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Yukawa textures in modular symmetric vacuum of magnetized orbifold models

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We study quark mass matrices derived from magnetized $T^2/\mathbb{Z}_2$ orbifold models. Yukawa matrices at three modular fixed points, $\tau = i$, $e^{2\pi i/3}$, and $i\infty$, are invariant under S-, ST-, and T-transformations. We study these invariances on the $T^2/\mathbb{Z}_2$ twisted orbifold. We find that Yukawa matrices have a kind of texture structure, although those at $\tau = i\infty$ are not realistic. We classify the Yukawa textures at $\tau = i$ and $e^{2\pi i/3}$. Moreover we investigate the conditions such that the quark mass matrix constructed by Yukawa textures becomes approximately a rank-one matrix, which is favorable to lead to hierarchical masses between the third generation and the others. It is found that realistic quark mass matrices can be obtained around the S-invariant and ST-invariant vacua. As an illustrative example, we show the realization of the quark mass ratios and mixing based on Fritzch and Fritzch–Xing mass matrices.

Subject Index B29, B41, B43, B55

1. Introduction

The origin of flavor structure such as the mass hierarchy and flavor mixing is one of the unsolved mysteries in present-day particle physics. In the Standard Model (SM), quark flavor observables have been described by ten real parameters: six quark masses, three mixing angles, and one CP-violating phase. Similarly, lepton flavor observables need twelve real parameters: six lepton masses, three mixing angles, and three Dirac and Majorana CP-violating phases. To understand the origin of this large number of parameters, two types of approach, bottom-up and top-down, have been carried out. In the bottom-up approach, non-Abelian discrete flavor models have been proposed where S_N, A_N, $\Delta(3N^2)$, $\Delta(6N^2)$, and so on are assumed as flavor symmetries of quarks and leptons [1–6]. Then, such symmetries are broken by the vacuum expectation values (VEVs) of gauge singlet scalars, so-called flavons, but they become complicated.

Another bottom-up approach is to limit the number of parameters in the fermion mass matrices. For example, in Ref. [7] Fritzch proposed the idea of texture-zero for quark mass matrices where some of entries are zero, and this was extended in Ref. [8] as the Fritzch–Xing mass matrix (for a review, see Ref. [9]). Moreover, several types of texture structures were studied [10]. Actually, four phenomenologically viable zero textures of Hermitian quark mass matrices have been investigated, and it has been found that there are several possibilities (see, e.g., Ref. [11] and references therein).

On the other hand, superstring theory is a promising candidate for the unified theory. Superstring theory predicts ten dimensions. Low-energy effective field theory of superstring theory can be described by ten-dimensional (10D) super-Yang–Mills theory. Compactification of 10D

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superstring theory as well as super-Yang–Mills theory can lead to a variety of phenomena in particle physics, e.g. the flavor structure. Torus and orbifold compactifications with magnetic flux background are among the simplest, but have interesting structure. They lead to four-dimensional chiral theory, and the generation number is determined by the size of the magnetic fluxes [12–15]. Furthermore, their Yukawa couplings depend on moduli and can be suppressed. Indeed, realistic mass matrices can be realized [16–19].

One important aspect is that the torus compactification and its orbifolding have the modular symmetry $\Gamma \equiv SL(2,\mathbb{Z})$ as well as $\bar{\Gamma} \equiv SL(2,\mathbb{Z})/\mathbb{Z}_2$, which is a geometrical symmetry. Moreover, zero-mode wavefunctions in magnetized torus and orbifold models transform non-trivially under the modular symmetry [20–26]. In this context the modular symmetry is regarded as the flavor symmetry. Indeed, three-generation magnetized orbifold models lead to covering groups of A_4, S_4, A_5, $\Delta(98)$, and $\Delta(384)$ with center extensions as flavor symmetries [25]. In addition, Yukawa couplings also transform non-trivially under the modular symmetry. In this sense, the modular symmetry is not a simple symmetry, under which coupling constants and masses are invariant, but Yukawa couplings are spurion fields, which transform non-trivially under the modular symmetry.

Recently, the modular symmetry has been attracting attention from the bottom-up approach. Interestingly, the finite modular subgroups Γ_N for $N = 2, 3, 4, 5$ are isomorphic to S_3, A_4, S_4, and A_5, respectively [27]. Motivated by this point and string compactification, in the bottom-up approach flavor models with Γ_N have been studied intensively to lead to realistic quark and lepton mass matrices (see, e.g., Refs. [28–76]). In these modular flavor symmetric models, Yukawa couplings as well as masses are modular forms, which are functions of the modulus τ. When we choose proper values of τ, we can realize quark and lepton masses and their mixing angles as well as CP phases. Stabilization of the modulus τ has also been studied. The modulus can be stabilized at fixed points, $\tau = i$, $e^{2\pi i/3}$, with a certain probability [77–79]. The $\mathbb{Z}_2$ and $\mathbb{Z}_3$ residual symmetries remain at these fixed points $\tau = i$ and $e^{2\pi i/3}$, respectively, and they are generated by S and ST, while at the fixed point $\tau = i\infty$, T-symmetry remains. Because of the residual symmetries, mass matrices have specific patterns. Indeed, realistic results were obtained at nearby fixed points [49,52,59,71,72].

In this paper we revisit the structure of Yukawa matrices in magnetized orbifold models. Generic string compactifictions including magnetized models lead to more than one candidate for the Higgs modes, which have the same quantum numbers under the $SU(3) \times SU(2) \times U(1)$ SM gauge group and can couple with quarks and leptons. They are massless at the perturbative level. They may gain mass terms by non-perturbative effects, i.e. the μ-term in supersymmetric models, and the lightest direction of multi-Higgs modes may be determined. However, such analyses are not straightforward in explicit models, and the lightest direction is not clear. Thus, in the analysis of Refs. [16–19], the lightest direction is parametrized in the multi-Higgs field space. Using those parameters, the possibility of deriving realistic quark masses and mixing angles was examined. We follow the same procedure. In addition, we emphasize the modular symmetry of Higgs modes. Multi-Higgs modes are a (reducible) multiplet of the modular symmetry in magnetized orbifold models. As mentioned above, the $\mathbb{Z}_2$ ($\mathbb{Z}_3$) residual symmetries generated by S (ST) remain at these fixed points $\tau = i$ ($\tau = e^{2\pi i/3}$). Each Higgs mode has a definite $\mathbb{Z}_2$ ($\mathbb{Z}_3$) charge at $\tau = i$ ($e^{2\pi i/3}$). We can realize a specific pattern of Yukawa matrix at these fixed points of τ, depending on $\mathbb{Z}_N$ charges of Higgs modes. That is, texture structures are realized; we classify them here. We show that S-invariant vacua at $\tau = i$ and ST-invariant vacua at $\tau = e^{2\pi i/3}$ are useful to realize a large hierarchy in quark masses. However, we need small deviations from S-invariant and ST-invariant vacua to derive realistic results fixing $\tau = i$ and $\tau = e^{2\pi i/3}$. For example, the Fritzch mass matrix and the Fritzch–Xing mass matrix can be realized from these textures by taking appropriate Higgs VEV directions.

The paper is organized as follows. In Sect. 2 we review the zero-mode wavefunctions and Yukawa couplings on the torus and orbifold with magnetic fluxes. In Sect. 3 we review the three-generation fermion models on the orbifold. In Sect. 4 we study and classify the structure of Yukawa matrices at three modular fixed points. In Sect. 5 we show the condition such that quark mass matrices become rank-one matrices, and hence a large hierarchy of quarks is realized. In Sect. 6 we give examples of numerical studies for the quark mass matrices in our models. Section 7 concludes this study. In Appendix A and B we give the proofs of the rank-one conditions shown in Sect. 5.

2. Orbifold compactification with magnetic fluxes

The 10D super-Yang–Mills theory is the low-energy effective theory of superstring theory. We compactify the six dimensions, which includes the orbifold $T^2/\mathbb{Z}_2$ and four-dimensional compact space. We assume the flavor structure originated from $T^2/\mathbb{Z}_2$, although four-dimensional compact space may contribute to an overall factor of Yukawa matrices. Thus, we concentrate on the two-dimensional orbifold $T^2/\mathbb{Z}_2$ with magnetic flux, and give a review of zero-mode wavefunctions and Yukawa couplings on these backgrounds [13–15].

2.1 *Torus compactification*

First, we briefly review zero-mode wavefunctions on magnetized T^2 [12]. For simplicity, we concentrate on a $U(1)$ background magnetic flux given by

$$F = dA = \frac{\pi i M}{\mathrm{Im}\,\tau} dz \wedge d\bar{z}, \tag{1}$$

where z is the complex coordinate on T^2 and τ is the complex structure modulus. The flux M is induced by the vector potential one-form

$$A = \frac{\pi M}{\mathrm{Im}\,\tau} \mathrm{Im}((\bar{z} + \bar{\zeta}) dz). \tag{2}$$

In what follows, we consider the vanishing Wilson line $\zeta = 0$. Then, the torus identification $z \sim z + m + n\tau$, $m, n \in \mathbb{Z}$, gives the Dirac quantization condition, $M \in \mathbb{Z}$. Furthermore, the two-dimensional spinor with $U(1)$ unit charge $q = 1$, $\psi = (\psi_+, \psi_-)^{\mathrm{T}}$, must fulfill the boundary conditions

$$\psi(z+1) = \exp\left\{ i\pi M \frac{\mathrm{Im}\, z}{\mathrm{Im}\,\tau} \right\} \psi(z), \qquad \psi(z+\tau) = \exp\left\{ i\pi M \frac{\mathrm{Im}(\bar{\tau} z)}{\mathrm{Im}\,\tau} \right\} \psi(z). \tag{3}$$

By solving the massless Dirac equation, $i \not{D} \psi = 0$, under the above conditions, it is found that only positive (negative) chiral zero-mode wavefunctions have the $|M|$ number of degenerate solutions for $M > 0 (M < 0)$; the jth zero mode is expressed as

$$\begin{aligned}\psi_+^{j,|M|}(z,\tau) &= \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi |M| z \frac{\mathrm{Im} z}{\mathrm{Im}\tau}} \sum_{\ell \in \mathbb{Z}} e^{i\pi |M| \tau \left(\frac{j}{|M|} + \ell\right)^2} e^{2\pi i |M| z \left(\frac{j}{|M|} + \ell\right)} \\ &= \left(\frac{|M|}{\mathcal{A}^2}\right)^{1/4} e^{i\pi |M| z \frac{\mathrm{Im} z}{\mathrm{Im}\tau}} \vartheta \begin{bmatrix} \frac{j}{|M|} \\ 0 \end{bmatrix} (|M| z, |M| \tau),\end{aligned} \tag{4}$$

$$\psi_-^{j,|M|}(z,\tau) = \left(\psi_+^{-j,|M|}(z,\tau)\right)^*, \qquad j = 0, 1, \ldots, |M|-1, \tag{5}$$

where $\mathcal{A}$ denotes the area of T^2, and ϑ denotes the Jacobi theta function defined by

$$\vartheta\begin{bmatrix} a \\ b \end{bmatrix}(\nu,\tau) = \sum_{\ell\in\mathbb{Z}} e^{\pi i(a+\ell)^2\tau} e^{2\pi i(a+\ell)(\nu+b)}. \tag{6}$$

This function has the property

$$\begin{aligned}\vartheta\begin{bmatrix} \frac{j}{M_1} \\ 0 \end{bmatrix}(\nu_1, M_1\tau) \times \vartheta\begin{bmatrix} \frac{k}{M_2} \\ 0 \end{bmatrix}(\nu_2, M_2\tau) = &\sum_{m\in\mathbb{Z}_{M_1+M_2}} \vartheta\begin{bmatrix} \frac{j+k+M_1 m}{M_1+M_2} \\ 0 \end{bmatrix}(\nu_1+\nu_2, (M_1+M_2)\tau) \\ &\times \vartheta\begin{bmatrix} \frac{M_2 j - M_1 k + M_1 M_2 m}{M_1 M_2(M_1+M_2)} \\ 0 \end{bmatrix} \\ &\times (\nu_1 M_2 - \nu_2 M_1, M_1 M_2(M_1+M_2)\tau).\end{aligned} \tag{7}$$

Consequently, we find the normalization and product expansions of the zero modes:

$$\int d^2 z \psi_\pm^{i,|M|}(z,\tau)\left(\psi_\pm^{j,|M|}(z,\tau)\right)^* = (2\mathrm{Im}\,\tau)^{-1/2}\delta_{i,j}, \tag{8}$$

$$\psi_\pm^{i,|M_1|}(z,\tau)\cdot\psi_\pm^{j,|M_2|}(z,\tau) = \sum_{k\in\mathbb{Z}_{|M_1|+|M_2|}} Y^{ijk}\psi_\pm^{k,|M_1|+|M_2|}(z,\tau), \tag{9}$$

where

$$\begin{aligned}Y^{ijk} &= \int d^2 z \psi_\pm^{i,|M_1|}(z,\tau)\psi_\pm^{j,|M_2|}(z,\tau)\left(\psi_\pm^{k,|M_1|+|M_2|}(z,\tau)\right)^* \\ &= \mathcal{A}^{-1/2}\left|\frac{M_1 M_2}{M_1+M_2}\right|^{1/4}\vartheta\begin{bmatrix} \frac{|M_2|i-|M_1|j+|M_1 M_2|k}{|M_1 M_2(M_1+M_2)|} \\ 0 \end{bmatrix}(0, |M_1 M_2(M_1+M_2)|\tau).\end{aligned} \tag{10}$$

Hereafter, we omit the chirality sign $\pm$ from the zero modes.

To end this subsection, we also give a review of the modular symmetry for wavefunctions [23]. The modular group $\Gamma = SL(2,\mathbb{Z})$ is generated by two generators, S- and T-transformations, and defined as

$$\Gamma \equiv \langle S, T \mid S^2 = Z,\ S^4 = (ST)^3 = Z^2 = \mathbb{I}\rangle. \tag{11}$$

Then, the modular transformation for (z,τ) is given by

$$S : (z,\tau) \to \left(-\frac{z}{\tau}, -\frac{1}{\tau}\right), \qquad T : (z,\tau) \to (z, \tau+1), \tag{12}$$

and under these two transformations the wavefunctions in Eqs. (4) and (5) behave as the modular forms of weight 1/2 transformed by $\widetilde{\Gamma}_{2|M|}$:

$$\psi^{j,|M|}(\widetilde{\gamma}(z,\tau)) = \widetilde{J}_{1/2}(\widetilde{\gamma},\tau)\sum_{k=0}^{|M|-1}\widetilde{\rho}(\widetilde{\gamma})_{jk}\psi^{k,|M|}(z,\tau), \qquad \widetilde{\gamma}\in\widetilde{\Gamma}, \tag{13}$$

where $\widetilde{J}_{1/2}(\widetilde{\gamma},\tau)$ is the automorphy factor, $\widetilde{\Gamma}$ is the double covering group of Γ generated by two generators, $\widetilde{S}$- and $\widetilde{T}$-transformations (which are the double covering of S and T), and defined as

$$\widetilde{\Gamma} \equiv \langle \widetilde{S}, \widetilde{T} \mid \widetilde{S}^2 = \widetilde{Z},\ \widetilde{S}^4 = (\widetilde{S}\widetilde{T})^3 = \widetilde{Z}^2,\ \widetilde{S}^8 = (\widetilde{S}\widetilde{T})^6 = \widetilde{Z}^4 = \mathbb{I},\ \widetilde{Z}\widetilde{T} = \widetilde{T}\widetilde{Z}\rangle, \tag{14}$$

Table 1. The numbers of zero modes on the magnetized $T^2/\mathbb{Z}_2$ twisted orbifold.

$\lvert M\rvert$	1	2	3	4	5	6	7	8	9	10	11	12
$\mathbb{Z}_2$-even	1	2	2	3	3	4	4	5	5	6	6	7
$\mathbb{Z}_2$-odd	0	0	1	1	2	2	3	3	4	4	5	5

and $\widetilde{\rho}$ is the unitary representation of $\widetilde{\Gamma}_{2|M|}$ generated by the following $\widetilde{S}$- and $\widetilde{T}$-transformations:

$$\widetilde{\rho}(\widetilde{S})_{jk} = e^{i\pi/4}\frac{1}{\sqrt{|M|}}e^{2\pi i\frac{jk}{|M|}}, \qquad \widetilde{\rho}(\widetilde{T})_{jk} = e^{i\pi\frac{j^2}{|M|}}\delta_{j,k}. \tag{15}$$

$\widetilde{\Gamma}_{2|M|}$ is defined as

$$\widetilde{\Gamma}_{2|M|} \equiv \langle \widetilde{S}, \widetilde{T} \mid \widetilde{S}^2 = \widetilde{Z},\ \widetilde{S}^4 = (\widetilde{S}\widetilde{T})^3 = \widetilde{Z}^2 = -\mathbb{I},\ \widetilde{Z}\widetilde{T} = \widetilde{T}\widetilde{Z},\ \widetilde{T}^{2M} = \mathbb{I}\rangle. \tag{16}$$

That is, $\widetilde{\rho}$ satisfies the following algebraic relations:

$$\begin{aligned}\widetilde{\rho}(\widetilde{S})^2 &= \widetilde{\rho}(\widetilde{Z}),\\ \widetilde{\rho}(\widetilde{S})^4 &= [\widetilde{\rho}(\widetilde{S})\widetilde{\rho}(\widetilde{T})]^3 = \widetilde{\rho}(\widetilde{Z})^2 = -\mathbb{I},\\ \widetilde{\rho}(\widetilde{Z})\widetilde{\rho}(\widetilde{T}) &= \widetilde{\rho}(\widetilde{T})\widetilde{\rho}(\widetilde{Z}),\\ \widetilde{\rho}(\widetilde{T})^{2M} &= \mathbb{I}.\end{aligned} \tag{17}$$

We note that T-transformation for the wavefunctions can be defined with the vanishing Wilson line only if $M \in 2\mathbb{Z}$ for consistency with the boundary conditions. The T-transformation can be consistent for non-vanishing Wilson lines when $M \in 2\mathbb{Z} + 1$ [25].

2.2 *Orbifold compactification*

Second, we briefly review zero-mode wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold with magnetic flux M [13]. The $T^2/\mathbb{Z}_2$ twisted orbifold is obtained by further identifying the $\mathbb{Z}_2$ twisted point $-z$ with z, i.e. $z \sim -z$. In addition to the torus boundary conditions in Eq. (3), the wavefunctions on the magnetized $T^2/\mathbb{Z}_2$ twisted orbifold are required to fulfill

$$\psi_{T^2/\mathbb{Z}_2^m}(-z) = (-1)^m \psi_{T^2/\mathbb{Z}_2^m}(z), \qquad m \in \mathbb{Z}_2. \tag{18}$$

Hence, they can be expressed by the wavefunctions on magnetized T^2; actually, the zero modes are expressed as

$$\begin{aligned}\psi^{j,|M|}_{T^2/\mathbb{Z}_2^m}(z) &= \mathcal{N}^j\left(\psi^{j,|M|}_{T^2}(z) + (-1)^m\psi^{j,|M|}_{T^2}(-z)\right)\\ &= \mathcal{N}^j\left(\psi^{j,|M|}_{T^2}(z) + (-1)^m\psi^{|M|-j,|M|}_{T^2}(z)\right),\end{aligned} \tag{19}$$

where

$$\mathcal{N}^j = \begin{cases} 1/2 & (j = 0,\ |M|/2),\\ 1/\sqrt{2} & \text{(otherwise)}.\end{cases} \tag{20}$$

In Table 1 we show the number of zero modes on the magnetized $T^2/\mathbb{Z}_2$ twisted orbifold for vanishing discrete Wilson lines and Sherk–Schwarz phases.[1]

Next, we review the modular symmetry of zero modes on the orbifold. The zero modes in Eq. (19) behave as modular forms of weight 1/2 transformed by $\widetilde{\Gamma}_{2|M|}$ under the modular

[1] For zero modes with non-vanishing discrete Wilson lines and Sherk–Schwarz phases, see Refs. [14,15].

transformation:

$$\psi^{j,|M|}_{T^2/\mathbb{Z}_2^m}(\widetilde{\gamma}(z,\tau)) = \widetilde{J}_{1/2}(\widetilde{\gamma},\tau)\sum_k \widetilde{\rho}_{T^2/\mathbb{Z}_2^m}(\widetilde{\gamma})_{jk}\psi^{k,|M|}_{T^2/\mathbb{Z}_2^m}(z,\tau), \tag{21}$$

where $\widetilde{\rho}_{T^2/\mathbb{Z}_2^m}$ is the unitary representation of $\widetilde{\Gamma}_{2|M|}$ generated by the following $\widetilde{S}$- and $\widetilde{T}$-transformations:

$$\widetilde{\rho}_{T^2/\mathbb{Z}_2^0}(\widetilde{S})_{jk} = \mathcal{N}^j\mathcal{N}^k\frac{4e^{\pi i/4}}{\sqrt{|M|}}\cos\left(\frac{2\pi jk}{|M|}\right), \qquad \widetilde{\rho}_{T^2/\mathbb{Z}_2^0}(\widetilde{T})_{jk} = e^{i\pi\frac{j^2}{|M|}}\delta_{j,k},$$
$$\widetilde{\rho}_{T^2/\mathbb{Z}_2^1}(\widetilde{S})_{jk} = \mathcal{N}^j\mathcal{N}^k\frac{4ie^{\pi i/4}}{\sqrt{|M|}}\sin\left(\frac{2\pi jk}{|M|}\right), \quad \widetilde{\rho}_{T^2/\mathbb{Z}_2^1}(\widetilde{T})_{jk} = e^{i\pi\frac{j^2}{|M|}}\delta_{j,k}. \tag{22}$$

We again note that the T-transformation is consistent for vanishing discrete Wilson lines only if $M \in 2\mathbb{Z}$. The T-transformation can be consistent for non-vanishing discrete Wilson lines when $M \in 2\mathbb{Z}+1$ [25].

3. Three-generation models

3.1 *Classification for three-generation models*

In this subsection we review the classification of the three-generation models which lead to non-vanishing Yukawa coupling in the $T^2/\mathbb{Z}_2$ twisted orbifolds (for details, see Refs. [80,81]). Yukawa coupling for four-dimensional effective theory is given by the overlap integral of zero modes on the orbifold:

$$Y^{ijk} = \int_{6D} d^6z\psi^i_L(z)\psi^j_R(z)\left(\psi^k_H(z)\right)^*, \tag{23}$$

where ψ^i_L, ψ^j_R, and ψ^k_H are zero modes for left-handed fermion, right-handed fermion, and Higgs fields. We focus on the case where the flavor structure comes from only $T^2/\mathbb{Z}_2$, although another four-dimensional compact space contributes an overall factor of Yukawa matrices. Then, Yukawa couplings relevant to the flavor structure are written as

$$Y^{ijk}_{T^2/\mathbb{Z}_2} = \int_{T^2/\mathbb{Z}_2} d^2z\psi^{i,|M_L|}_{T^2/\mathbb{Z}_2^\ell}(z)\psi^{j,|M_R|}_{T^2/\mathbb{Z}_2^m}(z)\left(\psi^{k,|M_H|}_{T^2/\mathbb{Z}_2^n}(z)\right)^*, \tag{24}$$

where M_L, M_R, and M_H are the magnetic fluxes for left-handed fermion, right-handed fermion, and Higgs fields, respectively. To preserve the gauge invariance, these fluxes must satisfy the following flux condition:

$$|M_H| = \big||M_L| \pm |M_R|\big|. \tag{25}$$

Moreover, the Yukawa coupling in Eq. (24) should be invariant under a $\mathbb{Z}_2$ twist. Thus, non-vanishing Yukawa coupling must satisfy the following $\mathbb{Z}_2$ parity condition:

$$\ell + m + n = 0 \ (\mathrm{mod}\ 2). \tag{26}$$

By these flux and parity conditions, the flux and parity for Higgs fields are fixed once we choose ones for left- and right-handed fermions such that three generations of fermions are realized. In Table 2 we show all the possible three-generation models with non-vanishing Yukawa couplings when $|M_H| = ||M_L| + |N_R||$. Here, we ignore the three-generation models with flux $|M_H| = ||M_L| - |N_R||$ because such models do not lead to realistic results.

Table 2. Possible three-generation models with non-vanishing Yukawa couplings on the $T^2/\mathbb{Z}_2$ twisted orbifold when $|M_H| = ||M_L| + |M_R||$. There are additional possible models obtained by left (L) and right (R) flipping, although they are omitted from this table.

M_L (parity)	M_R (parity)	M_H (parity)	No. Higgs modes	Model name
4 (even)	4 (even)	8 (even)	5	4-4-8, (e,e,e), 5H
4 (even)	5 (even)	9 (even)	5	4-5-9, (e,e,e), 5H
5 (even)	5 (even)	10 (even)	6	5-5-10, (e,e,e), 6H
4 (even)	7 (odd)	11 (odd)	5	4-7-11, (e,o,o), 5H
4 (even)	8 (odd)	12 (odd)	5	4-8-12, (e,o,o), 5H
5 (even)	7 (odd)	12 (odd)	5	5-7-12, (e,o,o), 5H
5 (even)	8 (odd)	13 (odd)	6	5-8-13, (e,o,o), 6H
7 (odd)	7 (odd)	14 (even)	8	7-7-14, (o,o,e), 8H
7 (odd)	8 (odd)	15 (even)	8	7-8-15, (o,o,e), 8H
8 (odd)	8 (odd)	16 (even)	9	8-8-16, (o,o,e), 9H

3.2 *Yukawa couplings*

Here, we review how to calculate Yukawa couplings in the three-generation models. First of all, we calculate ones on the torus given by

$$Y^{ijk}_{T^2} = \int_{T^2} d^2z \psi^{i,|M_L|}_{T^2}(z)\psi^{j,|M_R|}_{T^2}(z)\left(\psi^{k,|M_H|}_{T^2}(z)\right)^*. \tag{27}$$

Using the normalization in Eq. (8) and the product expansion in Eq. (9), we find that

$$\begin{aligned} Y^{ijk}_{T^2} &= (2\mathcal{A}\mathrm{Im}\,\tau)^{-1/2}\left|\frac{M_L M_R}{M_H}\right|^{1/4} \\ &\quad\times \sum_{m=0}^{|M_H|-1} \vartheta\begin{bmatrix} \frac{|M_R|i-|M_L|j+|M_LM_R|m}{|M_LM_RM_H|} \\ 0 \end{bmatrix}(0, |M_LM_RM_H|\tau)\cdot\delta_{i+j-k,|M_H|\ell-|M_L|m} \\ &= c\sum_{m=0}^{|M_H|-1} \eta_{|M_R|i-|M_L|j+|M_LM_R|m}\cdot\delta_{i+j-k,|M_H|\ell-|M_L|m}, \end{aligned} \tag{28}$$

where $\ell \in \mathbb{Z}$, $c = (2\mathcal{A}\mathrm{Im}\,\tau)^{-1/2}\left|\frac{M_LM_R}{M_H}\right|^{1/4}$, and we have used the notation

$$\eta_N = \vartheta\begin{bmatrix} \frac{N}{M} \\ 0 \end{bmatrix}(0, M\tau), \qquad M = |M_LM_RM_H|. \tag{29}$$

Then, Yukawa couplings on the $T^2/\mathbb{Z}_2$ twisted orbifold can be expressed by ones on the torus, because zero modes on the orbifold can be expressed by ones on the torus. Inserting zero modes on the orbifold in Eq. (19) to Yukawa couplings on the orbifold in Eq. (24), we find that

$$Y^{ijk}_{T^2/\mathbb{Z}_2} = \sum_{i',j',k'} O^{ii',|M_L|}_{\ell} O^{jj',|M_R|}_{m} O^{kk',|M_H|}_{n} Y^{i'j'k'}_{T^2}, \tag{30}$$

where

$$O^{jk,M}_m = \mathcal{N}^j\left(\delta_{j,k} + (-1)^m\delta_{j,M-k}\right). \tag{31}$$

We also study the modular symmetry of Yukawa couplings on the orbifold. Since Yukawa couplings are written by the overlap integral of zero modes, from the transformation law for

Table 3. The number of each $\mathbb{Z}_4$ eigenstate in wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold at $\tau = i$. η denotes the eigenvalues of the $\mathbb{Z}_4$ twist. The S-transformation eigenstates and eigenvalues are the same as those for $\mathbb{Z}_4$.

$\mathbb{Z}_2$ parity, no. generations	No. $\mathbb{Z}_4(S)$ eigenstates			
	$\eta = 1$	$\eta = -1$	$\eta = i$	$\eta = -i$
even, $2n$	n	n	0	0
even, $2n+1$	$n+1$	n	0	0
odd, $2n$	0	0	n	n
odd, $2n+1$	0	0	$n+1$	n

zero modes we find that Yukawa couplings are transformed as

$$Y^{ijk}_{T^2/\mathbb{Z}_2}(\widetilde{\gamma}\tau) = \widetilde{J}_{1/2}(\widetilde{\gamma},\tau)\widetilde{J}_{1/2}(\widetilde{\gamma},\tau)\widetilde{J}^*_{1/2}(\widetilde{\gamma},\tau)\widetilde{\rho}_{T^2/\mathbb{Z}_2^\ell}(\widetilde{\gamma})_{ii'}\widetilde{\rho}_{T^2/\mathbb{Z}_2^m}(\widetilde{\gamma})_{jj'}\widetilde{\rho}^*_{T^2/\mathbb{Z}_2^n}(\widetilde{\gamma})_{kk'}Y^{i'j'k'}_{T^2/\mathbb{Z}_2}(\tau). \tag{32}$$

4. Yukawa textures by modular symmetry

In this section we study the restrictions on Yukawa matrices by modular symmetry. We will see that modular symmetry at its fixed points restricts the structure of the Yukawa matrices, and then the Yukawa matrices have a kind of texture structure. The fixed points for the modular transformation are: as follows:

I. $\tau = i$ is invariant under S-transformation.
II. $\tau = e^{2\pi i/3} \equiv \omega$ is invariant under ST-transformation.
III. $\tau = i\infty$ ($\mathrm{Im}\,\tau = \infty$) is invariant under T-transformation.

Hereafter, we investigate the structure of Yukawa matrices at these three fixed points. We note that we write Yukawa matrices on the $T^2/\mathbb{Z}_2$ twisted orbifold as Y^{ijk} instead of $Y^{ijk}_{T^2/\mathbb{Z}_2}$.

4.1 *S-invariance*

Only if $\tau = i$, the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold can be expanded by $\mathbb{Z}_4$ twist eigenstates (for $\mathbb{Z}_4$ twist eigenstates, see Refs. [14,15,23,82]). The $\mathbb{Z}_4$ twist is defined by the following transformation of the complex coordinate on T^2:

$$z \to iz. \tag{33}$$

The numbers of the $\mathbb{Z}_4$ eigenstates in the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold are shown in Table 3. Note that the S-transformation eigenstates and eigenvalues are the same as for $\mathbb{Z}_4$; under S-transformation the wavefunctions on the $\mathbb{Z}_4$ eigenbasis are transformed by a diagonalized matrix composed of $\mathbb{Z}_4$ eigenvalues.

At $\tau = i$, Yukawa matrices are invariant under S-transformation because S: $\tau = -1/\tau$. This S-invariance is written as

$$Y^{ijk} = \widetilde{J}_{1/2}(\widetilde{S},i)\widetilde{\rho}_L(\widetilde{S})_{ii'} \cdot \widetilde{J}_{1/2}(\widetilde{S},i)\widetilde{\rho}_R(\widetilde{S})_{jj'} \cdot (\widetilde{J}_{1/2}(\widetilde{S},i)\widetilde{\rho}_H(\widetilde{S})_{kk'})^* \cdot Y^{i'j'k'}, \tag{34}$$

with

$$\widetilde{J}_{1/2}(\widetilde{S},\tau) = (-\tau)^{1/2}. \tag{35}$$

On the $\mathbb{Z}_4$ eigenstates, that is, on S-transformation eigenstates, the transformation matrix, $\widetilde{\rho}(\widetilde{S})$, is given by a diagonalized matrix composed of $\mathbb{Z}_4$ eigenvalues. The numbers of $\mathbb{Z}_4$ eigenvalues

Table 4. The structures of Yukawa matrices for each S-eigenstate Higgs mode. The Yukawa matrices are S-transformation eigenstates and are restricted to two types of structure by S-invariance. The symbol "$*$" denotes non-zero matrix elements.

$\mathbb{Z}_2$ parities of (L, R, H)	Structures of Yukawa matrices for each S-eigenstate Higgs mode			
	1	-1	i	$-i$
(even, even, even)	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$	None	None
(even, odd, odd)	None	None	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$
(odd, even, odd)	None	None	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$
(odd, odd, even)	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$	None	None

in the diagonalized matrix can be read from Table 3. Then, the S-invariance in Eq. (34) restricts the structure of the Yukawa matrices to two types, as shown in Table 4.

As a simple example, we show a restriction on the Yukawa matrices in the model "4-4-8, (e,e,e), 5H" in Table 2. Five Higgs modes in this model, whose flux is eight and parity is even, are transformed by

$$\widetilde{J}_{1/2}(\widetilde{S}, i)\widetilde{\rho}_H(\widetilde{S}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{36}$$

under S-transformation. On the other hand, three generations of fermions, whose flux is four and parity is even, are transformed by

$$\widetilde{J}_{1/2}(\widetilde{S}, i)\widetilde{\rho}_L(\widetilde{S}) = \widetilde{J}_{1/2}(\widetilde{S}, i)\widetilde{\rho}_R(\widetilde{S}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{37}$$

Then, the S-invariance on the Yukawa matrices is written as

$$Y^{ijk} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{ii'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}_{jj'} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}^*_{kk'} Y^{i'j'k'}. \tag{38}$$

Thus, the Yukawa matrices for S-invariant Higgs modes, Y^{ij0}, Y^{ij1}, and Y^{ij2}, and those for S-variant Higgs modes, Y^{ij3} and Y^{ij4}, are restricted to the following two structures, respectively:

$$Y^{ij0,1,2} = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad Y^{ij3,4} = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}, \tag{39}$$

Table 5. The number of each $\mathbb{Z}_6$ eigenstate in wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold at $\tau = e^{2\pi i/3} = \omega$. η denotes the eigenvalues ofthe $\mathbb{Z}_6$ twist. The ST-transformation eigenstates are the same as those for $\mathbb{Z}_6$. The ST-transformation eigenvalues are given by the squares of the $\mathbb{Z}_6$ eigenvalues.

$\mathbb{Z}_2$ parity, no. generations	No. $\mathbb{Z}_6$ eigenstates					
	$\eta = 1$	$\eta = \omega^{1/2}$	$\eta = \omega$	$\eta = \omega^{3/2}$	$\eta = \omega^2$	$\eta = \omega^{5/2}$
even, $3n$	n	0	n	0	n	0
even, $3n+1$	$n+1$	0	n	0	n	0
even, $3n+2$	$n+1$	0	$n+1$	0	n	0
odd, $3n$	0	n	0	n	0	n
odd, $3n+1$	0	$n+1$	0	n	0	n
odd, $3n+2$	0	$n+1$	0	$n+1$	0	n

where the symbol "$*$" denotes non-zero matrix elements.

4.2 *ST-invariance*

Only if $\tau = e^{2\pi i/3} \equiv \omega$ and flux $M =$ even, the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold can be expanded by $\mathbb{Z}_6$ twist eigenstates (for $\mathbb{Z}_6$ twist eigenstates, see Refs. [14,15,23,82]). The $\mathbb{Z}_6$ twist is defined by the following transformation of the complex coordinate on T^2:

$$z \to e^{\pi i/3} z. \tag{40}$$

The numbers of $\mathbb{Z}_6$ eigenstates in the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold are shown in Table 5. Note that the ST-transformation eigenstates are the same as those for $\mathbb{Z}_6$. The ST-transformation eigenvalues are given by the squares of the $\mathbb{Z}_6$ eigenvalues since ST-transformation at its fixed point is equivalent to a $\mathbb{Z}_3$ twist. Under the ST-transformation, hence, the wavefunctions on the $\mathbb{Z}_6$ eigenbasis are transformed by a diagonalized matrix composed of the squares of the $\mathbb{Z}_6$ eigenvalues.

At $\tau = \omega$, the Yukawa matrices are invariant under the ST-transformation because ST: $\tau = -1/(\tau + 1)$. Only if the fluxes M_L, M_R, and M_H are all even integers is this ST-invariance written as

$$Y^{ijk} = \widetilde{J}_{1/2}(\widetilde{ST}, \omega)\widetilde{\rho}_L(\widetilde{ST})_{ii'} \cdot \widetilde{J}_{1/2}(\widetilde{ST}, \omega)\widetilde{\rho}_R(\widetilde{ST})_{jj'} \cdot (\widetilde{J}_{1/2}(\widetilde{ST}, \omega)\widetilde{\rho}_H(\widetilde{ST})_{kk'})^* \cdot Y^{i'j'k'}, \tag{41}$$

with

$$\widetilde{J}_{1/2}(\widetilde{ST}, \tau) = (-(\tau + 1))^{1/2}. \tag{42}$$

On the $\mathbb{Z}_6$ eigenstates, i.e. on ST-transformation eigenstates, the transformation matrix, $\widetilde{\rho}(\widetilde{ST})$, is given by a diagonalized matrix composed of the squares of $\mathbb{Z}_6$ eigenvalues. The numbers of $\mathbb{Z}_6$ eigenvalues in the diagonalized matrix can be read from Table 5. Then, the ST-invariance in Eq. (41) restricts the Yukawa matrices to three types of structure, as shown in Table 6.

As a simple example, we show a restriction on the Yukawa matrices in the model "4-4-8, (e,e,e), 5H." The five Higgs modes in this model, whose flux is eight and parity is even, are transformed by

$$\widetilde{J}_{1/2}(\widetilde{ST}, \omega)\widetilde{\rho}_H(\widetilde{ST}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega \end{pmatrix} \tag{43}$$

Table 6. The structures of the Yukawa matrices for each ST-eigenstate Higgs mode. The Yukawa matrices are ST-transformation eigenstates and then are restricted to three types of structure by ST-invariance. The symbol "$*$" denotes non-zero matrix elements.

$\mathbb{Z}_2$ parities of (L, R, H)	Structures of Yukawa matrices for each ST-eigenstate Higgs mode		
	1	ω^2	ω
All patterns	$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$

under ST-transformation. On the other hand, the three-generation fermions, whose flux is four and parity is even, are transformed by

$$\widetilde{J}_{1/2}(\widetilde{ST}, \omega)\widetilde{\rho}_L(\widetilde{ST}) = \widetilde{J}_{1/2}(\widetilde{ST}, \omega)\widetilde{\rho}_R(\widetilde{ST}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}. \quad (44)$$

Then, the ST-invariance on the Yukawa matrices is written as

$$Y^{ijk} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}_{ii'} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}_{jj'} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega \end{pmatrix}^*_{kk'} Y^{i'j'k'}. \quad (45)$$

Thus, the Yukawa matrices for ST-invariant Higgs, Y^{ij0} and Y^{ij1}, those for ω-eigenstate Higgs, Y^{ij2} and Y^{ij3}, and those for ω^2-eigenstate Higgs, Y^{ij4}, are restricted to the following three structures, respectively:

$$Y^{ij0,1} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}, \qquad Y^{ij2,3} = \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}, \qquad Y^{ij4} = \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}. \quad (46)$$

4.3 *T-invariance*

Only if the flux $M =$ even, the wavefunctions on the $T^2/\mathbb{Z}_2$ twisted orbifold can be expanded by T-transformation eigenstates.

At $\mathrm{Im}\,\tau = \infty$, the Yukawa matrices are invariant under the T-transformation because T: $\tau = \tau + 1$. Only if the fluxes M_L, M_R, and M_H are all even integers is this T-invariance written as

$$Y^{ijk} = \widetilde{J}_{1/2}(\widetilde{T}, i\infty)\widetilde{\rho}_L(\widetilde{T})_{ii'} \cdot \widetilde{J}_{1/2}(\widetilde{T}, i\infty)\widetilde{\rho}_R(\widetilde{T})_{jj'} \cdot (\widetilde{J}_{1/2}(\widetilde{T}, i\infty)\widetilde{\rho}_H(\widetilde{T})_{kk'})^* \cdot Y^{i'j'k'}, \quad (47)$$

with

$$\widetilde{J}_{1/2}(\widetilde{T}, \tau) = 1, \qquad \widetilde{\rho}(\widetilde{T})_{jk} = e^{i\pi j^2/M}\delta_{j,k}. \quad (48)$$

This leads to

$$Y^{ijk} = Y^{ijk}\exp\left[\pi i\left(\frac{i^2}{M_L} + \frac{j^2}{M_R} - \frac{k^2}{M_H}\right)\right], \quad (49)$$

and we find the non-zero elements condition,

$$\left(\frac{i^2}{M_L} + \frac{j^2}{M_R} - \frac{k^2}{M_H}\right) \bmod 2 = 0, \qquad \text{otherwise } Y^{ijk} = 0, \quad (50)$$

Table 7. The number of each texture structure matrix in three-generation models. The first column shows the three-generation models classified and named in Table 2. The other columns show the number of each texture at $\tau = i$ and $\tau = \omega$. The values in parentheses denote the eigenvalues of corresponding Higgs modes under S- (at $\tau = i$) and ST-transformation (at $\tau = \omega$).

	No. of each texture at $\tau = i$		No. of each texture at $\tau = \omega$		
Three-generation models	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}$
4-4-8, (e,e,e), 5H	3 (1)	2 (− 1)	2 (1)	2 (ω^2)	1 (ω)
4-5-9, (e,e,e), 5H	3 (1)	2 (− 1)	None	None	None
5-5-10, (e,e,e), 6H	3 (1)	3 (− 1)	None	None	None
4-7-11, (e,o,o), 5H	3 (i)	2 (− i)	None	None	None
4-8-12, (e,o,o), 5H	3 (i)	2 (− i)	2 (1)	2 (ω^2)	1 (ω)
5-7-12, (e,o,o), 5H	3 (i)	2 (− i)	None	None	None
5-8-13, (e,o,o), 6H	3 (i)	3 (− i)	None	None	None
7-7-14, (o,o,e), 8H	4 (− 1)	4 (1)	None	None	None
7-8-15, (o,o,e), 8H	4 (− 1)	4 (1)	None	None	None
8-8-16, (o,o,e), 9H	4 (− 1)	5 (1)	3 (1)	3 (ω^2)	3 (ω)

which makes almost elements of the Yukawa matrices vanish. For example, in the model "4-4-8, (e,e,e), 5H," only three combinations of indices,

$$(i, j, k) = (0, 0, 0),\ (1, 1, 2),\ (2, 2, 4), \tag{51}$$

can satisfy the non-zero element condition in Eq. (50), and the Yukawa matrices are restricted to the following four structures:

$$Y^{ij0} = \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad Y^{ij2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & 0 \end{pmatrix},\quad Y^{ij4} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & * \end{pmatrix},\quad Y^{ij1,3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{52}$$

We cannot realize flavor mixing from these Yukawa matrices. Similarly, in other three-generation models we cannot realize mass matrices for the up and down sectors consistent with observations. Therefore, hereafter we avoid discussion of T-invariance in Yukawa matrices.

4.4 *Classification for textures in three-generation models*

To end this section we classify the number of each texture structure in three-generation models on the $T^2/\mathbb{Z}_2$ twisted orbifold. We show the result in Table 7. Note that we ignore the textures by T-invariance at $\mathrm{Im}\,\tau = \infty$.

5. Rank-one structures in the mass matrix

Once the lightest Higgs field develops its VEV, the Yukawa couplings give a fermion mass term:

$$M^{ij} = Y^{ijk} \langle H^k \rangle, \tag{53}$$

where we have assumed that $\langle H^k \rangle$ are given by the direction of the lightest Higgs mode. By using texture structures, we investigate here the Higgs VEV direction such that the quark mass matrix has rank one. Since quark mass ratios have a large hierarchy, we can approximately regard it as a rank-one matrix:

$$\begin{pmatrix} m_u & & \\ & m_c & \\ & & m_t \end{pmatrix} = m_t \begin{pmatrix} \mathcal{O}(10^{-6}) & & \\ & \mathcal{O}(10^{-3}) & \\ & & 1 \end{pmatrix} \sim m_t \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}, \tag{54}$$

$$\begin{pmatrix} m_d & & \\ & m_s & \\ & & m_b \end{pmatrix} = m_b \begin{pmatrix} \mathcal{O}(10^{-4}) & & \\ & \mathcal{O}(10^{-2}) & \\ & & 1 \end{pmatrix} \sim m_b \begin{pmatrix} 0 & & \\ & 0 & \\ & & 1 \end{pmatrix}. \tag{55}$$

Thus, the mass ratios consistent with observations would be realized near the Higgs VEV directions, leading to a rank-one quark mass matrix. In other words, if there is no direction leading to a rank-one mass matrix, it is difficult to reproduce the observed values of quark mass ratios. In this section we show the conditions where such a rank-one mass matrix can be realized by textures in the three-generation magnetized orbifold models.

5.1 *Higgs VEV directions at* $\tau = i$

In this subsection we investigate the Higgs VEV directions leading to a rank-one fermion mass matrix at $\tau = i$. In this case, the fermion mass matrix can be expanded by textures as

$$M^{ij} = \sum_m \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}^{ijm} \langle H^m \rangle + \sum_n \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix}^{ijn} \langle H^n \rangle. \tag{56}$$

Suppose that non-vanishing elements have generic values, but not specific relations among elements. Then, a rank-one matrix can be realized in the following cases:

I. If the mass matrix includes three or more of

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in the S-eigenstate directions.

II. In addition to case I, if the mass matrix is symmetric (non-symmetric) and includes one (two) or more of

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist not in the S-eigenstate directions, too.

III. If the mass matrix is symmetric and includes two or more of both types of textures, then the Higgs VEV directions leading to rank one exist not in the S-eigenstate directions.

IV. If the mass matrix is non-symmetric and includes two or more of

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix}$$

Table 8. The Higgs VEV directions leading to a rank-one mass matrix at $\tau = i$.

Three-generation models	Higgs VEV directions leading to rank one
4-4-8, (e,e,e), 5H	S-invariant, not S-eigenstate
4-5-9, (e,e,e), 5H	S-invariant, not S-eigenstate
5-5-10, (e,e,e), 6H	S-invariant, not S-eigenstate
4-7-11, (e,o,o), 5H	i eigenstate, not S-eigenstate
4-8-12, (e,o,o), 5H	i eigenstate, not S-eigenstate
5-7-12, (e,o,o), 5H	i eigenstate, not S-eigenstate
5-8-13, (e,o,o), 6H	i eigenstate, $-i$ eigenstate, not S-eigenstate
7-7-14, (o,o,e), 8H	-1 eigenstate, not S-eigenstate
7-8-15, (o,o,e), 8H	S-invariant, -1 eigenstate, not S-eigenstate
8-8-16, (o,o,e), 9H	-1 eigenstate, not S-eigenstate

and three or more of

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist not in the S-eigenstate directions.

V. If the mass matrix is non-symmetric and includes three or more of

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in the S-eigenstate directions.

Proofs of the above are presented in Appendix A. We show which Higgs VEV directions leading to rank one exist in the three-generation models in Table 8. There are four models where rank-one directions exist on S-invariant directions. In these four models, we have the possibility of realizing a realistic quark mass matrix if we assume an almost S-invariant vacuum.

5.2 *Higgs VEV directions at $\tau = \omega$*

In this subsection we investigate the Higgs VEV directions leading to a rank-one fermion mass matrix at $\tau = \omega$. In this case, the fermion mass matrix can be expanded by textures as

$$M^{ij} = \sum_{\ell} \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix}^{ij\ell} \langle H^{\ell} \rangle + \sum_{m} \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix}^{ijm} \langle H^{m} \rangle + \sum_{n} \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix}^{ijn} \langle H^{n} \rangle. \quad (57)$$

Suppose that non-vanishing elements have generic values, but not specific relations among the elements. Then, a rank-one matrix can be realized in the following cases:

I. If the mass matrix is symmetric (non-symmetric) and includes two (three) or more of

$$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in the ST-invariant directions.

Table 9. Higgs VEV directions leading to a rank-one mass matrix at $\tau = \omega$.

Three-generation models	Higgs VEV directions leading to rank one
4-4-8, (e,e,e), 5H	ST-invariant, ω^2 eigenstate, not ST-eigenstate
4-8-12, (e,o,o), 5H	not ST-eigenstate
8-8-16, (o,o,e), 9H	ST-invariant, ω^2 eigenstate, ω eigenstate, not ST-eigenstate

II. If the mass matrix is symmetric (non-symmetric) and includes two (three) or more of

$$\begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in the ST-eigenstate directions corresponding to eigenvalue ω^2.

III. If the mass matrix is symmetric (non-symmetric) and includes two (three) or more of

$$\begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix},$$

then the Higgs VEV directions leading to rank one exist in the ST-eigenstate directions corresponding to eigenvalue ω.

IV. If the mass matrix is symmetric (non-symmetric) and includes one (two) or more of two types of textures and two (one) or more of another type of texture, then the Higgs VEV directions leading to rank one exist not in the ST-eigenstate directions.

V. If a non-symmetric mass matrix includes three or more of two types of textures, then the Higgs VEV directions leading to rank one exist not in the ST-eigenstate directions.

Proofs of the above are presented in Appendix B. We show which Higgs VEV directions leading to rank one exist in three-generation models in Table 9. Note that we omit three-generation models including odd integral flux since the ST-transformation for Yukawa couplings cannot be defined with vanishing Wilson lines. There are two models where rank-one directions exist on ST-invariant directions. In these two models, we have the possibility of realizing a realistic quark mass matrix if we assume an almost ST-invariant vacuum.

6. Numerical example: model "4-4-8, (e,e,e), 5H"

In this section we study the model "4-4-8, (e,e,e), 5H." We assume that both up and down sectors correspond to this model. Then we show examples to realize the quark masses and mixing angles.

6.1 *Yukawa matrices*

Here we show the Yukawa matrices in the model "4-4-8, (e,e,e), 5H." Table 10 shows the zero-mode assignments for left-handed fermions L, right-handed fermions R, and the Higgs fields H.

This model has five zero modes for Higgs fields. The Yukawa couplings $Y^{ijk}L^iR^jH^k$ are given by

$$Y^{ijk}H^k = Y^{ij0}H^0 + Y^{ij1}H^1 + Y^{ij2}H^2 + Y^{ij3}H^3 + Y^{ij4}H^4,$$

Table 10. Zero-mode wavefunctions in the "4-4-8, (e,e,e), 5H" model.

	$L^i(\lambda^{ab})$	$R^j(\lambda^{ca})$	$H^k(\lambda^{bc})$
0	$\psi_{T^2}^{0,4}$	$\psi_{T^2}^{0,4}$	$\psi_{T^2}^{0,8}$
1	$\frac{1}{\sqrt{2}}\left(\psi_{T^2}^{1,4}+\psi_{T^2}^{3,4}\right)$	$\frac{1}{\sqrt{2}}\left(\psi_{T^2}^{1,4}+\psi_{T^2}^{3,4}\right)$	$\frac{1}{\sqrt{2}}\left(\psi_{T^2}^{1,8}+\psi_{T^2}^{7,8}\right)$
2	$\psi_{T^2}^{2,4}$	$\psi_{T^2}^{2,4}$	$\frac{1}{\sqrt{2}}\left(\psi_{T^2}^{2,8}+\psi_{T^2}^{6,8}\right)$
3			$\frac{1}{\sqrt{2}}\left(\psi_{T^2}^{3,8}+\psi_{T^2}^{5,8}\right)$
4			$\psi_{T^2}^{4,8}$

where

$$Y^{ij0} = c_{4\text{-}4\text{-}8}\begin{pmatrix} X_0 & & \\ & X_1 & \\ & & X_2 \end{pmatrix}, \quad Y^{ij1} = c_{4\text{-}4\text{-}8}\begin{pmatrix} & X_3 & \\ X_3 & & X_4 \\ & X_4 & \end{pmatrix},$$

$$Y^{ij2} = c_{4\text{-}4\text{-}8}\begin{pmatrix} & & \sqrt{2}X_1 \\ & \frac{1}{\sqrt{2}}(X_0+X_2) & \\ \sqrt{2}X_1 & & \end{pmatrix}, \quad Y^{ij3} = c_{4\text{-}4\text{-}8}\begin{pmatrix} & X_4 & \\ X_4 & & X_3 \\ & X_3 & \end{pmatrix},$$

$$Y^{ij4} = c_{4\text{-}4\text{-}8}\begin{pmatrix} X_2 & & \\ & X_1 & \\ & & X_0 \end{pmatrix}, \tag{58}$$

with

$$\begin{aligned} X_0 &= \eta_0 + 2\eta_{32} + \eta_{64}, \\ X_1 &= \eta_8 + \eta_{24} + \eta_{40} + \eta_{56}, \\ X_2 &= 2(\eta_{16} + \eta_{48}), \\ X_3 &= \eta_4 + \eta_{28} + \eta_{36} + \eta_{60}, \\ X_4 &= \eta_{12} + \eta_{20} + \eta_{44} + \eta_{52}. \end{aligned}$$

Here, we have used the notation

$$\eta_N = \vartheta\begin{bmatrix} \frac{N}{128} \\ 0 \end{bmatrix}(0, 128\tau).$$

Under modular transformation, these Yukawa couplings Y^{ijk} are transformed as follows:

$$Y^{ijk} \xrightarrow{\gamma} \left(\widetilde{J}_{1/2}(\widetilde{\gamma},\tau)\widetilde{\rho}_4^{ii'}(\widetilde{\gamma})\right)\left(\widetilde{J}_{1/2}(\widetilde{\gamma},\tau)\widetilde{\rho}_4^{jj'}(\widetilde{\gamma})\right)\left(\widetilde{J}_{1/2}(\widetilde{\gamma},\tau)\widetilde{\rho}_8^{kk'}(\widetilde{\gamma})\right)^* Y^{i'j'k'}, \tag{59}$$

where $\widetilde{\gamma} \in \widetilde{\Gamma}$ and the unitary representations $\widetilde{\rho}_4$ and $\widetilde{\rho}_8$ are generated by

$$\widetilde{\rho}_4(\widetilde{S}) = \frac{e^{\pi i/4}}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & \sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}, \quad \widetilde{\rho}_8(\widetilde{S}) = \frac{e^{\pi i/4}}{2\sqrt{2}}\begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} & 1 \\ \sqrt{2} & \sqrt{2} & 0 & -\sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 0 & -2 & 0 & \sqrt{2} \\ \sqrt{2} & -\sqrt{2} & 0 & \sqrt{2} & -\sqrt{2} \\ 1 & -\sqrt{2} & \sqrt{2} & -\sqrt{2} & 1 \end{pmatrix}, \tag{60}$$

$$\widetilde{\rho}_4(\widetilde{T}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{\pi i/4} & 0 \\ 0 & 0 & -1 \end{pmatrix}, \ \widetilde{\rho}_8(\widetilde{T}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\pi i/8} & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -e^{\pi i/8} & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{61}$$

In what follows we assume that both up and down Yukawa matrices for quarks are given by Eq. (58). We also assume that the Higgs VEV directions for the up and down sectors are independent. Otherwise, we cannot derive realistic results. In particular, the quark mixing can be realized by taking different Higgs VEV directions for the up and down sectors.

6.2 *Quark flavors at $\tau = i$*

In this subsection we present numerical studies on the model "4-4-8, (e,e,e), 5H" at $\tau = i$ where the Yukawa matrices are restricted by S-invariance. First, we assume that the vacuum is S-invariant. Then we search the Higgs VEV directions leading to a rank-one quark mass matrix on an S-invariant vacuum. The rank-one matrix is favorable in the limit that we neglect the masses of the first and second generations. However, we need a small deviation from the S-invariant vacuum to realize non-vanishing masses of two light generations.[2] That is, we could realize quark masses and mixing angles at a point close to the S-invariant vacuum. As an illustrative example, we show that the Fritzch–Xing mass matrix can be realized on such a vacuum. We also show some numerical results.

6.2.1 *S-invariance and rank-one directions.* At $\tau = i$, S-transformations for Yukawa couplings in Eq. (60) are diagonalized into

$$O_4^T \widetilde{\rho}_4(\widetilde{S}) O_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad O_8^T \widetilde{\rho}_8(\widetilde{S}) O_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix}, \tag{62}$$

where O_4 and O_8 are orthogonal matrices to diagonalize $\widetilde{\rho}_4$ and $\widetilde{\rho}_8$. These diagonalizations are consistent with the transformation in Eq. (38). Note that there are degrees of freedom on the choice of S-transformation eigenbasis because of its degeneracy. Without loss of generality, it is possible to choose the S-transformation eigenbasis such that the Yukawa matrices

$$\hat{Y}^{ijk} = \left[O_4^T\right]^{ii'} \left[O_4^T\right]^{jj'} \left[O_8^T\right]^{kk'} Y^{i'j'k'} \tag{63}$$

are expressed as

[2]On rank-one directions, we can also realize small but non-zero up (down) and charm (strange) quark masses by slightly shifting the value of the modulus τ from fixed points instead of the shifting of the directions of Higgs VEVs.

$$\hat{Y}^{ij0} = \begin{pmatrix} 1.00 & -0.0839 & 0 \\ -0.0839 & 0.00704 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{Y}^{ij1} = \begin{pmatrix} -0.0572 & -0.248 & 0 \\ -0.248 & -0.943 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{Y}^{ij2} = \begin{pmatrix} 0.0683 & -0.301 & 0 \\ -0.301 & 0.281 & 0 \\ 0 & 0 & 0.844 \end{pmatrix}, \quad \hat{Y}^{ij3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -0.636 \\ 0 & -0.636 & 0 \end{pmatrix},$$

$$\hat{Y}^{ij4} = \begin{pmatrix} 0 & 0 & 0.602 \\ 0 & 0 & -0.158 \\ 0.602 & -0.158 & 0 \end{pmatrix}. \tag{64}$$

As shown in Table 8, this model has the Higgs VEV directions leading to a rank-one mass matrix in both S-invariant and not S-eigenstate directions. In our numerical studies, we assume an almost S-invariant vacuum. We calculate the absolute values of the Cabibbo–Kobayashi–Maskawa (CKM) matrix elements as well as the mass ratios of the quarks near the S-invariant Higgs VEV direction which lead to a rank-one mass matrix. On the S-transformation eigenbasis in Eq. (64), we can find that one such S-invariant Higgs VEV direction is given by

$$\langle \hat{H}^k \rangle \equiv \left[O_8^T\right]^{kk'} \langle H^{k'} \rangle = (1, 0, 0, 0, 0). \tag{65}$$

6.2.2 *Illustrative example: Fritzch–Xing mass matrix.* In the model "4-4-8, (e,e,e), 5H," the mass matrix is symmetric. Here, we assume a mass matrix such as

$$M_u = \begin{pmatrix} A & B & 0 \\ B & D & C \\ 0 & C & 0 \end{pmatrix}, \qquad M_d = \begin{pmatrix} A' & B' & 0 \\ B' & D' & C' \\ 0 & C' & 0 \end{pmatrix}, \tag{66}$$

where A–D and A'–D' are real values. Such mass matrices can be realized by the appropriate linear combination of the Yukawa matrices in Eq. (64). Note that we have used the flavor basis such that the (1,1) entry is the largest. For convenience, we redefine the mass matrix for the up sector, M_u, as

$$M_u \to M_u^{(h)} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} M_u \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & C & 0 \\ C & D & B \\ 0 & B & A \end{pmatrix}. \tag{67}$$

In the same way, we can obtain

$$M_d^{(h)} = \begin{pmatrix} 0 & C' & 0 \\ C' & D' & B' \\ 0 & B' & A' \end{pmatrix} \tag{68}$$

for the down sector. These redefined mass matrices are the so-called Fritzch–Xing mass matrices.[3]

Now we can realize quark masses and mixing angles based on the Fritzch–Xing mass matrix. To realize the Fritzch–Xing mass matrix, we first parametrize the Higgs VEV direction by polar coordinates (θ, ϕ) as

$$\left\langle \hat{H}^k_{u,d} \right\rangle = v_{u,d}(\cos\theta_{u,d}, \sin\theta_{u,d}\cos\phi_{u,d}, 0, \sin\theta_{u,d}\sin\phi_{u,d}, 0). \tag{69}$$

[3]The Fritzch–Xing mass matrix can be obtained by another type of string compactification [83–85].

Table 11. The mass ratios of the quarks and the absolute values of the CKM matrix elements at $\tau = i$ under the Higgs vacuum in Eq. (71). Comparison values of mass ratios are shown in Ref. [86]. Those of the CKM matrix elements are shown in Ref. [87].

	Obtained values	Comparison values
$(m_u, m_c, m_t)/m_t$	$(2.16 \times 10^{-6}, 8.13 \times 10^{-3}, 1)$	$(5.58 \times 10^{-6}, 2.69 \times 10^{-3}, 1)$
$(m_d, m_s, m_b)/m_b$	$(2.02 \times 10^{-3}, 4.10 \times 10^{-2}, 1)$	$(6.86 \times 10^{-4}, 1.37 \times 10^{-2}, 1)$
$\lvert V_{\rm CKM}\rvert \equiv \left\lvert (U_L^u)^\dagger U_L^d \right\rvert$	$\begin{pmatrix} 0.973 & 0.233 & 0.000\,550 \\ 0.233 & 0.973 & 0.008\,48 \\ 0.002\,51 & 0.00812 & 1.00 \end{pmatrix}$	$\begin{pmatrix} 0.974 & 0.227 & 0.003\,61 \\ 0.226 & 0.973 & 0.0405 \\ 0.008\,54 & 0.0398 & 0.999 \end{pmatrix}$

Note that we take the third and fifth VEVs into zero to construct the Fritzch–Xing mass matrix. Then, the quark mass matrices take the forms in Eq. (66).

Next, to realize the quark flavors at $\tau = i$, we choose the following parameters:

$$\begin{cases} (\theta_u, \phi_u) = (0.008\,38, -0.0251), \\ (\theta_d, \phi_d) = (-0.0427, 0.346). \end{cases} \tag{70}$$

The Higgs VEV direction is given by

$$\begin{cases} \langle \hat{H}_u^k \rangle = v_u(1.00, 0.008\,38, 0, -0.000\,211, 0), \\ \langle \hat{H}_d^k \rangle = v_d(0.999, -0.0402, 0, -0.0145, 0), \end{cases} \tag{71}$$

which are the directions very close to the rank-one in Eq. (65). Then, the mass matrices for up and down quarks are given by

$$M_u^{ij} = \hat{Y}^{ijk} \langle \hat{H}_u^k \rangle = \begin{pmatrix} 1.00 & -8.60 \times 10^{-2} & 0 \\ -8.60 \times 10^{-2} & -8.53 \times 10^{-4} & 1.34 \times 10^{-4} \\ 0 & 1.34 \times 10^{-4} & 0 \end{pmatrix}, \tag{72}$$

$$M_d^{ij} = \hat{Y}^{ijk} \langle \hat{H}_d^k \rangle = \begin{pmatrix} 1.00 & -7.39 \times 10^{-2} & 0 \\ -7.39 \times 10^{-2} & 4.49 \times 10^{-2} & 9.20 \times 10^{-3} \\ 0 & 9.20 \times 10^{-3} & 0 \end{pmatrix}. \tag{73}$$

We can obtain the mass ratios of the quarks and the absolute values of the CKM matrix elements shown in Table 11.

6.3 *Quark flavors at* $\tau = \omega$

In this subsection we show another numerical example on the model "4-4-8, (e,e,e), 5H" at $\tau = \omega$, where the Yukawa matrices are restricted by ST-invariance. First, we assume that the vacuum is ST-invariant. Then, we search the Higgs VEV directions leading to a rank-one quark mass matrix on an ST-tranformation invariant vacuum. The rank-one matrix is favorable in the limit that we neglect the masses of the first and second generations. However, as in the studies at $\tau = i$, we need a small deviation from the ST-invariant vacuum to realize non-vanishing masses of two light generations. That is, we could realize quark masses and mixing angles at a point close to the ST-invariant vacuum. As an illustrating example, we show that the Fritzch mass matrix can be realized on such a vacuum. We also show some numerical results.

6.3.1 *ST-invariance and rank-one directions.* At $\tau = \omega$, ST-transformations for Yukawa couplings which are given by a product of Eqs. (60) and (61) are diagonalized into

$$U_4^\dagger \widetilde{\rho}_4(\widetilde{ST}) U_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}, \qquad U_8^\dagger \widetilde{\rho}_8(\widetilde{ST}) U_8 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \omega^2 & 0 & 0 \\ 0 & 0 & 0 & \omega^2 & 0 \\ 0 & 0 & 0 & 0 & \omega \end{pmatrix}, \tag{74}$$

where U_4 and U_8 are unitary matrices to diagonalize $\widetilde{\rho}_4$ and $\widetilde{\rho}_8$. These diagonalizations are consistent with the transformation in Eq. (45). Note that there are degrees of freedom on the choice of ST-transformation eigenbasis because of its degeneracy. Without loss of generality, it is possible to choose an ST-transformation eigenbasis such that the Yukawa matrices

$$\hat{Y}^{ijk} = \left[U_4^\dagger\right]^{ii'} \left[U_4^\dagger\right]^{jj'} \left[U_8^T\right]^{kk'} Y^{i'j'k'} \tag{75}$$

are expressed as

$$\hat{Y}^{ij0} = \begin{pmatrix} 0.9535 + 0.043\,57i & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{Y}^{ij1} = \begin{pmatrix} 0.2852 - 0.1027i & 0 & 0 \\ 0 & 0 & 0.8093 - 0.000\,5968i \\ 0 & 0.8093 - 0.000\,5968i & 0 \end{pmatrix},$$

$$\hat{Y}^{ij2} = \begin{pmatrix} 0 & -0.6454 - 0.064\,36i & 0 \\ -0.6454 - 0.064\,36i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{Y}^{ij3} = \begin{pmatrix} 0 & 0.1615 + 0.1576i & 0 \\ 0.1615 + 0.1576i & 0 & 0 \\ 0 & 0 & -0.6802 - 0.5248i \end{pmatrix},$$

$$\hat{Y}^{ij4} = \begin{pmatrix} 0 & 0 & 0.4039 + 0.080\,34i \\ 0 & 0.1607 - 0.8077i & 0 \\ 0.4039 + 0.080\,34i & 0 & 0 \end{pmatrix}. \tag{76}$$

As shown in Table 9, this model has the Higgs VEV directions leading to a rank-one mass matrix in both ST-invariant and ω^2-eigenstate directions. In our numerical studies, we assume an almost ST-invariant vacuum. We calculate the absolute values of the CKM matrix elements as well as the mass ratios of the quarks close to the ST-invariant Higgs VEV direction which lead to a rank-one mass matrix. On the ST-transformation eigenbasis in Eq. (76), we can find that one such ST-invariant Higgs VEV is given by

$$\langle \hat{H}^k \rangle \equiv \left[U_8^\dagger\right]^{kk'} \langle H^{k'} \rangle = (1, 0, 0, 0, 0). \tag{77}$$

6.3.2 *Illustrative example: Fritzch mass matrix.* Here, we assume a mass matrix such as

$$M_u = \begin{pmatrix} A & B & 0 \\ B & 0 & C \\ 0 & C & 0 \end{pmatrix}, \qquad M_d = \begin{pmatrix} A' & B' & 0 \\ B' & 0 & C' \\ 0 & C' & 0 \end{pmatrix}, \tag{78}$$

where A–C and $A^{'}$–$C^{'}$ are complex values. Such mass matrices can be realized by the appropriate linear combination of the Yukawa matrices in Eq. (76). Note again that we have used the flavor basis such that the (1,1) entry is the largest. For convenience, we redefine the mass matrix for the up sector, M_u, as

$$M_u \to M_u^{(h)} \equiv \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} M_u \begin{pmatrix} e^{ix} & 0 & 0 \\ 0 & e^{iy} & 0 \\ 0 & 0 & e^{iz} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & Ce^{iy} & 0 \\ Ce^{iz} & 0 & Be^{ix} \\ 0 & Be^{iy} & Ae^{ix} \end{pmatrix}, \tag{79}$$

where x, y, and z are fixed by

$$x = -\mathrm{Arg}(A), \qquad y = \mathrm{Arg}(A) - 2\mathrm{Arg}(B), \qquad z = -\mathrm{Arg}(A) + 2\mathrm{Arg}(B) - 2\mathrm{Arg}(C). \tag{80}$$

Then, the redefined mass matrix is given by

$$M_u^{(h)} = \begin{pmatrix} 0 & Ce^{i\mathrm{Arg}(A) - 2i\mathrm{Arg}(B)} & 0 \\ (Ce^{i\mathrm{Arg}(A) - 2i\mathrm{Arg}(B)})^* & 0 & Be^{-i\mathrm{Arg(A)}} \\ 0 & (Be^{-i\mathrm{Arg(A)}})^* & |A| \end{pmatrix}, \tag{81}$$

and this is a Hermitian matrix. In the same way, we can obtain the Hermitian mass matrix for the down sector:

$$M_d^{(h)} = \begin{pmatrix} 0 & C'e^{i\mathrm{Arg}(A') - 2i\mathrm{Arg}(B')} & 0 \\ (C'e^{i\mathrm{Arg}(A') - 2i\mathrm{Arg}(B')})^* & 0 & B'e^{-i\mathrm{Arg(A')}} \\ 0 & (B'e^{-i\mathrm{Arg(A')}})^* & |A'| \end{pmatrix}. \tag{82}$$

These redefined mass matrices are the so-called Fritzch mass matrices.

Here we realize quark masses and mixing angles based on the Fritzch mass matrix. To obtain the Fritzch mass matrices, we first parametrize the Higgs VEV direction by polar coordinates (θ, ϕ) as

$$\langle \hat{H}^k_{u,d} \rangle = v_{u,d}(\cos\theta_{u,d}, \sin\theta_{u,d}\cos\phi_{u,d}, \sin\theta_{u,d}\sin\phi_{u,d}, 0, 0). \tag{83}$$

Note that we take the fourth and fifth VEVs into zero to construct the Fritzch mass matrix. Then, the quark mass matrices take the forms in Eq. (78) and they can always be rewritten as Fritzch mass matrices by the appropriate transformations.

Next, to realize the quark masses and mixing angles at $\tau = \omega$, we choose the following parameters:

$$\begin{cases} (\theta_u, \phi_u) = (0.078\,54, 1.574), \\ (\theta_d, \phi_d) = (0.1414, 1.558). \end{cases} \tag{84}$$

The Higgs VEV direction is given by

$$\begin{cases} \langle \hat{H}^k_u \rangle = v_u(0.9969, -0.000\,2465, 0.078\,46, 0, 0), \\ \langle \hat{H}^k_d \rangle = v_d(0.9900, 0.001\,771, 0.1409, 0, 0), \end{cases} \tag{85}$$

which are the directions close to the rank-one in Eq. (77). Then, the mass matrices for up and down quarks are given by

$$\begin{aligned} M_u^{ij} &= \hat{Y}^{ijk}\langle \hat{H}^k_u \rangle \\ &= \begin{pmatrix} 0.9505 + 0.043\,46i & -0.050\,64 - 0.005\,050i & 0 \\ -0.050\,64 - 0.005\,050i & 0 & -(1.995 - 0.001\,471i) \times 10^{-4} \\ 0 & -(1.995 - 0.001\,471i) \times 10^{-4} & 0 \end{pmatrix}, \end{aligned} \tag{86}$$

Table 12. The mass ratios of the quarks and the absolute values of the CKM matrix elements at $\tau = \omega$ under the vacuum alignments of Higgs fields in Eq. (85). Comparison values of mass ratios are shown in Ref. [86]. Those of the CKM matrix elements are shown in Ref. [87].

	Obtained values	Comparison values
$(m_u, m_c, m_t)/m_t$	$(1.52 \times 10^{-5}, 2.86 \times 10^{-3}, 1)$	$(5.58 \times 10^{-6}, 2.69 \times 10^{-3}, 1)$
$(m_d, m_s, m_b)/m_b$	$(2.37 \times 10^{-4}, 9.41 \times 10^{-3}, 1)$	$(6.86 \times 10^{-4}, 1.37 \times 10^{-2}, 1)$
$\lvert V_{\text{CKM}}\rvert \equiv \lvert(U_L^u)^\dagger U_L^d\rvert$	$\begin{pmatrix} 0.974 & 0.228 & 0.00292 \\ 0.228 & 0.973 & 0.0421 \\ 0.00677 & 0.0416 & 0.999 \end{pmatrix}$	$\begin{pmatrix} 0.974 & 0.227 & 0.00361 \\ 0.226 & 0.973 & 0.0405 \\ 0.00854 & 0.0398 & 0.999 \end{pmatrix}$

$$M_d^{ij} = \hat{Y}^{ijk}\langle \hat{H}_d^k\rangle$$
$$= \begin{pmatrix} 0.9445 + 0.04296i & -0.09093 - 0.009068i & 0 \\ -0.09093 - 0.009068i & 0 & (1.433 - 0.001057) \times 10^{-3} \\ 0 & (1.433 - 0.001057) \times 10^{-3} & 0 \end{pmatrix}. \tag{87}$$

We can obtain the mass ratios of the quarks and the absolute values of the CKM matrix elements as shown in Table 12.

As a result, we can obtain realistic quark mass ratios and mixing on the model “4-4-8, (e,e,e), 5H” at both $\tau = i$ and $\tau = \omega$ by choosing appropriate Higgs VEV directions. As illustrative examples we have used the Fritzch and Fritzch–Xing mass matrices, but we can obtain realistic values of quark masses and mixing angles with other mass matrix forms around S-invariant and ST-invariant vacua. It is also possible to study other three-generation magnetized orbifold models.

7. Conclusion

In this paper we have studied the forms of Yukawa matrices in magnetized orbifold models. In particular, we focused on the forms at three modular fixed points, $\tau = i$, ω, and $i\infty$. Consequently, we found that Yukawa matrices have a kind of texture structure, although those at $\tau = i\infty$ are not realistic. We have therefore classified the Yukawa textures at $\tau = i$ and ω.

By choosing appropriate Higgs VEV directions, the Yukawa textures classified in this paper can lead to a mass matrix whose rank is one. A rank-one mass matrix is favorable in the limit that we neglect the masses of the first and second generations. We have also investigated the conditions such that the quark mass matrix constructed by Yukawa textures becomes a rank-one matrix. Then we found that rank-one directions exist on S-invariant and ST-invariant vacua in several three-generation models. Thus it is possible to realize the large hierarchy of quark masses if we assume that the vacuum has approximate S-invariance or ST-invariance. These invariances need to break slightly to shift the Higgs VEV directions from rank-one directions, since the first- and second-generation quarks have small but non-zero masses.

We have presented numerical studies on the model “4-4-8, (e,e,e), 5H” at both $\tau = i$ and ω, and assumed almost S-invariant and ST-invariant vacua to reproduce the quark masses and mixing angles. As illustrative examples we have shown that Fritzch–Xing and Fritzch mass matrices can be realized from Yukawa textures at $\tau = i$ and ω, respectively. Also, other forms of quark mass matrices can lead to realistic mass ratios of quarks and values of the CKM matrix elements around the S- and ST-invariant vacua. Other three-generation magnetized orbifold models are similarly interesting.

We can extend our studies to the realization of lepton flavors. The charged lepton masses are given by the Dirac mass matrix as quarks, but we need to study Majorana masses for the neutrino sector. For example, in Ref. [88] Majorana masses for right-handed neutrinos induced by non-perturbative effects of D-brane instanton effects were studied systematically in magnetized orbifold models. We will also study it and examine the realization of both quark and lepton flavors elsewhere.

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Appendix A. Proof: Rank-one conditions at $\tau = i$

Here we prove the conditions such that the mass matrix becomes rank one at $\tau = i$. As shown in Sect. 5.1, there are five conditions, denoted I, II, III, IV, and V, to realize a rank-one mass matrix. Under each condition, we show the existence of Higgs VEVs $\langle H^k \rangle = v^k$ such that the mass matrix $M^{ij} = Y^{ijk} v^k$ becomes rank one. Here and hereafter, we use c_k, $k \in \mathbb{Z}$, as any constant value.

Table A1 shows the forms of the rank-one mass matrices realized on each condition. It shows there are two (I), three (II (symmetric), III), four (II (non-symmetric)), and two (V) equations in

Table A1. Rank-one mass matrices realized on each condition. The second column shows one of the realized rank-one matrices whose elements satisfy Eqs. (A1)–(A4) to realize rank one; of course, other rank-one matrices can be constructed. The third column shows the textures included in each condition.

	Rank-one matrix	Textures included
I	$\begin{pmatrix} M^{00} & M^{01} & 0 \\ M^{10} & M^{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \times 3$
II	$\begin{pmatrix} M^{00} & M^{01} & M^{02} \\ M^{10} & M^{11} & M^{12} \\ M^{20} & M^{21} & M^{22} \end{pmatrix}$	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \times 3, \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \times \begin{cases} 1 \text{ (symmetric)} \\ 2 \text{ (non-symmetric)} \end{cases}$
III	$\begin{pmatrix} M^{00} & M^{01} & M^{02} \\ M^{10} & M^{11} & M^{12} \\ M^{20} & M^{21} & M^{22} \end{pmatrix}$	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \times 2, \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \times 2$
IV	$\begin{pmatrix} M^{00} & M^{01} & M^{02} \\ M^{10} & M^{11} & M^{12} \\ M^{20} & M^{21} & M^{22} \end{pmatrix}$	$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \times 2, \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \times 3$
V	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ M^{20} & M^{21} & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} \times 3$

each condition as follows:

$$\text{I}: \quad \frac{M^{00}}{M^{10}} = \frac{M^{01}}{M^{11}}, M^{22} = 0, \tag{A1}$$

$$\text{II (symmetric), III}: \quad \frac{M^{00}}{M^{10}} = \frac{M^{01}}{M^{11}} = \frac{M^{02}}{M^{12}}, \quad \frac{M^{00}}{M^{20}} = \frac{M^{02}}{M^{22}}, \tag{A2}$$

$$\text{II (non-symmetric), IV}: \quad \frac{M^{00}}{M^{10}} = \frac{M^{01}}{M^{11}} = \frac{M^{02}}{M^{12}}, \frac{M^{00}}{M^{20}} = \frac{M^{01}}{M^{21}} = \frac{M^{02}}{M^{22}}, \tag{A3}$$

$$\text{V}: \quad M^{02} = M^{12} = 0. \tag{A4}$$

In what follows we check that the above equations are satisfied by the textures on each condition shown in Table A1. Note that then the normalization condition of Higgs VEVs, $\sum_k |v^k|^2 = \langle H \rangle^2$, is also satisfied.

A1. Condition I

In this condition, the mass matrix can be expanded as

$$M^{ij} = Y^{ijk} v^k = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^0 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^1 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^2, \tag{A5}$$

where the Yukawa matrices Y^{ijk} correspond to S-even textures. The rank-one equations in Eq. (A1) require the following conditions:

$$M^{22} = Y^{22k} v^k = 0, \tag{A6}$$

$$M^{00}M^{11} - M^{01}M^{10} = (Y^{00k}v^k)(Y^{11k}v^k) - (Y^{01k}v^k)(Y^{10k}v^k) = 0. \tag{A7}$$

Equation (A6) means that v^2 is given by a linear combination of v^0 and v^1. Then, Eq. (A7) becomes a quadratic equation for $v^1/v^0 \in \mathbb{C}$, and we can always find a solution to this equation. Thus, we can obtain (v^0, v^1, v^2) satisfying the normalization condition and rank-one condition.

A2. Condition II (symmetric), III

First we consider the condition II (symmetric). In this condition, the mass matrix can be expanded as

$$M^{ij} = Y^{ijk} v^k = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^0 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^1 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} v^3, \tag{A8}$$

where the Yukawa matrices Y^{ij0}, Y^{ij1}, and Y^{ij2} correspond to S-even textures and Y^{ij3} corresponds to S-odd. The rank-one equations in Eq. (A2) require the following conditions:

$$Y^{123}(Y^{000} + Y^{001}(v^1/v^0) + Y^{002}(v^2/v^0)) = Y^{023}(Y^{100} + Y^{101}(v^1/v^0) + Y^{102}(v^2/v^0)), \tag{A9}$$

$$Y^{123}(Y^{010} + Y^{011}(v^1/v^0) + Y^{012}(v^2/v^0)) = Y^{023}(Y^{110} + Y^{111}(v^1/v^0) + Y^{112}(v^2/v^0)), \tag{A10}$$

$$(v^0)^2(Y^{220} + Y^{221}(v^1/v^0) + Y^{222}(v^2/v^0))(Y^{000} + Y^{001}(v^1/v^0) + Y^{002}(v^2/v^0)) = Y^{023}Y^{203}(v^3)^2. \tag{A11}$$

Equations (A9) and (A10) are linear equations for (v^1/v^0) and (v^2/v^0), and we can always find the solutions. Equation (A11) leads to $v^0 = c_1 v^3$, and v^3 is determined by the normalization condition. Thus, we can obtain (v^0, v^1, v^2, v^3) satisfying the normalization condition and rank-one condition.

Next, we consider condition III. In this condition, the mass matrix can be expanded as

$$M^{ij} = Y^{ijk}v^k = \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^0 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^1 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} v^3, \quad \text{(A12)}$$

where the Yukawa matrices Y^{ij0} and Y^{ij1} correspond to S-even textures, and Y^{ij2} and Y^{ij3} correspond to S-odd textures. The rank-one equations in Eq. (A2) require the following conditions:

$$(Y^{000} + Y^{001}(v^1/v^0))(Y^{110} + Y^{111}(v^1/v^0)) = (Y^{010} + Y^{011}(v^1/v^0))(Y^{100} + Y^{101}(v^1/v^0)), \quad \text{(A13)}$$

$$(Y^{000} + Y^{001}(v^1/v^0))(Y^{122} + Y^{123}(v^3/v^2)) = (Y^{022} + Y^{023}(v^3/v^2))(Y^{100} + Y^{101}(v^1/v^0)), \quad \text{(A14)}$$

$$(Y^{000}v^0 + Y^{001}v^1)(Y^{222} + Y^{223}(v^3/v^2)) = v^2(Y^{202} + Y^{203}(v^3/v^2))(Y^{022} + Y^{023}(v^3/v^2)). \quad \text{(A15)}$$

Equation (A13) is a quadratic equation for $v^1/v^0 \in \mathbb{C}$, and it is possible to find the solution $v^1 = c_1 v^0$. Equation (A14) is a linear equation for $v^3/v^2 \in \mathbb{C}$, and the solution $v^3 = c_2 v^2$ exists. Equation (A15) leads to the solution $v^0 = c_3 v^2$, and v^2 is determined by the normalization condition. Thus, we can obtain (v^0, v^1, v^2, v^3) satisfying the normalization condition and rank-one condition.

A3. Condition II (non-symmetric), IV

First, we consider condition II (non-symmetric). In this condition, the mass matrix can be expanded as

$$M^{ij} = Y^{ijk}v^k$$
$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & 0 \\ 0 & 0 & * \end{pmatrix} v^0 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^1 + \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} v^3 + \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & 0 \end{pmatrix} v^4, \quad \text{(A16)}$$

where the Yukawa matrices Y^{ij0}, Y^{ij1}, and Y^{ij2} correspond to S-even textures, and Y^{ij3} and Y^{ij4} correspond to S-odd textures. Note that we have chosen two of the three Higgs bases corresponding to S-invariant textures, and two fermion bases corresponding to S-invariant states to make the (1,1), (1,2), and (2,1) elements of the first Yukawa matrix zero. The rank-one equations in Eq. (A3) require the following conditions:

$$\frac{Y^{001} + Y^{002}(v^2/v^1)}{Y^{101} + Y^{102}(v^2/v^1)} = \frac{Y^{011} + Y^{012}(v^2/v^1)}{Y^{110}(v^0/v^1) + Y^{111} + Y^{112}(v^2/v^1)}, \quad \text{(A17)}$$

$$\frac{Y^{001} + Y^{002}(v^2/v^1)}{Y^{101} + Y^{102}(v^2/v^1)} = \frac{Y^{023} + Y^{024}(v^4/v^3)}{Y^{123} + Y^{124}(v^4/v^3)}, \quad \text{(A18)}$$

$$\frac{Y^{001} + Y^{002}(v^2/v^1)}{Y^{203} + Y^{204}(v^4/v^3)} = \frac{Y^{011} + Y^{012}(v^2/v^1)}{Y^{213} + Y^{214}(v^4/v^3)}, \quad \text{(A19)}$$

$$(v^1/v^3)\frac{Y^{001}+Y^{002}(v^2/v^1)}{Y^{203}+Y^{204}(v^4/v^3)}=(v^3/v^1)\frac{Y^{023}+Y^{024}(v^4/v^3)}{Y^{220}(v^0/v^1)+Y^{221}+Y^{222}(v^2/v^1)}. \quad \text{(A20)}$$

Equation (A17) means that (v^0/v^1) is determined by (v^2/v^1). Equations (A18) and (A19) lead to

$$(v^2/v^1)=\frac{c_1+c_2(v^4/v^3)}{c_3+c_4(v^4/v^3)}=\frac{c_5+c_6(v^4/v^3)}{c_7+c_8(v^4/v^3)}. \quad \text{(A21)}$$

This is a quadratic equation for $(v^4/v^3)\in\mathbb{C}$ and it is possible to find the solution. That is, we can obtain (v^4/v^3), (v^2/v^1), and (v^0/v^1). Then, Eq. (A20) leads to $v^3=c_9v^1$, and v^1 is determined by the normalization condition. Thus, we can obtain $(v^0, v^1, v^2, v^3, v^4)$ satisfying the normalization condition and rank-one condition.

Next, we consider condition IV. In this condition, similar to Eq. (A16), the mass matrix can be expanded as

$$\begin{aligned}M^{ij}&=Y^{ijk}v^k\\&=\begin{pmatrix}0&0&0\\0&*&0\\0&0&*\end{pmatrix}v^0+\begin{pmatrix}*&*&0\\*&*&0\\0&0&*\end{pmatrix}v^1+\begin{pmatrix}0&0&*\\0&0&*\\*&*&0\end{pmatrix}v^2+\begin{pmatrix}0&0&*\\0&0&*\\*&*&0\end{pmatrix}v^3+\begin{pmatrix}0&0&*\\0&0&*\\*&*&0\end{pmatrix}v^4,\end{aligned} \quad \text{(A22)}$$

where the Yukawa matrices Y^{ij0} and Y^{ij1} correspond to S-even textures, and Y^{ij2}, Y^{ij3}, and Y^{ij4} correspond to S-odd textures. The rank-one equations in Eq. (A3) require the following conditions:

$$\frac{Y^{001}}{Y^{101}}=\frac{Y^{011}}{Y^{110}(v^0/v^1)+Y^{111}}, \quad \text{(A23)}$$

$$\frac{Y^{001}}{Y^{101}}=\frac{Y^{022}+Y^{023}(v^3/v^2)+Y^{024}(v^4/v^2)}{Y^{122}+Y^{123}(v^3/v^2)+Y^{124}(v^4/v^2)}, \quad \text{(A24)}$$

$$\frac{Y^{001}}{Y^{202}+Y^{203}(v^3/v^2)+Y^{204}(v^4/v^2)}=\frac{Y^{011}}{Y^{212}+Y^{213}(v^3/v^2)+Y^{214}(v^4/v^2)}, \quad \text{(A25)}$$

$$(v^1/v^2)\frac{Y^{001}}{Y^{202}+Y^{203}(v^3/v^2)+Y^{204}(v^4/v^2)}=(v^2/v^1)\frac{Y^{022}+Y^{023}(v^3/v^2)+Y^{024}(v^4/v^2)}{Y^{220}(v^0/v^1)+Y^{221}}. \quad \text{(A26)}$$

Equation(A23) determines (v^0/v^1). Equations (A24) and (A25) determine (v^3/v^2) and (v^4/v^2). Then, Eq. (A26) leads to $v^1=c_1v^2$, and v^2 is determined by the normalization condition. Thus, we can obtain $(v^0, v^1, v^2, v^3, v^4)$ satisfying the normalization condition and rank-one condition.

A4. Condition V

In this condition, the mass matrix can be expanded as

$$M^{ij}=Y^{ijk}v^k=\begin{pmatrix}0&0&*\\0&0&*\\*&*&0\end{pmatrix}v^0+\begin{pmatrix}0&0&*\\0&0&*\\*&*&0\end{pmatrix}v^1+\begin{pmatrix}0&0&*\\0&0&*\\*&*&0\end{pmatrix}v^2, \quad \text{(A27)}$$

where the Yukawa matrices Y^{ijk} correspond to S-odd textures. The rank-one equations in Eq. (A4) require the following conditions:

$$M^{02}=Y^{020}v^0+Y^{021}v^1+Y^{022}v^2=0, \quad \text{(A28)}$$

Table B1. Rank-one mass matrices realized on each condition. The second column shows one of the realized rank-one matrices whose elements satisfy Eqs. (B1)–(B6) to realize rank one; of course, other rank-one matrices can be constructed. The third column shows the textures included in each condition.

	Rank-one matrix	Textures included
I	$\begin{pmatrix} M^{00} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \times \begin{cases} 2\ \text{(symmetric)} \\ 3\ \text{(non-symmetric)} \end{cases}$
II	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M^{22} \end{pmatrix}$	$\begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \times \begin{cases} 2\ \text{(symmetric)} \\ 3\ \text{(non-symmetric)} \end{cases}$
III	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & M^{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \times \begin{cases} 2\ \text{(symmetric)} \\ 3\ \text{(non-symmetric)} \end{cases}$
IV	$\begin{pmatrix} M^{00} & M^{01} & M^{02} \\ M^{10} & M^{11} & M^{12} \\ M^{20} & M^{21} & M^{22} \end{pmatrix}$	$\begin{cases} \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \times 2, \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \times 1, \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \times \begin{cases} 1\ \text{(symmetric)} \\ 2\ \text{(non-symmetric)} \end{cases} \\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \times 1, \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \times \begin{cases} 1\ \text{(symmetric)} \\ 2\ \text{(non-symmetric)} \end{cases}, \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \times 2 \\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \times \begin{cases} 1\ \text{(symmetric)} \\ 2\ \text{(non-symmetric)} \end{cases}, \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \times 2, \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \times 1 \end{cases}$
V	$\begin{cases} \begin{pmatrix} M^{00} & 0 & 0 \\ M^{10} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} M^{00} & 0 & 0 \\ 0 & 0 & 0 \\ M^{20} & 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 & 0 \\ M^{10} & 0 & 0 \\ M^{20} & 0 & 0 \end{pmatrix} \end{cases}$	$\begin{matrix} \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \times 3, \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \times 3 \\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} \times 3, \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \times 3 \\ \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} \times 3, \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} \times 3 \end{matrix}$

$$M^{12} = Y^{120}v^0 + Y^{121}v^1 + Y^{122}v^2 = 0. \tag{A29}$$

Equation (A28) means that v^2 is given by a linear combination of v^0 and v^1. Then, Eq. (A29) leads to $v^1 = c_1 v^0$, and v^0 is determined by the normalization condition. Thus, we can obtain (v^0, v^1, v^2) satisfying the normalization condition and rank-one condition.

Appendix B. Proof: Rank one conditions at $\tau = \omega$

As shown in Sect. 5.2, there are five conditions, denoted I, II, III, IV, and V, to realize a rank-one mass matrix at $\tau = \omega$. We prove these rank-one conditions in a similar way to Appendix A.

Table B1 shows the form of the rank-one mass matrices realized for each condition. The table shows that there are one (I, II, III (symmetric)), two (I, II, III (non-symmetric)), and four (IV, V) equations in each condition:

$$\mathrm{I}: \quad M^{12} = M^{21} = 0, \tag{B1}$$

$$\mathrm{II}: \quad M^{01} = M^{10} = 0, \tag{B2}$$

$$\text{III}: \quad M^{02} = M^{20} = 0, \tag{B3}$$

$$\text{IV (symmetric)}: \quad \frac{M^{00}}{M^{10}} = \frac{M^{01}}{M^{11}} = \frac{M^{02}}{M^{12}}, \quad \frac{M^{00}}{M^{20}} = \frac{M^{02}}{M^{22}}, \tag{B4}$$

$$\text{IV (non-symmetric)}: \quad \frac{M^{00}}{M^{10}} = \frac{M^{01}}{M^{11}} = \frac{M^{02}}{M^{12}}, \quad \frac{M^{00}}{M^{20}} = \frac{M^{01}}{M^{21}} = \frac{M^{02}}{M^{22}}, \tag{B5}$$

$$\text{V}: \quad \begin{cases} M^{12} = M^{21} = M^{01} = M^{22} = 0, \\ M^{12} = M^{21} = M^{02} = M^{11} = 0, \\ M^{01} = M^{22} = M^{02} = M^{11} = 0. \end{cases} \tag{B6}$$

B1. Conditions I, II, III

Here we prove only condition I, because conditions II and III can be proved in a similar way. In condition I, the mass matrix can be expanded as

$$M^{ij} = Y^{ijk} v^k = \begin{cases} \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 & \text{(symmetric)}, \\ \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^2 & \text{(non-symmetric)}, \end{cases} \tag{B7}$$

where the Yukawa matrices Y^{ijk} correspond to ST-invariant textures. The rank-one equations in Eq. (B1) require the following conditions:

$$\begin{cases} M^{12} = M^{21} = Y^{120}v^0 + Y^{121}v^1 = 0 & \text{(symmetric)}, \\ \begin{cases} M^{12} = Y^{120}v^0 + Y^{121}v^1 + Y^{122}v^2 s = 0, \\ M^{21} = Y^{210}v^0 + Y^{211}v^1 + Y^{212}v^2 = 0 \end{cases} & \text{(non} - \text{symmetric)}. \end{cases} \tag{B8}$$

These are linear equations for v^k, and we can find their solutions and the normalization condition. Thus, we can obtain v^k satisfying the normalization condition and rank-one condition.

B2. Condition IV (symmetric)

Here we prove only one of the three condition IV (symmetric) cases in Table B1, because the other two cases can be proved in a similar way. We prove the first case, in which the mass matrix can be expanded as

$$M^{ij} = Y^{ijk} v^k = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} v^2 + \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} v^3, \tag{B9}$$

where the Yukawa matrices Y^{ij0} and Y^{ij1} correspond to ST-invariant textures, Y^{ij2} corresponds to an ω^2-eigenstate texture, and Y^{ij3} corresponds to an ω-eigenstate texture. Note that we have chosen two Higgs bases corresponding to ST-invariant textures to make the (1,1) elements of the first Yukawa matrix zero. The rank-one equations in Eq. (B4) require the following conditions:

$$\frac{Y^{001}v^1}{Y^{102}v^2} = \frac{Y^{012}v^2}{Y^{113}v^3}, \tag{B10}$$

$$\frac{Y^{001}v^1}{Y^{203}v^3} = \frac{Y^{023}v^3}{Y^{222}v^2}, \tag{B11}$$

$$\frac{Y^{001}v^1}{Y^{102}v^2} = \frac{Y^{023}v^3}{Y^{120}v^0 + Y^{121}v^1}. \tag{B12}$$

Equations (B10) and (B11) lead to $v^1 = c_1 v^3$ and $v^2 = c_2 v^3$. Then, Eq. (B12) leads to $v^3 = c_3 v^0$, and v^0 is determined by the normalization condition. Thus, we can obtain (v^0, v^1, v^2, v^3) satisfying the normalization condition and rank-one condition.

B3. Condition IV (non-symmetric)

Here we prove only one of the three condition IV (non-symmetric) cases in Table B1, because the other two cases can be proved in a similar way. We prove the first case, in which, similar to Eq. (B9), the mass matrix can be expanded as

$$\begin{aligned} M^{ij} &= Y^{ijk}v^k \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} v^2 + \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} v^3 + \begin{pmatrix} 0 & 0 & * \\ 0 & * & 0 \\ * & 0 & 0 \end{pmatrix} v^4, \end{aligned} \tag{B13}$$

where the Yukawa matrices Y^{ij0} and Y^{ij1} correspond to ST-invariant textures, Y^{ij2} and Y^{ij3} correspond to ω^2-eigenstate textures, and Y^{ij4} corresponds to an ω-eigenstate texture. The rank-one equations in Eq. (B5) require the following conditions:

$$\frac{Y^{001}v^1}{Y^{102}v^2 + Y^{103}v^3} = \frac{Y^{012}v^2 + Y^{013}v^3}{Y^{114}v^4}, \tag{B14}$$

$$\frac{Y^{001}v^1}{Y^{102}v^2 + Y^{103}v^3} = \frac{Y^{024}v^4}{Y^{120}v^0 + Y^{121}v^1}, \tag{B15}$$

$$\frac{Y^{001}v^1}{Y^{204}v^4} = \frac{Y^{012}v^2 + Y^{013}v^3}{Y^{210}v^0 + Y^{211}v^1}, \tag{B16}$$

$$\frac{Y^{001}v^1}{Y^{204}v^4} = \frac{Y^{024}v^4}{Y^{223}v^3}. \tag{B17}$$

Equations (B14) and (B15) lead to

$$(v^4)^2/v^1 = v^3(c_1(v^2/v^3) + c_2)(c_3(v^0/v^1) + c_4)), \tag{B18}$$

and Eq. (B17) leads to

$$(v^4)^2/v^1 \propto v^3. \tag{B19}$$

From these two relations, we obtain

$$c_1(v^2/v^3) + c_2 \propto \frac{1}{c_3(v^0/v^1) + c_4}. \tag{B20}$$

On the other hand, Eqs. (B15) and (B16) lead to

$$(Y^{102}(v^2/v^3) + Y^{103})(Y^{210}(v^0/v^1) + Y^{211}) \propto (Y^{012}(v^2/v^3) + Y^{013})(Y^{120}(v^0/v^1) + Y^{121}). \tag{B21}$$

Combining both results, we obtain

$$\left(\frac{c_5}{c_3(v^0/v^1)+c_4}+c_6\right)(Y^{210}(v^0/v^1)+Y^{211}) \propto \left(\frac{c_7}{c_3(v^0/v^1)+c_4}+c_8\right)(Y^{120}(v^0/v^1)+Y^{121}). \tag{B22}$$

This is a quadratic equation for $(v^0/v^1) \in \mathbb{C}$, and it is possible to find the solution $v^0 = c_9 v^1$. Consequently, Eq. (B20) leads to $v^2 = c_{10}v^3$ and Eq. (B14) leads to

$$(v^3)^2/v^4 \propto v^1. \tag{B23}$$

This relation and Eq. (B19) lead to $v^3 = c_{11}v^4$ and $v^1 = c_{12}v^4$, and v^4 is determined by the normalization condition. Thus, we can obtain $(v^0, v^1, v^2, v^3, v^4)$ satisfying the normalization condition and rank-one condition.

B4. Condition V

Here we prove only one of the three condition V cases in Table B1, because the other two cases can be proved in a similar way. We prove the first case, in which we can choose the ST-eigenbasis on wavefunctions such that the mass matrix is expanded as

$$\begin{aligned} M^{ij} = Y^{ijk}v^k = &\begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^0 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^1 + \begin{pmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{pmatrix} v^2 \\ &+ \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} v^3 + \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} v^4 + \begin{pmatrix} 0 & * & 0 \\ * & 0 & 0 \\ 0 & 0 & * \end{pmatrix} v^5, \end{aligned} \tag{B24}$$

where the Yukawa matrices Y^{ij0}, Y^{ij1}, and Y^{ij2} correspond to ST-invariant textures, and Y^{ij3}, Y^{ij3} and Y^{ij5} correspond to ω^2-eigenstate textures. The rank-one equations in Eq. (B6) require the following conditions:

$$Y^{120}v^0 + Y^{121}v^1 + Y^{122}v^2 = 0, \tag{B25}$$

$$Y^{210}v^0 + Y^{211}v^1 + Y^{212}v^2 = 0, \tag{B26}$$

$$Y^{013}v^3 + Y^{014}v^4 + Y^{015}v^5 = 0, \tag{B27}$$

$$Y^{223}v^3 + Y^{224}v^4 + Y^{225}v^5 = 0. \tag{B28}$$

There are four linear equations for the six VEVs $(v^0, v^1, v^2, v^3, v^4, v^5)$. Thus, we can obtain $(v^0, v^1, v^2, v^3, v^4, v^5)$ satisfying the normalization condition and rank-one condition.

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