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On Some Aspects of the Courant-Type Algebroids, the Related Coadjoint Orbits and Integrable Systems

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Abstract: Poisson structures related to affine Courant-type algebroids are analyzed, including those related with cotangent bundles on Lie-group manifolds. Special attention is paid to Courant-type algebroids and their related R structures generated by suitably defined tensor mappings. Lie–Poisson brackets that are invariant with respect to the coadjoint action of the loop diffeomorphism group are created, and the related Courant-type algebroids are described. The corresponding integrable Hamiltonian flows generated by Casimir functionals and generalizing so-called heavenly-type differential systems describing diverse geometric structures of conformal type in finite dimensional Riemannian manifolds are described.

Keywords: Lie algebroid; Courant algebroid; Poisson structure; Grassmann algebra; differentiation; coadjoint orbits; Hamiltonian systems; invariants; integrability

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1. Introduction

As mathematical object Lie algebroids [1,2] are an *unrecognized* part of the folklore of differential geometry. They have been introduced repeatedly in differential geometry since the early 1950s, as well as in physics and algebra, under a wide variety of names, chiefly as infinitesimal invariants associated with geometric structures. In connection theory, they have been used as a means of treating de Rham cohomology with algebraic methods, as invariants of foliations and pseudogroups of various types, in symplectic and Poisson geometry and in a more algebraic setting as algebroids of differential operators associated with vector bundles and withinfinitesimal actions of Lie groups. Algebroid structures have recently found diverse applications in the geometry of Poisson [3–5] and Lagrangian manifolds [6] in mechanical sciences [7] and other branches of modern applied and theoretical research. We wish to highlight the original work reported in [8] with respect to so-called contrast (potential) functions in statistical and information geometry provided by Lie groupoids and Lie algebroids. Important theoretical aspects of homology and modular classes were studied in [9]. It is also worth mentioning interesting aspects of algebroid theory studied in [10] concerning the homomorphism Chern-Weyl transformations of algebroids and in [11] concerning the cohomology isomorphism subject to the piecewise morphism restriction of transitive Lie algebroids. New aspects of Lie algebroid theory were demonstrated in [3], where the authors showed how some Lie algebroid operations, in particular the Courant brackets on the doubles of Lie bialgebroids, can be realized in a natural way in the tangent spaces of reductive homogeneous spaces. A special realization of the doubled Lie algebroid was proposed in [12] and named for T.J. Courant, who implicitly devised the standard prototype of Courant algebroids through his discovery of a skew symmetric bracket on the doubled tangent–cotangent bundle (called the Courant bracket today) and deeply studied so-called Dirac manifolds and related Dirac structures. Here, we would like to mention Courant's characteristic theorem concerning dual Lie algebroid bundles. The dual bundle of a Lie algebroid is a Poisson manifold such that the Poisson bracket of linear functions remains linear. Furthermore, any vector bundle with such a Poisson bundle is a dual to a Lie algebroid, and its Poisson structure is inherited as such. Motivated by these results, the authors of [13] analyzed many interesting properties of Courant algebroids and the related Poisson–Lie T duality; in particular, they extended the known results to a much wider class of dualities, including cases with gauging, in addition to presenting an illustration of the use of the formalism to provide new classes of special solutions to modified type-two supergravity equations in symmetric spaces. Other interesting properties of Courant algebroids were studied in [12,14–19] within which the authors proposed a Lie algebroid on the loop space pinned down to the Lie algebroid on the manifold. The authors conjectured that this construction, as applied to the Dirac structure, should give rise to the Lie algebroid of symmetries specifying special σ models. A strikingly new face of algebroid theory related to the construction of integrable hierarchies was recently presented in [5]. The authors observed interesting connections between algebroid structures and Frölicher–Nijenhuis bicomplexes and Lauricella biflat F manifolds, which string theorists believe could have important applications in topological quantum field theory.

Inspired in part by these algebro-theoretical studies of differential geometric structures associated with Courant algebroids, we observed that some of their interesting properties can be studied in more detail both from symplectic and Lie-algebraic points of view. Therefore, we provided an instructive example of a Courant algebroid, considering a semisimple Lie group and its Lie algebra at the unity element, consisting of the corresponding left invariant vector fields. Within this algebraic setting, rigged with the canonical symplectic structure mapping as an anchor, we considered the cohomology group of this Lie group and showed that the related Lie algebroid reduces to the Courant algebroid, similarly to the result reported in [3]. Moreover, this construction proved to be naturally generalizable to cases in which the canonical symplectic mapping is replaced by some Lie algebra homomorphism that can be realized within the well-known Marsden–Weinstein reduction scheme [20–22] applied to a suitably constructed Hamiltonian group action. Another approach to constructing Courant algebroids with rich differential–geometric properties is based on the effective Adler–Kostant–Symes-type scheme [23–26] for the construction of Poisson structures in coadjoint orbits, in particular its version based on the R-structure approach associated with a specially defined tensor mapping and the related canonical Lie–Poisson bracket in the dual space.

We also paid attention to some differential geometric and symplectic properties of a special Courant-type algebroid foliation and analyzed the structure of related Hamiltonian flows. We showed that the Courant-type algebroid foliations, equipped with two compatible external differentials, generate a finite set of commuting Hamiltonian flows, realizing a classical Magri-type recursion scheme. As we were interested in Courant-type algebroids related to the loop diffeomorphism group, we constructed compatible pairs of Poisson brackets and the related integrable Hamiltonian flows within the classical Adler–Kostant–Symes scheme [23–25,27,28] the, suitably generalizing [29–32] so-called [33] heavenly-type differential systems, describing diverse geometric structures of the conformal type on finite-dimensional Riemannian manifolds.

2. A Lie Algebroid and Its Courant Reduction and Realization

We first recall the classical definition of a Lie algebroid and its reduction to the Courant algebroid first suggested in [3].

Definition 1. *Let M be a manifold. A Lie algebroid* $((E; [[.,.]], \rho, M))$ *on M or with a base of M is a vector bundle* $(E \to M)$*, together with a bracket* $([[.,.]] : \Gamma(E) \times \Gamma(E) \to \Gamma(E)$ *on the module*

$\Gamma(E)$) of global sections of E and a vector bundle morphism ($\rho : E \to T(M)$) from E to the tangent bundle ($T(M)$) of M called the anchor of E such that

(i) The bracket on $\Gamma(E)$ is $\mathbb{R}$-bilinear and skew-symmetric and satisfies the Jacobi identity;
(ii) $[[\alpha, f\beta]] = f[[\alpha,\beta]] + \rho(\alpha)f\,\beta$ for all $\alpha,\beta \in \Gamma(E)$ and all smooth functions ($f \in \mathcal{D}(M)$);
(iii) $\rho([[\alpha,\beta]]) = [\rho(\alpha), \rho(\beta)]$ for all $\alpha, \beta \in \Gamma(E)$.

The anchor (ρ) ties the bracket on $\Gamma(E)$ to the vector field structure on M as a module over $\mathcal{D}(M)$, and the algebra of smooth functions is $f : M \to \mathbb{R}$.

Consider the product ($T(M) \ltimes T^*(M)$) of tangent $T(M)$ and its cotangent ($T^*(M)$) bundles over the manifold (M). Then, the *canonical Courant bracket* [34] on the $\mathcal{D}(M)$ module ($\mathcal{A}(M) := T^*(M) \times T(M) \simeq (T(M) \times T^*(M))^*$) is defined as

$$[[(\alpha, a), (\beta, b)]] := (L_a\beta - L_b\alpha + \tfrac{1}{2}d(\alpha(b) - \beta(a)), [a, b]) \tag{1}$$

for any $(\alpha, a), (\beta, b) \in T^*(M) \times T(M)$, satisfying [20,35,36] the usual Jacobi identity.

Definition 2. *The bundle $\mathcal{A}(M) = T^*(M) \times T(M)$, jointly with the bracket (1) and the natural morphism projection mapping ($\rho : \mathcal{A}(M) \to T(M)$) is called the Courant algebroid.*

Let us now assume that the cotangent space ($T^*(M)$) is endowed with its own Poisson structure ($P : T^*(M) \to T(M)$). Then, by definition, $a := P\alpha$, $b := P\beta \in -(T(M))$, and one can easily observe that the Courant bracket (1) becomes its second-term identity, reducing to the next bracket in the cotangent space ($T^*(M)$):

$$\begin{aligned}[[\alpha,\beta]] &= L_{P\alpha}\beta - L_{P\beta}\alpha + \tfrac{1}{2}d(\alpha(P\beta) - \beta(P\alpha)) \\ &= i_{P\beta}d\alpha - i_{P\alpha}d\beta - \tfrac{1}{2}d(\alpha(P\beta) - \beta(P\alpha))\end{aligned} \tag{2}$$

for any $\alpha, \beta \in \Lambda^1(M)$, satisfying the Jacobi identity. Thus, the triple $(T^*(M); [[\cdot,\cdot]], P)$ becomes a Lie algebroid with *anchor* $P : T^*(M) \to T(M)$, which is considered a Lie algebra morphism:

$$P[[\alpha,\beta]] = [P\alpha, P\beta], \tag{3}$$

which is satisfied for any $\alpha, \beta \in T^*(M) \simeq \Lambda^1(M)$.

A Lie Group, the Hamilton Group Action and the Related Lie–Courant Algebroid Construction

As an instructive example of the construction described above, we consider a semisimple Lie group (G) and its Lie algebra ($\mathcal{G} \simeq T_e(G)$) at the unity element ($e \in G$) consisting of the left invariant vector fields on G. Assume that [20,21,28,36,37] $\Omega : \mathcal{G} \to \mathcal{G}^*$ is a symplectic structure on G, allowing for the construction of the adjoint left-invariant vector fields as $X_\alpha := \Omega^{-1}\alpha$, $X_\beta := \Omega^{-1}\beta \in \mathcal{G}$, subject to which the related Poisson bracket—

$$[[\alpha,\beta]] := i_{[X_\alpha, X_\beta]}\Omega \tag{4}$$

—satisfies the Jacobi identity. The latter, in particular, means that the constructed object ($(\mathcal{G}^*; [[.,.]], \Omega^{-1}, G)$) is also a reduced Lie algebroid. Moreover, the Lie bracket (4), owing to the Cartan representation of the Lie derivative ($L_X = i_X d + di_X$, $X \in \mathcal{G}$), [20,21,35,36] can be rewritten as

$$
\begin{aligned}
[[\alpha,\beta]](Z) &= \Big(i_{[X_\alpha,X_\beta]}\Omega\Big)(Z) = [L_{X_\alpha}, i_{X_\beta}]\Omega(Z)\\
&= L_{X_\alpha} i_{X_\beta}\Omega(Z) - i_{X_\beta} L_{X_\alpha}\Omega(Z)\\
&= i_{X_\alpha} d i_{X_\beta}\Omega(Z) + d\Big(i_{X_\alpha} i_{X_\beta}\Omega\Big)(Z) - i_{X_\beta} i_{X_\alpha} d\Omega(Z) - i_{X_\beta} d i_{X_\alpha}\Omega(Z)\\
&= i_{X_\alpha} d i_{X_\beta}\Omega(Z) - i_{X_\beta} d i_{X_\alpha}\Omega(Z) + d\big(\Omega(X_\beta, X_\alpha)\big)(Z)\\
&= X_\alpha\Omega(X_\beta, Z) - Z\Omega(X_\beta, X_\alpha) - \Omega(X_\beta,[X_\alpha,Z]) - X_\beta\Omega(X_\alpha,Z) + Z\Omega(X_\alpha,X_\beta) +\\
&\qquad + \Omega(X_\alpha,[X_\beta,Z]) + d\big(\Omega(X_\beta,X_\alpha)\big)(Z)\\
&= -\Omega(X_\beta,[X_\alpha,Z]) + \Omega(X_\alpha,[X_\beta,Z]) - d\big(\Omega(X_\alpha,X_\beta)\big)(Z)\\
&= \Big(ad^*_{X_\beta}(i_{X_\alpha}\Omega) - ad^*_{X_\alpha}\Big(i_{X_\beta}\Omega\Big)\Big)(Z) - d\big(\Omega(X_\alpha,X_\beta)\big)(Z)\\
&= \Big(ad^*_{\Omega^{-1}\beta}\alpha - ad^*_{\Omega^{-1}\alpha}\beta\Big)(Z) - 1/2d\Big(\alpha(\Omega^{-1}\beta) - \beta(\Omega^{-1}\alpha)\Big)(Z),
\end{aligned}
\tag{5}
$$

where we made use of the invariance conditions ($Z\alpha(X) = 0 = Z\beta(X)$) for arbitrary $\alpha, \beta \in \mathcal{G}^*$ and $X, Z \in \mathcal{G}$. Furthermore, $ad^* : \mathcal{G} \times \mathcal{G}^* \to \mathcal{G}^*$ denotes the natural coadjoint action of the Lie algebra ($\mathcal{G}$) in the adjoint space ($\mathcal{G}^*$). The obtained expression (5) on $\mathcal{G}^*$ can be rewritten as

$$[[\alpha,\beta]] = ad^*_{\rho(\beta)}\alpha - ad^*_{\rho(\alpha)}\beta - \tfrac{1}{2}d[\alpha(\rho(\beta)) - \beta(\rho(\alpha))], \tag{6}$$

where $\rho = \Omega^{-1} : \mathcal{G}^* \to \mathcal{G}$ denotes the corresponding anchor mapping the subject to the reduced Courant–Lie algebroid ($\mathcal{G}^*; [[.,.]], \rho, G$). Assume now that we are given a Lie algebroid ($\mathcal{G}^*_h; [[.,.]], \rho_h, G_h$) whose anchor ($\rho_h : \mathcal{G}^* \to \mathcal{G}$) is a Lie algebra homomorphism not necessarily related to a symplectic structure on G_h and *a priori* satisfying the Poisson bracket (6). Then, our inverse problem consists of describing at least sufficient conditions on the anchor ($\rho_h : \mathcal{G}^* \to \mathcal{G}$) under which the bracket (6) satisfies the Jacobi condition.

As a simple guiding construction for our Courant algebroids, let us consider the cohomology group ($H^1(G;\mathbb{C})$) of the Lie group (G) and observe that for any $\{\alpha\}, \{\beta\} \in H^1(G;\mathbb{C})$, $\alpha, \beta \in \mathcal{G}^*$, the Poisson bracket (6) reduces to the next Poisson bracket,

$$[[\{\alpha\},\{\beta\}]] = \{ad^*_{\rho(\beta)}\alpha - ad^*_{\rho(\alpha)}\beta\}, \tag{7}$$

on $H^1(G;\mathbb{C})$, satisfying the Jacobi condition. Thus, the cohomology group ($H^1(G;\mathbb{C})$) is simultaneously the Lie algebra with respect to the Lie product (7), satisfying the induced property:

$$\Omega^{-1} \to \Omega_h^{-1} : H^1(G;\mathbb{C}) \to \mathcal{G}/\mathcal{H} \simeq T(G/H),$$

where $H \subset G$ denotes the normal *Hamiltonian subgroup* of G, whose Lie algebra ($\mathcal{H} \subset \mathcal{G}$) consists of the Hamiltonian shifts ($\Omega(h) \in \mathcal{H}$) for all closed elements ($h \in \mathcal{G}^*$, $dh = 0$). The latter makes it possible to construct the reduced Lie algebroid $(T^*(G_h); [[.,.]]_h, \rho_h, G_h)$ with the following Lie bracket

$$[[\tilde{\alpha},\tilde{\beta}]]_h = ad^*_{\rho_h(\tilde{\beta})}\tilde{\alpha} - ad^*_{\rho_h(\tilde{\alpha})}\tilde{\beta} \tag{8}$$

for any $\tilde{\alpha}, \tilde{\beta} \in T^*(G_h)$, where $G_h := G/H$, and the anchor is $\rho_h := \Omega_h^{-1} : T^*(G_h) \to \mathcal{G}/\mathcal{H}$.

3. Courant-Type Algebroids and the Related *R* Structures

In what follows, we deal with the semidirect product-bundle Lie algebra ($\mathcal{A}^*(M) := T(M) \ltimes T^*(M)$), whose Lie product is defined as

$$[(a,\alpha),(b,\beta)] := ([a,b], ad^*_b\alpha - ad^*_a\beta) \tag{9}$$

for any $(a, \alpha), (b, \beta) \in \mathcal{A}^*(M)$, satisfying the Jacobi identity. Moreover, the Lie algebra $(\mathcal{A}^*(M))$ proves to be metrized [38,39] with respect to the dual *ad*-invariant pairing:

$$((a, \alpha)|(b, \beta)) := \alpha(b) + \beta(a), \tag{10}$$

and as for arbitrary $(\alpha, a), (\beta, b)$ and $(c, \gamma) \in \mathcal{A}^*(M)$, the identity

$$([(a, \alpha), (b, \beta)]|(c, \gamma)) = (a, \alpha)|[(b, \beta), (c, \gamma)]) \tag{11}$$

holds. Now, we take into account that the cotangent space $(\mathcal{A}(M)^* \simeq T(M) \times T^*(M))$ possesses the canonical Lie–Poisson structure defined by means of the following bracket

$$[[((l, p)|X), ((l, p)|Y)]]_{Lie} = ((l, p)|[X, Y]), \tag{12}$$

for all $X, Y \in \mathcal{A}(M)^*$ and any fixed element $((l, p) \in \mathcal{A}(M))$. To construct a Courant-type algebroid $(\mathcal{A}(M))$, let us take a tensor element $(r \in \mathcal{A}^*(M) \otimes \mathcal{A}^*(M))$ jointly with the related linear mapping $(r : \mathcal{A}(M) \to \mathcal{A}^*(M))$ and define the following bracket in the bundle $\mathcal{A}(M)$ for any $(\alpha, a), (\beta, b) \in \mathcal{A}(M)$:

$$[[(\alpha, a), (\beta, b)]]_r := ad^*_{r((\beta,b))}(\alpha, a) - ad^*_{r((\alpha,a))}(\beta, b) \tag{13}$$

The following proposition describing the conditions to be imposed on the tensor element holds:
$r \in \mathcal{A}^*(M) \otimes \mathcal{A}^*(M)$.

Proposition 1. *Let a tensor element* $(r \in \mathcal{A}^*(M) \otimes \mathcal{A}^*(M))$ *allow for direct sum splitting:* $r = k \oplus \eta$, *where* $k \in \mathcal{A}^*(M) \overset{Sym}{\otimes} \mathcal{A}^*(M)$ *is its symmetric non-degenerate part and* $\eta \in \mathcal{A}^*(M) \wedge \mathcal{A}^*(M)$ *is its antisymmetric non-degenerate part. If the related mapping* $(D := k \circ \eta^{-1} : \mathcal{A}^*(M) \to \mathcal{A}^*(M))$ *is a derivation of the algebra* $\mathcal{A}^*(M)$*, that is,*

$$D[(a, \alpha), (b, \beta)] = [D(a, \alpha), (b, \beta)] + [(a, \alpha), D(b, \beta)] \tag{14}$$

for any $(a, \alpha), (b, \beta) \in \mathcal{A}^*(M)$*, then the mapping* $\rho_r = pr_1 \circ r : \mathcal{A}(M) \to T(M)$

$$\rho_r[[(\alpha, a), (\beta, b)]]_r = [\rho_r(\alpha, a), \rho_r(\beta, b)] \tag{15}$$

defines a Lie algebra homomorphism, and the triple $(\mathcal{A}(M); [[., .]], \rho_r)$ *is a Courant-type algebroid.*

In addition, the following important corollaries are inferred from the reasoning presented above: if the invertible tensor $(r \in \mathcal{A}(M) \otimes \mathcal{A}(M))$ satisfies the formulated conditions (14) and (15), then the corresponding triple $(\mathcal{A}(M)^*, [[\cdot, \cdot]], \rho)$ is a generalized Courant-type algebroid with the related anchor morphism $(\rho : \mathcal{A}(M)^* \to T(M))$ defined by means of the composition $(\rho = pr_1 \circ r)$ with the projection mapping $(pr_1 : T(M) \times T^*(M) \to T(M))$ on the first component. Moreover, the following deformed commutator structure determines the corresponding R structure on $\mathcal{A}(M)$, $R = D^{-1}$ for any $(a, \alpha), (b, \beta) \in \mathcal{A}(M)$, subject to which the second Lie bracket $([[\cdot, \cdot]]_D)$ on $\mathcal{A}(M)$ also satisfies the Jacobi identity.

$$[[(a, \alpha), (b, \beta)]]_R := [[R(a, \alpha), (b, \beta)]] + [[(a, \alpha), R(b, \beta)]] \tag{16}$$

The latter makes it possible to construct the deformed Lie–Poisson bracket in the space $(\mathcal{A}(M)^*)$ for all $X, Y \in \mathcal{A}(M)$ and any fixed element $((l, p) \in \mathcal{A}(M)^*)$, subject to which the Casimir functionals of the Lie bracket (12) naturally generate [3,5,28,37–41] an infinite hierarchy of commuting, completely integrable Hamiltonian systems.

$$[[((l, p)|X), ((l, p)|Y))]]_R = ((l, p)|[[X, Y]]_R) \tag{17}$$

Below, we present the realization of the scheme described above and construct a Courant algebroid in the double bundle $(\tilde{\mathcal{G}}^* \times \tilde{\mathcal{G}})$ over the loop group $(\tilde{G} = \{\mathbb{C} \supset \mathbb{S}^1 \to G\})$ related [42] to a semisimple Lie group (G).

3.1. A Loop Group and the Related Hamiltonian Group Action

Now, consider a semisimple matrix Lie group (G), where $\widetilde{G} := \tilde{G}_+ \times \tilde{G}_- is$ the product of the loop subgroups ($\tilde{G}_+$ and $\tilde{G}$) defined by the continuous loop-group mappings $(\{\mathbb{C} \supset \mathbb{S}^1 \to G\})$ holomorphically extended on the interior $(\mathbb{D}^1_+ \subset \mathbb{C})$ and the exterior $(\mathbb{D}^1_- \subset \mathbb{C})$ of the centrally located unit disk $(\mathbb{D}^1 \subset \mathbb{C}^1)$, respectively, such that for any $\tilde{g}(\lambda) \in \tilde{G}_+, \lambda \in \mathbb{D}^1_-, \lim_{\lambda\to\infty} g(\lambda) = Id \in \tilde{G}_-$. There exists a Lie subalgebra splitting

$$\tilde{\mathcal{G}} \simeq \widetilde{\mathcal{G}}_+ \oplus \widetilde{\mathcal{G}}_- \tag{18}$$

in the direct sum of the subalgebras $(\tilde{\mathcal{G}}_\pm)$ of the left-invariant holomorphic vector fields of the subgroups $(\tilde{G}_\pm)$ in domains $\mathbb{D}^1_\pm \subset \mathbb{C}^1$, where for any $a(\lambda) \in \tilde{\mathcal{G}}_-$, the $\lim_{\lambda\to\infty} a(\lambda) = 0$. Based on the usual approach [37,39], one constructs the centrally extended loop Lie algebra $(\hat{\mathcal{G}} := \prod_{x\in\mathbb{R}/\{2\pi\mathbb{R}\}} \tilde{\mathcal{G}}_x \oplus \mathbb{C})$ jointly with the adjoint loop group $(\hat{G} := \prod_{x\in\mathbb{R}/\{2\pi\mathbb{R}\}} \tilde{G}_x)$ action, which is defined as

$$g : (S, c) \to (gSg^{-1},\ c + (g^{-1}\partial g/\partial x|S)). \tag{19}$$

for any $g \in \hat{\mathcal{G}}$, where $(T, c) \in \hat{\mathcal{G}}$, and $(\cdot|\cdot) : \hat{\mathcal{G}} \times \hat{\mathcal{G}} \to \mathbb{C}$ is the Killing-type non-degenerate symmetric bilinear form on $\tilde{\mathcal{G}}$:

$$(A|B) := \operatorname{res} \int_0^{2\pi} \operatorname{tr}(A(x;\lambda)B(x;\lambda)) = (B|A), \tag{20}$$

for any $A, B \in \hat{\mathcal{G}}$, where "tr" is the usual matrix trace, allowing for the useful identification of $\hat{\mathcal{G}}^* \simeq \hat{\mathcal{G}}$. The bilinear form (20) carries the hereditary property of ad invariance, that is, $(A|[B,C]) = ([A,B]|C)$ for any elements (A, B and $C \in \hat{\mathcal{G}}$). In the canonically symplectic functional-phase space $(M := T^*(\hat{\mathcal{G}}) \simeq \hat{\mathcal{G}} \times \hat{\mathcal{G}}^*)$, one can construct the corresponding Liouville 1-form,

$$\alpha^{(1)}(S, c; l, k) = (l|dS) + kdc, \tag{21}$$

whose exterior derivative gives the symplectic structure on the constructed functional manifold (M):

$$\omega^{(2)}(S, c; l, k) := d\alpha^{(1)}(S, c; l, k) = (dl| \wedge dS) + dk \wedge dc. \tag{22}$$

As in (19), we extend the group $\hat{G}$ action in the whole phase space (M), as by definition,

$$g : (l, k) \to (glg^{-1} - k\partial g/\partial x g^{-1}, k) \tag{23}$$

for any $(l, k) \in \hat{\mathcal{G}}^*$ and $g \in \hat{G}$, representing the coadjoint loop group $\hat{G}$ action in the adjoint linear space $(\hat{\mathcal{G}}^*)$. The following lemma is important for what follows.

Lemma 1. *The $\hat{G}$-group actions (19) and (23) in the symplectic phase space (M) are symplectic and Hamiltonian.*

Proof. It is easy to verify that the canonical Liouville 1-form (21) on the manifold (M) is $\hat{G}$-invariant:

$$\begin{aligned} g^*\alpha^{(1)}(S, c; l, k) &= (glg^{-1} - k\partial g/\partial x g^{-1}|gdSg^{-1}) + k(dc + (g^{-1}\partial g/\partial x|dS)) \\ &= (glg^{-1} + gdSg^{-1}) - k(\partial g/\partial x g^{-1}|gdSg^{-1}) \\ &\quad + kdc + k(g^{-1}\partial g/\partial x|dS) = (l|dS) + kdc = \alpha^{(1)}(S, c; l, k). \end{aligned} \tag{24}$$

According to (24), owing to the expression (22), one obtains the symplectic-form invariance

$$g^*\omega^{(2)}(S,c;l,k) = \omega^{(2)}(S,c;l,k) \tag{25}$$

for any element $((S,c;l,k) \in M)$. To define the Hamiltonian $\hat{G}$ action on the symplectic manifold (M) we take the group flow $(g(t) := \exp(tV))$ for $t \in \mathbb{R}$, $V \in \hat{\mathcal{G}}$ and find the corresponding vector field $(K_V : M \to T(M))$ in the phase space (M):

$$\begin{aligned} K_V(S,c;l,k) :&= \tfrac{d}{dt}(g(t)Sg(t)^{-1}|c + (g(t)^{-1}\partial g/\partial x(t)|S); g(t)lg(t)^{-1} \\ &- k\partial g(t)/\partial x g(t)^{-1}, k)\Big|_{t=0} = ([V,S], (\partial V/\partial x|S)[V,l] - k\partial V/\partial x, 0), \end{aligned} \tag{26}$$

driven by a Hamiltonian function $(H_V : M \to \mathbb{C})$, owing to the determining relationship $(-dH_V = i_{K_V}\omega^{(2)})$:

$$\begin{aligned} -dH_V &= -(\partial H/\partial l|dl) - (\partial H_V/\partial S|dS) + \partial H_V/\partial k dk + \partial H_V/\partial c dc \\ &= ([V,l] - k\partial V/\partial x|dS) - (dl|[V,S]) - (\partial V/\partial x|S)dk. \end{aligned}$$

As a consequence of (27), one obtains the following for any point $((S,k;l,c) \in M)$.

$$\begin{aligned} &\partial H_V/\partial l = [V,S], \qquad \partial H_V/\partial S = k\partial V/\partial x - [V,l], \\ &\partial H_V/\partial k = (\partial V/\partial x|S), \qquad \partial H_V/\partial c = 0 \end{aligned}$$

From (27), it follows that

$$H_V = ([S,l] - k\partial S/\partial x|V) := (V|p(S,c;l,k)|V) \tag{27}$$

is linear with respect to the generating element $(V \in \tilde{\mathcal{G}})$. This means that the loop-group $\tilde{G}$ action on the symplectic manifold (M) is Hamiltonian by definition [20–22]. There is a corresponding mapping $(p : M \to \hat{\mathcal{G}}^*)$, where

$$p(S,c;l,k) = [S,l] - k\partial S/\partial x, \tag{28}$$

is called the momentum mapping [20–22,37], which, in general, is fixed to the phase space (M) within the classical Marsden–Weinstein reduction scheme [20,21]. We now proceed to describing the related symplectic structure on the ξ- level submanifold for a fixed element $(\xi \in \hat{\mathcal{G}}^*)$.

$$M_\xi := \left\{(S,c;l,k) \in M : \quad [S,l] - k\partial S/\partial x = \xi \in \hat{\mathcal{G}}^*\right\} \tag{29}$$

The related isotropy subgroup $(\hat{G}_\xi \subset \hat{G})$ is defined via the condition $\mathrm{Ad}^*_g\xi = 0$ for all elements $(g \in \hat{G}_\xi)$. If we additionally assume that $[\xi, l] = 0$, one easily obtains $\hat{G}_\xi \simeq \hat{G}$. To proceed further, some additional properties of the submanifold $(M_\xi \subset M)$ are needed, which are described in the next section. □

3.2. The Marsden–Weinstein Reduction and Related Courant Algebroid Structure

In this section, we describe the reduced submanifold $(M_\xi \subset M)$ parameterized by the corresponding phase-space $(\bar{M}_\xi := M_\xi/\hat{G}_\xi)$ points. It is known [20,21,43] that this parameterization uniquely determines the points $((\bar{S},\bar{c};\bar{l},\bar{k}) \in M_\xi \subset M,)$ which are invariant with respect to the constructed loop-group $\hat{G}$ action ((19) and (23)). The latter makes it possible [20,21,37,43–47] to construct in the phase space $(\bar{M}_\xi)$ the reduced and suitably non-degenerate symplectic structure ensuing from the symplectic structure on the submanifold (M_ξ). Let us consider the point $((\bar{S}, \bar{c}; \bar{l}, \bar{k}) \in M_\xi)$ where the elements $\bar{S} \in \hat{\mathcal{G}}_\xi$, $\bar{k} \in \mathbb{C}$ satisfy the following differential expressions according to the definition of (29):

$$[\bar{S}, \bar{l}] - \bar{k}\partial\bar{S}/\partial x, \qquad \partial\bar{k}/\partial x = 0, \tag{30}$$

for all $x \in \mathbb{S}^1$. Now, consider a Hamiltonian vector field $(-\bar{k}\partial/\partial\tau, \tau \in \mathbb{C})$ on the submanifold (M_ξ) generated by the element $X = \bar{l} \in \hat{\mathcal{G}}^*$, owing to the following expressions:

$$-\bar{k}\partial\bar{S}/\partial\tau = [\bar{l}, \bar{S}] = -[\bar{S}, \bar{l}] = -\bar{k}\partial\bar{S}/\partial x, \qquad -\bar{k}\partial\bar{l}/\partial\tau = \bar{k}\partial\bar{l}/\partial x. \tag{31}$$

In particular, from (31), it follows that the equality $(\frac{\partial}{\partial\tau} = \frac{\partial}{\partial x})$ holds in the reduced phase space ($\bar{M}_\xi$). Moreover, to compute the evolution of the element $\bar{c} \in \mathbb{C}$ with respect to this vector field ($\partial/\partial\tau$ on $\bar{M}_\xi$:)

$$\begin{aligned} -\bar{k}\partial\bar{c}/\partial\tau &= (\partial\bar{l}/\partial x|\bar{S}) = -(\bar{l}|\partial\bar{S}/\partial x) \\ &= -(\bar{l}|\bar{k}^{-1}[\bar{S}, \bar{l}]) = \bar{k}^{-1}([\bar{l}, \bar{l}]|\bar{S}) = 0, \end{aligned} \tag{32}$$

coinciding with the *a priori* accepted condition $(\partial\bar{c}/\partial x = 0)$ for any $x \in \mathbb{S}^1$. Reasoning similarly as above, for the vector field $\partial/\partial t$, $t \in \mathbb{C}$, in the reduced phase space ($\bar{M}_\xi$) generated by the Lie algebra element $(q(\bar{l}) \in \hat{\mathcal{G}}_\xi)$, depending on the basis element $\bar{l} \in \hat{\mathcal{G}}^*_\xi$, one obtains

$$\begin{aligned} &\partial\bar{S}/\partial t = [q(\bar{l}), \bar{S}], && \partial\bar{l}/\partial t = [q(\bar{l}), \bar{l}] - \bar{k}\partial\bar{l}/\partial x, \\ &\partial\bar{c}/\partial t = (\partial q/\partial x(\bar{l})|\bar{T}), && \partial\bar{k}/\partial t = 0. \end{aligned} \tag{33}$$

The latter, in particular, means that the flows ($\partial/\partial t$ and $\partial/\partial x$) in the reduced phase space ($\bar{M}_\xi$) possess a countable set $(\gamma_n(\bar{l}) := \mathrm{tr}\bar{S}^n(\bar{l}),\ n \in \mathbb{Z})$ of conservation lows, where by definition, the element $\bar{S}(\bar{l}) \in \hat{\mathcal{G}}_\xi$ satisfies the determining equation for a given element $(\bar{l} \in \hat{\mathcal{G}}^*_\xi)$ for all $x \in \mathbb{S}^1$:

$$-\bar{k}\partial\bar{S}/\partial x(\bar{l}) = [\bar{l}, \bar{S}(\bar{l})] \tag{34}$$

From Equations (34), it follows that in the reduced phase space ($\bar{M}_\xi$),

$$\begin{aligned} \partial\bar{c}/\partial t &= (\partial q(\bar{l})/\partial x|\bar{S}) = \bar{k}^{-1}([q(\bar{l}), \bar{l}] - \partial\bar{l}/\partial t|\bar{S}) = \\ &= \bar{k}^{-1}([q(\bar{l}), \bar{l}]|\bar{S}) - \bar{k}^{-1}(\partial\bar{l}/\partial t|\bar{S}) = \bar{k}^{-1}(\bar{l}) - \bar{k}^{-1}(\partial\bar{l}/\partial t|\bar{S}) \\ &= -\bar{k}^{-1}(\bar{l}|\partial\bar{S}/\partial t) - \bar{k}^{-1}(\partial\bar{l}/\partial t|\bar{S}) \\ &= -\bar{k}^{-1}\tfrac{\partial}{\partial t}(\bar{l}|\bar{S}). \end{aligned}$$

Thus, from the t evolution (35) of the parameter $(\bar{c} \in \mathbb{C})$, one obtains the following constraint on the reduced phase space ($\bar{M}_\xi$) subject to the vector field ($\partial/\partial t$) generated by the element $q(\bar{l}) \in \hat{\mathcal{G}}_\xi$.

$$\bar{c} = -\bar{k}^{-1}(\bar{l}|\bar{S}) \tag{35}$$

Moreover, as it is easy to observe, these two vector fields ($\partial/\partial\tau$ and $\partial/\partial t$) in the reduced phase space ($\bar{M}_\xi$) commute:

$$[\partial/\partial t, \partial/\partial\tau] = 0. \tag{36}$$

The latter makes it possible to derive the following differential relationship from the conditions of (36):

$$-\bar{k}\partial\bar{F}/\partial x = \bar{l}\,\bar{F}, \tag{37}$$

imposed on the reduced matrix $(\bar{l} \in \hat{\mathcal{G}}^*_\xi)$ adjoint with the linear evolution Equation (34) and augmented with the following compatible differential equation for the matrix $\bar{F} \in \hat{\mathcal{G}}_\xi$:

$$\bar{F}_t = q(\bar{l})\,\bar{F} \tag{38}$$

Equations (37) and (38) represent the well-known [22,37,39–41,48–50] generalized Lax-type spectral problem, allowing for the investigation of the mentioned above differential relationships by means of either the inverse scattering or the inverse spectral transform method [39,49–51], as well as by using algebraic geometry methods [49,50] and their modern Lie-algebraic generalizations [48].

To achieve this aim more analytically, we need to describe the evolution of the vector field ($\partial/\partial t$) in the reduced phase space ($\bar{M}_\xi$) in more detail, subject to its dependence on the phase-space element ($\bar{l} \in \hat{\mathcal{G}}^*_\xi$). As the vector fields $\frac{\partial}{\partial t}$ and $\partial/\partial x$ satisfy the commutativity condition (36) on the reduced manifold (M_ξ), we apply the classical Marsden–Weinstein reduction scheme [43] to our symplectic manifold (M), with the fixed-moment mapping value ($\xi \in \hat{\mathcal{G}}^*$) for computing the resulting Poisson bracket ($\{(X|\bar{S}),(Y|\bar{S})\}_\xi$) of the smooth functions $(X|\bar{S})$ and $(Y|\bar{S}) \in \mathcal{D}(\bar{M}_\xi)$ in the reduced phase space ($\bar{M}_\xi$) parameterized by arbitrarily chosen $X, Y \in \hat{\mathcal{G}}^*_\xi$. It can be shown [28,37,45,52] that this Poisson bracket on $\bar{M}_\xi$ expressed as

$$\{(X|\bar{S}),(Y|\bar{S})\}_\xi = \{(X|\bar{S}),(Y|\bar{S})\}\big|_{\bar{M}_\xi} - (\xi|[V_X, V_Y])\big|_{\bar{M}_\xi}, \tag{39}$$

where, by definition, the mappings $V_X, V_Y : \bar{M}_\xi \to \hat{\mathcal{G}}_\xi$ are the solutions to the following relationship:

$$(\xi|[Z, V_X]) = K_Z(X|\bar{S}), \qquad (\xi|[Z, V_Y]) = K_Z(Y|\bar{S}), \tag{40}$$

which holds for any $Z \in \hat{\mathcal{G}}^*_\xi$. The functions $(X|\bar{S}),(Y|\bar{S}) \in \mathcal{D}(\bar{M}_\xi)$ can be extended to those in the whole phase space (M_ξ) in such a way that their restrictions on the submanifold ($M_\xi \subset M$) are $\hat{G}$-invariant. To apply the Marsden–Weinstein reduction scheme, (we construct such a group element $g(l) \in \hat{G}$) that for an arbitrarily chosen $l \in \hat{\mathcal{G}}^*_\xi$, the following gauge expression holds and satisfies the normalization condition ($g(\bar{l}) = \mathrm{Id} \in \hat{G}$):

$$l = g(l)\bar{l}g(l)^{-1} - \bar{k}\partial g/\partial x(l)g(l)^{-1} \tag{41}$$

Having now considered the function

$$f_X := (g(l)Xg(l)^{-1}|S), \tag{42}$$

for some elements suitably extended on the whole manifold (M_ξ) (elements $X \in \hat{\mathcal{G}}^*_\xi$ and $S \in \hat{\mathcal{G}}_\xi$), one can observe that $f_X|_{\bar{M}_\xi} = (X|\bar{S})$, which is, by construction, $\hat{G}$-invariant. The latter means that $f_X \in \mathcal{D}(M_\xi)$ for any $l \in \hat{\mathcal{G}}^*_\xi$ and for any $a \in \hat{G}_\xi \simeq \hat{G}$

$$\begin{aligned} a \circ f_X :&= (g(a \circ l)Xg(a \circ l)^{-1}|a \cdot S) \\ &= (ag(l)Xg(l)^{-1} \cdot a^{-1}|aSa^{-1}) \\ &= (g(l)Xg(l)^{-1}|S) = f_X, \end{aligned} \tag{43}$$

where we use the invariance property ($g(a \circ l) = ag(l),\ l \in \hat{\mathcal{G}}^*_\xi$) following from expressions (41) and (23):

$$\begin{aligned} a \circ l = ala^{-1} - \bar{k}\partial a/\partial x a^{-1} &= a(g(l)\bar{l}g(l)^{-1} - \bar{k}\partial g/\partial x(l)g(l)^{-1})a^{-1} - \bar{k}\partial a/\partial x a^{-1} \\ &= ag(l)\bar{l}(ag(l))^{-1} - \bar{k}a\partial g/\partial x(l)g(l)^{-1}a^{-1} - \bar{k}\partial a/\partial x a^{-1} \\ &= ag(l)\bar{l}(ag(l))^{-1} - \bar{k}(a\partial g(l))/\partial x(ag(l))^{-1} \\ &= g(a \circ l)\bar{l}g(a \circ l)^{-1} - \bar{k}\partial g/\partial x(a \circ l)g(a \circ l)^{-1}, \end{aligned} \tag{44}$$

which holds for any $a \in \hat{G}_\xi$ and $l \in \hat{\mathcal{G}}_\xi^*$. Returning to the Poisson bracket (39), we can replace the functions $(X|\bar{S})$ and $(Y|\bar{S}) \in \mathcal{D}(\bar{M}_\xi)$ with their $\hat{G}_\xi$-invariant extensions ($f_X \in \mathcal{D}(M_\xi)$). Before calculating the corresponding Poisson bracket

$$\{\bar{f}_X, \bar{f}_Y\}_\xi = \{\bar{f}_X, \bar{f}_Y\}|_{\bar{M}_\xi} - (\xi|[V_X, V_Y]) = \{f_X, f_Y\}|_{\bar{M}_\xi} - K_{V_X} f_Y|_{\bar{M}_\xi}, \tag{45}$$

where $K_{V_X} : M_\xi \to T(M_\xi)$ is the vector field generated on M_ξ by the element $V_X \in \hat{\mathcal{G}}_\xi$, we need to calculate the action $(K_Z f_Y)$ for any element $(Z \in \hat{\mathcal{G}}_\xi^*)$. From (43), one easily finds that

$$\begin{aligned} K_Z f_Y &= \tfrac{d}{d\varepsilon}\big(g(\exp(\varepsilon Z) \circ l) Y g(\exp(\varepsilon Z) \circ l)^{-1} | \exp(\varepsilon Z) S \exp(-\varepsilon Z)\big)|_{\varepsilon=0} \\ &= (g(l)[g(l)^{-1} g'(l)([Z, l] - \bar{k}\partial Z/\partial x) - g(l)^{-1} Z g(l), Y] g(l)^{-1} | S). \end{aligned} \tag{46}$$

holds on the submanifold (M_ξ). Thus, in the reduced phase space $(\bar{M}_\xi)$, the general expression (46) implies

$$K_{V_X} f_Y|_{\bar{M}_\xi} = ([g'(\bar{l}) \cdot ([V_x, \bar{l}] - \bar{k}\tfrac{\partial}{\partial x} V_x) - V_X, Y]|\bar{S}). \tag{47}$$

Thus, owing to the relationships $\{f_X, f_Y\} = -\omega^{(2)}(K_{V_X}, K_{V_Y})$ and (47), the Poisson bracket (45) becomes

$$\begin{aligned} \{(X|\bar{S}), (Y|\bar{S})\}_\xi &= ([g'(\bar{l})(Y), X] + [Y, g'(\bar{l})(X)]|\bar{S}) \\ &\quad - \Big([g'(\bar{l})([V_X, \bar{l}] - \bar{k}\tfrac{\partial}{\partial x} V_X) - V_X, Y]|\bar{S}\Big) \\ &= ([g'(\bar{l})(Y), X] + [Y, g'(\bar{l})(X)]|\bar{S}), \end{aligned} \tag{48}$$

taking into account that, owing to (40) and (47):

$$\Big([g'(\bar{l})([V_X, \bar{l}] - \bar{k}\tfrac{\partial}{\partial x} V_X) - V_X, Y]|\bar{S}\Big) = K_{V_X} f_Y = (\xi|[K_{V_X}, V_Y])\big|_{\xi=0} = 0.$$

Now, one can rewrite the Poisson bracket (48) as

$$\{(X|\bar{S}), (Y|\bar{S})\}_\xi = ([X, Y]_{\mathrm{D}}|\bar{S}), \tag{49}$$

where, by definition, the classical D structure is introduced in the linear space $(\hat{\mathcal{G}}_\xi^*)$:

$$[X, Y]_{\mathrm{D}} := ad^*_{\mathrm{D}(X)} Y - ad^*_{\mathrm{D}(Y)} X, \tag{50}$$

where $X, Y \in \hat{\mathcal{G}}_\xi^*$, and the linear homomorphism $(\mathrm{D} : \hat{\mathcal{G}}_\xi^* \to \hat{\mathcal{G}}_\xi)$ is defined as

$$\mathrm{D}(X) := -g'(\bar{l})(X). \tag{51}$$

The constructed mapping (51) should obviously satisfy [28,52] the well-known Jacobi-type condition for any X, Y and $Z \in \hat{\mathcal{G}}_\xi^*$.

$$([X, [\mathrm{D}(Y), \mathrm{D}(Z)] - \mathrm{D}[Y, Z]_{\mathrm{D}}]|\bar{S}) + ([X, \{(\bar{S}|Y), (\bar{S}|Z)\}_\xi]|\bar{S}) + \text{cycles} = 0 \tag{52}$$

We now take into account that the mapping $g : \hat{\mathcal{G}}_\xi^* \to \hat{G}$ satisfies relationship (41), implying [53] the following determining differential expression for any $X \in \hat{\mathcal{G}}_\xi^*$ depending on a chosen reduction $(\hat{\mathcal{G}} \ni l \to \bar{l} \in \hat{\mathcal{G}}_\xi^*)$:

$$[g'(\bar{l})(X), \bar{l}] - \bar{k}\tfrac{\partial}{\partial x} g'(\bar{l})(X) = X \tag{53}$$

The D mapping introduced above (51) satisfies the following additional relationship:

$$S(l) = g(l)\bar{S}(\bar{l})g(l)^{-1}, \tag{54}$$

which can be obtained from the group $\hat{G}$ action on element $S(\bar{l}) \in \hat{\mathcal{G}}_{\xi}$ and following naturally from (41). From the differentiation of (54) with respect to $l \in \hat{\mathcal{G}}^*_{\xi}$ at point $l \to \bar{l} \in \hat{\mathcal{G}}^*_{\xi}$, one obtains the following additional commutator expression for an arbitrary element ($X \in \hat{\mathcal{G}}^*_{\xi}$):

$$S'(\bar{l})(X) = [g'(\bar{l})(X), \bar{S}(\bar{l})] \tag{55}$$

Moreover, since matrix (54) satisfies relationship (34), its differentiation with respect to $\bar{l} \in \hat{\mathcal{G}}^*_{\xi}$ entails the following differential expression:

$$\bar{k}\frac{\partial}{\partial x}S'(\bar{l})(X) + [\bar{l}, S'(\bar{l})(X)] = [\bar{S}(\bar{l}), X], \tag{56}$$

which holds for any $X \in \hat{\mathcal{G}}^*_{\xi}$. Now, we take into account that the matrix $\bar{S}(\bar{l})(z;\lambda) \in \hat{\mathcal{G}}_{\xi}$ coincides exactly with the monodromy matrix ($\bar{F}(z+2\pi;z;\lambda)$, $z \in \mathbb{R}$, $\lambda \in \mathbb{C}$) of the linear periodic problem (37). Using direct calculations, one can verify the following commutator tensor expression:

$$\begin{aligned}
&\{\bar{S}(\bar{l})(z;\lambda) \overset{\otimes}{,} \bar{S}(\bar{l})(z;\mu)\}_{\xi} = \qquad (57)\\
&= \int_z^{z+2\pi} dx \int_z^{z+2\pi} dy \{F(z+2\pi, x;\lambda)\bar{l}(x;\lambda)F(x,z;\lambda) \overset{\otimes}{,} F(z+2\pi, y;\mu)\bar{l}(y;\mu)F(y,z;\mu)\}_{\xi} = \\
&= \int_z^{z+2\pi} dx \int_z^{z+2\pi} dy \{(F(z+2\pi, x;\lambda)\otimes \mathbb{I})(\bar{l}(x;\lambda)\otimes \mathbb{I})(F(x,z;\lambda)\otimes \mathbb{I}, \qquad (58)\\
&\mathbb{I}\otimes F(z+2\pi, y;\mu)(\mathbb{I}\otimes \bar{l}(y;\mu))(\mathbb{I}\otimes F(y,z;\mu))\}_{\xi} = \\
&= \int_z^{z+2\pi} dx \int_z^{z+2\pi} dy F(z+2\pi, x;\lambda)\otimes F(z+2\pi, y;\mu)\{\bar{l}(x;\lambda) \overset{\otimes}{,} \bar{l}(y;\mu)\}_{\xi} F(x,z;\lambda)\otimes F(y,z;\mu) \\
&\int_z^{z+2\pi} dx \int_z^{z+2\pi} dy F(z+2\pi, x;\lambda)\otimes F(z+2\pi, y;\mu)\bar{\omega}(\lambda,\mu;x,y)F(x,z;\lambda)\otimes F(y,z;\mu), \qquad (59)
\end{aligned}$$

where $z \in \mathbb{S}^1$, $\lambda, \mu \in \mathbb{C}$, and, by definition,

$$\{\bar{l}(x;\lambda) \overset{\otimes}{,} \bar{l}(y;\mu)\}_{\xi} := \bar{\omega}(\lambda,\mu;x,y) = \sum_{i,k=0}^{N} \bar{\omega}_{ik}(\lambda,\mu;x,y)(\partial/\partial x)^i(\partial/\partial y)^k \delta(x-y). \tag{60}$$

for any $x, y \in \mathbb{S}^1$. Here, the locally defined functional matrices ($\bar{\omega}_{ik}(\lambda,\mu;x,y) \in \hat{\mathcal{G}}^*_{\xi} \otimes \hat{\mathcal{G}}^*_{\xi}$, $i,k = \overline{0,N}$) are assumed to satisfy the antisymmetry property for all $i,k = \overline{1,N}$, $x,y \in \mathbb{S}^1$, $\lambda,\mu \in \mathbb{C}$,

$$P\bar{\omega}_{ik}(\lambda,\mu;x,y)P = -\bar{\omega}_{ki}(\mu,\lambda;x,y), \tag{61}$$

and the permutation operator ($P : \hat{\mathcal{G}}^*_{\xi} \otimes \hat{\mathcal{G}}^*_{\xi} \to \hat{\mathcal{G}}^*_{\xi} \otimes \hat{\mathcal{G}}^*_{\xi}$) acts as $PX \otimes YP := Y \otimes X$ for any $X, Y \in \hat{\mathcal{G}}^*_{\xi}$. Similarly to calculations reported in [39,54,55], (73) yields

$$\begin{aligned}
\{\bar{S}(z;\lambda) \overset{\otimes}{,} \bar{S}(z;\mu)\}_{\xi} = &\int_z^{z+2\pi} dx \bar{F}(z+2\pi, x;\lambda)\otimes \\
&\otimes \bar{F}(z+2\pi, x;\mu)\bar{\Omega}(\lambda,\mu;x)\bar{F}(x,z;\lambda)\otimes \bar{F}(x,z;\mu)
\end{aligned} \tag{62}$$

for a matrix ($\bar{\Omega}(\lambda,\mu;x) \in \hat{\mathcal{G}}_{\xi}^{*} \otimes \hat{\mathcal{G}}_{\xi}^{*}$) for all $\lambda,\mu \in \mathbb{C}, x \in \mathbb{S}^1$, depending only on the element ($\bar{l} \in \hat{\mathcal{G}}_{\xi}^{*}$). It is worth observing that the Poisson bracket (48) can be rewritten equivalently as

$$\{\bar{S}(\bar{l})(\lambda) \overset{\otimes}{,} \bar{S}(\bar{l})(\mu)\}_{\xi} = [\bar{R}(\lambda,\mu), \bar{S}(\bar{l})(\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes \bar{S}(\bar{l})(\mu)], \tag{63}$$

where $\bar{R} \in \hat{\mathcal{G}}_{\xi}^{*} \otimes \hat{\mathcal{G}}_{\xi}^{*}$ denotes a tensor mapping corresponding to the linear homomorphism ($\mathrm{D} : \tilde{\mathcal{G}}_{\xi}^{*} \to \tilde{\mathcal{G}}_{\xi}$) constructed above. Expression (63) allows for the follow compact representation:

$$\{\bar{S}(z;\lambda) \overset{\otimes}{,} \bar{S}(z;\mu)\}_{\xi} = R(\lambda,\mu;z)\bar{S}(z;\lambda) \otimes \bar{S}(z;\mu) - \bar{S}(z;\lambda) \otimes \bar{S}(z;\mu)R(\lambda,\mu;z), \tag{64}$$

if the kernel ($R(\lambda,\mu;x)$) of the tensor ($R \in \hat{\mathcal{G}}_{\xi}^{*} \otimes \hat{\mathcal{G}}_{\xi}^{*}$) satisfies the following differential relationship for all $x \in \mathbb{S}^1$ and $\lambda,\mu \in \mathbb{C}$:

$$\tfrac{\partial}{\partial x} R(\lambda,\mu;x) + [R(\lambda,\mu;x),\ \bar{l}(x;\lambda) \otimes \mathbb{I} + \mathbb{I} \otimes \bar{l}(x;\mu)] = \bar{\Omega}(\lambda,\mu;x). \tag{65}$$

Moreover, to define the related mapping ($R : \hat{\mathcal{G}}_{\xi} \to \hat{\mathcal{G}}_{\xi}$) as

$$RA := \operatorname*{res}_{\mu=0} \int_0^{2\pi} dy R(\lambda,\mu;y)\delta(x-y)A(y;\mu) \tag{66}$$

for any $A \in \hat{\mathcal{G}}_{\xi}$, the differential relationship (65) can be easily rewritten in the following operator form:

$$-(A|\partial R/\partial x B) + (\bar{l}|[A,B]_R) = (A|RB), \tag{67}$$

which holds for any $A, B \in \hat{\mathcal{G}}_{\xi}$, where the expression

$$[A,B]_R := [-R^* A, B] + [A, RB] \tag{68}$$

denotes a new so-called deformed Lie structure in the linear space ($\hat{\mathcal{G}}_{\xi}^{*}$). The result (67) can be also used to rewrite the Poisson bracket (64) as

$$\begin{aligned} \{(X|\bar{S}(\bar{l})),(Y|\bar{S}(\bar{l}))\}_{\xi} &= (\bar{l}|[\bar{F}X\bar{F}_{2\pi}, \bar{F}Y\bar{F}_{2\pi}]_R) - \left(\bar{F}X\bar{F}_{2\pi}|\tfrac{\partial}{\partial x}R(\bar{F}Y\bar{F}_{2\pi})\right) \\ = (\bar{l}|[\nabla(X|\bar{S})(\bar{l}), \nabla(Y|\bar{S})(\bar{l})]_R) &+ \left(\tfrac{\partial}{\partial x}\nabla(X|\bar{S})(\bar{l})|R(\nabla(Y|\bar{S})(\bar{l}))\right) \\ &- \left(R^*(\nabla(X|\bar{S})(\bar{l}))|\tfrac{\partial}{\partial x}(\nabla(Y|\bar{S})(\bar{l}))\right), \end{aligned} \tag{69}$$

where, by definition, $\bar{F} := \bar{F}(\bar{l})(x,y;\lambda)$, $\bar{F}_{2\pi} := \bar{F}(\bar{l})(y+2\pi,x;\lambda) \in \bar{G}$, $x,y \in \mathbb{S}^1, \lambda \in \mathbb{C}$, and the gradients ($\nabla(X|\bar{S})(\bar{l})$ and $\nabla(Y|\bar{S})(\bar{l}) \in \hat{\mathcal{G}}_{\xi}^{*}$) are defined in the standard way as

$$(Z|\nabla f(\bar{l})) := \tfrac{d}{d\varepsilon} f(\bar{l}+\varepsilon Z)\Big|_{\varepsilon=0} \tag{70}$$

for any smooth functional ($f \in \mathcal{D}(\hat{\mathcal{G}}_{\xi}^{*})$) and arbitrary $Z \in \hat{\mathcal{G}}_{\xi}^{*}$. It is also easy to observe that under the assumed antisymmetry condition described above ($R^* = -R$), the right-hand side of (69) coincides with the Lie–Poisson bracket [37,39,43,48,50] for functionals $(X|\hat{\mathcal{G}}_{\xi}^{*})$ and $(Y|\hat{\mathcal{G}}_{\xi}^{*}) \in \mathcal{D}(\hat{\mathcal{G}}_{\xi}^{*})$ in the adjoint space ($\hat{\mathcal{G}}_{\xi}^{*}$) with respect to the deformed commutator structure ($[\cdot,\cdot]_R$) on the centrally extended Lie algebra $\hat{\mathcal{G}}_{\xi}$: for any $(A,\alpha),(B,\beta) \in \hat{\mathcal{G}}_{\xi}^{*}$, $\alpha,\beta \in \mathbb{C}$, with the Lie bracket

$$[(A,\alpha),(B,\beta)]_R := ([A,B]_R, (R(B)|\partial A/\partial x) - (R(A)|\partial B/\partial x)). \tag{71}$$

Thus, owing to (68), the classical R structure

$$[A, B]_R := [RA, B] + [A, RB] \tag{72}$$

on the Lie algebra ($\hat{\mathcal{G}}_\xi$) under the conditions formulated above based on the mapping $R : \hat{\mathcal{G}}_\xi \to \hat{\mathcal{G}}_\xi$ generates a new Lie algebra structure on $\hat{\mathcal{G}}^*_\xi$. The obtained result can be formulated as the following proposition.

Proposition 2. *The constructed reduced canonical Poisson structure in the phase space ($\bar{M}_\xi$) for the monodromy matrix ($\bar{S}(\bar{l}) \in \hat{\mathcal{G}}_\xi$) exactly coincides with the corresponding classical Lie–Poisson bracket on the centrally extended Lie algebra ($\hat{\mathcal{G}}_\xi$) subject to the antisymmetric R structure (71).*

The Poisson bracket (72), satisfying the Jacobi identity, makes it possible to define a dual Lie–Poisson bracket in the adjoint space ($\hat{\mathcal{G}}^*_\xi$) via the following canonical expression at an element ($\bar{l} \in \hat{\mathcal{G}}^*_\xi$):

$$\{(\bar{l}|A), (\bar{l}|B)\}_\xi = (\bar{l}|[[A, B]_R) + (\partial A/\partial x|RB) - (\partial B/\partial x|RA) \tag{73}$$

If the antisymmetry property for the mapping ($R : \hat{\mathcal{G}}_\xi \to \hat{\mathcal{G}}_\xi$) does not hold, the generated Lie–Poisson-type bracket in the functional space ($\mathcal{D}(\hat{\mathcal{G}}^*_\xi)$) can be defined as follows, owing to (69): for any $f, g \in \mathcal{D}(\hat{\mathcal{G}}^*_\xi)$, the bracket is

$$\{f(\bar{l}), g(\bar{l})\}_\xi := (\bar{l}|[\nabla f(\bar{l}), \nabla g(\bar{l})]_R) + \left(\tfrac{\partial}{\partial x}\nabla f(\bar{l})|R\nabla g(\bar{l})\right) - \left((R\nabla f(\bar{l}))|\tfrac{d}{dx}\nabla g(\bar{l})\right) \tag{74}$$

where the generalized R structure ($[\cdot,\cdot]_R$) on $\hat{\mathcal{G}}_\xi$ is given by expression (68). The results obtained above can be summarized as the following theorem.

Theorem 1. *The Poisson bracket (48) in the reduced phase space ($\bar{M}_\xi$) represented by a* D *structure (49) in the linear space ($\hat{\mathcal{G}}_\xi$) naturally generated by the gauge transformation (41) reduced on the element $\bar{l} \in \hat{\mathcal{G}}^*_\xi$ is uniquely defined on $\bar{M}_\xi$ and generates the anchor mapping($\rho_\xi := \mathrm{D} : \hat{\mathcal{G}}^*_\xi \to \hat{\mathcal{G}}_\xi$) in the bundle ($\hat{\mathcal{G}}_\xi \times \hat{\mathcal{G}}_\xi$), satisfying the Lie algebra homomorphism property for any $X, Y \in \hat{\mathcal{G}}^*_\xi$, thus determining a Courant type algebroid.*

$$\mathrm{D}[X, Y]_\mathrm{D} = [\mathrm{D}X, \mathrm{D}Y] \tag{75}$$

Proof. One needs only to argue the homomorphism property (75) following from the dual tensor form of the Poisson bracket (48) in the space ($\hat{\mathcal{G}}_\xi \otimes \hat{\mathcal{G}}_\xi$), which holds for arbitrary $\lambda, \mu \in \mathbb{C}$, where $R(\lambda, \mu) \in \hat{\mathcal{G}}_\xi \otimes \hat{\mathcal{G}}_\xi$ denotes the tensor form of the D mapping ($\mathrm{D} : \hat{\mathcal{G}}^*_\xi \to \hat{\mathcal{G}}_\xi$) constructed above. As this mapping ($\mathrm{D} : \hat{\mathcal{G}}^*_\xi \to \hat{\mathcal{G}}_\xi$) is uniquely represented as a tensor ($D \in \hat{\mathcal{G}}_\xi \otimes \hat{\mathcal{G}}_\xi$), the latter can be split into non-degenerate symmetric ($\mu \in \hat{\mathcal{G}}_\xi \otimes \hat{\mathcal{G}}_\xi$) and antisymmetric ($\eta \in \hat{\mathcal{G}}_\xi \otimes \hat{\mathcal{G}}_\xi$) parts. We now take into account that the composed mapping ($R := \eta \circ \mu^{-1} : \hat{\mathcal{G}}_\xi \to \hat{\mathcal{G}}_\xi$) generates the second Lie bracket

$$[V_1, V_2]_R := [RV_1, V_2] + [V_1, RV_2], \tag{76}$$

exactly coinciding with that of (72), owing to relationship (52), thus proving the homomorphism property (75). □

It is also worth remarking that the trace operation applied to the Poisson bracket (63) causes it to vanish in the phase space ($\bar{M}_\xi$) for the functionals $\mathrm{tr}\bar{S}(\bar{l})(\lambda)$ and $\mathrm{tr}\bar{S}(\bar{l})(\mu)$ for arbitrary $\lambda, \mu \in \mathbb{C}$, describing the complete set of Casimir functionals [39,48] of the coadjoint action of the isotropy group ($\hat{G}_\xi$) in the adjoint space $\hat{\mathcal{G}}^*_\xi$.

4. Remarks on the Courant-Type Algebroid Foliation and Related Hamiltonian Flows

Let $(E;[[\cdot,\cdot]],\rho)$, $E := \mathcal{A}(M)$ be a Courant-type algebroid over a manifold (M) for which the characteristic space $(\rho(E_x) \subset T_x(M), x \in M)$ is involutive and finitely generated. On the algebroid $(E;[[\cdot,\cdot]],\rho)$, the external differential of $d_E : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$ is naturally defined, where $\Lambda(E^*) := \oplus_{k\in\mathbb{Z}_+} \wedge^k(E^*)$, as follows:

$$\begin{aligned}\left(d_E\alpha^{(k)}\right)(A_0,\ldots,A_k) : = &\sum_{i=0}^{k}(-1)^i\rho(A_i)_E\alpha^{(k)}((A_0,\ldots,\hat{A}_i,\ldots,A_k)\\ &+\sum_{i<j=0}^{k}(-1)^{i+j}\alpha^{(k)}([[A_i,A_j]],A_0,\ldots,\hat{A}_i,\ldots,\hat{A}_j,\ldots,A_k),\end{aligned} \tag{77}$$

for $\alpha^{(k)} \in \Gamma(\Lambda^k(E^*))$ and arbitrary $A_i \in \Gamma(E)$, $i = \overline{0,k}$, and $k \in \mathbb{Z}_+$, satisfying the natural algebraic cohomology complex property $(d_Ed_E = 0)$. Differential (77) makes it possible to also determine the Lie derivative along a section $(A \in \Gamma(E))$.

$$\mathcal{L}_A^{(E)} := i_Ad_E + d_Ei_A \tag{78}$$

It is obvious that for $E = T^*(M) \times T(M)$, $\rho := pr_{T(M)}$ and $A \in \Gamma(T^*(M) \times T(M))$, the external differential (d_E) transits into the usual differential $(d \circ pr_{\Lambda(M)} : T(M) \times \Lambda(M) \to \Lambda(M))$.

Let a two-form $\omega^{(2)} \in \Gamma(\Lambda^2(E))$ be closed and invariant with respect to a vector $(K \in \Gamma(E))$, that is,

$$d_E\omega^{(2)} = 0, \qquad \mathcal{L}_K^{(E)}\omega^{(2)} = 0 \tag{79}$$

on M. The latter simply means that there exists a locally defined smooth function $(H_1 : \Gamma(E^*) \to \mathbb{R})$ such that

$$i_K\omega^{(2)} = -d_EH_1. \tag{80}$$

If the function $H_1 : \Gamma(E^*) \to \mathbb{R}$ is defined globally, then the vector $K \in \Gamma(E)$ is called a *Hamiltonian flow* in bundle E.

We now consider a coordinate vector set $(e := \{e^i \in \Gamma(E^*) : i = \overline{1,m}\})$ in the vector bundle $(\Gamma(E))$ and the corresponding basis of its differentials $(\{d_Ee^i \in \Gamma(T^*(E^*)) : i = \overline{1,m}\})$ and take a linear and invertible subject to the second component mapping $(Q_E : \Gamma(E^*) \times \Gamma(T^*(E^*)) \to \Gamma(T^*(E^*)))$, determining the following elements by means of mappings $(Q_E(e)_j \in \mathrm{End}\,\Gamma(E^*), j = \overline{1,m})$.

$$d_E^*e = \sum_{j=\overline{1,m}} Q_E(e)_jd_Ee^j \in \Gamma(T^*(E^*)) \tag{81}$$

Expression (81) makes it possible to define a second external differential $(d_E^* : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*)))$, satisfying the algebraic cohomology complex property $(d_E^*d_E^* = 0)$ and anticommuting to the external deformed differential $(d_E : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*)))$:

$$d_Ed_E^* + d_E^*d_E = 0. \tag{82}$$

As the symplectic two-form $\omega^{(2)} \in \Gamma(\Lambda^2(E^*))$ is, by definition, d_E-closed $(d_E\omega^{(2)} = 0)$, we also assume its d_E^*-closedness, that is, $d_E^*\omega^{(2)} = 0$.

Definition 3. *External derivations (), $d_E, d_E^* : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$ satisfying the properties $d_Ed_E = 0$, $d_E^*d_E^* = 0$ and $d_Ed_E^* + d_E^*d_E = 0$ are called compatible.*

We additionally on the symplectic two-form $\omega^{(2)} \in \Gamma(\Lambda^2(E^*))$ its K invariance with respect to the related Lie derivative:

$$\mathcal{L}_K^{(E,*)}\omega^{(2)} := 0 \tag{83}$$

$$\mathcal{L}_K^{(E,*)} := i_K d_E^* + d_E^* i_K. \tag{84}$$

One easily derives that the following expression holds for some smooth mapping ($H_1, H_2 : \Gamma(E^*) \to \mathbb{R}$).

$$i_K\omega^{(2)} = -d_E^* H_1 = -d_E H_2 \tag{85}$$

Moreover, multiplying the right-hand side of the equality $-d_E^* H_1 = -d_E H_2$ by the differential $d_E^* : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$, one easily finds that

$$-d_E^* d_E^* H_1 = d_E(d_E^* H_2) = 0, \tag{86}$$

meaning that there exists a smooth mapping ($H_3 : \Gamma(E^*) \to \mathbb{R}$) such that

$$d_E^* H_2 = d_E H_3. \tag{87}$$

The relationship obtained above (85) can be recurrently continued and equivalently rewritten [56,57] as the modified *Lenard–Magri-type* mapping ($Q_E^{(*)} : \Gamma(T^*(E^*)) \to \Gamma(T^*(E^*))$), acting as

$$d_E H_{j+1} := Q_E^{(*)}(d_E H_j) = d_E^* H_j \tag{88}$$

for the set of smooth mappings ($H_j : \Gamma(E^*) \to \mathbb{R}$, $j = \overline{1,m}$, $H_2 = H, d_E^* H_m = 0$) generated by the second external differential ($d_E^* : \Gamma(\Lambda(E^*) \to \Gamma(\Lambda(E^*))$, which is defined by expression (81).

It is also easy to observe that there exists a one-form $\beta_1^{(1)} \in \Gamma(\Lambda^1(E))$ satisfying the condition $d_E\beta_1^{(1)} = \omega_1^{(2)} := \omega^{(2)}$ and generating the second closed two-form

$$\omega_2^{(2)} := d_E \circ Q_E^{(*)}(\beta_1^{(1)}) \in \Gamma(\Lambda^2(E^*)),$$

which is both d_E- and d_E^*-closed. The latter makes it possible to construct a countable hierarchy of symplectic structures ($\omega_j^{(2)} \in \Gamma(\Lambda^2(E^*)), j = \overline{1,m}$) such that

$$\omega_{j+1}^{(2)} = d_E \circ Q_E(\beta_j^{(1)}), \qquad \omega_j^{(2)} := d_E\beta_j^{(2)}$$

for $j = \overline{1,m}$, satisfying the next recurrent relationship jointly with the next countable hierarchy of differential relationships for all integers $j = \overline{1,m}$:

$$\omega_{j+1}^{(2)} = d_E \circ Q_E^{(*)} \circ d_E^{-1}(\omega_j^{(2)}), \qquad d_E \circ Q_E^{(*)}(\beta_m^{(1)}) = 0, \tag{89}$$

$$i_K\omega_j^{(2)} = -d_E H_{j+1} \tag{90}$$

Moreover, one can easily verify that all Hamiltonian functions constructed above ($H_j : \Gamma(E^*) \to \mathbb{R}, j = \overline{1,m}$) commuting with each other, that is,

$$\{\{H_j, H_i\}\}_s = 0 \tag{91}$$

for all $i, j = \overline{1,m}$ with respect to the following Poisson brackets:

$$\{\{f,g\}\}_s := \omega_s^{(2)}(K_f, K_g), \qquad i_{K_f}\omega_s^{(2)} = -d_E f, \qquad i_{K_f}\omega_s^{(2)} = -d_E f, \tag{92}$$

defined for $s = \overline{1,m}$ and arbitrary smooth functions ($f, g : \Gamma(E^*) \to \mathbb{R}$). The properties of the Courant-type algebroid foliation over the manifold (M) described above can be summarized as the next theorem.

Theorem 2. *The Courant-type algebroid foliation* ($(E; [[\cdot,\cdot]], \rho)$, $E = \mathcal{A}(M)$)*, endowed with two compatible external differentials* ($d_E, d_E^* : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$)*, generates a finite set of commuting Hamiltonian flows* ($K_j : \Gamma(E^*) \to \Gamma(E)$, $j = \overline{1,m}$) *in a coordinate set* ($e \in \Gamma(E^*)$) *of the bundle* $\Gamma(E)$*, that is,*

$$[[K_j, K_i]] = 0 \tag{93}$$

for all $i, j = \overline{1,m}$*, where*

$$K_j(e) = \{\{H_1, e\}\}_j \tag{94}$$

and $K_1 := K \in \Gamma(E)$.

It is worth observing that the external deformed differential ($d_E^* : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$) defined by means of relationship (81) and satisfying the constraint $d_E^* d_E^* = 0$ can be interpreted as a flat connection ($d_E^* : \Gamma(E^*) \to T^*(E^*) \otimes \Gamma(E^*)$ on $\Gamma(E)$), acting as

$$\begin{aligned} d_E^* A = \sum_{j=\overline{1,m}} [d_E(e_j(e))e^j(A) + e_j(e)\left(d_E e^j\right)(A)] \\ + \sum_{i,j=\overline{1,m}} e_i(e) Q_E(e)^i_j \left(d_E e^j\right)(A) \end{aligned} \tag{95}$$

on any vector $A := \sum_{j=1}^m e_j(e) A^j \in \Gamma(E)$, where $A^j := e^j(A), j = \overline{1,m}$. As it is easy to check, the curvature two-form

$$\begin{aligned} \Omega^{(2)} : = d_E^* d_E^* = d_E\left(\textstyle\sum_{j=\overline{1,m}} Q_E(e)_j d_E e^j\right) \\ + \left(\sum_{j=\overline{1,m}} Q_E(e)_j d_E e^j\right) \wedge \left(\sum_{j=\overline{1,m}} Q_E(e)_j d_E e^j\right) = 0, \end{aligned} \tag{96}$$

meaning that the connection $d_E^* : \Gamma(E^*) \to T^*(E^*) \otimes \Gamma(E^*)$ on $\Gamma(E)$ is flat.

It is interesting to look at a special case [5] of the external differential ($d_E^* : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$) defined by means of the following expression:

$$\begin{aligned} \left(\tilde{d}_E^* \alpha^{(k)}\right)(A_0, \dots, A_k) : = \sum_{i=0}^{k} (-1)^i \rho(TA_i)_E \alpha^{(k)}((A_0, \dots, \hat{A}_i, \dots, A_k) \\ + \sum_{i<j=0}^{k} (-1)^{i+j} \alpha^{(k)}([[A_i, A_j]]_T, A_0, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_k), \end{aligned} \tag{97}$$

where $T : \Gamma(E) \to T(E)$ is a linear homomorphism, and the commutator $[[\cdot,\cdot]]_T$ on $\Gamma(E)$ is calculated according to the following expression for arbitrary $X, Y \in \Gamma(E)$:

$$[[X, Y]]_T = [[TX, Y]] + [[X, TY]] - T[[X, Y]] \tag{98}$$

Then, the external differential (97) satisfies the algebraic cohomology complex property ($\tilde{d}_E^* \tilde{d}_E^* = 0$) if there the following Nijenhuis constraint ($T : \Gamma(E) \to T(E)$) holds:

$$T[[X, Y]]_T = [[TX, TY]] \tag{99}$$

for all $X, Y \to \Gamma(E)$ and is imposed on the homomorphism $T : \Gamma(E) \to T(E)$. Moreover, the external differential (97) satisfies the usual anticommutativity property

$$\tilde{d}_E^* d_E + d_E \tilde{d}_E^* = 0$$

and can be identified under some natural conditions imposed on the homomorphism $T : \Gamma(E) \to T(E)$ and the corresponding mappings ($(Q_E(e)_j \in \text{End}\,\Gamma(E^*), j = \overline{1,m})$, with the external differentiation $d_E^* : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$. The pair of external differentiations $\tilde{d}_E^*, d_E : \Gamma(\Lambda(E^*)) \to \Gamma(\Lambda(E^*))$, together, form the corresponding Frölicher–Nijenhuis-compatible bicomplex [5], generating a countable hierarchy of commuting bi-Hamiltonian flows on $\Gamma(E^*)$ as described above. The construction described above is also related to a Lauricella [58] problem of bi-flat F manifolds, which are interesting for applications in topological quantum field theory, as first demonstrated in [59–62].

5. The Loop Diffeomorphisms Group ($\widetilde{Diff}(\mathbb{T}^n)$), the Courant-Type Algebroid ($(\mathcal{A}(\tilde{G})^*, [[\cdot,\cdot]], \tilde{r})$) and the Related Integrable Hamiltonian Flows

Let us consider the product $\widetilde{Diff}_+(\mathbb{T}^n) \times \widetilde{Diff}_-(\mathbb{T}^n)$, $n \in \mathbb{Z}_+$, where $\widetilde{Diff}_\pm(\mathbb{T}^n)$ are subgroups of the loop diffeomorphism group ($\widetilde{Diff}(\mathbb{T}^n) := \{\mathbb{C} \supset \mathbb{S}^1 \to Diff(\mathbb{T}^n)\}$) of the torus ($\mathbb{T}^n$) holomorphically extended in the interior ($\mathbb{D}^1_+ \subset \mathbb{C}$) and exterior ($\mathbb{D}^1_- \subset \mathbb{C}$) regions of the unit's centrally located disk ($\mathbb{D}^1 \subset \mathbb{C}^1$), respectively, such that for any $\tilde{g}(\lambda) \in \widetilde{Diff}_-(\mathbb{T}^n)$, $\lambda \in \mathbb{D}^1_-$, $\tilde{g}(\infty) = 1 \in Diff(\mathbb{T}^n)$. The corresponding Lie subalgebra ($\widetilde{diff}(\mathbb{T}^n) \simeq \widetilde{diff}_+(\mathbb{T}^n) \oplus \widetilde{diff}_-(\mathbb{T}^n)$) is a direct sum of the subalgebras ($\widetilde{diff}_\pm(\mathbb{T}^n) \simeq \widetilde{Vect}_\pm(\mathbb{T}^n)$) of the loop subgroups ($\widetilde{Diff}_\pm(\mathbb{T}^n)$) of vector fields on $\mathbb{S}^1 \times \mathbb{T}^n$ extended holomorphically, respectively in regions $\mathbb{D}^1_\pm \subset \mathbb{C}^1$, where for any $\tilde{a}(\lambda) \in \widetilde{diff}_-(\mathbb{T}^n)$, the value is $\tilde{a}(\infty) = 0$.

We now proceed to studying Courant-type algebroids and the related integrable Hamiltonian flows within the classical Adler–Kostant–Symes-type scheme [23–26]. Let us consider the related affine Courant-type algebroid $\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n)) \simeq (\widetilde{diff}(\mathbb{T}^n)^* \ltimes \widetilde{diff}(\mathbb{T}^n), \{\cdot,\cdot\}, \tilde{r})$, where, by definition, the affine Lie algebra is $\mathcal{A}(\widetilde{Diff}(\mathbb{T}^n)) := \widetilde{diff}(\mathbb{T}^n) \ltimes \widetilde{diff}(\mathbb{T}^n)^*$, where the invariant anchor morphism is $\tilde{r} : \widetilde{diff}_\pm(\mathbb{T}^n)^* \to \widetilde{diff}_\pm(\mathbb{T}^n)$, $\tilde{r} := \tilde{k} \oplus \tilde{\eta}$ such that the linear mappings generated by the related derivation $\tilde{D} := \tilde{k} \circ \tilde{\eta}^{-1} : \mathcal{A}(\widetilde{Diff}(\mathbb{T}^n)) \to \mathcal{A}(\widetilde{Diff}(\mathbb{T}^n))$, are the corresponding projectors on the subalgebras ($\mathcal{A}_\pm(\widetilde{Diff}(\mathbb{T}^n)) \subset \mathcal{A}(\widetilde{Diff}(\mathbb{T}^n))$).

$$\tilde{P}_\pm := I \pm \tilde{D}^{-1}/2 : \mathcal{A}(\widetilde{Diff}(\mathbb{T}^n)) \to \mathcal{A}_\pm(\widetilde{Diff}(\mathbb{T}^n)), \tag{100}$$

It is worth observing that if the mapping ($\tilde{k} : \widetilde{diff}_\pm(\mathbb{T}^n)^* \to \widetilde{diff}_\pm(\mathbb{T}^n)$) is invertible and the dual mapping ($\tilde{R} := \tilde{\eta} \circ \tilde{k}^{-1} : \widetilde{diff}_\pm(\mathbb{T}^n) \to \widetilde{diff}_\pm(\mathbb{T}^n)$) is such that mappings

$$\tilde{P}_\pm := I \pm \tilde{R}/2 : \mathcal{A}(\widetilde{Diff}(\mathbb{T}^n)) \to \mathcal{A}_\pm(\widetilde{Diff}(\mathbb{T}^n)) \tag{101}$$

are projectors, then the following commutator structure defines the linear space ($\mathcal{A}(\widetilde{Diff}(\mathbb{T}^n))$) of the deformed Lie structure.

$$[A, B]_{\tilde{R}} := [\tilde{R}A, B] + [A, \tilde{R}B] \tag{102}$$

The following theorem, justifying the relationship between the Lie–Poisson bracket in the space ($\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))$) and the corresponding Courant structure, holds.

Theorem 3. *The adjoint space ($\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))$) is Poissonian with respect to the following deformed Lie–Poisson structure at $(\tilde{l}, \tilde{p}) \in \mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))$ for any $\tilde{X}, \tilde{Y} \in \mathcal{A}(\widetilde{Diff}(\mathbb{T}^n))$:*

$$\{(\tilde{l}, \tilde{p})(\tilde{X}), (\tilde{l}, \tilde{p})(\tilde{Y})\}_D = (\tilde{l}, \tilde{p})([\tilde{X}, \tilde{Y}]_D) \tag{103}$$

Let $I(\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))) := \{\gamma \in D(\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))) : ad^*_{\text{grad}\gamma(\tilde{l})}(\tilde{l}, \tilde{p}) = 0, (\tilde{l}, \tilde{p}) \in \mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))\}$ denote the set of Casimir functionals in the space ($\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))$) invariant with respect to the canonical coadjoint action of the loop diffeomorphism group $(\widetilde{Diff}(\mathbb{T}^n))$ on $\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))$. Then, any countable sequence of independent functionals $(\gamma_j \in I(\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n))), j \in \mathbb{N})$ commute with each other with respect to the deformed Lie–Poisson structure $(\{\cdot,\cdot\}_D)$ and generate an infinite hierarchy of completely integrable [20,37,39] Hamiltonian systems in the space $(\mathcal{A}^*(\widetilde{Diff}(\mathbb{T}^n)))$ with respect to the evolution parameters $(t_j \in \mathbb{R}, j \in \mathbb{N})$.

$$\partial(\tilde{l},\tilde{p})/\partial t_j = -ad^*_{\tilde{D}^{-1}\mathrm{grad}\gamma_j(\tilde{l},\tilde{p})}(\tilde{l},\tilde{p}) \tag{104}$$

As naturally follows from [29–31,44,46], the integrable Hamiltonian systems (104) constructed above suitably generalize so-called [33] heavenly-type differential systems, describing diverse geometric structures of conformal types on finite dimensional Riemannian manifolds. We plan to conduct detailed investigations of some such structures related to so-called WDVV associativity equations [46,63,64] and the related Dubrovin-type affine connections [59,60].

6. Conclusions

We provided constructions of Courant algebroids related to semisimple Lie groups and showed that the related Lie algebroid reduces to the Courant algebroid, as similarly described in [3]. Moreover, this construction proved to be naturally generalizable in the case in which the canonical symplectic mapping is replaced by a Lie algebra homomorphism. We also devised an approach to constructing Courant algebroids with rich differential–geometric properties, making use of the powerful Adler–Kostant–Symes scheme for Poisson structures in coadjoint orbits, in particular its version based on the R structure and associated with a specially defined tensor mapping, providing the canonical Lie–Poisson bracket in the dual space. We also studied some differential geometric and symplectic properties of a special Courant-type algebroid foliation and analyzed the algebraic structure of related Hamiltonian flows. In particular, we showed that the Courant-type algebroid foliation, being equipped with two compatible external differentials, generates a finite set of commuting Hamiltonian flows, realizing the classical Magri-type recursion scheme. We also conducted a detailed analysis of Courant-type algebroids related to the loop diffeomorphism group and constructed compatible pairs of Poisson brackets and the related integrable Hamiltonian flows within the classical Adler–Kostant–Symes scheme, suitably generalizing the so-called heavenly-type differential systems describing diverse geometric structures of conformal types on finite dimensional Riemannian manifolds.

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