} \tag{10}$$

Here c_0–c_{10} are the group constants and $f^i(t)$, $i = 0, 1, 2, 3$, and $\varphi(t)$ are smooth arbitrary functions. The group constants are connected with the parameters of system (1)–(5) by the classifying equations

$$\begin{aligned}
&\big(\beta_T(c_7 + 3c_4) + \beta_C c_8\big)\mathbf{g} + \beta_T G\boldsymbol{c} = 0,\\
&\big(\beta_T c_9 + \beta_C(c_{10} + 3c_4)\big)\mathbf{g} + \beta_C G\boldsymbol{c} = 0,\\
&\big(\beta_T(c_7 + c_4) + \beta_C c_8\big)\mathbf{e} + \beta_T E\boldsymbol{c} = 0,\\
&\big(\beta_T c_9 + \beta_C(c_{10} + c_4)\big)\mathbf{e} + \beta_C E\boldsymbol{c} = 0,\\
&E^*\boldsymbol{c} = 0, \quad D_T\beta_C\big(\mathrm{e}_1(c_4 + c_{10}) - \mathrm{e}_2 c_1 - \mathrm{e}_3 c_2\big) = 0,\\
&D_T(c_{10} - c_7) + (\chi - D)c_8 = 0, \quad D_T c_9 = 0, \quad (\chi - D)c_9 = 0,
\end{aligned} \tag{11}$$

where

$$\boldsymbol{c} = \begin{pmatrix} c_1\\ c_2\\ c_3 \end{pmatrix}, \quad G = \begin{pmatrix} -\mathrm{g}_2 & -\mathrm{g}_3 & 0\\ \mathrm{g}_1 & 0 & -\mathrm{g}_3\\ 0 & \mathrm{g}_1 & \mathrm{g}_2 \end{pmatrix},$$

$$E = \begin{pmatrix} -\mathrm{e}_2 & -\mathrm{e}_3 & 0\\ \mathrm{e}_1 & 0 & -\mathrm{e}_3\\ 0 & \mathrm{e}_1 & \mathrm{e}_2 \end{pmatrix}, \quad E^* = \begin{pmatrix} \mathrm{e}_1^2 + \mathrm{e}_2^2 & \mathrm{e}_2\mathrm{e}_3 & -\mathrm{e}_1\mathrm{e}_3\\ \mathrm{e}_2\mathrm{e}_3 & \mathrm{e}_1^2 + \mathrm{e}_3^2 & \mathrm{e}_1\mathrm{e}_2\\ -\mathrm{e}_1\mathrm{e}_3 & \mathrm{e}_1\mathrm{e}_2 & \mathrm{e}_2^2 + \mathrm{e}_3^2 \end{pmatrix}.$$

Table 1. Symmetry classification of the governing equations

Basic Lie algebra			$X_0,\ H_i,\ H_0,\ H_\Phi,\ U_T,\ U_C$			
D_T	β_T	β_C	$\mathbf{g}\times\mathbf{e}\neq 0$	$\mathbf{g}\times\mathbf{e}=0$, $\mathbf{g}\neq 0$	$\mathbf{g}=0$	Additional generators
0	0	$\neq 0$	T^1	T^1, X_R	T^1, X_R, Z_C	$T^2\ (D=\chi)$
0	$\neq 0$	0	C^1	C^1, X_R	C^1, X_R, Z_T	$C^2\ (D=\chi)$
0	$\neq 0$	$\neq 0$	—	X_R	$X_R,\ Z$	$R_1, R_2\ (D=\chi)$
$\neq 0$	0	$\neq 0$	—	X_R	$X_R,\ Z$	—
$\neq 0$	$\neq 0$	0	L	L, X_R	$L,\ X_R,$ $Z_T^*\ (D\neq\chi)$	$Z\ (D=\chi)$
$\neq 0$	$\neq 0$	$\neq 0$	—	X_R	$X_R,$ $Z\ (D_T\neq D_T^*)$	$R_1,$ $Z_C^*\ (D_T=D_T^*)$

$$
\begin{gathered}
X_0=\frac{\partial}{\partial t},\quad X_{ij}=x^i\frac{\partial}{\partial x^j}-x^j\frac{\partial}{\partial x^i}+u^i\frac{\partial}{\partial u^j}-u^j\frac{\partial}{\partial u^i},\quad H_0(f^0(t))=f^0(t)\frac{\partial}{\partial p},\\
H_i(f^i(t))=f^i(t)\frac{\partial}{\partial x^i}+f^i_t(t)\frac{\partial}{\partial u^i}-\rho_0 x^i f^i_{tt}(t)\frac{\partial}{\partial p},\quad i,j=1,2,3\ \ (i<j),\\
H_\Phi(\varphi(t))=\varphi(t)\frac{\partial}{\partial\Phi},\qquad Z_0=2t\frac{\partial}{\partial t}+\sum_{i=1}^{3}\left(x^i\frac{\partial}{\partial x^i}-u^i\frac{\partial}{\partial u^i}\right)-2p\frac{\partial}{\partial p},\\
U_T=-\rho_0\beta_T\big(\mathrm{g}_1x^1+\mathrm{g}_2x^2+\mathrm{g}_3x^3\big)\frac{\partial}{\partial p}+\frac{\partial}{\partial T}+\beta_T\big(\mathrm{e}_1x^1+\mathrm{e}_2x^2+\mathrm{e}_3x^3\big)\frac{\partial}{\partial\Phi},\\
U_C=-\rho_0\beta_C\big(\mathrm{g}_1x^1+\mathrm{g}_2x^2+\mathrm{g}_3x^3\big)\frac{\partial}{\partial p}+\frac{\partial}{\partial C}+\beta_C\big(\mathrm{e}_1x^1+\mathrm{e}_2x^2+\mathrm{e}_3x^3\big)\frac{\partial}{\partial\Phi},\\
T^1=T\frac{\partial}{\partial T},\qquad T^2=C\frac{\partial}{\partial T},\qquad C^1=C\frac{\partial}{\partial C},\qquad C^2=T\frac{\partial}{\partial C},\\
Z_T=Z_0-T^1,\qquad Z_C=Z_0-C^1,\qquad Z=Z_0-T^1-C^1,\\
Z_T^*=Z_T+\frac{D_T}{D-\chi}C^2,\quad Z_C^*=Z_C-\frac{D_T}{D-\chi}C^2,\quad L=(D_TT+(D-\chi)C)\frac{\partial}{\partial C},\\
R_1=T^1-\frac{\beta_T}{\beta_C}C^2,\quad R_2=C^1-\frac{\beta_C}{\beta_T}T^2,\quad X_R=\mathrm{e}_3X_{12}-\mathrm{e}_2X_{13}+\mathrm{e}_1X_{23}.
\end{gathered}
\tag{12}
$$

Using formulas (10) and equations (11) we can find the Lie symmetry algebras admitted by the governing equations depending upon the values of parameters. The results of the symmetry classification are given in Table 1. The generators X_0, H_i, H_0, H_Φ, U_T and U_C are admitted by the system independently of the values of parameters and construct the basic Lie algebra. Possible extensions of this algebra are also presented in Table 1. The values of parameters D_T, β_T and β_C are specified in the first three columns. The generators, which are admitted when the vectors **g** and **e** are noncollinear, collinear, and when **g** is equal to zero are presented in the next three columns. The additional generators admitted in the cases of $D = \chi$ or $D_T = D_T^*$ (where $D_T^* = \beta_T(D-\chi)/\beta_C$) are presented in the seventh column. Note that in the case $D_T \neq 0$, $\beta_T \neq 0$, $\beta_C = 0$ and $\mathbf{g} = 0$ the equations admit the generator Z_T^* when $D \neq \chi$. Similarly, if D_T, β_T and β_C are nonzero and $\mathbf{g} = 0$, then the generator Z is admitted when $D_T \neq D_T^*$.

We now describe the one-parameter transformation subgroups that correspond to the generators (12). These subgroups are obtained by solving the corresponding Lie equation for each generator. The transformations generated by X_0, X_{ij}, H_i and H_0 are well-known [6] and are not presented here. The generator H_Φ corresponds to addition of an arbitrary function of time to the amplitude of the oscillations of the pressure, Φ, while the generator X_R corresponds to rotation in the plane perpendicular to the vector $\mathbf{e}$ (X_R is admitted when the vectors $\mathbf{e}$ and $\mathbf{g}$ are collinear or $\mathbf{g} = 0$). The transformations induced by other generators have the form

$$
\begin{aligned}
U_T:&\quad \widetilde{p} = p - a\rho_0\beta_T(\mathrm{g}_1x^1 + \mathrm{g}_2x^2 + \mathrm{g}_3x^3), \quad \widetilde{T} = T + a,\\
&\quad \widetilde{\Phi} = \Phi + a\beta_T(\mathrm{e}_1x^1 + \mathrm{e}_2x^2 + \mathrm{e}_3x^3);\\
U_C:&\quad \widetilde{p} = p - a\rho_0\beta_C(\mathrm{g}_1x^1 + \mathrm{g}_2x^2 + \mathrm{g}_3x^3), \quad \widetilde{C} = C + a,\\
&\quad \widetilde{\Phi} = \Phi + a\beta_C(\mathrm{e}_1x^1 + \mathrm{e}_2x^2 + \mathrm{e}_3x^3);\\
T^1:&\quad \widetilde{T} = e^aT; \qquad T^2: \quad \widetilde{T} = T + aC;\\
C^1:&\quad \widetilde{C} = e^aC; \qquad C^2: \quad \widetilde{C} = C + aT;\\
Z_0:&\quad \widetilde{t} = e^{2a}t, \quad \widetilde{x}^i = e^ax^i, \quad \widetilde{u}^i = e^{-a}u^i, \quad \widetilde{p} = e^{-2a}p, \quad i = 1,2,3;\\
R_1:&\quad \widetilde{T} = e^aT, \quad \widetilde{C} = C + \frac{\beta_T}{\beta_C}(1 - e^a)T;\\
R_2:&\quad \widetilde{C} = e^aC, \quad \widetilde{T} = T + \frac{\beta_C}{\beta_T}(1 - e^a)C;\\
L:&\quad \widetilde{C} = \left(C + \frac{D_T}{D-\chi}T\right)e^{a(D-\chi)} - \frac{D_T}{D-\chi}T.
\end{aligned}
$$

Here a is a real parameter (every subgroup has its own parameter). The variables not mentioned in the above formulas remain unchanged. The transformations generated by Z, Z_T, Z_C, Z_T^*, Z_C^* are obtained by extending the one-parameter subgroup corresponding to the generator Z_0 by the following transformations:

$$
\begin{aligned}
&Z_T: \ \widetilde{T} = e^{-a}T; \quad Z_C: \ \widetilde{C} = e^{-a}C; \quad Z: \ \widetilde{T} = e^{-a}T, \ \widetilde{C} = e^{-a}C;\\
&Z_T^*: \ \widetilde{T} = e^{-a}T, \ \widetilde{C} = C + \frac{aD_T}{D-\chi}T;\\
&Z_C^*: \ \widetilde{C} = e^{-a}C + \frac{D_T}{D-\chi}(e^{-a} - 1)T.
\end{aligned}
$$

Note that the generators X_0, X_{ij}, H_i and H_0 are admitted by many models of continuum mechanics, while the other generators are specific for the equations of vibrational convection.

In this work the equivalence transformations of parameters were not taken into account when performing the symmetry classification. These transformations are usually used to simplify the arbitrary elements entering into the equations. When arbitrary elements are constants, the aim is to set as many constants as possible to zero or to unity. However, the equations so obtained do not possess the necessary physical parameters and to use them to construct physically meaningful solutions with the help of symmetries is not convenient. In contrast to this approach the classification presented here shows the dependence of symmetry properties upon physical effects incorporated into the model.

Finally it should be noted that many analytical solutions of equations (1)–(5) presented in [2] turn out to be invariant or partially invariant with respect to the subgroups of the admissible transformation group. The symmetry analysis performed in this work allows a systematic investigation the equations of invariant submodels for vibrational convection and the construction of their solutions.

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[1] Ovsyannikov L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[2] Gershuni G.Z., Lyubimov D.V., *Thermal Vibrational Convection*, Wiley, New York, 1998.

[3] Mialdun A., Ryzhkov I.I., Melnikov D.E. and Shevtsova V., Experimental evidence of thermal vibrational convection in non-uniformly heated fluid in a reduced gravity environment, *Phys. Rev. Lett.*, 2008, V.101, 084501, 4 pp.

[4] Shevtsova V., Ryzhkov I.I., Melnikov D., Gaponenko Y. and Mialdun A., Experimental and theoretical study of vibration-induced convection in low gravity, *J. Fluid Mech.*, 2010, V.648, 53–82.

[5] Andreev V.K. and Ryzhkov I.I., Symmetry analysis and exact solutions of the thermal diffusion equations, *Differ. Equ.*, 2005, V.41, 508–517.

[6] Ryzhkov I.I., Symmetry analysis of equations for convection in binary mixture, *J. Siberian Fed. Univ.*, 2008, V.1, 410–431.

[7] Rodionov A.A. and Stepanova I.V., Group classification of model equations of convection with the buoyancy force, *J. Comp. Techn.*, 2008, V.13, no. 5, 61–69.

[8] Ryzhkov I.I., Invariant solutions of the thermal-diffusion equations for a binary mixture in the case of plane motion, *J. Appl. Mech. and Techn. Phys.*, 2006, V.47, no. 1, 79–90.

[9] Ryzhkov I.I., On double diffusive convection with Soret effect in a vertical layer between co-axial cylinders, *Phys. D*, 2008, V.215, 191–200.

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Reduction operators of variable coefficient semilinear diffusion equations with an exponential source

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Reduction operators (called also nonclassical or Q-conditional symmetries) of variable coefficient semilinear reaction-diffusion equations with exponential source $f(x)u_t = (g(x)u_x)_x + h(x)e^{mu}$ are investigated using the algorithm involving a mapping between classes of differential equations, which is generated by a family of point transformations. Special attention is paid to check whether reduction operators are inequivalent to Lie symmetry operators. The derived reduction operators are applied to construction of closed-form solutions.

1 Introduction

Various processes in nature are successfully modeled by nonlinear systems of partial differential equations (PDEs). In order to study the behaviour of these processes, it is important to know solutions of corresponding model equations. Lie symmetries and the classical reduction method present a powerful and algorithmic technique for the construction of solutions (of systems) of PDEs [14, 16]. In [2] Bluman and Cole introduced a new method to find solutions of PDEs. It was called "non-classical" to emphasize its difference from the classical method of Lie reduction. A precise and rigorous definition of nonclassical invariance was firstly formulated in [6] as "a generalization of the Lie definition of invariance" (see also [26]). Subsequently operators satisfying the nonclassical invariance criterion were called nonclassical symmetries, conditional symmetries and Q-conditional symmetries by different authors [4, 5, 7, 12]. All names are in use till now. See, e.g., [11, 15, 20] for comprehensive reviews of the subject. Following Ref. [19] we call nonclassical symmetries *reduction operators*. The necessary definitions, including ones of equivalence of reduction operators, and statements relevant for this paper are collected in [24].

The problem of finding reduction operators for a PDE reduces to the integration of an overdetermined system of nonlinear PDEs. The complexity increases essentially in the case of classification problem of reduction operators for a class of PDEs having nonconstant arbitrary elements.

The experience of classification of Lie symmetries for classes of variable coefficient PDEs shows that the usage of equivalence and gauging transformations can essentially simplify the group classification problem and even be a crucial point in solving the problem [8, 22, 24]. The above transformations are of major importance for studying reduction operators since under their classification one needs to overcome much more essential obstacles then those arising under the classification of Lie symmetries.

In [24] we propose an algorithm involving mapping between classes for finding reduction operators of the variable coefficient reaction-diffusion equations with power nonlinearity

$$f(x)u_t = (g(x)u_x)_x + h(x)u^m, \tag{1}$$

where f, g and h are arbitrary smooth functions of the variable x and m is an arbitrary constant such that $fgh \neq 0$ and $m \neq 0, 1$. In [23] reduction operators of the equations from class (1) with $m \neq 2$ were investigated using this algorithm. The case $m = 2$ was not systematically considered since it is singular from the Lie symmetry point of view and needs an additional mapping between classes (see [24] for more details). Nevertheless all the reduction operators constructed in [23] for the general case $m \neq 0, 1, 2$ are also fit for the values $m = 0, 1, 2$.

In this paper we implement the same technique to find reduction operators of the variable coefficient reaction-diffusion equations with exponential nonlinearity

$$f(x)u_t = (g(x)u_x)_x + h(x)e^{mu}. \tag{2}$$

Here f, g and h are arbitrary smooth functions of the variable x, $fgh \neq 0$ and m is an arbitrary nonvanishing constant.

The structure of this paper is as follows. For the convenience of readers section 2 contains a short review of results obtained in [21] and used here, namely, in this section the necessary information concerning equivalence transformations, the mapping of class (2) to the so-called "imaged" class and the group classification of equations from the imaged class is collected. Moreover all additional equivalence transformations connecting the cases of Lie symmetry extensions (cf. Table 1) are first found and presented therein. As a result the classifications of Lie symmetry extensions up to all admissible point transformations in the imaged and, therefore, initial classes are also obtained. Section 3 is devoted to the description of the original algorithm for finding reduction operators of equations from class (2) using a mapping between classes generated by a family of point transformations. The results of section 4 are completely new and concern the investigation of reduction operators for equations from the imaged class. Reduction operators obtained in an explicit form are used for the construction of solutions of equations from both the imaged and initial classes.

2 Lie symmetries and equivalence transformations

Class (2) has complicated transformational properties. An indicator of this is that it possesses the nontrivial generalized extended equivalence group, which does not coincide with its usual equivalence group, cf. Theorem 1 below. To produce the group classification of class (2) it is necessary to gauge arbitrary elements of this class with equivalence transformations and a subsequent mapping of it onto a simpler class [21, 24]. It appears that the preimage set of each equation from the imaged class is a biparametric family of equations from the initial class (2). Moreover preimages of the same equation belong to the same orbit of the equivalence group of the initial class. It allows one to look only for the simplest representative of the preimage to obtain its symmetries, solution etc., and then to reproduce these results for a two-parametric family of equations from the initial class using equivalence transformations.

Theorem 1. *The generalized extended equivalence group* $\hat{G}^{\sim}_{\rm exp}$ *of class* (2) *consists of the transformations*

$$\tilde t=\delta_1 t+\delta_2,\quad \tilde x=\varphi(x),\quad \tilde u=\delta_3 u+\psi(x),$$
$$\tilde f=\frac{\delta_0\delta_1}{\varphi_x}f,\quad \tilde g=\delta_0\varphi_x g,\quad \tilde h=\frac{\delta_0\delta_3}{\varphi_x}\exp\left(-\frac{m}{\delta_3}\,\psi\right)h,\quad \tilde m=\frac{m}{\delta_3},$$

where φ *is an arbitrary nonconstant smooth function of* x, $\psi=\delta_4\int\frac{dx}{g(x)}+\delta_5$ *and* δ_j, $j=0,1,\dots,5$, *are arbitrary constants such that* $\delta_0\delta_1\delta_3\neq0$.

Corollary 1. *The usual equivalence group of class* (2) *is the subgroup of* $\hat{G}^{\sim}_{\rm exp}$ *singled out by the condition* $\delta_4=0$.

The transformations from $\hat{G}^{\sim}_{\rm exp}$ associated with varying the parameter δ_0 in fact do not change equations from class (2) and hence form the gauge equivalence group of this class. The values of arbitrary elements connected by a such transformation correspond to different representations of the same equation.

Analogously to the power case we firstly map class (2) onto its subclass

$$f(x)u_t=(f(x)u_x)_x+h(x)e^u \tag{3}$$

(we omit tildes over the variables) using the family of equivalence transformations parameterized by the arbitrary elements f, g and m,

$$\tilde t=\mathrm{sign}(f(x)g(x))\,t,\quad \tilde x=\int\left|\frac{f(x)}{g(x)}\right|^{\frac12}dx,\quad \tilde u=m\,u. \tag{4}$$

The new arbitrary elements are expressed via the old ones in the following way:

$$\tilde f(\tilde x)=\tilde g(\tilde x)=\mathrm{sign}(g(x))|f(x)g(x)|^{\frac12},\quad \tilde h(\tilde x)=m\left|\frac{g(x)}{f(x)}\right|^{\frac12}h(x),\quad \tilde m=1.$$

The next step is to change the dependent variable in class (3):

$$v(t,x) = u(t,x) + G(x), \quad \text{where} \quad G(x) = \ln|f(x)^{-1}h(x)|. \tag{5}$$

Finally we obtain the class

$$v_t = v_{xx} + F(x)v_x + \varepsilon e^{\,v} + H(x), \tag{6}$$

where $\varepsilon = \operatorname{sign}(f(x)h(x))$ and the new arbitrary elements F and H are expressed via the arbitrary elements of class (6) according to the formulas

$$F = f_x f^{-1} \quad \text{and} \quad H = -G_{xx} - G_x F. \tag{7}$$

All results on Lie symmetries and solutions of class (6) can be extended to class (3) by the inversion of transformation (5).

The arbitrary elements f and h of class (6) are expressed via the functions F and H in the following way:

$$\begin{aligned} &f = c_0 \exp\left(\textstyle\int F dx\right), \quad h = \varepsilon c_0 \exp\left(\textstyle\int F dx + G\right), \\ &\text{where} \quad G = \textstyle\int e^{-\int F dx}\left(c_1 - \int H e^{\int F dx} dx\right) dx + c_2. \end{aligned} \tag{8}$$

Here c_0, c_1 and c_2 are arbitrary constants, $c_0 \neq 0$. The constant c_0 is inessential and can be set to the unity by an obvious gauge equivalence transformation. The equations from class (3), that have the same image in class (6) with respect to transformation (5), i.e. the arbitrary elements of which are given by (8) and differ only by values of constants c_1 and c_2, are $\hat{G}^{\sim}_{\exp}$-equivalent. The equivalence transformation

$$\tilde{t} = t, \quad \tilde{x} = x, \quad \tilde{u} = u + c_1 \textstyle\int e^{-\int F dx} dx + c_2 \tag{9}$$

maps an equation (6) having f and h of the form (8) with $c_1^2 + c_2^2 \neq 0$ to the one with $c_1 = c_2 = 0$. Hence up to $\hat{G}^{\sim}_{\exp}$-equivalence we can consider, without loss of generality, only equations from class (3) that have the arbitrary elements determined by (8) with $c_1 = c_2 = 0$.

Theorem 2. *The generalized extended equivalence group $G^{\sim}_{\exp}$ of class* (6) *coincides with its usual equivalence group and is formed by the transformations*

$$\begin{aligned} &\tilde{t} = \delta_1^2 t + \delta_2, \quad \tilde{x} = \delta_1 x + \delta_3, \quad \tilde{v} = v - \ln\delta_1^2, \\ &\tilde{F} = \delta_1^{-1} F, \quad \tilde{H} = \delta_1^{-2} H, \end{aligned}$$

where δ_j, $j = 1, 2, 3$, are arbitrary constants, $\delta_1 \neq 0$.

The *kernel* of the maximal Lie invariance algebras of equations from class (6) is the one-dimensional algebra $\langle\partial_t\rangle$. It means that any equation from class (6) is invariant with respect to translations by t and there are no more common Lie symmetries.

Theorem 3. $G^{\sim}_{\rm exp}$*-inequivalent cases of extension of the maximal Lie invariance algebras in class* (6) *are exhausted by those presented in Table 1.*

Table 1. The group classification of the class $v_t = v_{xx} + F(x)v_x + \varepsilon e^{\,v} + H(x)$.

N	$F(x)$	$H(x)$	Basis of $A^{\max}$
0	$\forall$	$\forall$	∂_t
1	$\alpha x^{-1}+\mu x$	$\beta x^{-2}+2\mu$	$\partial_t,\ e^{-2\mu t}(\partial_t-\mu x\partial_x+2\mu\partial_v)$
2	αx^{-1}	βx^{-2}	$\partial_t,\ 2t\partial_t+x\partial_x-2\partial_v$
3	μx	γ	$\partial_t,\ e^{-\mu t}\partial_x$
4	λ	γ	$\partial_t,\ \partial_x$
5	μx	2μ	$\partial_t,\ e^{-\mu t}\partial_x,\ e^{-2\mu t}(\partial_t-\mu x\partial_x+2\mu\partial_v)$
6	λ	0	$\partial_t,\ \partial_x,\ 2t\partial_t+(x-\lambda t)\partial_x-2\partial_v$

Here $\lambda\in\{0,1\} \bmod G^{\sim}_{\rm exp}$, $\mu=\pm1 \bmod G^{\sim}_{\rm exp}$; α,β and γ are arbitrary constants, $\alpha^2+\beta^2\neq0$. We also have $\gamma\neq2\mu$ and $\gamma\neq0$ in Cases 3 and 4, respectively.

The corresponding results on group classification of class (2) up to $\hat G^{\sim}_{\rm exp}$-equivalence is given in Table 3 of [21].

Additional equivalence transformations between $G^{\sim}_{\rm exp}$-inequivalent cases of Lie symmetry extension are also constructed. The pairs of point-equivalent cases from Table 1 and the corresponding transformations are exhausted by the following:

$$
\begin{gathered}
1\mapsto\tilde2,\quad 5\mapsto\tilde6|_{\tilde\lambda=0}\colon\ \tilde t=\frac{1}{2\mu}e^{2\mu t},\quad \tilde x=e^{\mu t}x,\quad \tilde v=v-2\mu t,\\
4\mapsto\tilde4|_{\tilde\lambda=0},\quad 6\mapsto\tilde6|_{\tilde\lambda=0}\colon\ \tilde t=t,\quad \tilde x=x+\lambda t,\quad \tilde v=v.
\end{gathered}
\tag{10}
$$

The inequivalence of other different cases of Table 1 can be proved using differences in properties of the corresponding maximal Lie invariance algebras, which should coincide for similar equations. Thus the dimensions of the maximal Lie invariance algebras are one, three and two in the general case (Case 0), Cases 5 and 6 and the other cases, respectively. In contrast to Cases 1–3, the algebra of Case 4 is commutative. The derivative of the algebra of Case 3 has the zero projection onto the space of t and this is not the case for Cases 1 and 2. Possession of the zero (resp. nonzero) projection onto the space of t is an invariant characteristic of Lie algebras of vector fields in the space of the variables t, x and v with respect to point transformations connecting a pair of evolution equations since for any such transformation the expression of the transformed t is well known to depend only on t [9, 13].

A more difficult problem is to prove that there are no more additional equivalences within a parameterized case of Table 1. (In fact all the cases are parameterized.) This needs at least a preliminary study of form-preserving [9] (or admissible [17, 18]) transformations. In contrast to transformations from the corresponding equivalence group, which transform each equation from the class $\mathcal{L}$ of differential equations under consideration to an equation from the same class, a form-preserving transformation should transform at least a single equation from $\mathcal{L}$ to an equation from the same class. The notion of admissible transformations is a formalization of the notion of form-preserving transformations. The set of admissible transformations of the class $\mathcal{L}$ is formed by the triples each of which consists of the tuples of arbitrary elements corresponding to the initial and target equations and a point transformation connecting these equations. It is obvious that each transformation from the equivalence group generates a family of admissible transformations parameterized by arbitrary elements of the class $\mathcal{L}$.

A preliminary description of the set of admissible transformations of the class (6), which is sufficient for our purpose, is given by the following statements.

Proposition 1. *Any admissible point transformation in the class* (6) *has the form*

$$\tilde{t} = T(t), \quad \tilde{x} = \delta\sqrt{T_t}\,x + X(t), \quad \tilde{v} = v - \ln T_t,$$

where $\delta = \pm 1$ and T and X are arbitrary smooth functions of t such that $T_t > 0$. The corresponding values of the arbitrary elements are related via the formulas

$$\tilde{F} = \frac{\delta}{\sqrt{T_t}}F - \frac{\delta}{2}\frac{T_{tt}}{\sqrt{T_t}^3}x - \frac{X_t}{T_t}, \quad \tilde{H} = \frac{1}{T_t}H - \frac{T_{tt}}{{T_t}^2}.$$

Corollary 2. *Only equations from the class* (6) *the arbitrary elements of which have the form*

$$F = \mu x + \lambda + \frac{\alpha}{x+\kappa}, \quad H = \gamma + \frac{\beta}{(x+\kappa)^2}, \tag{11}$$

where α, β, γ, κ and μ are constants, possess admissible transformations that are not generated by transformations from the equivalence group $G^{\sim}_{\rm exp}$. The subclass of the class (6)*, singled out by condition* (11)*, is closed under any admissible transformation within class* (6)*. The (constant) parameters of the representation* (11) *are transformed by an admissible transformation in the following way:*

$$\tilde{\alpha} = \alpha, \quad \tilde{\beta} = \beta, \qquad \tilde{\kappa} = \delta\sqrt{T_t}\kappa - X \quad \textit{if} \quad (\alpha,\beta) \neq (0,0),$$
$$\tilde{\gamma} = \frac{\gamma}{T_t} - \frac{T_{tt}}{T_t^2}, \quad \tilde{\mu} = \frac{\mu}{T_t} - \frac{1}{2}\frac{T_{tt}}{T_t^2}, \quad \tilde{\lambda} = -\tilde{\mu}X - \frac{X_t}{T_t} + \frac{\delta\lambda}{\sqrt{T_t}}.$$

In particular $T_{tt} = 0$ if $\gamma \neq 2\mu$.

Finally we can formulate the assertion on group classification with respect to the set of admissible transformations.

Theorem 4. *Up to point equivalence cases of extension of the maximal Lie invariance algebras in class* (6) *are exhausted by Cases* 0*,* 2*,* 3*,* $4_{\lambda=0}$ *and* $6|_{\lambda=0}$ *of Table 1.*

3 Algorithm of finding reduction operators via mappings between classes

At first we adduce the definition of nonclassical symmetries [7, 19, 26], adapting it for the case of one second-order PDE with two independent variables, relevant for this paper. Consider a second-order differential equation $\mathcal{L}$ of the form $L(t,x,u_{(2)})=0$ for the unknown function u of the two independent variables t and x, where $u_{(2)}=(u,u_t,u_x,u_{tt},u_{tx},u_{xx})$. Let Q be a first-order differential operator of the general form

$$Q=\tau(t,x,u)\partial_t+\xi(t,x,u)\partial_x+\eta(t,x,u)\partial_u,\quad (\tau,\xi)\neq(0,0).$$

Definition 1. The differential equation $\mathcal{L}$ is called *conditionally invariant* with respect to an operator Q if the relation

$$Q_{(2)}L(t,x,u_{(2)})\big|_{\mathcal{L}\cap\mathcal{Q}^{(2)}}=0 \tag{12}$$

holds, which is called the *conditional (or nonclassical) invariance criterion.* Then Q is called a *conditional symmetry* (or nonclassical symmetry, Q-conditional symmetry or reduction operator) of the equation $\mathcal{L}$.

The symbol $Q_{(2)}$ stands for the standard second prolongation of Q (see e.g. [14, 16]). $\mathcal{Q}^{(2)}$ is the manifold determined in the second-order jet space by the differential consequences of the characteristic equation $Q[u]:=\eta-\tau u_t-\xi u_x=0$, which have, as differential equations, orders not greater than two.

It was proven in [26] that a differential equation $\mathcal{L}$ is conditionally invariant with respect to the operator Q if and only if the Ansatz constructed with this operator reduces the equation $\mathcal{L}$. That is why it seems natural to call operators of conditional (nonclassical) symmetries *reduction operators*.

Here we present the algorithm of application of equivalence transformations, gauging of arbitrary elements and mappings between classes of equations to classification of reduction operators of class (2)

1. Firstly we gauge class (2) to subclass (3) constrained by the condition $f=g$. Then class (3) is mapped to the imaged class (6) by transformation (5).
2. Reduction operators should be classified up to the equivalence relations generated by the corresponding equivalence groups or even by the whole sets of admissible transformations. As the singular case $\tau=0$ is “no-go” [10, 25], only the regular case $\tau\neq0$ (reduced to the case $\tau=1$) should be considered. Operators equivalent to Lie symmetry ones should be neglected.
3. It is well-known (see e.g. [1, 3]) that the equations from the imaged class (6) with $F=0$ and $H=\text{const}$, and therefore all equations similar to them with respect to point transformations, possess no regular reduction operators that are inequivalent to Lie symmetry operators. This is why all the above equations should be excluded from consideration.

4. Preimages of the nonclassical symmetries obtained and of equations admitting them should be found using the inverses of gauging transformations and the push-forwards by these inverses on the sets of operators.

Reduction operators of equations from class (3) are easily found from reduction operators of corresponding equations from (6) using the formula

$$\tilde{Q} = \tau\partial_t + \xi\partial_x + (\eta - \xi G_x)\,\partial_u. \tag{13}$$

Here τ, ξ and η, respectively, are the coefficients of ∂_t, ∂_x and ∂_v in a reduction operator of an equation from class (6). The function G is defined in (8).

In [23, 24] we discussed two ways to use mappings between classes of equations in the investigation of reduction operators and their usage to find solutions. The preferable way is based on the implementation of reductions in the imaged class and preimaging of the obtained solutions instead of preimaging the corresponding reduction operators.

4 Reduction operators and solutions

Following the above algorithm we look for $G^{\sim}_{\rm exp}$-inequivalent reduction operators with nonvanishing coefficient of ∂_t for the equations from the imaged class (6). Up to the usual equivalence of reduction operators we need to consider only the operators of the form

$$Q = \partial_t + \xi(t,x,v)\partial_x + \eta(t,x,v)\partial_v.$$

Applying conditional invariance criterion (12) to equation (6) we obtain a third-degree polynomial of v_x with coefficients depending on t, x and v which has to identically equal zero. Separation respect to different powers of v_x results in the following determining equations for the coefficients ξ and η:

$$\begin{aligned}
&\xi_{vv} = 0, \qquad \eta_{vv} = 2(\xi_{xv} - \xi\xi_v - F\xi_v),\\
&\xi_t - \xi_{xx} + 2\xi_x\xi + 3\xi_v\,(H + \varepsilon e^v) + 2\eta_{vx} - 2\xi_v\eta + F\xi_x + \xi F_x = 0,\\
&\eta_t - \eta_{xx} + 2\xi_x\eta = \xi H_x + F\eta_x + (2\xi_x - \eta_v)H + \varepsilon e^v\,(\eta + 2\xi_x - \eta_v)\,.
\end{aligned} \tag{14}$$

Integration of the first two equations of (14) gives us the expressions for ξ and η with an explicit dependence on v:

$$\xi = av + b, \quad \eta = -\frac{1}{3}a^2v^3 + (a_x - ab - aF)v^2 + cv + d, \tag{15}$$

where $a = a(t,x)$, $b = b(t,x)$, $c = c(t,x)$ and $d = d(t,x)$ are smooth functions of t and x.

Substituting the expressions (15) for ξ and η into the third and forth equations of (14) and collecting the coefficients of different powers of v in the resulting equations, we derive the conditions $a = c = 0$, $d = -2b_x$ and two classifying equations, which contain both the coefficient $b=b(t,x)$ and the arbitrary elements $F=F(x)$ and $H=H(x)$. Summarising the above consideration we have the following assertion.

Proposition 2. *Any regular reduction operator of an equation from the imaged class* (6) *is equivalent to an operator of the form*

$$Q=\partial_t+b\partial_x-2b_x\partial_v, \tag{16}$$

where the coefficient $b=b(t,x)$ satisfies the overdetermined system of partial differential equations

$$\begin{aligned} &b_t-b_{xx}+2bb_x+Fb_x+bF_x=0,\\ &bH_x+2b_xH-4bb_{xx}-2(Fb)_{xx}-2Fb_{xx}=0 \end{aligned} \tag{17}$$

with the corresponding values of the arbitrary elements $F=F(x)$ and $H=H(x)$.

The second equation of (17) can be written in the more compact form

$$4(b+F)b_{xx}=2Kb_x+K_xb,$$

where $K=H-2F_x$, which is more convenient for the study of compatibility.

Analogously to the power case, we were not able to completely study all the cases of integration of system (17) depending upon values of F and H. This is why we try to solve this system under different additional constraints imposed either on b or on (F,H).

The most interesting results are obtained for the constraint $b_t=0$. Then F and H are expressed, after a partial integration of (17), via the function $b=b(x)$ that leads to the following statement.

Theorem 5. *For any nonvanishing smooth function $b=b(x)$ the equation from the class* (6) *with the arbitrary elements*

$$F=\frac{1}{b}\left(b_x+k_1-b^2\right), \quad H=\frac{2}{b^2}\left(k_2+b_x(k_1-b^2)+bb_{xx}\right), \tag{18}$$

where k_1 and k_2 are constants, admits the reduction operator (16) *with the same b.*

An Ansatz constructed by the reduction operator (16) with $b_t=0$ has the form

$$v=z(\omega)-2\ln|b|, \quad \text{where} \quad \omega=t-\int\frac{dx}{b}.$$

The substitution of the Ansatz into equation (6) leads to the reduced ODE

$$z_{\omega\omega}-k_1z_\omega+\varepsilon e^{\,z}+2k_2=0. \tag{19}$$

For $k_1=0$ the general solution of (19) is written in the implicit form

$$\int(c_1-4k_2z-2\varepsilon e^{\,z})^{-\frac{1}{2}}\,dz=\pm(\omega+c_2). \tag{20}$$

Up to similarity of solutions of equation (6) the constant c_2 is inessential and can be set to equal zero by a translation of ω, which is always induced by a translation of t.

Setting additionally $k_2 = 0$ in (20), we are able to integrate (20) in closed form and to write explicitly the general solution of (19). If $\varepsilon = 1$, then $c_1 > 0$ and (20) gives the following expression for e^z:

$$e^z = \frac{2s_1^2}{\cosh^2(s_1\omega + s_2)}.$$

Here and below $s_1 = \sqrt{|c_1|}/2$ and $s_2 = c_2 s_1$. If $\varepsilon = -1$, the integration leads to

$$e^z = \begin{cases} \dfrac{2s_1^2}{\sinh^2(s_1\omega + s_2)}, & c_1 > 0, \\ \dfrac{2s_1^2}{\cos^2(s_1\omega + s_2)}, & c_1 < 0, \\ \dfrac{2}{(\omega + c_2)^2}, & c_1 = 0. \end{cases}$$

As a result, for the equation from class (6) of the form

$$v_t = v_{xx} + \frac{1}{b}\left(b_x - b^2\right) v_x + \varepsilon e^v + \frac{2}{b}\left(b_{xx} - bb_x\right) \tag{21}$$

with $\varepsilon = -1$, we construct three families of closed-form solutions

$$\begin{aligned} v &= -2\ln\left|\frac{\sqrt{2}}{2s_1}\, b\sinh\left(s_1 t - s_1\int\frac{dx}{b} + s_2\right)\right|, \\ v &= -2\ln\left|\frac{\sqrt{2}}{2s_1}\, b\cos\left(s_1 t - s_1\int\frac{dx}{b} + s_2\right)\right|, \\ v &= -2\ln\left|\frac{\sqrt{2}}{2}\, b\left(t - \int\frac{dx}{b} + c_2\right)\right|, \end{aligned} \tag{22}$$

where s_1, s_2 and c_2 are arbitrary constants, $s_1 \neq 0$. Also we obtain a family of solutions

$$v = -2\ln\left|\frac{\sqrt{2}}{2s_1}\, b\cosh\left(s_1 t - s_1\int\frac{dx}{b} + s_2\right)\right| \tag{23}$$

of the equation (21) with $\varepsilon = 1$.

We continue the consideration by studying whether the equations from class (6) possessing nontrivial Lie symmetry properties, i.e. having the maximal Lie invariance algebras of dimension two or three, have nontrivial (i.e. inequivalent to Lie ones) regular reduction operators. It has been already remarked that constant coefficient equations from class (6) do not admit such reduction operators [1, 3]. Hence it is needless to consider Cases 4 and 6 of Table 1 as well as Case 5 connected with Case 6 by point transformation (10). As Case 1 reduces to Case 2 with the same transformation (10), we have to study only two cases, namely Cases 2 and 3. We substitute the pairs of values of the parameter-functions F and H corresponding to Cases 2 and 3 into system (17) in order to find relevant values for b. We ascertain that $b_t = 0$ is a necessary condition for existing non-Lie regular reduction operators for equations with the above values of (F, H). This is why we can use equations (18) instead of (17) for further studying.

The investigation of Case 3 of Table 1 leads to the conclusion that there are no non-Lie regular reduction operators for this case.

The functions F and H presented in Case 2 of Table 1 satisfy (18) if and only if $\beta = 2(1-\alpha)$, i.e., they have the form $F = \alpha x^{-1}$, $H = 2(1-\alpha)x^{-2}$, and $k_1 = k_2 = 0$. The corresponding value of b is $b = -(1+\alpha)x^{-1}$. Hence $\alpha \neq -1$ since otherwise $b = 0$. Substituting the derived form of the function b into the formulas (22) and (23), we find that the equation

$$v_t = v_{xx} + \frac{\alpha}{x}v_x + \varepsilon e^v + \frac{2(1-\alpha)}{x^2} \tag{24}$$

has the families of solutions

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2s_1 x}\cosh\left(s_1 t + \frac{s_1 x^2}{2(1+\alpha)} + s_2\right)\right|$$

if $\varepsilon = 1$ and

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2s_1 x}\sinh\left(s_1 t + \frac{s_1 x^2}{2(1+\alpha)} + s_2\right)\right|,$$

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2s_1 x}\cos\left(s_1 t + \frac{s_1 x^2}{2(1+\alpha)} + s_2\right)\right|,$$

$$v = -2\ln\left|\frac{\sqrt{2}(1+\alpha)}{2\,x}\left(t + \frac{x^2}{2(1+\alpha)} + c_2\right)\right|$$

if $\varepsilon = -1$. Recall that s_1, s_2 and c_2 are arbitrary constants with $s_1 \neq 0$.

As a representative of the preimage of equation (24) with respect to the transformation (5) we can choose the equation

$$x^\alpha u_t = (x^\alpha u_x)_x + \varepsilon x^{\alpha+2}e^u. \tag{25}$$

Solutions of this equation can be easily constructed from the above solutions of equation (24) using the transformation $u = v - 2\ln|x|$. If $\alpha = 1$, the chosen equation (25) can be replaced, e.g., by $xu_t = (xu_x)_x + \varepsilon x e^u$ which is just another representation of equation (24).

Non-Lie solutions of the equation

$$v_t = v_{xx} + \left(\frac{\alpha}{x} + \mu x\right) v_x + \varepsilon e^v + \frac{2(1-\alpha)}{x^2} + 2\mu,$$

where $\alpha \neq -1$ (Case 1 of Table 1), can be easily obtained from exact solutions of the equation (25) using the transformation (10). The corresponding reduction operator has the form (16) with $b = -(1+\alpha)x^{-1} - \mu x$.

We also prove the following assertions.

Proposition 3. *Equations from class* (6) *with* $F = \mathrm{const}$ *or* $H = \mathrm{const}$ *may admit only nontrivial regular reduction operators that are equivalent to operators of the form* (16)*, where the function* b *does not depend upon the variable* t*.*

Proposition 4. *Any reduction operator of an equation from class* (6)*, having the form* (16) *with* $b_{xx} = 0$*, is equivalent to a Lie symmetry operator of this equation.*

The proofs of these propositions are quite cumbersome and will be presented elsewhere.

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[1] Arrigo D.J., Hill J.M. and Broadbridge P., Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source, *IMA J. Appl. Math.*, 1994, V.52, 1–24.

[2] Bluman G.W. and Cole J.D., The general similarity solution of the heat equation, *J. Math. Mech.*, 1969, V.18, 1025–1042.

[3] Clarkson P.A. and Mansfield E.L., Symmetry reductions and exact solutions of a class of nonlinear heat equations, *Phys. D*, 1994, V.70, 250–288.

[4] Fushchych W.I., Shtelen W.M. and Serov N.I., *Simmetrijnyj Analiz i Tochnye Resheniya Nelinejnykh Uravnenij Matematicheskoj Fiziki*, Kyiv, Naukova Dumka, 1989 (in Russian). Translated and extended version: *Symmetry Analysis and Exact Solutions of Equations of Nonlinear Mathimatical Physics*, Dordrecht, Kluwer Academic Publishers, 1993.

[5] Fushchich W.I., Shtelen W.M., Serov M.I. and Popovych R.O., Q-conditional symmetry of the linear heat equation, *Proc. Acad. Sci. Ukraine*, 1992, no. 12, 28–33.*

[6] Fushchich W.I. and Tsyfra I.M., On a reduction and solutions of the nonlinear wave equations with broken symmetry, *J. Phys. A: Math. Gen.*, 1987, V.20, L45–L48.*

[7] Fushchych W.I. and Zhdanov R.Z., Conditional symmetry and reduction of partial differential equations, *Ukrainian Math. J.*, 1992, V.44, 875–886.*

[8] Ivanova N.M., Popovych R.O. and Sophocleous C., Group analysis of variable coefficient diffusion-convection equations. I. Enhanced group classification, *Lobachevskii J. Math.*, 2010, V.31, 100–122.

[9] Kingston J.G. and Sophocleous C., On form-preserving point transformations of partial differential equations, *J. Phys. A: Math. Gen.*, 1998, V.31, 1597–1619.

[10] Kunzinger M. and Popovych R.O., Singular reduction operators in two dimensions, *J. Phys. A: Math. Theor.*, 2008, V.41, 505201, 24 pp., arXiv:0808.3577.

[11] Kunzinger M. and Popovych R.O., Is a nonclassical symmetry a symmetry?, Proc. of Fourth International Workshop "Group Analysis of Differential Equations and Integrable Systems" (2008, October 26–30, Protaras, Cyprus), 2009, 107–120, arXiv:0903.0821.

[12] Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A: Math. Gen.*, 1989, V.22, 2915–2924.

[13] Magadeev B.A., On group classification of nonlinear evolution equations, *Algebra i Analiz*, 1993, V.5, 141–156 (in Russian); English translation in *St. Petersburg Math. J.*, 1994, V.5, 345–359.

[14] Olver P., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[15] Olver P.J. and Vorob'ev E.M., Nonclassical and conditional symmetries, in *CRC Handbook of Lie Group Analysis of Differential Equations*, Vol. 3, Editor N.H. Ibragimov, Boca Raton, Florida, CRC Press, 1996, 291–328.

[16] Ovsiannikov L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[17] Popovych R.O., Classification of admissible transformations of differential equations, Collection of Works of Institute of Mathematics (Kyiv, Ukraine), 2006, V.3, no. 2, 239–254.**

[18] Popovych R.O., Kunzinger M. and Eshraghi H., Admissible point transformations and normalized classes of nonlinear Schrodinger equations, *Acta Appl. Math.*, 2010, V.109, 315–359, arXiv:math-ph/0611061.

[19] Popovych R.O., Vaneeva O.O. and Ivanova N.M., Potential nonclassical symmetries and solutions of fast diffusion equation, *Phys. Lett. A*, 2007, V.362, 166–173, arXiv:math-ph/0506067.

[20] Saccomandi G., A personal overview on the reduction methods for partial differential equations, *Note Mat.*, 2004, V.23, no. 2, 217–248.

[21] Vaneeva O., Group classification via mapping between classes: an example of semilinear reaction-diffusion equations with exponential nonlinearity, *Proc. of the 5th Math. Phys. Meeting: Summer School and Conf. on Modern Mathematical Physics (July 6–17, 2008, Belgrade, Serbia)*, 2009, 463–471, arXiv:0811.2587.

[22] Vaneeva O.O., Johnpillai A.G., Popovych R.O. and Sophocleous C., Enhanced group analysis and conservation laws of variable coefficient reaction-diffusion equations with power nonlinearities, *J. Math. Anal. Appl.*, 2007, V.330, 1363–1386, arXiv:math-ph/0605081.

[23] Vaneeva O.O., Popovych R.O. and Sophocleous C., Reduction operators of variable coefficient semilinear diffusion equations with a power source, *Proc. of 4th International Workshop "Group Analysis of Differential Equations and Integrable Systems" (2008, October 26–30, Protaras, Cyprus)*, 2009, 191–209, arXiv:0904.3424.

[24] Vaneeva O.O., Popovych R.O. and Sophocleous C., Enhanced group analysis and exact solutions of variable coefficient semilinear diffusion equations with a power source, *Acta Appl. Math.*, 2009, V.106, 1–46, arXiv:0708.3457.

[25] Zhdanov R.Z. and Lahno V.I., Conditional symmetry of a porous medium equation, *Phys. D*, 1998, V.122, 178–186.

[26] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, 101–123, arXiv:math-ph/0207023.

* Available at http://www.imath.kiev.ua/∼ fushchych

** Available at http://www.imath.kiev.ua/∼ appmath/Collections/collection2006.pdf

On symmetries and conservation laws for a system of hydrodynamic type describing relaxing media

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We find the Lie point symmetries and conservation laws for a (1+1)-dimensional system of hydrodynamic type describing relaxing media which was first studied by V.A. Vladimirov [*Rep. Math. Phys.*, 2008, V.61, 381–400].

Vladimirov [7] (see also [1]) undertook analytical and numerical studies of the system of hydrodynamic type describing relaxing media,

$$\begin{aligned} &u_t + p_x = \gamma, \\ &v_t - u_x = 0, \\ &\tau p_t + \frac{\chi}{v^2} u_x = \frac{\kappa}{v} - p, \end{aligned} \tag{1}$$

where t is time, x is mass coordinate, u is mass velocity, v is specific volume, p is pressure and γ is acceleration of the external force, κ and χ/τ are squares of the equilibrium and “frozen” sound velocities, respectively (so $\tau \neq 0$ by assumption). Inter alia in [7] it was shown that compacton-like solutions for (1) can be found among the set of traveling-wave solutions. In view of the investigation of the stability properties of such solutions undertaken in [7] it is interesting to study the conservation laws of (1), as their existence helps the stability analysis and the symmetries, in order to find further solutions for (1).

Computing the Lie point symmetries of (1) yields the following assertion.

Theorem 1. *The most general Lie point symmetry of* (1) *is a linear combination of the operators*

$$\begin{aligned} &\mathbf{v}_1 = \frac{\partial}{\partial t}, \quad \mathbf{v}_2 = \frac{\partial}{\partial x}, \quad \mathbf{v}_3 = \frac{\partial}{\partial u}, \quad \mathbf{v}_4 = p\frac{\partial}{\partial p} + x\frac{\partial}{\partial x} - v\frac{\partial}{\partial v}, \\ &\mathbf{v}_5 = \exp(-t/\tau)\frac{\partial}{\partial p}, \quad \mathbf{v}_6 = \exp(-t/\tau)\left(\tau\frac{\partial}{\partial u} + x\frac{\partial}{\partial p}\right) \end{aligned} \tag{2}$$

with the commutation relations (all remaining commutators vanish)

$$\begin{aligned} &[\mathbf{v}_1, \mathbf{v}_5] = -\frac{1}{\tau}\mathbf{v}_5, \quad [\mathbf{v}_1, \mathbf{v}_6] = -\frac{1}{\tau}\mathbf{v}_6, \quad [\mathbf{v}_2, \mathbf{v}_4] = \mathbf{v}_2, \\ &[\mathbf{v}_2, \mathbf{v}_6] = \mathbf{v}_5, \quad [\mathbf{v}_4, \mathbf{v}_5] = -\mathbf{v}_5. \end{aligned} \tag{3}$$

The symmetries $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_4$ were found in [7]; those remaining are new. Note that the Lie algebra spanned by the vector fields (2) is solvable; actually it is a direct sum of the one-dimensional abelian Lie algebra $\mathfrak{g}_1$, spanned by $\mathbf{v}_3$, and a five-dimensional solvable Lie algebra which is isomorphic to the algebra $A_{5,19}$ in the notation of [6] (as an aside note that all real five-dimensional Lie algebras were first classified in [4]). The vector field $\mathbf{v}_1$ is the time-translation symmetry, $\mathbf{v}_2$ is the x-translation symmetry, $\mathbf{v}_3$ reflects the invariance of (1) under the shift of u by a constant, $\mathbf{v}_4$ is the scaling symmetry, $\mathbf{v}_5$ expresses the invariance of (1) under the shift of p by a constant times $\exp(-t/\tau)$ and $\mathbf{v}_6$ corresponds to the invariance of (1) under the simultaneous shift of p by a constant times $\exp(-t/\tau)x$ and of u by the same constant times $\tau\exp(-t/\tau)$.

Using the above symmetries enables us to find a number of particular solutions for (1). For instance,

$$\begin{aligned}
u &= \gamma t + C_1 + C_2\tau\exp(-t/\tau) + C_3\kappa t,\\
v &= -1/(C_3 x),\\
p &= x(C_2\exp(-t/\tau) - C_3\kappa),
\end{aligned}$$

where C_i are arbitrary constants ($C_3 \neq 0$), is the most general solution of (1) which is invariant under the symmetry $\mathbf{v}_4$.

Exponentiating the vector fields (2) allows us (cf. e.g. [5]) to proliferate known solutions of (1):

Corollary 1. *If $u = U(x,t)$, $v = V(x,t)$ and $p = P(x,t)$ is a solution of (1), then so is*

$$\begin{aligned}
u &= U(\exp(c_4)x + c_2, t + c_1) + c_3 + c_6\tau\exp(-t/\tau),\\
v &= \exp(-c_4)V(\exp(c_4)x + c_2, t + c_1),\\
p &= \exp(c_4)P(\exp(c_4)x + c_2, t + c_1) + c_5\exp(-t/\tau) + c_6 x\exp(-t/\tau),
\end{aligned}$$

where $c_1, \ldots, c_6$ are arbitrary constants.

Theorem 2. *The system (1) possesses the following local conservation laws:*

$$\begin{aligned}
D_t(u) &= D_x(\gamma x - p),\\
D_t(v) &= D_x(u),\\
D_t(tu + xv) &= D_x(xu - tp + \gamma tx),
\end{aligned} \tag{4}$$

where D_t and D_x are total derivatives.

We conjecture that for generic values of the parameters γ, κ, χ and τ system (1) has no generalized symmetries other than those given in (2) and no local conservation laws inequivalent to (4). This conjecture is supported by the direct computation of symmetries and cosymmetries of order up to three using the software *Jets* [3].

If we define the potentials w_1, w_2, w_3 associated with (4) by the relations

$$\begin{aligned}&(w_1)_x = u, \quad (w_1)_t = \gamma x - p\\&(w_2)_x = v, \quad (w_2)_t = u,\\&(w_3)_x = tu + xv, \quad (w_3)_t = xu - tp + \gamma tx,\end{aligned} \tag{5}$$

then we find that (1) supplemented by (5) also admits a nonlocal conservation law

$$(w_2)_t = (w_1)_x. \tag{6}$$

The corresponding potential w_4 is defined by the system

$$(w_4)_x = w_2, \quad (w_4)_t = w_1.$$

However, it is readily verified that we have

$$tw_1 + xw_2 - w_3 - w_4 = \text{const},$$

i.e., the potential w_4 is a function of the rest (up to an inessential constant) and thus is not really independent. In the terminology of [2] the conservation law (6) is induced by the local conservation laws (4).

Proposition 1. *If $\chi = \kappa\tau$, then system* (1), *possesses an additional conservation law of special form,*

$$D_t\left(\exp(t/\tau)\left(p - \frac{\kappa}{v}\right)\right) = 0, \tag{7}$$

i.e., the quantity $q = \exp(t/\tau)\,(p - \kappa/v)$ is an integral of motion for (1).

In fact (7) implies that the system (1) for $\chi = \kappa\tau$ has infinitely many conservation laws of the form

$$D_t\left(\rho(x, q, D_x(q), D_x^2(q), \ldots)\right) = 0, \tag{8}$$

where ρ is an arbitrary smooth function of its arguments. However, these conservation laws bear virtually no essential new information in comparison with (7).

When one passes to new dependent variables u, v, q, the system (1) with $\chi = \kappa\tau$ takes the form

$$u_t = \gamma - \exp(-t/\tau)q_x - \exp(-t/\tau)\kappa v_x/v^2, \quad v_t = u_x, \quad q_t = 0. \tag{9}$$

The last equation is decoupled and yields $q = q_0(x)$, where q_0 is an arbitrary smooth function of its argument. The second equation is solved by introducing the potential $w = w_2$, see (5). Thus (9) is reduced to a single second-order equation for w,

$$w_{tt} = \gamma - \frac{dq_0(x)}{dx}\exp(-t/\tau) - \exp(-t/\tau)\frac{\kappa w_{xx}}{w_x^2}, \tag{10}$$

from which u and v are recovered as

$$u = w_t, \quad v = w_x.$$

Proposition 2. *If $\kappa = 0$, then the first conservation law in (4) is equivalent to*

$$D_t\left(u - \gamma t - \tau p_x - \frac{\chi v_x}{v^2}\right) = 0, \tag{11}$$

i.e., the quantity $r = u - \gamma t - \tau p_x - \chi v_x/v^2$ is an integral of motion for (1).

Just as in the previous case (11) implies that (1) for $\kappa = 0$ has infinitely many conservation laws of the form

$$D_t\left(\rho(x, r, D_x(r), D_x^2(r), \ldots)\right) = 0, \tag{12}$$

where ρ is an arbitrary smooth function of its arguments. Obviously these conservation laws again bring no substantially new information in comparison with (11).

Note that for $\kappa = 0$ the extended system which consists of (1) and of the first pair of equations of (5) admits an additional conservation law,

$$D_t\left(w_1 - \gamma t x - \tau p + \frac{\chi}{v}\right) = 0, \tag{13}$$

i.e., $s = w_1 - \gamma t x - \tau p + \chi/v$ is a nonlocal integral of motion.

When one passes to new dependent variables r, v, p, the system (1) with $\kappa = 0$ takes the form

$$\begin{aligned} r_t &= 0,\\ v_t &= \frac{\chi v_{xx}}{v^2} + \tau p_{xx} - \frac{2\chi v_x^2}{v^3} - \tau r_x,\\ p_t &= -\frac{\chi^2 v_{xx}}{\tau v^4} - \frac{\chi p_{xx}}{v^2} + \frac{2\chi^2 v_x^2}{\tau v^5} + \frac{\chi r_x}{v^2} - \frac{p}{\tau}. \end{aligned} \tag{14}$$

Again the first equation is readily solved to yield $r = r_0(x)$, where r_0 is an arbitrary smooth function of x, and substituting this into the remaining two equations we gives a two-component second-order system,

$$\begin{aligned} v_t &= \frac{\chi v_{xx}}{v^2} + \tau p_{xx} - \frac{2\chi v_x^2}{v^3} - \tau\frac{dr_0(x)}{dx},\\ p_t &= -\frac{\chi^2 v_{xx}}{\tau v^4} - \frac{\chi p_{xx}}{v^2} + \frac{2\chi^2 v_x^2}{\tau v^5} + \frac{\chi}{v^2}\frac{dr_0(x)}{dx} - \frac{p}{\tau}. \end{aligned} \tag{15}$$

We have deliberately left aside the degenerate case for which $\kappa = \chi = 0$ because then (1) becomes linear and its general solution is readily found to be

$$\begin{aligned} u &= u_0(x) + \gamma t + \tau\exp(-t/\tau)p_0(x),\\ v &= v_0(x) + \frac{du_0(x)}{dx} + \tau\exp(-t/\tau)\frac{dp_0(x)}{dx},\\ p &= p_0(x)\exp(-t/\tau), \end{aligned}$$

where u_0, v_0, and p_0 are arbitrary smooth functions of x.

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[1] Danylenko V.A., Sorokina V.V. and Vladimirov V.A., On the governing equations in relaxing media models and self-similar quasiperiodic solutions, *J. Phys. A: Math. Gen.*, 1993, V.26, 7125–7135.

[2] Kunzinger M. and Popovych R.O., Potential conservation laws, *J. Math. Phys.*, 2008, V.49, 103506, 34 pp.

[3] Marvan M., Jets, A Software for Differential Calculus on Jet Spaces and Diffieties, online at `http://jets.math.slu.cz/Download/Documentation/Common/JetGuide/JetGuide.pdf`

[4] Mubarakzyanov G.M., Classification of real structures of Lie algebras of fifth order, *Izv. Vyssh. Uchebn. Zaved. Mat.*, 1963, no. 3 (34), 99–106 (in Russian).

[5] Olver P.J., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1993.

[6] Patera J., Sharp R.T., Winternitz P. and Zassenhaus H., Invariants of real low dimension Lie algebras, *J. Math. Phys.*, 1976, V.17, 986–994.

[7] Vladimirov V.A., Compacton-like solutions of the hydrodynamic system describing relaxing media, *Rep. Math. Phys.*, 2008, V.61, 381–400.

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Reduction of multidimensional wave equations to two-dimensional equations: investigation of possible reduced equations

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We study possible Lie and nonclassical reductions of multidimensional wave equations and the special classes of reduced equations possible – their symmetries and equivalence classes. Such an investigation allows one to find many new conditional and hidden symmetries of the original equations.

1 Why nonlinear wave equation

We study Lie and nonclassical reductions of multidimensional wave equations and special classes of reduced equations possible – their symmetries and equivalence classes as well as the types of reduced equations which represent interesting classes of two-dimensional equations – parabolic, hyperbolic and elliptic. This paper continues the discussion in [1].

Ansätze and methods used for reduction of the d'Alembert (n-dimensional wave) equation can be also used for arbitrary Poincaré-invariant equations. Below we show that this seemingly simple and partial problem involves many important aspects in the studies of the partial differential equation (PDE).

The topic we consider demonstrates relations of the symmetry methods (see e.g. [2, 3]) to other aspects of investigation of PDEs – compatibility of systems of equations, methods of finding general solutions (e.g. by means of hodograph transformations).

The methods we used were not fully algorithmic – it was necessary to decide when to switch methods and many hypotheses had to be tested.

We consider the multidimensional wave equation

$$\Box u = F(u), \qquad \Box \equiv \partial_{x_0}^2 - \partial_{x_1}^2 - \cdots - \partial_{x_n}^2, \quad u = u(x_0, x_1, \ldots, x_n)$$

It seems to have been thoroughly studied and almost trivial. We list only some papers in which solutions of this equation are studied specifically – [4, 12].

However, this equation appears to have many new facets and ideas to discover. We observe that investigation of hyperbolic equations, both with respect to their conditional symmetry and classification, is considerably more difficult than the same problem for equations in which at least for one variable the partial derivatives have only lower order than the order of equation.

2 Reduction of nonlinear wave equations – ansatz

We found conditions of reduction of the multidimensional wave equation

$$\Box u = F(u),$$

by means of the ansatz with two new independent variables.

$$u = \varphi(y, z), \tag{1}$$

where y, z are new variables. Henceforth n is the number of independent spatial variables in the initial d'Alembert equation.

Reduction conditions for such an ansatz are a system of the d'Alembert equations and three equations of Hamiltonian type

$$\begin{aligned} &y_\mu y_\mu = r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z), \\ &\Box y = R(y, z), \quad \Box z = S(y, z). \end{aligned} \tag{2}$$

We proved necessary conditions for compatibility of such a system of conditions for reduction (see [1]). However, the resulting conditions and reduced equations needed further research.

3 General background

There are two major methods of reduction of PDEs to ODEs or PDEs with a lower number of independent variables:"

Symmetry reduction to equations with a lower number of independent variables or to ordinary differential equations (for the algorithms see e.g. the books by Ovsyannikov [2] or Olver [3]).

"Direct method" (giving wider classes of solutions than the symmetry reduction) was proposed by P. Clarkson and M. Kruskal [13]). See more detailed investigation of the direct reduction and conditional symmetry in [4, 13–19]. This method for the majority of equations results in considerable difficulties as it requires investigation of compatibility and solution of cumbersome conditions of reduction of the initial equation.

These conditions for reduction are much more difficult for investigation and solution in the case of equations containing second and/or higher derivatives for all independent variables and for multidimensional equations – e.g. in the situation of nonlinear wave equations.

We would like to point out once more that the problem we consider has two specific difficulties. Firstly it is always more technically difficult to work with hyperbolic equations such as the nonlinear wave equation than with parabolic ones (such as evolution equations). Secondly normally the methods and algorithms for working with reductions and solutions are designed and applied for a limited number of variables – usually two or three. Here we work with an arbitrary number of variables although we limit the number of variables for specific examples.

4 Compatibility of the conditions for reduction: summary

A similar problem was considered previously for an ansatz with one independent variable

$$u = \varphi(y), \tag{3}$$

where y is a new independent variable.

Compatibility analysis of the d'Alembert–Hamilton system

$$\Box u = F(u), \quad u_\mu u_\mu = f(u) \tag{4}$$

in the three-dimensional space was done by Collins [20].

The sufficient conditions of reduction of the wave equation to an ordinary differential equation (ODE) and the general solution of the system (4) in the case of three spatial dimensions were found by Fushchych, Zhdanov and Revenko [21]. For a discussion of previous results in this area see [22]. It is evident that the d'Alembert–Hamilton system (4) may be reduced by local transformations to the form

$$\Box u = F(u), \quad u_\mu u_\mu = \lambda, \quad \lambda = 0, \pm 1. \tag{5}$$

Statement [23]. *For the system* (5) ($u = u(x_0, x_1, x_2, x_3)$) *to be compatible it is necessary and sufficient that the function F have the following form:*

$$F = \frac{\lambda}{N(u + C)}, \quad N = 0, 1, 2, 3.$$

Ansätze of the type (1) for some particular cases were studied in [24–27].

5 Transformations of compatibility conditions

Substitution of the ansatz $u = \varphi(y, z)$ into the equation $\Box u = F(u)$ leads to the following equation (see [1]):

$$\varphi_{yy} y_\mu y_\mu + 2\varphi_{yz} z_\mu y_\mu + \varphi_{zz} z_\mu z_\mu + \varphi_y \Box y + \varphi_z \Box z = F(\varphi) \tag{6}$$
$$\left(y_\mu = \frac{\partial y}{\partial x_\mu}, \quad \varphi_y = \frac{\partial \varphi}{\partial y} \right),$$

whence we get a system of equations:

$$\begin{aligned} &y_\mu y_\mu = r(y, z), \quad y_\mu z_\mu = q(y, z), \quad z_\mu z_\mu = s(y, z), \\ &\Box y = R(y, z), \quad \Box z = S(y, z). \end{aligned} \tag{7}$$

System (7) is a condition of reduction for the multidimensional wave equation (1) to the two-dimensional equation (6) by means of ansatz $u = \varphi(y, z)$.

The system of equations (7), depending upon the sign of the expression $rs-q^2$, may be reduced by local transformations to one of the following types:
1) elliptic case: $rs-q^2>0$, $v=v(y,z)$ is a complex-valued function,

$$\begin{aligned}&\Box v = V(v,v^*), \quad \Box v^* = V^*(v,v^*),\\ &v^*_\mu v_\mu = h(v,v^*), \quad v_\mu v_\mu = 0, \quad v^*_\mu v^*_\mu = 0\end{aligned} \tag{8}$$

(the reduced equation is of elliptic type);
2) hyperbolic case: $rs-q^2<0$, $v=v(y,z)$, $w=w(y,z)$ are real functions,

$$\begin{aligned}&\Box v = V(v,w), \quad \Box w = W(v,w),\\ &v_\mu w_\mu = h(v,w), \quad v_\mu v_\mu = 0, \quad w_\mu w_\mu = 0\end{aligned} \tag{9}$$

(the reduced equation is of hyperbolic type);
3) parabolic case: $rs-q^2=0$, $r^2+s^2+q^2\neq 0$, $v(y,z)$, $w(y,z)$ are real functions,

$$\begin{aligned}&\Box v = V(v,w), \quad \Box w = W(v,w),\\ &v_\mu w_\mu = 0, \quad v_\mu v_\mu = \lambda \ (\lambda=\pm 1), \quad w_\mu w_\mu = 0\end{aligned} \tag{10}$$

(if $W\neq 0$, then the reduced equation is of parabolic type);
4) first-order equations: $(r=s=q=0)$, $y\to v$, $z\to w$

$$\begin{aligned}&v_\mu v_\mu = w_\mu w_\mu = v_\mu w_\mu = 0,\\ &\Box v = V(v,w), \quad \Box w = W(v,w).\end{aligned} \tag{11}$$

Elliptic case.

Theorem 1. *System* (8) *is compatible if and only if*

$$V = \frac{h(v,v^*)\partial_{v^*}\Phi}{\Phi}, \quad \partial_{v^*} \equiv \frac{\partial}{\partial v^*},$$

where Φ *is an arbitrary function for which the following condition is satisfied*

$$(h\partial_{v^*})^{n+1}\Phi = 0.$$

The function h may be represented in the form $h=1/R_{vv^*}$, where R is an arbitrary sufficiently smooth function and R_v, R_{v^*} are partial derivatives with respect to the respective variables.

Then the function Φ may be represented in the form $\Phi = \sum\limits_{k=0}^{n+1} f_k(v)R_v^k$, where $f_k(v)$ are arbitrary functions and

$$V = \frac{\sum\limits_{k=1}^{n+1} k f_k(v) R_v^k}{\sum\limits_{k=0}^{n+1} f_k(v) R_v^k}.$$

The respective reduced equation has the form

$$h(v,v^*)\left(2\phi_{vv^*}+\phi_v\frac{\partial_{v^*}\Phi}{\Phi}+\phi_{v^*}\frac{\partial_v\Phi^*}{\Phi^*}\right)=F(\phi). \tag{12}$$

The equation (12) may also be rewritten as an equation with two real independent variables ($v=\omega+\theta$, $v^*=\omega-\theta$):

$$\widetilde{h}(\omega,\theta)(\phi_{\omega\omega}+\phi_{\theta\theta})+\Omega(\omega,\theta)\phi_\omega+\Theta(\omega,\theta)\phi_\theta=F(\phi).$$

Hyperbolic case.

Theorem 2. *System* (9) *is compatible if and only if*

$$V=\frac{h(v,w)\partial_w\Phi}{\Phi},\quad W=\frac{h(v,w)\partial_v\Psi}{\Psi},$$

where the functions Φ *and* Ψ *are arbitrary functions for which the following conditions are satisfied*

$$(h\partial_v)^{n+1}\Psi=0,\quad (h\partial_w)^{n+1}\Phi=0.$$

The function h may be presented in the form $h=1/R_{vw}$, where R is an arbitrary sufficiently smooth function and R_v, R_w are partial derivatives with respect to the respective variables. Then the functions Φ and Ψ may be represented in the form

$$\Phi=\sum_{k=0}^{n+1}f_k(v)R_v^k,\quad \Psi=\sum_{k=0}^{n+1}g_k(w)R_w^k,$$

where $f_k(v)$ and $g_k(w)$ are arbitrary functions,

$$V=\frac{\sum\limits_{k=1}^{n+1}kf_k(v)R_v^k}{\sum\limits_{k=0}^{n+1}f_k(v)R_v^k},\quad W=\frac{\sum\limits_{k=1}^{n+1}kg_k(w)R_w^k}{\sum\limits_{k=0}^{n+1}g_k(w)R_w^k}.$$

The respective reduced equation has the form

$$h(v,w)\left(2\phi_{vw}+\phi_v\frac{\partial_w\Phi}{\Phi}+\phi_w\frac{\partial_v\Psi}{\Psi}\right)=F(\phi). \tag{13}$$

The equation (13) may also be rewritten as a standard wave equation ($v=\omega+\theta$, $w=\omega-\theta$):

$$\widetilde{h}(\omega,\theta)(\phi_{\omega\omega}-\phi_{\theta\theta})+\Omega(\omega,\theta)\phi_\omega+\Theta(\omega,\theta)\phi_\theta=F(\phi).$$

Parabolic case.

Theorem 3. *System* (10) *is compatible if and only if*

$$V = \frac{\lambda \partial_v \Phi}{\Phi}, \quad \partial_v^{n+1}\Phi = 0, \quad W \equiv 0.$$

Equation $\Box u = F(u)$ cannot be reduced to a parabolic equation by means of the ansatz $u = \varphi(y, z)$ – in this case one of the variables enters the reduced ordinary differential equation of the first order as a parameter.

System (11) is compatible only in the case $V = W \equiv 0$, that is, the reduced equation may be only an algebraic equation $F(u)$=0. Thus we cannot reduce equation $\Box u = F(u)$ by means of the ansatz $u = \varphi(y, z)$ to a first-order equation.

Proof of the theorems above is done by means of the well-known Hamilton–Cayley theorem in accordance to which a matrix is a root of its characteristic polynomial.

6 Reduction and conditional symmetry

Solutions obtained by the direct reduction are related to symmetry properties of the equation – Q-conditional symmetry of this equation (symmetries of such type are also called nonclassical or non-Lie symmetries. It is also possible to see from previous papers that symmetry of the two-dimensional reduced equations is often wider than symmetry of the initial equation, that is, the reduction to two-dimensional equations allows one to find new non-Lie solutions and hidden symmetries of the initial equation (see e.g. papers by Abraham-Schrauner and Leach [28, 29]). The Hamiltonian equation may also be considered, irrespective of the reduction problem, as an additional condition for the d'Alembert equation that allows extending the symmetry of this equation.

Consider the wave equation in two spatial dimensions. Reduction of $\Box u = F(u)$ by our ansatz $u = \varphi(v, w)$ means Q-conditional invariance this equation under the operator

$$Q = \partial_{x_0} + \tau_1(x_0, x_1, x_2)\partial_{x_1} + \tau_2(x_0, x_1, x_2)\partial_{x_2}.$$

This equivalence of reduction and Q-conditional symmetry was proved by Zhdanov, Tsyfra and Popovych [18]. New variables, v and w, are invariants of the operator Q:

$$Qv = Qw = 0.$$

7 Study of the reduced equations

Equivalence of quasilinear wave equations is well studied, but we consider a particular class of such equations.

We consider the reduced equation of the form

$$h(v,w)\left(2\phi_{vw}+\phi_v\frac{\partial_w\Phi}{\Phi}+\phi_w\frac{\partial_v\Psi}{\Psi}\right)=F(\phi),$$

where Φ and Ψ are arbitrary functions for which the following conditions are satisfied

$$(h\partial_v)^{n+1}\Psi=0,\quad (h\partial_w)^{n+1}\Phi=0.$$

Equivalence transformations of the reduced equations are only of the type

$$h(v,w)\to k(v)l(w)h(v,w),\quad v\leftrightarrow w;\quad \phi\to a\phi+b.$$

There are special additional equivalence groups only for special forms of the function F. Special class of the reduced equations – $h(v,w)=k(v)l(w)$; in this case the equations can be reduced to the case $h(v,w)=\text{const}$. All symmetry reductions have $h(v,w)=\text{const}$ and linear Φ and Ψ.

We have a quite narrow equivalence group of the reduced equation as we actually took a single representative of an equivalence class of hyperbolic reduced equations.

Description of all possible reductions involves classification of the reductions found and nomination of certain inequivalent representatives. Any classification problem is a description of equivalence classes under certain equivalence relations.

Selection of an equivalence group for classification may be in principle arbitrary, but as a rule one of the following is selected: either the symmetry group of the conditions describing the initial limited class or the group of automorphisms of some general class.

There is a generally accepted method for classification of symmetry reductions – by subalgebras inequivalent up to conjugacy. This method does not work for general reductions and we have to choose another method of classification.

Another important note is that, if we do classification in several steps, we have to consider commutativity and associativity of classification conditions (e.g. under some equivalence group) adopted at each step.

8 Example: Solutions for the two-dimensional case

We will look for parametric solutions for the system

$$\Box v=V(v,w),\quad \Box w=W(v,w),$$
$$v_\mu w_\mu=h(v,w),\quad v_\mu v_\mu=0,\quad w_\mu w_\mu=0,\quad \mu=0,1,2.$$

Firstly we construct parametric or explicit solutions for the equations $w_\mu w_\mu=0$, $v_\mu v_\mu=0$ and then use them to find solutions of other equations.

Rank 0. General solution of the equations $v_\mu v_\mu = 0$, $w_\mu w_\mu = 0$

$$v = A_\mu x_\mu + B, \quad w = C_\mu x_\mu + D,$$
$$A_\mu A_\mu = 0, \quad C_\mu C_\mu = 0.$$

p, q are parametric functions on x, A_μ ($\mu = 1,2$), B, C_μ ($\mu = 1,2$), D are arbitrary constants up to conditions. In this case h is constant, $\Box v = \Box w = 0$ and we have solutions that can be obtained by symmetry reduction.

Rank 1. General solution of the equations $v_\mu v_\mu = 0$, $w_\mu w_\mu = 0$.

$$v = A_\mu(p)x_\mu + B(p), \quad w = C_\mu(q)x_\mu + D(q),$$
$$A^\mu_p x_\mu + B_p = 0, \quad C^\mu_q x_\mu + D_q = 0, \quad A_\mu A_\mu = 0, \quad C_\mu C_\mu = 0.$$

p, q are parametric functions on x, A_μ ($\mu = 1,2$), B, C_μ ($\mu = 1,2$) and D are arbitrary functions up to conditions.

Rank 2. General solution of the equations $v_\mu v_\mu = 0$, $w_\mu w_\mu = 0$.

$$v = A_\mu(p_1,p_2)x_\mu + B(p_1,p_2), \quad w = C_\mu(q_1,q_2)x_\mu + D(q_1,q_2),$$
$$A^\mu_{p_k} x_\mu + B_{p_k} = 0, \quad C^\mu_{q_k} x_\mu + D_{q_k} = 0, \quad A_\mu A_\mu = 0, \quad C_\mu C_\mu = 0.$$

p, q are parametric functions on x, A_μ ($\mu = 1,2$), B, C_μ ($\mu = 1,2$) and D are arbitrary functions up to conditions.

It is easy to prove that that for $v_\mu w_\mu = h(v,w)$ solutions of $v_\mu v_\mu = 0$ and $w_\mu w_\mu = 0$ should have the same rank. Further we can find partial parametric solutions taking the same parameter functions p for v and w. This way we have new non-Lie solutions with hidden infinite symmetry. (For a definition of hidden symmetry see [28].)

It is well-known [30] that the general solution of the system (4) with $F = f = 0$, $n = 1,2$, can be written as

$$u = A_\mu(p_1,p_2)x_\mu + B(p_1,p_2),$$
$$A^\mu_{p_k} x_\mu + B_{p_k} = 0, \quad A_\mu A_\mu = 0, \quad A^\mu_{p_k} A^\mu_{p_m} = 0.$$

Similarly we can construct a parametric solution for (9) with $V = W = 0, h =$ const.

$$v = A_\mu(p_1,p_2)x_\mu + B(p_1,p_2),$$
$$A^\mu_{p_k} x_\mu + B_{p_k} = 0, \quad A_\mu A_\mu = 0, \quad A^\mu_{p_k} A^\mu_{p_m} = 0,$$
$$w = C_\mu(p_1,p_2)x_\mu + D(p_1,p_2),$$
$$C^\mu_{p_k} x_\mu + D_{p_k} = 0, \quad C_\mu C_\mu = 0, \quad C^\mu_{p_k} C^\mu_{p_m} = 0, \quad A_\mu C_\mu = \mathrm{const}.$$

The operator of Q-conditional symmetry that gives such an ansatz has the form

$$Q = \partial_0 + \tau_1\partial_1 + \tau_2\partial_2,$$
$$\tau_1 = \frac{C_0A_2 - A_0C_2}{A_1C_2 - A_2C_1}, \quad \tau_2 = \frac{C_0A_1 - A_0C_1}{A_1C_2 - A_2C_1}.$$

9 Conclusion

The topic we discuss is closely related to majority of main ideas in the symmetry analysis of a PDE – direct reduction of the PDE; conditional symmetry; Q-conditional symmetry; finding solutions directly using nonlocal transformations; group classification of equations and systems of equations.

Our general problem – study of reductions of the nonlinear wave equation (and of other equations in general) – requires several classifications up to equivalence on the way.

At each step we have to define correctly the criteria of equivalence and check commutativity and associativity of these equivalence conditions or otherwise take into account a lack of such properties.

10 Further research

1. Study of Lie and conditional symmetry of the system of the conditions for reduction.
2. Investigation of Lie and conditional symmetry of the reduced equations. Finding closed-form solutions of the reduced equations.
3. Finding of places of previously found solutions on the general equivalence map.
4. Relation of the equivalence group of the class of the reduced equations with symmetry of the initial equation.
5. Finding and investigation of compatibility conditions and classes of the reduced equations for other types of equations, in particular, for Poincaré–invariant scalar equations.

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[1] Yehorchenko I.A., Reduction of non-linear d'Alembert equations to two-dimensional equations, *Proc. of 4th International Workshop "Group Analysis of Differential Equations and Integrable Systems" (2008, October 26–30, Protaras, Cyprus)*, 2009, 243–253.

[2] Ovsiannikov L.V., *Group Analysis of Differential Equations*, Academic Press, New York, 1982.

[3] Olver P., *Applications of Lie Groups to Differential Equations*, Springer-Verlag, New York, 1986.

[4] Fushchych W.I., Shtelen W.M. and Serov N.I., *Symmetry Analysis and Exact Solutions of Nonlinear Equations of Mathematical Physics*, Kluwer Publishers, Dordrecht, 1993.

[5] Bluman G.W. and Kumei S., *Symmetries and Differential Equations*, Springer Verlag, New York, 1989.

[6] Grundland A., Harnad J. and Winternitz P., Symmetry reduction for nonlinear relativistically invariant equations, *J. Math. Phys.*, 1984, V.25, 791–807.

[7] Tajiri M., Some remarks on similarity and soliton solutions of nonlinear Klein–Gordon equations, *J. Phys. Soc. Japan*, 1984, V.53, 3759–3764.

[8] Patera J., Sharp R.T., Winternitz P. and Zassenhaus H., Subgroups of the Poincaré group and their invariants, *J. Math. Phys.*, 1976, V.17, 977–985.

[9] Fushchych W.I. and Serov N.I., The symmetry and some exact solutions of the nonlinear many-dimensional Liouville, d'Alembert and eikonal equation, *J. Phys. A: Math. Gen.*, 1983, V.16, 3645–3656.

[10] Fushchych W.I. and Barannyk A.F., On exact solutions of the nonlinear d'Alembert equation in Minkowski space $R(1,n)$, *Reports of Acad. Sci. of Ukr. SSR, Ser. A*, 1990, no. 6, 31–34; Fushchych W.I., Barannyk L.F. and Barannyk A.F., *Subgroup Analysis of Galilei and Poincaré Groups and Reduction of Non-Linear Equations*, Naukova Dumka, Kyiv, 1991 (in Russian); Fushchych W.I. and Barannyk A.F., Maximal subalgebras of rank $n-1$ of the algebra $\mathrm{AP}(1,n)$, and reduction of nonlinear wave equations. I, *Ukrainian Math. J.*, 1990, V.42, 1394–1402; II, *Ukrainian Math. J.*, 1990, V.42, 1528–1534.

[11] Fushchych W.I., Ansatz–95, *J. Nonlinear Math. Phys.*, 1995, V.2, 216–235.

[12] Fushchych W.I. and Barannyk A.F., On a new method of construction of solutions for non-linear wave equations, *Reports of the Nat. Acad. Sci. of Ukraine*, 1996, no. 10, 48–51.

[13] Clarkson P.A. and Kruskal M., New similarity reductions of the Boussinesq equations, *J. Math. Phys.*, 1989, V.30, 2201–2213.

[14] Clarkson P.A. and Mansfield E.L., Algorithms for the nonclassical method of symmetry reductions, *SIAM J. Appl. Math.*, 1994, V.54, 1693–1719, arXiv:solv-int/9401002.

[15] Olver P., Direct reduction and differential constraints, *Proc. Roy. Soc. London Ser. A*, 1994, V.444, 509–523.

[16] Olver P.J. and Rosenau P., The construction of special solutions to partial differential equations, *Phys. Lett. A*, 1986, V.114, 107–112.

[17] Levi D. and Winternitz P., Non-classical symmetry reduction: example of the Boussinesq equation, *J. Phys. A: Math. Gen.*, 1989, V.22, 2915–2924.

[18] Zhdanov R.Z., Tsyfra I.M. and Popovych R.O., A precise definition of reduction of partial differential equations, *J. Math. Anal. Appl.*, 1999, V.238, 101–123.

[19] Cicogna G., A discussion on the different notions of symmetry of differential equations, *Proceedings of Fifth International Conference "Symmetry in Nonlinear Mathematical Physics" (June 23–29, 2003, Kyiv)*, Institute of Mathematics, Kyiv, 2004, V.50, 77–84.

[20] Collins S.B., Complex potential equations. I, *Math. Proc. Camb. Phil. Soc.*, 1976, V.80, 165–187; Collins S.B., All solutions to a nonlinear system of complex potential equations, *J. Math. Phys.*, 1980, V.21, 240–248.

[21] Fushchych W.I., Zhdanov R.Z. and Revenko I.V., General solutions of nonlinear wave equation and eikonal equation, *Ukrainian Math. J.*, 1991, V.43, 1364–1379.

[22] Cieciura G. and Grundland A., A certain class of solutions of the nonlinear wave equation, *J. Math. Phys.*, 1984, V.25, 3460–3469.

[23] Fushchych W.I., Zhdanov R.Z. and Yegorchenko I.A., On reduction of the nonlinear many-dimensional wave equations and compatibility of the d'Alembert–Hamilton system, *J. Math. Anal. Appl.*, 1991, V.160, 352–360.

[24] Barannyk A. and Yuryk I., On some exact solutions of nonlinear wave equations, *Proceedings of the Second International Conference "Symmetry in Nonlinear Mathematical Physics" (7–13 July 1997, Kyiv)*, 1997, Institute of Mathematics, Kyiv, V.1, 1997, 98–107.

[25] Zhdanov R. and Panchak O., New conditional symmetries and exact solutions of the nonlinear wave equation, *J. Phys. A: Math. Gen.*, 1998, V.31, 8727–8734.

[26] Euler M. and Euler N., Symmetries for a class of explicitly space- and time-dependent (1+1)-dimensional wave equations, *J. Nonlinear Math. Phys.*, 1997, V.1, 70–78.

[27] Anco S.C. and Liu S., Exact solutions of semilinear radial wave equations in n dimensions, *J. Math. Anal. Appl.*, 2004, V.297, 317–342, arXiv:math-ph/0309049.

[28] Abraham-Shrauner B., Hidden symmetries and nonlocal group generators for ordinary differential equations, *IMA J. Appl. Math.*, 1996, V.56, 235–252; Abraham-Shrauner B., Hidden symmetries, first integrals and reduction of order of nonlinear ordinary differential equations, *J. Nonlinear Math. Phys.*, 2002, V.9, suppl. 2, 1–9.

[29] Abraham-Shrauner B. and Leach P.G.L., Hidden symmetries of nonlinear ordinary differential equations, in *Exploiting Symmetry in Applied and Numerical Analysis, Lectures in Applied Mathematics*, American Mathematical Society, 1993, V.29, 1–10.

[30] Sobolev S.L., Functional-invariant solutions to the wave equation, *Tr. Steklov Phys. Mat. Inst.*, 1934, V.5, 259–264 (in Russian).

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Notes

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