



Correlators in the supereigenvalue model in the Ramond sector

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ABSTRACT

We investigate the supereigenvalue model in the Ramond sector. We prove that its partition function can be obtained by acting on elementary functions with exponents of the given operators. The Virasoro constraints for this supereigenvalue model are presented. The remarkable property of these bosonic constraint operators is that they obey the Witt algebra and null 3-algebra. The compact expression of correlators can be derived from these Virasoro constraints.

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1. Introduction

Matrix models play important roles in physics and mathematics. Generally speaking they are quantum field theories where the field is an $N \times N$ real or complex matrix. Supereigenvalue models can be regarded as supersymmetric generalizations of matrix models. They have attracted considerable attention [1–11]. The supereigenvalue model in the Ramond sector is given by [9]

$$Z = \int d^N z d^N \theta \Delta_R(z, \theta)^\beta e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N V_R(z_a, \theta_a)}, \quad (1)$$

where $d^N z d^N \theta = \prod_{a=1}^N dz_a d\theta_a$, N is even, z_a are positive real variables, θ_a are Grassmann variables, $\Delta_R(z, \theta)$ is the Vandermonde-like determinant,

$$\Delta_R(z, \theta) = \prod_{1 \leq a < b \leq N} (z_a - z_b - \frac{1}{2}(z_a + z_b) \frac{\theta_a \theta_b}{\sqrt{z_a z_b}}), \quad (2)$$

and

$$V_R(z, \theta) = V_B(z) + V_F(z) \frac{\theta}{\sqrt{z}}, \quad V_B(z) = \sum_{k=0}^{\infty} t_k z^k, \quad V_F(z) = \sum_{k=0}^{\infty} \xi_k z^k, \quad (3)$$

ξ_k are Grassmann coupling constants, $V_B(z)$ and $V_F(z)$ are the bosonic and fermionic potentials, respectively.

The various constraints for matrix models have been constructed, such as Virasoro constraints [12–15], $W_{1+\infty}$ constraints [16,17] and Ding-Iohara-Miki constraints [18,19]. They are useful in analyzing the structures of matrix models. For the partition function (1), it is known that there are the super Virasoro constraints [9]

$$L_n Z = \frac{1}{16} \delta_{n,0} Z, \quad G_n Z = 0, \quad n \in \mathbb{N}, \quad (4)$$

where

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$$L_n = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_{n+k}} + \sum_{k=0}^{\infty} (k + \frac{n}{2}) \xi_k \frac{\partial}{\partial \xi_{k+n}} + \frac{\hbar^2}{2} \sum_{k=0}^n \frac{\partial}{\partial t_{n-k}} \frac{\partial}{\partial t_k} + \frac{\hbar^2}{4} n \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial \xi_n} + \frac{\hbar^2}{2} \sum_{k=1}^{n-1} k \frac{\partial}{\partial \xi_{n-k}} \frac{\partial}{\partial \xi_k} - \frac{\hbar}{2\sqrt{\beta}} (1-\beta)(n+1) \frac{\partial}{\partial t_n} + \frac{1}{16} \delta_{n,0}, \quad (5)$$

$$G_n = \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial \xi_{n+k}} + \sum_{k=0}^{\infty} \xi_k \frac{\partial}{\partial t_{k+n}} + \frac{\hbar^2}{2} \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial t_n} + \hbar^2 \sum_{k=1}^n \frac{\partial}{\partial \xi_k} \frac{\partial}{\partial t_{n-k}} - \frac{\hbar}{\sqrt{\beta}} (1-\beta)(n + \frac{1}{2}) \frac{\partial}{\partial \xi_n}. \quad (6)$$

The operators (5) and (6) obey the super Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n}, \quad (7a)$$

$$[L_m, G_n] = \frac{m-2n}{2} G_{m+n}, \quad (7b)$$

$$\{G_m, G_n\} = 2L_{m+n} - \frac{1}{8} \delta_{m+n,0}. \quad (7c)$$

Recently a formal supereigenvalue model in the Ramond sector is investigated [11]

$$\check{Z} = \int \prod_{a=1}^{2N} dz_a d\theta_a \Delta(z, \theta) e^{-\frac{N}{t} \sum_{a=1}^{2N} (z_a^2 + V_B(z_a^2) + V_F(z_a^2) \theta_a)}, \quad (8)$$

where

$$\Delta(z, \theta) = \prod_{1 \leq a < b \leq 2N} (z_a^2 - z_b^2 - \frac{\theta_a \theta_b}{2} (z_a^2 + z_b^2)), \quad (9)$$

and the bosonic variables z_a are integrated from $-\infty$ to $+\infty$. To calculate the correlation functions of the model (8), the recursive formalism has been derived. It was found that the correlation functions obtained from the recursion formalism have no poles at the irregular ramification point due to a supersymmetric correction.

The partition functions of various matrix models can be obtained by acting on elementary functions with exponents of the given operators, such as Gaussian Hermitian and complex matrix models and the given W operators called W -representations [20–23]. For the case of supersymmetric generalizations, to our best knowledge, it has not been reported so far in the existing literature. In this letter, we investigate the supereigenvalue model in the Ramond sector and derive its W -representations. We also give the correlators in this matrix model.

2. Generation of the supereigenvalue model in the Ramond sector by $\hat{W}$ -operator

Let us consider the supereigenvalue model in the Ramond sector

$$\bar{Z} = \frac{1}{\Lambda} \int d^N z d^N \theta \Delta_R(z, \theta)^\beta e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N (V_R(z_a, \theta_a) + z_a)}, \quad (10)$$

which can be obtained by taking the shift $t_1 \rightarrow t_1 + 1$ in the bosonic potential $V_B(z)$ of (1), the normalization factor Λ is given by

$$\Lambda = \int d^N z d^N \theta \Delta_R(z, \theta)^\beta e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N z_a}. \quad (11)$$

We note that the partition function (10) is invariant under

$$z_a \rightarrow z_a + \epsilon \sum_{n=0}^{\infty} (n+1) t_{n+1} z_a^{n+1}, \quad \theta_a \rightarrow \theta_a + \epsilon \sum_{n=0}^{\infty} \frac{n(n+1)}{2} t_{n+1} z_a^n \theta_a, \quad (12)$$

with an infinitesimal bosonic parameter ϵ . It leads to the bosonic loop equation

$$\begin{aligned} \sum_{n=0}^{\infty} (n+1) t_{n+1} < -\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N z_a^{n+1} - \frac{\sqrt{\beta}}{\hbar} \sum_{k=0}^{\infty} (k + \frac{n}{2}) \xi_k \sum_{a=1}^N z_a^{k+n} \frac{\theta_a}{\sqrt{z_a}} - \frac{\sqrt{\beta}}{\hbar} \sum_{k=1}^{\infty} k t_k \sum_{a=1}^N z_a^{n+k} + \frac{\beta}{2} \sum_{k=0}^n \sum_{a,b=1}^N z_a^{n-k} z_b^k \\ + \frac{\beta}{4} n \sum_{a,b=1}^N z_b^n \frac{\theta_a \theta_b}{\sqrt{z_a z_b}} + \frac{\beta}{2} \sum_{k=1}^{n-1} \sum_{a,b=1}^N k z_a^{n-k} z_b^k \frac{\theta_a \theta_b}{\sqrt{z_a z_b}} + \frac{1-\beta}{2} (n+1) \sum_{a=1}^N z_a^n >= 0, \end{aligned} \quad (13)$$

where the expectation value is taken with respect to the partition function (10). The loop equation (13) can be derived by applying the following differential operators to the partition function (10)

$$(\hat{W}_1 + \hat{D}_1) \bar{Z} = 0, \quad (14)$$

where

$$\begin{aligned}
\hat{D}_1 &= \sum_{k=1}^{\infty} k t_k \frac{\partial}{\partial t_k}, \\
\hat{W}_1 &= \sum_{n,k=1}^{\infty} n k t_n t_k \frac{\partial}{\partial t_{n+k-1}} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} n(k + \frac{n-1}{2}) t_n \xi_k \frac{\partial}{\partial \xi_{n+k-1}} + \frac{\hbar^2}{2} \sum_{n=1}^{\infty} \sum_{k=0}^{n-1} n t_n \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k-1}} \\
&\quad + \frac{\hbar^2}{4} \sum_{n=1}^{\infty} n(n-1) t_n \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial \xi_{n-1}} + \frac{\hbar^2}{2} \sum_{n=3}^{\infty} \sum_{k=1}^{n-2} n k t_n \frac{\partial}{\partial \xi_{n-k-1}} \frac{\partial}{\partial \xi_k} - \frac{\hbar}{2\sqrt{\beta}} (1-\beta) \sum_{n=1}^{\infty} n^2 t_n \frac{\partial}{\partial t_{n-1}}.
\end{aligned} \tag{15}$$

The partition function (10) is also invariant under

$$z_a \rightarrow z_a + \epsilon \sum_{n=0}^{\infty} (n+1) \xi_{n+1} z_a^n \sqrt{z_a} \theta_a, \quad \theta_a \rightarrow \theta_a - \epsilon \sum_{n=0}^{\infty} (n+1) \xi_{n+1} z_a^n \sqrt{z_a}, \tag{16}$$

which leads to another bosonic loop equation

$$\begin{aligned}
\sum_{n=0}^{\infty} (n+1) \xi_{n+1} &< -\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N z_a^{n+1} \frac{\theta_a}{\sqrt{z_a}} - \frac{\sqrt{\beta}}{\hbar} \sum_{k=0}^{\infty} k t_k \sum_{a=1}^N z_a^{k+n} \frac{\theta_a}{\sqrt{z_a}} - \frac{\sqrt{\beta}}{\hbar} \sum_{k=0}^{\infty} \xi_k \sum_{a=1}^N z_a^{k+n} \\
&+ \frac{\beta}{2} \sum_{a,b=1}^N \frac{\theta_a}{\sqrt{z_a}} z_b^n + \beta \sum_{k=1}^n \sum_{a,b=1}^N z_a^k \frac{\theta_a}{\sqrt{z_a}} z_b^{n-k} + (1-\beta)(n + \frac{1}{2}) \sum_{a=1}^N z_a^n \frac{\theta_a}{\sqrt{z_a}} >= 0.
\end{aligned} \tag{17}$$

Similarly, (17) can be also obtained by applying the following differential operators to the partition function

$$(\hat{W}_2 + \hat{D}_2) \bar{Z} = 0, \tag{18}$$

where

$$\begin{aligned}
\hat{D}_2 &= \sum_{k=1}^{\infty} k \xi_k \frac{\partial}{\partial \xi_k}, \\
\hat{W}_2 &= \sum_{n,k=1}^{\infty} n k t_k \xi_n \frac{\partial}{\partial \xi_{n+k-1}} + \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} n \xi_n \xi_k \frac{\partial}{\partial t_{n+k-1}} + \frac{\hbar^2}{2} \sum_{n=1}^{\infty} n \xi_n \frac{\partial}{\partial \xi_0} \frac{\partial}{\partial t_{n-1}} \\
&\quad + \hbar^2 \sum_{n=2}^{\infty} \sum_{k=1}^{n-1} n \xi_n \frac{\partial}{\partial \xi_k} \frac{\partial}{\partial t_{n-k-1}} - \frac{\hbar}{\sqrt{\beta}} (1-\beta) \sum_{n=1}^{\infty} n(n - \frac{1}{2}) \xi_n \frac{\partial}{\partial \xi_{n-1}}.
\end{aligned} \tag{19}$$

Combining (14) and (18), we have

$$(\hat{W} + \hat{D}) \bar{Z} = 0, \tag{20}$$

where $\hat{D} = \hat{D}_1 + \hat{D}_2$, $\hat{W} = \hat{W}_1 + \hat{W}_2$ and their commutation relation is

$$[\hat{D}, \hat{W}] = \hat{W}. \tag{21}$$

Since the partition function (10) only depends on even numbers of the fermionic variables, it can be formally expanded as

$$\begin{aligned}
\bar{Z} &= \sum_{s=0}^{\infty} \bar{Z}^{(s)} = e^{-\frac{\sqrt{\beta}}{\hbar} N t_0} \left[1 - \frac{\sqrt{\beta}}{\hbar} C_{k_1} t_{k_1} + \frac{1}{2!} \left(\frac{\sqrt{\beta}}{\hbar} \right)^2 C_{k_1, k_2} t_{k_1} t_{k_2} - \frac{1}{2!} \left(\frac{\sqrt{\beta}}{\hbar} \right)^2 C^{s_1, s_2} \xi_{s_1} \xi_{s_2} \right. \\
&\quad \left. - \frac{1}{3!} \left(\frac{\sqrt{\beta}}{\hbar} \right)^3 C_{k_1, k_2, k_3} t_{k_1} t_{k_2} t_{k_3} + \frac{1}{2!} \left(\frac{\sqrt{\beta}}{\hbar} \right)^3 C^{s_1, s_2} t_{k_1} \xi_{s_1} \xi_{s_2} + \dots \right],
\end{aligned} \tag{22}$$

where

$$\bar{Z}^{(s)} = e^{-\frac{\sqrt{\beta}}{\hbar} N t_0} \left[\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{\frac{m(m+1)}{2}} \left(-\frac{\sqrt{\beta}}{\hbar} \right)^{n+m}}{n! m!} \sum_{\substack{k_1 + \dots + k_n + s_1 + \dots + s_m = s \\ k_1, \dots, k_n \geq 1 \\ s_1, \dots, s_m \geq 0}} C_{k_1, \dots, k_n}^{s_1, \dots, s_m} t_{k_1} \dots t_{k_n} \xi_{s_1} \dots \xi_{s_m} \right], \tag{23}$$

m is even and the coefficients $C_{k_1, \dots, k_n}^{s_1, \dots, s_m}$ are the correlators defined by

$$C_{k_1, \dots, k_n}^{s_1, \dots, s_m} = \frac{1}{\Lambda} \int d^N z d^N \theta \Delta_R(z, \theta)^\beta e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N z_a} \sum_{\substack{a_1, \dots, a_n=1 \\ b_1, \dots, b_m=1}}^N z_{a_1}^{k_1} \dots z_{a_n}^{k_n} z_{b_1}^{s_1} \frac{\theta_{b_1}}{\sqrt{z_{b_1}}} \dots z_{b_m}^{s_m} \frac{\theta_{b_m}}{\sqrt{z_{b_m}}}. \tag{24}$$

For the cases of $m = 0$ and $n = 0$ in (24), respectively, we denote

$$C_{k_1, \dots, k_n} = \frac{1}{\Lambda} \int d^N z d^N \theta \Delta_R(z, \theta)^\beta e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N z_a} \sum_{a_1, \dots, a_n=1}^N z_{a_1}^{k_1} \dots z_{a_n}^{k_n}, \quad (25)$$

and

$$C^{s_1, \dots, s_m} = \frac{1}{\Lambda} \int d^N z d^N \theta \Delta_R(z, \theta)^\beta e^{-\frac{\sqrt{\beta}}{\hbar} \sum_{a=1}^N z_a} \sum_{b_1, \dots, b_m=1}^N z_{b_1}^{s_1} \frac{\theta_{b_1}}{\sqrt{z_{b_1}}} \dots z_{b_m}^{s_m} \frac{\theta_{b_m}}{\sqrt{z_{b_m}}}. \quad (26)$$

Due to the properties of the fermionic variables, we have

$$C_{k_1, \dots, k_n}^{s_1, \dots, s_m} = 0, \quad m > N, \quad (27)$$

and

$$C_{k_1, \dots, k_n}^{s_1, \dots, s_i, \dots, s_j, \dots, s_m} = 0, \quad s_i = s_j. \quad (28)$$

The operator $\hat{D}$ acting on $\bar{Z}^{(s)}$ gives

$$\hat{D} \bar{Z}^{(s)} = s \bar{Z}^{(s)}. \quad (29)$$

By means of (20), (21) and (29), we obtain

$$\hat{W} \bar{Z}^{(s)} = -(s+1) \bar{Z}^{(s+1)}. \quad (30)$$

The partition function (10) is graded by the total (t, ξ) -degree. From (29) and (30), we see that the $\hat{D}$ and $\hat{W}$ are indeed the operators preserving and increasing the grading, respectively. In terms of the operator $\hat{W}$, (22) can be rewritten as

$$\begin{aligned} \bar{Z} &= \bar{Z}^{(0)} - \hat{W} \bar{Z}^{(0)} + \frac{1}{2!} \hat{W}^2 \bar{Z}^{(0)} - \frac{1}{3!} \hat{W}^3 \bar{Z}^{(0)} + \dots \\ &= e^{-\hat{W}} \cdot e^{-\frac{\sqrt{\beta}}{\hbar} N t_0}. \end{aligned} \quad (31)$$

It indicates that the supereigenvalue model in the Ramond sector can be obtained by acting on elementary functions with exponents of the given bosonic operators $\hat{W}$.

For the $(l+1)$ -th power of $\hat{W}$, it can be formally expressed as

$$\hat{W}^{l+1} = \sum_{a,b,c,d=0}^{2(l+1)} \sum_{i_1, \dots, i_d=0}^{\infty} \sum_{j_1, \dots, j_b=0}^{\infty} \sum_{\substack{k_1+\dots+k_c+\rho \\ k_1, \dots, k_c \geq 1 \\ s_1, \dots, s_d \geq 0}} \hat{P}_{(i_1, \dots, i_d | j_1, \dots, j_b)}^{(k_1, \dots, k_c | s_1, \dots, s_d)} t_{k_1} \dots t_{k_c} \xi_{s_1} \dots \xi_{s_d} \frac{\partial}{\partial t_{i_1}} \dots \frac{\partial}{\partial t_{i_d}} \frac{\partial}{\partial \xi_{j_1}} \dots \frac{\partial}{\partial \xi_{j_b}}, \quad (32)$$

where $\rho = \sum_{\mu=1}^a i_\mu + \sum_{v=1}^b j_v + l+1$, the coefficients $\hat{P}_{(i_1, \dots, i_d | j_1, \dots, j_b)}^{(k_1, \dots, k_c | s_1, \dots, s_d)}$ are polynomials with respect to $i_\mu, j_v, k_{\bar{\mu}}$ and $s_{\bar{v}}$, $\bar{\mu} = 1, \dots, c$, $\bar{v} = 1, \dots, d$.

Substituting (32) into (31), comparing the coefficients of $t_{k_1} \dots t_{k_n} \xi_{s_1} \dots \xi_{s_m}$ with $\sum_{\mu=1}^n k_\mu + \sum_{v=1}^m s_v = l+1$, $k_\mu \geq 1$, $s_v \geq 0$ in (31) and (22), we obtain

$$\begin{aligned} &\frac{(-1)^{l+1}}{(l+1)!} e^{-\frac{\sqrt{\beta}}{\hbar} N t_0} \sum_{\alpha=1}^{2(l+1)} \sum_{\sigma_1, \sigma_2} \left(-\frac{\sqrt{\beta}}{\hbar} N\right)^\alpha (-1)^{\tau(\sigma_2(s_1), \dots, \sigma_2(s_m))} \hat{P}_{\underbrace{(0, \dots, 0)}_\alpha}^{(\sigma_1(k_1), \dots, \sigma_1(k_n) | \sigma_2(s_1), \dots, \sigma_2(s_m))} \\ &= \frac{(-1)^{\frac{m(m+1)}{2}} \left(-\frac{\sqrt{\beta}}{\hbar}\right)^{n+m}}{n!m!} e^{-\frac{\sqrt{\beta}}{\hbar} N t_0} \sum_{\sigma_1, \sigma_2} (-1)^{\tau(\sigma_2(s_1), \dots, \sigma_2(s_m))} C_{\sigma_1(k_1), \dots, \sigma_1(k_n)}^{\sigma_2(s_1), \dots, \sigma_2(s_m)} \\ &= \frac{(-1)^{\frac{m(m+1)}{2}} \left(-\frac{\sqrt{\beta}}{\hbar}\right)^{n+m}}{n!m!} e^{-\frac{\sqrt{\beta}}{\hbar} N t_0} \lambda_{(k_1, \dots, k_n)} \lambda_{(s_1, \dots, s_m)} C_{k_1, \dots, k_n}^{s_1, \dots, s_m}, \end{aligned} \quad (33)$$

where σ_1 denotes all the distinct permutations of $(k_1, \dots, k_n)$, σ_2 is all the distinct permutations of $(s_1, \dots, s_m)$ and its inverse number is denoted as $\tau(\sigma_2(s_1), \dots, \sigma_2(s_m))$, $\lambda_{(k_1, \dots, k_n)}$ and $\lambda_{(s_1, \dots, s_m)}$ are the numbers of distinct permutations of $(k_1, \dots, k_n)$ and $(s_1, \dots, s_m)$, respectively.

Then we obtain the correlators from (33)

$$C_{k_1, \dots, k_n}^{s_1, \dots, s_m} = \frac{(-1)^{l+1+\frac{m(m+1)}{2}} n!m! \left(-\frac{\hbar}{\sqrt{\beta}}\right)^{n+m}}{(l+1)! \lambda_{(k_1, \dots, k_n)} \lambda_{(s_1, \dots, s_m)}} \sum_{\alpha=1}^{2(l+1)} \left(-\frac{\sqrt{\beta}}{\hbar} N\right)^\alpha P_{\underbrace{(0, \dots, 0)}_\alpha}^{(k_1, \dots, k_n | s_1, \dots, s_m)}, \quad (34)$$

where $P_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(k_1, \dots, k_n | s_1, \dots, s_m)} = \sum_{\sigma_1, \sigma_2} (-1)^{\tau(\sigma_2(s_1), \dots, \sigma_2(s_m))} \hat{P}_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(\sigma_1(k_1), \dots, \sigma_1(k_n) | \sigma_2(s_1), \dots, \sigma_2(s_m))}$, $\sum_{\mu=1}^n k_{\mu} + \sum_{\nu=1}^m s_{\nu} = l + 1$, $k_{\mu} \geq 1$ and $s_{\nu} \geq 0$.

When particularized to the $m = 0$ and $n = 0$ cases in (34), respectively, we have

$$C_{k_1, \dots, k_n} = \frac{(-1)^{l+1} n! (-\frac{\hbar}{\sqrt{\beta}})^n}{(l+1)! \lambda_{(k_1, \dots, k_n)}} \sum_{\alpha=1}^{2(l+1)} (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} P_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(k_1, \dots, k_n)}, \quad (35)$$

$$C^{s_1, \dots, s_m} = \frac{(-1)^{l+1+\frac{m(m+1)}{2}} m! (-\frac{\hbar}{\sqrt{\beta}})^m}{(l+1)! \lambda_{(s_1, \dots, s_m)}} \sum_{\alpha=1}^{2(l+1)} (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} P_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(s_1, \dots, s_m)}. \quad (36)$$

For examples, let us list some correlators.

(I) When $l = 0$ in (32), we have

$$P_{(0,0)}^{(1)} = \frac{\hbar^2}{2}, \quad P_{(0)}^{(1)} = -\frac{\hbar}{2\sqrt{\beta}}(1-\beta), \quad P_{(0)}^{(1,0)} = 1. \quad (37)$$

Substituting (37) into (35) and (36), we obtain

$$\begin{aligned} C_1 &= \frac{1}{\lambda_{(1)}} \left[-NP_{(0)}^{(1)} + \frac{\sqrt{\beta}}{\hbar} N^2 P_{(0,0)}^{(1)} \right] = \frac{\hbar}{2\sqrt{\beta}} N \tilde{N}, \\ C^{1,0} &= -\frac{2\hbar}{\sqrt{\beta} \lambda_{(1,0)}} NP_{(0)}^{(1,0)} = -\frac{\hbar}{\sqrt{\beta}} N, \end{aligned} \quad (38)$$

where $\lambda_{(1)} = 1$, $\lambda_{(1,0)} = 2$, $\tilde{N} = \beta N + (1 - \beta)$.

(II) When $l = 1$ in (32), we have

$$\begin{aligned} P_{(0,0,0,0)}^{(1,1)} &= \frac{\hbar^4}{4}, & P_{(0,0,0)}^{(1,1)} &= -\frac{\hbar^3}{2\sqrt{\beta}}(1-\beta), & P_{(0,0)}^{(1,1)} &= \frac{\hbar^2}{4\beta}(1-\beta)^2 + \frac{\hbar^2}{2}, \\ P_{(0,0,0)}^{(2)} &= \hbar^4, & P_{(0)}^{(1,1)} &= -\frac{\hbar}{2\sqrt{\beta}}(1-\beta), & P_{(0,0)}^{(2)} &= \frac{2\hbar^3}{\sqrt{\beta}}(1-\beta), \\ P_{(0,0,0)}^{(1|1,0)} &= \hbar^2, & P_{(0)}^{(2)} &= \frac{\hbar^2}{\beta}(1-\beta)^2 + \frac{\hbar^2}{2}, & P_{(0,0)}^{(1|1,0)} &= -\frac{\hbar}{\sqrt{\beta}}(1-\beta), \\ P_{(0,0)}^{(2,0)} &= 3\hbar^2, & P_{(0)}^{(2,0)} &= -\frac{4\sqrt{\beta}}{\hbar}(1-\beta), & P_{(0)}^{(1|1,0)} &= 2. \end{aligned} \quad (39)$$

Substituting (39) into (34), (35) and (36), we obtain

$$\begin{aligned} C_2 &= -\frac{\hbar}{2\sqrt{\beta} \lambda_{(2)}} \sum_{\alpha=1}^3 (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} P_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(2)} = \frac{\hbar^2}{4\beta} N(2\tilde{N}^2 + \beta), \\ C_{1,1} &= \frac{\hbar^2}{\beta \lambda_{(1,1)}} \sum_{\alpha=1}^4 (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} P_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(1,1)} = \frac{\hbar^2}{4\beta} \tilde{N} N(\tilde{N} N + 2), \\ C^{2,0} &= -\frac{\hbar^2}{\beta \lambda_{(2,0)}} \sum_{\alpha=1}^2 (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} P_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(2,0)} = -\frac{\hbar^2}{2\beta} N(3\tilde{N} + 1 - \beta), \\ C_1^{1,0} &= \frac{(\frac{\hbar}{\sqrt{\beta}})^3}{\lambda_{(1)} \lambda_{(1,0)}} \sum_{\alpha=1}^3 (-\frac{\sqrt{\beta}}{\hbar} N)^{\alpha} P_{\underbrace{(0, \dots, 0)}_{\alpha}}^{(1|1,0)} = -\frac{\hbar^2}{2\beta} N(N\tilde{N} + 2), \end{aligned} \quad (40)$$

where $\lambda_{(2)} = \lambda_{(1,1)} = 1$, $\lambda_{(2,0)} = 2$.

(III) When $l = 2$ in (32), by direct calculations, it is easy to obtain the precise expression of the 3-th power of $\hat{W}$. Then we have the final results from (34)

$$\begin{aligned} C_3 &= \frac{1}{8} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N[5\tilde{N}^3 + (1-\beta)\tilde{N}^2 + 10\beta\tilde{N} + 3\beta(1-\beta)], \\ C_{1,2} &= \frac{1}{8} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N(2N\tilde{N}^3 + 8\tilde{N}^2 + \beta N\tilde{N} + 4\beta), \\ C^{2,1} &= \frac{1}{4} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N(-2\tilde{N}^2 + \beta N\tilde{N} + \beta), \end{aligned}$$

$$\begin{aligned}
C^{3,0} &= \frac{1}{4} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N [-10\tilde{N}^2 - 9(1-\beta)\tilde{N} - 5\beta - 3(1-\beta)^2], \\
C_{1,1,1} &= \frac{1}{8} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N \tilde{N} (N\tilde{N} + 2)(N\tilde{N} + 4), \\
C_1^{2,0} &= \frac{1}{4} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N [-3N\tilde{N}^2 - 12\tilde{N} - (1-\beta)N\tilde{N} - 4(1-\beta)], \\
C_2^{1,0} &= \frac{1}{4} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N [-2N\tilde{N}^2 - 13\tilde{N} - 3(1-\beta)], \\
C_{1,1}^{1,0} &= \frac{1}{4} \left(\frac{\hbar}{\sqrt{\beta}} \right)^3 N (-N^2\tilde{N}^2 - 6N\tilde{N} - 8).
\end{aligned} \tag{41}$$

3. Virasoro constraints for the supereigenvalue model in the Ramond sector

It is known that the partition function (1) is invariant under two pairs of the changes of integration variables ($z_a \rightarrow z_a + \epsilon z_a^{n+1}$, $\theta_a \rightarrow \theta_a + \frac{1}{2} \epsilon n z_a^n \theta_a$) and ($z_a \rightarrow z_a + z_a^n \sqrt{z_a} \theta_a \delta$, $\theta_a \rightarrow \theta_a + z_a^n \sqrt{z_a} \delta$), where ϵ and δ are the infinitesimal bosonic and fermionic constants, respectively. These invariances, respectively, lead to the bosonic and fermionic loop equations which give the super Virasoro constraints (4). Taking the shift $t_1 \rightarrow t_1 + 1$ in (4), we have the super Virasoro constraints for (10)

$$\bar{L}_n \bar{Z} = \frac{1}{16} \delta_{n,0} \bar{Z}, \quad \bar{G}_n \bar{Z} = 0, \quad n \in \mathbb{N}. \tag{42}$$

The super Virasoro algebra (7) still holds for the constraint operators $\bar{L}_n$ and $\bar{G}_n$.

From the super Virasoro constraints (42), the recursive formulas for correlators can be obtained. In principle, we can calculate the correlators step by step from the recursive formulas. However, the compact expression of correlators (34) can not be derived from them.

Let us introduce the bosonic operators

$$\hat{L}_l = \hat{W}^l (\hat{W} + \hat{D}), \quad l \in \mathbb{N}. \tag{43}$$

These operators are different from $\bar{L}_n$. They obey not only the Witt algebra (7a), but also the null Witt 3-algebra [24]

$$[\hat{L}_{l_1}, \hat{L}_{l_2}, \hat{L}_{l_3}] := \hat{L}_{l_1} [\hat{L}_{l_2}, \hat{L}_{l_3}] - \hat{L}_{l_2} [\hat{L}_{l_1}, \hat{L}_{l_3}] + \hat{L}_{l_3} [\hat{L}_{l_1}, \hat{L}_{l_2}] = 0. \tag{44}$$

The action of the operators (43) on the partition function (10) leads to the Virasoro constraints

$$\hat{L}_l \bar{Z} = 0. \tag{45}$$

Recently similar Virasoro constraints without the Grassmann variables have been presented for the Gaussian Hermitian matrix model and they have been used to derive the correlators of the matrix model [25].

Let us first consider the Virasoro constraints (45) with $l = 0$, i.e., (20). Substituting (22) into (20), by collecting the coefficients of t_1^l and setting to zero, we obtain

$$C_1 = \frac{\hbar}{2\sqrt{\beta}} N \tilde{N}, \tag{46}$$

and the recursive relations

$$C_{\underbrace{1, \dots, 1}_{l+1}} = \frac{\hbar}{2\sqrt{\beta}} (N\tilde{N} + 2l) C_{\underbrace{1, \dots, 1}_l}. \tag{47}$$

From (47), it is easy to obtain

$$C_{\underbrace{1, \dots, 1}_{l+1}} = \left(\frac{\hbar}{2\sqrt{\beta}} \right)^{l+1} \prod_{j=0}^l (N\tilde{N} + 2j). \tag{48}$$

We observe that it is difficult to give the precise expression of $P_{\alpha}^{\overbrace{(1, \dots, 1)}^{l+1}, \overbrace{(0, \dots, 0)}^{\alpha}}$ from $\hat{W}^{l+1}$. However, by taking $n = l + 1$ and $k_1 = \dots = k_n = 1$ in (35) and using (48), we obtain

$$\begin{aligned}
P_{\alpha}^{\overbrace{(1, \dots, 1)}^{l+1}, \overbrace{(0, \dots, 0)}^{\alpha}} &= \frac{1}{2^{l+1}} \left(-\frac{\hbar}{\sqrt{\beta}} \right)^{\alpha} \left[\sum_{\substack{2i+j=\alpha-2 \\ 0 \leq i, j \leq l}} \beta^{i+1} (1-\beta)^j + \sum_{\substack{2i+j=\alpha-1 \\ 0 \leq i, j \leq l}} \beta^i (1-\beta)^{j+1} \right] \\
&\cdot \sum_{\substack{1 \leq r_1 < r_2 < \dots < r_{l+j} \leq l \\ r_0=1}} \frac{2^{l-(i+j)} \cdot l!}{\prod_{k=0}^{i+j} r_k}, \quad \alpha = 1, \dots, 2(l+1).
\end{aligned} \tag{49}$$

Let us collect the coefficients of $t_1^l \xi_0 \xi_1$ in (20) and set to zero, we have

$$C^{0,1} = \frac{\hbar}{\sqrt{\beta}} N, \quad (50)$$

and the recursive relations

$$\underbrace{C_{1,\dots,1}^{0,1}}_l = \frac{\hbar}{\sqrt{\beta}(l+1)} [l(l+1) + \frac{1}{2} N \tilde{N}] \underbrace{C_{1,\dots,1}^{0,1}}_{l-1} + N \underbrace{C_{1,\dots,1}}_l. \quad (51)$$

Substituting (48) into (51) we obtain

$$\underbrace{C_{1,\dots,1}^{0,1}}_l = 2N \left(\frac{\hbar}{2\sqrt{\beta}} \right)^{l+1} \prod_{j=1}^l (N \tilde{N} + 2j). \quad (52)$$

Proceeding the similar procedure for the case of the coefficients of $t_1^l t_2$ in (20), we have

$$C_2 = \frac{\hbar}{4\sqrt{\beta}} (2\tilde{N} C_1 + \beta C^{0,1}) = \frac{\hbar^2}{4\beta} N (2\tilde{N}^2 + \beta), \quad (53)$$

and the recursive relations

$$\underbrace{C_{2,1,\dots,1}}_l = \frac{\hbar}{\sqrt{\beta}} \frac{l}{l+2} (l+3 + \frac{1}{2} N \tilde{N}) \underbrace{C_{2,1,\dots,1}}_{l-1} + \frac{\hbar}{\sqrt{\beta}(l+2)} \left(\frac{\beta}{2} \underbrace{C_{1,\dots,1}^{0,1}}_l + 2\tilde{N} \underbrace{C_{1,\dots,1}}_{l+1} \right). \quad (54)$$

Substituting (48) and (52) into (54), we obtain

$$\underbrace{C_{2,1,\dots,1}}_l = \left(\frac{\hbar}{2\sqrt{\beta}} \right)^{l+2} N (2\tilde{N}^2 + \beta) \prod_{j=1}^l (N \tilde{N} + 2j + 2). \quad (55)$$

Comparing (52), (55) with (34), we obtain

$$P_{\underbrace{(1,\dots,1)_{|0,1)}}_{\alpha}} = P_{\underbrace{(2,1,\dots,1)_{|1)}}_{\alpha}} = 0, \quad \alpha = 2(l+1), \quad (56)$$

and

$$\begin{aligned} P_{\underbrace{(1,\dots,1)_{|0,1)}}_{\alpha}} &= \frac{(-1)^\alpha (l+1)}{2^l} \left(\frac{\hbar}{\sqrt{\beta}} \right)^{\alpha-1} \sum_{\substack{2i+j=\alpha-1 \\ 0 \leq i, j \leq l}} \beta^i (1-\beta)^j \sum_{\substack{1 \leq r_1 < \dots < r_{i+j} \leq l \\ r_0=1}} \frac{2^{l-(i+j)} \cdot l!}{\prod_{k=0}^{i+j} r_k}, \\ P_{\underbrace{(2,1,\dots,1)_{|1)}}_{\alpha}} &= \frac{l(l+1)}{2^{l+1}} \left(\frac{-\hbar}{\sqrt{\beta}} \right)^{\alpha+1} \left[\sum_{\substack{2i+j=\alpha-3 \\ 0 \leq i, j \leq l-1}} 2\beta^{i+2} (1-\beta)^j + \sum_{\substack{2i+j=\alpha-2 \\ 0 \leq i, j \leq l-1}} 4\beta^{i+1} (1-\beta)^{j+1} \right. \\ &\quad \left. + \sum_{\substack{2i+j=\alpha-1 \\ 0 \leq i, j \leq l-1}} (2(1-\beta)^2 + \beta) \beta^i (1-\beta)^j \right] \sum_{\substack{2 \leq r_1 < \dots < r_{i+j} \leq l \\ r_0=1}} \frac{2^{l-1-(i+j)} \cdot l!}{\prod_{k=0}^{i+j} r_k}, \end{aligned} \quad (57)$$

for $\alpha = 1, \dots, 2l+1$.

We have derived the special correlators from (20). It is known that the compact expression of correlators (34) can not be derived from the super Virasoro constraints (42). However, it should be pointed out that the special correlators (48), (52) and (55) can be still obtained from (42).

Let us consider the case of (45) with $l \neq 0$. By means of (20) and (21), (45) can be rewritten as

$$\hat{W}^{l+1} \bar{Z} = (-1)^{l+1} \prod_{j=0}^l (\hat{D} - j) \bar{Z}. \quad (58)$$

Substituting (32) into (58), by collecting the coefficients of $t_{k_1} \dots t_{k_n} \xi_{s_1} \dots \xi_{s_m}$ with $\sum_{\mu=1}^n k_\mu + \sum_{\nu=1}^m s_\nu = l+1$ and setting to zero, we may also derive the correlators (34).

We have achieved the desired correlators from the Virasoro constraints (45). Unlike the operators $\bar{L}_n$ in (42), the remarkable property of the constraint operators (43) is that these bosonic operators yield the higher algebraic structures. It should be noted that the closure of the super algebra does not hold for (43) and the fermionic operators $\bar{G}_n$ in (42).

4. Summary

We have investigated the supereigenvalue model in the Ramond sector and proved that its partition function can be obtained by acting on elementary functions with exponents of the $\hat{W}$ operators. In terms of the operators $\hat{D}$ and $\hat{W}$ preserving and increasing the grading, respectively, we have constructed the Virasoro constraints for this supereigenvalue model, where the constraint operators obey the Witt algebra and null 3-algebra. The compact expression of correlators (34) can be derived from these Virasoro constraints. It should be noted that this desired result can not be derived from the well known super Virasoro constraints (42). For the supereigenvalue model in the Neveu-Schwarz sector, whether its partition function can be expressed in terms of W -representation still deserves further study.

We have only constructed the Virasoro constraints for the supereigenvalue model (10). The remarkable property of these bosonic constraint operators is that they yield the higher algebraic structures. It is certainly worth to construct the super (Virasoro) constraints for supereigenvalue models, where the super higher algebraic structures hold for the bosonic and fermionic constraint operators. It would be interesting to study further properties of supereigenvalue models from these constraints.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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