CERN-THESIS-2013-356

Gauge Anomalies and Neutrino Seesaw Models

Luís Manuel Neves Cebola

Thesis to obtain the Master of Science Degree in

Engineering Physics

Examination Committee

Chairperson: Prof. Doutor Mário João Martins Pimenta
Supervisor: Prof. Doutor Ricardo Jorge Gonzalez Felipe
Member of the Committee: Prof. Doutor Gustavo da Fonseca Castelo Branco

November 2013

Acknowledgements

I would like to start by thanking my supervisor, Professor Ricardo González Felipe for his guidance, patience and support throughout the development of this work. I emphasize his partnership which motivated me to work every day and allowed me to reach the high quality levels present in this thesis, and also his constant care when discussing unknown topics for me, always working side by side with me.

I am very grateful to Professor David Emmanuel-Costa for his interesting ideas and hints that truly boosted the realization of this project. Without his help and skills I would not have come so far.

I acknowledge CFTP - Centro de Física Teórica de Partículas for giving me the opportunity of being a member in this group, which presented me with all the necessary tools and facilities to conclude my work. I also thank CFTP for integrating me in the project CERN/FP/123580/2011, entitled "Física para além do Modelo Standard na Era do LHC".

Throughout these incredible five years, I improved not only my knowledge of physics, but also my own self. I am indebted to all my friends who contributed to this process, particularly to my roommate Jonathan Ribeiro, my regular collaborator Tomás Cruz and those that also belong to CFTP: António Coutinho, Pedro Boavida and João Penedo. Thanks to my colleagues from Tocha, who provided me fun, friendship and hope in my projects since I was a kid. Without them, I would not have been so successful in either electronic music, or particle physics.

Last, but certainly not least, I would like to thank my parents, my grandmother Leonor and my soulmate Diana. My family was the most supportive and encouraging one I could ever expect. I am forever indebted to Diana for balancing me between stiff nights behind the decks, backbreaking days working on physics and a nice, pleasant and happy life. This thesis is dedicated to these amazing four people.

This project was supported by *Fundação para a Ciência e Tecnologia*, under the grant CERN/FP/123580/2011.

Resumo

Apesar do sucesso do Modelo Padrão no que concerne a previsões teóricas, há diversos resultados experimentais que este não consegue explicar, havendo portanto razões para acreditar na existência de nova física além deste. As oscilações de neutrinos e consequentemente as suas massas são exemplo disso mesmo.

Experimentalmente sabe-se que essas massas são bastante pequenas quando comparadas com as massas das partículas do Modelo Padrão. Entre várias possibilidades teóricas para explicar estas massas muito pequenas, o mecanismo de *seesaw* é um modelo simples e bem motivado. Na versão mínima deste modelo são introduzidas partículas pesadas que desacoplam da teoria no universo primordial.

Para que uma teoria seja consistente as simetrias clássicas devem ser preservadas ao nível quântico, de forma a que não ocorram anomalias. O cancelamento das mesmas leva a constrangimentos nos parâmetros da teoria. Uma solução interessante é modificar a simetria de gauge de forma a que haja cancelamento das anomalias.

Nesta tese apresentamos uma pequena revisão de alguns conceitos do Modelo Padrão, relevantes para os aspectos supracitados. De seguida discutimos a implementação de uma nova simetria, livre de anomalias, e a respectiva ligação com a estrutura de sabor da matriz de massa dos neutrinos, obtida através do mecanismo de *seesaw*. Discutimos ainda a possibilidade de distinguir diferentes simetrias de gauge e diferentes tipos de mecanismo de *seesaw* em aceleradores.

Palavras-chave: Anomalias de Gauge, Física dos Neutrinos, Mecanismo *Seesaw*, Quantização da Carga, Simetrias de Gauge.

Abstract

Despite the success of the Standard Model concerning theoretical predictions, there are several experimental results that cannot be explained and there are reasons to believe that there exists new physics beyond it. Neutrino oscillations, and hence their masses, are examples of this.

Experimentally it is known that neutrinos masses are quite small, when compared to all Standard Model particle masses. Among the theoretical possibilities to explain these tiny masses, the seesaw mechanism is a simple and well-motivated framework. In its minimal version, heavy particles are introduced that decouple from the theory in the early universe.

To build consistent theories, classical symmetries need to be preserved at quantum level, so that there are no anomalies. The cancellation of these anomalies leads to constraints in the parameters of the theory. One attractive solution is to realize the anomaly cancellation through the modification of the gauge symmetry.

In this thesis we present a short review of some features of the Standard Model, relevant to the aspects mentioned above. We then discuss the implementation of new anomaly free gauge symmetries and their connection with the flavour structure of the neutrino mass matrix obtained through the seesaw mechanism. The possibility of distinguishing different gauge symmetries and seesaw realizations at collider experiments is also addressed.

Keywords: Charge Quantization, Gauge Anomalies, Gauge Symmetries, Neutrino Physics, Seesaw Mechanism.

Contents

List of Acronyms

ABJ Adler-Bell-Jackiw

AWI Axial Ward Identity

CERN European Organization for Nuclear Research

CKM Cabibbo-Kobayashi-Maskawa

FCNC Flavour Changing Neutral Currents

GUT Grand Unified Theories

LHC Large Hadron Collider

PMNS Pontecorvo-Maki-Nakagawa-Sakata

QED Quantum Electrodynamics

QFT Quantum Field Theory

SM Standard Model

SSB Spontaneous Symmetry Breaking

VEV Vacuum Expectation Value

VWI Vectorial Ward Identity

WI Ward Identity

List of Figures

List of Tables

Outline

The present thesis is divided in four structural chapters. The first chapter is dedicated to a review of the Standard Model of particle physics, with major emphasis, of course, in the subjects vital to the study here exposed. This chapter has a pivotal role in this work, in that it will allow us to present the reader with the language, notation and conventions we will use further on.

Chapter 2 comprises the introduction and discussion of the concept of gauge anomalies and their cancellations. The chiral gauge anomaly is addressed, with the specific case of the Standard Model being shown as the archetypal anomaly-free theory.

The third chapter starts with a brief review of neutrino masses, manifest due to their established oscillations, and types I, II and III seesaw mechanisms. It then moves to a discussion of how the anomaly-free conditions are modified and the electric charge quantization is realized within these minimal extensions to the SM. The chapter thus ends with a phenomenological study of feasible flavour structures of the effective neutrino mass matrix, with a particular focus on the valid two-zero texture realizations of type I and/or type III seesaw mechanisms.

Finally, in Chapter 4, the approach of cancelling gauge anomalies and the constraints that ensue is employed to the study of an Abelian extension to the gauge group of the SM with an extra $U(1)_X$ gauge symmetry. The allowed charge assignments under this new gauge symmetry are studied, in the context of either two or three additional right-handed neutrino singlets of fermion triplets. The phenomenological constraints on these theories are then inspected, with a further discussion on the possibility of distinguishing different charge assignments and neutrino textures at collider experiments.

Part of Chapter 3 and the whole of Chapter 4 are summarized in Ref. [1].

Chapter 1

Standard Model

The Standard Model (SM) of particle physics is a theory concerning three of the fundamental forces/interactions in Universe, electromagnetic, weak, and strong interactions, which mediate the dynamics of the known subatomic particles [2]. It is described by a Lagrangian, which predicts very accurately many of the experimentally verified phenomena (e.g. anomalous magnetic moment [3] and weak boson masses [4]).

In July/2012, LHC experiments at CERN announced the discovery of a Higgs-like particle [5, 6]. This is one of the most important discoveries in particle physics and a triumph for the SM because the Higgs boson is one of its main ingredients. Hence, in 2013, Higgs and Englert were awarded with the Nobel prize in Physics for the theoretical discovery of the mechanism that is at the origin of the masses of all the SM particles.

In this chapter we review a few concepts of the SM that are relevant to the topics of this thesis, namely the electroweak interactions, the Higgs mechanism and the gauge symmetries of the SM Lagrangian.

1.1 Quantum Electrodynamics

Back in 1928, Paul Dirac obtained the well-known Dirac equation,

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\psi = \left(i\not{\partial} - m\right)\psi = 0\,, \tag{1.1}$$

describing a field ψ with spin-$\frac{1}{2}$, where γ^{μ} are the Dirac matrices. It was the first theory to account fully for relativity in the context of quantum mechanics, being consistent with both the principles of quantum mechanics and the theory of special relativity [7, 8]. Although this equation describes the hydrogen spectrum completely, it needs further improvements to understand other quantum phenomena.

It became clear that quantization of fields provides the correct way to deal with fundamental particle interactions (precision tests on quantum electrodynamics (QED) point exactly this [9]). The main quantity in a quantum or classical field theory is the Lagrangian $\mathcal{L}\left(\phi_i, \partial_{\mu}\phi_i\right)$, which allows to obtain the

Euler-Lagrange equations through the principle of stationary action:

$$\begin{aligned}
&\delta S = \delta \int d^4x \mathcal{L} = 0 \Leftrightarrow \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i}\delta\phi_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\delta(\partial_\mu\phi_i)\right] = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)}\right]\delta\phi_i = 0 \\
&\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial\left(\partial_\mu \phi_i\right)} = 0\,.
\end{aligned} \tag{1.2}$$

Given a specific $\mathcal{L}$, Eq. (1.2) provides the equation of motion. As a trivial example, the Lagrangian

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi\,, \tag{1.3}$$

where $\overline{\psi} = \psi^\dagger\gamma^0$, leads to the Dirac equation, Eq. (1.1).

Another important feature of a quantum field theory (QFT) is the underlying symmetries that are present in the Lagrangian of the theory. From Eq. (1.3) it is clear that the transformation

$$\psi \to \psi' = \psi e^{i\alpha}\,, \tag{1.4}$$

with α constant throughout space-time, leaves the Lagrangian invariant:

$$\delta\mathcal{L} = \mathcal{L}' - \mathcal{L} = 0\,. \tag{1.5}$$

Since α is constant, this is called a global invariance.

It is important to realize that an arbitrary change in the phase of the field (wave function) does not affect the theory, but it is hard to conceive a real experiment where the phase is equal in different laboratories/measurements. In another description, we are allowed to choose the phase convention locally, so the Lagrangian of the theory is locally invariant and the theory is a gauge theory. The constraint of being locally invariant enforces the derivative ∂_μ to change into a covariant derivative D_μ which include new fields. These fields are exchanged when particles interact, providing a quantum concept of force.

Under the local transformation

$$\psi \to \psi' = \psi e^{i\alpha(x)}\,, \tag{1.6}$$

the Lagrangian (1.3) should become

$$\mathcal{L} = \overline{\psi}\left(iD\!\!\!\!/ - m\right)\psi\,, \tag{1.7}$$

in order to be gauge invariant, since, by definition

$$D_\mu\psi \to \left(D_\mu\psi\right)' = e^{i\alpha(x)}D_\mu\psi\,. \tag{1.8}$$

To check the transformation on the new field we construct $D_\mu = \partial_\mu + A_\mu$, so that

$$\begin{aligned}
\mathcal{L}_A &= \overline{\psi}\left(iD\!\!\!\!/ - m\right)\psi = \overline{\psi}\left(i\partial\!\!\!/ + iA\!\!\!/ - m\right)\psi \to \mathcal{L}'_A \\
\mathcal{L}'_A &= \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi + i\overline{\psi}\gamma^\mu\psi\partial_\mu\alpha + \overline{\psi}\left(iA\!\!\!/'\right)\psi\,.
\end{aligned} \tag{1.9}$$

Due to gauge invariance $\delta\mathcal{L}_A = 0$, then

$$A'_\mu = A_\mu - i\partial_\mu\alpha \Leftrightarrow \delta A_\mu = -i\partial_\mu\alpha\,. \tag{1.10}$$

To obtain the correct interaction with the electromagnetic field we change the coupling constant, so $A_\mu \to -ieA_\mu$ and identify A_μ as the photon [10]. The complete QED Lagrangian still needs the photon kinetic term, which is also gauge invariant

$$\mathcal{L}_{kinetic} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \tag{1.11}$$

where $F^{\mu\nu}$ is the electromagnetic tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \, . \tag{1.12}$$

The full Lagragian is[1]

$$\mathcal{L}_{QED} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \overline{\psi} \left(i\partial\!\!\!/ - m \right) \psi + e\overline{\psi}\gamma^\mu \psi A_\mu \, . \tag{1.13}$$

From a theoretical point of view, gauge invariance provides an explanation for the photon to be massless, since a mass term as $A^\mu A_\mu$ is not allowed. However, it is the photon that physically ensures consistency in the theory for different phases over space-time. Although Maxwell formulation of electrodynamics had already a gauge symmetry, only in the 40's the importance of gauge theories and their connection with QED was noticed [11–13].

Understanding QED as $U(1)$ quantum gauge group theory with electric charge e as the group coupling is fundamental to analyse the gauge group of the SM. Nevertheless, to comprise weak interactions, one needs to go further.

1.2 Electroweak Interactions

In 1934 Fermi firstly proposed the weak interaction theory to describe β decay, introducing the neutrino to satisfy the energy conservation principle [14],

$$n \to p + e \quad \xrightarrow{Fermi} \quad n \to p + e + \nu_e \, . \tag{1.14}$$

In order to explain this interaction, Fermi proposed the Lagrangian

$$\mathcal{L}_\beta = \frac{G_\beta}{\sqrt{2}} \left(\overline{\psi}_p \gamma^\mu \psi_n \right) \left(\overline{\psi}_e \gamma^\nu \psi_{\nu_e} \right) g_{\mu\nu} + \text{H.c.} = \frac{G_\beta}{\sqrt{2}} J_h^\mu J_l^\nu g_{\mu\nu} + \text{H.c.} \, , \tag{1.15}$$

where

$$J_h^\mu = \overline{\psi}_p \gamma^\mu \psi_n \;\; , \; J_l^\mu = \overline{\psi}_e \gamma^\mu \psi_{\nu_e} \, , \tag{1.16}$$

are the hadronic and leptonic parts of the current, respectively.

Despite the similarities with QED, Lee and Yang stated that weak interaction should violate parity [15] (QED does not). This fact became clear experimentally, and it was also verified that neutrinos have negative helicity [16],

$$\psi_{\nu_e} \to \frac{1-\gamma_5}{2} \psi_{\nu_e} = \psi_{L\nu_e} \, , \tag{1.17}$$

where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. Since we can write

$$\psi = \frac{1-\gamma_5}{2}\psi + \frac{1+\gamma_5}{2}\psi = P_L\psi + P_R\psi = \psi_L + \psi_R \, , \tag{1.18}$$

where

$$\begin{aligned} P_L &= \frac{1-\gamma_5}{2} \, , \text{ and} \\ P_R &= \frac{1+\gamma_5}{2} \, , \end{aligned} \tag{1.19}$$

[1] There are also other terms that are gauge invariant, e.g. the gauge fixing term, but we will not present them explicitly.

then the leptonic part of the current changes to

$$J_l^\mu = \overline{\psi}_e \gamma^\mu \psi_{L\nu_e} = \overline{\psi}_{Le} \gamma^\mu \psi_{L\nu_e} \,, \tag{1.20}$$

because the cross terms with ψ_{Re} vanish. Due to the same property of helicity projectors, a Dirac mass term for neutrinos also vanishes[2]

$$\mathcal{L}_{Dirac\,mass} = -m\overline{\nu_e}\nu_e = -m\overline{\nu_{Le}}\nu_{Le} = 0\,. \tag{1.21}$$

In 1958, Feynman and Gell-Mann proposed a different Lagrangian that summarizes the Fermi weak interaction theory [17]

$$\mathcal{L}_F = 2\sqrt{2} G_F J^\mu J_\mu^\dagger \,, \tag{1.22}$$

where, J^μ has again a leptonic and a hadronic part. The leptonic current is

$$J_l^\mu = \overline{e}_L \gamma^\mu \nu_{Le} + \overline{\mu}_L \gamma^\mu \nu_{L\mu} + \overline{\tau}_L \gamma^\mu \nu_{L\tau} \,, \tag{1.23}$$

although, by the time the theory was suggested, τ was still unknown. For the hadronic part of the current, the problem is harder to address due to the strong interactions. Experimental tests reveal that decays with $|\Delta S| = 0$, for example,

$$n \to p\,\mathrm{e}^-\,\overline{\nu}_e \,, \tag{1.24}$$

have an amplitude similar to the leptonic processes, and decays with $|\Delta S| = 1$, for example,

$$\Lambda \to p\,\mathrm{e}^-\,\overline{\nu}_e \,, \tag{1.25}$$

have a much smaller amplitude. However, if we consider the squared-sum of these amplitudes, weak universality is nearly restored. As proposed by Cabibbo in 1963, we can write the hadronic part as [18]

$$J_h^\mu = \left(\overline{d_L}\cos\theta_c + \overline{s_L}\sin\theta_c\right)\gamma^\mu u_L \,, \tag{1.26}$$

defined by an angle θ_c, the Cabibbo angle. Since Fermi weak theory is constructed in a similar way to QED, it is clear that a new bosonic field W_μ should be present in analogy with the photon. Due to electric charge conservation and the charge assignments

$$Q_u = \frac{2}{3}\,,\;\; Q_d = Q_s = -\frac{1}{3}\,,\;\; Q_e = Q_\mu = Q_\tau = -1\,,\;\; Q_{\nu_e} = Q_{\nu_\mu} = Q_{\nu_\tau} = 0\,, \tag{1.27}$$

the new W boson is charged ($Q_{W^+} = 1\,,\;\; Q_{W^-} = -1$). Then the Lagrangian (1.22) becomes

$$\mathcal{L}_W = \frac{g_W}{\sqrt{2}} J^\mu W_\mu^- + \text{H.c.} = \frac{g_W}{\sqrt{2}}\left(J^\mu W_\mu^- + J^{\dagger\mu} W_\mu^+\right)\,. \tag{1.28}$$

Despite the fact that some features present in QED are analogous to those of the weak interactions, there are several differences concerning the W boson. While the photon is massless, W must have a high mass, because weak interactions have a very short length. The process $e^- + e^+ \to W^- + W^+$ violates unitarity, since the longitudinal polarization of the W leads to a cross section that grows with the center-of-mass energy, whereas the similar QED process $e^- + e^+ \to \gamma + \gamma$ poses no problem because the photon is

[2]From here on, we label fermionic fields ψ_f simply with f.

massless. Finally, the QED gauge group is simply $U(1)$, while the gauge group of electroweak interactions is not so straightforward to attain.

If we combine the left-handed fields $\nu_{L\alpha}$ and $e_{L\alpha}$ into an $SU(2)$ doublet,

$$\ell_{L\alpha} = \begin{pmatrix} \nu_{L\alpha} \\ e_{L\alpha} \end{pmatrix} , \tag{1.29}$$

with $\alpha = 1, 2, 3$ labelling fermions of the first, second and third family respectively, the leptonic part of the current becomes

$$J_l^\mu = \sum_{\alpha=1}^{3} \overline{e_{L\alpha}} \gamma^\mu \nu_{L\alpha} = \sqrt{2} \sum_{\alpha=1}^{3} \overline{\ell_{L\alpha}} T^- \gamma^\mu \ell_{L\alpha} \, . \tag{1.30}$$

The leptonic part of the Lagrangian reads

$$\mathcal{L}_W = g_W \sum_{\alpha=1}^{3} \overline{\ell_{L\alpha}} T^- W_\mu^- \gamma^\mu \ell_{L\alpha} + \text{H.c.} = g_W \sum_{\alpha=1}^{3} \overline{\ell_{L\alpha}} \left(T^- W_\mu^- + T^+ W_\mu^+ \right) \gamma^\mu \ell_{L\alpha} \, , \tag{1.31}$$

where

$$T^- = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} , \; T^+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \tag{1.32}$$

which can be related with the generators of $SU(2)$ in the fundamental representation through

$$T^\pm = \frac{T^1 \pm i T^2}{\sqrt{2}} \, , \; T^i = \frac{\sigma^i}{2} \, , \; \left[T^i, T^j \right] = i \varepsilon^{ijk} T^k \, , \tag{1.33}$$

where σ^i are the Pauli matrices and ε^{ijk} is the Levi-Civita tensor. The presence of two $SU(2)$ generators gives us a hint for the gauge group of the electroweak theory. If we naively consider $SU(2)$, the demand of a third generator T^3 would lead to a neutral current

$$\sum_{\alpha=1}^{3} \overline{\ell_{L\alpha}} T^3 W_\mu^3 \gamma^\mu \ell_{L\alpha} = \sum_{\alpha=1}^{3} W_\mu^3 \left(\overline{\nu_{L\alpha}} \gamma^\mu \nu_{L\alpha} - \overline{e_{L\alpha}} \gamma^\mu e_{L\alpha} \right) , \tag{1.34}$$

which cannot be identified with the electromagnetic current because it involves the neutrino ($Q_{\nu_\alpha} = 0$) and only left-handed fields. In 1961, S.L. Glashow proposed a model with four vectorial bosons to describe both weak and electromagnetic interactions [19], which is now known as the SM electroweak interaction. The proposed gauge group was the right one, $SU(2) \otimes U(1)$, however, the universality in the intensity of leptonic and hadronic currents (neglecting effects from the Cabibbo angle θ_c), which points towards a gauge theory, was only discovered later. Moreover, since the Higgs mechanism had not yet been discovered, gauge theories did not get too much attention at the time.

It was only in 1967-1968 that Weinberg [20] and Salam [21] applied the spontaneous symmetry breaking (SSB) in the electroweak gauge theory in order to generate mass for gauge bosons. Since in 1972 the consistency of the theory (i.e. that it preserves unitarity and it is renormalizable) was proved by 't Hooft and Veltman [22, 23], it became clear that the gauge group of the SM electroweak sector is indeed $SU(2) \otimes U(1)$, and that the gauge bosons acquire mass through the Higgs mechanism.

1.3 Higgs Mechanism

The renowned Higgs mechanism was in fact discovered in 1964 by three independent groups, Higgs [24], Brout and Englert [25] and Guralnik, Hagen and Kibble [26].

As we already discussed, a Lagrangian with terms that are not gauge invariant explicitly breaks the underlying symmetry. On the other hand, SSB occurs when the ground state is not invariant but the Lagrangian is still symmetric under the gauge group. The Higgs mechanism in the SM introduces only one complex $SU(2)$ doublet, the scalar Higgs field H, that realizes the electroweak SSB.

The relevant parts of the SM Lagrangian for this mechanism are

$$\begin{aligned}\mathcal{L}_{Higgs} &= (D_\mu H)^\dagger (D^\mu H) - V(H)\,, \\ \mathcal{L}_{gauge} &= -\frac{1}{4} W^{i\mu\nu} W^i_{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu}\,,\end{aligned} \tag{1.35}$$

where the scalar potential $V(H)$ is

$$V(H) = \mu^2 H^\dagger H + \lambda \left(H^\dagger H\right)^2 . \tag{1.36}$$

The field tensors are

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_W \varepsilon^{ijk} W^j_\mu W^k_\nu\,, \tag{1.37}$$

and

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\,, \tag{1.38}$$

where W^i_μ, $(i = 1, 2, 3)$ are the fields associated with $SU(2)$ and B_μ is the field associated with $U(1)$. The covariant derivative acting on H is

$$D_\mu H = \left(\partial_\mu - i g_W W^i_\mu T^i - i g_Y B_\mu Y_H\right) H\,, \tag{1.39}$$

where g_W and g_Y are, respectively, $SU(2)$ and $U(1)$ coupling constants and Y_H is the Higgs field hypercharge.

One can read the mass directly from the bilinear term in the theory (as we try to do for the photon in Eq. (1.13), but this procedure is valid only if the vacuum expectation value (VEV) of the fields is null[3]. If $\mu^2 > 0$, the ground state arises when the VEV of H is null and the mass spectrum can be read directly. The interesting case, however, occurs when $\mu^2 < 0$. Since we can always perform a rotation to obtain the Higgs VEV

$$\langle 0 | H | 0 \rangle = \langle H \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix}\,,\; v^2 = -\frac{\mu^2}{\lambda}\,, \tag{1.40}$$

which minimizes the potential V, then we can parametrize

$$H = \frac{e^{i\frac{T^i \xi^i}{v}}}{\sqrt{2}} \begin{pmatrix} 0 \\ h + v \end{pmatrix}\,, \tag{1.41}$$

with $\langle h \rangle = 0$. Due to gauge invariance, it is always possible to fix the gauge such that the Goldstone bosons are absent [27]. This is the so-called unitary gauge, where

$$H \to H' = e^{-i\frac{T^i \xi^i}{v}} H = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ h + v \end{pmatrix}\,, \tag{1.42}$$

which imposes the gauge field transformations

$$W^i_\mu \to W'^i_\mu = W^i_\mu - \frac{1}{v g_W} \partial_\mu \xi^i + \frac{1}{v} \varepsilon^{ijk} \xi^j W^k_\mu\,. \tag{1.43}$$

[3]Lorentz invariance of the ground state constrains all but the Higgs field to have zero VEV.

Replacing these fields in Eq. (1.35), the field tensors remain unchanged, while for the covariant derivative and the potential we get

$$
\begin{aligned}
&(D_\mu H)^\dagger (D^\mu H) = \\
&\frac{1}{2}\left\{\partial_\mu h\,\partial^\mu h + \frac{1}{4}(v+h)^2\left[g_W^2\left(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu}\right) + \left(g_W W_\mu^3 - g_Y B_\mu\right)\left(g_W W^{3\mu} - g_Y B^\mu\right)\right]\right\}, \\
&V(H) = \frac{\mu^2}{2}(v+h)^2 + \frac{\lambda}{4}(v+h)^4 \,,
\end{aligned}
\tag{1.44}
$$

respectively. The relevant mass terms are

$$
\begin{aligned}
&(D_\mu H)^\dagger (D^\mu H) = \\
&\cdots + \frac{1}{2}\left(\frac{1}{4}v^2\right)\left[g_W^2\left(W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu}\right) + \left(g_W W_\mu^3 - g_Y B_\mu\right)\left(g_W W^{3\mu} - g_Y B^\mu\right)\right] \,,
\end{aligned}
\tag{1.45}
$$

and

$$
V(H) = \cdots + \left(\frac{1}{2}\mu^2 + \frac{3}{2}\lambda v^2\right)h^2 = \cdots - \frac{1}{2}\left(2\mu^2\right)h^2 \,. \tag{1.46}
$$

Therefore, we obtain a scalar field h with mass

$$
m_h = \sqrt{-2\mu^2}\,, \tag{1.47}
$$

from Eq. (1.46), and cross terms between the gauge bosons from Eq. (1.45). To correctly analyse the mass spectrum, one needs to diagonalize the symmetric mass matrix $\mathbf{M}^2$

$$
\mathbf{M}^2 = \frac{1}{4}v^2 \begin{pmatrix} g_W^2 & 0 & 0 & 0 \\ 0 & g_W^2 & 0 & 0 \\ 0 & 0 & g_W^2 & -g_W g_Y \\ 0 & 0 & -g_W g_Y & g_Y^2 \end{pmatrix}, \tag{1.48}
$$

which have the following eigenvalues

$$
\frac{1}{4}v^2 g_W^2\,; \quad \frac{1}{4}v^2 g_W^2\,; \quad \frac{1}{4}v^2\left(g_W^2 + g_Y^2\right)\,; \quad 0\,. \tag{1.49}
$$

The respective normalized eigenvectors are

$$
\begin{aligned}
&(1,0,0,0)\,, \\
&(0,1,0,0)\,, \\
&\left(0,0,\frac{g_W}{\sqrt{g_W^2+g_Y^2}},-\frac{g_Y}{\sqrt{g_W^2+g_Y^2}}\right), \\
&\left(0,0,\frac{g_Y}{\sqrt{g_W^2+g_Y^2}},\frac{g_W}{\sqrt{g_W^2+g_Y^2}}\right).
\end{aligned}
\tag{1.50}
$$

We know that the photon is massless. Then, labelling it with A_μ as in QED, and denoting the heavier boson with Z_μ, we can write[4]

$$
\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} c_W & -s_W \\ s_W & c_W \end{pmatrix} \begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}, \tag{1.51}
$$

[4]Since there are no cross-terms for W_μ^1 and W_μ^2 they are elements of the new basis as well.

where

$$s_W = \sin\theta_W = \frac{g_Y}{\sqrt{g_W^2 + g_Y^2}}\,, \text{ and } c_W = \cos\theta_W = \frac{g_W}{\sqrt{g_W^2 + g_Y^2}}\,. \tag{1.52}$$

It is clear now that electroweak SSB occurs due to the Higgs VEV, which leads to the vanishing of the scalar fields (Goldstone bosons) in the presence of gauge bosons. There are several gauges where Goldstone bosons do not disappear. Yet in the unitary gauge, these fields are absorbed by gauge bosons, which become massive. In the SM, the three scalar fields introduced in Eq. (1.41) match the longitudinal polarization of the three massive bosons, two W bosons and one Z boson. The complete mass spectrum is

$$m_h = \sqrt{-2\mu^2}\,, \quad m_W = \frac{v}{2} g_W\,, \quad m_Z = \frac{v}{2}\sqrt{g_W^2 + g_Y^2}\,, \quad m_A = 0\,. \tag{1.53}$$

Since weak interaction is described in terms of a charged and a neutral current, one usually write the W bosons as

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i W_\mu^2\right)\,, \tag{1.54}$$

because

$$W_\mu^+ T^+ + W_\mu^- T^- = W_\mu^1 T^1 + W_\mu^2 T^2\,. \tag{1.55}$$

Using this definition and Eq. (1.51), we are able to write the covariant derivative after SSB

$$\begin{gathered} D_\mu H = \left[\partial_\mu - i g_W\left(W_\mu^+ T^+ + W_\mu^- T^- + s_W A_\mu T^3 + c_W Z_\mu T^3\right) - i g_Y Y_H\left(c_W A_\mu - s_W Z_\mu\right)\right] H = \\ \left[\partial_\mu - i g_W\left(W_\mu^+ T^+ + W_\mu^- T^-\right) - i\frac{g_Y\, g_W}{\sqrt{g_W^2 + g_Y^2}}\left(T^3 + Y_H\right) A_\mu - i\frac{g_W^2 T^3 - g_Y^2 Y_H}{\sqrt{g_W^2 + g_Y^2}} Z_\mu\right] H\,, \end{gathered} \tag{1.56}$$

where we identify the electric charge from QED Lagrangian Eq. (1.13) as

$$e = \frac{g_Y\, g_W}{\sqrt{g_W^2 + g_Y^2}}\,, \quad Q = T^3 + Y\,. \tag{1.57}$$

The electric charge operator Q is the conserved generator ($Q.H = 0$) related with the massless gauge boson A_μ. Thus, the electroweak gauge group is spontaneously broken to the QED one.

Finally, under the new gauge group, electromagnetic, charged and neutral weak interactions are manifestly present if we write the covariant derivative as

$$D_\mu H = \left[\partial_\mu - i g_W\left(W_\mu^+ T^+ + W_\mu^- T^-\right) - i Q e A_\mu - i\frac{g_W}{c_W}\left(T^3 - s_W^2 Q\right) Z_\mu\right] H\,. \tag{1.58}$$

As we stated, the Higgs mechanism is the connecting piece between the electroweak gauge group and massive bosons, allowing the unified theory of weak and electromagnetic forces to be renormalizable. Nevertheless, to understand the full SM gauge group, we need to introduce the strong interactions.

1.4 Gauge Symmetries and Particle Content

In the same year that the Higgs mechanism was proposed (1964), a quark model describing the known hadrons was also put forward by Gell-Mann [28]. The model relies on the internal symmetry of $SU(3)$, in which baryons and mesons are composite particles, made up of three quarks and a quark and anti-quark pair, respectively.

There is no experimental evidence for particles composed of leptons to exist. Therefore, the new underlying (strong) force is expected to act solely on quarks. The strong force holds them together in a similar way as atoms and molecules are held together by the electromagnetic force, so there is also an equivalent for the electric charge, the colour charge. Quarks are introduced in 6 flavours (u, d, s, c, b, t), each in the fundamental representation of the gauge group $SU(3)$ (triplet). This fact suggests that the underlying charge of the symmetry must have three kinds of values, commonly related with the three primary colours (red, green and blue).

The strong or colour force carriers, called gluons, are the eight gauge bosons associated with the eight generators of $SU(3)$. As we can see from Eq. (1.58), gluons do not couple with the Higgs field, allowing them to be massless and the corresponding gauge group to remain unbroken, even after SSB occurs.

We can finally write the full SM gauge group, before and after SSB

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \xrightarrow{SSB} SU(3)_C \otimes U(1)_Q \,. \tag{1.59}$$

Any quark of flavour f can be combined in an $SU(3)$ triplet as

$$f = \begin{pmatrix} f^r \\ f^g \\ f^b \end{pmatrix} , \tag{1.60}$$

and left-handed components of up and down quarks can be combined in an $SU(2)$ doublet

$$q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix} . \tag{1.61}$$

With this definition, we can write down the full SM particle content as

$$\text{Fermions} \begin{cases} \text{Quarks} \begin{cases} q_{L\alpha} = \begin{pmatrix} u_{L\alpha} \\ d_{L\alpha} \end{pmatrix} = \left[\begin{pmatrix} u_L \\ d_L \end{pmatrix} , \begin{pmatrix} c_L \\ s_L \end{pmatrix} , \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right] \\ u_{R\alpha} = [u_R, c_R, t_R] \\ d_{R\alpha} = [d_R, s_R, b_R] \end{cases} \\ \\ \text{Leptons} \begin{cases} \ell_{L\alpha} = \begin{pmatrix} \nu_{L\alpha} \\ e_{L\alpha} \end{pmatrix} = \left[\begin{pmatrix} \nu_{Le} \\ e_L \end{pmatrix} , \begin{pmatrix} \nu_{L\mu} \\ \mu_L \end{pmatrix} , \begin{pmatrix} \nu_{L\tau} \\ \tau_L \end{pmatrix} \right] \\ e_{R\alpha} = [e_R, \mu_R, \tau_R] \end{cases} \end{cases} \tag{1.62}$$

$$\text{Bosons} \begin{cases} \text{Gauge} \begin{cases} SU(3) : G^a_\mu \,,\; a = 1, \dots 8 \\ SU(2) : W^i_\mu \,,\; i = 1, 2, 3 \\ U(1) : B_\mu \end{cases} \\ \\ \text{Higgs} \begin{cases} H = \begin{pmatrix} H^+ \\ H^0 \end{pmatrix} \end{cases} \end{cases}$$

After SSB, we usually describe the Higgs and $SU(2)_L \otimes U(1)_Y$ bosons as

$$W^1_\mu, W^2_\mu, W^3_\mu, B_\mu, H \xrightarrow{SSB} W^\pm_\mu, Z_\mu, A_\mu, h\,, \tag{1.63}$$

and the remaining fields as in Eq. (1.62).

Under the SM gauge group, an $SU(3)$ triplet transforms according to

$$f \to f' = e^{-i\frac{\lambda^a}{2}\theta^a} f\,, \tag{1.64}$$

where $\lambda^a, a = 1, \ldots 8$ are the Gell-Mann matrices. An $SU(2)$ doublet transforms according to

$$f \to f' = e^{-iT^i\omega^i} f\,, \tag{1.65}$$

and every field with hypercharge Y_f transforms as

$$f \to f' = e^{-i\alpha Y_f} f\,. \tag{1.66}$$

These transformation properties allow us to write down the covariant derivative for fermions as

$$\begin{aligned}
D_\mu q_{L\alpha} &= \left[\partial_\mu - ig_s G^a_\mu \frac{\lambda^a}{2} - ig_W \left(W^+_\mu T^+ + W^-_\mu T^-\right) - ie\,QA_\mu - i\frac{g_W}{c_W}\left(T^3 - s^2_W Q\right) Z_\mu\right] q_{L\alpha}\,, \\
D_\mu \ell_{L\alpha} &= \left[\partial_\mu - ig_W \left(W^+_\mu T^+ + W^-_\mu T^-\right) - ie\,QA_\mu - i\frac{g_W}{c_W}\left(T^3 - s^2_W Q\right) Z_\mu\right] \ell_{L\alpha}\,, \\
D_\mu u_{R\alpha} &= \left(\partial_\mu - ig_s G^a_\mu \frac{\lambda^a}{2} - ie\,QA_\mu + i\tan\theta_W e\,QZ_\mu\right) u_{R\alpha}\,, \\
D_\mu d_{R\alpha} &= \left(\partial_\mu - ig_s G^a_\mu \frac{\lambda^a}{2} - ie\,QA_\mu + i\tan\theta_W e\,QZ_\mu\right) d_{R\alpha}\,, \\
D_\mu e_{R\alpha} &= \left(\partial_\mu - ie\,QA_\mu + i\tan\theta_W e\,QZ_\mu\right) e_{R\alpha}\,.
\end{aligned} \tag{1.67}$$

Finally, we are able to describe the kinetic terms for fermions

$$\mathcal{L}_{fermion} = \overline{q_{L\alpha}} i\not{D} q_{L\alpha} + \overline{\ell_{L\alpha}} i\not{D} \ell_{L\alpha} + \overline{u_{R\alpha}} i\not{D} u_{R\alpha} + \overline{d_{R\alpha}} i\not{D} d_{R\alpha} + \overline{e_{R\alpha}} i\not{D} e_{R\alpha}\,. \tag{1.68}$$

From Eq. (1.35), we can also generalize the kinetic terms for gauge bosons

$$\mathcal{L}_{gauge} = -\frac{1}{4} B^{\mu\nu} B_{\mu\nu} - \frac{1}{4} W^{i\mu\nu} W^i_{\mu\nu} - \frac{1}{4} G^{a\mu\nu} G^a_{\mu\nu}\,, \tag{1.69}$$

with

$$\begin{aligned}
B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu\,, \\
W^i_{\mu\nu} &= \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_W \varepsilon^{ijk} W^j_\mu W^k_\nu\,,\; i = 1, 2, 3\,, \\
G^a_{\mu\nu} &= \partial_\mu G^a_\nu - \partial_\nu G^a_\mu + g_s f^{abc} G^b_\mu G^c_\nu\,,\; a = 1, \ldots 8\,,
\end{aligned} \tag{1.70}$$

where g_s is the $SU(3)$ coupling constant and f^{abc} are group structure constants. The Higgs part of the Lagrangian ($\mathcal{L}_{Higgs}$) remains the same.

With the introduction of the Higgs mechanism and the SM gauge group we understand how gauge bosons become massive and formally describe particle interactions through the exchange of these bosons. Nevertheless, it is necessary to include further terms in the Lagrangian to describe massive fermions. If we include a Dirac mass term as in the QED Lagrangian given in Eq. (1.13), we obtain

$$-m_f \overline{f} f = -m_f \left(\overline{f_R} f_L + \overline{f_L} f_R\right)\,, \tag{1.71}$$

which is not invariant under $SU(2)_L \otimes U(1)_Y$, since singlets and doublets of $SU(2)_L$ transform differently and may have different hypercharge assignments.

Invariant mass terms for fermions can be constructed by requiring them to be massless before SSB. Since, experimentally, left and right-handed components have the same electric charge, the Dirac mass term (1.71) is invariant under $U(1)_Q$. Therefore, one can include Yukawa interactions between fermions and the Higgs field and demand that the SSB mechanism gives the mass terms for fermions. The correct gauge invariant Yukawa Lagrangian is

$$\mathcal{L}_{Yukawa} = -\mathbf{Y}_u^{\alpha\beta}\, \overline{q_{L\alpha}} \widetilde{H} u_{R\beta} - \mathbf{Y}_d^{\alpha\beta}\, \overline{q_{L\alpha}} H d_{R\beta} - \mathbf{Y}_e^{\alpha\beta}\, \overline{\ell_{L\alpha}} H e_{R\beta} + \text{H.c.}\,, \tag{1.72}$$

where $\widetilde{H} = i\sigma^2 H^*$ and $\mathbf{Y}_{u,d,e}$ are respectively the up quark, down quark and charged-lepton Yukawa couplings matrices. These terms are clearly invariant under $SU(3)_C \otimes SU(2)_L$ and constrain the hypercharge assignments of fermion fields. After SSB takes place each term can be split into a mass term and an interaction one

$$\begin{aligned} \mathcal{L}_{mass} &= -\mathbf{m}_u^{\alpha\beta}\, \overline{u_{L\alpha}} u_{R\beta} - \mathbf{m}_d^{\alpha\beta}\, \overline{d_{L\alpha}} d_{R\beta} - \mathbf{m}_e^{\alpha\beta}\, \overline{e_{L\alpha}} e_{R\beta} + \text{H.c.}\,, \\ \mathcal{L}_{h\,int} &= -\frac{1}{\sqrt{2}} \mathbf{Y}_u^{\alpha\beta}\, \overline{u_{L\alpha}} u_{R\beta}\, h - \frac{1}{\sqrt{2}} \mathbf{Y}_d^{\alpha\beta}\, \overline{d_{L\alpha}} d_{R\beta}\, h - \frac{1}{\sqrt{2}} \mathbf{Y}_e^{\alpha\beta}\, \overline{e_{L\alpha}} e_{R\beta}\, h + \text{H.c.}\,, \end{aligned} \tag{1.73}$$

with $\mathbf{m}_{u,d,e} = \frac{v}{\sqrt{2}} \mathbf{Y}_{u,d,e}$. These mass matrices are arbitrary 3×3 complex matrices which mix fermions from different families. To find the right mass spectrum we need to diagonalize these matrices and rotate the interaction states to the physical ones. This is exactly what Cabibbo angle describes, the mismatch between these states, although at the time, only three quarks were known.

Since we have been working in the interaction basis, we should replace the fermion label f for f' in the SM Lagrangian and analyse the mass terms in the physical basis f. We make the unitary transformations

$$f'_L = \mathbf{L}_f f_L\,,\, f'_R = \mathbf{R}_f f_R\,, \tag{1.74}$$

in such a way that leads to the diagonalization of the mass matrices

$$\mathbf{L}_f^\dagger \mathbf{m}_f \mathbf{R}_f = \mathbf{d}_f = \text{diag}\left(m_{f_1}, m_{f_2}, m_{f_3}\right)\,. \tag{1.75}$$

In the SM context, neutrinos are massless since we cannot construct a gauge invariant mass term, due to the absence of right-handed fields $\nu_{R\alpha}$. This absence also allows for a redefinition of the lepton fields that make them diagonal in the interaction and mass terms. If we choose $\nu'_{L\alpha} = (\mathbf{L}_e)^{\alpha\beta}\, \nu_{L\beta}$ and perform the transformations for charged leptons given in Eq. (1.74), all interactions become diagonal for lepton families, preserving lepton family numbers. Thus, for leptons, there is no need to distinguish between physical states f and interaction states f', as long as neutrinos remain massless.

For quarks, we cannot proceed in the same way. Applying the same transformations (1.74) into Eq. (1.67), the neutral currents remain unchanged

$$\begin{aligned} \mathcal{L}_{NC} &= \sum_f \overline{f'} \gamma^\mu \left[e\,QA_\mu + \frac{g_W}{c_W}\left(g_V - g_A\gamma^5\right) Z_\mu \right] f' \Leftrightarrow \\ \mathcal{L}_{NC} &= \sum_f \overline{f} \gamma^\mu \left[e\,QA_\mu + \frac{g_W}{c_W}\left(g_V - g_A\gamma^5\right) Z_\mu \right] f\,, \end{aligned} \tag{1.76}$$

	Fields					
	H	$q_{L\alpha}$	$\ell_{L\alpha}$	$u_{R\alpha}$	$d_{R\alpha}$	$e_{R\alpha}$
$U(1)_Y$	$\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{2}$	$\frac{2}{3}$	$-\frac{1}{3}$	-1
$SU(2)_L$	**2**	**2**	**2**	**1**	**1**	**1**
$SU(3)_C$	**1**	**3**	**1**	**3**	**3**	**1**

Table 1.1: Representations under the SM gauge group and hypercharge assignments of the Higgs boson and the SM fermions.

where

$$g_V = \frac{1}{2}T^3 - s_W^2 Q \, , \, g_A = \frac{1}{2}T^3 \tag{1.77}$$

are the Z_μ gauge boson vectorial and axial coupling, respectively. However, with these transformations, the charged current becomes

$$\begin{aligned} \mathcal{L}_{CC} &= \frac{g_W}{\sqrt{2}} \left(\overline{u'_{L\alpha}} \gamma^\mu d'_{L\alpha} + \overline{\nu'_{L\alpha}} \gamma^\mu e'_{L\alpha} \right) W_\mu^+ + \text{H.c.} \Leftrightarrow \\ \mathcal{L}_{CC} &= \frac{g_W}{\sqrt{2}} \left(\overline{u_{L\alpha}} \, \mathbf{V}_{CKM}^{\alpha\beta} \gamma^\mu d_{L\beta} + \overline{\nu_{L\alpha}} \gamma^\mu e_{L\alpha} \right) W_\mu^+ + \text{H.c.} \, , \end{aligned} \tag{1.78}$$

where

$$\mathbf{V}_{CKM} = \mathbf{L}_u^\dagger \mathbf{L}_d = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \tag{1.79}$$

is the Cabibbo-Kobayashi-Maskawa quark mixing matrix that generalizes the Cabibbo angle for three families. This matrix was proposed by Kobayashi and Maskawa [29]. The usual parametrization is given by the Particle Data Group [30] in terms of three mixing angles and one Dirac CP-violating phase.

It is clear that, due to the presence of left and right-handed quark fields, we can construct up-quark and down-quark mass terms, but we are not able to simultaneously diagonalize mass and interaction terms.

Finally, the complete SM Lagrangian is the sum of four terms[5]

$$\mathcal{L}_{SM} = \mathcal{L}_{fermion} + \mathcal{L}_{gauge} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} \, , \tag{1.80}$$

written in the same physical basis with the appropriate introduction of the quark mixing matrix $\mathbf{V}_{CKM}$. It is also worth mentioning that, at tree level, flavour can only be changed by charged currents. Therefore, there are no flavour changing neutral currents (FCNC) at tree level in the SM.

The particle group representations is another important aspect that allows us to construct gauge invariant terms in the SM Lagrangian. Since the Higgs boson does not couple to the photon, from Eq. (1.57), it must have $Y_H = \frac{1}{2}$. The hypercharge assignments of the SM fermions and their representations are summarized in Table 1.1, which allows the Yukawa Lagrangian to be invariant. In the next chapter we discuss these assignments from another point of view.

[5]There are also the gauge fixing and Faddev-Popov (or ghost) terms, which are not presented explicitly here.

Chapter 2

Gauge Anomalies

Feynman diagrams are really useful to understand interactions and their radiative corrections, where loops appear. These corrections are fundamental in any QFT to explain experimental results, since they alter the constants of the theory and, ultimately, the interactions themselves [10, 31]. In some theories, corrections can be even more significant, breaking the underlying symmetries present in the classic equations of motion. Anomalies appear when symmetries of the classical Lagrangian are not invariant of the functional integral or the path integral formulation of the theory. If this is a gauge or local symmetry, we have then a gauge anomaly.

In this chapter, we discuss the chiral gauge anomaly, and hence, the possibility of violation of the Ward Identities (WI) [32]. The SM as an example of an anomaly-free theory is also addressed.

2.1 Ward Identities

If we consider the simple Lagrangian

$$\mathcal{L} = \overline{\psi}(i\not{\partial} + \not{A} + \not{B}\gamma_5 - m)\psi\,, \tag{2.1}$$

through the Euler-Lagrange Eq. (1.2) one can obtain the following relations:

$$\begin{aligned}
&\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \Leftrightarrow i\partial_\mu \overline{\psi}\gamma^\mu - \overline{\psi}(\not{A} + \not{B}\gamma_5 - m) = 0 \Leftrightarrow \overline{\psi}(i\overleftarrow{\not{\partial}} - \not{A} - \not{B}\gamma_5 + m) = 0\,,\\
&\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \overline{\psi})} - \frac{\partial \mathcal{L}}{\partial \overline{\psi}} = 0 \Leftrightarrow (i\not{\partial} + \not{A} + \not{B}\gamma_5 - m)\psi = 0\,.
\end{aligned} \tag{2.2}$$

Under Lorentz transformations, there are five different elements classified according to their transformation properties:

$$\begin{aligned}
\text{Scalar: } & S = \overline{\psi}\psi\,,\\
\text{Vector: } & j_\mu = \overline{\psi}\gamma_\mu\psi\,,\\
\text{Tensor: } & T_{\mu\nu} = \overline{\psi}\gamma_\mu\gamma_\nu\psi\,,\\
\text{Axial vector: } & j^5_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi\,,\\
\text{Pseudoscalar: } & P = \overline{\psi}\gamma_5\psi\,.
\end{aligned} \tag{2.3}$$

Using these definitions, we can compute the conservation of the vector and axial currents

$$\begin{aligned}
&\partial^\mu j_\mu = \partial^\mu \overline{\psi}\gamma_\mu \psi + \overline{\psi}\gamma_\mu \partial^\mu \psi = \overline{\psi}\overleftarrow{\not\partial}\psi + \overline{\psi}\not\partial\psi = 0\,, \\
&\Rightarrow \partial^\mu j_\mu = 0\,. \\
&\partial^\mu j^5_\mu = \overline{\psi}(\overleftarrow{\not\partial}\gamma_5 + \not\partial\gamma_5)\psi = \overline{\psi}(\overleftarrow{\not\partial}\gamma_5 - \gamma_5\not\partial)\psi = i\overline{\psi}\left[\left(-\not A - \not B\gamma_5 + m\right)\gamma_5 - \gamma_5\left(\not A + \not B\gamma_5 - m\right)\right]\psi\,, \\
&\Rightarrow \partial^\mu j^5_\mu = 2mi\overline{\psi}\gamma_5\psi = 2miP\,.
\end{aligned} \tag{2.4}$$

In this classical computation, it is straightforward to realize that the vector current is conserved and that the axial current is conserved only in the massless case ($m = 0$). To analyse quantum effects in these conservation laws, one needs to perform similar calculations in the corresponding QFT, based on the same Lagrangian.

In Section 1.1 we stated that the Lagrangian is the main quantity in any QFT, but a more complete description is in fact given by the path integral formulation, introduced by Feynman. It is useful to perform computations by means of the functional integral, which can be written as

$$Z = \int \mathcal{D}f \exp\left[i\int d^4x\, \mathcal{L}(f)\right]\,, \tag{2.5}$$

where f represents the field or fields of the underlying QFT and $\mathcal{D}f$ stands for all the possible field configurations. The correlation or Green's functions are a fundamental tool in any QFT calculation. They can be obtained through the functional integral

$$\langle 0\,|T\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)|\,0\rangle = \frac{\int \mathcal{D}f \exp\left[i\int d^4x\,\mathcal{L}(f)\right]\mathcal{O}_1(x_1)\ldots\mathcal{O}_n(x_n)}{\int \mathcal{D}f \exp\left[i\int d^4x\,\mathcal{L}(f)\right]}\,, \tag{2.6}$$

where T is the time-ordering operator and $\mathcal{O}_i(x_i)$ is a field operator. These correlation functions are the time-ordered VEV of the respective operators. In the spirit of perturbation theory, it is clear that they are closely related to propagators.

A trivial example is the Green's function associated with the propagation of a Dirac particle between two points in space-time (x_1 and x_2) in the free theory (vacuum). This is simply the Feynman free propagator for spin-$\frac{1}{2}$ particles:

$$\left\langle 0\left|T\psi(x_1)\overline{\psi}(x_2)\right|0\right\rangle = \frac{\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \exp\left[i\int d^4x\,\mathcal{L}(\psi)\right]\psi(x_1)\overline{\psi}(x_2)}{\int \mathcal{D}\psi\mathcal{D}\overline{\psi} \exp\left[i\int d^4x\,\mathcal{L}(\psi)\right]} = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip.(x_1-x_2)}}{\not p - m}\,. \tag{2.7}$$

For scalar particles within the same context, we obtain

$$\langle 0\,|T\phi(x_1)\phi(x_2)|\,0\rangle = \frac{\int \mathcal{D}\phi \exp\left[i\int d^4x\,\mathcal{L}(\phi)\right]\phi(x_1)\phi(x_2)}{\int \mathcal{D}\phi \exp\left[i\int d^4x\,\mathcal{L}(\phi)\right]} = i\int \frac{d^4p}{(2\pi)^4}\frac{e^{-ip.(x_1-x_2)}}{p^2 - m^2}\,. \tag{2.8}$$

In a classical field theory, for every local symmetry of the Lagrangian, there is a conserved current (Noether theorem). At quantum level, the integral functional contains the analogous version of this theorem. Thus, the quantum conservation laws constrain the correlation functions. In 1950, Ward realized this fact and derived a relation between the exact electron propagator S_0 and the QED vertex Γ^μ [33],

$$-\,S_0(p)\Gamma^\mu(p,p)S_0(p) = \frac{1}{i}\frac{\partial S_0(p)}{\partial p_\mu}\,, \tag{2.9}$$

which is generalized in 1957 by Takahashi [34]

$$-\,ik_\mu S_0(p+k)\Gamma^\mu(p+k,p)S_0(p) = S_0(p+k) - S_0(p)\,. \tag{2.10}$$

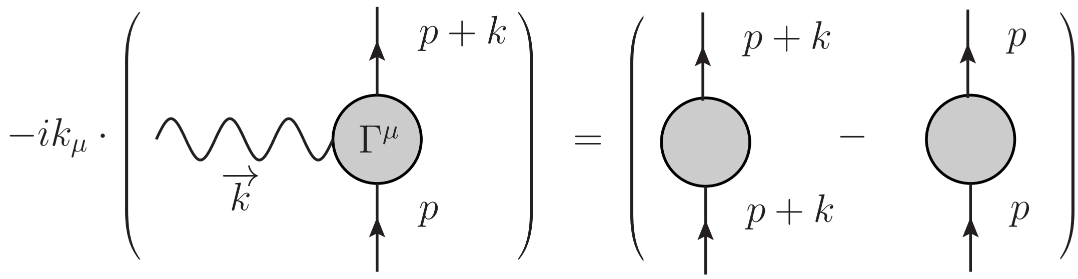

Figure 2.1: Diagrammatic description of an application of the Ward-Takahashi identity in QED. On the left hand-side the contraction between the photon polarization vector was replaced by its momentum. The vertex and the propagators are the exact ones.

Equation (2.9) is known as the WI and Eq. (2.10) is known as the generalized WI. The latter is represented in Fig. 2.1 and it is also an example of the application of the Ward-Takahashi identity in QED. These relations between Green's functions are a consequence of the gauge invariance of the theory, and they need to be preserved in order to account for the renormalizability of the theory [22, 32, 35].

Even though the SM is a non-Abelian gauge theory, it is beyond the scope of this thesis to discuss the non-Abelian generalization of Ward-Takahashi identities (usually called as Ward-Takahashi-Slavnov-Taylor identities) and the subtleties of non-Abelian gauge anomalies. Instead, it is more useful for our purposes to calculate the $U(1)$ chiral anomaly and discuss the resulting constraints for gauge theories through simple but general arguments. Thus, for any physically possible scattering process, we will refer to WI simply as

$$k_\mu \mathcal{M}^\mu(k) = 0\,. \tag{2.11}$$

The amplitude for some process of the Abelian theory, involving an external gauge boson with momentum k, and the polarization vector of the gauge boson $\epsilon_\mu(k)$ is $\mathcal{M}(k) = \epsilon_\mu(k)\mathcal{M}^\mu(k)$.

2.2 Adler-Bell-Jackiw Anomaly

In 1969, Adler [36] and, independently, Bell and Jackiw [37] derived an anomalous term present in the divergence of the axial current. This is known as the Adler-Bell-Jackiw (ABJ) anomaly or the Abelian chiral anomaly. To discuss this point, we need to introduce the following Green's functions:

$$G_{\mu\nu\lambda} = \left\langle 0 \left| T j_\mu(x) j_\nu(y) j^5_\lambda(z) \right| 0 \right\rangle\,, \tag{2.12}$$

$$G_{\mu\nu} = \left\langle 0 \left| T j_\mu(x) j_\nu(y) P(z) \right| 0 \right\rangle\,. \tag{2.13}$$

Naively we can consider

$$\partial^\mu_x G_{\mu\nu\lambda} = \partial^\nu_y G_{\mu\nu\lambda} = 0\,, \tag{2.14}$$

and

$$\partial^\lambda_z G_{\mu\nu\lambda} = 2miG_{\mu\nu}\,, \tag{2.15}$$

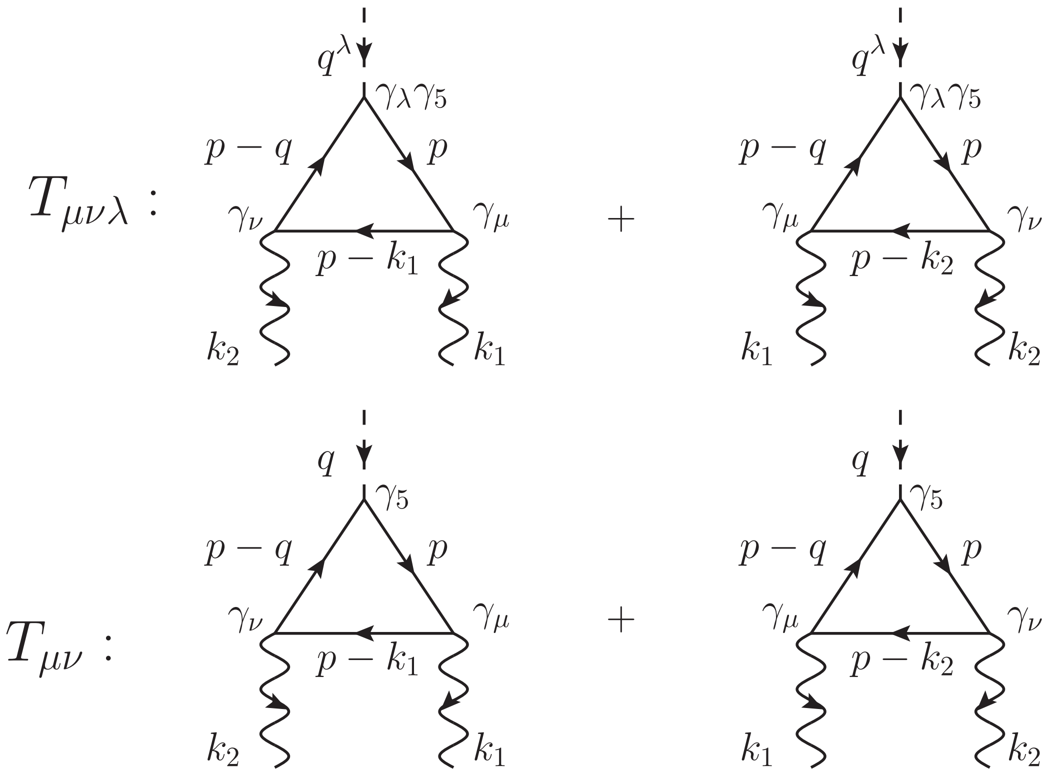

Figure 2.2: Triangle diagrams with vertices vector-vector-axial and vector-vector-pseudoscalar.

since

$$\begin{aligned}
&\partial_x^\mu \langle 0\,|T j_\mu(x) O_1(y_1)\dots O_n(y_n)|\,0\rangle = \langle 0\,|T \partial_x^\mu j_\mu(x) O_1(y_1)\dots O_n(y_n)|\,0\rangle + \\
&\sum_{i=1}^{n} \langle 0\,\big|T\,[j_0(x),\,O_i(y_i)]\,\delta(x^0-y_i^0) O_1(y_1)\dots O_{i-1}(y_{i-1}) O_{i+1}(y_{i+1})\dots O_n(y_n)\big|\,0\rangle \\
&\Leftrightarrow \partial_x^\mu \langle 0\,|T j_\mu(x) O_1(y_1)\dots O_n(y_n)|\,0\rangle \simeq \langle 0\,|T \partial_x^\mu j_\mu(x) O_1(y_1)\dots O_n(y_n)|\,0\rangle\,,
\end{aligned} \tag{2.16}$$

where we neglect the contact terms $\delta(x^0-y_i^0)$ because they are meaningless in our calculations.

In the spirit of WI given in Eq. (2.11), we treat the relation given in Eq. (2.14) as the vector WI (VWI) and the relation given in Eq. (2.15) as the axial WI (AWI)[1]. The violation of these identities is manifestly present in perturbation theory, since as we shall see below, it is impossible to simultaneously verify both VWI and AWI.

In momentum space, these quantities are described through Fourier transformations as

$$T^{\mu\nu\lambda} = i\int d^4x d^4y d^4z\, e^{i(xk_1+yk_2-zq)} G^{\mu\nu\lambda}\,, \tag{2.17}$$

$$T^{\mu\nu} = i\int d^4x d^4y d^4z\, e^{i(xk_1+yk_2-zq)} G^{\mu\nu}\,. \tag{2.18}$$

They are depicted in Fig. 2.2, where we consider only triangle diagrams, without the external bosons.

We have then the VWI

$$k_{1,2}^{\mu,\nu} T_{\mu\nu\lambda} = \int d^4x d^4y d^4z\, e^{i(xk_1+yk_2-zq)} \partial_{x,y}^{\mu,\nu} G_{\mu\nu\lambda} = 0\,, \tag{2.19}$$

[1] We call this identity AWI just for the purpose of labelling it, but it is only a WI in the massless case, as can be seen from Eq (2.11).

and the AWI

$$q^\lambda T_{\mu\nu\lambda} = \int d^4x d^4y d^4z \, e^{i(xk_1+yk_2-zq)} \partial_z^\lambda G_{\mu\nu\lambda} \,, \tag{2.20}$$

which can be also identified as

$$q^\lambda T_{\mu\nu\lambda} = 2mi \int d^4x d^4y d^4z \, e^{i(xk_1+yk_2-zq)} G_{\mu\nu} = 2m T_{\mu\nu} \,, \tag{2.21}$$

where $q = k_1 + k_2$.

Now, we are able to compute the amplitudes $T_{\mu\nu}$ and $T_{\mu\nu\lambda}$ (the propagator is given in Eq. (2.7); for a review on Feynman rules see e.g. Ref. [38])

$$T_{\mu\nu\lambda} = -i \int \frac{d^4p}{(2\pi)^4} \mathrm{tr} \left[\frac{i}{\not{p} - m} \gamma_\lambda \gamma_5 \frac{i}{\not{p} - \not{q} - m} \gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m} \gamma_\mu \right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \,, \tag{2.22}$$

$$T_{\mu\nu} = -i \int \frac{d^4p}{(2\pi)^4} \mathrm{tr} \left[\frac{i}{\not{p} - m} \gamma_5 \frac{i}{\not{p} - \not{q} - m} \gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m} \gamma_\mu \right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \,. \tag{2.23}$$

The trace and the negative sign appear due to the fermion loop.

If we write

$$\not{q} \gamma_5 = \gamma_5 (\not{p} - \not{q} - m) + (\not{p} - m) \gamma_5 + 2m\gamma_5 \,, \tag{2.24}$$

and apply it in $q^\lambda T_{\mu\nu\lambda}$, we obtain

$$\begin{aligned} q^\lambda T_{\mu\nu\lambda} = & - \int \frac{d^4p}{(2\pi)^4} \mathrm{tr} \left[\frac{1}{\not{p} - m} \gamma_5 \gamma_\nu \frac{1}{\not{p} - \not{k}_1 - m} \gamma_\mu \right] \\ & - \int \frac{d^4p}{(2\pi)^4} \mathrm{tr} \left[\gamma_5 \frac{1}{\not{p} - \not{q} - m} \gamma_\nu \frac{1}{\not{p} - \not{k}_1 - m} \gamma_\mu \right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} + 2m T_{\mu\nu} \,. \end{aligned} \tag{2.25}$$

Due to the permutation in momenta and vertices, we can rearrange the terms to get

$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + R_{\mu\nu} \,, \tag{2.26}$$

where

$$R_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{tr} \left[\frac{1}{\not{p} - \not{k}_2 - m} \gamma_5 \gamma_\nu \frac{1}{\not{p} - \not{q} - m} \gamma_\mu - \frac{1}{\not{p} - m} \gamma_5 \gamma_\nu \frac{1}{\not{p} - \not{k}_1 - m} \gamma_\mu \right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \,. \tag{2.27}$$

For the AWI to hold, it is mandatory that $R_{\mu\nu} = 0$. Since $q = k_1 + k_2$, if we perform the shift $p \to p + k_2$ in the first term inside the trace, then the two terms cancel each other. Clearly, in the interchange $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$ the shift is $p \to p + k_1$. However, this procedure is only valid when the integral is convergent[2], which is not the case of $R_{\mu\nu}$. In Minkwoski space, if $\int d^4x \, f(x)$ is divergent, we can write (see Appendix A.1 for details)

$$\Delta(a) = \int d^4x \big[f(x+a) - f(x) \big] \approx \int d^4x \, a^\mu \partial_\mu f(x) = 2\pi^2 i a_\mu \lim_{r\to\infty} r^\mu r^2 f(r) \,. \tag{2.28}$$

[2]If the integral is divergent, this problem may change the Ward identity given in Eq. (2.11). A more complete discussion can be found in section 7.4 of Ref. [10].

Applying this result in $R_{\mu\nu}$, we find

$$\begin{aligned} R_{\mu\nu} =& \int \frac{d^4p}{(2\pi)^4}\left[f_{\mu\nu}(p-k_2) - f_{\mu\nu}(p)\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ =& -2\pi^2 i k_2^\tau \lim_{p\to\infty} p_\tau p^2 \frac{\mathrm{tr}\left[(\not{p}+m)\gamma_5\gamma_\nu(\not{p}-\not{k}_1+m)\gamma_\mu\right]}{(2\pi)^4\left[p^2-m^2\right]\left[(p-k_1)^2-m^2\right]} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} . \end{aligned} \tag{2.29}$$

Using the γ_5 properties, we can compute the trace as (see Appendix A.2 for details)

$$\mathrm{tr}\left[\gamma_5(-\not{p}+m)\gamma_\nu(\not{p}-\not{k}_1+m)\gamma_\mu\right] = -4i\varepsilon_{\beta\nu\alpha\mu}p^\beta k_1^\alpha \,. \tag{2.30}$$

In the limit $p \to \infty$, the remaining powers are the higher ones, so $R_{\mu\nu}$ reduces to

$$R_{\mu\nu} = -\frac{1}{2\pi^2}\varepsilon_{\beta\nu\alpha\mu}k_1^\alpha k_2^\tau \lim_{p\to\infty}\frac{p^\beta p_\tau}{p^2} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} . \tag{2.31}$$

Now, if we take the symmetric limit

$$g^{\beta\tau}\lim_{p\to\infty}\frac{p_\beta p_\tau}{p^2} = 1 \Leftrightarrow \lim_{p\to\infty}\frac{p_\beta p_\tau}{p^2} = \frac{g_{\beta\tau}}{4} \,, \tag{2.32}$$

and realizing that the interchanges $k_1 \leftrightarrow k_2$ and $\mu \leftrightarrow \nu$ contribute in the same amount, we finally obtain

$$R_{\mu\nu} = -\frac{1}{4\pi^2}\varepsilon_{\beta\nu\alpha\mu}k_1^\alpha k_2^\beta = \frac{1}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta . \tag{2.33}$$

From this result, it is clear that the AWI changes to an anomalous term. Nevertheless, it is not yet the result we have been looking for. Through this calculation we explicitly verify that the result depends on the shift performed. One may wonder how a global shift in the momentum p, running the loop in $T_{\mu\nu\lambda}$, changes the value of the amplitude. This mathematical ambiguity can be evaluated in a simple way if we perform the shift $p \to p + a$, where $a = \alpha k_1 + (\alpha - \beta)k_2$. So, computing the difference between amplitudes

$$\Delta_{\mu\nu\lambda}(a) = T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0) \,, \tag{2.34}$$

where $T_{\mu\nu\lambda}(a)$ and $T_{\mu\nu\lambda}(0)$ are the shifted and the original amplitudes respectively, we obtain

$$\begin{aligned} \Delta_{\mu\nu\lambda}(a) =& -\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{1}{\not{p}+\not{a}-m}\gamma_\lambda\gamma_5\frac{1}{\not{p}+\not{a}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}+\not{a}-\not{k}_1-m}\gamma_\mu\right] \\ &+ \int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m}\gamma_\mu\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} . \end{aligned} \tag{2.35}$$

From our previous considerations, we can rewrite

$$\begin{aligned} \Delta_{\mu\nu\lambda}(a) =& -\int \frac{d^4p}{(2\pi)^4}\left[f_{\mu\nu\lambda}(p+a) - f_{\mu\nu\lambda}(p)\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} \\ =& -2\pi^2 i a^\tau \lim_{p\to\infty} p_\tau p^2 \frac{\mathrm{tr}\left[(\not{p}+m)\gamma_\lambda\gamma_5(\not{p}-\not{q}+m)\gamma_\nu(\not{p}-\not{k}_1+m)\gamma_\mu\right]}{(2\pi)^4\left[p^2-m^2\right]\left[(p-q)^2-m^2\right]\left[(p-k_1)^2-m^2\right]} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} . \end{aligned} \tag{2.36}$$

Picking up the higher powers of the momentum p, we obtain

$$\Delta_{\mu\nu\lambda}(a) = -2\pi^2 i a^\tau \lim_{p\to\infty} p_\tau p^2 \frac{\mathrm{tr}\left[\not{p}\gamma_\lambda\gamma_5\not{p}\gamma_\nu\not{p}\gamma_\mu\right]}{(2\pi)^4 p^6} + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix} , \tag{2.37}$$

which can be reduced by computing the trace (see Appendix A.2 for details)

$$\mathrm{tr}\left[\not{p}\gamma_\lambda\gamma_5\not{p}\gamma_\nu\not{p}\gamma_\mu\right] = 4ip^2 p^\beta \varepsilon_{\lambda\nu\beta\mu}\,, \tag{2.38}$$

and taking the symmetric limit given in Eq. (2.32),

$$\Delta_{\mu\nu\lambda}(a) = \frac{1}{8\pi^2}\varepsilon_{\lambda\nu\beta\mu}a^\beta + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}. \tag{2.39}$$

Finally we get

$$\begin{aligned}\Delta_{\mu\nu\lambda}(a) &= \frac{1}{8\pi^2}\left[\varepsilon_{\mu\nu\lambda\tau}(\alpha k_1 + (\alpha-\beta)k_2)^\tau + \varepsilon_{\nu\mu\lambda\tau}(\alpha k_2 + (\alpha-\beta)k_1)^\tau\right] \\ &= \frac{\beta}{8\pi^2}\varepsilon_{\mu\nu\lambda\tau}(k_1-k_2)^\tau\,,\end{aligned} \tag{2.40}$$

which can be contracted with q^λ in order to check the effect of the shift in the AWI

$$q^\lambda T_{\mu\nu\lambda}(a) = q^\lambda \Delta_{\mu\nu\lambda}(a) + q^\lambda T_{\mu\nu\lambda}(0) = 2mT_{\mu\nu} + \frac{1-\beta}{4\pi^2}\varepsilon_{\mu\nu\alpha\tau}k_1^\alpha k_2^\tau\,. \tag{2.41}$$

This result leads an anomalous term when $\beta \neq 1$, meaning that the AWI does not hold. When $\beta = 1$, the AWI is verified, yet to check the real value for β we need to perform similar calculations for the VWI. Starting from

$$k_1^\mu T_{\mu\nu\lambda}(0) = -\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{1}{\not{p}-m}\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m}\not{k}_1\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}, \tag{2.42}$$

and applying the transformations

$$\begin{aligned}\not{k}_1\frac{1}{\not{p}-m} &= 1 - (\not{p}-\not{k}_1-m)\frac{1}{\not{p}-m}\,, \\ \frac{1}{\not{p}-\not{q}-m}\not{k}_1 &= -1 + \frac{1}{\not{p}-\not{q}-m}(\not{p}-\not{k}_2-m)\,,\end{aligned} \tag{2.43}$$

in the VWI, we get

$$\begin{aligned}k_1^\mu T_{\mu\nu\lambda}(0) &= -\int \frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{q}-m}\gamma_\nu\frac{1}{\not{p}-\not{k}_1-m} - \gamma_\lambda\gamma_5\frac{1}{\not{p}-\not{k}_2-m}\gamma_\nu\frac{1}{\not{p}-m}\right] \\ &= -\int \frac{d^4p}{(2\pi)^4}\left[f_{\nu\lambda}(p-k_1) - f_{\nu\lambda}(p)\right] \\ &= 2\pi^2 i k_1^\tau \lim_{p\to\infty} p_\tau p^2 \frac{\mathrm{tr}\left[\gamma_5(\not{p}-\not{k}_2+m)\gamma_\nu(\not{p}+m)\gamma_\lambda\right]}{(2\pi)^4\left[(p-k_2)^2-m^2\right]\left[p^2-m^2\right]} \\ &= -\frac{1}{8\pi^2}\varepsilon_{\beta\nu\alpha\lambda}k_1^\alpha k_2^\beta\,.\end{aligned} \tag{2.44}$$

Now, we need to complete this result with the possible effect of the shift $p \to p + a$

$$\begin{aligned}k_1^\mu T_{\mu\nu\lambda}(a) &= k_1^\mu \Delta_{\mu\nu\lambda}(a) + k_1^\mu T_{\mu\nu\lambda}(0) \\ &= \frac{1}{8\pi^2}\left[\beta\varepsilon_{\alpha\nu\lambda\tau}k_1^\alpha(k_1-k_2)^\tau - \varepsilon_{\tau\nu\alpha\lambda}k_1^\alpha k_2^\tau\right] \\ &= -\frac{1+\beta}{8\pi^2}\varepsilon_{\nu\lambda\alpha\tau}k_1^\alpha k_2^\tau.\end{aligned} \tag{2.45}$$

It is then clear that is impossible to verify both VWI and AWI. For the VWI to hold (conservation of vector current), the correct value should be $\beta = -1$. This implies that the AWI does not hold (non-conservation of the axial current), yielding an anomalous term[32]. Therefore,

$$k_1^\mu T_{\mu\nu\lambda} = 0\,,\;\; k_2^\nu T_{\mu\nu\lambda} = 0\,,\;\; q^\lambda T_{\mu\nu\lambda} = 2mT_{\mu\nu} + \mathcal{A}_{\mu\nu}\,, \tag{2.46}$$

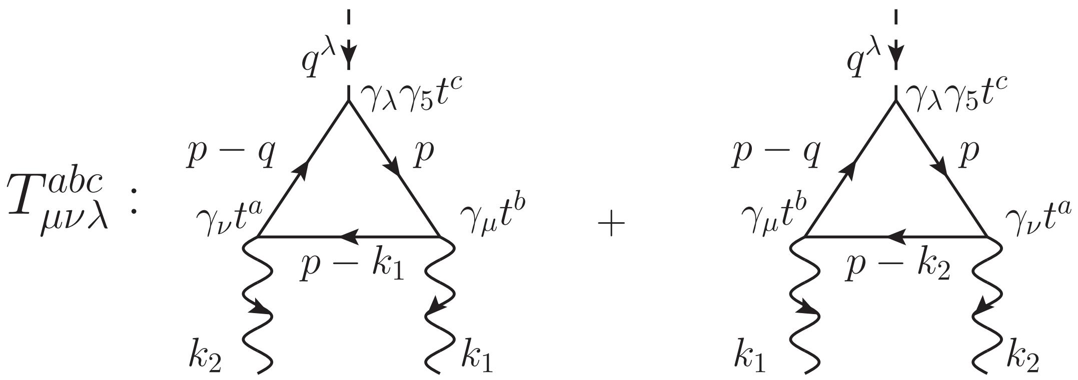

Figure 2.3: Triangle diagrams with vertices vector-vector-axial for non-Abelian gauges.

where $k_2^\nu T_{\mu\nu\lambda} = 0$ is a direct consequence of the symmetry present in the triangle diagram (or in the computations that lead to the verification of the VWI), and

$$\mathcal{A}_{\mu\nu} = \frac{1}{2\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \,. \tag{2.47}$$

This is precisely the result discovered by Adler, Bell and Jackiw that we mentioned in the beginning of this section. They considered the possibility that the divergence of the axial current produces two photons

$$\partial^\mu j^5_\mu = 2miP + \mathcal{A} = 2miP - \frac{e^2}{16\pi^2}\varepsilon_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} \,. \tag{2.48}$$

From experimental results (e.g. the pion decay into two photons), it is clear that the VWI holds and that the ABJ anomaly appears, which justifies our choice for β.

Since this is a physical and verified result, the ambiguity in β cannot appear. This is a consequence of the regularization scheme we use, which is purely mathematical and does not have any physical meaning. Instead, if we use other regulators, such as Pauli-Villars regularization [39] or dimensional regularization by t'Hooft-Veltman [23, 35], the VWI is automatically verified. Our regulator with $\beta = -1$ preserves the VWI, but others do not. When two regulators give different results, it is usual to choose the one that verifies the WI, postulating that the underlying symmetry of the WI is a fundamental aspect of the theory [10].

The complete discussion of non-Abelian gauge anomalies is far more complex than the calculation for the Abelian one presented here. Nevertheless, we can reproduce this calculation including the non-Abelian generators in the vertices (from Feynman rules, the group generator t^a modifies the vertex Γ^μ to $\Gamma^\mu t^a$). The respective diagram is depicted in Fig. 2.3 and we can compute the amplitude as

$$T^{abc}_{\mu\nu\lambda} = -i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5 t^c\frac{i}{\not p - \not q - m}\gamma_\nu t^a\frac{i}{\not p - \not k_1 - m}\gamma_\mu t^b\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \\ a \leftrightarrow b \end{pmatrix} . \tag{2.49}$$

Since the gamma matrices commute with group generators ($[\gamma^\mu, t^a] = [\gamma_5, t^a] = 0$), we can write

$$T^{abc}_{\mu\nu\lambda} = -i\int\frac{d^4p}{(2\pi)^4}\mathrm{tr}\left[\frac{i}{\not p - m}\gamma_\lambda\gamma_5\frac{i}{\not p - \not q - m}\gamma_\nu\frac{i}{\not p - \not k_1 - m}\gamma_\mu\right]\mathrm{tr}_{\mathcal{R}}\left[t^c t^a t^b\right] + \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \\ a \leftrightarrow b \end{pmatrix} , \tag{2.50}$$

where $\mathrm{tr}_{\mathcal{R}}\left[t^c t^a t^b\right]$ stands for the trace of the group generators in the representation $\mathcal{R}$ of the fields. This modification leads to a change in the anomalous term $A_{\mu\nu}$ given in Eq. (2.47),

$$\begin{aligned}\mathcal{A}^{abc}_{\mu\nu} &= \frac{1}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \mathrm{tr}_{\mathcal{R}}\left[t^c t^a t^b\right] + \frac{1}{4\pi^2}\varepsilon_{\nu\mu\alpha\beta}k_2^\alpha k_1^\beta \mathrm{tr}_{\mathcal{R}}\left[t^c t^b t^a\right] \\ &= \frac{1}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \mathrm{tr}_{\mathcal{R}}\left[t^c t^a t^b + t^c t^b t^a\right] \\ &= \frac{1}{4\pi^2}\varepsilon_{\mu\nu\alpha\beta}k_1^\alpha k_2^\beta \mathrm{tr}_{\mathcal{R}}\left[\left\{t^a, t^b\right\} t^c\right] \, .\end{aligned} \tag{2.51}$$

Despite the subtleties of non-Abelian gauge anomalies, the commutation relation between gamma matrices and group generators still holds in general computations for these anomalies [40]. Therefore, since other loops contributing to the anomaly are proportional to $\mathrm{tr}_{\mathcal{R}}\left[\left\{t^a, t^b\right\} t^c\right]$, the relevant anomaly-free condition is

$$\mathrm{tr}_{\mathcal{R}}\left[\left\{t^a, t^b\right\} t^c\right] = 0\, , \tag{2.52}$$

when summed over all fermions of the theory. We recall that different fermion fields contribute additively to the anomaly if they couple to gauge bosons [41].

The other important aspect that should be mentioned is the fact that left and right-handed fermions contribute with an opposite sign. When an anomaly is present, usually the VWI is verified but an anomalous AWI appears. Thus, if we have an amplitude without the presence of γ_5 we expect no anomalies.

We stated in Section 1.2 that fermions can be decomposed into two orthogonal projections (ψ_L and ψ_R), which satisfy $P_R + P_L = 1$ and $P_R - P_L = \gamma_5$. It is now straightforward to realize that only the γ_5 part contributes to the anomaly and that the projectors P_R and P_L contribute with opposite sign [42]. The relation with chiral fields becomes then clear if we write

$$\overline{f_L} i \not{D}_L f_L + \overline{f_R} i \not{D}_R f_R = \overline{f} i \not{D}_L P_L f + \overline{f} i \not{D}_R P_R f\, , \tag{2.53}$$

where $\not{D}_L$ and $\not{D}_R$ are the covariant derivative for left and right-handed fields, respectively. Therefore, it is possible to describe the amplitude in terms of non-chiral fields and, instead, include the projectors in the propagators and vertices to account for chirality in computations. This argument is proved by Bardeen, calculating the opposite contribution of left and right-handed fermions in non-Abelian gauge anomalies [43].

The complete discussion of anomalies may alter the presented arguments due to the large spectrum of gauge theories combined with the space-time dimensions of the theory. Nevertheless, these arguments are quite general when dealing with the SM and its minimal extensions.

2.3 Anomaly Cancellation in the Standard Model

When we discussed the chiral anomaly, we did not mention the gauge group of the Lagrangian given in Eq. (2.1). Instead we derived the conservation laws and computed the anomaly directly from Feynman diagrams. These calculations do not change whether the external bosons are gauge bosons or not, but the type of the anomaly does. Since our goal was to point out the existence of chiral anomalies, we did not refer to this aspect before. However it is fundamental to distinguish between global and gauge anomalies in any gauge theory. A global anomaly appears if a global symmetry is anomalously broken, only implying that classical selection rules are not obeyed in the respective QFT and classically forbidden processes may actually occur. The Abelian anomaly, which breaks the symmetry under the global chiral transformation, is an example of this type. On the other hand, gauge anomalies occur when the external fields are gauge bosons, leading to an inconsistent theory [23].

QED, described in Section 1.1, is a very well-known example of an anomaly-free and consistent theory, since the possible gauge anomalies have photons as external bosons. The vertex in QED does not include γ_5 and thus, there is no anomalous terms in triangle diagrams.

For a gauge theory to be consistent, the contributions to anomalies of the different chiral fermions should cancel each other, satisfying Eq. (2.52). If the gauge group is a direct product of G_i ($i = 1, \ldots n$) factors, each being a simple or $U(1)$ group, the $G_i G_j G_k$ anomaly can be computed through the corresponding triangle diagram that couples to the gauge bosons associated with these gauge groups. We then need to check $(n+2)!/\left[3!\,(n-1)!\right]$ different anomaly conditions.

Similar to QED, the SM is an anomaly-free theory, which have chiral fermions as a main ingredient. Since there are three distinct G_i in the SM gauge group, namely $SU(3)_C$, $SU(2)_L$ and $U(1)_Y$, we need to check ten different possibilities. Recalling the SM Lagrangian and the fermion representations given in Table 1.1, we obtain for the Abelian anomaly

$$\begin{aligned}
\left[U(1)_Y\right]^3 : & \sum_{f_L} \mathrm{tr}_{\mathcal{R}} \left[\{g_Y Y_{f_L}, g_Y Y_{f_L}\}\, g_Y Y_{f_L}\right] - \sum_{f_R} \mathrm{tr}_{\mathcal{R}} \left[\{g_Y Y_{f_R}, g_Y Y_{f_R}\}\, g_Y Y_{f_R}\right] = \\
& 2g_Y^3 \left(\sum_{f_L} Y_{f_L}^3 - \sum_{f_R} Y_{f_R}^3\right) = 2g_Y^3\, n_G \left(6Y_q^3 + 2Y_\ell^3 - 3Y_u^3 - 3Y_d^3 - 1Y_e^3\right) = \\
& 2g_Y^3\, n_G \left[6\left(\frac{1}{6}\right)^3 + 2\left(-\frac{1}{2}\right)^3 - 3\left(\frac{2}{3}\right)^3 - 3\left(-\frac{1}{3}\right)^3 - 1\,(-1)^3\right] = 0\,,
\end{aligned} \tag{2.54}$$

where n_G is the number of generations ($n_G = 3$ in the SM) and Y_{f_L}, Y_{f_R}, Y_q, Y_ℓ, Y_u, Y_d and Y_e are the hypercharges of f_L, f_R, $q_{L\alpha}$, $\ell_{L\alpha}$, $u_{R\alpha}$, $d_{R\alpha}$ and $e_{R\alpha}$ respectively. We keep n_G explicit in the calculation to clarify that anomaly cancellation occurs between quarks and leptons within each generation. In fact, this holds for all the gauge anomalies of the SM [44–46]. The coefficients multiplying the hypercharges account for the representations of the respective fields in the SM. Thus, since these representations and hypercharge assignments do not discriminate generations, it is clear that the cancellation of an anomaly must occur within each generation.

To simplify the notation we can write the $\left[U(1)_Y\right]^3$ anomaly as

$$\left[U(1)_Y\right]^3 : \mathrm{tr}_{\mathcal{R}} \left[\{g_Y Y_f, g_Y Y_f\}\, g_Y Y_f\right] \to 2g_Y^3 \sum_f Y_f^3 = 0\,, \tag{2.55}$$

keeping in mind that fermions with opposite chirality contribute with opposite signs. With this short notation we can now proceed to the calculation of the other nine anomalies.

For the $\left[U(1)_Y\right]^2 SU(2)_L$ anomaly we get

$$\left[U(1)_Y\right]^2 SU(2)_L : \mathrm{tr}_{\mathcal{R}} \left[\{g_Y Y_f, g_Y Y_f\}\, g_W T^c\right] \to g_Y^2 g_W\, \mathrm{tr}_{\mathcal{R}} \left[\{Y_f, Y_f\}\right] \mathrm{tr}_{\mathcal{R}} \left[T^c\right] = 0\,. \tag{2.56}$$

Due to the direct product, for distinct factors G_i the trace factorizes into the trace of each generator. Also, when discussing the SM gauge group, we use T^i as $SU(2)$ group generators and $\lambda^a/2$ as $SU(3)$

group generators. Regardless of the fermionic content of the SM, these generators are traceless and, therefore, the $\left[U(1)_Y\right]^2 SU(2)_L$ anomaly vanishes.

Similarly, we get direct cancellation of another four anomalies, $\left[U(1)_Y\right]^2 SU(3)_C$, $\left[SU(2)_L\right]^2 SU(3)_C$, $U(1)_Y SU(2)_L SU(3)_C$ and $SU(2)_L \left[SU(3)_C\right]^2$.

Due to the properties of Pauli matrices, there is another anomaly whose cancellation is independent of the fermionic content of the SM

$$\left[SU(2)_L\right]^3 : \mathrm{tr}_{\mathcal{R}}\left[\left\{g_W T^a, g_W T^b\right\} g_W T^c\right] = g_W^3\, \mathrm{tr}_{\mathcal{R}}\left[\frac{1}{2}\delta^{ab} T^c\right] \rightarrow \frac{g_W^3}{2}\delta^{ab}\, \mathrm{tr}_{\mathcal{R}}\left[T^c\right] = 0\,. \tag{2.57}$$

The cancellation of the $\left[SU(3)_C\right]^3$ anomaly is quite simple

$$\left[SU(3)_C\right]^3 : \mathrm{tr}_{\mathcal{R}}\left[\left\{g_s \frac{\lambda^a}{2}, g_s \frac{\lambda^b}{2}\right\} g_s \frac{\lambda^c}{2}\right] \rightarrow \frac{g_s^3}{2} d^{abc}\, n_G \sum_{quarks} = \frac{g_s^3}{2} d^{abc}\, n_G\,(2-1-1) = 0\,. \tag{2.58}$$

The only fermions that couple to gluons are the quarks, which appear symmetrically in the left and right sectors. The coefficient $d^{abc}/2$ (d^{abc} is the symmetric structure constants of $SU(3)$) is the correct one, but it is irrelevant since left and right-handed quarks couple with opposite sign. So, this symmetric structure leads to cancellation of this anomaly.

Now, there are only two anomalies left. For the $U(1)_Y \left[SU(3)_C\right]^2$ we obtain

$$\begin{aligned} U(1)_Y \left[SU(3)_C\right]^2 : \mathrm{tr}_{\mathcal{R}}\left[\left\{g_s \frac{\lambda^a}{2}, g_s \frac{\lambda^b}{2}\right\} g_Y Y_f\right] &= g_s^2 g_Y\, \mathrm{tr}_{\mathcal{R}}\left[\frac{1}{3}\delta^{ab}\,\mathbf{I}_3 Y_f + d^{abx} T^x Y_f\right] \\ \rightarrow g_s^2 g_Y\, n_G \sum_{quarks} Y_f &= g_s^2 g_Y\, n_G \left[2\left(\frac{1}{6}\right) - 1\left(\frac{2}{3}\right) - 1\left(-\frac{1}{3}\right)\right] = 0\,, \end{aligned} \tag{2.59}$$

and for the $U(1)_Y \left[SU(2)_L\right]^2$ we get

$$\begin{aligned} U(1)_Y \left[SU(2)_L\right]^2 : \mathrm{tr}_{\mathcal{R}}\left[\left\{g_W T^a, g_W T^b\right\} g_Y Y_f\right] &= g_W^2 g_Y\, \mathrm{tr}_{\mathcal{R}}\left[\frac{1}{2}\delta^{ab}\,\mathbf{I}_2 Y_f\right] \\ \rightarrow g_W^2 g_Y\, n_G \sum_{f_L} Y_f &= g_W^2 g_Y\, n_G \left[3\left(\frac{1}{6}\right) + 1\left(-\frac{1}{2}\right)\right] = 0\,, \end{aligned} \tag{2.60}$$

where $\mathbf{I}_n$ is the identity matrix in n dimensions.

The SM accounts for three fundamental forces in nature, but a complete theory should include the well-known gravitational force. For the purpose of studying gravitational anomalies we need to consider anomalies under local Lorentz transformations, which can be considered $SO(4)$ gauge transformations in Euclidean space with four dimensions [47, 48]. From all the possible mixed gauge-gravitational and pure gravitational anomalies, the only non-trivial triangle anomaly is the mixed $U(1)_Y$-gravitational anomaly, $U(1)_Y \left[SO(4)\right]^2$, usually simply denoted as $U(1)_Y$. This argument follows from the similar properties shared between $SO(4)$ and $SU(2)$, which imply that purely gravitational anomalies ($\left[SO(4)\right]^3$) and $\left[SO(4)\right] G_1 G_2$ anomalies, with $G_i \neq SO(4)$, automatically vanish [42].

From our previous considerations, therefore, in the SM minimally coupled to gravity in four dimensions, the only anomaly that we need to check explicitly is $U(1)_Y$. Since all particles couple universally to gravity, the relevant cancellation condition is easily verified

$$\begin{aligned} &U(1)_Y : \mathrm{tr}_{\mathcal{R}}\left[\left\{t^a_{SO(4)}, t^b_{SO(4)}\right\} g_Y Y_f\right] \rightarrow g_G^2\, g_Y \sum_f Y_f = \\ &g_G^2\, g_Y\, n_G \left[6\left(\frac{1}{6}\right) + 2\left(-\frac{1}{2}\right) - 3\left(\frac{2}{3}\right) - 3\left(-\frac{1}{3}\right) - 1\,(-1)\right] = 0\,, \end{aligned} \tag{2.61}$$

where g_G is a constant that account for gravity effects and does not affect the cancellation of the anomaly.

In conclusion, the SM is an anomaly-free theory, even when we minimally extend it to couple with gravity in four dimensions[3]. Throughout this proof, the hypercharge assignments (and particle representations) that we applied in the anomaly-free conditions were already fixed. Rather than checking if the SM is anomaly free, given a set of hypercharges, we could impose the anomaly-free conditions and verify whether or not these constraints lead to the uniqueness of hypercharges. If there is a single solution, then electric charge (hypercharge) is quantized [50–52].

From the ten conditions, six automatically vanish (including gravity, fifteen out of twenty). However there are only three (four) relevant equations because the $[SU(3)_C]^3$ anomaly does not constrain the hypercharges. Therefore, considering mixed-gravitational anomaly and family universal assignments, we get

$$\begin{aligned} [U(1)_Y]^3 &: 2g_Y^3\, n_G \left(6Y_q^3 + 2Y_\ell^3 - 3Y_u^3 - 3Y_d^3 - Y_e^3\right) = 0\,, \\ U(1)_Y\,[SU(2)_L]^2 &: g_W^2 g_Y\, n_G\,(3Y_q + Y_\ell) = 0\,, \\ U(1)_Y\,[SU(3)_C]^2 &: g_s^2 g_Y\, n_G\,(2Y_q - Y_u - Y_d) = 0\,, \\ U(1)_Y &: g_G^2\, g_Y\, n_G\,(6Y_q + 2Y_\ell - 3Y_u - 3Y_d - Y_e) = 0\,. \end{aligned} \tag{2.62}$$

From these equations, we clearly see that the anomaly-free conditions express the rescaling invariance present in the SM Lagrangian. If we perform the changes $g_Y \to \alpha g_Y$ and $Y_i \to Y_i/\alpha$ the Lagrangian $(i g_Y B_\mu Y)$ remains invariant. We can solve this system as a function of only one hypercharge, e.g. Y_q,

$$Y_e = -6Y_q\,,\ Y_\ell = -3Y_q\,,\ Y_u = 4Y_q\,,\ Y_d = -2Y_q\,. \tag{2.63}$$

Yet, there is no charge quantization since we cannot relate Y_q and Y_H from these equations. Using rescaling invariance (overall factor) we can fix one of these parameters (e.g. $Y_H = \frac{1}{2}$), however there is yet one free hypercharge. Therefore, to obtain charge quantization additional input is needed.

From experiments, it is very well-known that left and right-handed fermions have the same electric charge [31]. If we use these constraints and the relation between hypercharge and electric charge given in Eq. (1.57) we obtain charge quantization. Another possibility is to use the gauge invariance of the Yukawa part of the SM Lagrangian, which yields the same result. Indeed, after the diagonalization of the Yukawa coupling matrices given in Eq. (1.72), the $U(1)_Y$ gauge invariance leads to the constraints

$$-Y_q - Y_H + Y_u = 0\,,\ -Y_q + Y_H + Y_d = 0\,,\ -Y_\ell + Y_H + Y_e = 0\,. \tag{2.64}$$

From these equations and the anomaly-free conditions we obtain

$$Y_q = \frac{Y_H}{3}\,,\ Y_u = \frac{4Y_H}{3}\,,\ Y_d = -\frac{2Y_H}{3}\,,\ Y_\ell = -Y_H\,,\ Y_e = -2Y_H\,. \tag{2.65}$$

Then, using the freedom of the overall factor, we can fix $Y_H = \frac{1}{2}$ to obtain the hypercharge quantization with the assignments given in Table 1.1.

[3] In models with other number of dimensions, the relevant group may not be $SO(4)$. If another group stands for the gauge part of a minimal extension of the SM, then, purely gravitational anomalies (and others) could appear. For a more complete derivation of the SM anomalies, as well as a comprehensive discussion of gravitational effects on anomalies, see e.g. Ref. [49].

To conclude this chapter, we remark that, considering non-universal hypercharges, there is no in the SM. In this case, the relevant (generalized) anomaly-free conditions are

$$\begin{aligned}
[U(1)_Y]^3 : \sum_{i=1}^{n_G} \left(6Y_{q_i}^3 + 2Y_{\ell_i}^3 - 3Y_{u_i}^3 - 3Y_{d_i}^3 - Y_{e_i}^3\right) = 0\,, \\
U(1)_Y\,[SU(2)_L]^2 : \sum_{i=1}^{n_G} (3Y_{q_i} + Y_{\ell_i}) = 0\,, \\
U(1)_Y\,[SU(3)_C]^2 : \sum_{i=1}^{n_G} (2Y_{q_i} - Y_{u_i} - Y_{d_i}) = 0\,, \\
U(1)_Y : \sum_{i=1}^{n_G} (6Y_{q_i} + 2Y_{\ell_i} - 3Y_{u_i} - 3Y_{d_i} - Y_{e_i}) = 0\,,
\end{aligned} \tag{2.66}$$

containing fifteen free parameters. Even with the gauge invariance constraints

$$-Y_{q_i} - Y_H + Y_{u_i} = 0\,,\ -Y_{q_i} + Y_H + Y_{d_i} = 0\,,\ -Y_{\ell_i} + Y_H + Y_{e_i} = 0\,,\ i = 1, 2, 3\,, \tag{2.67}$$

that also include Y_H, there are only thirteen equations, which do not yield a unique solution for the sixteen hypercharges. Thus, there is no electric charge quantization in the SM.

Chapter 3

Neutrinos and Seesaw Mechanisms

Neutrino oscillation experiments have firmly established the existence of neutrino masses and lepton mixing, implying that new physics beyond the SM is required to account for these observations [53, 54].

The fact that neutrino masses are tiny constitutes a puzzling aspect of nowadays particle physics. One of the most appealing theoretical frameworks to understand the smallness of neutrino masses is the so-called seesaw mechanism (for recent reviews see e.g. [55, 56]). In this context, the tree-level exchanges of new heavy states generate an effective neutrino mass matrix at low energies. Three simple possibilities consist of the addition of singlet right-handed neutrinos (type I seesaw), colour-singlet $SU(2)_L$-triplet scalars (type II) or $SU(2)_L$-triplet fermions (type III).

In this chapter, we briefly review neutrino oscillations and mass generation through the three types of seesaw mechanism. We then discuss how the anomaly-free conditions are modified and electric charge quantization is realized in these minimal SM extensions. Finally, we study phenomenologically viable and predictive flavour structures of the effective neutrino mass matrix. In particular, we look for all possible type I and/or Type III seesaw realizations of two-zero textures of the effective neutrino mass matrix compatible with the experimental data.

3.1 Neutrino Oscillations and Masses

In 1957, the idea of neutrino oscillations was proposed by Bruno Pontecorvo, considering that neutrino-antineutrino transitions may occur [57]. Although such transition has not been experimentally verified, it was at the origin of a theory explaining neutrino flavour oscillations. The first evidence of this phenomena occur in 1968 when experiments with the aim of measuring the flux of solar neutrinos found results suggesting the disappearance of electron-neutrinos (ν_e) [58]. Gribov and Pontecorvo realized that this disappearance was easily explained in terms of neutrino oscillations [59, 60].

The main concept of these oscillations is very similar to the transitions that change quark flavour. When neutrinos take part in weak interactions they are created with a specific lepton flavour (ν_e, ν_μ, ν_τ), although there is a non-zero probability of being in a different flavour state when they are measured. This means that the initial state is not an eigenstate/stationary state but a superposition of them. The

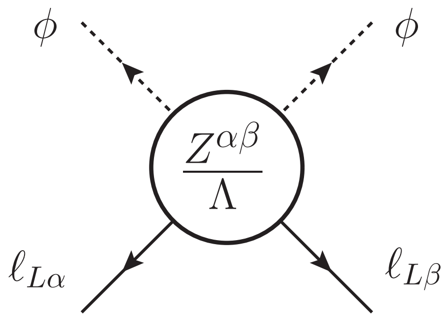

Figure 3.1: Feynman diagram for the $d = 5$ Weinberg operator. The effective couplings arise at low energy after the decoupling of heavy states. This diagram should be interpreted as a four-point interaction.

similarities with the Cabibbo angle and with the quark mixing matrix were firstly realized by Maki, Nakagawa, and Sakata, introducing the neutrino mixing matrix [61]. For three generations of neutrinos, the mismatch between interaction basis and mass basis is given by

$$|\nu'_\alpha\rangle = \sum_{i=1}^{3} \mathbf{U}_{\alpha i}\, |\nu_i\rangle \ , \tag{3.1}$$

where $\mathbf{U}$ is a 3×3 unitary matrix and $|\nu_i\rangle$ is a neutrino mass eigenstate.

As explained in Section 1.4, we can rotate the neutrino fields freely due to the absence of a mass term, matching the interaction and mass bases. However, since experiments confirm a non-zero probability transition, which is directly related with the neutrino mass squared differences, neutrinos cannot be massless. Hence, one of the greatest interests concerning neutrino oscillations is the necessity of the introduction of a mass term and a mixing matrix for neutrinos, which are not present in the original SM. Since right-handed neutrino fields (ν_R) are not included in the SM, no Dirac mass can be written down. The possible low-energy mass term that can be constructed after the electroweak SBB is the Majorana one, $\mathbf{m}_\nu^{\alpha\beta}\, \overline{\nu_{L\alpha}}\, \nu_{L\beta}^c$ (see Appendix A.2), where $\mathbf{m}_\nu^{\alpha\beta}$ is a symmetric matrix since it is contracted with the symmetric quantity $\overline{\nu_{L\alpha}}\, \nu_{L\beta}^c$. Nevertheless, this Majorana mass term cannot be generated by nonperturbative effects nor in higher loop corrections because it violates the lepton number ($\Delta L = 2$) and therefore, the $B - L$ symmetry, which is exact and non-anomalous in the SM [49]. Furthermore, there is no renormalizable invariant term that could account for the interaction $\overline{\ell_L}\, \ell_L^c$. Then, neutrinos are strictly massless in the SM.

In fact, since neutrinos are massive, the SM should be considered an effective theory and it is necessary to extend it with non-renormalizable terms that generate neutrino masses through new physics. The lowest order non-renormalizable operator, which generates Majorana neutrino masses after SSB, is the unique $d = 5$ Weinberg operator [62, 63] (the respective diagram is depicted in Fig. 3.1)

$$\mathcal{L}_{Weinberg} = -\frac{z^{\alpha\beta}}{\Lambda} \left(\overline{\ell_{L\alpha}} \widetilde{H}\right) C \left(\overline{\ell_{L\beta}} \widetilde{H}\right)^T + \text{H.c.} \xrightarrow{SSB} -\frac{1}{2} \mathbf{m}_\nu^{\alpha\beta}\, \overline{\nu_{L\alpha}}\, \nu_{L\beta}^c + \text{H.c.} + \cdots \ , \tag{3.2}$$

where $\mathbf{m}_\nu^{\alpha\beta} = v^2 z^{\alpha\beta}/\Lambda$ is 3×3 effective neutrino mass matrix, $z^{\alpha\beta}$ are complex constants and Λ is the new high-energy physics cutoff scale.

In this context, we can write the lepton mass terms in the interaction basis as

$$\mathcal{L}_{lep.mass} = -\mathbf{m}_e^{\alpha\beta}\,\overline{e'_{L\alpha}}e'_{R\beta} - \frac{1}{2}\mathbf{m}_\nu^{\alpha\beta}\,\overline{\nu'_{L\alpha}}\,\nu'^{c}_{L\beta} + \text{H.c.}\,. \tag{3.3}$$

In order to diagonalize these matrices we use the unitary transformations given in Eq. (1.74),

$$e'_{L\alpha} = \mathbf{L}_e^{\alpha\beta} e_{L\beta}\,,\; e'_{R\alpha} = \mathbf{R}_e^{\alpha\beta} e_{R\beta}\,,\; \nu'_{L\alpha} = \mathbf{L}_\nu^{\alpha i}\nu_{Li}\,, \tag{3.4}$$

which lead to

$$\mathbf{L}_e^\dagger \mathbf{m}_e \mathbf{R}_e = \mathbf{d}_e = \text{diag}\left(m_e, m_\mu, m_\tau\right)\,,\; \mathbf{L}_\nu^\dagger \mathbf{m}_\nu \mathbf{L}_\nu^* = \mathbf{d}_n = \text{diag}\left(m_1, m_2, m_3\right)\,, \tag{3.5}$$

where $\mathbf{d}_e$, $\mathbf{d}_n$ are the diagonal mass matrices and m_i $(i = 1, 2, 3)$ is the mass of the light neutrino ν_i. With these transformations the charged current becomes

$$\begin{aligned}\mathcal{L}_{CC} &= \frac{g_W}{\sqrt{2}}\left(\overline{u'_{L\alpha}}\gamma^\mu d'_{L\alpha} + \overline{\nu'_{L\alpha}}\gamma^\mu e'_{L\alpha}\right)W^+_\mu + \text{H.c.} \Leftrightarrow \\ \mathcal{L}_{CC} &= \frac{g_W}{\sqrt{2}}\left(\overline{u_{L\alpha}}\,\mathbf{V}_{CKM}^{\alpha\beta}\gamma^\mu d_{L\beta} + \overline{\nu_{Li}}\,\mathbf{U}_{PMNS}^{\dagger i\alpha}\gamma^\mu e_{L\alpha}\right)W^+_\mu + \text{H.c.}\,,\end{aligned} \tag{3.6}$$

where

$$\mathbf{U}_{PMNS} = \mathbf{L}_e^\dagger \mathbf{L}_\nu = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3}\end{pmatrix}\,. \tag{3.7}$$

The unitary matrix $\mathbf{U}_{PMNS}$ is known as the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) leptonic mixing matrix and is usually parametrized by the Particle Data Group as [30]

$$\mathbf{U}_{PMNS} = \mathbf{V}\mathbf{P}\,,\; \mathbf{P} = \text{diag}\left(1, e^{i\frac{\alpha_1}{2}}, e^{i\frac{\alpha_2}{2}}\right)\,, \tag{3.8}$$

where α_1 and α_2 represent the phases associated with the Majorana character of neutrinos, and $\mathbf{V}$ is parametrized as $\mathbf{V}_{CKM}$. Therefore, the matrix $\mathbf{U}_{PMNS}$ contains three mixing angles, one Dirac CP-violating phase, and two or zero Majorana phases whether neutrinos are Majorana or Dirac particles, respectively.

Instead of using the Weinberg operator, we could obviously extend the SM in a natural way by including right-handed neutrinos to generate Dirac masses through Yukawa couplings, as for the other SM fermions. However, to explain the smallness of neutrino masses ($\lesssim 1$ eV), the term $\mathbf{Y}_\nu^{\alpha i}\overline{\ell_{L\alpha}}\widetilde{H}\nu_{Ri}$ is widely regarded as unsatisfactory because it requires $\mathbf{Y}_\nu \lesssim 10^{-11}$, which is unnatural since it is much smaller than the SM couplings $\mathbf{Y}_{u,d,e}$ (all known couplings are between $Y_{electron} \sim 10^{-6}$ and $Y_{top} \sim 1$). The effective Weinberg operator solves this problem elegantly, because the scale Λ could be high enough to account for the tiny neutrino masses ($\mathbf{m}_\nu \sim v^2/\Lambda \lesssim 1$ eV, with $\Lambda \gtrsim 10^{14}$ GeV).

To finalize this section, we address the problem of charge quantization if neutrinos were Dirac particles [64, 65]. Considering right-handed neutrinos as $SU(3)_C \otimes SU(2)_L$ singlets with hypercharge Y_ν, one obtains the anomaly-free conditions

$$
\begin{aligned}
[U(1)_Y]^3 &: \sum_{i=1}^{n_G} \left(6Y_{q_i}^3 + 2Y_{\ell_i}^3 - 3Y_{u_i}^3 - 3Y_{d_i}^3 - Y_{e_i}^3 - Y_{\nu_i}^3\right) = 0\,,\\
U(1)_Y\,[SU(2)_L]^2 &: \sum_{i=1}^{n_G} (3Y_{q_i} + Y_{\ell_i}) = 0\,,\\
U(1)_Y\,[SU(3)_C]^2 &: \sum_{i=1}^{n_G} (2Y_{q_i} - Y_{u_i} - Y_{d_i}) = 0\,,\\
U(1)_Y &: \sum_{i=1}^{n_G} (6Y_{q_i} + 2Y_{\ell_i} - 3Y_{u_i} - 3Y_{d_i} - Y_{e_i} - Y_{\nu_i}) = 0\,,
\end{aligned}
\tag{3.9}
$$

which are automatically satisfied if we impose the SM hypercharge assignments and fix $Y_{\nu_i} = 0$. From the gauge invariance of the Yukawa Lagrangian, one can obtain twelve constraints, nine of them are given in Eq. (2.67), and the other three are

$$
-Y_{\ell_i} - Y_H + Y_{\nu_i} = 0\,,\; i = 1, 2, 3\,. \tag{3.10}
$$

Since we have nineteen free parameters (sixteen from the SM plus three Y_{ν_i}), the system cannot yield a unique solution and, therefore, there is no charge quantization as in the SM.

As seen before, when family universal hypercharges are considered, charge is quantized within the SM context; however, in this minimal extension, this is not the case. The anomaly-free conditions are now

$$
\begin{aligned}
[U(1)_Y]^3 &: 6Y_q^3 + 2Y_\ell^3 - 3Y_u^3 - 3Y_d^3 - Y_e^3 - Y_\nu^3 = 0\,,\\
U(1)_Y\,[SU(2)_L]^2 &: 3Y_q + Y_\ell = 0\,,\\
U(1)_Y\,[SU(3)_C]^2 &: 2Y_q - Y_u - Y_d = 0\,,\\
U(1)_Y &: 6Y_q + 2Y_\ell - 3Y_u - 3Y_d - Y_e - Y_\nu = 0\,,
\end{aligned}
\tag{3.11}
$$

and the constraints from Yukawa couplings become

$$
-Y_q - Y_H + Y_u = 0\,,\; -Y_q + Y_H + Y_d = 0\,,\; -Y_\ell + Y_H + Y_e = 0\,,\; -Y_\ell - Y_H + Y_\nu = 0\,, \tag{3.12}
$$

which lead to two free parameters (e.g. Y_q and Y_H). Hence, there is no unique solution when we fix $Y_H = \frac{1}{2}$. We conclude that charge is not quantized if neutrinos are Dirac particles, not even when the condition of family universal charges is assumed.

3.2 Seesaw Mechanisms

In what follows, we shall discuss how to obtain the Weinberg operator at low energies through the seesaw mechanism by introducing new heavy particles in the high energy theory. Since these heavy particles can have masses comparable to the scale in grand unified theories (GUT) [30], $\Lambda_{GUT} \sim 10^{16}$ GeV, they can explain the smallness of neutrino masses, making the seesaw mechanism a well-motivated framework. It is also simple because the tree-level exchange of heavy particles within this context generate the Weinberg operator without breaking the SM gauge group.

The different seesaw mechanisms contain the high-energy physics that dictates the couplings of the interaction term $\widetilde{H}\widetilde{H}\overline{\ell}\,\overline{\ell}$, as presented in Fig. 3.1, which in turn generates the neutrino mass matrix $\mathbf{m}_\nu$ after the electroweak SSB.

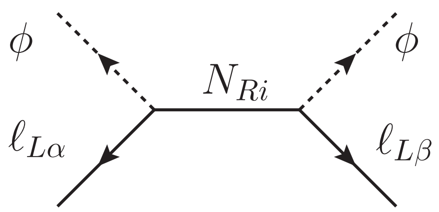

Figure 3.2: Exchange interactions of heavy particles N_{Ri} (introduced in the context of type I seesaw) that generate the Weinberg operator at low energy.

3.2.1 Type I Seesaw

To generate type I seesaw, n_R right-handed neutrino fields ν_{Ri} with the respective gauge group representations and hypercharge assignments $\sim (\mathbf{1},\mathbf{1},0)$ are introduced [66–70]. The respective Lagrangian is

$$\mathcal{L}_I = \mathcal{L}_{SM} + \frac{i}{2}\overline{\nu_{Ri}}\partial\!\!\!/\nu_{Ri} - \mathbf{Y}_\nu^{\alpha i}\overline{\ell_{L\alpha}}\widetilde{H}\nu_{Ri} - \frac{1}{2}\mathbf{M}_R^{ij}\,\overline{\nu_{Ri}^c}\nu_{Rj} + \text{H.c.}\,, \tag{3.13}$$

where $\mathbf{Y}_\nu$ is a $3 \times n_R$ complex Yukawa coupling matrix and $\mathbf{M}_R$ is a $n_R \times n_R$ symmetric matrix.

The heavy neutrino fields ν_{Ri} are the ones exchanged to generate the effective Weinberg operator, as depicted in Fig. 3.2. To easily analyse the respective Feynman diagram we work in the basis where right-handed neutrinos are mass eigenstates, diagonalizing $\mathbf{M}_R$ through

$$\nu_{Ri} = \mathbf{R}_R^{ij}N_{Rj}\,,\ \mathbf{R}_R^T\mathbf{M}_R\mathbf{R}_R = \mathbf{d}_R = \text{diag}\left(M_1,\dots M_{n_R}\right)\,. \tag{3.14}$$

We can now write the Lagrangian in this basis as

$$\mathcal{L}_I = \mathcal{L}_{SM} + i\overline{N_{Ri}}\partial\!\!\!/N_{Ri} - \mathbf{Y}_R^{\alpha i}\overline{\ell_{L\alpha}}\widetilde{H}N_{Ri} - \frac{1}{2}\mathbf{d}_R^{ij}\,\overline{N_{Ri}^c}N_{Rj} + \text{H.c.}\,, \tag{3.15}$$

where $\mathbf{Y}_R = \mathbf{Y}_\nu\mathbf{R}_R$, and calculate the amputated diagram. Comparing the two diagrams in Figs. 3.1 and 3.2, we see that the relevant part for the effective couplings are the two vertices and the N_{Ri} propagator. Reading the vertex and the propagator directly from the Lagrangian (3.15), we can write

$$\frac{z^{\alpha\beta}}{\Lambda} \propto \mathbf{Y}_R^{\alpha i}\frac{1}{p\!\!\!/ - M_i}\mathbf{Y}_R^{\beta i}\,. \tag{3.16}$$

Since the mass M_i of the neutrino N_{Ri} is much larger than the electroweak scale ($M_i \gg p$), we get

$$\frac{z^{\alpha\beta}}{\Lambda} \simeq -\mathbf{Y}_R^{\alpha i}\frac{1}{M_i}\mathbf{Y}_R^{\beta i} = -\mathbf{Y}_R^{\alpha i}\frac{1}{\mathbf{d}_R^{ii}}\mathbf{Y}_R^{\beta i} = -\mathbf{Y}_R^{\alpha i}\left(\mathbf{d}_R^{ii}\right)^{-1}\mathbf{Y}_R^{\beta i}\,. \tag{3.17}$$

Using Eq. (3.14), one can write

$$\begin{aligned}\frac{z^{\alpha\beta}}{\Lambda} &\simeq -\,\mathbf{Y}_R^{\alpha i}\left(\mathbf{d}_R^{ii}\right)^{-1}\mathbf{Y}_R^{\beta i} = -\left(\mathbf{Y}_R\mathbf{d}_R^{-1}\mathbf{Y}_R^T\right)^{\alpha\beta}\\ &= -\left(\mathbf{Y}_\nu\mathbf{R}_R\left(\mathbf{R}_R\right)^{-1}\left(\mathbf{M}_R\right)^{-1}\left(\mathbf{R}_R^T\right)^{-1}\mathbf{R}_R^T\mathbf{Y}_\nu^T\right)^{\alpha\beta} = -\left(\mathbf{Y}_\nu\mathbf{M}_R^{-1}\mathbf{Y}_\nu^T\right)^{\alpha\beta}\,.\end{aligned} \tag{3.18}$$

Finally, directly from Eq. (3.2), we obtain the desired effective mass matrix of the light neutrinos

$$\mathbf{m}_\nu = -v^2\mathbf{Y}_\nu\mathbf{M}_R^{-1}\mathbf{Y}_\nu^T = -\mathbf{m}_D\mathbf{M}_R^{-1}\mathbf{m}_D^T\,, \tag{3.19}$$

with $\mathbf{m}_D = v\mathbf{Y}_\nu$.

Our approach is very simple and a precise calculation should include the other factors that have been neglected, such as the different possible channels to realize the diagram (s- and t-channels) and higher orders in perturbation theory. Nevertheless, in the heavy mass limit, our computation has enough details to clarify the flavour structure of the neutrino mass matrix and its dependence on $\mathbf{m}_D$ and $\mathbf{M}_R$.

To conclude, we discuss the anomaly-free conditions within this minimal SM extension where neutrinos have a Majorana character. If we regard the SM as a low-energy effective theory, a more complete theory including right-handed neutrinos and the type I seesaw mechanism could be renormalizable. Even though n_R fields are included, if the gauge group remains the SM one, the anomaly-free conditions are satisfied without changes. This follows from the fact that right-handed neutrinos transform trivially under the SM gauge group. If one considers the right-handed neutrinos as colour and $SU(2)_L$ singlets but with arbitrary hypercharges Y_{ν_i}, the generalized gauge and mixed gauge gravitational anomaly-free conditions then become

$$\begin{aligned}
[U(1)_Y]^3 &: \sum_{i=1}^{n_G} \left(6Y_{q_i}^3 + 2Y_{\ell_i}^3 - 3Y_{u_i}^3 - 3Y_{d_i}^3 - Y_{e_i}^3\right) - \sum_{i=1}^{n_R} Y_{\nu_i}^3 = 0\,,\\
U(1)_Y\,[SU(2)_L]^2 &: \sum_{i=1}^{n_G} (3Y_{q_i} + Y_{\ell_i}) = 0\,,\\
U(1)_Y\,[SU(3)_C]^2 &: \sum_{i=1}^{n_G} (2Y_{q_i} - Y_{u_i} - Y_{d_i}) = 0\,,\\
U(1)_Y &: \sum_{i=1}^{n_G} (6Y_{q_i} + 2Y_{\ell_i} - 3Y_{u_i} - 3Y_{d_i} - Y_{e_i}) - \sum_{i=1}^{n_R} Y_{\nu_i} = 0\,,
\end{aligned} \tag{3.20}$$

which contain $15 + n_R$ free parameters. From the type I seesaw Lagrangian given in Eq. (3.13), besides the SM constraints

$$- Y_{q_i} - Y_H + Y_{u_i} = 0\,,\ -Y_{q_i} + Y_H + Y_{d_i} = 0\,,\ -Y_{\ell_i} + Y_H + Y_{e_i} = 0\,,\ i = 1, 2, 3\,, \tag{3.21}$$

we obtain the additional constraints

$$- Y_{\ell_i} - Y_H + Y_{\nu_j} = 0\,,\ Y_{\nu_j} + Y_{\nu_j} = 0\,,\ i = 1, 2, 3\,,\ j = 1, \dots n_R\,. \tag{3.22}$$

These last equations force $Y_{\nu_i} = 0$ and impose $Y_{\ell_\alpha} = -Y_H$, which lead to thirteen free parameters and the thirteen equations already present in the SM. Nevertheless, one can check that the system does not yield a viable solution for charge quantization since four variables are still free. The main departure from the case with Dirac neutrinos appears when we consider family universal charges. Since the Majorana mass term enforces $Y_\nu = 0$, from the point of view of anomalies, we are left with the SM case (the constraint $-Y_\ell - Y_H + Y_\nu = -Y_\ell - Y_H = 0$ does not bring an additional independent equation). Therefore, when one considers family universal charges, electric charge quantization only occurs if neutrinos are Majorana particles [65].

3.2.2 Type II Seesaw

The minimal type II framework requires the introduction of a scalar triplet $\hat{\Delta} = \{\hat{\Delta}_1, \hat{\Delta}_2, \hat{\Delta}_3\}$ with the respective gauge group representations and hypercharge assignments $\sim (\mathbf{1}, \mathbf{3}, 1)$ [71–75]. Since $\hat{\Delta}$ is a triplet **3**, the adjoint representation of $SU(2)$, it transforms as

$$\hat{\Delta}' = e^{-i\,T^a\omega_a}\hat{\Delta}\,,\ (T^a)^{ij} = -i\varepsilon^{aij}\,. \tag{3.23}$$

To easily address the construction of gauge invariant terms that contain $\hat{\Delta}$, one may use the Pauli matrices to write the scalar field as

$$\tilde{\Delta} = \hat{\Delta}_i\frac{\sigma^i}{2} = \frac{1}{2}\begin{pmatrix} \hat{\Delta}_3 & \hat{\Delta}_1 - i\hat{\Delta}_2 \\ \hat{\Delta}_1 + i\hat{\Delta}_2 & -\hat{\Delta}_3 \end{pmatrix}\,. \tag{3.24}$$

In the basis where T^3 is a 3×3 diagonal matrix, it is possible to relate the hypercharge $Y_\Delta = 1$ with the electric charge of these components of the 2×2 matrix representation, namely [53]

$$\tilde{\Delta} = \begin{pmatrix} \frac{\Delta^+}{\sqrt{2}} & -\Delta^{++} \\ \Delta^0 & -\frac{\Delta^+}{\sqrt{2}} \end{pmatrix} \Rightarrow Q\left(\tilde{\Delta}\right) = \begin{pmatrix} \frac{\Delta^+}{\sqrt{2}} & -2\Delta^{++} \\ 0 & -\frac{\Delta^+}{\sqrt{2}} \end{pmatrix}\,. \tag{3.25}$$

Therefore, $\tilde{\Delta}$ transforms according to $\tilde{\Delta}' = \mathbf{U}\tilde{\Delta}\mathbf{U}^\dagger$, where $\mathbf{U}$ is an unitary matrix. However, the term related with the Weinberg operator, $\overline{\ell_L}\tilde{\Delta}^\dagger\ell_L^c = \overline{\ell_L}\tilde{\Delta}^\dagger C\overline{\ell_L}^T$, and the term $\widetilde{H}^T\tilde{\Delta}\widetilde{H}$ are not gauge invariant because

$$\begin{aligned} \overline{\ell_L'}\tilde{\Delta}'^\dagger C\overline{\ell_L'}^T &= \overline{\ell_L}\mathbf{U}^\dagger\mathbf{U}\tilde{\Delta}^\dagger\mathbf{U}^\dagger\mathbf{U}^* C\overline{\ell_L}^T = \overline{\ell_L}\tilde{\Delta}^\dagger\mathbf{U}^\dagger\mathbf{U}^* C\overline{\ell_L}^T\,, \\ \widetilde{H}'^T\tilde{\Delta}'\widetilde{H}' &= \widetilde{H}^T\mathbf{U}^T\mathbf{U}\tilde{\Delta}\mathbf{U}^\dagger\mathbf{U}\widetilde{H} = \widetilde{H}^T\mathbf{U}^T\mathbf{U}\tilde{\Delta}\widetilde{H}\,. \end{aligned} \tag{3.26}$$

To generate an invariant term of this type, one needs to rotate the field through the Pauli matrix σ^2, as we do with the Higgs field. Therefore

$$\Delta = i\sigma^2\tilde{\Delta} = \begin{pmatrix} \Delta^0 & -\frac{\Delta^+}{\sqrt{2}} \\ -\frac{\Delta^+}{\sqrt{2}} & \Delta^{++} \end{pmatrix}\,, \tag{3.27}$$

which transforms as $\Delta' = \mathbf{U}^*\Delta\mathbf{U}^\dagger$, because $i\sigma^2\mathbf{U} = \mathbf{U}^* i\sigma^2$. This modification leads to the terms $\overline{\ell_L}\Delta^\dagger\ell_L^c$ and $\widetilde{H}^T\Delta\widetilde{H}$, which are invariant under the SM gauge group. Hence, the extended Lagrangian is

$$\mathcal{L}_{II} = \mathcal{L}_{SM} + \mathrm{tr}\left[(D_\mu\Delta)^\dagger(D^\mu\Delta)\right] + M_\Delta^2\mathrm{tr}\left[\Delta^\dagger\Delta\right] - \left(\mathbf{Y}_\Delta^{\alpha\beta}\overline{\ell_{L\alpha}}\Delta^\dagger\ell_{L\beta}^c - \mu\widetilde{H}^T\Delta\widetilde{H} + \text{H.c.}\right) + \cdots\,, \tag{3.28}$$

where $\mathbf{Y}_\Delta$ is a 3×3 symmetric matrix, M_Δ is the mass of Δ and μ is a coupling constant. There are also other interaction terms with Δ and H that do not affect our lowest order approximation (low energy limit).

Introducing only one scalar triplet, the previous Lagrangian is already written in the mass basis, thus one can repeat the process done for type I seesaw. Comparing the diagram depicted in Fig. 3.3 with the Weinberg operator diagram in Fig. 3.1, we get

$$\frac{z^{\alpha\beta}}{\Lambda} \propto \mathbf{Y}_\Delta^{\alpha\beta}\frac{1}{p^2 - M_\Delta^2}(-\mu) \simeq \frac{\lambda}{M_\Delta}\mathbf{Y}_\Delta^{\alpha\beta}\,, \tag{3.29}$$

where $\lambda = \frac{\mu}{M_\Delta}$ is an adimensional parameter. Then, after the electroweak SSB

$$\mathbf{m}_\nu = \frac{v\lambda}{M_\Delta}\mathbf{Y}_\Delta\,. \tag{3.30}$$

If we consider n_Δ scalar fields instead of just one, the effective mass matrix becomes

$$\mathbf{m}_\nu = \sum_{i=1}^{n_\Delta}\frac{v\lambda_i}{M_{\Delta_i}}\mathbf{Y}_{\Delta_i}\,,\ \lambda_i = \frac{\mu_i}{M_{\Delta_i}}\,, \tag{3.31}$$

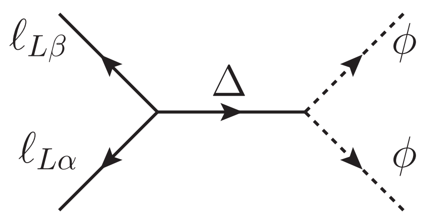

Figure 3.3: Exchange interactions of a heavy particle Δ (introduced in the context of type II seesaw) that generate the Weinberg operator at low energy.

in first order approximation.

Despite the complexity of the generalized procedure with more than one scalar, as we shall see in the next section, the minimal case contain less free parameters than type I (and type III) seesaw. This follows from the fact that the effective mass matrix for light neutrinos is uniquely determined by the flavour structure of $\mathbf{Y}_\Delta$, which has a direct correspondence between the high- and low-energy parameters ($\mathbf{m}_\nu \propto \mathbf{Y}_\Delta$). Under these considerations, the type II seesaw is a more economical framework than type I or type III.

Finally we address the charge quantization problem. The new field(s) is a scalar triplet, hence it does contribute to the anomalies and we basically get the same constraints as in the SM case. The gauge invariance leads to the new constraints

$$-2Y_{\ell_i} - Y_{\Delta_j} = 0\,,\ -2Y_H + Y_{\Delta_j} = 0\,,\ -Y_{\Delta_j} + Y_{\Delta_j} = 0\,,\ i = 1,2,3\,,\ j = 1,\dots n_\Delta\,, \tag{3.32}$$

when considering the addition of n_Δ scalars. Exactly as in type I, these equations impose $Y_{\Delta_j} = 2Y_H$ and $Y_{\ell_\alpha} = -Y_H$. Furthermore, the SM constraints are general regardless of the seesaw type, thus we arrive at the same conclusion of the type I case. Charge is quantized within the context of seesaw type II if family universal charges are assumed.

Since for our purposes, namely the study of anomaly-free gauge extensions of the SM and their connection with the flavour structure of $\mathbf{m}_\nu$, the type II seesaw mechanism does not lead to relevant constraints, we shall not consider it further in this work.

3.2.3 Type III Seesaw

In order to generate the effective neutrino mass matrix within the context of type III seesaw, one includes n_Σ fermion triplets Σ_{Ri} to the SM particle content [76]. The respective gauge group representations and hypercharge assignments are $\sim (\mathbf{1}, \mathbf{3}, 0)$. From the relation between electric charge and hypercharge ($Q = T^3 + Y$) and following a procedure analogous to the type II construction of $\tilde{\Delta}$, one obtains directly

$$\tilde{\Sigma}_{Ri} = \begin{pmatrix} \frac{\Sigma^0_{Ri}}{\sqrt{2}} & -\Sigma^+_{Ri} \\ \Sigma^-_{Ri} & -\frac{\Sigma^0_{Ri}}{\sqrt{2}} \end{pmatrix} \Rightarrow Q\left(\tilde{\Sigma}_{Ri}\right) = \begin{pmatrix} 0 & -\Sigma^+_{Ri} \\ -\Sigma^-_{Ri} & 0 \end{pmatrix}\,. \tag{3.33}$$

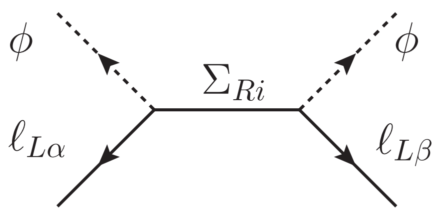

Figure 3.4: Exchange interactions of heavy particles Σ_{Ri} (introduced in the context of type III seesaw) that generate the Weinberg operator at low energy.

Usually, one redefines the field and writes it as

$$\Sigma_{Ri} = \begin{pmatrix} \Sigma^0_{Ri} & \sqrt{2}\Sigma^+_{Ri} \\ \sqrt{2}\Sigma^-_{Ri} & -\Sigma^0_{Ri} \end{pmatrix}\,, \tag{3.34}$$

to construct a mass term and the respective extended type III Lagrangian, which is

$$\mathcal{L}_{III} = \mathcal{L}_{SM} + \frac{i}{2}\mathrm{tr}\left(\overline{\Sigma_{Ri}}\not{D}\Sigma_{Ri}\right) - \mathbf{Y}_T^{\alpha i}\overline{\ell_{L\alpha}}\Sigma_{Ri}\widetilde{H} - \frac{1}{2}\mathbf{M}_\Sigma^{ij}\mathrm{tr}\left(\overline{\Sigma^c_{Ri}}\Sigma_{Rj}\right) + \mathrm{H.c.}\,, \tag{3.35}$$

where $\mathbf{Y}_T$ is a $3\times n_\Sigma$ complex Yukawa coupling matrix and $\mathbf{M}_\Sigma$ is a $n_\Sigma \times n_\Sigma$ symmetric matrix. This mass term includes Majorana masses for the fields $\left(\Sigma^0_{Ri}\right)^c + \Sigma^0_{Ri}$ and Dirac masses for the fields $\left(\Sigma^+_{Ri}\right)^c + \Sigma^-_{Ri}$.

As in the other two types of seesaw mechanisms, a new interaction is responsible for the effective Weinberg operator, namely the one presented in Fig. 3.4. To obtain the effective neutrino mass matrix one can extract the vertex and propagator from the Lagrangian, however, looking closely to the type I Lagrangian given in Eq. (3.13), we identify similar terms by simply replacing $\{\nu_{Ri}, \mathbf{Y}_\nu, \mathbf{M}_R\}$ by $\{\Sigma_{Ri}, \mathbf{Y}_T, \mathbf{M}_\Sigma\}$. Therefore, it is straightforward to conclude that

$$\frac{z^{\alpha\beta}}{\Lambda} \simeq -\left(\mathbf{Y}_T\mathbf{M}_\Sigma^{-1}\mathbf{Y}_T^T\right)^{\alpha\beta} \Rightarrow \mathbf{m}_\nu = -v^2\mathbf{Y}_T\mathbf{M}_\Sigma^{-1}\mathbf{Y}_T^T = -\mathbf{m}_T\mathbf{M}_\Sigma^{-1}\mathbf{m}_T^T\,, \tag{3.36}$$

with $\mathbf{m}_T = v\mathbf{Y}_T$.

As in the case of type I seesaw, the extra fermion content added to the SM changes the anomaly-free conditions. While the anomalies concerning $SU(3)_C$ and $U(1)_Y$ are simple, because Σ is a colour singlet and its hypercharge (Y_σ) is zero, those with $SU(2)_L$ have some peculiar details. In the adjoint representation of $SU(2)$, one can write the group generators as $(T^a)^{ij} = -i\varepsilon^{aij}$, hence $\mathrm{tr}_\mathcal{R}\,[T^a] = 0$ and the $G_iG_jSU(2)_L$ anomalies, $G_{i,j} \neq SU(2)_L$, still vanish in this minimal SM extension. The $G_i\,[SU(2)_L]^2$ anomalies, $G_i \neq SU(2)_L$, vanish automatically (recall their properties given in Section 2.3). For the $[SU(2)_L]^3$ anomaly, considering only the new contributions, we get

$$\begin{aligned}
[SU(2)_L]^3 : \mathrm{tr}_\mathcal{R}\left[\left\{g_WT^a, g_WT^b\right\}g_WT^c\right] &= (-i)^3\,g_W^3\,\mathrm{tr}_\mathcal{R}\left[\left(\varepsilon^{aij}\varepsilon^{bjk} + \varepsilon^{bij}\varepsilon^{ajk}\right)\varepsilon^{ckl}\right] = \\
&ig_W^3\left(\varepsilon^{aij}\varepsilon^{bjk} + \varepsilon^{bij}\varepsilon^{ajk}\right)\varepsilon^{cki} = ig_W^3\left(\delta^{ib}\delta^{ak} - \delta^{ik}\delta^{ab} + \delta^{ia}\delta^{bk} - \delta^{ik}\delta^{ba}\right)\varepsilon^{cki} = \\
&ig_W^3\left(\varepsilon^{cab} - 0 + \varepsilon^{cba} - 0\right) = 0\,.
\end{aligned} \tag{3.37}$$

In order to address the possibility of quantized charges, one needs to calculate the $U(1)_Y\,[SU(2)_L]^2$

anomaly with an arbitrary Y_σ. Therefore we get

$$
\begin{aligned}
U(1)_Y\left[SU(2)_L\right]^2 : \mathrm{tr}_{\mathcal{R}}\left[\left\{g_W T^a, g_W T^b\right\} g_Y Y_f\right] &= (-i)^2\, g_W^2 g_Y\, \mathrm{tr}_{\mathcal{R}}\left[\left(\varepsilon^{aij}\varepsilon^{bjk}+\varepsilon^{bij}\varepsilon^{ajk}\right)\delta^{kl} Y_f\right] = \\
&- Y_f\, g_W^2 g_Y\left(\varepsilon^{aij}\varepsilon^{bjk}+\varepsilon^{bij}\varepsilon^{ajk}\right)\delta^{ki} = -Y_f\, g_W^2 g_Y\left(\varepsilon^{aij}\varepsilon^{bji}+\varepsilon^{bij}\varepsilon^{aji}\right) = \\
&2Y_f\, g_W^2 g_Y\left(\varepsilon^{aij}\varepsilon^{bij}\right) = 4\delta^{ab} Y_f\, g_W^2 g_Y\,,
\end{aligned}
\tag{3.38}
$$

summed over all the fermions that transforms as an $SU(2)_L$ triplet plus the contribution computed in Eq. (2.60). Since the hypercharge is arbitrary in this context, one obtains the generalized anomaly-free conditions

$$
\begin{aligned}
\left[U(1)_Y\right]^3 &: \sum_{i=1}^{n_G}\left(6Y_{q_i}^3 + 2Y_{\ell_i}^3 - 3Y_{u_i}^3 - 3Y_{d_i}^3 - Y_{e_i}^3\right) - \sum_{i=1}^{n_\Sigma} 3Y_{\sigma_i}^3 = 0\,, \\
U(1)_Y\left[SU(2)_L\right]^2 &: \sum_{i=1}^{n_G}\left(3Y_{q_i} + Y_{\ell_i}\right) - \sum_{i=1}^{n_\Sigma} 4Y_{\sigma_i} = 0\,, \\
U(1)_Y\left[SU(3)_C\right]^2 &: \sum_{i=1}^{n_G}\left(2Y_{q_i} - Y_{u_i} - Y_{d_i}\right) = 0\,, \\
U(1)_Y &: \sum_{i=1}^{n_G}\left(6Y_{q_i} + 2Y_{\ell_i} - 3Y_{u_i} - 3Y_{d_i} - Y_{e_i}\right) - \sum_{i=1}^{n_\Sigma} 3Y_{\sigma_i} = 0\,,
\end{aligned}
\tag{3.39}
$$

which contain $15 + n_\Sigma$ free parameters. If we consider the anomaly-free conditions obtained from the type I seesaw with the replacement $n_R \to n_\Sigma$ and $Y_{\nu_i} \to Y_{\sigma_i}$, only one equation remains unchanged. However, with these replacements, the gauge invariance of the type III Lagrangian imposes exactly the same constraints as the type I Lagrangian

$$
-Y_{\ell_i} - Y_H + Y_{\sigma_j} = 0\,,\ Y_{\sigma_j} + Y_{\sigma_j} = 0\,,\ i = 1,2,3\,,\ j = 1,\dots n_\Sigma\,. \tag{3.40}
$$

Finally, if family universal charges are assumed, one concludes that the electric charge is quantized in the framework of type I, II or III seesaw models.

3.3 Zero Textures for the Neutrino Mass Matrix and their Seesaw Realization

Despite the simplicity of the seesaw mechanism in explaining the smallness of the neutrino masses, the corresponding high-energy theory usually contains many more free parameters than those required at low energies. We recall that the effective neutrino mass matrix $\mathbf{m}_\nu$ can be written in terms of only nine physical parameters: 3 light neutrino masses and 3 mixing angles + 3 phases, that parametrize the PMNS mixing matrix.

For instance, the type I seesaw Lagrangian given in Eq. (3.13) with n_R right-handed neutrino fields, contains altogether $7n_R - 3$ free parameters. Therefore, in the SM extended with $n_R = 3$ there are 18 parameters in the neutrino sector at high energies: 3 heavy Majorana masses and 9 moduli + 6 phases needed to specify the 3×3 Yukawa coupling matrix $\mathbf{Y}_\nu$. Nevertheless, only 15 parameters are independent in what respects the neutrino mass matrix $\mathbf{m}_\nu = -\mathbf{m}_D \mathbf{M}_R^{-1} \mathbf{m}_D^T$ since the 3 heavy Majorana masses can be absorbed into $\mathbf{Y}_\nu$ by rescaling the appropriate couplings. For the minimal case[1], with $n_R = 2$, there are 11 parameters: 2 heavy Majorana masses and 6 moduli + 3 phases that define the 3×2 Yukawa coupling matrix. Of these, the 2 heavy masses can be absorbed, thus reducing the effective number to 9 parameters. The same parameter counting holds for the type III seesaw with the replacements $n_R \to n_\Sigma$, $\mathbf{Y}_\nu \to \mathbf{Y}_T$ and $\mathbf{M}_R \to \mathbf{M}_\Sigma$.

It then becomes clear that for a high energy seesaw theory to be predictive the number of free parameters should be somehow reduced. A well-motivated framework is provided by the so-called zero textures of the Yukawa coupling matrices. In some cases, such zeros also propagate to the low energy neutrino mass matrix, implying relations among the neutrino observables. These textures can be obtained, for instance, in the presence of flavour symmetries or additional local gauge symmetries.

The neutrino mass matrix $\mathbf{m}_\nu$ is a symmetric matrix with six independent entries. There are $6!/[n!(6-n)!]$ different textures, each containing n independent texture zeros. Since each matrix entry is a complex number, there are $2n$ constraints. It can be shown that any pattern of $\mathbf{m}_\nu$ with more than two independent zeros ($n > 2$) is not compatible with current neutrino oscillation data. Clearly, one-zero textures in $\mathbf{m}_\nu$ have much less predictability than the two-zero textures. Their phenomenological implications have been studied in Refs. [77–81] and we shall not discuss them any further here.

For $n = 2$, there are fifteen two-zero textures of $\mathbf{m}_\nu$, which can be classified into six categories $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F})$:

$$
\begin{gathered}
\mathbf{A}_1 : \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & * \end{pmatrix}, \ \mathbf{A}_2 : \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}; \\
\mathbf{B}_1 : \begin{pmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix}, \ \mathbf{B}_2 : \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ * & * & 0 \end{pmatrix}, \ \mathbf{B}_3 : \begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}, \ \mathbf{B}_4 : \begin{pmatrix} * & * & 0 \\ * & * & * \\ 0 & * & 0 \end{pmatrix}; \\
\mathbf{C} : \begin{pmatrix} * & * & * \\ * & 0 & * \\ * & * & 0 \end{pmatrix}; \\
\mathbf{D}_1 : \begin{pmatrix} * & * & * \\ * & 0 & 0 \\ * & 0 & * \end{pmatrix}, \ \mathbf{D}_2 : \begin{pmatrix} * & * & * \\ * & * & 0 \\ * & 0 & 0 \end{pmatrix}; \\
\mathbf{E}_1 : \begin{pmatrix} 0 & * & * \\ * & 0 & * \\ * & * & * \end{pmatrix}, \ \mathbf{E}_2 : \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & 0 \end{pmatrix}, \ \mathbf{E}_3 : \begin{pmatrix} 0 & * & * \\ * & * & 0 \\ * & 0 & * \end{pmatrix};
\end{gathered}
\tag{3.41}
$$

[1]For a type I (type III) seesaw mechanism alone, consistency with neutrino oscillation data requires $n_R \geq 2$ ($n_\Sigma \geq 2$). Aside from this constraint, the number of right-handed neutrinos (fermion triplets) is arbitrary.

$\mathbf{M}_{R,\Sigma}$	$\mathbf{D}_2\,,\begin{pmatrix}0&*&*\\ *&*&0\\ *&0&0\end{pmatrix}$	$\mathbf{D}_1\,,\begin{pmatrix}0&*&*\\ *&0&0\\ *&0&*\end{pmatrix}$	$\mathbf{B}_4\,,\begin{pmatrix}*&*&0\\ *&0&*\\ 0&*&0\end{pmatrix}$	$\mathbf{B}_3\,,\begin{pmatrix}*&0&*\\ 0&0&*\\ *&*&0\end{pmatrix}$
$\mathbf{m}_\nu$	$\mathbf{A}_1$	$\mathbf{A}_2$	$\mathbf{B}_3$	$\mathbf{B}_4$

Table 3.1: Viable type I (type III) seesaw realizations of two-zero textures of the effective neutrino mass matrix $\mathbf{m}_\nu$ when $n_R = 3$ ($n_\Sigma = 3$) and the Dirac-neutrino Yukawa mass matrix $\mathbf{m}_D$ ($\mathbf{m}_T$) is diagonal. All cases belong to the permutation set $\mathcal{P}_1$.

$$\mathbf{F}_1:\ \begin{pmatrix}*&0&0\\ 0&*&*\\ 0&*&*\end{pmatrix},\ \mathbf{F}_2:\ \begin{pmatrix}*&0&*\\ 0&*&0\\ *&0&*\end{pmatrix},\ \mathbf{F}_3:\ \begin{pmatrix}*&*&0\\ *&*&0\\ 0&0&*\end{pmatrix};$$

the symbol "$*$" denotes a nonzero matrix element. In the flavour basis, where the charged-lepton mass matrix $\mathbf{m}_e$ is diagonal ($\mathbf{m}_e = \mathbf{d}_e$), only seven patterns, to wit $\mathbf{A}_{1,2}$, $\mathbf{B}_{1,2,3,4}$ and $\mathbf{C}$ [82], are compatible with the present neutrino oscillation data [83].

Since any ordering of the charged leptons in the flavour basis is allowed, any permutation transformation acting on the above patterns is permitted, provided that it leaves $\mathbf{m}_e$ diagonal. In particular, the following permutation sets can be constructed:

$$\begin{aligned}\mathcal{P}_1 &\equiv (\mathbf{A}_1, \mathbf{A}_2, \mathbf{B}_3, \mathbf{B}_4, \mathbf{D}_1, \mathbf{D}_2),\\ \mathcal{P}_2 &\equiv (\mathbf{B}_1, \mathbf{B}_2, \mathbf{E}_3),\\ \mathcal{P}_3 &\equiv (\mathbf{C}, \mathbf{E}_1, \mathbf{E}_2),\\ \mathcal{P}_4 &\equiv (\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3).\end{aligned} \tag{3.42}$$

Starting from any pattern belonging to a particular set, one can obtain any other pattern in the same set by permutations.

Our aim is to look for possible type I and/or Type III seesaw realizations of two-zero textures of the neutrino mass matrix $\mathbf{m}_\nu$ compatible with the experimental data, i.e. that lead to a pattern $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{B}_1$, $\mathbf{B}_2$, $\mathbf{B}_3$, $\mathbf{B}_4$ or $\mathbf{C}$. We restrict our analysis to the cases with $n_R + n_\Sigma \leq 4$.

We start by searching for solutions with $n_R = 3$ ($n_\Sigma = 3$) and the Dirac-Yukawa mass matrices $\mathbf{m}_D$ ($\mathbf{m}_T$) diagonal, i.e. $\mathbf{m}_{D,T} = \mathrm{diag}\,(*,*,*)$, so that the zero texture of $\mathbf{m}_\nu$ is the same as $\mathbf{M}_R^{-1}$ ($\mathbf{M}_\Sigma^{-1}$). In Table 3.1, we present all viable type I (type III) seesaw realizations found in this case. All patterns belong to the permutation set $\mathcal{P}_1$.

If instead we assume that $\mathbf{m}_D$ ($\mathbf{m}_T$) belongs to a permutation set $\mathcal{P}_i\,(i = 1, 2, 3, 4)$, then the viable solutions are those in Table 3.2. As can be seen from the table, only matrices $\mathbf{m}_D$ ($\mathbf{m}_T$) and $\mathbf{m}_\nu$ contained in $\mathcal{P}_1$ are allowed, sharing always the same pattern, i.e. exhibiting "parallel" structures.

To obtain neutrino mass matrices of type $\mathbf{C}$, belonging to the permutation set $\mathcal{P}_3$, matrices $\mathbf{m}_{D,T}$ (and $\mathbf{M}_{R,\Sigma}$) with two and four zeros, for $n_{R,\Sigma} = 2$ and $n_{R,\Sigma} = 3$, respectively, are required. In Table 3.3

$\mathbf{m}_{D,T}$	$\mathbf{M}_{R,\Sigma}$	$\mathbf{m}_\nu$
$\mathbf{A}_1$	$\mathbf{A}_1\,,\ \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix},\ \begin{pmatrix} 0 & 0 & * \\ 0 & * & * \\ * & * & 0 \end{pmatrix},\ \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$	$\mathbf{A}_1$
$\mathbf{A}_2$	$\mathbf{A}_2\,,\ \begin{pmatrix} 0 & * & 0 \\ * & 0 & * \\ 0 & * & * \end{pmatrix},\ \begin{pmatrix} 0 & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix},\ \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$	$\mathbf{A}_2$
$\mathbf{B}_3$	$\mathbf{B}_3\,,\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{pmatrix},\ \begin{pmatrix} * & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix},\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$	$\mathbf{B}_3$
$\mathbf{B}_4$	$\mathbf{B}_4\,,\ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & 0 \end{pmatrix},\ \begin{pmatrix} * & * & 0 \\ * & 0 & * \\ 0 & * & 0 \end{pmatrix},\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$	$\mathbf{B}_4$

Table 3.2: Viable type I (type III) seesaw realizations of two-zero textures of $\mathbf{m}_\nu$ when $n_R = 3$ ($n_\Sigma = 3$) and assuming that $\mathbf{m}_D$ ($\mathbf{m}_T$) belongs to a permutation set $\mathcal{P}_i\,(i = 1, 2, 3, 4)$. Only matrices $\mathbf{m}_D$ ($\mathbf{m}_T$) and $\mathbf{m}_\nu$ contained in $\mathcal{P}_1$ are allowed, sharing always the same pattern.

we present all viable type I (type III) seesaw realizations that lead to the two-zero pattern $\mathbf{C}$ in the effective neutrino mass matrix $\mathbf{m}_\nu$. The cases with $n_R = 2$ ($n_\Sigma = 2$) and $n_R = 3$ ($n_\Sigma = 3$) are considered. From the table, we conclude that with only two right-handed singlet (fermion triplet) neutrinos, there are only two possible constructions, both leading to a massless neutrino ($\det \mathbf{C} = 0$). In fact, these are the only solutions that yield a pattern consistent with neutrino oscillation data. We did not find any texture of type $\mathbf{A}_i$ or $\mathbf{B}_i$. For $n_R = 3$ ($n_\Sigma = 3$), besides the $\mathbf{C}$-pattern, there exist several combinations of matrices $\mathbf{m}_{D,T}$ and $\mathbf{M}_{R,\Sigma}$ (not displayed in the table) that lead to the viable patterns $\mathbf{A}_{1,2}$ and $\mathbf{B}_{3,4}$.

In the framework of a single type seesaw, textures $\mathbf{B}_{1,2}$, belonging to the permutation set $\mathcal{P}_2$, cannot be obtained. They can only be realized in the context of mixed seesaw schemes. In Table 3.4, several patterns leading to neutrino mass matrices of type $\mathbf{B}_{1,2}$ through a mixed seesaw with two right-handed neutrinos and two fermion triplets ($n_R = n_\Sigma = 2$) are shown. The solutions correspond to cases where the Dirac-Yukawa mass matrices $\mathbf{m}_{D,T}$ contain the maximum of allowed vanishing matrix elements, i.e. four zeros. We remark that in the mixed cases with $n_R = 2, n_\Sigma = 1$ and $n_R = 1, n_\Sigma = 2$ there are viable

$\mathbf{m}_{D,T}$	$\mathbf{M}_{R,\Sigma}$	$\mathbf{m}_\nu$
$\begin{pmatrix} * & * \\ 0 & * \\ * & 0 \end{pmatrix}$, $\begin{pmatrix} * & * \\ * & 0 \\ 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix}$	$\mathbf{C}$ with $\det \mathbf{C} = 0$
$\begin{pmatrix} * & * & * \\ 0 & 0 & * \\ * & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$	$\mathbf{C}$
$\begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ * & 0 & 0 \\ 0 & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$	
$\begin{pmatrix} * & * & * \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$, $\begin{pmatrix} * & * & * \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$	

Table 3.3: Viable type I (type III) seesaw realizations that lead to the two-zero pattern $\mathbf{C}$ in $\mathbf{m}_\nu$. The cases with $n_R = 2$ ($n_\Sigma = 2$) and $n_R = 3$ ($n_\Sigma = 3$) are displayed.

patterns as well, but they only generate neutrino mass matrices of type $\mathbf{A}_{1,2}$, $\mathbf{B}_{3,4}$ and $\mathbf{C}$, and, therefore, are not presented in Table 3.4.

From the above analyses it turns out that are several possibilities of realizing two-zero textures in the effective neutrino mass matrix obtained through the seesaw mechanism. One attractive possibility is to impose these zeros through the modification of the SM gauge symmetry. Next, we consider Abelian extensions of the SM based on an extra $U(1)_X$ gauge symmetry, where X is an arbitrary linear combination of the baryon number and the individual lepton numbers.

<table>
<tr><th>$\mathbf{m}_D$</th><th>$\mathbf{M}_R$</th><th>$\mathbf{m}_T, \mathbf{M}_\Sigma$</th><th>$\mathbf{m}_\nu$</th></tr>
<tr><td>$\begin{pmatrix} 0 & 0 \\ 0 & * \\ * & 0 \end{pmatrix}$</td><td>$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$</td><td rowspan="2">$\begin{pmatrix} 0 & * \\ * & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ or $\begin{pmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$</td><td rowspan="2">$\mathbf{B}_1$</td></tr>
<tr><td>$\begin{pmatrix} 0 & 0 \\ * & 0 \\ 0 & * \end{pmatrix}$</td><td>$\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$</td></tr>
<tr><td>$\begin{pmatrix} 0 & 0 \\ 0 & * \\ * & 0 \end{pmatrix}$</td><td>$\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$</td><td rowspan="2">$\begin{pmatrix} 0 & * \\ 0 & 0 \\ * & 0 \end{pmatrix}, \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ or $\begin{pmatrix} * & 0 \\ 0 & 0 \\ 0 & * \end{pmatrix}, \begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$</td><td rowspan="2">$\mathbf{B}_2$</td></tr>
<tr><td>$\begin{pmatrix} 0 & 0 \\ * & 0 \\ 0 & * \end{pmatrix}$</td><td>$\begin{pmatrix} 0 & * \\ * & * \end{pmatrix}$</td></tr>
</table>

Table 3.4: Examples of type I/III mixed seesaw realizations with two right-handed neutrinos and two fermion triplets ($n_R = n_\Sigma = 2$) that lead to a neutrino mass matrix of type $\mathbf{B}_{1,2}$. The solutions correspond to cases where the 3×2 Dirac-Yukawa mass matrices $\mathbf{m}_D$ and $\mathbf{m}_T$ contain the maximum of allowed vanishing elements, i.e. four zeros.

Chapter 4

Anomaly-free Gauge Symmetries and Neutrino Flavour Models

Abelian symmetries naturally arise in a wide variety of grand unified and string theories. One of the interesting features of such theories is their richer phenomenology, when compared with the SM (for reviews, see e.g. Refs. [84, 85]). In particular, the spontaneous breaking of additional gauge symmetries leads to new massive neutral gauge bosons which, if kinematically accessible, could be detectable at the Large Hadron Collider (LHC). Clearly, the experimental signatures of these theories crucially depend on whether or not the SM particles have nontrivial charges under the new gauge symmetry. Assuming that the SM fermions are charged under the new gauge group, and that the new gauge boson has a mass around the TeV scale, one expects some effects on the LHC phenomenology.

In the context of neutrino seesaw models, the implications of anomaly-free constraints based on the gauge structure $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \otimes U(1)_X$ have been widely studied in the literature [86–91]. In particular, assuming family universal charges, it was shown in Ref. [91] that type I and type III seesaw mechanisms cannot be simultaneously realized, unless the new $U(1)_X$ symmetry is a replica of the standard hypercharge or new fermions are added to the theory. Models based on gauge symmetries that are linear combinations of the baryon number B and the individual lepton flavour numbers L_α ($\alpha = e, \mu, \tau$) have also been extensively discussed [92–98]. From the phenomenological viewpoint many aspects of the latter symmetries are similar to those of the $B-L$ symmetry, with $L = \sum_\alpha L_\alpha$ being the lepton number.

In this chapter, we consider Abelian extensions of the SM based on an extra $U(1)_X$ gauge symmetry, with $X \equiv a\,B - \sum_\alpha b_\alpha L_\alpha$ being an arbitrary linear combination of the baryon number B and the individual lepton numbers L_α. Our purpose is to perform a systematic study, thus complementing previous works on several aspects.

In Section 4.1, by requiring cancellation of gauge anomalies, we study the allowed charge assignments under the new gauge symmetry, when two or three right-handed neutrino singlets or fermion triplets are added to the SM particle content. We then discuss in Section 4.2 the phenomenological constraints on these theories, requiring consistency with current neutrino oscillation data. In particular, by extending the SM with a minimal extra fermion and scalar content, we study how the new gauge symmetry can lead to predictive two zero textures in the effective neutrino mass matrix. We also briefly address the possibility of distinguishing different charge assignments (gauge symmetries) and neutrino textures at collider experiments.

4.1 Anomaly Constraints on the Extended Gauge Group

We consider a renormalizable theory containing the SM particles plus a minimal extra fermionic and scalar content, so that light neutrinos acquire seesaw masses. We include n_R singlet right-handed neutrinos ν_R and n_Σ color-singlet $SU(2)$-triplet fermions Σ to implement type-I and type-III seesaw mechanisms, respectively. Besides the SM Higgs doublet H that gives masses to quarks and leptons, a complex scalar singlet field S is introduced in order to give Majorana masses to ν_R and Σ.

We assume that each fermion field f have a charge x_f under the new $U(1)_X$ gauge symmetry. For quarks, a family universal charge assignment is assumed, while leptons are allowed to have non-universal X charges.

As we have seen in previous sections, in the presence of extra fermion degrees of freedom, the anomaly conditions may change. Furthermore, when we extend the gauge group, for instance by including a $U(1)_X$ Abelian symmetry, extra conditions should be satisfied to render the theory free of the $U(1)_X$ anomalies. Following the same line of reasoning of Section 2.3, we obtain the system of constraints

$$
\begin{aligned}
U(1)_X\,[SU(3)_C]^2 &: n_G\,(2\,x_q - x_u - x_d) = 0\,,\\
U(1)_X\,[SU(2)_L]^2 &: \frac{3n_G}{2}x_q + \frac{1}{2}\sum_{i=1}^{n_G} x_{\ell i} - 2\sum_{i=1}^{n_\Sigma} x_{\sigma i} = 0\,,\\
U(1)_X\,[U(1)_Y]^2 &: n_G\left(\frac{x_q}{6} - \frac{4x_u}{3} - \frac{x_d}{3}\right) + \sum_{i=1}^{n_G}\left(\frac{x_{\ell i}}{2} - x_{ei}\right) = 0\,,\\
[U(1)_X]^2\,U(1)_Y &: n_G\left(x_q^2 - 2x_u^2 + x_d^2\right) + \sum_{i=1}^{n_G}\left(-x_{\ell i}^2 + x_{ei}^2\right) = 0\,,\\
[U(1)_X]^3 &: n_G\left(6\,x_q^3 - 3\,x_u^3 - 3\,x_d^3\right) + \sum_{i=1}^{n_G}\left(2\,x_{\ell i}^3 - x_{ei}^3\right) - \sum_{i=1}^{n_R} x_{\nu i}^3 - 3\sum_{i=1}^{n_\Sigma} x_{\sigma i}^3 = 0\,,\\
U(1)_X &: n_G\,(6x_q - 3x_u - 3x_d) + \sum_{i=1}^{n_G}(2\,x_{\ell i} - x_{ei}) - \sum_{i=1}^{n_R} x_{\nu i} - 3\sum_{i=1}^{n_\Sigma} x_{\sigma i} = 0\,.
\end{aligned}
\tag{4.1}
$$

Since the anomaly equations are nonlinear and contain many free parameters, some assumptions are usually made to obtain simple analytic solutions. For instance, in family universal models, universal charges are assigned so that anomaly cancellation is satisfied within each family. Family universality is nevertheless not necessarily required and non-universal solutions can be equally found [99]. Assuming, for instance, a non-universal purely leptonic gauge symmetry with $x_{\ell i} = x_{ei}$ and $n_R = n_\Sigma = n_G = 3$, the

n_R	n_Σ	Anomaly constraints	Symmetry generator X
2	0	$b_i + b_j = 3a$, $b_k = 0$	$B - 3L_j - b'_i(L_i - L_j)$
		$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$
0	2	$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$
2	1	$b_i + b_j = 3a$, $b_k = 0$	$B - 3L_j - b'_i(L_i - L_j)$
		$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$
1	2	$b_i + b_j = 0$, $b_k = 3a$	$B - 3L_k - b'_i(L_i - L_j)$
		$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$
3	0	$b_i + b_j + b_k = 3a$	$(B - L) + (1 - b'_i)(L_i - L_j) + (1 - b'_k)(L_k - L_j)$
		$b_i + b_j + b_k = 0$	$-b'_i(L_i - L_k) - b'_j(L_j - L_k)$
0	3	$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$
3	1	$b_i + b_j = 3a$, $b_k = 0$	$B - 3L_j - b'_i(L_i - L_j)$
		$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$
1	3	$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$
2	2	$b_i + b_j = 0$, $b_k = 0$	$L_i - L_j$

Table 4.1: Anomaly-free solutions for minimal type I and/or type III seesaw realizations and their symmetry generators. In all cases, $i \neq j \neq k$ and $b'_i \equiv b_i/a$. Cases with $a = 0$ correspond to a purely leptonic symmetry.

anomaly equations (4.1) lead to the following charge constraints:

$$\begin{aligned} &x_{e1} + x_{e2} + x_{e3} = 0, \\ &x_{\nu 1} + x_{\nu 2} + x_{\nu 3} = 0, \\ &x_{\sigma 1} + x_{\sigma 2} + x_{\sigma 3} = 0, \\ &x_{e1}x_{e2}x_{e3} - x_{\nu 1}x_{\nu 2}x_{\nu 3} - 3\, x_{\sigma 1}x_{\sigma 2}x_{\sigma 3} = 0\,. \end{aligned} \tag{4.2}$$

This system of equations has an infinite number of integer solutions. For example, with the charge assignment $(x_{\ell 1}, x_{\ell 2}, x_{\ell 3}) = (x_{e1}, x_{e2}, x_{e3}) = (1, 2, -3)$, one can have the solutions $(x_{\nu 1}, x_{\nu 2}, x_{\nu 3}) = (-1, -3, 4)$ and $(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}) = (1, 2, -3)$, or $(x_{\nu 1}, x_{\nu 2}, x_{\nu 3}) = (1, 3, -4)$ and $(x_{\sigma 1}, x_{\sigma 2}, x_{\sigma 3}) = (-1, -1, 2)$, among many others.

We shall consider models where

$$X \equiv a\, B - \sum_{i=1}^{n_G} b_i L_i \tag{4.3}$$

is an arbitrary linear combination of the baryon number B and individual lepton numbers L_i, simultaneously allowing for the existence of right-handed neutrinos and fermion triplets that participate in the seesaw mechanism to generate Majorana neutrino masses. Under the gauge group $U(1)_X$, the charge for the quarks q_L, u_R, d_R, is universal,

$$x_q = x_u = x_d = a/3\,, \tag{4.4}$$

while the charged leptons ℓ_{Li}, e_{Ri} have the family non-universal charge assignment

$$x_{\ell i} = x_{ei} = -b_i\,, \tag{4.5}$$

with all b_i different. The latter condition guarantees that the charged lepton mass matrix is always diagonal (i.e. it is defined in the charged lepton flavour basis), assuming that the SM Higgs is neutral under the new gauge symmetry. The right-handed neutrinos ν_R and/or the triplets Σ are allowed to have any charge assignment $-b_k$, where $k = 1 \ldots n_G$.

Substituting the $U(1)_X$ charge values given in Eqs. (4.4) and (4.5) into the anomaly equations (4.1), we obtain the constraints

$$\begin{aligned}
&\sum_{k\leq n_\Sigma} b_k = 0, \\
&\sum_{i=1}^{n_G} b_i = \sum_{j\leq n_R} b_j = n_G\, a, \\
&\sum_{i=1}^{n_G} b_i^3 - \sum_{j\leq n_R} b_j^3 - 3\sum_{k\leq n_\Sigma} b_k^3 = 0\,.
\end{aligned} \tag{4.6}$$

The solutions of this system of equations and the corresponding symmetry generators X are presented in Table 4.1, for minimal type I and type III seesaw realizations with $n_R + n_\Sigma \leq 4$. We note that in the absence of right-handed neutrinos only purely leptonic ($a = 0$) gauge symmetry extensions are allowed. This is a direct consequence of the second constraint in Eq. (4.6). Given the charge assignments, one can identify the maximal gauge group corresponding to each solution. For instance, when $n_R = 3$ and $n_\Sigma = 0$, the maximal anomaly-free Abelian gauge group extension is $U(1)_{B-L} \times U(1)_{L_e-L_\mu} \times U(1)_{L_\mu-L_\tau}$ [98].

4.2 Phenomenological Constraints

4.2.1 Neutrino Mass Matrix and Texture Zeros from the Gauge Symmetry

For our study, besides the usual SM Yukawa interactions, the relevant Lagrangian terms in the context of (minimal) type I and type III seesaw models are

$$\begin{aligned}
&\mathbf{Y}_\nu^{\alpha i}\overline{\ell_{L\alpha}}\widetilde{H}\nu_{Ri} + \frac{1}{2}\mathbf{m}_R^{ij}\,\overline{\nu_{Ri}^c}\nu_{Rj} + \mathbf{Y}_1^{ij}\,\overline{\nu_{Ri}^c}\nu_{Rj}S + \mathbf{Y}_2^{ij}\,\overline{\nu_{Ri}^c}\nu_{Rj}S^* \\
&\mathbf{Y}_T^{\alpha i}\overline{\ell_{L\alpha}}\Sigma_{Ri}\widetilde{H} + \frac{1}{2}\mathbf{m}_\Sigma^{ij}\mathrm{tr}\left(\overline{\Sigma_{Ri}^c}\Sigma_{Rj}\right) + \mathbf{Y}_3^{ij}\mathrm{tr}\left(\overline{\Sigma_{Ri}^c}\Sigma_{Rj}\right)S + \mathbf{Y}_4^{ij}\mathrm{tr}\left(\overline{\Sigma_{Ri}^c}\Sigma_{Rj}\right)S^* + \text{H.c.}\,.
\end{aligned} \tag{4.7}$$

We assume that the SM Higgs doublet is neutral under the $U(1)_X$ gauge symmetry, and that the complex singlet scalar field S has a $U(1)_X$ charge equal to x_s. Here $\mathbf{Y}_{1,2}$ are $n_R \times n_R$ symmetric matrices, while $\mathbf{Y}_{3,4}$ are $n_\Sigma \times n_\Sigma$ symmetric matrices.

Notice that, in general, the $U(1)_X$ symmetry does not forbid bare Majorana mass terms for the right-handed neutrinos and fermion triplets. For matrix entries with $X = 0$, such terms are allowed. In turn, entries with $X \neq 0$ are permitted in the presence of the singlet scalar S, charged under $U(1)_X$. The latter gives an additional contribution to the Majorana mass terms once S acquires a VEV.

Since a universal $U(1)_X$ charge is assigned to quarks (see Eq. (4.4)), the new gauge symmetry does not impose any constraint on the quark mass matrices. However, our choice of a non-universal charge assignment for charged leptons given in Eq. (4.5), with all b_i different, forces the charged lepton mass matrix to be diagonal. Thus, leptonic mixing depends exclusively on the way that neutrinos mix. As discussed in Section 3.2, the effective neutrino mass matrix $\mathbf{m}_\nu$ is obtained after the decoupling of the

Symmetry generator X	$\lvert x_s \rvert$	$\mathbf{M}_R$	$\mathbf{m}_\nu$
$B + L_e - L_\mu - 3L_\tau$	2	$\mathbf{D}_2$	$\mathbf{A}_1$
$B + 3L_e - L_\mu - 5L_\tau$	2	$(\mathbf{M}_R)_{11} = (\mathbf{M}_R)_{23} = (\mathbf{M}_R)_{33} = 0$	
$B + 3L_e - 6L_\tau$	3		
$B + 9L_e - 3L_\mu - 9L_\tau$	6		
$B + L_e - 3L_\mu - L_\tau$	2	$\mathbf{D}_1$	$\mathbf{A}_2$
$B + 3L_e - 5L_\mu - L_\tau$	2	$(\mathbf{M}_R)_{11} = (\mathbf{M}_R)_{22} = (\mathbf{M}_R)_{23} = 0$	
$B + 3L_e - 6L_\mu$	3		
$B + 9L_e - 9L_\mu - 3L_\tau$	6		
$B - L_e + L_\mu - 3L_\tau$	2	$\mathbf{B}_4$	$\mathbf{B}_3$
$B - L_e + 3L_\mu - 5L_\tau$	2	$(\mathbf{M}_R)_{13} = (\mathbf{M}_R)_{22} = (\mathbf{M}_R)_{33} = 0$	
$B + 3L_\mu - 6L_\tau$	3		
$B - 3L_e + 9L_\mu - 9L_\tau$	6		
$B - L_e - 3L_\mu + L_\tau$	2	$\mathbf{B}_3$	$\mathbf{B}_4$
$B - L_e - 5L_\mu + 3L_\tau$	2	$(\mathbf{M}_R)_{12} = (\mathbf{M}_R)_{22} = (\mathbf{M}_R)_{33} = 0$	
$B - 6L_\mu + 3L_\tau$	3		
$B - 3L_e - 9L_\mu + 9L_\tau$	6		

Table 4.2: Anomaly-free $U(1)$ gauge symmetries that lead to phenomenologically viable two-zero textures of the neutrino mass matrix $\mathbf{m}_\nu$ in a type I seesaw framework with 3 right-handed neutrinos. In all cases, the Dirac-neutrino mass matrix $\mathbf{m}_D$ is diagonal and the charge assignment $x_{\nu i} = x_{\ell i} = x_{ei} = -b_i$ is verified. The solutions belong to the permutation set $\mathcal{P}_1$. For a mixed type I/III seesaw scenario with $n_R = 3$ and $n_\Sigma = 1$ only the solutions with $|x_s| = 3$ remain viable.

heavy right-handed neutrinos and fermion triplets. In the presence of both (type I and type III) seesaw mechanisms it reads as

$$\mathbf{m}_\nu \simeq -\mathbf{m}_D\, \mathbf{M}_R^{-1}\, \mathbf{m}_D^T - \mathbf{m}_T\, \mathbf{M}_\Sigma^{-1}\, \mathbf{m}_T^T\,, \tag{4.8}$$

where, according to Eq. (4.7),

$$\begin{aligned} \mathbf{m}_D &= \mathbf{Y}_\nu \langle H \rangle, \quad \mathbf{M}_R = \mathbf{m}_R + 2\mathbf{Y}_1 \langle S \rangle + 2\mathbf{Y}_2 \langle S^* \rangle\,, \\ \mathbf{m}_T &= \mathbf{Y}_T \langle H \rangle, \quad \mathbf{M}_\Sigma = \mathbf{m}_\Sigma + 2\mathbf{Y}_3 \langle S \rangle + 2\mathbf{Y}_4 \langle S^* \rangle\,. \end{aligned} \tag{4.9}$$

In what follows we restrict our analysis to minimal seesaw scenarios with $n_R + n_\Sigma \leq 4$. The requirement that charged leptons are diagonal ($b_1 \neq b_2 \neq b_3$) imposes strong constraints on the matrix textures of $\mathbf{m}_D$ and $\mathbf{m}_T$. Indeed, considering either a type I or a type III seesaw framework, only those matrices with a single nonzero element per column are allowed. Furthermore, matrices with a null row or column are excluded since they lead to a neutrino mass matrix with determinant equal to zero, not belonging to any pattern of those given in Eq. (3.41)[1]

[1] Mixed type I/III seesaw mechanisms can relax this constraint.

We look for anomaly-free $U(1)_X$ gauge symmetries that lead to phenomenologically viable two-zero textures of the neutrino mass matrix $\mathbf{m}_\nu$, namely to patterns $\mathbf{A}_{1,2}$, $\mathbf{B}_{1,2,3,4}$ and $\mathbf{C}$ given in Eq. (3.41). Solutions were found only within a type I seesaw framework with three right-handed neutrinos, or in a mixed type I/III seesaw scenario with three right-handed neutrinos and one fermion triplet. In Table 4.2 we show the allowed solutions, for the cases when the Dirac-neutrino mass matrix $\mathbf{m}_D$ is diagonal, which implies the charge assignment $x_{\nu i} = -b_i$. All the solutions belong to the permutation set $\mathcal{P}_1$ [see Eq. (3.42)]. We remark that, for each pattern of $\mathbf{m}_\nu$, there are another 20 solutions corresponding to matrices $\mathbf{m}_D$ with 6 zeros (i.e. permutations of the diagonal matrix) and their respective charge assignments. Thus, all together there exist 96 viable solutions. No other anomaly-free solutions are obtained in our minimal setup. Solutions leading to $\mathbf{M}_R = \mathbf{D}_1, \mathbf{D}_2, \mathbf{B}_3, \mathbf{B}_4$ have been recently considered in Ref. [98]. The remaining solutions, to our knowledge, are new in this context. For a mixed type I/III seesaw with $n_R = 3$ and $n_\Sigma = 1$, only the set of solutions with $|x_s| = 3$ in Table 4.2 are allowed, since the anomaly equations imply that the b_k coefficient associated to the fermion triplet charge is always zero.

Notice also that, starting from any pattern given in Table 4.2, other patterns in the table can be obtained by permutations of the charged leptons. For instance, starting from the symmetry generators that lead to the $\mathbf{A}_1$ pattern, those corresponding to $\mathbf{A}_2$ and $\mathbf{B}_3$ are obtained by $\mu \leftrightarrow \tau$ and $e \leftrightarrow \mu$ exchange, respectively. Similarly, the $\mathbf{B}_4$ texture can be obtained from $\mathbf{A}_2$ through the $e \leftrightarrow \tau$ exchange.

4.2.2 Scalar Sector

The VEV of the scalar S breaks the $U(1)_X$ symmetry spontaneously, giving a contribution to the masses of the right-handed neutrinos and fermion triplets. The scalar potential, including the Higgs potential given in Eq. (1.36), reads as

$$V = \mu^2 H^\dagger H + \lambda (H^\dagger H)^2 + \mu_S^2 S^\dagger S + \lambda_S (S^\dagger S)^2 + \beta (S^\dagger S)(H^\dagger H)\,, \tag{4.10}$$

with $\mu^2 < 0$ and $\mu_S^2 < 0$ to generate the VEVs $\langle H \rangle = v/\sqrt{2}$ and $\langle S \rangle = v_S/\sqrt{2}$; $\lambda, \lambda_S > 0$ and $\beta^2 < 4\lambda\lambda_S$ for V to be positive-definitive. In the unitary gauge, the charged and pseudoscalar neutral components of H are absorbed by the $W^\pm$ and Z gauge bosons, respectively, while the pseudoscalar component of S is absorbed by the new Z'. In the physical basis, where

$$H = \begin{pmatrix} 0 \\ \dfrac{h+v}{\sqrt{2}} \end{pmatrix}, \quad S = \frac{s + v_S}{\sqrt{2}}\,, \tag{4.11}$$

the potential has the form

$$V = \lambda v^2 h^2 + \lambda_S v_S^2 s^2 + \beta v v_S h s + \frac{1}{4}\lambda h^4 + \frac{1}{4}\lambda_S s^4 + \lambda v h^3 + \lambda_S v_S s^3 + \frac{1}{4}\beta h^2 s^2 + \frac{1}{2}\beta v h s^2 + \frac{1}{2}\beta v_S h^2 s\,. \tag{4.12}$$

The mass matrix for the neutral scalars h and s is given by

$$\mathbf{M}^2 = \begin{pmatrix} 2\lambda v^2 & \beta v v_S \\ \beta v v_S & 2\lambda_S v_S^2 \end{pmatrix}, \tag{4.13}$$

leading to the mass eigenstates $\phi_{1,2}$,

$$\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} h \\ s \end{pmatrix} , \tag{4.14}$$

with

$$\tan 2\theta = \frac{\beta v v_S}{\lambda_S v_S^2 - \lambda v^2} . \tag{4.15}$$

The masses are

$$m_{1,2}^2 = \lambda v^2 + \lambda_S v_S^2 \mp \sqrt{(\lambda_S v_S^2 - \lambda v^2)^2 + \beta^2 v_S^2 v^2} . \tag{4.16}$$

In the limit $v_S \gg v$ and $\lambda_S v_S^2 \gg \lambda v^2$, one obtains

$$m_1^2 \simeq 2\left(\lambda - \frac{\beta^2}{4\lambda_S}\right) v^2, \quad m_2^2 \simeq 2\lambda_S\, v_S^2 , \tag{4.17}$$

and

$$\theta \simeq \frac{\beta v}{2\lambda_S v_S} . \tag{4.18}$$

The mass of the new Z' gauge boson is

$$m_{Z'} = |x_s|\, g_X v_S , \tag{4.19}$$

where g_X is the $U(1)_X$ gauge coupling. An indirect constraint on $m_{Z'}$ comes from analyses of LEP2 precision electroweak data [100]:

$$\frac{m_{Z'}}{g_X} = |x_s|\, v_S \gtrsim 13.5 \text{ TeV} . \tag{4.20}$$

Thus, depending on the charge x_s, different lower bounds on the breaking scale of the $U(1)_X$ gauge symmetry are obtained. For the anomaly-free scalar charges given in Table 4.2, namely $|x_s| = 2, 3, 6$, one obtains the bounds $v_S \gtrsim 6.75$ TeV, 4.5 TeV, and 2.25 TeV, respectively. To put limits on the Z' mass, the gauge coupling strength must be known. Assuming, for definiteness, $g_X \sim 0.1$, the bound in Eq. (4.20) implies $m_{Z'} \gtrsim 1.4$ TeV. Such masses could be probed through the search of dilepton Z' resonances at the final stage of the LHC, with a center-of-mass energy $\sqrt{s} = 14$ TeV and integrated luminosity $L \simeq 100$ fb^{-1} [101, 102]. Recent searches for narrow high-mass dilepton resonances at the LHC ATLAS [103] and CMS [104] experiments have already put stringent lower limits on extra neutral gauge bosons. In particular, from the analysis of pp collisions at $\sqrt{s} = 8$ TeV, corresponding to an integrated luminosity of about 20 fb^{-1}, these experiments have excluded at 95% C.L. a sequential SM Z' (i.e. a gauge boson with the same couplings to fermions as the SM Z boson) lighter than 3 TeV.

Electroweak precision data severely constrain any mixing with the ordinary Z boson [85]. The $Z - Z'$ mixing may appear either due to the presence of Higgs bosons which transforms nontrivially under the SM gauge group and the new $U(1)_X$ Abelian gauge symmetry or via kinetic mixing in the Lagrangian [105]. The mass mixing is not induced in our case because the SM Higgs doublet is neutral under $U(1)_X$, while kinetic mixing may be avoided (up to one loop), if $U(1)_Y$ and $U(1)_X$ are orthogonal [106]. Although a detailed analysis of the $Z - Z'$ mixing is beyond the scope of our work, it is worth noting that, in general, it imposes additional restrictions on these models. For simplicity, hereafter we assume that mixing is negligible and restrict ourselves to the case with no mixing.

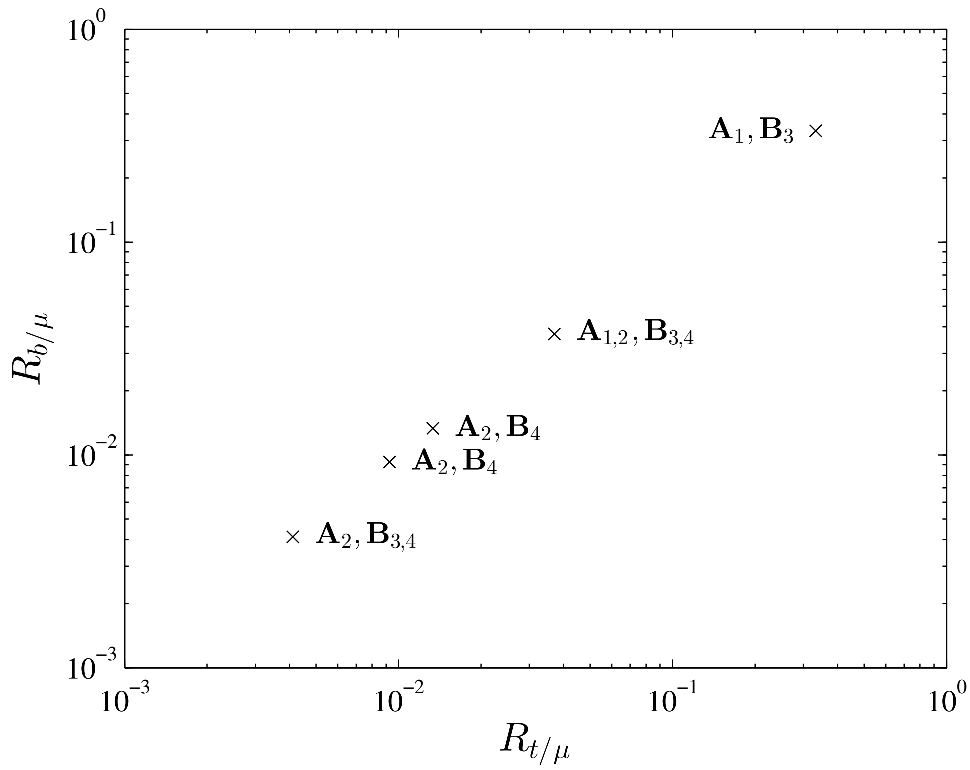

Figure 4.1: $R_{t/\mu} - R_{b/\mu}$ branching ratio plane for the anomaly-free solutions of Table 4.2, leading to neutrino mass matrix patterns of type $\mathbf{A}_{1,2}$ and $\mathbf{B}_{3,4}$.

4.2.3 Gauge Sector and Flavour Model Discrimination

For the effects due to the new gauge symmetry to be observable, the seesaw scale should be low enough. One expects a phenomenology similar to the case with a minimal $B-L$ scalar sector [107]. Nevertheless, by studying the Z' resonance and its decay products, one could in principle distinguish the generalized $U(1)_X$ models from the minimal $B-L$ model.

Due to their low background and neat identification, leptonic final states give the cleanest channels for the discovery of a new neutral gauge boson. In the limit that the fermion masses are small compared with the Z' mass, the Z' decay width into fermions is approximately given by

$$\Gamma(Z' \to f\overline{f}) \simeq \frac{{g'}^2}{24\pi} m_{Z'} \left(x_{fL}^2 + x_{fR}^2\right) , \tag{4.21}$$

where x_{fL} and x_{fR} are the $U(1)_X$ charges for the left and right chiral fermions, respectively. Moreover, the decays of Z' into third-generation quarks, $pp \to Z' \to b\,\bar{b}$ and $pp \to Z' \to t\,\bar{t}$ can be used to discriminate between different models, having the advantage of reducing the theoretical uncertainties [108, 109]. In particular, the branching ratios $R_{b/\mu}$ and $R_{t/\mu}$ of quarks to $\mu^+\mu^-$ production,

$$\begin{aligned} R_{b/\mu} &= \frac{\sigma(pp \to Z' \to b\,\bar{b})}{\sigma(pp \to Z' \to \mu^+\mu^-)} \simeq 3K_b \frac{x_q^2 + x_d^2}{x_{\ell 2}^2 + x_{e2}^2} \,, \\ R_{t/\mu} &= \frac{\sigma(pp \to Z' \to t\,\bar{t})}{\sigma(pp \to Z' \to \mu^+\mu^-)} \simeq 3K_t \frac{x_q^2 + x_u^2}{x_{\ell 2}^2 + x_{e2}^2} \,, \end{aligned} \tag{4.22}$$

could serve as discriminators. The $K_{b,t} \sim \mathcal{O}(1)$ factors incorporate the QCD and QED next-to-leading-

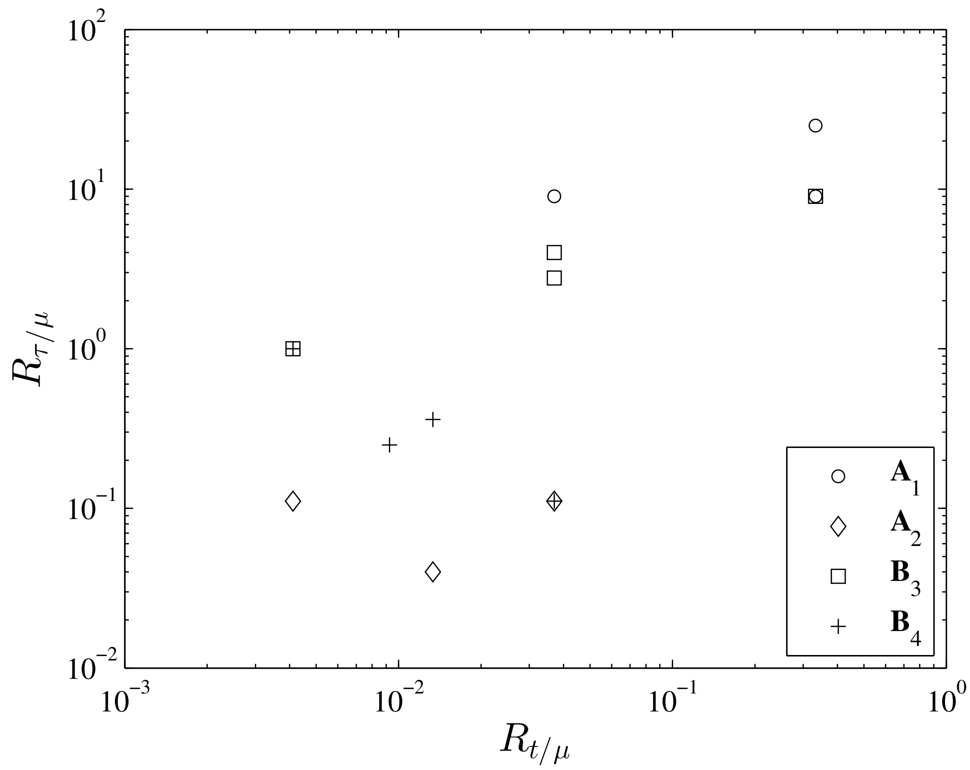

Figure 4.2: $R_{t/\mu} - R_{\tau/\mu}$ branching ratio plane for the anomaly-free solutions of Table 4.2, leading to neutrino mass matrix patterns of type $\mathbf{A}_{1,2}$ and $\mathbf{B}_{3,4}$.

order correction factors. Substituting the quark and charged-lepton $U(1)_X$ charges given in Eqs. (4.4) and (4.5), we obtain

$$R_{b/\mu} \simeq \frac{K_b}{3}\frac{a^2}{b_2^2}\,, \quad R_{t/\mu} \simeq \frac{K_t}{3}\frac{a^2}{b_2^2}\,, \tag{4.23}$$

yielding $R_{b/\mu} \simeq R_{t/\mu}$. Fig. 4.1 shows the $R_{t/\mu} - R_{b/\mu}$ branching ratio plane for the anomaly-free solutions given in Table 4.2, which lead to the viable neutrino mass matrix patterns $\mathbf{A}_{1,2}$ and $\mathbf{B}_{3,4}$, with two independent zeros. As can be seen from the figure, the solutions split into five different points in the plane, which correspond to the allowed values of the b_2 coefficient, $|b_2| = 1, 3, 5, 6, 9$, assuming $a = 1$. The allowed $\mathbf{m}_\nu$ patterns are shown at each point.

The ratio $R_{\tau/\mu}$ of the branching fraction of $\tau^+\tau^-$ to $\mu^+\mu^-$ has also proven to be useful for understanding models with preferential couplings to Z' [109]. It is approximately given in our case by

$$R_{\tau/\mu} = \frac{\sigma(pp \to Z' \to \tau^+\tau^-)}{\sigma(pp \to Z' \to \mu^+\mu^-)} \simeq K_\tau \frac{x_{\ell 3}^2 + x_{e3}^2}{x_{\ell 2}^2 + x_{e2}^2} \simeq K_\tau \frac{b_3^2}{b_2^2}\,, \tag{4.24}$$

where in the last expression we have used the charge relation (4.5). Clearly, this ratio can be used to distinguish models with generation universality ($R_{\tau/\mu} \simeq 1$) from models with non-universal couplings, as those given in Table 4.2. The $R_{t/\mu} - R_{\tau/\mu}$ branching ratio plane is depicted in Fig. 4.2. In this case, the neutrino mass matrix patterns exhibit a clear discrimination in the plane, having overlap of two solutions in just three points.

In conclusion, by studying the decays of the Z' boson into leptons and third-generation quarks at collider experiments, it is possible to discriminate different gauge symmetries and the corresponding flavour structure of the neutrino mass matrix.

Chapter 5

Conclusions

The recent discovery of a Higgs-like particle at the LHC reinforces the great success of the SM as the effective low energy theory for the electroweak interactions. In spite of this, there remain a few aspects that cannot be explained within the SM. In particular, neutrino oscillation experiments have confirmed that neutrinos have non-vanishing masses and mix. The well-known seesaw mechanism is an appealing and economical theoretical framework to explain the tiny neutrino masses. In this context, the addition of new heavy particles (fermions or bosons) to the theory allows for the generation of an effective neutrino mass matrix at low energies. As is well known, theories that contain fermions with chiral couplings to the gauge fields suffer from anomalies and, to make them consistent, the chiral sector of the new theory should be arranged so that the gauge anomalies cancel. One attractive possibility is to realize the anomaly cancellation through the modification of the gauge symmetry.

In this thesis, after briefly reviewing the SM and some theoretical aspects of anomalies, we discussed the anomaly cancellation and electric charge quantization in three popular (type I, II and III) seesaw extensions of the SM. We have then studied how to reduce the number of high energy parameters in the neutrino sector so that the effective neutrino mass matrix, obtained through the seesaw in the presence of an Abelian local gauge symmetry, exhibits a two-zero texture.

We have considered extensions of the SM based on Abelian gauge symmetries that are linear combinations of the baryon number B and the individual lepton numbers $L_{e,\mu,\tau}$. In the presence of a type I and/or type III seesaw mechanisms for neutrino masses, we have then looked for all viable charge assignments and gauge symmetries that lead to cancellation of gauge anomalies and, simultaneously, to a predictive flavour structure of the effective Majorana neutrino mass matrix, consistent with present neutrino oscillation data. Our analysis was performed in the physical basis where the charged leptons are diagonal. This implies that the neutrino mass matrix patterns with two independent zeros, obtained via the seesaw mechanism, are directly linked to low-energy parameters. We recall that, besides three charged lepton masses, there are nine low-energy leptonic parameters (three neutrino masses, three mixing angles, and three CP violating phases). Two-zero patterns in the neutrino mass matrix imply four constraints on these parameters. Would we consider charge assignments that lead to nondiagonal charged leptons, then the predictability of our approach would be lost, since rotating the charged leptons to the diagonal basis would destroy, in most cases, the zero textures in the neutrino mass matrix.

Working in the charged lepton flavour basis, we have found that only a limited set of solutions are viable, namely those presented in Table 4.2, leading to two-zero textures of the neutrino mass matrix with a minimal extra fermion and scalar content. All allowed patterns were obtained in the framework of the type I seesaw mechanism with three right-handed neutrinos (or in a mixed type I/III seesaw framework with three right-handed neutrinos and one fermion triplet), extending the SM scalar sector with a complex scalar singlet field.

Finally, we briefly addressed the possibility of discriminating the different charge assignments (gauge symmetries) and seesaw realizations at the LHC. We have shown that the measurements of the ratios of third generation final states (τ, b, t) to μ decays of the new gauge boson Z' could be useful in distinguishing between different gauge symmetry realizations, as can be seen from Figs. 4.1 and 4.2. This analysis provides a complementary way of testing flavour symmetries and their implications for low-energy neutrino physics.

Appendix A

Mathematical Relations

A.1 Regularization with Shifting of Variables

In Euclidean space, if $\int d^n x\, f(x)$ is divergent, we can write

$$\Delta(a) = \int d^n x[f(x+a) - f(x)] \approx \int d^n x\, a^\mu \partial_\mu f(x)\,, \tag{A.1}$$

in first-order approximation.

From the generalized Gauss theorem, we have

$$\int_V d^n x\, \partial_\mu F^\mu(x) = \int_{S(V)} d^{n-1}x\, n_S^\mu F_\mu(x)\,. \tag{A.2}$$

The left side is a volume integral over the volume V and the right side is the surface integral over the closed boundary of the volume V, which is $S(V)$. On each point of the surface $S(V)$, n_S^μ is the outward pointing unit normal field. Since a^μ is constant throughout space, we obtain

$$\int_V d^n x\, \partial_\mu(a^\mu f(x)) = a_\mu \int_{S(V)} d^{n-1}x\, n_S^\mu f(x)\,. \tag{A.3}$$

If $\int_V d^n x$ stands for an integration over all space, one can perform a symmetrical integration over a sphere with n dimensions and then take the infinite limit of its radius r. For a sphere, $n_S^\mu = r^\mu/r$, therefore

$$\int d^n x\, a^\mu \partial_\mu f(x) = a_\mu \lim_{r\to\infty} \frac{r^\mu}{r} S^{n-1}(r) f(r)\,. \tag{A.4}$$

In order to apply this result in Minkowski space, we perform a Wick rotation because $x_4 = ix_0$, which leads to an overall i factor. In four dimensions, we have $S^3(r) = 2\pi^2 r^3$, and it is now straightforward to regulate divergent integrals in four-dimensional Minkowski space.

A.2 Properties of the Gamma Matrices

The gamma matrices (γ_μ) obey the anticommutation relation

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2g_{\mu\nu}\,, \tag{A.5}$$

and the fifth gamma matrix, defined as $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, obeys

$$\{\gamma_5, \gamma_\mu\} = \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0\,. \tag{A.6}$$

Using these properties, it is possible to obtain different traces identities. The relevant ones to our discussion are

$$\begin{aligned}
&\mathrm{tr}\left[\gamma_{\mu_1}...\,\gamma_{\mu_n}\right] = \mathrm{tr}\left[\gamma_5\gamma_{\mu_1}...\,\gamma_{\mu_n}\right] = \mathrm{tr}\left[\gamma_5\right] = \mathrm{tr}\left[\gamma_5\gamma_\mu\gamma_\nu\right] = 0\,,\ \text{for } n \text{ odd},\\
&\mathrm{tr}\left[\gamma_5\gamma_\beta\gamma_\nu\gamma_\alpha\gamma_\mu\right] = -4i\varepsilon_{\beta\nu\alpha\mu}\,,\\
&\mathrm{tr}\left[\gamma_5\gamma_\alpha\gamma_\lambda\gamma_\rho\gamma_\nu\gamma_\beta\gamma_\mu\right] = -4i\left[\delta_{\alpha\lambda}\varepsilon_{\rho\nu\beta\mu} - \delta_{\alpha\rho}\varepsilon_{\lambda\nu\beta\mu} + \delta_{\lambda\rho}\varepsilon_{\alpha\nu\beta\mu} + \delta_{\nu\beta}\varepsilon_{\alpha\lambda\rho\mu} - \delta_{\nu\mu}\varepsilon_{\alpha\lambda\rho\beta} - \delta_{\beta\mu}\varepsilon_{\alpha\lambda\rho\nu}\right]\,,
\end{aligned} \tag{A.7}$$

which directly lead to

$$\begin{aligned}
\mathrm{tr}\left[\gamma_5\not{p}\gamma_\nu\not{k}_1\gamma_\mu\right] &= -4i\varepsilon_{\beta\nu\alpha\mu}p^\beta k_1^\alpha\,,\\
\mathrm{tr}\left[\not{p}\gamma_\lambda\gamma_5\not{p}\gamma_\nu\not{p}\gamma_\mu\right] &= 4ip^2p^\beta\varepsilon_{\lambda\nu\beta\mu}\,.
\end{aligned} \tag{A.8}$$

In the Dirac space, the charge conjugation matrix C is an unitary matrix that obeys the relations

$$C\gamma^{\mu T} + \gamma^\mu C = C^T + C = 0\,. \tag{A.9}$$

For any Dirac spinor ψ, one can define $\psi^c = C\overline{\psi}^T$ so that $\overline{\psi^c} = -\psi^T C^\dagger = \psi^T C$.

Bibliography

[1] L. M. Cebola, D. Emmanuel-Costa, and R. G. Felipe, Phys. Rev. D **88**, 116008 (2013), arXiv:1309.1709 [hep-ph] .

[2] R. Oerter, *The Theory of Almost Everything: The Standard Model, the Unsung Triumph of Modern Physic* (Plume-Penguin Group, USA, 2006).

[3] G. W. Bennett *et al.* (Muon G-2 Collaboration), Phys. Rev. D **73**, 072003 (2006), arXiv:hep-ex/0602035 [hep-ex] .

[4] G. Arnison *et al.* (UA1 Collaboration), Phys. Lett. B **122**, 103 (1983).

[5] G. Aad *et al.* (ATLAS Collaboration), Phys. Lett. B **716**, 1 (2012), arXiv:1207.7214 [hep-ex] .

[6] S. Chatrchyan *et al.* (CMS Collaboration), Phys. Lett. B **716**, 30 (2012), arXiv:1207.7235 [hep-ex] .

[7] P. A. M. Dirac, Proc. R. Soc. A **117**, 610 (1928).

[8] P. A. M. Dirac, *The Principles of Quantum Mechanics* (Oxford University Press, London, 1958).

[9] T. Kinoshita, *History of Original Ideas and Basic Discoveries in Particle Physics*, Vol. 352 (Springer, US, 1996) pp. 9–26.

[10] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley Publishing Company, USA, 1995).

[11] W. Pauli, Rev. Mod. Phys. **13**, 203 (1941).

[12] J. Schwinger, Phys. Rev. **74**, 1439 (1948).

[13] H. A. Bethe, Phys. Rev. **72**, 339 (1947).

[14] E. Fermi, Z. Phys. **88**, 161 (1934).

[15] T. D. Lee and C. N. Yang, Phys. Rev. **104**, 254 (1956).

[16] C. S. Wu, E. Ambler, R. W. Hayward, D. D. Hoppes, and R. P. Hudson, Phys. Rev. **105**, 1413 (1957).

[17] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).

[18] N. Cabibbo, Phys. Rev. Lett. **10**, 531 (1963).

[19] S. L. Glashow, Nuclear Physics **22**, 579 (1961).

[20] S. Weinberg, Phys. Rev. Lett. **19**, 1264 (1967).

[21] A. Salam, Conf. Proc. **C680519**, 367 (1968).

[22] G. 't Hooft, Nucl. Phys. B **35**, 167 (1971).

[23] G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44**, 189 (1972).

[24] P. W. Higgs, Phys. Rev. Lett. **13**, 508 (1964).

[25] F. Englert and R. Brout, Phys. Rev. Lett. **13**, 321 (1964).

[26] G. S. Guralnik, C. R. Hagen, and T. W. B. Kibble, Phys. Rev. Lett. **13**, 585 (1964).

[27] E. S. Abers and B. W. Lee, Phys. Rept. **9**, 1 (1973).

[28] M. Gell-Mann, Phys. Lett. **8**, 214 (1964).

[29] M. Kobayashi and T. Maskawa, Prog. Theor. Phys. **49**, 652 (1973).

[30] J. Beringer *et al.* (Particle Data Group), Phys. Rev. D **86**, 010001 (2012).

[31] A. Zee, *Quantum Field Theory in a Nutshell* (Princeton University Press, 2010).

[32] R. A. Bertlmann, *Anomalies in Quantum Field Theory* (Clarendon Press - Oxford, 1996).

[33] J. C. Ward, Phys. Rev. **78**, 182 (1950).

[34] Y. Takahashi, Nuovo Cim. **6**, 371 (1957).

[35] G. 't Hooft, Nucl. Phys. B **61**, 455 (1973).

[36] S. L. Adler, Phys. Rev. **177**, 2426 (1969).

[37] J. S. Bell and R. Jackiw, Nuovo Cim. A **60**, 47 (1969).

[38] J. C. Romao and J. P. Silva, Int. J. Mod. Phys. A **27**, 1230025 (2012), arXiv:1209.6213 [hep-ph] .

[39] W. Pauli and F. Villars, Rev. Mod. Phys. **21**, 434 (1949).

[40] A. J. Bruce, *Anomalies in Quantum Field Theory*, Ph.D. thesis, University of Sussex (2005).

[41] J. Banks and H. Georgi, Phys. Rev. D **14**, 1159 (1976).

[42] H. Georgi and S. L. Glashow, Phys. Rev. D **6**, 429 (1972).

[43] W. A. Bardeen, Phys. Rev. **184**, 1848 (1969).

[44] D. J. Gross and R. Jackiw, Phys. Rev. D **6**, 477 (1972).

[45] C. Bouchiat, J. Iliopoulos, and P. Meyer, Phys. Lett. B **38**, 519 (1972).

[46] H. Georgi and S. L. Glashow, Phys. Rev. D **6**, 429 (1972).

[47] R. Delbourgo and A. Salam, Phys. Lett. B **40**, 381 (1972).

[48] L. Alvarez-Gaume and E. Witten, Nucl. Phys. B **234**, 269 (1984).

[49] A. Bilal, (2008), arXiv:0802.0634 [hep-th] .

[50] C. Q. Geng and R. E. Marshak, Phys. Rev. D **39**, 693 (1989).

[51] R. Foot, H. Lew, and R. R. Volkas, J. Phys. G **19**, 361 (1993), arXiv:hep-ph/9209259 [hep-ph] .

[52] R. Foot, G. C. Joshi, H. Lew, and R. R. Volkas, Mod. Phys. Lett. A **5**, 2721 (1990).

[53] A. Strumia and F. Vissani, arXiv:hep-ph/0606054 [hep-ph] .

[54] G. C. Branco, R. González Felipe, and F. R. Joaquim, Rev. Mod. Phys. **84**, 515 (2012), arXiv:1111.5332 [hep-ph] .

[55] R. N. Mohapatra, S. Antusch, K. S. Babu, G. Barenboim, M.-C. Chen, *et al.*, Rept. Prog. Phys. **70**, 1757 (2007), arXiv:hep-ph/0510213 [hep-ph] .

[56] H. Nunokawa, S. J. Parke, and J. W. F. Valle, Prog. Part. Nucl. Phys. **60**, 338 (2008), arXiv:0710.0554 [hep-ph] .

[57] B. Pontecorvo, Sov. Phys. JETP **6**, 429 (1957).

[58] J. Davis, Raymond, D. S. Harmer, and K. C. Hoffman, Phys. Rev. Lett. **20**, 1205 (1968).

[59] V. N. Gribov and B. Pontecorvo, Phys. Lett. B **28**, 493 (1969).

[60] B. Pontecorvo, Sov. Phys. JETP **26**, 984 (1968).

[61] Z. Maki, M. Nakagawa, and S. Sakata, Prog. Theor. Phys. **28**, 870 (1962).

[62] S. Weinberg, Phys. Rev. Lett. **43**, 1566 (1979).

[63] S. Weinberg, Phys. Rev. D **22**, 1694 (1980).

[64] E. Golowich and P. B. Pal, Phys. Rev. D **41**, 3537 (1990).

[65] K. S. Babu and R. N. Mohapatra, Phys. Rev. D **41**, 271 (1990).

[66] P. Minkowski, Phys. Lett. B **67**, 421 (1977).

[67] T. Yanagida, In Proceedings of the Workshop on the Baryon Number of the Universe and Unified Theories, Tsukuba, Japan, 13-14 Feb 1979 (1979).

[68] M. Gell-Mann, P. Ramond, and R. Slansky, In Supergravity, P. van Nieuwenhuizen and D. Z. Freedman (eds.), North Holland Publ. Co. , 315 (1979).

[69] S. L. Glashow, Quarks and Leptons, in Cargèse Lectures, eds. M. Lévy et al., Plenum, NY , 687 (1980).

[70] R. N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980).

[71] R. N. Mohapatra and G. Senjanovic, Phys. Rev. D **23**, 165 (1981).

[72] J. Schechter and J. W. F. Valle, Phys. Rev. D **22**, 2227 (1980).

[73] G. Lazarides, Q. Shafi, and C. Wetterich, Nucl. Phys. B **181**, 287 (1981).

[74] T. P. Cheng and L.-F. Li, Phys. Rev. D **22**, 2860 (1980).

[75] W. Konetschny and W. Kummer, Phys. Lett. B **70**, 433 (1977).

[76] R. Foot, H. Lew, X. G. He, and G. C. Joshi, Z. Phys. C **44**, 441 (1989).

[77] Z.-z. Xing, Phys. Rev. D **68**, 053002 (2003), arXiv:hep-ph/0305195 [hep-ph] .

[78] Z.-z. Xing, Phys. Rev. D **69**, 013006 (2004), arXiv:hep-ph/0307007 [hep-ph] .

[79] A. Merle and W. Rodejohann, Phys. Rev. D **73**, 073012 (2006), arXiv:hep-ph/0603111 [hep-ph] .

[80] Y. BenTov and A. Zee, Phys. Rev. D **84**, 073012 (2011), arXiv:1103.2616 [hep-ph] .

[81] E. I. Lashin and N. Chamoun, Phys. Rev. D **85**, 113011 (2012), arXiv:1108.4010 [hep-ph] .

[82] P. H. Frampton, S. L. Glashow, and D. Marfatia, Phys. Lett. B **536**, 79 (2002), arXiv:hep-ph/0201008 [hep-ph] .

[83] H. Fritzsch, Z.-z. Xing, and S. Zhou, JHEP **1109**, 083 (2011), arXiv:1108.4534 [hep-ph] .

[84] A. Leike, Phys. Rept. **317**, 143 (1999), arXiv:hep-ph/9805494 [hep-ph] .

[85] P. Langacker, Rev. Mod. Phys. **81**, 1199 (2009), arXiv:0801.1345 [hep-ph] .

[86] S. M. Barr, B. Bednarz, and C. Benesh, Phys. Rev. D **34**, 235 (1986).

[87] E. Ma, Mod. Phys. Lett. A **17**, 535 (2002), arXiv:hep-ph/0112232 [hep-ph] .

[88] S. M. Barr and I. Dorsner, Phys. Rev. D **72**, 015011 (2005), arXiv:hep-ph/0503186 [hep-ph] .

[89] J. C. Montero and V. Pleitez, Phys. Lett. B **675**, 64 (2009), arXiv:0706.0473 [hep-ph] .

[90] R. Adhikari, J. Erler, and E. Ma, Phys. Lett. B **672**, 136 (2009), arXiv:0810.5547 [hep-ph] .

[91] D. Emmanuel-Costa, E. T. Franco, and R. G. Felipe, Phys. Rev. D **79**, 115001 (2009), arXiv:0902.1722 [hep-ph] .

[92] E. Ma, Phys. Lett. B **433**, 74 (1998), arXiv:hep-ph/9709474 [hep-ph] .

[93] E. Ma and D. P. Roy, Phys. Rev. D **58**, 095005 (1998), arXiv:hep-ph/9806210 [hep-ph] .

[94] E. Ma, D. P. Roy, and U. Sarkar, Phys. Lett. B **444**, 391 (1998), arXiv:hep-ph/9810309 [hep-ph] .

[95] L. N. Chang, O. Lebedev, W. Loinaz, and T. Takeuchi, Phys. Rev. D **63**, 074013 (2001), arXiv:hep-ph/0010118 [hep-ph] .

[96] E. Salvioni, A. Strumia, G. Villadoro, and F. Zwirner, JHEP **1003**, 010 (2010), arXiv:0911.1450 [hep-ph] .

[97] J. Heeck and W. Rodejohann, Phys. Rev. D **85**, 113017 (2012), arXiv:1203.3117 [hep-ph] .

[98] T. Araki, J. Heeck, and J. Kubo, JHEP **1207**, 083 (2012), arXiv:1203.4951 [hep-ph] .

[99] J.-Y. Liu, Y. Tang, and Y.-L. Wu, J. Phys. G **39**, 055003 (2012), arXiv:1108.5012 [hep-ph] .

[100] LEP Collaborations (LEP, ALEPH, DELPHI, L3, OPAL, LEP Electroweak Working Group, SLD Electroweak Group, SLD Heavy Flavor Group), arXiv:hep-ex/0312023 .

[101] H.-S. Lee and E. Ma, Phys. Lett. B **688**, 319 (2010), arXiv:1001.0768 [hep-ph] .

[102] S. Godfrey and T. Martin, (2013), arXiv:1309.1688 [hep-ph] .

[103] ATLAS Collaboration, unpublished (2013), ATLAS-CONF-2013-017 .

[104] CMS Collaboration, unpublished (2013), CMS-PAS-EXO-12-061 .

[105] B. Holdom, Phys. Lett. B **166**, 196 (1986).

[106] W. Loinaz and T. Takeuchi, Phys. Rev. D **60**, 115008 (1999), arXiv:hep-ph/9903362 [hep-ph] .

[107] L. Basso, S. Moretti, and G. M. Pruna, Phys. Rev. D **83**, 055014 (2011), arXiv:1011.2612 [hep-ph] .

[108] S. Godfrey and T. A. W. Martin, Phys. Rev. Lett. **101**, 151803 (2008), arXiv:0807.1080 [hep-ph] .

[109] R. Diener, S. Godfrey, and T. A. W. Martin, Phys. Rev. D **83**, 115008 (2011), arXiv:1006.2845 [hep-ph] .