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Gromov–Witten Axioms for Symplectic Manifolds via Polyfold Theory

by

Wolfgang William Schmaltz

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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University of California, Berkeley

Committee in charge:

Professor Katrin Wehrheim, Chair
Professor Michael Hutchings
Professor Jonathan Shewchuk

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Abstract

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Wolfgang William Schmaltz

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Professor Katrin Wehrheim, Chair

Polyfold theory, as developed by Hofer, Wysocki, and Zehnder, is a relatively new approach to resolving transversality issues that arise in the study of J-holomorphic curves in symplectic geometry. This approach has recently led to a well-defined Gromov–Witten invariant for J-holomorphic curves of arbitrary genus, and for all closed symplectic manifolds. In this thesis we prove the Effective, Grading, Homology, Zero, Symmetry, Fundamental Class, and Divisor Gromov–Witten axioms for J-holomorphic curves of arbitrary genus, and for all closed symplectic manifolds.

To my father, Archie

Contents

List of Figures

Acknowledgments

I would like to thank my advisor, Katrin Wehrheim. This thesis would not be half as interesting or readable without their insights and feedback.

I would also like to thank all of my friends. Without them, life would not be worth living.

Chapter 1

Introduction

In 1985 Gromov published the paper “Pseudo holomorphic curves in symplectic manifolds,” laying the foundations for the modern study of pseudo holomorphic curves (also know as J-holomorphic curves) in symplectic topology [12]. In this paper, Gromov proved a compactness result for the moduli space of J-holomorphic curves in a fixed homology class. This paper contained antecedents to the modern notion of the Gromov–Witten invariants in the proofs of the nonsqueezing theorem and the uniqueness of symplectic structures on $\mathbb{C}P^2$.

Around 1988, inspired by Floer's study of gauge theory on three manifolds, Witten introduced the topological sigma model [8] [46]. The invariants of this model are the ‘k-point correlation functions,’ another precursor to the modern notion of the Gromov–Witten invariants. Witten also observed some of the relationships between these invariants and possible degenerations of Riemann surfaces [47]. Further precursors to the notion of the Gromov–Witten invariants can also be seen in McDuff's classification of symplectic ruled surfaces [31].

In 1993 Ruan gave a modern definition of the genus zero Gromov–Witten invariants for semipositive symplectic manifolds [41] [40]. At the end of 1993, Ruan and Tian established the associativity of the quantum product for semipositive symplectic manifolds, giving a mathematical basis to the composition law of Witten's topological sigma model [39].

In 1994 Kontsevich and Manin stated the Gromov–Witten axioms, given as a list of formal relations between the Gromov–Witten invariants [27]. At the time it was not possible for Kontsevich and Manin to give a proof of the relations they listed, due to the absence of a well-defined Gromov–Witten invariant for all cases. Hence they used to term ‘axiom’ with the presumed meaning ‘to take for assumption without proof’ / ‘to use as a premise for further reasoning’. And indeed, from these starting assumptions they were able to establish foundational results in enumerative geometry, answers to esoteric questions such as:

> *(Kontsevich's Recursion Formula).* Let $d \geq 1$. How many degree d rational curves in $\mathbb{C}P^2$ pass through $3d-1$ points in general position?

In this paper they moreover outlined some of the formal consequences of the axioms by demonstrating how to combine the invariants into a Gromov–Witten potential, and interpret the axioms as differential equations which the potential satisfies.

To varying extents, this work has predated the construction of well-defined Gromov–Witten invariant in symplectic geometry for J-holomorphic curves of arbitrary genus, and for all closed symplectic manifolds. Efforts to construct a well-defined Gromov–Witten invariant constitute an ever growing list of publications, including but not limited to the following: [28] [10] [11] [44] [5] [33] [34] [35] [25] [38]. A discussion of some of the difficulties inherent in these approaches can be found in [7]. Similarly, there have been several efforts to prove the Gromov–Witten axioms [10] [32] [4].

Over the past decade, Hofer, Wysocki, and Zehnder have developed a new approach to resolving transversality issues that arise in the study of J-holomorphic curves in symplectic geometry called *polyfold theory* [16] [24] [21] [22] [19] [17] [23] [18]. This approach has been successful in constructing a well-defined Gromov–Witten invariant [22].

Following this work, in this thesis we prove the Gromov–Witten axioms for the polyfold Gromov–Witten invariants.

1.1 The Polyfolds of Gromov–Witten Theory

Gromov–Witten theory is concerned with the study of the solution spaces consisting of J-holomorphic stable curves and using these solution spaces to define invariants for symplectic manifolds.

Given a closed symplectic manifold (Q, ω) with a compatible almost complex structure J and a Riemann surface (Σ, j), the solution set $\mathcal{M}_{g,k}(A; J)$ is defined as the set of smooth maps $u : (\Sigma, j) \to Q$ which satisfy the Cauchy–Riemann equation modulo reparametrization (where g is the genus of the Riemann surface (Σ, j), k is the number of marked points $\{z_1, \ldots, z_k\} \in \Sigma$, and $A \in H_2(Q; \mathbb{Z})$ is a fixed homology class such that $u_*[\Sigma] = A$), i.e.

$$\mathcal{M}_{g,k}(A; J) := \left\{ u : (\Sigma, j) \to Q \;\middle|\; \begin{array}{c} \overline{\partial}_J u := \frac{1}{2}(du + J \circ du \circ j) = 0 \\ \{z_1, \ldots, z_k\} \in \Sigma \\ u_*[\Sigma] = A \in H_2(Q; \mathbb{Z}) \end{array} \right\} \Big/ \begin{array}{l} u \sim u \circ \phi, \\ \phi \in \mathrm{Aut}(\Sigma, j). \end{array}$$

We will refer to an equivalence class of a smooth map solution to the Cauchy–Riemann equation as a *J-holomorphic curve*.

Gromov's compactness theorem states that given a sequence of J-holomorphic curves there exists a subsequence which 'weakly converges' to a 'cusp-curve' [12]. This was later refined in [26] into the 'stable map compactification'. Consequently, the set $\mathcal{M}_{g,k}(A; J)$ can be compactified by adding stable curves yielding a compact topological space $\overline{\mathcal{M}}_{g,k}(A; J)$. We call this space the *(unperturbed) Gromov–Witten solution space of genus g, k marked stable curves which represent the class A.*

In a set of small but often studied cases where the symplectic manifold (Q, ω) is 'semi-positive' it is possible to give this compact topological space the additional structure of a 'pseudocycle', which is suitable for defining an invariant. This is achieved via a perturbation of the almost complex structure J. The space of compatible almost complex structures $\mathcal{J}(Q,\omega)$ is nonempty and contractible, from which it can be shown that the invariant does not depend on the choice of J. However in general symplectic manifolds no $J \in \mathcal{J}(Q,\omega)$ can give sufficient transversality to yield a well-defined invariant. For a textbook treatment of this material, we refer to [32].

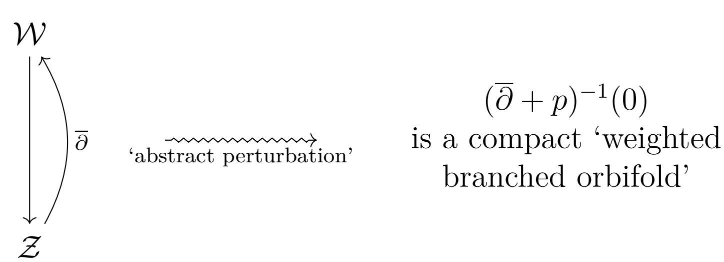
Polyfold theory, developed by Hofer, Wysocki, and Zehnder, is a relatively new approach to resolving transversality issues that arise in attempts to solve moduli space problems in symplectic geometry.

> *"Polyfold theory is what lets symplectic topologists argue as if compactified moduli spaces are zero sets of smooth sections"* — Katrin Wehrheim

Consider the following situation from classical differential topology. Suppose that M is a finite-dimensional manifold, E is a finite rank vector bundle over M, and s is a smooth section of this bundle. If the preimage of the zero section $s^{-1}(0)$ is compact, one can construct sections $p : M \to E$ such that $s+p$ is transverse to the zero section, and such that $(s+p)^{-1}(0)$ is a compact set. It follows that the perturbed zero set $(s+p)^{-1}(0)$ is a compact finite-dimensional smooth manifold.

The polyfold theoretic approach to solving a moduli space problem is to recast the problem as a generalization of the above setup. To do this, it seeks to construct a 'polyfold' $\mathcal{Z}$—a massive, infinite-dimensional ambient space, designed to contain an entire unperturbed compactified moduli space $\overline{\mathcal{M}}$ as a compact subset. We may furthermore construct a 'strong polyfold bundle' $\mathcal{W}$ over $\mathcal{Z}$, and moreover suppose that we have a 'scale smooth Fredholm section' of this bundle, $\overline{\partial} : \mathcal{Z} \to \mathcal{W}$, such that $\overline{\partial}^{-1}(0) = \overline{\mathcal{M}}$.

We can construct an 'abstract perturbation' p of this section such that $\overline{\partial}+p$ is transverse to the zero section and such that $(\overline{\partial}+p)^{-1}(0)$ is a compact set. In this way, we may take a scale smooth Fredholm section and 'regularize' the unperturbed moduli space yielding a perturbed moduli space $(\overline{\partial}+p)^{-1}(0)$ that will have the structure of a compact oriented 'weighted branched orbifold.' For a survey of the core ideas of the polyfold theory, we refer to [7].



Theorem 1.1.1. *[16] [24] [21] [22] Let (Q,ω) be a closed symplectic manifold, let $A \in H_2(Q;\mathbb{Z})$ and let $g,k \geq 0$ be integers. Then the set of genus g, k marked stable curves which*

represent the class A can be given a natural polyfold structure $\mathcal{Z}_{A,g,k}$ called the Gromov–Witten polyfold. The Cauchy–Riemann section $\overline{\partial}_J : \mathcal{Z}_{A,g,k} \to \mathcal{W}$ is an sc-smooth proper Fredholm section of a strong polyfold bundle $\mathcal{W}_{A,g,k}$ of index

$$Ind\,(\overline{\partial}_J) = 2c_1(A) + (dim_{\mathbb{R}} Q - 6)(1-g) + 2k.$$

There exists abstract perturbations p which give the perturbed Gromov–Witten solution spaces $\mathcal{S}_{A,g,k}(p) := (\overline{\partial}_J + p)^{-1}(0)$ the structure of a compact oriented weighted branched orbifold.

This approach has been successful in giving a well-defined Gromov–Witten invariant for curves of arbitrary genus, and for all closed symplectic manifolds.

Definition 1.1.2. [22, Theorem 1.12] Fix a closed symplectic manifold (Q,ω), let $A \in H_2(Q;\mathbb{Z})$, and let $g, k \geq 0$ be integers such that $2g + k \geq 3$. Consider the following diagram of continuous maps between the perturbed Gromov–Witten solution space $\mathcal{S}_{A,g,k}(p)$, the k-fold product manifold Q^k, and the Deligne–Mumford orbifold $\overline{\mathcal{M}}^{\log}_{g,k}$.

$$\begin{array}{ccc} \mathcal{S}_{A,g,k}(p) & \xrightarrow{ev_1 \times \cdots \times ev_k} & Q^k \\ \big\downarrow \pi & & \\ \overline{\mathcal{M}}^{log}_{g,k} & & \end{array}$$

Here ev_i is evaluation at the ith-marked point, and π is the projection map to the Deligne–Mumford space which forgets the stable map solution and stabilizes the resulting nodal Riemann surface by contracting unstable components.

Using this, the *polyfold Gromov–Witten invariant* is defined as the homomorphism

$$\mathrm{GW}^Q_{A,g,k} : H_*(Q;\mathbb{R})^{\otimes k} \otimes H_*(\overline{\mathcal{M}}^{\log}_{g,k};\mathbb{R}) \to \mathbb{R}$$

defined by

$$\mathrm{GW}^Q_{A,g,k}(\alpha_1,\ldots,\alpha_k;\beta) := \int_{\mathcal{S}_{A,g,k}(p)} ev_1^*\mathrm{PD}(\alpha_1) \wedge \cdots \wedge ev_k^*\mathrm{PD}(\alpha_k) \wedge \pi^*\mathrm{PD}(\beta)$$

where the integral is the ‘branched integration’ of [23]. This invariant does not depend on the choice of perturbation.

1.2 Main Results—The Gromov–Witten Axioms

With a general polyfold Gromov–Witten invariant in place, a natural question is: To what extent does this newly defined invariant satisfy traditional results of Gromov–Witten theory for symplectic manifolds? A natural place to begin is with verifying the Gromov–Witten axioms. In this thesis, we prove the following Gromov–Witten axioms for the polyfold

Gromov–Witten invariants, for curves of arbitrary genus, and for all closed symplectic manifolds. For the fundamental class and divisor axioms we will require an additional technical assumption on the homology classes $\beta \in H_*(\overline{\mathcal{M}}^{\log}_{g,k};\mathbb{R})$. This assumption is called the 'suborbifold representability condition' (Definition 7.3.11).

Definition 1.2.1 (Basic Classes)**.** [27, Equation 2.3] We say that (A, g, k) is a **basic class** if it is equal to one of the following: $(A, 0, 3)$, $(A, 1, 1)$, $(A, g \geq 2, 0)$. The point is, for such values of g and k we will have $\overline{\mathcal{M}}_{g,k-1} = \emptyset$ by definition.

Theorem 1.2.2 (Gromov–Witten Axioms)**.** *[27] Let (Q, ω) be a closed symplectic manifold. Let $A \in H_2(Q;\mathbb{Z})$ and let $g \geq 0$ and $k \geq 0$ be integers such that $2g + k \geq 3$.*
Effective Axiom: *If $\omega(A) < 0$ then $GW^Q_{A,g,k} = 0$.*
Grading Axiom: *If $GW^Q_{A,g,k}(\alpha_1, \dots, \alpha_k; \beta) \neq 0$ then*

$$\sum_{i=1}^{k}(2n - \deg(\alpha_i)) + (6g - 6 + 2k - \deg(\beta)) = 2c_1(A) + (2n - 6)(1 - g) + 2k.$$

Homology Axiom: *There exists a homology class*

$$\sigma_{A,g,k} \in H_{2c_1(A)+(2n-6)(1-g)+2k}(Q^k \times \overline{\mathcal{M}}_{g,k};\mathbb{R})$$

such that

$$GW^Q_{A,g,k}(\alpha_1, \dots, \alpha_k; \beta) = \langle p_1^* PD(\alpha_1) \smile \cdots \smile p_k^* PD(\alpha_k) \smile p_0^* PD(\beta), \sigma_{A,g,k}\rangle$$

where $p_i : Q^k \times \overline{\mathcal{M}}_{g,k} \to Q$ denotes the projection onto the ith factor and the map $p_0 : Q^k \times \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k}$ denotes the projection onto the last factor.
Zero Axiom: *If $A = 0, g = 0$ then $GW^Q_{0,0,k}(\alpha_1, \dots, \alpha_k; \beta) = 0$ whenever $\deg(\beta) > 0$, and*

$$GW^Q_{0,0,k}(\alpha_1, \dots, \alpha_k; [pt]) = \int_Q PD(\alpha_1) \wedge \cdots \wedge PD(\alpha_k)$$

Symmetry Axiom: *Fix a permutation $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$. Consider the permutation map $\sigma : \overline{\mathcal{M}}^{log}_{g,k} \to \overline{\mathcal{M}}^{log}_{g,k}$, $[\Sigma, j, M, D] \mapsto [\Sigma, j, M^\sigma, D]$ where $M = \{z_1, \dots, z_k\}$ and where $M^\sigma := \{z'_1, \dots, z'_k\}$, $z'_i := z_{\sigma(i)}$. Then*

$$GW^Q_{A,g,k}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)}; \sigma_*\beta) = (-1)^{N(\sigma;\alpha_i)} GW^Q_{A,g,k}(\alpha_1, \dots, \alpha_k; \beta)$$

where $N(\sigma; \alpha_i) := \sharp\{i < j \mid \sigma(i) > \sigma(j), \deg(\alpha_i)\deg(\alpha_j) \in 2\mathbb{Z} + 1\}$.
Fundamental Class Axiom *Consider the fundamental classes $[Q] \in H_{2n}(Q;\mathbb{Q})$ and $[\overline{\mathcal{M}}^{log}_{g,k}] \in H_{6g-6+2k}(\overline{\mathcal{M}}^{log}_{g,k};\mathbb{Q})$. Suppose $A \neq 0$ and that (A, g, k) is not basic. Then*

$$GW^Q_{A,g,k}(\alpha_1, \dots, \alpha_{k-1}, [Q]; [\overline{\mathcal{M}}^{log}_{g,k}]) = 0.$$

Suppose that $\beta \in H_*(\overline{\mathcal{M}}^{log}_{g,k-1};\mathbb{Q})$ *satisfies the suborbifold representability condition. Consider the* l*th-marked point doubling map* $s_l : \overline{\mathcal{M}}^{log}_{g,k-1} \to \overline{\mathcal{M}}^{log}_{g,k}$ *(see Definition 2.5.9). Then*

$$GW^Q_{A,g,k}(\alpha_1,\ldots,\alpha_{k-1},[Q];s_{l*}\beta) = GW^Q_{A,g,k-1}(\alpha_1,\ldots,\alpha_{k-1};\beta).$$

Divisor Axiom: *Suppose that* (A,g,k) *is not basic, and suppose that* $\beta \in H_*(\overline{\mathcal{M}}^{log}_{g,k-1};\mathbb{Q})$ *satisfies the suborbifold representability condition. Consider the* k*th-marked point forgetting map* $ft_k : \overline{\mathcal{M}}^{log}_{g,k} \to \overline{\mathcal{M}}^{log}_{g,k-1}$ *(see Definition 2.5.1). If* $\deg(\alpha_k) = 2n-2$ *then*

$$GW^Q_{A,g,k}(\alpha_1,\ldots,\alpha_k;PD(ft_k^*PD(\beta))) = (A\cdot\alpha_k)\ GW^Q_{A,g,k-1}(\alpha_1,\ldots,\alpha_{k-1};\beta),$$

where $A\cdot\alpha_k$ *is given by the homological intersection product.*

1.3 Strategy for Proving the Gromov–Witten Axioms

We outline the overarching strategy for proving the Gromov–Witten axioms. The Gromov–Witten axioms give relationships between the Gromov–Witten invariants. These relationships are determined by the geometry of the permutation maps and the marked point forgetting maps between the perturbed Gromov–Witten solution spaces.

However, without work, such maps will not persist after abstract perturbation. We develop a general approach for ensuring the persistence of a well-defined map between perturbed solution spaces in Chapter 3, by pulling back perturbations via a well-defined map on the ambient polyfolds. This approach allows us to define the permutation maps between the perturbed Gromov–Witten solution spaces.

Problems arise when we consider the kth-marked point forgetting map. In that case, there is no hope of defining a kth-marked point forgetting map on the Gromov–Witten polyfolds as constructed in [22]. To deal with this, in Chapter 4 we shall construct a new Gromov–Witten polyfold of *stable curves with constant destabilizing ghost components* which more naturally models the anticipated geometry of the Gromov–Witten solution space.

On this new Gromov–Witten polyfold, it is then possible to consider a well-defined kth-marked point forgetting map, and moreover to pullback perturbations via this map as we explain in Chapter 5.

We must then show that the invariants for this new Gromov–Witten polyfold coincide with those for the original Gromov–Witten polyfold as constructed in [22]. Key to resolving this problem is proving an invariance of domain result for branched orbifolds in Chapter 6.

Finally, in Chapter 7 we prove the Gromov–Witten axioms.

Pulling Back Abstract Perturbations

Without work, after perturbation maps will not persist between the perturbed Gromov–Witten solution spaces. To explain, consider the situation of the permutation map

$$\sigma : \overline{\mathcal{M}}^{\log}_{g,k} \to \overline{\mathcal{M}}^{\log}_{g,k},$$

which lifts to an sc-diffeomorphism of Gromov–Witten polyfolds

$$\sigma : \mathcal{Z}_{A,g,k} \to \mathcal{Z}_{A,g,k}.$$

In the semipositive situation, the set $\mathcal{J}(Q,\omega)$ gives a common space of perturbations for the source and target of the permutation map. We can therefore choose a common regular J and obtain a well-defined permutation map $\sigma : \overline{\mathcal{M}}_{g,k}(A;J) \to \overline{\mathcal{M}}_{g,k}(A;J)$.

In contrast, abstract perturbations are constructed using bump functions and choices of vectors in a strong polyfold bundle, which in general will not exhibit symmetry with regards to the labellings of the marked points. As a result, given a stable curve $x \in \mathcal{Z}_{A,g,k}$ which satisfies a perturbed equation $(\overline{\partial}_J + p)(x) = 0$ we cannot expect that $(\overline{\partial}_J + p)(\sigma(x)) = 0$, as the perturbations are not symmetric with regards to the permutation σ. Therefore, naively there does not exist a map $\sigma : \mathcal{S}_{A,g,k}(p) \to \mathcal{S}_{A,g,k}(p)$.

The natural approach for obtaining a well-defined map between perturbed solution spaces is to pullback an abstract perturbation. In Theorem 3.1.4 we establish a mild criteria under which a regular perturbation will pullback to a regular perturbation. Using this approach, we can easily show there exists a regular perturbation which pulls back to regular perturbation via the permutation map.

Problems Arise—The kth-Marked Point Forgetting Map

The kth-marked point forgetting map is, by far, the most difficult and complicated map to make compatible with perturbation of the Gromov–Witten solution spaces, as we immediately encounter numerous difficulties.

The construction of the smooth structure for the Deligne–Mumford orbifolds as described in [22] and [17] requires a choice: that of a 'gluing profile', which is a smooth diffeomorphism $\varphi : (0,1] \to [0,\infty)$. The logarithmic gluing profile is given by $\varphi_{\log}(r) = -\frac{1}{2}\log(r)$ and produces the classical holomorphic Deligne–Mumford orbifolds $\overline{\mathcal{M}}^{\log}_{g,k}$. There is also an exponential gluing profile, given by $\varphi_{\exp}(r) = e^{1/r} - e$ which produces Deligne–Mumford orbifolds $\overline{\mathcal{M}}^{\exp}_{g,k}$ which are only smooth orbifolds.

This use of nonstandard smooth structure has the following consequence.

> ***In general the map $ft_k : \overline{\mathcal{M}}^{exp}_{g,k} \to \overline{\mathcal{M}}^{exp}_{g,k-1}$ is continuous but not differentiable (Remark 2.5.8).***

Differentiability fails at points which are nodal Riemann surfaces which contain components S^2 which contain precisely 3 special points, two of which are nodal, and one of which is the kth-marked point.

> ***In general there does not exist a natural map ft_k on the Gromov–Witten polyfolds (Remark 2.6.4).***

The reason is that the stability condition imposed on stable curves in the polyfold $\mathcal{Z}_{A,g,k}$ may not hold in $\mathcal{Z}_{A,g,k-1}$ once the kth point is removed. The stability condition which is used to define the underlying sets of the Gromov–Witten polyfolds is defined as follows.

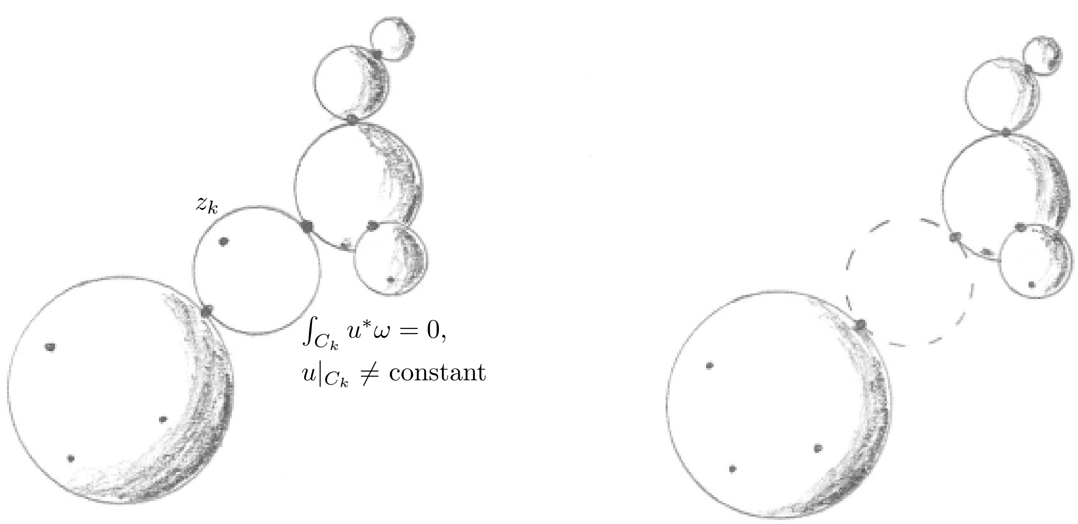
- For each connected component $C \subset \Sigma$ we require at least one of the following:

$$2 \cdot g_C + \sharp(M \cup |D|)_C \geq 3 \quad \text{or} \quad \int_C u^*\omega > 0,$$

  where g_C is the genus of C and $\sharp(M \cup |D|)_C$ is the number of marked and nodal points on the component C.

A stable curve in $\mathcal{Z}_{A,g,k}$ may contain a *destabilizing ghost component*, i.e. a component $C_k \simeq S^2$ with precisely 3 special points, one of which is the kth-marked point, and such that $\int_{C_k} u^*\omega = 0$, $u|_{C_k} =$ constant. In this case, after removal of the kth-marked point, the stability condition is no longer satisfied, and we cannot consider the resulting data as a stable curve in $\mathcal{Z}_{A,g,k-1}$.

Figure 1.1: There Does Not Exist a Natural kth-Marked Point Forgetting Map



We can remove the stable curves in $\mathcal{Z}_{A,g,k}$ for which the stability condition does not hold after forgetting the kth-marked point; thus we can attempt to restrict to a subset $\mathcal{Z}^{\text{const}}_{A,g,k} \subset \mathcal{Z}_{A,g,k}$ where we require a stronger stability condition.

- For each connected component $C \subset \Sigma$ we require at least one of the following:

$$2 \cdot g_C + \sharp(M \cup |D|)_C \geq 3 \quad \text{or} \quad \int_C u^*\omega > 0.$$

Additionally, if the kth-marked point z_k lies on a component C_k with

$$2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} = 3 \quad \text{and} \quad \int_{C_k} u^*\omega = 0$$

then we require that $u|_{C_k}$ is constant, hence necessarily $u|_{C_k} \equiv u(z_k)$.

By the energy identity, any map $u : C_k \to Q$ with $\overline{\partial}_J u = 0$ and $\int_{C_k} u^*\omega = 0$ must be constant. Hence it follows that the above subset is large enough to contain the entire unperturbed Gromov–Witten solution space, i.e. $\overline{\mathcal{M}}_{g,k}(A;J) \subset \mathcal{Z}^{\text{const}}_{A,g,k}$.

With this stability condition the kth-marked point forgetting map is well-defined on $\mathcal{Z}^{\text{const}}_{A,g,k}$. Consider the subspace topology on $\mathcal{Z}^{\text{const}}_{A,g,k} \subset \mathcal{Z}_{A,g,k}$, and the usual polyfold topology on $\mathcal{Z}_{A,g,k-1}$.

> ***In general the well-defined restriction*** $ft_k : \mathcal{Z}^{const}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}$ ***is not continuous (Proposition 2.6.5).***

Our proof of the axioms resolves these issues by constructing a new Gromov–Witten polyfold.

A New Gromov–Witten Polyfold

In Chapter 4 we construct a new Gromov–Witten polyfold $\mathcal{Z}^{ft}_{A,g,k}$ of stable curves with constant destabilizing ghost components C_k. This polyfold is designed to more naturally reflects the anticipated geometries of the Gromov–Witten solution spaces. This polyfold has the same underlying set considered above, i.e. $\mathcal{Z}^{ft}_{A,g,k} = \mathcal{Z}^{\text{const}}_{A,g,k}$, but it carries a *new* polyfold structure, with a *new* sc-smooth structure, and a *new* topology.

At a destabilizing ghost component which contains precisely two nodal points (in addition to the kth-marked point), this new gluing construction takes the two gluing parameters of two nodes, combines them into a single gluing length, and using this single gluing length to directly interpolate maps. At a destabilizing ghost component which contains precisely one nodal point and one other marked point (in addition to the kth-marked point), it forgets the gluing parameter, and relabels the remaining nodal point as a marked point.

The new Gromov–Witten polyfold fits within the following diagram of continuous maps

$$\begin{array}{ccc} \mathcal{Z}_{A,g,k} & & \\ \iota \uparrow & & \\ \mathcal{Z}^{ft}_{A,g,k} & \xrightarrow{ft_k} & \mathcal{Z}_{A,g,k-1} \end{array}$$

where ι is an inclusion map, and ft_k is the well-defined and continuous kth-marked point forgetting map.

We must still be careful, as we cannot pullback a smooth perturbation via ft_k and necessarily expect to get a smooth perturbation. This is because the Gromov–Witten polyfolds use the exponential Deligne–Mumford orbifolds in coordinate representations, and hence the best we can hope for is that the kth-point forgetting map $ft_k : \mathcal{Z}^{ft}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}$ will be continuous. In Chapter 5 we show that it is still possible to pullback a regular perturbation and obtain a regular perturbation via the kth-marked point forgetting map. This is accomplished by careful study of local expressions of the map ft_k and via a hands-on approach to defining a suitable perturbation to be pulled back.

Branched Orbifolds and Invariance of Domain

Having define a new Gromov–Witten polyfold, we must show that the Gromov–Witten invariants for this new Gromov–Witten polyfold coincide with the Gromov–Witten invariants for the original Gromov–Witten polyfold as constructed in [22]. This is an example of a more general problem—small modifications to the construction of a polyfold yield invariants which, a priori, we can not assume are equivalent.

In this situation, the approach of pulling back perturbations will not work; the strong polyfold bundles are not related via pullbacks. Moreover, the inclusion map we are considering will not be proper with respect to the polyfold topologies. Instead, we will construct perturbations which restrict in addition to satisfying certain properties (Proposition 6.2.2).

Ultimately, we can construct an abstract perturbation p_2 which restricts to a perturbation p_1 and which gives a well-defined continuous bijection between weighted branched orbifolds,

$$\iota : \mathcal{S}^{ft}_{A,g,k}(p_1) \to \mathcal{S}_{A,g,k}(p_2).$$

The usual approach of constructing an abstract perturbation ensures that $\mathcal{S}_{A,g,k}(p_2)$ is compact. However, this approach fails to guarantee that $\mathcal{S}^{ft}_{A,g,k}(p_1)$ will be compact.

Using only knowledge of the underlying topologies of both of these spaces, it is impossible to say anything more. The key to resolving this problem is understanding the additional structure that these spaces possess—the branched orbifold structure—and using this structure to prove an invariance of domain result. This result will allow us to assert that the above map is a homeomorphism. We explain these issues and how to resolve them in Chapter 6.

This approach will be used to establish equality of the Gromov–Witten invariants associated to polyfolds which have different choices of punctures at the marked points. It will also be used to establish equality of the Gromov–Witten invariants for polyfolds constructed with different strictly increasing sequences $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$.

The Polyfold Gromov–Witten Invariants as an Intersection Number

The polyfold Gromov–Witten invariants are defined in [22] via the branched integration described in [23]. The branched integral is useful for giving a well-defined definition of the polyfold Gromov–Witten invariants and moreover showing that they are, in fact, invariants and do not depend on choices. But they are not the best viewpoint for giving a proof of all of the axioms—the geometric relationships given by the kth-marked point forgetting map are difficult to phrase in terms of integrals.

A more useful viewpoint is to interpret the Gromov–Witten invariants as a count of curves via intersection theory. Such a definition is closest to the earliest interpretations of the Gromov–Witten invariants present in the literature, described in [41] as a 'finite sum, counted with multiplicity, of nonmultiple cover J-spheres in $\mathcal{M}^*_{(A,J)}$ which intersect representatives of given cycles in the symplectic manifold' (paraphrasing slightly).

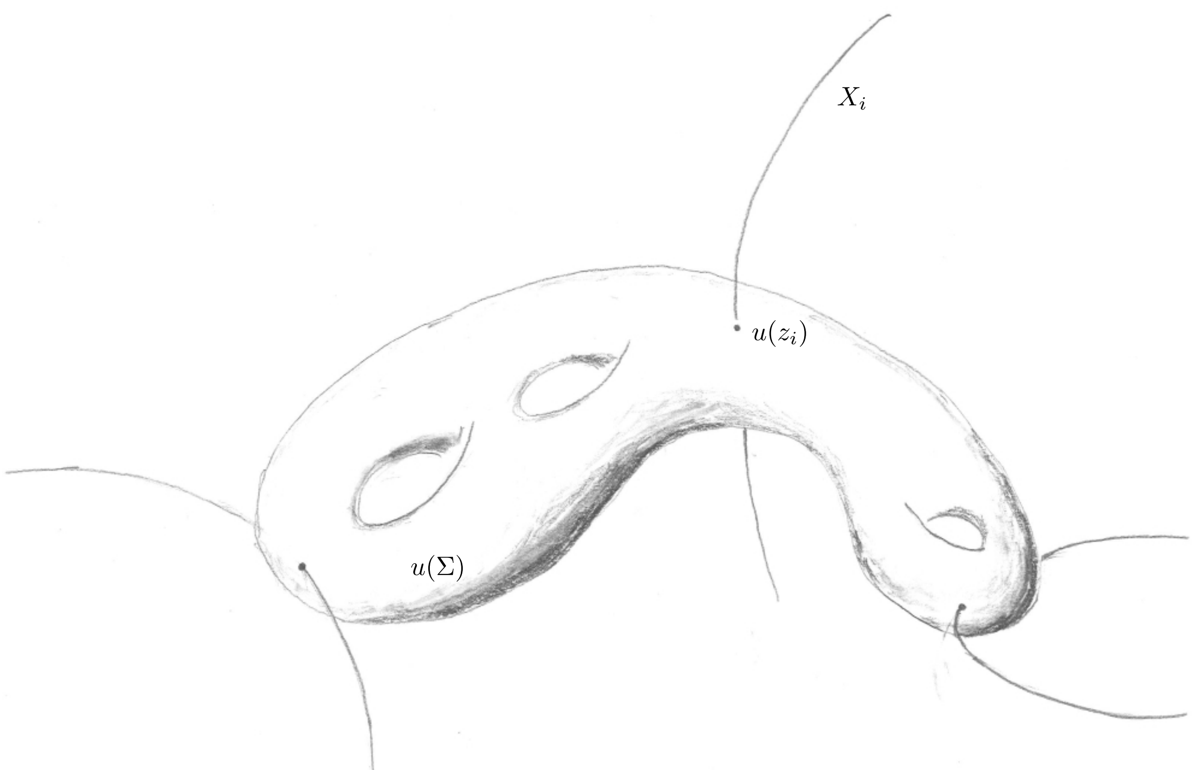
In Chapter 7 we show that compact oriented weighted branched orbifolds satisfy a well-defined intersection theory. This allows us to give an alternative definition of a polyfold Gromov–Witten invariant as an intersection number

$$\mathrm{GW}^Q_{A,g,k}(\alpha_1,\ldots,\alpha_k;\beta) := (ev_1 \times \cdots \times ev_k \times \pi)_*[\mathcal{S}_{A,g,k}(p)] \cdot (X_1 \times \cdots \times X_k \times B)$$

where $X_i \subset Q$ are closed submanifolds such that $[X_i] = \alpha_i$ and where $B \subset \overline{\mathcal{M}}^{\log}_{g,k}$ is a closed suborbifold such that $[B] = \beta$.

This definition aligns with the traditional geometric interpretation of the Gromov–Witten invariants as a 'count of curves', which at the ith-marked point passes through X_i and such that the image under the projection π is restricted to B.

Figure 1.2: Geometric Interpretation of a Gromov–Witten Invariant



Proving the Gromov–Witten Axioms

From the starting point in [22] of a well-defined polyfold Gromov–Witten invariant for arbitrary genus and for all closed symplectic manifolds, we have developed the general abstract tools necessary to give a proof of the Gromov–Witten axioms.

We can pullback abstract perturbations via maps, ensuring the persistence of well-defined maps between perturbed Gromov–Witten solution spaces. We can construct abstract perturbations which restrict, and via our invariance of domain result we can show that the Gromov–Witten invariants are stable under modifications to the polyfold construction. And we are armed with a workable interpretation of the Gromov–Witten invariant as an intersection number.

Everything is in its right place. In Chapter 7 we prove the Gromov–Witten axioms. The Effective, Grading, Homology, and Zero Axioms can be verified using facts about the definition of the Gromov–Witten invariant. To prove the Symmetry Axiom, we study the geometry of the permutation map between the perturbed Gromov–Witten solution spaces. To prove the Fundamental Class and Divisor Axioms, we study the geometry of the well-defined kth-marked point forgetting map between the perturbed Gromov–Witten solution spaces.

Chapter 2

Deligne–Mumford Orbifolds and Gromov–Witten Polyfolds

In this chapter we recall and summarize the constructions by Hofer, Wysocki, and Zehnder of the Deligne–Mumford orbifolds as well as the Gromov–Witten polyfolds. We will use the modern language of étale proper Lie groupoids to define orbifolds and polyfolds.

The notion of orbifold was first introduced by Satake [42], with further descriptions in terms of groupoids and categories by Haefliger [15][13][14], and Moerdijk [36][37]. Following this point of view, a polyfold may be viewed as a generalization of a (usually infinite-dimensional) orbifold, with some additional structure. This generalization of the étale proper Lie groupoid language to the polyfold context is due to Hofer, Wysocki, and Zehnder [21]. For full details in the present context, we will refer the reader to [21] for the abstract definitions of ep-groupoids in the polyfold context, [17] for the constructions of the Deligne–Mumford orbifolds, and [22] for the constructions of the Gromov–Witten polyfolds.

2.1 Orbifolds and Polyfolds

We take a moment to summarize the hierarchy of structures which leads to the definition of a polyfold.

Definition 2.1.1. An **sc-Banach space** consists of a Banach space E together with a decreasing sequence of linear subspaces

$$E = E_0 \supset E_1 \supset \cdots \supset E_\infty := \cap_{i\geq 0} E_i$$

such that the following two conditions are satisfied.

1. The inclusion operators $E_{m+1} \to E_m$ are compact.
2. E_∞ is dense in every E_i.

Definition 2.1.2. Consider an sc-Banach space E and consider an open subset $U \subset E$. An sc-smooth map $r : U \to U$ is called an **sc-smooth retraction** on U if $r \circ r = r$. A **local M-polyfold model (without boundary)** is a pair (O, E) consisting of an sc-Banach space E and a subset $O \subset E$ such that there exists an sc-smooth retraction $r : U \to U$ defined on an open subset $U \subset E$ such that $r(U) = O$. We call O, equipped with the subspace topology $O \subset E$, an **sc-retract**.

For notions of sc-differentiability for maps between sc-retracts, see Definition 2.4.1 and Definition 2.4.3 below.

In Gromov–Witten theory the compactification phenomena consist of nodal curves; hence our local M-polyfold models are without boundary and the ith-noded curves appear as interior points in the local M-polyfold models of codimension $2i$ (see [7, Remark 5.3.2]). Other cases such as Hamiltonian Floer theory (as first introduced in [9]) and Symplectic Field Theory (as first introduced in [6]) would require the inclusion of partial quadrants in order to deal with boundaries and corners (see [18, Definition 1.6, Definition 2.2] for details on local M-polyfold models with partial quadrants).

Definition 2.1.3. We say that a second countable paracompact topological space Z has an **M-polyfold structure** if every point $z \in Z$ has an open neighborhood $\hat{O}$ which is homeomorphic to an sc-retract O, and such that the induced transition maps between two sc-retracts are sc-smooth.

Remark 2.1.4. It is observed in [7, Example 4.1.8] that the real and complex Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ can be viewed as sc-Banach spaces with standard norm and the trivial sc-structures $(\mathbb{R}^n)_k = \mathbb{R}^n$ for $k \geq 0$ and $(\mathbb{C}^n)_k = \mathbb{C}^n$ for $k \geq 0$. It is furthermore observed in [7, Remark 4.2.2] that on finite-dimensional vector spaces with trivial sc-structure the notion of sc-differentiability is the same as classical differentiability. It follows that open subsets of $\mathbb{R}^n$ and $\mathbb{C}^m$ are special cases of sc-retracts, with the sc-retraction given by the identity. Hence, local M-polyfold models are a general enough definition to also give local models for finite dimensional manifolds and orbifolds.

In order to give the transition information on how our local models will fit together, we need the definition of an ep-groupoid.

Definition 2.1.5. A **groupoid** $(Z, \boldsymbol{Z})$ is a small category consisting of a set of objects Z, a set of morphisms $\boldsymbol{Z}$ which are all invertible, and the five structure maps (s, t, m, u, i) (the source, target, multiplication, unit, and inverse maps). An **ep-groupoid** is a groupoid $(Z, \boldsymbol{Z})$ together with M-polyfold structures on the object set Z as well as on the morphism set $\boldsymbol{Z}$ so that all the structure maps are sc-smooth maps (see Section 2.4) and which satisfy the following properties.

- **(étale)**. The source and target maps $s : \boldsymbol{Z} \to Z$ and $t : \boldsymbol{Z} \to Z$ are surjective local sc-diffeomorphisms.

- **(proper)**. For every point $z \in Z$, there exists an open neighborhood $V(z)$ so that the map $t : s^{-1}(\overline{V(z)}) \to Z$ is a proper mapping.

For a fixed object $z \in Z$ we denote the **isotropy group of** z by

$$\boldsymbol{G}(z) := \{\phi \in \boldsymbol{Z} \mid s(\phi) = t(\phi = z)\}.$$

The properness condition ensures that this is a finite group.

The **orbit space** of the ep-groupoid $(Z, \boldsymbol{Z})$,

$$|Z| := Z/\sim$$

is the quotient of the set of objects Z by the equivalence relation given by $z \sim z'$ if there exists a morphism $\phi \in \boldsymbol{Z}$ with $s(\phi) = z$ and $t(\phi) = z'$. It is equipped with the quotient topology defined via the map $Z \to |Z|, z \mapsto |z|$.

Definition 2.1.6. Let $\mathcal{Z}$ be a second countable paracompact topological space. A **polyfold structure** on $\mathcal{Z}$ consists of an ep-groupoid $(Z, \boldsymbol{Z})$ and a homeomorphism $|Z| \simeq \mathcal{Z}$.

It is useful to have in mind the local meaning of the above definition.

Proposition 2.1.7 (Natural Representation of $\boldsymbol{G}(x)$)**.** *[18, Theorem 7.1, Proposition 7.6] Let be an ep-groupoid $(Z, \boldsymbol{Z})$. Let $x \in Z$ with isotropy group $\boldsymbol{G}(x)$. Then for every open neighborhood V of x there exists an open neighborhood $U \subset V$ of x, a group homomorphism $\Phi : \boldsymbol{G}(x) \to \mathit{Diff}_{sc}(U)$, $g \mapsto \Phi(g)$, and a sc-smooth map $\Gamma : \boldsymbol{G}(x) \times U \to \boldsymbol{Z}$ such that the following holds.*

1. $\Gamma(g, x) = g$.
2. $s(\Gamma(g, y)) = y$ *and* $t(\Gamma(g, y)) = \Phi(g)(y)$ *for all* $y \in U$ *and* $g \in \boldsymbol{G}(x)$.
3. *If* $h : y \to z$ *is a morphism between points in* U*, then there exists a unique element* $g \in \boldsymbol{G}(x)$ *satisfying* $\Gamma(g, y) = h$*, i.e.,*

$$\Gamma : \boldsymbol{G}(x) \times U \to \{\phi \in \boldsymbol{Z} \mid s(\phi) \text{ and } t(\phi) \in U\}$$

is a bijection.

The data (Φ, Γ) *is called the* ***natural representation*** *of* $\boldsymbol{G}(x)$*. Moreover, consider the following topological spaces:*

- $\boldsymbol{G}(x)\backslash U$*, equipped with quotient topology via the projection* $U \to \boldsymbol{G}(x)\backslash U$*,*
- $U/\sim$*, where* $x \sim x'$ *for* $x, x' \in U$ *if there exists a morphism* $\phi, s(\phi) = x, t(\phi) = x'$*, equipped with the quotient topology via the projection* $U \to U/\sim$*,*
- $|U|$*, the image of* U *under the map* $Z \to |Z|$ *equipped with the subspace topology.*

Then these spaces are all naturally homeomorphic.

Defining an ep-groupoid involves making a choice of local structures. Taking an equivalence class of ep-groupoids makes our differentiable structure choice independent. The appropriate notion of equivalence in this category-theoretic context is a 'Morita-equivalence class' (see [21, Definition 3.2]).

Definition 2.1.8. A **polyfold** consists of a second countable paracompact topological space $\mathcal{Z}$ together with a Morita-equivalence class of polyfold structures $[(Z, \boldsymbol{Z})]$ on $\mathcal{Z}$.

An **orbifold** is a special case of a polyfold where the local models are given by $\mathbb{R}^n$, while a **holomorphic orbifold** has local models given by $\mathbb{C}^n$ with the additional requirement that the structure maps are holomorphic maps.

A **manifold** can also be seen as a special case of a polyfold where the local models are given by $\mathbb{R}^n$ and where every point in the polyfold structure has trivial isotropy.

Taking a Morita-equivalence class of a given polyfold structure (in the case of polyfolds) is analogous to taking a maximal atlas for a given atlas (in the usual definition of manifolds). Given distinct polyfold structures which define an orbifold or a polyfold, the method of proving they define the same Morita-equivalence class is by demonstrating that both polyfold structures possess a common refinement.

The scales of an sc-Banach space induce a filtration on the local M-polyfold models, which is moreover preserved by the structure maps s, t. Consequently, there is a well defined filtration on the orbit space which hence induces a filtration

$$\mathcal{Z} = \mathcal{Z}_0 \supset \mathcal{Z}_1 \supset \cdots \supset \mathcal{Z}_\infty = \cap_{k\geq 0} \mathcal{Z}_k$$

on the underlying topological space $\mathcal{Z}$.

Remark 2.1.9 (Notation). We use the convention of denoting both the ep-groupoid $(Z, \boldsymbol{Z})$ and its object set Z by the same letter Z. We will refer to the underlying set, the underlying topological space, or the polyfold by the letter $\mathcal{Z}$. Furthermore, we will write objects as '$z \in Z$', morphisms as '$\phi \in \boldsymbol{Z}$', and points as '$[z] \in \mathcal{Z}$' (due to the identification $|Z| \simeq \mathcal{Z}$).

2.2 The Deligne–Mumford Orbifolds

In this section we recall the construction by Hofer-Wysocki-Zehnder of the Deligne–Mumford orbifolds (abbreviated as DM-orbifolds and sometimes DM-spaces). We will describe the underlying set of the DM-orbifolds, discuss the significance of the choice of a 'gluing profile' and its effect on the orbifold structure, and define the 'good uniformizing families' which serve as local models for these orbifolds.

In the present context of Gromov–Witten theory, it is sufficient to refer the reader to the overview contained in [22]. For full details, we refer the reader to [17] for the full construction of these orbifolds (in particular, details on the ep-groupoid structure), and [19] for some necessary results on the smoothness of the transition maps between local coordinates.

The Underlying Set of the Deligne–Mumford Orbifolds

Definition 2.2.1. For fixed integers $g \geq 0$ and $k \geq 0$ which satisfy $2g+k \geq 3$, the underlying set of the Deligne–Mumford orbifold $\overline{\mathcal{M}}_{g,k}$ is defined as the set of equivalence classes of stable noded Riemann surfaces of arithmetic genus g and with k marked points

$$\overline{\mathcal{M}}_{g,k} := \{(\Sigma, j, M, D) \mid \ + \text{DM-stability condition}\}/\sim$$

with data as follows.

- (Σ, j) is a closed (possibly disconnected) Riemann surface.
- M consists of k ordered distinct marked points $z_1, \ldots, z_k \in \Sigma$.
- D consists of finitely many unordered nodal pairs $\{x, y\}$ with $x, y \in \Sigma$ and $x \neq y$. We require that two such pairs are disjoint. We require both elements of the pair to be distinct from M. We denote $|D| := \cup_{\{x,y\}\in D}\{x, y\} \subset \Sigma$, and we let $\#D := \frac{1}{2}\#|D|$, i.e. the number of pairs.
- Viewing each connected component $C \subset \Sigma$ as a vertex, and each nodal pair $\{x, y\}$ as an edge via the incidence relation $\{C_x, C_y\}$ if $x \in C_x,\ y \in C_y$, we obtain a graph T. We require that T be connected.
- The arithmetic genus g is given as:
$$g = \sum_C g_C + \text{number of cycles of the graph } T.$$
where the sum is taken over the finitely many connected components $C \subset \Sigma$, and where g_C is defined as the genus of the connected component C.
- For each connected component $C \subset \Sigma$ we require the following **DM-Stability Condition**:
$$2 \cdot g_C + \sharp(M \cup |D|)_C \geq 3$$
where $\sharp(M \cup |D|)_C := \sharp((M \cup |D|) \cap C)$, i.e. it is the number of marked and nodal points on the component C.
- The equivalence relation is given by $(\Sigma, j, M, D) \sim (\Sigma', j', M', D')$ if there exists a biholomorphism $\phi : (\Sigma, j) \to (\Sigma', j')$ such that $\phi(M) = M',\ \phi(|D|) = |D'|$, and which preserves the ordering of the marked points, and maps each pair of nodal points to a pair of nodal points.

The importance of the stability condition is the following: the group $\operatorname{Aut}(\Sigma, j, M, D)$ of biholomorphisms for a fixed (Σ, j, M, D) is finite if and only if the stability condition holds. We will refer to any point in $M \cup |D|$ as a **special point**. We call a tuple (Σ, j, M, D) which satisfies the DM-stability condition a **stable noded Riemann surface**.

Proposition 2.2.2. *[22, Proposition 2.4] The set $\overline{\mathcal{M}}_{g,k}$ has a natural second countable paracompact Hausdorff topology. With this topology, $\overline{\mathcal{M}}_{g,k}$ is a compact topological space.*

Local Models for the Deligne–Mumford Orbifolds

In order to define a smooth structure of the Deligne–Mumford spaces, we must first make a choice of a 'gluing profile' which we now define.

Definition 2.2.3. A **gluing profile** is a smooth diffeomorphism

$$\varphi : (0,1] \to [0,\infty).$$

We will be especially concerned with the following two gluing profiles; the **logarithmic gluing profile**

$$(0,1] \to [0,\infty), \quad r \mapsto \varphi_{\log}(r) := -\frac{1}{2\pi}\log(r).$$

and the **exponential gluing profile**

$$(0,1] \to [0,\infty), \quad r \mapsto \varphi_{\exp}(r) := e^{\frac{1}{r}} - e.$$

Construction of a Family of Riemann Surfaces

Consider a Riemann surface (Σ, j) with a nodal pair $\{x,y\}$. Associate to this pair a **gluing parameter** $a \in B_{\frac{1}{2}} = \{z \in \mathbb{C} \mid |z| < \frac{1}{2}\}$. We use the gluing profile to construct a family of Riemann surfaces parametrized by a in the following way.

- Choose small disk-like neighborhoods D_x of x and D_y of y, and identifications (via biholomorphisms) $D_x \setminus \{x\} \simeq \mathbb{R}^+ \times S^1$ and $D_y \setminus \{y\} \simeq \mathbb{R}^- \times S^1$. Moreover, the punctures x and $+\infty$ are identified, and likewise y and $-\infty$.
- Write the gluing parameter $a \neq 0$ in polar coordinates as

  $$a = r_a e^{i\theta_a}; \qquad r_a \in (0, \tfrac{1}{2}),\ \theta_a \in \mathbb{R}/2\pi\mathbb{Z}$$

  we use the gluing profile to define a **gluing length** given by

  $$R_a := \varphi(r_a) \in (\varphi(\tfrac{1}{2}), \infty).$$

- Delete the points $(R_a, +\infty) \times S^1 \subset \mathbb{R}^+ \times S^1$ and $(-\infty, -R_a) \times S^1 \subset \mathbb{R}^- \times S^1$ from $\mathbb{R}^+ \times S^1$ and $\mathbb{R}^- \times S^1$, and identify the remaining cylinders $[0, R_a] \times S^1$ and $[-R_a, 0] \times S^1$ via the map

  $$L_{R_a} : [0, R_a] \times S^1 \to [-R_a, 0] \times S^1; \qquad (s,t) \mapsto (s - R_a, t - \theta_a)$$

  We replace $D_x \sqcup D_y$ with the finite cylinder

  $$Z_a := [0, R_a] \times S^1 \simeq_{L_{R_a}} [-R_a, 0] \times S^1$$

  (For $a = 0$ we may define $Z_0 := \mathbb{R}^+ \times S^1 \sqcup \mathbb{R}^- \times S^1$, identifiable with $D_x \setminus \{x\} \sqcup D_y \setminus \{y\}$.)

This procedure yields a new Riemann surface defined by the quotient space

$$\Sigma_a := \frac{\Sigma \setminus (([R_a, \infty) \times S^1 \cup \{x\}) \cup (\{y\} \cup (-\infty, R_a] \times S^1))}{L_{R_a} : [0, R_a] \times S^1 \to [-R_a, 0] \times S^1}$$

and which carries a naturally induced complex structure. Alternatively, we can write Σ_a as the disjoint union

$$\Sigma_a = (\Sigma \setminus (D_x \sqcup D_y)) \sqcup Z_a$$

with complex structure

$$j(a) := \begin{cases} j & \text{on } \Sigma \setminus (D_x \sqcup D_y) \\ i & \text{on } Z_a \end{cases}$$

where i is the standard complex structure on the finite cylinder $Z_a = [0, R_a] \times S^1$. We can repeat this gluing construction for a Riemann surface with multiple nodes, hence we introduce the following definition.

Definition 2.2.4. Consider a Riemann surface (Σ, j) with nodal pairs $\{x_a, y_a\} \in D$. We may choose disjoint small disk like neighborhoods D_{x_a} and D_{y_a} at every node. Carrying out the above procedure at every node we obtain a **glued Riemann surface** defined by the quotient space

$$\Sigma_{\underline{a}} := \frac{\Sigma \setminus \sqcup_{a \in \underline{a}} (([R_a, \infty) \times S^1 \cup \{x_a\}) \cup (\{y_a\} \cup (-\infty, R_a] \times S^1))}{(L_{R_a} : [0, R_a] \times S^1 \to [-R_a, 0] \times S^1)_{a \in \underline{a}}}$$

and which carries a naturally induced complex structure. Here we denote $\underline{a} \in B_{\frac{1}{2}}^{\sharp D}$. Alternatively, we can write $\Sigma_{\underline{a}}$ as the disjoint union

$$\Sigma_{\underline{a}} := \left(\Sigma \setminus \cup_{\{x_a, y_a\} \in D} (D_{x_a} \sqcup D_{y_a})\right) \sqcup_{a \in \underline{a}} Z_a.$$

with complex structure

$$j(\underline{a}) := \begin{cases} j & \text{on } \Sigma \setminus \cup_{\{x_a, y_a\} \in D} (D_{x_a} \sqcup D_{y_a}) \\ i & \text{on } Z_a \text{ for every } a \in \underline{a} \end{cases}$$

where i is the standard complex structure on the finite cylinder Z_a.

Definition 2.2.5. [22, Definition 2.6], [20, Definition 2.15] Given a stable noded Riemann surface (Σ, j, M, D) with an associated small disk structure $\boldsymbol{D}$ at the marked and nodal points. Let v denote a parameter in an open subset V of a complex vector space E, where $\dim_{\mathbb{R}} E = 6g - 6 + 2\sharp M - 2\sharp D$. We call a smooth family $v \mapsto j(v)$ of complex structures $j(v)$ on Σ a **good complex deformation** if it satisfies the following.

- $j(0) = j$

- $j(v) = j$ on all small disks D_x associated to $\boldsymbol{D}$
- For every $v \in V$ the Kodaira-Spencer differential (see [22, p21-22] or [20, p15])

$$[Dj(v)] : H^1(\Sigma, j, M, D) \to H^1(\Sigma, j(v), M, D)$$

  is a complex linear isomorphism.

- There exists a natural action

$$\boldsymbol{G}(\Sigma, j, M, D) \times V \to V, \qquad (\phi, v) \mapsto \phi * v$$

  such that $\phi : (\Sigma, j(v)) \to (\Sigma, j(\phi * v))$ is a biholomorphism (where we view $\phi \in \boldsymbol{G}(\Sigma, j, M, D)$ as a map $\phi : \Sigma \to \Sigma$).

Definition 2.2.6. [22, Definition 2.12] Consider a stable noded Riemann surface (Σ, j, M, D), with isotropy group $\boldsymbol{G}(\Sigma, j, M, D)$. A **good uniformizing family** centered at (Σ, j, M, D) is a family of stable noded Riemann surfaces

$$(\underline{a}, v) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}) \quad \text{where} \quad (\underline{a}, v) \in O \subset (B_{\frac{1}{2}})^{\#D} \times E.$$

Explicitly, the stable noded Riemann surface $(\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}})$ is given by the following.

- The glued Riemann surface $\Sigma_{\underline{a}}$ is given by

$$\Sigma_{\underline{a}} = \Big(\Sigma \setminus \cup_{\{x_a, y_a\} \in D}(D_{x_a} \sqcup D_{y_a})\Big) \sqcup_{a \in \underline{a}} Z_a.$$

- The map $v \mapsto j(v)$ is a good complex deformation and induces the following complex structure on the glued Riemann surface $\Sigma_{\underline{a}}$

$$j(\underline{a}, v) := \begin{cases} j(v) & \text{on } \Sigma \setminus \cup_{\{x_a, y_a\} \in D}(D_{x_a} \sqcup D_{y_a}) \\ i & \text{on } Z_a \text{ for ever } a \in \underline{a}. \end{cases}$$

- The set of marked points $z_1, \ldots, z_k \in M_{\underline{a}}$ are given by the former marked points which by construction all lie in $\Sigma \setminus \cup_{\{x_a, y_a\} \in D}(D_{x_a} \sqcup D_{y_a})$.
- The set of nodal pairs $D_{\underline{a}}$ is obtained from D by deleting every nodal pair $\{x_a, y_a\} \in D$ for which $a \neq 0$.

In the above description of a good uniformizing family, movement of the marked points is determined by the variation of the complex structure j via the parameter v. Consider a stable noded Riemann surface (Σ, j, M, D) with a good complex deformation $v \mapsto j(v)$. Wiggle the marked points slightly, and obtain a new stable noded Riemann surface (Σ, j, M', D). Then there exists a parameter $v \in V$ such that there exists a biholomorphism $\phi : (\Sigma, j(v), M, D) \to (\Sigma, j, M', D)$.

Situations will arise where we will want to parametrize the movement of the kth-marked point directly (e.g. when we consider the kth-marked point forgetting maps in Section 2.5 and Chapter 5). Consider a stable noded Riemann surface (Σ, j, M, D) and suppose that the component C_k which contains the kth-marked point z_k remains stable after forgetting z_k. We parametrize a neighborhood of z_k by embedding a small disk via a holomorphic map

$$\varphi : (B_\epsilon, i) \to (\Sigma, j) \text{ so that } \varphi(0) = z_k \text{ and } \varphi(B_\epsilon) \subset D_{z_k}$$

where D_{z_k} is the small disk structure at z_k.

Definition 2.2.7. Given a stable noded Riemann surface (Σ, j, M, D) with an associated small disk structure $\boldsymbol{D}$ at the marked and nodal points. Let v denote a parameter in an open subset W of a complex vector space F, where $\dim_{\mathbb{R}} F = 6g - 6 + 2\sharp M - 2\sharp D - 2$. We call a smooth family $v \mapsto j(v)$ of complex structures $j(v)$ on Σ an **alternative good complex deformation** if it satisfies the following.

- $j(0) = j$
- $j(v) = j$ on all small disks D_x associated to $\boldsymbol{D}$
- For every $v \in W$ the Kodaira-Spencer differential

$$[Dj(v)] : H^1(\Sigma, j, M \setminus \{z_k\}, D) \to H^1(\Sigma, j(v), M \setminus \{z_k\}, D)$$

  is a complex linear isomorphism.
- There exists a natural action

$$\boldsymbol{G}(\Sigma, j, M, D) \times W \times B_\epsilon \to W \times B_\epsilon, \qquad (\phi, v, z) \mapsto \phi * (v, z)$$

  such that $\phi : (\Sigma, j(v)) \to (\Sigma, j(g * v))$ is a biholomorphism which furthermore satisfies $\phi(z_k) = \varphi(z)$.

Definition 2.2.8. Consider a stable noded Riemann surface (Σ, j, M, D), with isotropy group $\boldsymbol{G}(\Sigma, j, M, D)$. We define an **alternative good uniformizing family** centered at (Σ, j, M, D) as a family of stable noded Riemann surfaces

$$(\underline{a}, v, z) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \setminus \{z_k\})_{\underline{a}} \cup \{\varphi(z)\}_{\underline{a}}, D_{\underline{a}}) \quad \text{where} \quad (\underline{a}, v, z) \in O \subset (B_{\frac{1}{2}})^{\#D} \times F \times B_\epsilon.$$

Explicitly, the stable noded Riemann surface $(\Sigma_{\underline{a}}, j(\underline{a}, v), (M \setminus \{z_k\})_{\underline{a}} \cup \{\phi(z)\}_{\underline{a}}, D_{\underline{a}})$ is given by the following.

- The glued Riemann surface $\Sigma_{\underline{a}}$ is given by

$$\Sigma_{\underline{a}} = \left(\Sigma \setminus \cup_{\{x_a, y_a\} \in D} (D_{x_a} \sqcup D_{y_a})\right) \sqcup_{a \in \underline{a}} Z_a.$$

- The map $v \mapsto j(v)$ is an alternative good complex deformation and induces the following complex structure on the glued Riemann surface $\Sigma_{\underline{a}}$

$$j(\underline{a}, v) := \begin{cases} j(v) & \text{on } \Sigma \setminus \cup_{\{x_a, y_a\} \in D}(D_{x_a} \sqcup D_{y_a}) \\ i & \text{on } Z_a \text{ for ever } a \in \underline{a}. \end{cases}$$

- The marked points $z'_1, \ldots, z'_{k-1} \in (M \setminus \{z_k\})_{\underline{a}}$ are given by the former marked points which by construction all lie in $\Sigma \setminus \cup_{\{x_a, y_a\} \in D}(D_{x_a} \sqcup D_{y_a})$. The kth-marked point is parametrized by the map $\varphi : B_\epsilon \to \Sigma$ i.e.

$$z'_k = \varphi(z) \in \Sigma \setminus \cup_{\{x_a, y_a\} \in D}(D_{x_a} \sqcup D_{y_a}).$$

- The set of nodal pairs $D_{\underline{a}}$ is obtained from D by deleting every nodal pair $\{x_a, y_a\} \in D$ for which $a \neq 0$.

The Logarithmic and Exponential Deligne–Mumford Orbifolds

Using the good uniformizing families, we may define an ep-groupoid structure on the topological spaces $\overline{\mathcal{M}}_{g,k}$. Such an ep-groupoid structure necessarily depends on the choice of gluing profile.

Theorem 2.2.9. *[17, Theorem 2.25] Using the logarithmic gluing profile, we reproduce the classical Deligne–Mumford theory and obtain a complex orbifold we denote as $\overline{\mathcal{M}}^{log}_{g,k}$. Conversely, using the exponential gluing profile we obtain a smooth orbifold $\overline{\mathcal{M}}^{exp}_{g,k}$. Moreover, the identity map*

$$id : \overline{\mathcal{M}}^{exp}_{g,k} \to \overline{\mathcal{M}}^{log}_{g,k}$$

is smooth. The inverse map is continuous, but not differentiable.

2.3 The Gromov–Witten Polyfolds

In this section we recall the construction by Hofer, Wysocki, and Zehnder of the Gromov–Witten polyfolds (abbreviated as GW-polyfolds). We describe the underlying set of the GW-polyfolds, and recall the definition of the 'good uniformizing families of stabilizing maps' which serve as the local M-polyfold models. For the full construction of the GW-polyfolds, we refer the reader to [22], although necessary results on sc-smoothness of the retraction maps as well as the transition maps are contained in [19].

The Underlying Set of the Gromov–Witten Polyfolds

Definition 2.3.1. For a fixed homology class $A \in H_2(Q;\mathbb{Z})$, and for fixed integers $g \geq 0$ and $k \geq 0$, the underlying set of the GW-polyfold $\mathcal{Z}_{A,g,k}$ is defined as the set of stable curves with homology class A, arithmetic genus g, and k marked points

$$\mathcal{Z}_{A,g,k} := \{(\Sigma, j, M, D, u) \mid + \text{ stability condition*}\}/\sim$$

where (Σ, j, M, D) is a noded Riemann surface as in Definition 2.2.1 (where we do not require the DM-stability condition), with data as follows.

- $u : \Sigma \to M$ is a continuous map such that $u_*[\Sigma] = A \in H_2(Q)$.
- For each nodal pair $\{x, y\} \in D$ we have $u(x) = u(y)$.
- The map u is of class H^{3,δ_0} at the nodal points in $|D|$ and of class H^3_{loc} near the other points in Σ (see Definition 2.3.2 below).
- $\int_C u^*\omega \geq 0$ for each connected component $C \subset \Sigma$.
- For each connected component $C \subset \Sigma$ the following **stability condition*** holds. We require at least one of the following:

  $$2 \cdot g_C + \sharp(M \cup |D|)_C \geq 3 \quad \text{or} \quad \int_C u^*\omega > 0, \tag{2.1}$$

  where g_C is the genus of C and $\sharp(M \cup |D|)_C$ is the number of marked and nodal points on the component C.
- The equivalence relation is given by $(\Sigma, j, M, D, u) \sim (\Sigma', j', M', D', u')$ if there exists a biholomorphism $\phi : (\Sigma, j) \to (\Sigma', j')$ such that $u' \circ \phi = u$, in addition to $\phi(M) = M'$, $\phi(|D|) = |D'|$, and which preserves ordering and pairs.

We call a tuple (Σ, j, M, D, u) which satisfies these requirements a **stable map**, and call an equivalence class $[\Sigma, j, M, D, u]$ a **stable curve**.

The stability condition* implies the following: the group $\text{Aut}(\Sigma, j, M, D, u)$ of biholomorphisms for a fixed (Σ, j, M, D, u) is finite if and only if the stability condition holds.

Definition 2.3.2. [22, Definition 1.1] Consider a point $z \in \Sigma$, and let $D_z \subset \Sigma$ be a small disk neighborhood of z such that there exists a biholomorphism $\sigma : [0, \infty) \times S^1 \to D_z \setminus \{z\}$.

Let $u : \Sigma \to Q$ be a continuous map, let $m \geq 3$ be an integer, and let $\delta > 0$. We say that u is of **class** (m, δ) **at the puncture** $z \in \Sigma$, written $u \in H^{m,\delta}(D_z \setminus \{z\})$, if for a smooth chart $\varphi : U(u(0)) \to \mathbb{R}^{2n}$ of Q mapping $u(0)$ to 0 the map

$$v(s,t) = \varphi \circ u \circ \sigma(s,t),$$

which is defined for s large, has weak partial derivatives up to order m, which if weighted by $e^{\delta s}$, belong to $L^2([s_0, \infty) \times S^1, \mathbb{R}^{2n})$ if s_0 is sufficiently large.

We say that u is of **class m around the point** $z \in \Sigma$, if u belongs to the space $H^m_{\text{loc}}(D_z)$.

These definitions do not depend on the choices involved, like charts on Q and holomorphic polar coordinates on Σ.

For a fixed increasing sequence $(\delta_m) \subset (0, 2\pi)$ the filtration on $\mathcal{Z}_{A,g,k}$ is then given as follows. We say $[\Sigma, j, M, D, u]$ is of regularity $(m+3, \delta_m)$ if the map u is of class H^{3+m,δ_m} at the nodal points in $|D|$ and of class H^m_{loc} near the other points in Σ.

Remark 2.3.3. Consider a point $z \in \Sigma$ with a small disk neighborhood $D_z \subset \Sigma$ as above. Consider the spaces $H^{3,\delta_0}(D_z \setminus \{z\})$ and $H^3_{\text{loc}}(D_z)$. As it turns out, neither of these spaces contains the other.

Remark 2.3.4. By definition, the following sets of stable curves must be empty:

$$\mathcal{Z}_{0,0,2} = \mathcal{Z}_{0,0,1} = \mathcal{Z}_{0,0,0} = \mathcal{Z}_{0,1,0} = \emptyset$$

This is because the stability condition* (2.1) cannot be satisfied.

Remark 2.3.5 (Punctures at the Marked Points). Some situations will require that the map u to be of class H^{3,δ_0} at a fixed subset of the marked points, in addition to the nodal points. Allowing a puncture at an ith-marked is a global condition on our polyfold, and so we may add the following to the above conditions on the set $\mathcal{Z}_{A,g,k}$.

- Consider a subset $L \subset \{1, \ldots, k\}$. For all $i \in L$ we require the map u to be of class H^{3,δ_0} at the puncture at the ith-marked point.

The entirety of the GW-polyfold theory carries over without modification with this new condition added, specifically the abstract Fredholm perturbation theory and consequently the existence of well-defined GW-invariants for this polyfold. In Theorem 6.1.2, we prove that the GW-invariants defined by such GW-polyfolds do not depend on the choices of punctures at the marked points.

The Gluing Construction

Consider a stable map (Σ, j, M, D, u) with a nodal pair $\{x_a, y_a\} \in D$ with associated gluing parameter $a \in B_{\frac{1}{2}}$. Consider a section $\eta \in H^{3,\delta_0}_c(u^*TQ)$. In a neighborhood of the point $u(x_a) = u(y_a) \in Q$ choose a chart which maps $u(x_a) = u(y_a)$ to $0 \in \mathbb{R}^{2n}$. Choose a small disk structure at x_a and y_a such that there exist biholomorphisms between $D_{x_a} \setminus \{x_a\}$ and $\mathbb{R}^+ \times S^1$; and between $D_{y_a} \setminus \{y_a\}$ and $\mathbb{R}^- \times S^1$.

Localized to these coordinate neighborhoods, we may view the base map u as maps

$$u^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}, \qquad u^- : \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}$$

and likewise the section η as maps

$$\eta^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}, \qquad \eta^- : \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}.$$

Choose a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ which satisfies the following:

- $\beta(-s) + \beta(s) = 1 \quad \text{for all } s \in \mathbb{R}$
- $\beta(s) = 1 \quad \text{for all } s \leq -1$
- $\frac{d}{ds}\beta(s) < 0 \quad \text{for all } s \in (-1,1).$

Definition 2.3.6. [22, Section 2.4] For a given gluing parameter $a \in B_{\frac{1}{2}}$ the glued base map $\oplus_a(u^+,u^-) : Z_a \to \mathbb{R}^{2n}$ is defined by

$$\oplus_a(u^+,u^-)(s,t) := \begin{cases} \beta\left(s - \frac{R_a}{2}\right) \cdot u^+(s,t) + \left(1 - \beta\left(s - \frac{R_a}{2}\right)\right) \cdot u^-(s - R_a, t - \theta_a) & a \neq 0 \\ (u^+,u^-) & a = 0. \end{cases}$$

The glued section $\oplus_a(\eta^+,\eta^-) : Z_a \to \mathbb{R}^{2n}$ is defined in an identical manner.

We may assume our chart is chosen such that we may identify the Riemannian metric on Q with the Euclidean metric on $\mathbb{R}^{2n}$; hence we may identify the maps $\exp_{\oplus_a(u^+,u^-)} \oplus_a(\eta^+,\eta^-)$ and $\oplus_a(\exp_{(u^+,u^-)}(\eta^+,\eta^-))$ (see [22, Proposition 2.51]). The **glued map** $\oplus_a \exp_u(\eta) : \Sigma_a \to Q$ is defined by

$$\oplus_a \exp_u(\eta) := \begin{cases} \exp_u(\eta) & \text{on } \Sigma \setminus (D_{x_a} \sqcup D_{y_a}) \\ \oplus_a \exp_u(\eta) & \text{on } Z_a. \end{cases}$$

Remark 2.3.7. For $a \neq 0$ data is lost by this gluing procedure; this data can be kept track of via a complementary gluing procedure, called the **anti-gluing**, which we briefly describe. Define cylinders

$$C_a = \begin{cases} \mathbb{R} \times S^1 & \text{when } a \neq 0 \\ \emptyset & \text{when } a = 0 \end{cases}$$

For $a \neq 0$ define the anti-glued section as the map $\ominus_a(\eta^+,\eta^-) : C_a \to \mathbb{R}^{2n}$,

$$\begin{aligned} \ominus_a(\eta^+,\eta^-)(s,t) := &- \left(1 - \beta\left(s - \tfrac{R_a}{2}\right)\right) \cdot [\eta^+(s,t) - \mathrm{av}_a(\eta^+,\eta^-)] \\ &+ \beta\left(s - \tfrac{R_a}{2}\right) \cdot [\eta^-(s - R_a, t - \theta_a) - av_a(\eta^+,\eta^-)] \end{aligned}$$

where

$$\mathrm{av}_a(\eta^+,\eta^-) := \tfrac{1}{2}\left(\int_{S^1} \eta^+(\tfrac{R_a}{2},t)dt + \int_{S^1} \eta^-(-\tfrac{R_a}{2},t)dt\right).$$

For $a = 0$, $\ominus_0(\eta^+,\eta^-)$ is the unique map $\emptyset \to \mathbb{R}^{2n}$.

Local M-polyfold models for the Gromov–Witten Polyfolds

Consider a stable map (Σ, j, M, D, u). There exists 'good data' centered at this stable map, which in particular includes a 'stabilization' S of the noded, not necessarily stable, Riemann surface (Σ, j, M, D) (see [22, Definition 3.1, Lemma 3.2, Definition 3.6, Proposition 3.7]). The most important facts about a stabilization are the following:

- a stabilization consists of an unordered finite set of points $S \subset \Sigma$ which is disjoint from $M \cup |D|$, such that the nodal Riemann surface $(\Sigma, j, M \cup S, D)$ satisfies the DM-stability condition,
- at all points $z_s \in S$ the tangent map $Tu(z_s)$ is injective.

By the second condition, there exists a $2n-2$-dimensional complement $H_{u(z_s)}$,

$$Tu(z_s) \oplus H_{u(z_s)} = T_{u(z_s)}M$$

called the **linear constraint associated with the point** $z_s \in S$. We can identify a neighborhood of zero in $H_{u(z_s)}$ with an embedded submanifold of M via the exponential map.

Every map in an open neighborhood of the map u in an appropriate Banach space will have the property it intersects $H_{u(z_s)}$ precisely once, and intersects transversally. Requiring that this map intersects at the stabilized point z_s allows us to fix an automorphism of the underlying domain. This is a so-called 'local slice' construction, which yields a Banach manifold chart for a space of maps modulo reparametrization.

We can describe this chart precisely as follows. Define an sc-Banach space by

$$E_u = \{\eta \in H^{3,\delta_0}_c(u^*TM) \mid \eta(z_s) \in H_{u(z_s)} \text{ for } z_s \in S\}$$

which consists of sections along the smooth map $u : \Sigma \to M$ which satisfy the linear constraint $H_{u(z_s)}$ at the points $z_s \in \Sigma$. There exists a good uniformizing family

$$(\underline{a}, v) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \cup S)_{\underline{a}}, D_{\underline{a}}), \quad (\underline{a}, v) \in O$$

of stabilized noded Riemann surfaces. The parameters $(\underline{a}, v)$ give splicing parameters, and together with the gluing construction allow us to define a splicing core

$$K = \{(\underline{a}, v, \eta) \mid (\underline{a}, v) \in O, \eta \in E_u, \text{ and } \pi_{\underline{a}}(\eta) = \eta\}.$$

Definition 2.3.8. [22, Definition 3.9] Assume that we have good data and a stabilization centered at a stable map (Σ, j, M, D, u) with isotropy group $\boldsymbol{G}(\Sigma, j, M, D, u)$. A **good uniformizing family of stable maps** centered at (Σ, j, M, D, u) is a family of stable maps

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}} \exp_u(\eta)) \quad \text{where} \quad (\underline{a}, v, \eta) \in \mathcal{O}.$$

Here $\mathcal{O}$ is an open subset of the splicing core K. The family of stable noded Riemann surfaces $(\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}})$ is defined in the same manner as in Definition 2.2.6, while the glued map $\oplus_{\underline{a}} \exp_u(\eta) : \Sigma_{\underline{a}} \to Q$ is defined using the gluing construction of Definition 2.3.6 as

$$\oplus_{\underline{a}} \exp_u \eta := \begin{cases} \exp_u \eta & \text{on } \Sigma \setminus \cup_{\{x_a,y_a\}\in D}(D_{x_a} \sqcup D_{y_a}) \\ \oplus_a \exp_u \eta & \text{on } Z_a \text{ for every } a \in \underline{a}. \end{cases}$$

The Gromov–Witten Polyfolds

A topology on $\mathcal{Z}_{A,g,k}$ is induced from the topologies of the local M-polyfold models. It is proved in [22, Section 3.4] that this topology is a second countable, Hausdorff, paracompact topology. From there, we may construct a polyfold structure; this requires us to take a countable collection of good uniformizing families of stable maps which give a locally finite open cover of the underlying topological space $\mathcal{Z}_{A,g,k}$. Any two such polyfold structures constructed by this recipe will have a common refinement, and as such, define the same Morita equivalence class.

Theorem 2.3.9. *[22, Theorem 1.7] Having fixed the exponential gluing profile and a strictly increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$, the second countable paracompact topological space $\mathcal{Z}_{A,g,k}$ of stable curves possesses a natural equivalence class of polyfold structures.*

See [22, Remark 3.38] for elaboration on the meaning of ‘natural’ in this context.

2.4 sc-Calculus and sc-Differentiability Results

We begin this section by recalling some essential definitions and facts of the sc-calculus, as originally formulated in [16].

Definition 2.4.1. A map $f : U \to U'$ between two open subsets of sc-Banach spaces E and E' is called an **sc^0-map**, if $f(U_i) \subset U'_i$ for all $i \geq 0$ and if the induced maps $f : U_i \to U'_i$ are continuous. Furthermore, f is called an **sc^1-map**, or of **class sc^1** if the following conditions are satisfied.

- For every $x \in U_1$ there exists a bounded linear map $Df(x) \in \mathcal{L}(E_0, E'_0)$ satisfying for $h \in E_1$, with $x + h \in U_1$,

$$\frac{1}{\|h\|_1} \|f(x+h) - f(x) - Df(x)h\|_0 \to 0 \quad \text{as } \|h\|_1 \to 0.$$

- The tangent map $Tf : TU \to TU'$, defined by

$$Tf(x, h) = (f(x), Df(x)h),$$

  is an sc^0-map between the tangent spaces.

If $Tf : TU \to TU'$ is of class sc^1, then $f : U \to U'$ is called of class sc^2; inductively, the map $f : U \to E'$ is called of class sc^k if the sc^0-map $T^{k-1}f : T^{k-1}U \to T^{k-1}E'$ is of class sc^1. A map which is of class sc^k for every k is called **sc-smooth** or of class $\mathbf{sc}^\infty$. The basic building block which allows us to easily check the sc-differentiability of maps is the chain rule.

Proposition 2.4.2 (Chain rule). *[16, Theorem 2.16] Assume that E, F, and G are sc-smooth Banach spaces and $U \subset E$ and $V \subset F$ are open sets. Assume that $f : E \to F$, $g : V \to G$ are of class sc^1 and $f(U) = V$. Then the composition $g \circ f : U \to G$ is of class sc^1 and the tangent maps satisfy*

$$T(g \circ f) = Tg \circ Tf.$$

These definitions of sc-differentiability extend to local M-polyfolds models in the following way.

Definition 2.4.3. A map $f : O \to O'$ between two local M-polyfold models is of class sc^k if the composition $f \circ r : U \to E'$ is of class sc^k where $U \subset E$ is an open subset of the sc-Banach space E and where $r : U \to U$ is an sc-smooth retraction onto $r(U) = O$.

Given a map between two polyfolds, by writing the map in the local sc-coordinates of the M-polyfold models we can talk about its sc-differentiability. Moreover, because orbifolds and manifolds can be interpreted as finite-dimensional polyfolds, this definition also applies to maps between a polyfold and an orbifold or a manifold.

Maps Between Polyfolds

Having given an abstract definition of a polyfold in Section 2.1, we now define maps between them.

Definition 2.4.4. An $\mathbf{sc}^0$ **(or** $\mathbf{sc}^\infty$**) -functor** between two ep-groupoids

$$\hat{f} : (Z_1, \boldsymbol{Z}_1) \to (Z_2, \boldsymbol{Z}_2)$$

is a functor on groupoidal categories which moreover is an sc^0 (or sc^∞) map when considered on the object and morphism sets.

An sc^0-functor between two polyfold structures $(Z_1, \boldsymbol{Z}_1)$, $(Z_2, \boldsymbol{Z}_2)$ with underlying topological spaces $\mathcal{Z}_1$, $\mathcal{Z}_2$ induces a continuous map on the orbit spaces $|\hat{f}| : |Z|_1 \to |Z|_2$, and hence also induces a continuous map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, as illustrated in the following commutative diagram.

$$\begin{array}{ccc} |Z_1| & \xrightarrow{|\hat{f}|} & |Z_2| \\ \wr\! \mid & & \wr\! \mid \\ \mathcal{Z}_1 & \xrightarrow{f} & \mathcal{Z}_2 \end{array}$$

Definition 2.4.5 (Maps Between Polyfolds)**.** Consider two polyfold structures $(Z_1, \boldsymbol{Z}_1)$, $(Z_2, \boldsymbol{Z}_2)$ which are representatives for two polyfolds $\mathcal{Z}_1$, $\mathcal{Z}_2$. We define a **sc^0- (or sc^∞) map between polyfolds** as a continuous map

$$f : \mathcal{Z}_1 \to \mathcal{Z}_2$$

between the underlying topological spaces of the polyfolds, for which there exists an associated sc^0- (or sc^∞-) functor

$$\hat{f} : (Z_1, \boldsymbol{Z}_1) \to (Z_2, \boldsymbol{Z}_2).$$

such that $|\hat{f}|$ induces f.

Remark 2.4.6. From an abstract point of view a stronger notion of map is needed. This leads to the definition of ‘generalized maps’ between polyfold structures, following a category-theoretic localization procedure [21, Section 2.3]. Following this, a precise notion of map between two polyfolds is defined using an appropriate equivalence class of a given generalized map between two given polyfold structures [18, Definition 16.5].

With this in mind, taking an appropriate equivalence class of a given sc^0-functor between two given polyfold structures is sufficient for giving a well-defined map between two polyfolds. The following proposition follows directly from the definitions and summarizes our discussion, for further treatment we refer to [18, Remark 16.2].

sc-Differentiability for the Evaluation Maps

Proposition 2.4.7. *Consider the evaluation map at the ith-marked point, given by:*

$$\begin{aligned} ev_i : &\mathcal{Z}_{A,g,k} \to Q \\ &[\Sigma, j, M, D, u] \mapsto u(z_i) \end{aligned}$$

The evaluation map is sc-smooth.

Proof. We check the sc-smoothness at an arbitrary point $[\alpha] = [\Sigma, j, M, D, u] \in \mathcal{Z}_{A,g,k}$. Let $(a, v, \eta) \in \mathcal{O}$ be local sc-coordinates centered at a representative $\alpha = (\Sigma, j, M, D, u)$ of $[\alpha]$, and let U be local coordinates centered at $u(z_i) \in M$. Noticing that the gluing construction is the identity at the marked points, the local expression for a retraction composed with the evaluation map is

$$(\underline{a}, v, \eta) \mapsto (\underline{a}, v, \pi_{\underline{a}}(\eta)) \mapsto \exp_u(\eta)(z_i) = \exp_u(\eta).$$

which is sc-smooth. □

sc-Differentiability for the Projection Maps

Here we consider the projection maps from the GW-polyfolds to the Deligne–Mumford orbifolds. These maps are sometimes called 'forgetful maps' in the literature. These maps are defined by taking a stable curve and, after removing unstable components, associating the underlying stable domain

$$\pi : \mathcal{Z}_{A,g,k} \to \overline{\mathcal{M}}_{g,k}, \quad [\Sigma, j, M, D, u] \mapsto [(\Sigma, j, M, D)_{stab}].$$

We describe this process on the level of the underlying sets as follows. Take $[\alpha] \in \mathcal{Z}_{A,g,k}$, and take a representative (Σ, j, M, D, u) of $[\alpha]$. First forget the map u and consider, if it exists, a component C satisfying $2g_C + \sharp(M \cup |D|)_C < 3$. Then we have the following cases.

1. C is a sphere without marked points and with one nodal point, say x. Then we remove the sphere, the nodal point x and its partner y, where $\{x, y\} \in D$.
2. C is a sphere with two nodal points. In this case there are two nodal pairs $\{x, y\}$ and $\{x', y'\}$, where x and x' lie on the sphere. We remove the sphere and the two nodal pairs but add the nodal pair $\{y, y'\}$.
3. C is a sphere with one node and one marked point. In that case we remove the sphere but replace the corresponding nodal point on the other component by the marked point.

Once we have removed all unstable components in this manner, we end up with a stable noded marked Riemann surface we denote as $(\Sigma, j, M, D)_{stab}$.

Proposition 2.4.8. *The projection map*

$$\begin{aligned} \pi : &\mathcal{Z}_{A,g,k} \to \overline{\mathcal{M}}^{log}_{g,k} \\ &[(\Sigma, j, M, D, u)] \mapsto [(\Sigma, j, M, D)_{stab}] \end{aligned}$$

defined between the GW-polyfold and the logarithmic DM-orbifold is sc-smooth.

Proof. Consider an arbitrary point $[\alpha] = [\Sigma, j, M, D, u] \in \mathcal{Z}_{A,g,k}$. Given a representative $\alpha = (\Sigma, j, M, D, u)$ of $[\alpha]$, a good uniformizing family of stable maps gives us local sc-coordinates. The construction of a good uniformizing family of stable maps requires we add additional marked points (a stabilization S) to our underlying Riemann surface to make it stable. Thus, in addition to our good uniformizing family of stable maps

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}} \exp_u(\eta)),$$

we also have an associated good uniformizing family

$$(\underline{a}, v) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \cup S)_{\underline{a}}, D_{\underline{a}})$$

of stable Riemann surfaces. Crucially, both of these families use the exponential gluing profile in their construction.

We may write a local expression the projection map as a composition in the following way. First, we forget the coordinate η from $(\underline{a}, v, \eta)$, and keep only the coordinates for the stabilized Riemann surface $(\underline{a}, v)$ (this map is sc-smooth). We then forget all of the marked points in S, which we may write as a composition of maps forgetting a single marked point (as S is finite). Having forgotten the coordinate η, if our final target is the logarithmic Deligne–Mumford space, we compose with the following sequence of smooth maps

$$\overline{\mathcal{M}}^{\exp}_{g,k+s} \xrightarrow{id} \overline{\mathcal{M}}^{\log}_{g,k+s} \xrightarrow{ft} \overline{\mathcal{M}}^{\log}_{g,k+s-1} \xrightarrow{ft} \cdots \xrightarrow{ft} \overline{\mathcal{M}}^{\log}_{g,k}$$

which is smooth.

□

sc-Differentiability for the Permutation Maps

Here we consider the sc-differentiability of the permutation maps, which permute the marked points.

Proposition 2.4.9. *Fix a permutation $\sigma \in S_k$, $\sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$. Consider the permutation map*

$$\sigma : \mathcal{Z}_{A,g,k} \to \mathcal{Z}_{A,g,k}, \ [\Sigma, j, M, D, u] \mapsto [\Sigma, j, M^\sigma, D, u]$$

where $M = \{z_1, \ldots, z_k\}$ and where $M^\sigma := \{z'_1, \ldots, z'_k\}$, $z'_i := z_{\sigma(i)}$. This map is sc-smooth.

Proof. Consider a stable curve $[\Sigma, j, M, D, u]$ which maps to the stable curve $[\Sigma, j, M^\sigma, D, u]$. Let (Σ, j, M, D, u) and $(\Sigma, j, M^\sigma, D, u)$ be stable map representatives.

As described in Section 2.3, we may choose good data and a stabilization S for (Σ, j, M, D, u). This determines a good uniformizing family of stable maps centered at (Σ, j, M, D, u), given by

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}} \cup S_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}} \exp_u(\eta)).$$

We may use the same exact good data and stabilization S for $(\Sigma, j, M^\sigma, D, u)$. This determines a good uniformizing family of stable maps centered at $(\Sigma, j, M^\sigma, D, u)$, given by

$$(\underline{a}', v', \eta') \mapsto (\Sigma_{\underline{a}'}, j(\underline{a}', v'), M^\sigma_{\underline{a}'} \cup S_{\underline{a}'}, D_{\underline{a}'}, \oplus_{\underline{a}'} \exp_u(\eta')).$$

We now compare the good uniformizing families of stable maps, and deduce

$$\begin{aligned} &\hat{\sigma}(\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}} \cup S_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}} \exp_u(\eta)) \\ &= (\Sigma_{\underline{a}'}, j(\underline{a}', v'), M^\sigma_{\underline{a}'} \cup S_{\underline{a}'}, D_{\underline{a}'}, \oplus_{\underline{a}'} \exp_u(\eta')) \end{aligned}$$

precisely when $\underline{a}' = \underline{a}$, $v' = v$, and $\eta' = \eta$. Thus in the coordinates of such a good uniformizing family the permutation is the identity, and is sc-smooth. □

2.5 The kth-Marked Point Forgetting Maps on the Deligne–Mumford Spaces

Definition 2.5.1. Suppose that $2g+k>3$. We define the kth-marked point forgetting map

$$ft_k : \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k-1}$$

on the underlying sets of the DM-spaces as follows. Let $[\Sigma, j, M, D] \in \overline{\mathcal{M}}_{g,k}$, and let a representative be given by a stable noded Riemann surface (Σ, j, M, D). To define $ft_k([\Sigma, j, M, D])$ we distinguish three cases for the component C_k which contains the kth-marked point, $z_k \in C_k$.

The component $C_k \setminus \{z_k\}$ satisfies the DM-stability condition.

I. It follows that $2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} > 3$ and we define

$$ft_k([\Sigma, j, M, D]) = [\Sigma, j, M \setminus \{z_k\}, D].$$

The component $C_k \setminus \{z_k\}$ does not satisfy the DM-stability condition. It follows that $2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} = 3$. From $2g+k>3$ and the assumption that the set $\Sigma/\sim$ (where $x_a \sim y_a$ for nodal pairs $\{x_a, y_a\} \in D$) is connected, we can conclude

$$2 \cdot g(C_k) = 0, \qquad \sharp(M \cup |D|)_{C_k} = 3.$$

There are now two possibilities.

II. $z_k \in M_{C_k}$, and two nodal points $z_a, z_b \in |D|_{C_k}$ Then $ft_k([\Sigma, j, M, D])$ is defined as the equivalence class of the stable noded Riemann surface obtained from (Σ, j, M, D) as follows. Delete z_k, delete the component C_k, and delete the two nodal pairs. We add a new nodal pair $\{x_a, y_b\}$ given by two points of the former nodal pairs.

III. $z_i, z_k \in M_{C_k}$, and one nodal point $z_a \in |D|_{C_k}$ Then $ft_k([\Sigma, j, M, D])$ is defined as the equivalence class of the stable noded Riemann surface obtained from (Σ, j, M, D) as follows. Delete z_k, delete the component C_k, and delete the nodal pair. We add a new marked point z_i, given by the former nodal point which did not lie on C_k.

Remark 2.5.2. By definition, if $2g+(k-1)<3$ then the DM-space $\overline{\mathcal{M}}_{g,k-1} = \emptyset$; hence, one can also consider the trivially defined maps $ft_k : \overline{\mathcal{M}}_{g,k} \to \emptyset$.

Remark 2.5.3. By considering possible stable configurations, one can easily see that ft_k is given by case I whenever $[\Sigma, j, M, D]$ lies on the top dimensional stratum of the DM-space. Additionally, there will exist a point $[\Sigma, j, M, D]$ with the configuration specified by case II whenever

$$(g=0, k\geq 5), \qquad (g=1, k=1), \qquad (g\geq 2, k\geq 1).$$

Finally, there will exist a point $[\Sigma, j, M, D]$ with the configuration specified by case III whenever

$$(g = 0, k \geq 4), \qquad (g \geq 1, k \geq 2).$$

Remark 2.5.4 (The Universal Curve). The preimage of ft_k at a a point $[\Sigma, j, M, D] \in \overline{\mathcal{M}}_{g,k-1}$ consists of the Riemann surface Σ with nodes identified, i.e.

$$ft_k^{-1}([\Sigma, j, M, D]) \simeq \Sigma / \sim, \qquad \text{where } x_a \sim y_a \text{ for nodal pairs } \{x_a, y_a\} \in D.$$

According to such a description $\overline{\mathcal{M}}_{g,k}$ is sometimes called the **universal curve** over $\overline{\mathcal{M}}_{g,k-1}$. It is important to note that this is not a fiber bundle (the 'fibers' are not constant and can vary locally), nor is it a fibration (the 'fibers' are not homotopy equivalent). However, considered as homology classes, the fibers are homologous.

Local Expressions for the kth-Marked Point Forgetting Map on the Deligne–Mumford Spaces

Having defined ft_k on the underlying sets of the DM-spaces, we now write down local expressions for ft_k in the coordinates given by the (alternative) good uniformizing families.

Let $[\Sigma, j, M, D] \in \overline{\mathcal{M}}_{g,k}$, and let a representative be given by a stable noded Riemann surface (Σ, j, M, D).

Case I

The component $C_k \setminus \{z_k\}$ satisfies the DM-stability condition, hence $2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} > 3$ and

$$ft_k([\Sigma, j, M, D]) = [\Sigma, j, M \setminus \{z_k\}, D].$$

Let $(\underline{a}, v, z) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \setminus \{z_k\})_{\underline{a}} \cup \{\phi(z)\}_{\underline{a}}, D_{\underline{a}})$ be an alternative good uniformizing family centered at α. We may observe that $(\underline{a}, v) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \setminus \{z_k\})_{\underline{a}}, D_{\underline{a}})$ is a good uniformizing family centered at $\hat{ft}_k(\alpha)$

Therefore, considered as a map between stable noded Riemann surfaces, $\hat{ft}_k$ is given by

$$\hat{ft}_k : (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \setminus \{z_k\})_{\underline{a}} \cup \{\phi(z)\}_{\underline{a}}, D_{\underline{a}}) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \setminus \{z_k\})_{\underline{a}}, D_{\underline{a}})$$

Hence considered as a map between the coordinates given by the these good uniformizing families, a local expression for $\hat{ft}_k$ is given by

$$\hat{ft}_k : (\underline{a}, v, z) \mapsto (\underline{a}, v).$$

Case II

The component $C_k \setminus \{z_k\}$ does not satisfy the DM-stability condition, hence by the above $2 \cdot g(C_k) = 0, \sharp(M \cup |D|)_{C_k} = 3$. We suppose that $[\Sigma, j, M, D, u]$ is of case II, hence the component C_k contains the marked point z_k, and there exist two nodal pairs $\{x_a, y_a\}$ and $\{x_b, y_b\}$ such that $y_a, x_b \in C_k$.

Then $ft_k([\Sigma, j, M, D])$ is defined as the equivalence class of the stable noded Riemann surface obtained from $[\Sigma, j, M, D]$ as follows. Delete z_k, delete the component C_k, and delete the two nodal pairs. We add a new nodal pair $\{x_c, y_c\} := \{x_a, y_b\}$ given by two points of the former nodal pairs.

For simplicity, let us assume that $\{x_a, y_a\}$ and $\{x_b, y_b\}$ are the only nodal pairs on $[\Sigma, j, M, D]$ and hence $\{x_c, y_c\}$ is the only nodal pair on $ft_k([\Sigma, j, M, D])$. We make this assumption entirely on the basis of notational convenience and clarity. The general case can then easily be seen to follow from this description. Hence we assume $[\alpha] = [\Sigma, j, M, \{\{x_a, y_a\}, \{x_b, y_b\}\}]$ and that $ft_k([\alpha]) = [\Sigma \setminus C_k, j, M \setminus \{z_k\}, \{\{x_c, y_c\}\}]$.

There is a unique biholomorphism between $C_k \setminus \{y_a, x_b\}$ and $\mathbb{R} \times S^1$ which sends the marked point z_k to the point $(0, 0)$, the puncture y_a to $-\infty$, and the puncture x_b to $+\infty$. We may choose the small disk structure at y_a such that there is a biholomorphism between $D_{y_a} \setminus \{y_a\}$ and $\mathbb{R}^- \times S^1 \subset \mathbb{R} \times S^1$. Likewise, we choose the small disk structure at x_b such that there is a biholomorphism between $D_{x_b} \setminus \{x_b\}$ and $\mathbb{R}^+ \times S^1$.

Let $(a, b, v) \mapsto (\Sigma_{a,b}, j(a, b, v), M_{a,b}, \{\{x_a, y_a\}, \{x_b, y_b\}_{a,b})$ be a good uniformizing family centered at $(\Sigma, j, M, \{\{x_a, y_a\}, \{x_b, y_b\}\})$. Explicitly, this family is given by the following.

- The glued Riemann surface $\Sigma_{a,b}$ is given by
$$\Sigma_{a,b} = (\Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b})) \sqcup Z_a \sqcup Z_b.$$
- The good complex deformation is given by
$$j(a, b, v) = \begin{cases} j(v) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b}) \\ i & \text{on } Z_a \sqcup Z_b. \end{cases}$$
- The marked points $z_1, \ldots, z_{k-1} \in M_{a,b}$ are given by the former marked points $\{z_1, \ldots, z_{k-1}\} \subset \Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b})$. The marked point $z_k \in M_{a,b}$ is given by
$$z_k = \begin{cases} (0, 0) \in \mathbb{R} \times S^1 & \text{when } a = 0 \text{ and } b = 0 \\ (\varphi(r_a), \theta_a) \in Z_a & \text{when } a \neq 0 \text{ and } b = 0 \\ (0, 0) \in Z_b & \text{when } a = 0 \text{ and } b \neq 0 \\ (\varphi(r_a), \theta_a) \in Z_a (\text{equivalently, } (0, 0) \in Z_b) & \text{when } a \neq 0 \text{ and } b \neq 0. \end{cases}$$
- The set of nodal pairs $\{\{x_a, y_a\}, \{x_b, y_b\}_{a,b}$ is given by
$$\{\{x_a, y_a\}, \{x_b, y_b\}\}_{a,b} = \begin{cases} \{\{x_a, y_a\}, \{x_b, y_b\}\} & \text{when } a = 0 \text{ and } b = 0 \\ \{\{x_b, y_b\}\} & \text{when } a \neq 0 \text{ and } b = 0 \\ \{\{x_a, y_a\}\} & \text{when } a = 0 \text{ and } b \neq 0 \\ \emptyset & \text{when } a \neq 0 \text{ and } b \neq 0. \end{cases}$$

Let $(c, v') \mapsto ((\Sigma \setminus C_k)_c, j(c, v'), (M \setminus \{z_k\})_c, \{\{x_c, y_c\}\}_c)$ be a good uniformizing family centered at $\hat{ft}_k(\Sigma, j, M, \{\{x_a, y_a\}, \{x_b, y_b\}\}) = (\Sigma \setminus C_k, j, M \setminus \{z_k\}, \{\{x_c, y_c\}\})$. Explicitly, this family is given by the following.

- The glued Riemann surface $(\Sigma \setminus C_k)_c$ is given by

$$(\Sigma \setminus C_k)_c = (\Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c})) \sqcup Z_c$$

- The good complex deformation is the same as above on $\Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c})$, and is given by

$$j(a, b, v') = \begin{cases} j(v') & \text{on } \Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c}) \\ i & \text{on } Z_c \end{cases}$$

- The marked points $z_1, \ldots, z_{k-1} \in M_{a,b}$ are given by the former marked points $\{z_1, \ldots, z_{k-1}\} \subset \Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c})$.
- The set of nodal pairs $\{\{x_c, y_c\}\}_c$ is given by $\{\{x_c, y_c\}\}$ when $c = 0$ and $\emptyset$ when $c \neq 0$.

Comparing the good uniformizing families, we see

$$\hat{ft}_k(\Sigma_{a,b}, j(a, b, v), M_{a,b}, \{\{x_a, y_a\}, \{x_b, y_b\}\}_{a,b}) = ((\Sigma \setminus C_k)_c, j(c, v'), (M \setminus \{z_k\})_c, \{x_c, y_c\}_c)$$

precisely when $v' = v$ and when $c = a *_\varphi b$, where

$$a *_\varphi b := \begin{cases} \varphi^{-1}(\varphi(r_a) + \varphi(r_b)) e^{-2\pi i(\theta_a + \theta_b)} & \text{when } a \neq 0 \text{ and } b \neq 0 \\ 0 & \text{when } a = 0 \text{ or } b = 0. \end{cases}$$

Therefore, considered as a map between the coordinates given by the these good uniformizing families, a local expression for $\hat{ft}_k$ is given by

$$\hat{ft}_k : (a, b, v) \mapsto (a *_\varphi b, v).$$

Example 2.5.5. An easy illustration of the above case is to consider the map

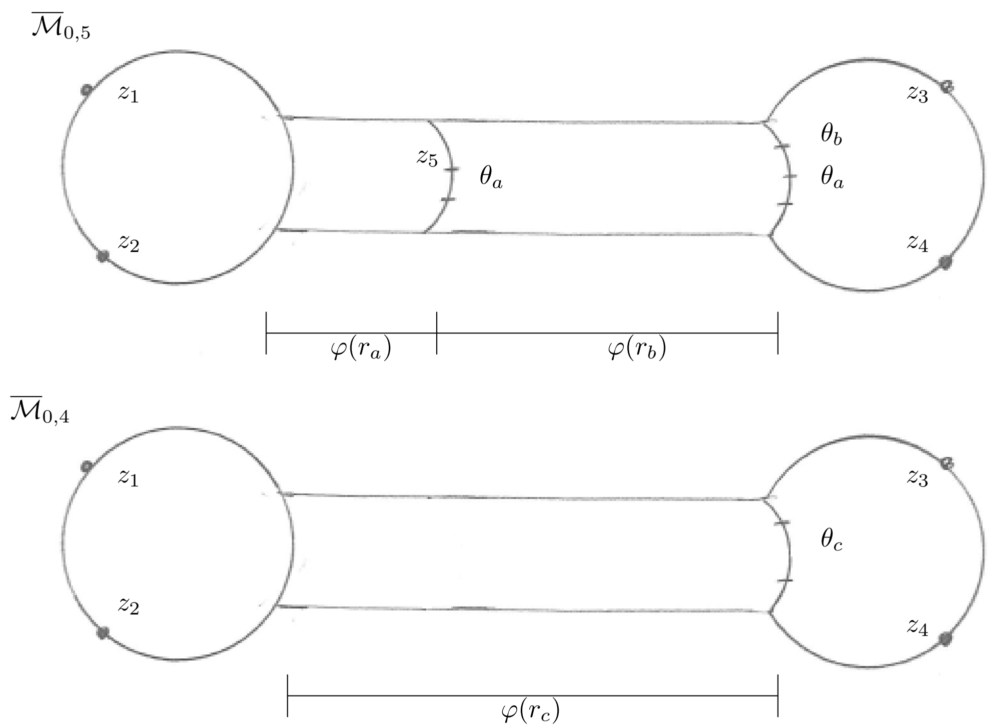
$$ft_5 : \overline{\mathcal{M}}_{0,5} \to \overline{\mathcal{M}}_{0,4}$$

as depicted in the figure below.

Case III

The component $C_k \setminus \{z_k\}$ does not satisfy the DM-stability condition, hence by the above $2 \cdot g(C_k) = 0, \sharp(M \cup |D|)_{C_k} = 3$. We suppose that $[\Sigma, j, M, D]$ is of case III, hence the component C_k contains the marked point z_k together with another marked point z_i, and there exist a nodal pairs $\{x_a, y_a\}$ such that $y_a \in C_k$.

Figure 2.1: The 5th-Marked Point Forgetting Map on the Deligne–Mumford Orbifolds



Then $ft_k([\Sigma, j, M, D])$ is defined as the equivalence class of the stable noded Riemann surface obtained from $[\Sigma, j, M, D]$ as follows. Delete the marked points z_i and z_k, delete the component C_k, and delete the nodal pair. We add a new marked point z_i, given by the former nodal point which did not lie on C_k, i.e. $\{x_a\}$.

For simplicity, let us assume that $\{x_a, y_a\}$ is the only nodal pair on $[\Sigma, j, M, D]$ and hence $ft_k([\Sigma, j, M, D])$ contains no nodal pairs. We make this assumption entirely on the basis of notational convenience and clarity. The general case can then easily be seen to follow from this description. Hence we assume $[\alpha] = [\Sigma, j, M, \{\{x_a, y_a\}\}]$ and that $ft_k([\alpha]) = [\Sigma \setminus C_k, j, M \setminus \{z_k\}, \emptyset]$.

There is a unique biholomorphism between $C_k \setminus \{y_a, z_i\}$ and $\mathbb{R} \times S^1$ which sends the marked point z_k to the point $(0, 0)$, the puncture y_a to $-\infty$, and the puncture at the marked point z_i to $+\infty$. We may choose the small disk structure at y_a such that there is a biholomorphism between $D_{y_a} \setminus \{y_a\}$ and $\mathbb{R}^- \times S^1 \subset \mathbb{R} \times S^1$.

Let $(a, v) \mapsto (\Sigma_a, j(a, v), M_a, \{\{x_a, y_a\}\}_a)$ be a good uniformizing family centered at $(\Sigma, j, M, \{\{x_a, y_a\}\})$. Explicitly, this family is given by the following.

- The glued Riemann surface Σ_a is given by

$$\Sigma_a = (\Sigma \setminus (D_{x_a} \sqcup C_k)) \sqcup \mathbb{R}^+ \times S^1 \sqcup \{+\infty\}.$$

- The good complex deformation is given by

$$j(a,v) = \begin{cases} j(v) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k) \\ i & \text{on } \mathbb{R}^+ \times S^1. \end{cases}$$

- The marked points $z_1, \ldots, \hat{z}_i, \ldots, z_{k-1} \in M_a$ (i.e. z_i is omitted) are given by the former marked points $\{z_1, \ldots, \hat{z}_i, \ldots, z_{k-1}\} \subset \Sigma \setminus (D_{x_a} \sqcup C_k)$. The marked point $z_k \in M_a$ is given by

$$z_k = \begin{cases} (0,0) \in \mathbb{R} \times S^1 & \text{when } a = 0 \\ (\varphi(r_a), \theta_a) \in \mathbb{R}^+ \times S^1) & \text{when } a \neq 0. \end{cases}$$

  Meanwhile, the marked point $z_i \in M_a$ is given by the point $\{+\infty\}$

- The set of nodal pairs $\{\{x_a, y_a\}\}_a$ is given by $\{\{x_a, y_a\}\}$ when $a = 0$ and $\emptyset$ when $a \neq 0$.

Let $(v') \mapsto (\Sigma \setminus C_k, j(v'), (M \setminus \{z_k\}), \emptyset)$ be a good uniformizing family centered at $\hat{ft}_k(\Sigma, j, M, \{\{x_a, y_a\}\}) = (\Sigma \setminus C_k, j, M \setminus \{z_k\}, \emptyset)$. Explicitly, this family is given by the following.

- The Riemann surface is $\Sigma \setminus C_k$. We note that we may identify the neighborhoods $D_{z_i} \setminus \{z_i\} \simeq D_{x_a} \setminus \{z_i\} \simeq \mathbb{R}^+ \times S^1$; this sends the marked point z_i to $+\infty$.

- The good complex deformation can be taken the same as above, i.e.

$$j(v') = \begin{cases} j(v') & \text{on } \Sigma \setminus D_{z_i} \\ i & \text{on } \mathbb{R}^+ \times S^1 \end{cases}$$

  using the identification of $D_{z_i} \setminus \{z_i\}$ with $\mathbb{R}^+ \times S^1$.

- The marked points $z_1, \ldots, \hat{z}_i, \ldots, z_{k-1} \in M$, while z_i is identified with $+\infty$ via the identification $D_{z_i} \setminus \{z_i\} \simeq \mathbb{R}^+ \times S^1$

- The set of nodal pairs is the empty set.

Comparing the good uniformizing families, we see

$$\hat{ft}_k(\Sigma_a, j(a,v), M_a, \{\{x_a, y_a\}\}_a) = (\Sigma \setminus C_k, j(v'), M \setminus \{z_k\}, \emptyset)$$

precisely when $v' = v$. It follows that a local expression for $\hat{ft}_k$ is given by

$$\hat{ft}_k : (a,v) \mapsto (v).$$

Proposition 2.5.6. *The kth-point forgetting map, considered on the Deligne–Mumford orbifolds constructed with the logarithimic gluing profile,*

$$ft_k : \overline{\mathcal{M}}^{log}_{g,k} \to \overline{\mathcal{M}}^{log}_{g,k-1},$$

is a smooth map. Moreover, it is holomorphic.

Proof. We check the local expressions of ft_k in coordinate charts. The only points $[\alpha] \in \overline{\mathcal{M}}^{\log}_{g,k}$ where the local form of ft_k will not be trivially holomorphic is when $[\alpha]$ contains a component S^2 with precisely 3 special points, one of which is the kth-marked point while the other two are nodal points. The inverse of $\varphi_{\log}(r) = -\frac{1}{2\pi}\log(r)$ is given by $\varphi^{-1}_{\log}(R) = e^{-2\pi R}$. Hence it follows that:

$$a *_{\log} b = r_a \cdot r_b e^{-2\pi i(\theta_a+\theta_b)} = a \cdot b$$

i.e. complex multiplication of the gluing parameters, a holomorphic map. □

The topology on the DM-space is independent of the choice of gluing profile, hence we immediately obtain the following corollary.

Corollary 2.5.7. *The kth-marked point forgetting map defined on the exponential Deligne–Mumford orbifolds,*

$$ft_k : \overline{\mathcal{M}}^{exp}_{g,k} \to \overline{\mathcal{M}}^{exp}_{g,k-1},$$

is continuous.

Remark 2.5.8. If $(g = 0, k \geq 5), (g = 1, k = 1), (g \geq 2, k \geq 1)$ then the kth-marked point forgetting map

$$ft_k : \overline{\mathcal{M}}^{\exp}_{g,k} \to \overline{\mathcal{M}}^{\exp}_{g,k-1}$$

is not C^1. Differentiability fails at points $[\alpha] \in \overline{\mathcal{M}}^{\exp}_{g,k}$ which contain precisely 3 special points, one of which is the kth-marked point while the other two are nodal points. A coordinate expression for ft_k at such a point is given by Case II above:

$$\hat{ft}_k : (a, b, v) \mapsto (a *_{\exp} b, v).$$

Hence consider the function

$$\begin{aligned} &\mathbb{C} \times \mathbb{C} \to \mathbb{C} \\ &(a, b) \mapsto a *_{\exp} b \end{aligned}$$

where once again

$$a *_{\exp} b := \begin{cases} \varphi^{-1}_{\exp}(\varphi_{\exp}(r_a) + \varphi_{\exp}(r_b))e^{-2\pi i(\theta_a+\theta_b)} & \text{when } a \neq 0 \text{ and } b \neq 0 \\ 0 & \text{when } a = 0 \text{ or } b = 0. \end{cases}$$

The inverse of $\varphi_{\exp}(r) = e^{1/r} - e$ is given by $\varphi^{-1}_{\exp}(R) = \frac{1}{\log(R+e)}$, and so for $a \neq 0$ and $b \neq 0$

$$a *_{\exp} b = \frac{1}{\log\left(e^{1/r_a} + e^{1/r_b} - e\right)} e^{-2\pi i(\theta_a+\theta_b)}.$$

This expression is not C^1. To see this, we rewrite the equation in rectangular coordinates as a function $F : \mathbb{R}^4 \to \mathbb{R}^2$,

$$F : (x_1, x_2, x_3, y_4) \mapsto \begin{cases} 0 & \text{when } x_1 = x_2 = 0 \text{ or } x_3 = x_4 = 0 \\ \frac{1}{\log\left(e^{1/r_a} + e^{1/r_b} - e\right)}(\cos(\vartheta_a + \vartheta_b), \sin(\vartheta_a + \vartheta_b)) & \text{otherwise} \end{cases}$$

where $r_a^2 = x_1^2 + x_2^2, r_b^2 = x_3^2 + x_4^2$ and $\vartheta_a = -2\pi \tan^{-1}(\frac{x_2}{x_1}), \vartheta_b = -2\pi \tan^{-1}(\frac{x_4}{x_3})$. We now compute the Jacobian matrix $\boldsymbol{J}$ of partial derivatives of F at $(0,0,0,0) \in \mathbb{R}^4$. From the above expression for F we see that $\frac{\partial F_i}{\partial x_j} = 0$ for all $i = 1,2$ and $j = 1,2,3,4$, hence $\boldsymbol{J}_{i,j} = 0$.

If F were differentiable, the Jacobian matrix would give the total derivative, and we could compute the directional derivative at $(0,0,0,0)$ via the equation

$$\nabla_v F = \boldsymbol{J} \cdot v = (0,0).$$

We may directly compute the directional derivative as follows. Let $v \in \mathbb{R}^4$ be a unit vector, we may write

$$v = r_1 \cos\theta_1 \tfrac{\partial}{\partial x_1} + r_1 \sin\theta_1 \tfrac{\partial}{\partial x_2} + r_2 \cos\theta_2 \tfrac{\partial}{\partial x_3} + r_2 \cos\theta_2 \tfrac{\partial}{\partial x_4}$$

where $r_1^2 + r_2^2 = 1$. Then

$$\begin{aligned}\nabla_v F &= \lim_{h\to 0} \frac{F(0+hv)}{h} \\ &= \lim_{h\to 0} \frac{1}{h \log\left(e^{1/(r_1 h)} + e^{1/(r_2 h)} - e\right)} (\cos(\theta_1+\theta_2), \sin(\theta_1+\theta_2)) \\ &= \min\{r_1, r_2\}(\cos(\theta_1+\theta_2), \sin(\theta_1+\theta_2))\end{aligned}$$

This contradicts the assumption that F was differentiable.

The lth-Marked Point Doubling Map

Definition 2.5.9. For $l \in \{1,\ldots,k-1\}$, we define the **lth-marked point doubling map**

$$s_l : \overline{\mathcal{M}}_{g,k-1} \to \overline{\mathcal{M}}_{g,k}$$

on the Deligne–Mumford spaces by introducing a new component S^2 which contains the marked points z_l and z_k. It can be explicitly described as follows. For an equivalence class of stable noded Riemann surfaces $[\Sigma, j, M, D] \in \overline{\mathcal{M}}_{g,k-1}$, we define $s_l([\Sigma, j, M, D]) = [\Sigma \sqcup S^2, \tilde{j}, \tilde{M}, \tilde{D}]$ by the following data.

- The complex structure is given by

$$\tilde{j} := \begin{cases} j & \text{on } \Sigma \\ i & \text{on } S^2 \end{cases}$$

  where i is the standard complex structure on S^2,

- The marked points $\{z_1, \ldots, \hat{z}_l, \ldots, z_{k-1}\} \in \tilde{M}$ are given by the former marked points, while the marked points z_l and z_k are given by two disjoint points on S^2,

- The set of nodal pairs $\tilde{D}$ is given by $D \sqcup \{x_a, y_a\}$ where $x_a \in \Sigma$ is given by the former lth-marked point while $y_a \in S^2$ is disjoint from $\{z_l, z_k\} \in S^2$.

This map is frequently called a section of $ft_k : \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k-1}$ due to the fact that $ft_k \circ s_l = \mathrm{id}_{\overline{\mathcal{M}}_{g,k-1}}$. For any choice of gluing profile, this map is a smooth embedding of the orbifold $\overline{\mathcal{M}}_{g,k-1}$ into $\overline{\mathcal{M}}_{g,k}$.

2.6 The kth-Marked Point Forgetting Map on the Gromov–Witten Polyfolds

Definition 2.6.1. Consider a stable curve $[\alpha] = [\Sigma, j, M, D, u]$ with k marked points, and let $\alpha = (\Sigma, j, M, D, u)$ be a stable map representative. We say that the stable curve $[\alpha]$ and the stable map α contains a **destabilizing ghost component** if the connected component $C_k \subset \Sigma$ with $z_k \in C_k$ satisfies

$$2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} = 3 \quad \text{and} \quad \int_{C_k} u^*\omega = 0.$$

We classify destabilizing ghost components as follows:

- $g(C_k) = 0, \quad \sharp(M \cup |D|)_{C_k} = 3$

Type I. $z_k \in M_{C_k}$, and two nodal points $z_a, z_b \in |D|_{C_k}$

Type II. $z_i, z_k \in M_{C_k}$, and one nodal point $z_a \in |D|_{C_k}$

Type III. $z_i, z_j, z_k \in M_{C_k}$. It follows that C_k is the only component and hence $[A] = 0$. This situation can only arise when $(A, g, k) = (0, 0, 3)$.

- $g(C_k) = 1, \quad \sharp(M \cup |D|)_{C_k} = 1$

Type IV. $z_k \in M_{C_k}$ and no other special points. It follows that C_k is the only component and hence $[A] = 0$. This situation can only arise when $(A, g, k) = (0, 1, 1)$.

Remark 2.6.2. By definition, the set of stable curves $\mathcal{Z}_{0,0,2} = \emptyset$ and $\mathcal{Z}_{0,1,0} = \emptyset$ (this is because the stability condition* (2.1) cannot be satisfied). It follows that in the fringe cases of III and IV there exist trivially well-defined and sc-smooth maps $ft_3 : \mathcal{Z}_{0,0,3} \to \emptyset$ and $ft_1 : \mathcal{Z}_{0,1,1} \to \emptyset$.

Remark 2.6.3. By considering possible configurations in the DM-spaces, one can show that a destabilizing ghost component of type I will always arise in the following situations

$$(A, g = 0, k \geq 5), \qquad (A, g = 1, k \geq 2), \qquad (A, g \geq 2, k \geq 1).$$

One can also show that a destabilizing ghost component of type II will always arise in the following situations

$$(A, g = 0, k \geq 4), \qquad (A, g \geq 1, k \geq 2).$$

The existence of destabilizing ghost components of type I and II are therefore very common.

Remark 2.6.4. Suppose that $(A, g, k) \neq (0,0,2),(0,0,1),(0,0,0),(0,1,0)$ and $(A, g, k) \neq (0,0,3),(0,1,1)$ (in these cases one can consider trivially well-defined kth-marked point forgetting maps, see Remark 2.3.4 and Remark 2.6.2, respectively).

There does not exist a natural definition of the kth-marked point forgetting map between the GW-polyfolds $\mathcal{Z}_{A,g,k}$ and $\mathcal{Z}_{A,g,k-1}$.

To see this, consider a stable curve $[\Sigma, j, M, D, u]$ which contains a destabilizing ghost component of type I. Suppose that $u|_{C_k} \neq$ constant. Then if we forget the kth-marked point, the component C_k no longer satisfies the stability condition* (2.1). We also cannot delete the component C_k, as this will led to a disconnected noded Riemann surface.

We can attempt to restrict to a subspace of $\mathcal{Z}_{A,g,k}$ on which there is a naturally defined kth-marked point forgetting map. In particular, consider the subset $\mathcal{Z}^{\text{const}}_{A,g,k} \subset \mathcal{Z}_{A,g,k}$ where if the kth-marked point z_k lies on a component C_k with

$$2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} = 3 \quad \text{and} \quad \int_{C_k} u^*\omega = 0$$

then we require that $u|_{C_k}$ is constant, hence necessarily $u|_{C_k} \equiv u(z_k)$.

Proposition 2.6.5. *Consider the subset $\mathcal{Z}^{const}_{A,g,k} \subset \mathcal{Z}_{A,g,k}$ just described, and equip the underlying set with the subspace topology. We can define a kth-marked point forgetting map*

$$ft_k : \mathcal{Z}^{const}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}.$$

If $\mathcal{Z}^{const}_{A,g,k}$ contains a destabilizing ghost component of type I or of type II this map is not continuous.

Proof. Failure of continuity occurs at stable curves $[\alpha] \in \mathcal{Z}^{\text{const}}_{A,g,k}$ which contain a component S^2 with precisely 3 special points, one of which is the kth-marked point, and such that $\int_{S^2} u^*\omega = 0$. We demonstrate lack of continuity by exhibiting a sequence which converges to such a stable curve $[\alpha]$ but for which the image of the sequence does not converge. For simplicity, we will assume that this $[\alpha]$ satisfies the following:

- constant on a region surrounding the component S^2 (this simplifies the local forms for ft_k since our reference curves are now constant),
- the two other special points on S^2 are both nodal points.

Consider an sc-Banach space which consists of a gluing parameter $a \in B_{\frac{1}{2}} \subset \mathbb{C}$ and of maps

$$\eta^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}, \qquad \eta^0 : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$$

which converge to asymptotic constants

$$\lim_{s\to\infty} \eta^+ = \lim_{s\to-\infty} \eta^0 = c \qquad \text{and} \qquad \lim_{s\to\infty} \eta^0 = c'.$$

On the level m we give this space the following norm:

$$\begin{aligned}|(a,\eta^+,\eta^0)|_m^2 =& |a|^2+|c|^2+|c'|^2+\sum_{|\alpha|\le m+3}\int_{\mathbb{R}^+\times S^1}|D^\alpha(\eta^+-c)|^2 e^{2\delta_m|s|}dsdt\\ &+\sum_{|\alpha|\le m+3}\int_{\mathbb{R}^-\times S^1}|D^\alpha(\eta^0-c)|^2 e^{2\delta_m|s|}dsdt\\ &+\sum_{|\alpha|\le m+3}\int_{\mathbb{R}^+\times S^1}|D^\alpha(\eta^0-c')|^2 e^{2\delta_m|s|}dsdt.\end{aligned}$$

We now construct the sequence. Choose a smooth cut-off function $\beta:\mathbb{R}\to[0,1]$ having the following properties:

- $\beta(s)=1$ for all $-\frac{1}{2}\le s\le\frac{1}{2}$
- $\beta(s)=0$ for all $-1\le s\le 1$

Choose a vector $v\in\mathbb{R}^{2n}$. Then define a vector field $\gamma:\mathbb{R}\times S^1\to\mathbb{R}^n$ by:

$$\gamma(s,t):=\beta(s)\cdot v$$

We may renormalize v by a constant so that:

$$\sum_{|\alpha|\le 3}\int_{\mathbb{R}\times S^1}|D^\alpha\gamma(s,t)|^2 e^{2\delta_0|s|}\ dsdt=1.$$

Then $\xi_n:=\frac{1}{\sqrt{n}}\gamma$ is a vector field such that

$$\sum_{|\alpha|\le 3}\int_{\mathbb{R}\times S^1}|D^\alpha\xi_n(s,t)|^2 e^{2\delta_0|s|}\ dsdt=\tfrac{1}{n}.$$

Now choose $a_n\in B_{\frac{1}{2}}$ small enough such that $e^{2\delta_0 R_{a_n}}>\frac{2n}{|v|^2}$ and moreover $a_n\neq 0$. It follows that:

$$|(a_n,0,\xi_n)|_0^2=|a_n|^2+\tfrac{1}{n}\to 0\qquad\text{as}\qquad n\to\infty.$$

On the other hand, we may consider a second sc-Banach space consisting of maps

$$\eta':\mathbb{R}^+\times S^1\to\mathbb{R}^{2n}$$

with asymptotic constant given by $\lim_{s\to\infty}\eta'=c''$ and with m level norm

$$|\eta'|_m^2=|c''|^2+\sum_{|\alpha|\le m+3}\int_{\mathbb{R}^+\times S^1}|D^\alpha(\eta'-c'')|^2 e^{2\delta_m|s|}dsdt.$$

Consider the sequence $\oplus_{a_n}(0,\xi_n)$, using the gluing procedure described in ([22], Section 2.4). In this norm:

$$\begin{aligned}|\oplus_{a_n}(0,\xi_n)|_0^2 &= \sum_{|\alpha|\le 3}\int_{\mathbb{R}^+\times S^1}|D^\alpha\xi_n(s-R_{a_n})|^2 e^{2\delta_0|s|}dsdt > \int_{[R_{a_n},R_{a_n}+\frac{1}{2}]\times S^1}|\tfrac{1}{\sqrt{n}}\cdot v|^2 e^{2\delta_0|s|}dsdt\\ &> \tfrac{1}{2}\cdot\tfrac{1}{n}\cdot|v|^2\cdot e^{2\delta_0 R_{a_n}}>1.\end{aligned}$$

for all n.

The topology of a neighborhood of $[\alpha]$ is determined by the norm of the first sc-Banach space, moreover, because the gluing parameters $a_n \neq 0$ the $(a_n, 0, \xi_n)$ give a sequence $x_n \in \mathcal{Z}^{\text{const}}_{A,g,k}$ which converges to $[\alpha]$. On the other hand, the topology of a neighborhood of $ft_k([\alpha])$ is determined by the norms of the second sc-Banach space; the $\oplus_{a_n}(0, \xi_n)$ give a sequence $y_n \in \mathcal{Z}_{A,g,k-1}$ such that $ft_k(x_n) = y_n$ and which does not converge. □

Chapter 3

Pulling Back Abstract Perturbations

In order to prove the Gromov–Witten axioms we need to study the geometry of the permutation and marked point forgetting maps between the perturbed Gromov–Witten solution spaces, and to do this, we must first show that these maps will persist after abstract perturbation. The natural approach for obtaining a well-defined map between perturbed solution spaces is to pullback an abstract perturbation. In Theorem 3.1.4 we establish mild criterion under with a regular perturbation will pullback to a regular perturbation. As an immediate application of this theorem, in Corollary 3.1.5 we show there exists a regular perturbation which pulls back to regular perturbation via the permutation map. In Section 3.7 we study the geometry of the permutation map on the perturbed Gromov–Witten solution spaces.

3.1 General Setup for Pulling Back Perturbations

Consider a *polyfold Fredholm problem* consisting of a strong polyfold bundle $\mathcal{W}$ over a polyfold $\mathcal{Z}$, together with a sc-smooth proper Fredholm section $\overline{\partial}$:

$$\mathcal{W} \underset{P}{\overset{\overline{\partial}}{\leftrightarrows}} \mathcal{Z},$$

(see Section 3.2). Associated to this polyfold Fredholm problem we can consider the **unperturbed solution set** defined by

$$\mathcal{S}(\overline{\partial}) := \{[x] \in \mathcal{Z} \mid \overline{\partial}([x]) = 0\} \subset \mathcal{Z}.$$

By means of abstract perturbation, there exist *regular* sc^+-multisections $\Lambda : \mathcal{W} \to \mathbb{Q}^+$, such that the **perturbed solution set** defined by

$$\mathcal{S}(\overline{\partial}, \Lambda) := \{[x] \in \mathcal{Z} \mid \Lambda \circ \overline{\partial}([x]) > 0\} \subset \mathcal{Z}.$$

has the additional structure of a compact oriented *weighted branched suborbifold* (see Definition 3.4.9 and Definition 6.4.4, respectively).

Consider an sc-smooth map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, and consider a strong polyfold bundle $\mathcal{W} \to \mathcal{Z}_2$ with an sc-smooth proper Fredholm section $\overline{\partial}$. We can pullback this bundle and section via the map f, and obtain the following commutative diagram:

$$\begin{array}{ccc} f^*\mathcal{W} & \xrightarrow[\mathrm{proj}_2]{} & \mathcal{W} \\ {\scriptstyle f^*\overline{\partial}}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle \overline{\partial}} \\ \mathcal{Z}_1 & \xrightarrow[f]{} & \mathcal{Z}_2, \end{array}$$

(see Section 3.5). Here proj_2 is defined on the underlying sets by the projection of $f^*\mathcal{W} \subset \mathcal{Z}_1 \times \mathcal{W}$ onto the second factor. Assume that the sc-smooth section $f^*\overline{\partial}$ is a proper Fredholm section; such an assumption is not automatic from the above setup, however it is natural in the context of the maps we will encounter.[1]

We can perturb the Fredholm section $\overline{\partial}$ and its pullback $f^*\overline{\partial}$ at the same time by means of a perturbation $\Lambda : \mathcal{W} \to \mathbb{Q}^+$ and its pullback perturbation $\mathrm{proj}_2^*\Lambda : f^*\mathcal{W} \to \mathbb{Q}^+$. As the next theorem demonstrates, for a small perturbation Λ we can achieve transversality for both polyfold Fredholm problems.

Theorem 3.1.1 (Simultaneous Transversality)**.** *There exists a perturbation $\Lambda : \mathcal{W}_2 \to \mathbb{Q}^+$ which pulls back to a perturbation $proj_2^*\Lambda : f^*\mathcal{W}_2 \to \mathbb{Q}^+$ such that $(\overline{\partial}, \Lambda)$ and $(f^*\overline{\partial}, proj_2^*\Lambda)$ are both transversal pairs (see Definition 3.4.3).*

We prove this in Section 3.6. In order to ensure that the perturbed solution set is compact, we require that our perturbations are 'small' in an appropriate sense. This is accomplished via a *pair which controls compactness* $(N, \mathcal{U})$ (see Definition 3.4.6). We would like to take the pullback of a pair which controls compactness. To do this, we first introduce the following topological condition on the map f which will be satisfied by many of the maps we will consider.

Definition 3.1.2. We say that the map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ satisfies the **topological pullback condition** if for all $[y] \in \mathcal{S}(\overline{\partial}) \subset \mathcal{Z}_2$ and for any open neighborhood $\mathcal{V} \subset \mathcal{Z}_1$ of the fiber $f^{-1}([y])$ there exists an open neighborhood $\mathcal{U}_{[y]} \subset \mathcal{Z}_2$ of $[y]$ such that $f^{-1}(\mathcal{U}_{[y]}) \subset \mathcal{V}$. Note that if $f^{-1}([y]) = \emptyset$, this implies that there exists an open neighborhood $\mathcal{U}_{[y]}$ of $[y]$ such that $f^{-1}(\mathcal{U}_{[y]}) = \emptyset$.

Theorem 3.1.3 (Simultaneous Compactness)**.** *Suppose that the map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ satisfies the topological pullback condition. Then there exists a pair $(N_2, \mathcal{U}_2)$ which controls the compactness of $\overline{\partial}$ such that the pair $(proj_2^* N_2, f^{-1}(\mathcal{U}_2))$ controls the compactness of $f^*\overline{\partial}$.*

It follows that if a perturbation $\Lambda : \mathcal{W} \to \mathbb{Q}^+$ satisfies $N_2[\Lambda] \leq 1$ and dom-supp$(\Lambda) \subset \mathcal{U}_2$, then its pullback $proj_2^*\Lambda : f^*\mathcal{W} \to \mathbb{Q}^+$ satisfies $proj_2^* N_2[proj_2^*\Lambda] \leq 1$ and dom-supp$(proj_2^*\Lambda) \subset f^{-1}(\mathcal{U}_2)$.

[1]An alternative to outright assuming that $f^*\overline{\partial}$ is a Fredholm section would be to formulate a precise notion of a 'Fredholm map' for a map between polyfolds, and then require that f is such a map. This would also be natural in the context of the maps we are considering.

We prove this in Section 3.6. Combining the methods used to prove Theorem 3.1.1 and Theorem 3.1.3, we obtain the following theorem.

Theorem 3.1.4. *We can construct a regular perturbation $\Lambda : \mathcal{W} \to \mathbb{Q}^+$ which pulls back to a regular perturbation $proj_2^*\Lambda$, i.e. both perturbations satisfy the following conditions.*

1. $(\overline{\partial}, \Lambda)$ *and* $(f^*\overline{\partial}, proj_2^*\Lambda)$ *are both transversal pairs.*

2. *There exist pairs* $(N_1, \mathcal{U}_1)$ *and* $(N_2, \mathcal{U}_2)$ *which control the compactness of* $f^*\overline{\partial}$ *and* $\overline{\partial}$ *respectively, and such that*

    - $N_2[\Lambda] \leq 1$ *and dom-supp*$(\Lambda) \subset \mathcal{U}_2$
    - $N_1[proj_2^*\Lambda] \leq 1$ *and dom-supp*$(proj_2^*\Lambda) \subset \mathcal{U}_1$.

The significance of this theorem is the following. Both perturbed solution sets $\mathcal{S}(f^*\overline{\partial}, \text{proj}_2^*\Lambda)$ and $\mathcal{S}(\overline{\partial}, \Lambda)$ have the structure of compact oriented weighted branched suborbifolds. Moreover, the restriction of f gives a well-defined continuous function between these perturbed solution spaces, i.e.

$$f|_{\mathcal{S}(f^*\overline{\partial}, \text{proj}_2^*\Lambda)} : \mathcal{S}(f^*\overline{\partial}, \text{proj}_2^*\Lambda) \to \mathcal{S}(\overline{\partial}, \Lambda).$$

Furthermore, f is **weight preserving** in the sense that the weight functions are related via pullback by the following equation $(\Lambda \circ \overline{\partial}) \circ f = \text{proj}_2^*\Lambda \circ f^*\overline{\partial}$.

As an immediate application, we may check that the permutation maps satisfy the hypothesis of Theorem 3.1.4, and obtain the following corollary.

Corollary 3.1.5 (Pulling Back Perturbations via the Permutation Maps)**.** *Fix a permutation* $\sigma \in S_k,\ \sigma : \{1, \ldots, k\} \to \{1, \ldots, k\}$. *Consider the permutation map*

$$\sigma : \mathcal{Z}_{A,g,k} \to \mathcal{Z}_{A,g,k},\ [\Sigma, j, M, D] \mapsto [\Sigma, j, M^\sigma, D]$$

where $M = \{z_1, \ldots, z_k\}$ *and where* $M^\sigma := \{z_1', \ldots, z_k'\}$, $z_i' := z_{\sigma(i)}$. *There exists a regular perturbation which pulls back to a regular perturbation via this permutation map.*

3.2 Polyfold Fredholm Problems

Definition 3.2.1. A **strong polyfold bundle** $\mathcal{W}$ **over the polyfold** $\mathcal{Z}$ consists of the following:

1. an sc-smooth surjective projection map between polyfolds $P : \mathcal{W} \to \mathcal{Z}$.

2. an associated sc-smooth functor between polyfold structures $\hat{P} : W \to Z$, and where W moreover has the structure of a 'strong bundle over the ep-groupoid Z' (see [18, Definition 8.4])

3. around every $[x] \in \mathcal{Z}$ there exists a local sc-trivialization of this bundle, consisting of a 'strong local bundle model' (see [18, Definition 2.23]).

The functor $\hat{P}$ induces a continuous map $|\hat{P}|$ on the orbit spaces which is related to the underlying continuous map P by the following commutative diagram.

$$\begin{array}{ccc} |W| & \xrightarrow{|\hat{P}|} & |Z| \\ \wr| & & \wr| \\ \mathcal{W} & \xrightarrow{P} & \mathcal{Z}. \end{array}$$

Given $[x] \in \mathcal{Z}$, a **local sc-trivialization** of this bundle consists of a representative $x \in Z$, an M-polyfold chart O_x containing x, and a strong local bundle model K over O_x

$$p : K \to O_x$$

where K has the structure of a 'local strong bundle retract' (see [18, Definition 2.23]). The fibers $p^{-1}(y) = K_y$ over $y \in O_x$ carry the structure of a Banach space.

As a local strong bundle retract, K comes equipped with a double filtration $K_{m,k}$ for $0 \leq k \leq m+1$. This double filtration is preserved by the structure maps, and hence the underlying topological space $\mathcal{W}$ is equipped with a double filtration

$$\mathcal{W}_{m,k}, \qquad \text{for } 0 \leq m \text{ and } 0 \leq k \leq m+1.$$

We moreover obtain polyfolds $\mathcal{W}[0]$ and $\mathcal{W}[1]$ with the filtrations $\mathcal{W}[0]_m = \mathcal{W}_{m,m}$ and $\mathcal{W}[1]_m = \mathcal{W}_{m,m+1}$. We clarify that the projection map between polyfolds is $P : \mathcal{W}[0] \to \mathcal{Z}$; when we refer to $\mathcal{W}$ as a polyfold we will generally mean with the underlying set $\mathcal{W}[0]$ and the filtration $\mathcal{W}[0]_m$.

Definition 3.2.2. An **sc-smooth Fredholm section** of the strong polyfold bundle $P : \mathcal{W} \to \mathcal{Z}$ consists of the following:

1. an sc-smooth map between polyfolds $\overline{\partial} : \mathcal{Z} \to \mathcal{W}$ which satisfies $P \circ \overline{\partial} = \mathrm{id}_{\mathcal{Z}}$, where $\mathrm{id}_{\mathcal{Z}}$ is the identity map on $\mathcal{Z}$
2. an associated sc-smooth functor between polyfold structures $\hat{\overline{\partial}} : Z \to W$ where $|\hat{\overline{\partial}}|$ induces $\overline{\partial}$
3. $\overline{\partial}$ is regularizing, meaning that if $[x] \in \mathcal{Z}_m$ and $\overline{\partial}([x]) \in \mathcal{W}_{m,m+1}$ then $[x] \in \mathcal{Z}_{m+1}$
4. in a given local sc-trivialization $p : K \to O$, at every smooth point $y \in O$ the germ $(\overline{\partial}, y)$ is a 'Fredholm germ' (see [22, Definition 2.44]).

We say that $\overline{\partial}$ is **proper** if the unperturbed solution set

$$\mathcal{S}(\overline{\partial}) := \{[x] \in \mathcal{Z} \mid \overline{\partial}([x]) = 0\} \subset \mathcal{Z},$$

is compact (considered as a subset of the underlying topological space of the polyfold $\mathcal{Z}$). Associated to the unperturbed solution set we can also define the **unperturbed solution subgroupoid**, with object set

$$S(\hat{\bar{\partial}}) := \{x \in Z \mid \hat{\bar{\partial}}(x) = 0\} \subset Z$$

and with morphisms given from the restriction of $\boldsymbol{Z}$ to this set (i.e. where both source and target of a morphism are in $S(\overline{\partial})$). It carries the subspace topology induced from the topologies on the object set Z and morphism set $\boldsymbol{Z}$.

3.3 The Gromov–Witten Polyfold Fredholm Problem

We now describe the polyfold Fredholm problem in the current situation of the GW-polyfolds. We start by recalling the definition of the strong polyfold bundle $\mathcal{W}_{A,g,k}$ over the GW-polyfold $\mathcal{Z}_{A,g,k}$. Consider the closed symplectic manifold (Q, ω) and fix a compatible almost complex structure J on Q so that $\omega \circ (\mathrm{id} \oplus J)$ is a Riemannian metric on Q.

Definition 3.3.1. The underlying set of the strong polyfold bundle $\mathcal{W}_{A,g,k}$ is defined as the set of equivalence classes

$$\mathcal{W}_{A,g,k} := \{(\Sigma, j, M, D, u, \xi)\}/\sim$$

with data as follows.

- (Σ, j, M, D, u) is a stable map representative of a stable curve $[\Sigma, j, M, D, u] \in \mathcal{Z}_{A,g,k}$.
- ξ is a continuous section along u such that the map

  $$\xi(z) : T_z\Sigma \to T_{u(z)}Q, \quad \text{for } z \in \Sigma$$

  is a complex anti-linear map.
- ξ is of class $(2, \delta_0)$ around the nodal points in $|D|$ and of class H^2_{loc} near the other points in Σ.
- The equivalence relation is given by $(\Sigma, j, M, D, u, \xi) \sim (\Sigma', j', M', D', u', \xi')$ if there exists a biholomorphism $\phi : (\Sigma, j) \to (\Sigma', j')$ which satisfies

  $$\xi' \circ T\phi = \xi$$

  in addition to $u' \circ \phi = u$, $\phi(M) = M'$, $\phi(|D|) = |D'|$, and which preserves ordering and pairs.

Similar to Definition 2.3.2, we say that ξ is of **class** (m, δ_0) **around a nodal point** $x \in |D|$ if, taking holomorphic polar coordinates σ around the point $x \in |D|$ and a chart ψ around $u(x)$ in Q, that the map

$$(s, t) \mapsto T\psi(u(\sigma(s, t)))\xi(\sigma(s, t))(\partial_s\sigma(s, t)),$$

which is defined for s large, has weak partial derivatives up to order 2, which if weighted by $e^{\delta_0 s}$, belongs to the space $L^2([s_0,\infty)\times S^1,\mathbb{R}^{2n})$ for s_0 large enough. We say that ξ is of **class** m **around the point** $z\in\Sigma$, if a similar coordinate expression belongs to the space H^m_{loc} near z.

The double filtration on $\mathcal{W}_{A,g,k}$ is given as follows. We say an element $[\Sigma,j,M,D,u,\xi]$ is of (bi)-regularity $((m+3,\delta_m),(k+2,\delta_k))$ for $0\le k\le m+1$ if the map u has regularity $(m+3,\delta_m)$ and the section ξ has regularity $(k+2,\delta_k)$.

Proposition 3.3.2. *[22, Theorem 1.10] The set $\mathcal{W}_{A,g,k}$ possesses a polyfold structure such that*

$$P:\mathcal{W}_{A,g,k}\to\mathcal{Z}_{A,g,k},\qquad [\Sigma,j,M,D,u,\xi]\mapsto[\Sigma,j,M,D,u]$$

defines a strong polyfold bundle over the GW-polyfold $\mathcal{Z}_{A,g,k}$.

Definition 3.3.3. The Cauchy–Riemann section $\overline{\partial}_J$ of the strong polyfold bundle $P:\mathcal{W}_{A,g,k}\to\mathcal{Z}_{A,g,k}$ is defined on the underlying sets by

$$\overline{\partial}_J([\Sigma,j,M,D,u])=[\Sigma,j,M,D,u,\tfrac{1}{2}(du+J(u)\circ du\circ j)].$$

Proposition 3.3.4. *[22, Theorem 1.11] The Cauchy–Riemann section $\overline{\partial}_J$ of the strong polyfold bundle $P:\mathcal{W}_{A,g,k}\to\mathcal{Z}_{A,g,k}$ is an sc-smooth Fredholm section. The Fredholm index of $\overline{\partial}_J$ is given by*

$$Ind\ (\overline{\partial}_J)=2c_1(A)+(dim_{\mathbb{R}}Q-6)(1-g)+2k.$$

As a set, the unperturbed solution set $\mathcal{S}(\overline{\partial}_J)$ is precisely the same as the stable map compactification of the Gromov–Witten moduli space as originally introduced in [26] and described in Chapter 5 of [32]. With the topology given by this stable map compactification $\mathcal{S}(\overline{\partial}_J)$ is a compact topological space. We must then show that this topology is equivalent to the the subspace topology induced by the underlying topological space of the Gromov–Witten polyfold $\mathcal{Z}_{A,g,k}$.

3.4 Polyfold Abstract Perturbations

In this section we review the abstract perturbation theory for polyfolds, originally developed in [21].

Definition 3.4.1. Consider a strong local bundle model $p:K\to O$. An **sc$^+$-section** is an sc-smooth map $s:O\to K[1]$ which satisfies $p\circ s=Id_O$

The significance of this definition is captured in the fact that if $(\overline{\partial},x)$ is a Fredholm germ and s is a germ of an sc$^+$-section around y, then $(\overline{\partial}+s,x)$ remains a Fredholm germ. This follows tautologically from the definition of a Fredholm germ (see the comment following [22, Definition 2.44]). Hence, we may view the relationship of Fredholm sections and sc$^+$-sections in the current theory as the analogs of Fredholm and compact operators in classical functional analysis.

Definition 3.4.2. We view $\mathbb{Q}^+ := \mathbb{Q} \cap [0, \infty)$ as an ep-groupoid, having only the identities as morphisms. An **sc$^+$-multisection** of a strong polyfold bundle $P : \mathcal{W} \to \mathcal{Z}$ consists of the following:

1. a function $\Lambda : \mathcal{W} \to \mathbb{Q}^+$
2. an associated functor $\hat{\Lambda} : W \to \mathbb{Q}^+$ where $|\hat{\Lambda}|$ induces Λ
3. at each $[x] \in \mathcal{Z}$ there exists a 'local section structure' for Λ, defined below.

Given $[x] \in \mathcal{Z}$, we define a **local section structure** for Λ at $[x]$ as follows. Consider a local sc-trivialization consisting of a representative $x \in Z$, an M-polyfold chart O_x containing x, and a strong local bundle model $p : K \to O_x$. Suppose moreover that O_x is invariant under the induced action by the isotropy group $\boldsymbol{G}(x)$.

Then there exist finitely many sc$^+$-sections $s_1, \ldots, s_k : O_x \to K$ (called **local sections**) with associated positive rational numbers $\sigma_1, \ldots, \sigma_k$ (called **weights**) which satisfy the following:

1. $\sum_{i=1}^k \sigma_i = 1$.
2. The local expression $\hat{\Lambda} : K \to \mathbb{Q}^+$ is related to the local sections and weights via the equation
$$\hat{\Lambda}(w) = \sum_{i \in \{1,\ldots,k | w = s_i(p(w))\}} \sigma_i$$
for all $w \in K$, where the empty sum has by definition the value 0.

We define the **domain support** of Λ as the subset of $\mathcal{Z}$ given by
$$\text{dom-supp}(\Lambda) = \text{cl}_{\mathcal{Z}}(\{[x] \in \mathcal{Z} \mid \text{there exists } [w] \in \mathcal{W}_{[x]} \setminus \{0\} \text{ for which } \Lambda([w]) > 0\}).$$

Associated to a sc-smooth Fredholm section and a sc$^+$-multisection, we can define the **perturbed solution subgroupoid** with object set
$$S(\hat{\overline{\partial}}, \hat{\Lambda}) := \{x \in Z \mid \hat{\Lambda}(\hat{\overline{\partial}}(x)) > 0\} \subset Z$$
and with morphisms given from the restriction of $\boldsymbol{Z}$ to this set. It is moreover equipped with a **weight functor** $\hat{\Lambda} \circ \hat{\overline{\partial}} : S(\hat{\overline{\partial}}, \hat{\Lambda}) \to \mathbb{Q}^+$. We remark that $S(\hat{\overline{\partial}}, \hat{\Lambda})$ can be encoded solely via this weight functor; such a description is closer to the language used in [21] and [18].

We also define the **perturbed solution set** by
$$\mathcal{S}(\overline{\partial}, \Lambda) := \{[x] \in \mathcal{Z} \mid \Lambda(\overline{\partial}([x])) > 0\} \subset \mathcal{Z}$$
which is equipped with a **weight function** $\Lambda \circ \overline{\partial} : \mathcal{S}(\overline{\partial}, \Lambda) \to \mathbb{Q}^+$. Moreover, the weight functor induces the weight function.

Transverse Perturbations

Definition 3.4.3. Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle (with $\hat{P} : W \to Z$ an associated strong polyfold bundle structure), $\overline{\partial}$ an sc-smooth Fredholm section (with $\hat{\overline{\partial}}$ an associated sc-smooth Fredholm section functor), and Λ an sc$^+$-multisection (with $\hat{\Lambda}$ an associated sc$^+$-multisection functor).

Consider a point $[x] \in \mathcal{Z}$. We say $(\overline{\partial}, \Lambda)$ is **transversal at** $[x]$ if, given a local sc$^+$-section structure for Λ at $[x]$, then the local expression $(\overline{\partial} - s_i)'(x) : T_x O_x \to K_x$ is surjective for all $i \in I$ with $\overline{\partial}(x) = s_i(x)$. We say that $(\overline{\partial}, \Lambda)$ is **transversal** if it is transversal at every $[x] \in \text{dom-supp}(\Lambda)$.

Theorem 3.4.4. *[21, Theorem 4.13]* [2] *Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, $\overline{\partial}$ an sc-smooth Fredholm section, and Λ an sc$^+$-multisection. If the pair $(\overline{\partial}, \Lambda)$ is transversal, then the perturbed solution set $\mathcal{S}(\overline{\partial}, \Lambda)$ carries in a natural way the structure of a weighted branched suborbifold (see Definition 6.4.4).*

Compactness of the Perturbed Solution Space

Definition 3.4.5. Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle (with $\hat{P} : W \to Z$ an associated strong polyfold bundle structure). We define an **auxiliary norm** as an sc^0-map

$$N : \mathcal{W}[1] \to [0, \infty)$$

where we are regarding $[0, \infty)$ as a smooth manifold (i.e. a polyfold with finite-dimensional local models and trivial isotropy). It has an associated **auxiliary norm sc^0-functor**

$$\hat{N} : W[1] \to [0, \infty)$$

where as usual $|\hat{N}|$ induces N. We require that $\hat{N}$ satisfies the following conditions.

1. For each $x \in Z$, the fiber $\hat{P}^{-1}(x)$ is a Banach space. The restriction of $\hat{N}$ to each fiber $P^{-1}(x)$ is a complete norm.

2. If (h_k) is a sequence in $W[1]$ such that $(\hat{P}(h_k))$ converges in Z to some x, and $\hat{N}(h_k) \to 0$, then $h_k \to 0_x$ in $W[1]$.

For $[x] \in \mathcal{Z}$ we define the **pointwise norm** of Λ with respect to the auxiliary norm N by

$$N[\Lambda]([x]) := \max\{N([w]) \mid [w] \in \mathcal{W}[1], \Lambda([w]) > 0, P([w]) = [x]\}$$

and moreover define the **norm** of Λ with respect to N by

$$N[\Lambda] := \sup_{[x] \in \mathcal{Z}} N[\Lambda]([x]).$$

[2] The original statement of this theorem carries the additional requirement that the perturbed solution set $\mathcal{S}(\overline{\partial}, \Lambda)$ is a compact set. This requirement is unnecessary, and is not used in the proof.

Definition 3.4.6. Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, let $\overline{\partial}$ be an sc-smooth proper Fredholm section, and let $N : \mathcal{W}[1] \to [0, \infty)$ be an auxiliary norm.

Consider an open neighborhood $\mathcal{U}$ of the unperturbed solution set $\mathcal{S}(\overline{\partial}) \subset \mathcal{Z}$ (i.e. $\mathcal{U}$ is open considered as a set of the underlying topological space $\mathcal{Z}$). We say that the pair $(N,\mathcal{U})$ **controls the compactness** of $\overline{\partial}$ provided the set

$$cl_{\mathcal{Z}}\{[x] \in \mathcal{U} \mid \overline{\partial}([x]) \in \mathcal{W}[1], N(\overline{\partial}([x])) \leq 1\} \subset \mathcal{Z}$$

is compact.

Remark 3.4.7. We may always shrink the neighborhood $\mathcal{U}$ of the unperturbed solution set; to be precise, suppose that $(N,\mathcal{U})$ is a pair which controls compactness, and let $\mathcal{U}'$ be an open set such that $\mathcal{S}(\overline{\partial}) \subset \mathcal{U}' \subset \mathcal{U}$. It is immediate from the above definition that the pair $(N,\mathcal{U}')$ also controls compactness.

Given a polyfold Fredholm problem, [21, Proposition 2.27] guarantees the existence of auxiliary norms. The existence of a pair which control compactness then follows from [21, Theorem 4.5] which states that given an auxiliary norm N there always exists an associated neighorhood $\mathcal{U}$, such that the pair $(N,\mathcal{U})$ controls compactness.

Theorem 3.4.8. *[21, Lemma 4.16] Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, let $\overline{\partial}$ be an sc-smooth proper Fredholm section, and let $(N,\mathcal{U})$ be a pair which controls compactness.*

If an sc$^+$-multisection Λ satisfies $N[\Lambda] \leq 1$ and dom-supp$(\Lambda) \subset \mathcal{U}$, then the perturbed solution set $\mathcal{S}(\overline{\partial}, \Lambda)$ is compact (considered with the subspace topology induced from the underlying topological space $\mathcal{Z}$).

Regular Perturbations and Compact Cobordism

Definition 3.4.9. [18, Corollary 15.1] Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, let $\overline{\partial}$ be an sc-smooth proper Fredholm section, and let $(N,\mathcal{U})$ be a pair which controls compactness.

Suppose an sc$^+$-multisection Λ satisfies both the requirements of Theorem 3.4.4 and Theorem 3.4.8, i.e.

- $(\overline{\partial}, \Lambda)$ is a transversal pair
- $N[\Lambda] \leq 1$ and dom-supp$(\Lambda) \subset \mathcal{U}$

We then say Λ is a **regular perturbation** of $\overline{\partial}$ with respect to the pair $(N,\mathcal{U})$.

Proposition 3.4.10. *[18, Corollary 15.1] Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, let $\overline{\partial}$ be an sc-smooth proper Fredholm section, and let $(N,\mathcal{U})$ be a pair which controls compactness. Then there exists regular perturbations Λ of $\overline{\partial}$ with respect to the pair $(N,\mathcal{U})$.*

It follows from Theorem 3.4.4 and Theorem 3.4.8 that the perturbed solution space $\mathcal{S}(\overline{\partial}, \Lambda)$ has the structure of a compact weighted branched suborbifold, with weight function given by $\Lambda \circ \overline{\partial} : \mathcal{S}(\overline{\partial}, \Lambda) \to \mathbb{Q}^+$.

Proposition 3.4.11. *[18, Corollary 15.1] Let $P : \mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, let $\overline{\partial}$ be an sc-smooth proper oriented Fredholm section, and let $(N_0,\mathcal{U}_0)$, $(N_1,\mathcal{U}_1)$ be two pairs which control compactness. Suppose that Λ_0 is a regular perturbation of $\overline{\partial}$ with respect to the pair $(N_0,\mathcal{U}_0)$, and likewise Λ_1 is a regular perturbation of $\overline{\partial}$ with respect to the pair $(N_1,\mathcal{U}_1)$. Consider the strong polyfold bundle $[0,1]\times\mathcal{W} \to [0,1]\times\mathcal{Z}$ and the sc-smooth proper oriented Fredholm section $\tilde{\overline{\partial}}$ defined by $(t,[z]) \mapsto (t,\overline{\partial}([z]))$.*

Then

1. *there exists an auxiliary norm N for $[0,1]\times\mathcal{W}$ which restricts to N_0 on $\{0\}\times\mathcal{W}$ and which restricts to N_1 on $\{1\}\times\mathcal{W}$,*
2. *there exists an open neighborhood $\mathcal{U}$ of $\mathcal{S}(\tilde{\overline{\partial}})$ such that $\mathcal{U}\cap(\{0\}\times\mathcal{Z}) = \mathcal{U}_0$ and $\mathcal{U}\cap(\{1\}\times\mathcal{Z}) = \mathcal{U}_1$,*

such that the pair $(N,\mathcal{U})$ controls the compactness of $\tilde{\overline{\partial}}$.

In addition, there exists a regular perturbation Λ of $\tilde{\overline{\partial}}$ with respect to the pair $(N,\mathcal{U})$, such that $\Lambda|_{\{0\}\times\mathcal{W}}$ can be identified with Λ_0 and likewise $\Lambda|_{\{1\}\times\mathcal{W}}$ can be identified with Λ_1.

3.5 Pullbacks of Polyfold Fredholm Problems

Consider a surjective continuous map $P : \mathcal{W} \to \mathcal{Z}_2$ and let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be a continuous map. The topological pullback $f^*\mathcal{W}$ is defined by

$$f^*\mathcal{W} = \{([x],[w]_{[y]}) \mid f([x]) = [y] = P([w]_{[y]})\} \subset \mathcal{Z}_1 \times \mathcal{W}$$

and equipped with the subspace topology. Projections onto each factor give the following commutative diagram.

$$\begin{array}{ccc} f^*\mathcal{W} & \xrightarrow{\text{proj}_2} & \mathcal{W} \\ \downarrow{\scriptstyle \text{proj}_1} & & \downarrow{\scriptstyle P} \\ \mathcal{Z}_1 & \xrightarrow{f} & \mathcal{Z}_2 \end{array}$$

Suppose moreover that $\mathcal{Z}_1$ and $\mathcal{W}$ are second countable paracompact topological spaces. It follows that the product $\mathcal{Z}_1 \times \mathcal{W}$ and hence $f^*\mathcal{W}$ are also second countable paracompact topological spaces.

Definition 3.5.1. Consider a strong polyfold bundle $P : \mathcal{W} \to \mathcal{Z}_2$, and let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be an sc-smooth map between polyfolds. Then the second countable paracompact topological space $f^*\mathcal{W}$ has the natural structure of a strong polyfold bundle $f^*\mathcal{W} \to \mathcal{Z}_2$. It carries a double filtration induced from $\mathcal{W}$. We call this a **pullback strong polyfold bundle**.

Given $[x] \in \mathcal{Z}_1$ there exists a local sc-trivialization of $f^*\mathcal{W}$ with the following description. Let $f([x]) = [y] \in \mathcal{Z}_2$ and consider a local sc-trivialization of $\mathcal{W}$ consisting of a representative y, an M-polyfold chart O_y containing y and a strong local bundle model

$$p : K \to O_y.$$

Let x be a representative of $[x]$, let O_x be an M-polyfold chart containing x, and moreover assume we have a local description $\hat{f}$ of f such that $\hat{f}(O_x) \subset O_y$ (shrinking O_x if necessary).

We may take the pullback $\hat{f}^*K$, yielding a commutative diagram.

$$\begin{array}{ccc} \hat{f}^*K & \xrightarrow{\text{proj}_2} & K \\ \downarrow{\scriptstyle \text{proj}_1} & & \downarrow{\scriptstyle p} \\ O_x & \xrightarrow{\hat{f}} & O_y \end{array}$$

It is proved in [16, Proposition 4.11] that $\hat{f}^*K$ has the structure of a local strong bundle retract. The pullback $\hat{f}^*K \to O_x$ is the desired local sc-trivialization.

We have the usual analog for the pullback of a section.

Definition 3.5.2. Given an sc-smooth section $\overline{\partial} : \mathcal{Z}_2 \to \mathcal{W}$ there exists a well-defined **pullback section** $f^*\overline{\partial} : \mathcal{Z}_1 \to f^*\mathcal{W}$. On the level of underlying sets, it is defined by

$$f^*\overline{\partial}([x]) = ([x], \overline{\partial} \circ f([x])).$$

Moreover, it is regularizing if $\overline{\partial}$ is regularizing.

We may also define the pullback of an sc$^+$-multisection.

Definition 3.5.3. Given an sc$^+$-multisection $\Lambda : \mathcal{W} \to \mathbb{Q}^+$ there exists a well-defined **pullback sc$^+$-multisection** $\text{proj}_2^*\Lambda : f^*\mathcal{W} \to \mathbb{Q}^+$. It consists of the following:

1. the function $\Lambda \circ \text{proj}_2 : f^*\mathcal{W} \to \mathbb{Q}^+$
2. the functor $\hat{\Lambda} \circ \text{proj}_2 : \hat{f}^*W \to \mathbb{Q}^+$
3. at each $[x] \in \mathcal{Z}_1$ there exists a 'pullback local section structure' for $\text{proj}_2^*\Lambda$, defined below.

Given $[x] \in \mathcal{Z}_1$, the local section structure for $\text{proj}_2^*\Lambda$ at $[x]$ is described as follows. Let $f([x]) = [y] \in \mathcal{Z}_2$, and consider a local sc-trivialization at a representative y consisting of a strong local bundle model $p : K \to O_y$. Consider the local section structure of Λ at $[y]$; consisting of local sections $s_1, \ldots, s_k : O_y \to K$ and associated weights $\sigma_1, \ldots, \sigma_k$.

Letting x be a representative of $[x]$, as above we have a pullback strong local bundle model $\hat{f}^*K \to O_x$. We may assume that O_x is invariant under the induced action by the isotropy group $\boldsymbol{G}(x)$. Then the pullback of the local sections $\hat{f}^*s_1, \ldots, \hat{f}^*s_k : O_x \to K$ with the associated weights $\sigma_1, \ldots, \sigma_k$ gives the local section structure for $\mathrm{proj}_2^*\Lambda$ at $[x]$.

Indeed, it tautologically follows from the original assumption that $s_1, \ldots, s_k$, $\sigma_1, \ldots, \sigma_k$ is a local section structure for Λ at $[y]$ that

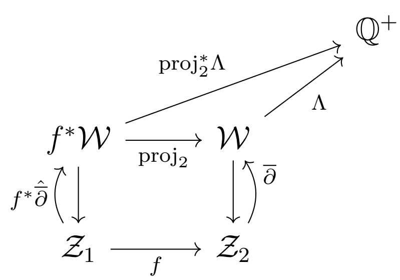
1. $\sum_{i=1}^k \sigma_i = 1$
2. the local expression $\mathrm{proj}_2^*\hat{\Lambda} : \hat{f}^*K \to \mathbb{Q}^+$ is related to the local sections and weights via the equation

$$\mathrm{proj}_2^*\hat{\Lambda}(x', w_{y'}) = \sum_{i \in \{1,\ldots,k \mid (x', w_{y'}) = \hat{f}^* s_i(\mathrm{proj}_1(x', w_{y'}))\}} \sigma_i$$

   for all $(x', w_{y'}) \in \hat{f}^*K$ (which necessarily satisfy $\hat{f}(x') = y' = p(w_{y'})$).

3.6 Explicit Construction of Regular Perturbations which Pullback to Regular Perturbations

Let $P : \mathcal{W} \to \mathcal{Z}_2$ be a strong polyfold bundle, $\overline{\partial}$ an sc-smooth proper Fredholm section, and Λ an sc$^+$-multisection. Consider an sc-smooth map $f : \mathcal{Z}_1 \to \mathcal{Z}_2$, which we use to pullback the bundle, section, and multisection. This is summarized in the following commutative diagram.



Moreover we assume that the sc-smooth section $f^*\overline{\partial}$ is a proper Fredholm section.

It is immediate that restriction gives a continuous function on the underlying perturbed solution sets

$$f|_{\mathcal{S}(f^*\overline{\partial}, \mathrm{proj}_2^*\Lambda)} : \mathcal{S}(f^*\overline{\partial}, \mathrm{proj}_2^*\Lambda) \to \mathcal{S}(\overline{\partial}, \Lambda).$$

as well as an sc-smooth functor between the perturbed solution subgroupoids

$$\hat{f}|_{S(\hat{f}^*\hat{\overline{\partial}}, \mathrm{proj}_2^*\hat{\Lambda})} : S(\hat{f}^*\hat{\overline{\partial}}, \mathrm{proj}_2^*\hat{\Lambda}) \to S(\hat{\overline{\partial}}, \hat{\Lambda})$$

Furthermore the weight functions and weight functors are related via pullback via the equations $f^*(\Lambda \circ \overline{\partial}) = \mathrm{proj}_2^*\Lambda \circ f^*\overline{\partial}$ and $\hat{f}^*(\hat{\Lambda} \circ \hat{\overline{\partial}}) = \mathrm{proj}_2^*\hat{\Lambda} \circ \hat{f}^*\hat{\overline{\partial}}$.

Achieving Simultaneous Transversality - Proof of Theorem 3.1.1

Proof. We now give an explicit construction of an sc$^+$-multisection $\Lambda : \mathcal{W} \to \mathbb{Q}^+$ such that $(\overline{\partial}, \Lambda)$ and $(f^*\overline{\partial}, \mathrm{proj}_2^*\Lambda)$ are both transversal pairs. Our approach is based on the general position argument of [18, Theorem 15.4].

Local Construction: Consider a point $[x_0] \in \mathcal{S}(\overline{\partial}_1) \subset \mathcal{Z}_1$ which maps to $[y_0] = f([x_0]) \in \mathcal{S}(\overline{\partial}_2) \subset \mathcal{Z}_2$. Consider a local sc-trivialization of $f^*\mathcal{W}$ at $[x_0]$ and a local trivialization of $\mathcal{W}$ at $[y_0]$ consisting of the following:

1. a representative y_0 of $[y_0]$, an M-polyfold chart O_{y_0} containing y_0,and a strong local bundle model $K \to O_{y_0}$
2. a representative x_0 of $[x_0]$, an M-polyfold chart O_{x_0} containing x_0, and a strong local bundle model $\hat{f}^*K \to O_{x_0}$.

Moreover, we may assume that $\hat{f}(x_0) = y_0$ and the local expression $\hat{f} : O_{x_0} \to O_{y_0}$ is well-defined. This data gives the following commutative diagram of maps.

$$\begin{array}{ccc} \hat{f}^*K & \xrightarrow{\mathrm{proj}_2} & K \\ {\scriptstyle \hat{f}^*\hat{\overline{\partial}}}\uparrow\downarrow & & \downarrow\uparrow{\scriptstyle \hat{\overline{\partial}}} \\ O_{x_0} & \xrightarrow{\hat{f}} & O_{y_0} \end{array}$$

The fibers at x_0 and at y_0 can be identified via the bijection $\mathrm{proj}_2|_{\hat{f}^*K_{x_0}} : \hat{f}^*K_{x_0} \to K_{y_0}$. Using this identification we can choose smooth vectors $v_1, \ldots, v_m \in K_{y_0}$ such that

- $\mathrm{span}\{v_1, \ldots, v_m\} \oplus \overline{\partial}'(y_0)(T_{y_0}O_{y_0}) = K_{y_0}$
- $\mathrm{span}\{\mathrm{proj}_2^{-1}(v_1), \ldots, \mathrm{proj}_2^{-1}(v_m)\} \oplus \hat{f}^*\overline{\partial}'(x_0)(T_{x_0}O_{x_0}) = K_{x_0}$.

For every i we use [18, Lemma 5.3] to define a sc$^+$-section $s^i : O_{y_0} \to K$ such that $s^i(y_0) = v_i$ and such that $\mathrm{supp}\, s^i \subset V \subset O_{y_0}$, where V is a $\boldsymbol{G}(y_0)$-invariant open neighborhood of y_0.

Consider the locally defined sc-Fredholm sections

- $(\underline{t}, y) \mapsto \hat{\overline{\partial}}(y) - \sum_{i=1}^{m} t_i \cdot s^i(y), \qquad \underline{t} = (t_1, \ldots, t_m) \in B_\epsilon^m \subset \mathbb{R}^m,\ y \in O_{y_0},$
- $(\underline{t}, x) \mapsto (\hat{f}^*\hat{\overline{\partial}})(x) - \sum_{i=1}^{m} t_i \cdot (\hat{f}^*s^i)(x), \qquad \underline{t} = (t_1, \ldots, t_m) \in B_\epsilon^m \subset \mathbb{R}^m,\ x \in O_{x_0}$

and observe that the linearizations at $(0, y_0)$ and at $(0, x_0)$ are both surjective.

Using the action of the isotropy group $\boldsymbol{G}(y_0)$ we obtain a collection of sc$^+$-sections $\{g \cdot (\Sigma t_i s^i)\}_{g \in \boldsymbol{G}(y_0)}$, each with weight $\frac{1}{\sharp \boldsymbol{G}(y_0)}$ which together give a local section structure for a well-defined sc$^+$-multisection $\lambda_0^t : B_\epsilon^m \times \mathcal{W} \to \mathbb{Q}^+$. By Definition 3.5.3, the collection $\{\hat{f}^*(g \cdot (\Sigma t_i s^i))\}_{g \in \boldsymbol{G}(y_0)}$, each with weight $\frac{1}{\sharp \boldsymbol{G}(y_0)}$ give a local section structure for the pullback sc$^+$-multisection $\mathrm{proj}_2^*\lambda_0^{\underline{t}} : B_\epsilon^m \times f^*\mathcal{W} \to \mathbb{Q}^+$. Furthermore, recall the functor $\hat{f}$ induces a group homomorphism between the isotropy groups $\hat{f} : \boldsymbol{G}(x_0) \to \boldsymbol{G}(y_0)$; we now observe the isotropy group $\boldsymbol{G}(x_0)$ acts on the collection $\{\Sigma t_i \hat{f}^*(g \cdot s^i)\}_{g \in \boldsymbol{G}(y_0)}$ as follows: $h*(\Sigma t_i \hat{f}^*(g \cdot s^i)) = \Sigma t_i \hat{f}^*(\hat{f}(h) \cdot g \cdot s^i)$. Lastly, since both $\hat{\overline{\partial}}$ and $\hat{f}^*\hat{\overline{\partial}}$ are invariant under the action of the isotropy groups $\boldsymbol{G}(y_0)$ and $\boldsymbol{G}(x_0)$ respectively, it follows that the linearization of $\hat{\overline{\partial}} - \sum t_i(g \cdot s^i)$ is surjective for all $g \in \boldsymbol{G}(y_0)$ and the linearization of $\hat{f}^*\hat{\overline{\partial}} - \sum t_i \hat{f}^*(\hat{f}(h) \cdot g \cdot s^i)$ is surjective for all $h \in \boldsymbol{G}(x_0)$.

Surjectivity is an open condition, hence there exists an open neighborhood $\mathcal{V}_{[y_0]} \subset \mathcal{Z}_2$ of $[y_0]$ such that the pair $(\overline{\partial}, \lambda_0^{t_0})$ is transversal for all $[y] \in \mathcal{V}_{[y_0]}$, likewise there exists an open neighborhood $\mathcal{V}_{[x_0]} \subset \mathcal{Z}_1$ of $[x_0]$ such that the pair $(f^*\overline{\partial}, \mathrm{proj}_2^*\lambda_0^{t_0})$ is transversal for all $[x] \in \mathcal{V}_{[x_0]}$.

Local to Global: We can cover the compact set $\mathcal{S}(f^*\overline{\partial})$ by a finite collection of open sets $\mathcal{V}_{[x_i]} \subset \mathcal{Z}_1$ of the above form.The corresponding collection of open sets $\mathcal{V}_{[y_i]} \subset \mathcal{Z}_2$ is not yet a cover of $\mathcal{S}(\overline{\partial})$ (there may be $[y] \in \mathcal{S}(\overline{\partial})$ such that $f^{-1}([y]) = \emptyset$). But the above recipe also shows how to construct sc$^+$-multisections at $[y_j] \in \mathcal{S}(\overline{\partial})$, $f^{-1}([y_j]) = \emptyset$, such that $(\overline{\partial}, \lambda_j^{\underline{t}})$ is transversal for all $[y] \in \mathcal{V}_{[y_j]}$.

We may take a finite sum of sc$^+$-multisections constructed in this way, and hence obtain

- $\Lambda^{\underline{t}} := \oplus_i \lambda^{\underline{t}_i} : B_\epsilon^l \times \mathcal{W} \to \mathbb{Q}^+$ such that $(\overline{\partial}, \Lambda^{\underline{t}})$ is a transversal pair,
- $\mathrm{proj}_2^*\Lambda^{\underline{t}} : B_\epsilon^l \times f^*\mathcal{W} \to \mathbb{Q}^+$ such that $(f^*\overline{\partial}, \mathrm{proj}_2^*\Lambda^{\underline{t}})$ is a transversal pair.

It follows by [18, Theorem 15.2] that the universal solution sets, defined by

- $\mathcal{S}(\overline{\partial}, \Lambda^{\underline{t}}; \underline{t} \in B_\epsilon^l) := \{(\underline{t}, [z]) \in B_\epsilon^l \times \mathcal{Z}_2 \mid \Lambda^{\underline{t}}(\overline{\partial}([z])) > 0\} \subset B_\epsilon^l \times \mathcal{Z}_2$,
- $\mathcal{S}(f^*\overline{\partial}, \mathrm{proj}_2^*\Lambda^{\underline{t}}; \underline{t} \in B_\epsilon^l) := \{(\underline{t}, [z]) \in B_\epsilon^l \times \mathcal{Z}_1 \mid \mathrm{proj}_2^*\Lambda^{\underline{t}}(f^*\overline{\partial}([z])) > 0\} \subset B_\epsilon^l \times \mathcal{Z}_1$.

have the structure of weighted branched suborbifolds (see Definition 6.4.4).

Simultaneous Regular Value: Consider the projections from the universal solution sets to the parameter space $B_\epsilon^l \subset \mathbb{R}^l$.

$$\begin{array}{ccc} \mathcal{S}(f^*\overline{\partial}, \mathrm{proj}_2^*\Lambda^{\underline{t}}; \underline{t} \in B_\epsilon^l) & \xrightarrow{\quad f \quad} & \mathcal{S}(\overline{\partial}, \Lambda^{\underline{t}}; \underline{t} \in B_\epsilon^l) \\ & \searrow \quad \swarrow & \\ & B_\epsilon^l & \end{array}$$

For any $(t, [y]) \in \mathcal{S}(\overline{\partial}, \Lambda^{\underline{t}}; \underline{t} \in B_\epsilon^l)$ consider a representative $(t, y) \in S(\hat{\overline{\partial}}, \hat{\Lambda}^{\underline{t}}; \underline{t} \in B_\epsilon^l)$; there exists an open neighborhood $U \subset S(\hat{\overline{\partial}}, \hat{\Lambda}^{\underline{t}}; \underline{t} \in B_\epsilon^l)$ of (t, y) consisting of a finite collection of finite-dimensional manifolds, $U = \cup N_i$. The restriction of the projection defines maps $N_i \to B_\epsilon^l$; by Sard's theorem, the complement of the set of regular values for this map is of measure zero. A countable number of open sets U covers $S(\hat{\overline{\partial}}, \hat{\Lambda}^{\underline{t}}; \underline{t} \in B_\epsilon^l)$.

Likewise, for any $(t,[x]) \in \mathcal{S}(f^*\overline{\partial}, \text{proj}_2^*\Lambda^{\underline{t}}; \underline{t} \in B_\epsilon^l)$ consider a representative $(t,x) \in S(\hat{f}^*\hat{\overline{\partial}}, \text{proj}_2^*\hat{\Lambda}^{\underline{t}}; \underline{t} \in B_\epsilon^l)$; there exists an open neighborhood V of (t,x) consisting of a finite collection of finite-dimensional manifolds M_i. The restriction of the projection defines maps $M_i \to B_\epsilon^l$; by Sard's theorem, the complement of the set of regular values for this map is of measure zero. A countable number of open sets V covers $S(\hat{f}^*\hat{\overline{\partial}}, \text{proj}_2^*\hat{\Lambda}^{\underline{t}}; \underline{t} \in B_\epsilon^l)$.

A countable union of sets of measure zero is also a set of measure zero. Hence we may find $\underline{t}_0 \in B_\epsilon^l$ which is a regular value for any of the restricted projections just considered. Such a 'simultaneous' regular value yields the desired sc$^+$-multisections. □

Remark 3.6.1. In the following subsection, it will be important that $\Lambda^{\underline{t}_0}$ is controlled by a pair $(N_2, \mathcal{U}_2)$ which controls the compactness of $\overline{\partial}$. This is achieved in the above local construction by requiring that the sc$^+$-multisections $\lambda_i^{\underline{t}_i} : B_\epsilon^l \times \mathcal{W} \to \mathbb{Q}^+$ satisfy the following conditions:

- $N_2[\lambda_i^{\underline{t}_i}] \leq 1$,
- dom-supp$(\lambda_i^{\underline{t}_i}) \subset \mathcal{U}_2$.

That such an sc$^+$-multisection exists follows from [18, Lemma 5.3]. By choosing a regular value $|\underline{t}_0|_\infty << 1$ we then ensure that the sum satisfies $N_2[\Lambda^{\underline{t}_0}] = N_2[\oplus \lambda^{\underline{t}_0}] \leq 1$. It also follows that dom-supp$(\Lambda^{\underline{t}_0}) \subset \mathcal{U}_2$.

Achieving Simultaneous Compactness

In this subsection we demonstrate how to take the pullback of pairs which control compactness.

Proposition 3.6.2. *Let $N_2 : \mathcal{W}[1] \to [0,\infty)$ be an auxiliary norm and let $\hat{N}_2 : W[1] \to [0,\infty)$ be an associated sc^0-functor. The pullback of N_2, given by*

$$\text{proj}_2^* N_2 : f^*\mathcal{W}[1] \to [0,\infty)$$

and the associated functor

$$\text{proj}_2^* \hat{N}_2 : \hat{f}^* W[1] \to [0,\infty)$$

together define an auxiliary norm on the pullback strong polyfold bundle $f^*\mathcal{W} \to \mathcal{Z}_1$.

Proof. This is immediate from the definitions. In particular, condition 2 of Definition 3.4.5 can be checked as follows. Let (x_k, w_k) be a sequence in $\hat{f}^*W[1]$, such that x_k converges to x in Z_1, and suppose $\text{proj}_2^*\hat{N}_2(x_k, w_k) \to 0$. Then w_k is a sequence in $W[1]$ such that $\hat{f}(x_k)$ converges to $\hat{f}(x)$ in Z_2, and $\hat{N}_2(w_k) = \text{proj}_2^*\hat{N}_2(x_k, w_k) \to 0$, and hence $w_k \to 0_{\hat{f}(x)}$. Thus $(x_k, w_k) \to (x, 0_{\hat{f}(x)})$, as required. □

Proof of Theorem 3.1.3

Proof. Let $(N_2, \mathcal{V}_2)$ be a pair which controls the compactness of $\overline{\partial}$. By the previous proposition we know that the pullback $\text{proj}_2^* N_2 : f^*\mathcal{W}[1] \to [0,\infty)$ is in fact an auxiliary norm. We may then apply [21, Proposition 4.5] to assert the existence of a neighborhood $\mathcal{U}' \subset \mathcal{Z}_1$ of $\mathcal{S}(f^*\overline{\partial})$ such that the pair $(\text{proj}_2^* N_2, \mathcal{U}')$ controls the compactness of $f^*\overline{\partial}$.

At every $[y] \in \mathcal{S}(\overline{\partial})$, observe that $f^{-1}([y]) \subset \mathcal{S}(f^*\overline{\partial}) \subset \mathcal{U}'$. We can use the topological pullback condition 3.1.2 to choose a neighborhood $\mathcal{U}_{[y]}$ such that $f^{-1}(\mathcal{U}_{[y]}) \subset \mathcal{U}' \subset \mathcal{Z}_1$ and moreover such that $\mathcal{U}_{[y]} \subset \mathcal{V}_2 \subset \mathcal{Z}_2$. We define an open neighborhood by $\mathcal{U}_2 := \cup_i \mathcal{U}_{[y]_i}$ for every $[y]_i \in \mathcal{S}(\overline{\partial})$.

Then $(N_2, \mathcal{U}_2)$ is the desired pair. Indeed, $\mathcal{U}_2$ is an open neighborhood of the unperturbed solution set $\mathcal{S}(\overline{\partial})$. And $\mathcal{U}_2 \subset \mathcal{V}_2$ since for every $[y]_i$ we have $\mathcal{U}_{[y]_i} \subset \mathcal{V}_2$. Hence we have $\mathcal{S}(\overline{\partial}) \subset \mathcal{U}_2 \subset \mathcal{V}_2$ therefore it follows from Remark 3.4.7 that $(N_2, \mathcal{U}_2)$ controls the compactness of $\overline{\partial}$.

Observe that $\mathcal{S}(f^*\overline{\partial}) = f^{-1}(\mathcal{S}(\overline{\partial})) \subset f^{-1}(\mathcal{U}_2)$. By the construction of $\mathcal{U}_2$ we have $f^{-1}(\mathcal{U}_2) \subset \mathcal{U}'$. Hence we have $\mathcal{S}(f^*\overline{\partial}) \subset f^{-1}(\mathcal{U}_2) \subset \mathcal{U}'$ therefore it follows from Remark 3.4.7 that $(\text{proj}_2^* N_2, f^{-1}(\mathcal{U}_2))$ controls the compactness of $f^*\overline{\partial}$. □

3.7 Pulling Back Perturbations via the Permutation Maps

Proof of Corollary 3.1.5

Proof. Consider the pullback via σ of the polyfold Fredholm problem given by the strong polyfold bundle $\mathcal{W}_{A,g,k} \to \mathcal{Z}_{A,g,k}$ and the Cauchy–Riemann section $\overline{\partial}_J$, as illustrated in the below commutative diagram.

$$\begin{array}{ccc} \sigma^*\mathcal{W}_{A,g,k} & \xrightarrow[\text{proj}_2]{} & \mathcal{W}_{A,g,k} \\ {\scriptstyle \sigma^*\overline{\partial}_J} \uparrow\downarrow & & \downarrow\uparrow {\scriptstyle \overline{\partial}_J} \\ \mathcal{Z}_{A,g,k} & \xrightarrow[\sigma]{} & \mathcal{Z}_{A,g,k} \end{array}$$

The map σ is a homeomorphism when considered on the underlying topological spaces, and hence satisfies the topological pullback condition 3.1.2. Thus we have the required setup to apply Theorem 3.1.4 and obtain compatible perturbations Λ and $\text{proj}_2^*\Lambda$. □

Structure of the Permutation Map on the Perturbed Gromov–Witten Solution Spaces

As a result of Corollary 3.1.5, he permutation map restricts to a well-defined map between compact oriented weighted branched orbifolds,

$$\sigma|_{\mathcal{S}_{A,g,k}(\overline{\partial}_J, \text{proj}_2^*\Lambda)} : \mathcal{S}_{A,g,k}(\overline{\partial}_J, \text{proj}_2^*\Lambda) \to \mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda).$$

Considered on the underlying topological spaces, this map is a homeomorphism. Considered on the branched ep-subgroupoid structures, the associated functor

$$\hat{\sigma}|_{S_{A,g,k}(\hat{\overline{\partial}}_J,\mathrm{proj}_2^*\hat{\Lambda})} : S_{A,g,k}(\hat{\overline{\partial}}_J, \mathrm{proj}_2^*\hat{\Lambda}) \to S_{A,g,k}(\hat{\overline{\partial}}_J, \hat{\Lambda})$$

is a local diffeomorphism, and moreover is injective. The restricted permutation map σ and its associated functor $\hat{\sigma}$ are both weight preserving, i.e. $(\Lambda \circ \overline{\partial}_J) \circ \sigma = \mathrm{proj}_2^*\Lambda \circ \overline{\partial}_J$ and $(\hat{\Lambda} \circ \hat{\overline{\partial}}_J) \circ \hat{\sigma} = \mathrm{proj}_2^*\hat{\Lambda} \circ \hat{\overline{\partial}}_J$.

Remark 3.7.1. In [43] we will show that this functor is orientation preserving with respect to the orientations of the local branching structures.

Chapter 4

The Gromov–Witten Polyfold of Stable Curves with Constant Destabilizing Ghost Components

In general, there does not exist a well-defined map between the Gromov–Witten polyfolds $\mathcal{Z}_{A,g,k}$ and $\mathcal{Z}_{A,g,k-1}$ which forgets the kth-marked point. Even if we restrict to a subset $\mathcal{Z}^{\text{const}}_{A,g,k} \subset \mathcal{Z}_{A,g,k}$ where the kth marked point is defined, Proposition 2.6.5 shows that this map is not continuous.

In this chapter, we show how to give the set of stable curves with constant destabilizing ghost components a *new* polyfold structure, with a *new* sc-smooth structure, and a *new* topology. This new Gromov–Witten polyfold $\mathcal{Z}^{ft}_{A,g,k}$ has a modified gluing construction, designed to more accurately anticipate the geometry of the desired solution space. When the destabilizing ghost component is of type I, it interpolates the gluing parameters surrounding a ghost component directly. When the destabilizing ghost component is of type II, it forgets the gluing parameter, and relabels the remaining nodal point as a marked point.

This new polyfold carries the full abstract perturbation theory developed in [21]. We thus obtain well-defined Gromov–Witten invariants for this new polyfold, see Theorem 4.1.5. That these invariants coincide with the original polyfold Gromov–Witten invariants constructed in [22] is proved in Theorem 6.1.1.

Throughout this chapter, we will assume that $(A, g, k) \neq (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 0)$ and $(A, g, k) \neq (0, 0, 3), (0, 1, 1)$ as these cases have already been dealt with, see Remark 2.3.4 and Remark 2.6.2, respectively.

4.1 The Gromov–Witten Polyfold of Stable Curves with Constant Destabilizing Ghost Components

Definition 4.1.1. Suppose $(A, g, k) \neq (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 0), (0, 0, 3), (0, 0, 1)$. The new Gromov–Witten polyfold $\mathcal{Z}^{ft}_{A,g,k}$ is defined as the set of stable curves with constant destabilizing ghost components

$$\mathcal{Z}^{ft}_{A,g,k} := \{(\Sigma, j, M, D, u) \mid +ft\text{-stability condition}^*\}/\sim$$

where (Σ, j, M, D) is a connected noded Riemann surface (where we do not require the DM-stability condition), and which satisfies the same conditions as Definition 2.3.1 - except here we replace the stability condition* with the following.

- For each connected component $C \subset \Sigma$ the following ft-**stability condition** holds. We require at least one of the following:

$$2 \cdot g_C + \sharp(M \cup |D|)_C \geq 3 \quad \text{or} \quad \int_C u^*\omega > 0.$$

  Additionally, if the kth-marked point z_k lies on a component C_k with

$$2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} = 3 \quad \text{and} \quad \int_{C_k} u^*\omega = 0$$

  then we require that $u|_{C_k}$ is constant, hence necessarily $u|_{C_k} \equiv u(z_k)$.

- We require that u be of class $(3, \delta_0)$ at all marked points $\{z_1, \ldots z_{k-1}\}$. We require that u be of class H^3_{loc} at the marked point z_k (see Definition 2.3.2).

To understand why we require the second condition see Remark 4.2.8. We call a tuple (Σ, j, M, D, u) which satisfies the above a **stable map with constant destabilizing ghost component** C_k, and call an equivalence class satisfying the above a **stable curve with constant destabilizing ghost component** C_k.

Theorem 4.1.2. *The set $\mathcal{Z}^{ft}_{A,g,k}$ is a polyfold. In particular, this means:*

1. *we can give $\mathcal{Z}^{ft}_{A,g,k}$ a second countable paracompact topology,*

2. *we can give the topological space $\mathcal{Z}^{ft}_{A,g,k}$ a natural polyfold structure.*

For our new polyfold, we can describe an analogous GW-polyfold Fredholm problem as in Section 3.3.

Definition 4.1.3. The underlying set of the strong polyfold bundle $\mathcal{W}^{ft}_{A,g,k}$ is defined as an equivalence class of tuples $(\Sigma, j, M, D, u, \xi)$, similar to Definition 3.3.1 with data as follows.

- ξ is a continuous section along u such that the map

$$\xi(z) : T_zC \to T_{u(z)}Q, \qquad z \in C \subset \Sigma,$$

  is a complex anti-linear map, where $C \subset \Sigma$ is a connected component which is not a destabilizing ghost component, $C \neq C_k$,

- At a destabilizing ghost component $C_k \subset \Sigma$ we do not define a map ξ.

We have the following analog of [22, Theorem 4.6].

Theorem 4.1.4. *The set $\mathcal{W}^{ft}_{A,g,k}$ possesses a polyfold structure such that*

$$P : \mathcal{W}^{ft}_{A,g,k} \to \mathcal{Z}^{ft}_{A,g,k}, \qquad [\Sigma, j, M, D, u, \xi] \mapsto [\Sigma, j, M, D, u]$$

defines a strong polyfold bundle over the new GW-polyfold $\mathcal{Z}^{ft}_{A,g,k}$.

The Cauchy–Riemann section $\overline{\partial}_J$ of the strong polyfold bundle $P : \mathcal{W}^{ft}_{A,g,k} \to \mathcal{Z}^{ft}_{A,g,k}$ is a proper sc-smooth Fredholm section. The Fredholm index of $\overline{\partial}_J$ is given by

$$Ind\ (\overline{\partial}_J) = 2c_1(A) + (dim_{\mathbb{R}} Q - 6)(1 - g) + 2k.$$

We prove this in Section 4.5. By Proposition 3.4.10 there exists regular perturbations Λ of the Cauchy–Riemann section $\overline{\partial}_J$ with respect to a pair $(N,\mathcal{U})$. Hence the perturbed solution space

$$\mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \Lambda) := \{[x] \in \mathcal{Z}^{ft}_{A,g,k} \mid \Lambda \circ \overline{\partial}_J > 0\} \subset \mathcal{Z}^{ft}_{A,g,k}$$

has the structure of a compact weighted branched suborbifold, with weight function given by $\Lambda \circ \overline{\partial}_J : \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \Lambda) \to \mathbb{Q}^+$. We may integrate over the perturbed solution space to obtain a Gromov–Witten invariant, obtaining an analog of [22, Theorem 1.12].

Theorem 4.1.5 (Gromov–Witten Invariants for the New Polyfold)**.** *Let $\alpha_1, \ldots, \alpha_k \in H_*(Q; \mathbb{R})$, and let $\beta \in H_*(\overline{\mathcal{M}}^{log}_{g,k}; \mathbb{R})$. We define $GW^Q_{A,g,k}(\alpha_1, \ldots, \alpha_k; \beta)$ as the branched integral*

$$\int_{(\mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J,\Lambda),\Lambda\circ\overline{\partial}_J)} \bigwedge_{i=1}^{k} ev_i^* PD(\alpha_i) \wedge \pi^* PD(\beta)$$

where PD denotes the Poincaré dual.

4.2 New Gluing Construction at the Destabilizing Ghost Components

Consider a stable map $\alpha = (\Sigma, j, M, D, u)$ which satisfies the new stability condition, and suppose the marked point z_k lies on a destabilizing component C_k, hence

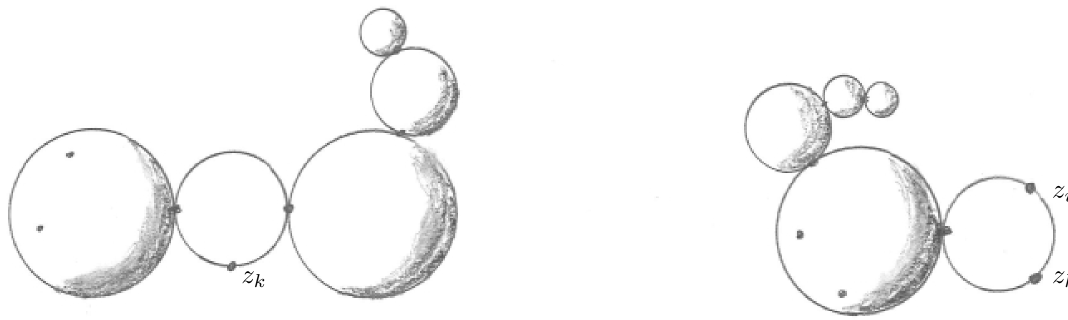
$$2 \cdot g(C_k) + \sharp(M \cup |D|)_{C_k} = 3 \qquad \text{and} \qquad \int_{C_k} u^*\omega = 0$$

and thus we require that $u|_{C_k} \equiv u(z_k)$.

By the classification of destabilizing ghost components in Definition 2.6.1 and by the assumption $(A, g, k) \neq (0, 0, 3), (0, 0, 1)$ we have the following two possibilities.

- The stable map α contains a **destabilizing ghost component of type I**. The ghost component $C_k \setminus \{z_k\}$ carries precisely two nodal points. In this case there are two nodal pairs $\{x_a, y_a\}$ and $\{x_b, y_b\}$ where $\{y_a\}, \{x_b\}$ lie on C_k.
- The stable map α contains a **destabilizing ghost component of type II**. The ghost component $C_k \setminus \{z_k\}$ carries precisely one nodal point and one other marked point. In this case there is a nodal pair $\{x_a, y_a\}$ where $\{y_a\}$ lies on C_k and a marked point z_i with $i \neq k$ which lies on C_k.

Figure 4.1: Destabilizing Ghost Components



Destabilizing Ghost Component of Type I Destabilizing Ghost Component of Type II

In what follows, we will define new gluing constructions for these cases, designed to more accurately model the expected behavior of the GW-moduli spaces on regions near a destabilizing ghost component. This new gluing procedure remains identical to the Deligne–Mumford gluing of the underlying Riemann surface Σ. This new gluing is a modification of the gluing construction originally given in [22, Section 2.4]. In order to prove sc-smoothness of the expressions for gluing and anti-gluing it will be important to use the exponential gluing profile given by $\varphi_{\exp}(r) = e^{1/r} - e$.

New Gluing at Destabilizing Ghost Components of Type I

Consider a stable map $\alpha = (\Sigma, j, M, D, u)$ which has a destabilizing ghost component of type I. Hence the component C_k contains the marked point z_k, and there exist two nodal pairs $\{x_a, y_a\}$ and $\{x_b, y_b\}$ such that $y_a, x_b \in C_k$. There is a unique biholomorphism between $C_k \setminus \{y_a, x_b\}$ and $\mathbb{R} \times S^1$ which sends the marked point z_k to the point $(0, 0)$, the puncture y_a to $-\infty$, and the puncture x_b to $+\infty$. We may choose the small disk structure at y_a such that there is a biholomorphism between $D_{y_a} \setminus \{y_a\}$ and $\mathbb{R}^- \times S^1 \subset \mathbb{R} \times S^1$. Likewise, we choose the small disk structure at x_b such that there is a biholomorphism between $D_{x_b} \setminus \{x_b\}$ and $\mathbb{R}^+ \times S^1 \subset \mathbb{R} \times S^1$.

Consider a section $\eta \in H^{3,\delta_0}_c(u^*TQ)$. In a neighborhood of the point $u(x_a) = u(y_a) \in Q$ choose a chart which maps $u(x_a) = u(y_a)$ to $0 \in \mathbb{R}^{2n}$. Localized to these coordinate neighborhoods, we may view the base map u as maps

$$u^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}, \qquad u^- : \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}$$

and likewise the section η as maps

$$\eta^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}, \qquad \eta^- : \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}$$

(using the identification $T_{u^\pm(s,t)}\mathbb{R}^{2n} = \mathbb{R}^{2n}$.)

We recall the construction of a glued Riemann surface of Definition 2.2.4 for our current situation. We change the notation and coordinates slightly, in order to define the new gluing in a straightforward way. The underlying glued Riemann surfaces are still identical to those constructed in Definition 2.2.4 however.

Consider the Riemann surface Σ. Write the gluing parameters $a \neq 0$ or $b \neq 0$ in polar coordinates as

$$a = r_a e^{i\theta_a} \text{ where } r_a \in (0, \tfrac{1}{2}),\ \theta_a \in \mathbb{R}/2\pi\mathbb{Z}; \qquad b = r_b e^{i\theta_b} \text{ where } r_b \in (0, \tfrac{1}{2}),\ \theta_b \in \mathbb{R}/2\pi\mathbb{Z}.$$

We replace $D_{x_a} \sqcup C_k \sqcup D_{y_b}$ with the glued cylinder

$$Z_{a,b} := \begin{cases} [0, R_a + R_b] \times S^1 & \text{when } a \neq 0, b \neq 0 \\ \mathbb{R}^+ \times S^1 \sqcup \mathbb{R}^- \times S^1 & \text{when } a \neq 0, b = 0 \text{ or } a = 0, b \neq 0 \\ \mathbb{R}^+ \times S^1 \sqcup \mathbb{R} \times S^1 \sqcup \mathbb{R}^- \times S^1 & \text{when } a = 0, b = 0. \end{cases}$$

We thus obtain the glued Riemann surface

$$\Sigma_{a,b} := \Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b}) \sqcup Z_{a,b}.$$

We now describe the new gluing procedure used to obtain maps on the glued cylinders $Z_{a,b}$.

Choose a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ which satisfies the following:

- $\beta(-s) + \beta(s) = 1 \quad \text{for all } s \in \mathbb{R}$
- $\beta(s) = 1 \quad \text{for all } s \leq -1$
- $\frac{d}{ds}\beta(s) < 0 \quad \text{for all } s \in (-1,1).$

Definition 4.2.1. For given gluing parameters $(a,b) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}}$, $a \neq 0$, $b \neq 0$, define the **new glued base map** $\oplus^{ft}_{a,b}(u^+,u^-) : Z_{a,b} \to \mathbb{R}^{2n}$ by

$$\oplus^{ft}_{a,b}(u^+,u^-)(s,t) := \beta\left(s - \tfrac{R_a+R_b}{2}\right)\cdot u^+(s,t) + \left((1 - \beta\left(s - \tfrac{R_a+R_b}{2}\right)\right)\cdot u^-(s - R_a - R_b, t - \theta_a - \theta_b)$$

For other values of a and b we define $\oplus^{ft}_{a,b}(u^+,u^-)$: by

$$\oplus^{ft}_{a,b}(u^+,u^-) := \begin{cases} (u^+,u^-) : \mathbb{R}^+ \times S^1 \sqcup \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}, & a \neq 0, b = 0 \text{ or } a = 0, b \neq 0 \\ (u^+,0,u^-) : \mathbb{R}^+ \times S^1 \sqcup \mathbb{R} \times S^1 \sqcup \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}, & a = 0, b = 0 \end{cases}$$

where $0 : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is the constant map to $0 \in \mathbb{R}^{2n}$.

Denote by E the vector space of pairs (η^+,η^-) of continuous maps

$$\eta^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^{2n}$$

satisfying

$$\lim_{s\to\infty} h^+(s,t) =: h^+(\infty) = h^-(-\infty) := \lim_{s\to-\infty} h^-(s,t)$$

uniformly in t, where $c := h^\pm(\pm\infty)$ are the asymptotic constants in $\mathbb{R}^{2n}$.

In order to describe data (via the new anti-gluing) that would otherwise be lost in the gluing procedure, we define corresponding cylinders $C_{a,b}$ by

$$C_{a,b} := \begin{cases} \mathbb{R} \times S^1 & \text{when } a \neq 0, b \neq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Definition 4.2.2. For given gluing parameters $(a,b) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}}$, $a \neq 0$, $b \neq 0$, define the **new glued section** $\oplus^{ft}_{a,b}(\eta^+,\eta^-) : Z_{a,b} \to \mathbb{R}^{2n}$ by

$$\oplus^{ft}_{a,b}(\eta^+,\eta^-)(s,t) := \beta\left(s - \tfrac{R_a+R_b}{2}\right)\cdot u^+(s,t) + \left((1 - \beta\left(s - \tfrac{R_a+R_b}{2}\right)\right)\cdot u^-(s-R_a-R_b, t-\theta_a-\theta_b)$$

For other values of a and b we define $\oplus^{ft}_{a,b}(\eta^+,\eta^-)$ by

$$\oplus^{ft}_{a,b}(\eta^+,\eta^-) := \begin{cases} (\eta^+,\eta^-) : \mathbb{R}^+ \times S^1 \sqcup \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}, & a \neq 0, b = 0 \text{ or } a = 0, b \neq 0 \\ (\eta^+,c,\eta^-) : \mathbb{R}^+ \times S^1 \sqcup \mathbb{R} \times S^1 \sqcup \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}, & a = 0, b = 0 \end{cases}$$

where $c : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is the constant map to the asymptotic constant $c = \eta^\pm(\pm\infty) \in \mathbb{R}^{2n}$.

For $a \neq 0$ and $b \neq 0$ define the **new anti-glued section** as the map $\ominus^{ft}_{a,b}(\eta^+,\eta^-)$: $C_{a,b} \to \mathbb{R}^{2n}$,

$$\begin{aligned}\ominus^{ft}_{a,b}(\eta^+,\eta^-)(s,t) := &- \left(1 - \beta\left(s - \tfrac{R_a+R_b}{2}\right)\right) \cdot [\eta^+(s,t) - \mathrm{av}_{a,b}(\eta^+,\eta^-)] \\ &+ \beta\left(s - \tfrac{R_a+R_b}{2}\right) \cdot [\eta^-(s - R_a - R_b, t - \theta_a - \theta_b) - av_{a,b}(\eta^+,\eta^-)]\end{aligned}$$

where

$$\mathrm{av}_{a,b}(\eta^+,\eta^-) := \tfrac{1}{2}\left(\int_{S^1} \eta^+(\tfrac{R_a+R_b}{2},t)dt + \int_{S^1} \eta^-(-\tfrac{R_a+R_b}{2},t)dt\right).$$

For other values of a and b we define $\ominus^{ft}_{a,b}(\eta^+,\eta^-)$ as the unique map $\emptyset \to \mathbb{R}^{2n}$.

Remark 4.2.3. As in [22, p60-61], there are analagous new hat gluings and new hat antiglu-ings, used to define the polyfold strong bundles.

Definition 4.2.4. We may assume our chart is chosen such that we may identify the Riemannian metric on Q with the Euclidean metric on $\mathbb{R}^{2n}$; hence we may identify the maps $\exp_{\oplus^{ft}_{a,b}(u^+,u^-)} \oplus^{ft}_{a,b}(\eta^+,\eta^-)$ and $\oplus^{ft}_{a,b}(\exp_{(u^+,u^-)}(\eta^+,\eta^-))$ (see [22, Proposition 2.51]). The **new glued map at a destabilizing ghost component of type I** $\oplus^{ft}_{a.b} \exp_u(\eta) : \Sigma_{a,b} \to Q$ is defined by

$$\oplus^{ft}_{a,b} \exp_u(\eta) := \begin{cases} \exp_u(\eta) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_a}) \\ \oplus^{ft}_{a,b} \exp_u(\eta) & \text{on } Z_{a,b}. \end{cases}$$

Remark 4.2.5. We will sometimes use the abbreviations $R_{a,b} := \varphi(|a|)+\varphi(|b|)$, $\vartheta_{a,b} := \vartheta_a+\vartheta_b$.

If $(a, b) \in \mathbb{C}^2$ are gluing parameters with $a \neq 0$, $b \neq 0$, and $R_{a,b}$ the associated gluing length, we introduce the translated function $\beta_{a,b} : \mathbb{R} \to \mathbb{R}$ defined by

$$\beta_{a,b}(s) := \beta\left(s - \frac{R_{a,b}}{2}\right).$$

The average is the number

$$\mathrm{av}_{a,b}(h^+, h^-) := \frac{1}{2}\left([h^+]_{R_{a,b}} + [h^-]_{R_{a,b}}\right)$$

where

$$[h^\pm]_{R_{a,b}} := \int_{S^1} h^\pm\left(\pm\frac{R_{a,b}}{2}, t\right)\, dt.$$

New Gluing at Destabilizing Ghost Components of Type II

Consider a stable map $\alpha = (\Sigma, j, M, D, u)$ which has a destabilizing ghost component of type II. Hence the component C_k contains the marked point z_k together with another marked point z_i, and there exist a nodal pairs $\{x_a, y_a\}$ such that $y_a \in C_k$. There is a unique biholomorphism between $C_k \setminus \{y_a, z_i\}$ and $\mathbb{R} \times S^1$ which sends the marked point z_k to the point $(0, 0)$, the puncture y_a to $-\infty$, and the puncture at the marked point z_i to $+\infty$. We may choose the small disk structure at y_a such that there is a biholomorphism between $D_{y_a} \setminus \{y_a\}$ and $\mathbb{R}^- \times S^1 \subset \mathbb{R} \times S^1$.

Consider a section $\eta \in H^{3,\delta_0}_c(u^*TQ)$. In a neighborhood of the point $u(x_a) = u(y_a) \in Q$ choose a chart which maps $u(x_a) = u(y_a)$ to $0 \in \mathbb{R}^{2n}$. Localized to these coordinate neighborhoods, we may view the base map u as maps

$$u^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}$$

and likewise the section η as the map

$$\eta^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n}, \qquad \eta^- : \mathbb{R}^- \times S^1 \to \mathbb{R}^{2n}$$

(using the identification $T_{u^\pm(s,t)}\mathbb{R}^{2n} = \mathbb{R}^{2n}$.)

There is a single gluing parameter $a \in B_{\frac{1}{2}}$ associated to the node $\{x_a, y_a\}$. When $a \neq 0$ we may represent a as $a = |a| \cdot e^{-2\pi i \vartheta_a}$. We use this gluing parameter to parametrize movement of the marked point z_k as follows:

- For $a = 0$, the marked point z_k is given by $(0,0)$ on the component $\mathbb{R} \times S^1$, while the marked point z_i is given by the puncture at $+\infty$.
- For $a \neq 0$ the marked point z_k is given by $(R_a, \vartheta_a) \in \mathbb{R}^+ \times S^1$, while the marked point z_i is again given by the puncture at $+\infty$ in $\mathbb{R}^+ \times S^1$.

Definition 4.2.6. In this case we use a single gluing parameter which serves to parametrize movement of the marked point z_k; no interpolation of maps on $\mathbb{R}^\pm \times S^1$ is necessary. For a given gluing parameter $a \in B_{\frac{1}{2}}$, define the **new glued base map** $\oplus_a^{ft}(u^+) : Z_a \to \mathbb{R}^{2n}$ by

$$\oplus_a^{ft}(u^+)(s,t) := \begin{cases} u^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n} & \text{when } a \neq 0 \\ (u^+, 0) : \mathbb{R}^+ \times S^1 \sqcup \mathbb{R} \times S^1 \to \mathbb{R}^{2n} & \text{when } a = 0 \end{cases}$$

where $0 : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is the constant map to $0 \in \mathbb{R}^{2n}$.

Likewise, for a given gluing parameter $a \in B_{\frac{1}{2}}$, define the **new glued section** $\oplus_a^{ft}(\eta^+) : Z_a \to \mathbb{R}^{2n}$ by

$$\oplus_a^{ft}(\eta^+)(s,t) := \begin{cases} \eta^+ : \mathbb{R}^+ \times S^1 \to \mathbb{R}^{2n} & \text{when } a \neq 0 \\ (\eta^+, c) : \mathbb{R}^+ \times S^1 \sqcup \mathbb{R} \times S^1 \to \mathbb{R}^{2n} & \text{when } a = 0 \end{cases}$$

where $c : \mathbb{R} \times S^1 \to \mathbb{R}^{2n}$ is the constant map to the asymptotic constant $c = \eta^+(\infty) \in \mathbb{R}^{2n}$.

Definition 4.2.7. We may assume our chart is chosen such that we may identify the Riemannian metric on Q with the Euclidean metric on $\mathbb{R}^{2n}$. The **new glued map at a destabilizing ghost component of type II** $\oplus_a^{ft} \exp_u(\eta) : \Sigma_a \to Q$ is defined by

$$\oplus_a^{ft} \exp_u(\eta) := \begin{cases} \exp_u(\eta) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k) \\ \oplus_a^{ft} \exp_u(\eta) = \exp_u(\eta) & \text{on } Z_a. \end{cases}$$

Remark 4.2.8. By Definition 2.3.1 because x_a is a nodal point, the map $u : \Sigma \to Q$ is of class $(3, \delta_0)$ when restricted to the region $D_{x_a} \setminus \{x_a\}$. After forgetting the marked point z_k and deleting the ghost component C_k, we delete the nodal pair $\{x_a, y_a\}$ relabel x_a as the new marked point z_i. We are left with a map that is of class $(3, \delta_0)$ at the new marked point z_i. Hence, in order to get a well-defined map

$$ft_k : \mathcal{Z}^{ft}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}$$

we must require regularity $(3, \delta_0)$ at punctures at the marked points $\{1, \ldots, k-1\}$ for both of these GW-polyfolds, $\mathcal{Z}^{ft}_{A,g,k}$ and $\mathcal{Z}_{A,g,k-1}$.

Definition 4.2.9. Assume that we have good data and a stabilization centered at a stable map (Σ, j, M, D, u) with isotropy group $\boldsymbol{G}(\Sigma, j, M, D, u)$. We can define **good uniformizing family of stable maps associated to the new gluing** centered at (Σ, j, M, D, u) as a family of stable maps

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta)) \quad \text{where} \quad (\underline{a}, v, \eta) \in \mathcal{O}.$$

Here $\mathcal{O}$ is an open subset of a splicing core K. The family of stable noded Riemann surfaces $(\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}})$ is defined in the same manner as in Definition 2.2.6.

If the stable map (Σ, j, M, D, u) contains a destabilizing ghost component of type I, we define the new glued map $\oplus_{\underline{a}} \exp_u(\eta) : \Sigma_{\underline{a}} \to Q$ using the gluing constructions of Definition 2.3.6 and Definition 4.2.4 by the following:

$$\oplus^{ft}_{\underline{a}} \exp_u(\eta) := \begin{cases} \exp_u(\eta) & \text{on } \Sigma \setminus \cup_{\{x_a, y_a\} \in D} (D_{x_a} \sqcup D_{y_a}) \\ \oplus^{ft}_{a,b} \exp_u(\eta) & \text{on } Z_{a,b} \\ \oplus_{a'} \exp_u(\eta) & \text{on } Z_{a'} \text{ for all other gluing paramters } a' \in \underline{a}. \end{cases}$$

If the stable map (Σ, j, M, D, u) contains a destabilizing ghost component of type II, we define the new glued map $\oplus_{\underline{a}} \exp_u(\eta) : \Sigma_{\underline{a}} \to Q$ using the gluing constructions of Definition 2.3.6 and Definition 4.2.7 by the following:

$$\oplus^{ft}_{\underline{a}} \exp_u(\eta) := \begin{cases} \exp_u(\eta) & \text{on } \Sigma \setminus \cup_{\{x_a, y_a\} \in D} (D_{x_a} \sqcup D_{y_a}) \\ \exp_u(\eta) & \text{on } Z_a \\ \oplus_{a'} \exp_u(\eta) & \text{on } Z_{a'} \text{ for all other gluing paramters } a' \in \underline{a}. \end{cases}$$

4.3 Technical Construction of the New Polyfold

In order to carry over the GW-polyfold theory of [22] there are three essential technical results that we must verify:

- (Analog of [22, Theorem 2.50]) Proposition 4.3.1 which concerns the sc-smoothness of our new gluing construction
- (Analog of [22, Theorem 3.13]) Proposition 4.3.10 which concerns the sc-smooth compatibility of our newly added M-polyfold charts
- (Analog of [22, Theorem 3.22]) Proposition 4.3.13 a compactness statement for the morphisms between two M-polyfold charts.

sc-Smoothness of the New Gluing

Consider the projection map

$$\begin{aligned} \pi_{a,b} : & E \to E \\ & (\xi^+, \xi^-) \to (\eta^+, \eta^-) \end{aligned}$$

which is uniquely defined by the equations:

$$\oplus_{a,b}^{ft}(\eta^+, \eta^-) = \oplus_{a,b}^{ft}(\xi^+, \xi^-) \quad \text{and} \quad \ominus_{a,b}^{ft}(\eta^+, \eta^-) = 0.$$

Abbreviate $\beta_{a,b} = \beta(s^+ - \frac{R_{a,b}}{2})$ and $\gamma_{a,b} = \beta_{a,b}^2(s^+) + (1 - \beta_{a,b}(s^+))^2$. We may write down the following explicit formulas:

$$\begin{aligned} \eta^+(s^+, t^+) = &\left(1 - \frac{\beta_{a,b}}{\gamma_{a,b}}\right) \cdot \mathrm{av}_{a,b}(\xi^+, \xi^-) + \frac{\beta_{a,b}^2}{\gamma_{a,b}} \cdot \xi^+(s^+, t^+) \\ &+ \frac{\beta_{a,b}(1-\beta_{a,b})}{\gamma_{a,b}} \cdot \xi^-(s^+ - R_{a,b}, t^+ - \vartheta_{a,b}) \end{aligned} \tag{4.1}$$

for $(s^+, t^+) \in R^+ \times S^1$. A similar calculation leads to the following formula for η^-:

$$\begin{aligned} \eta^-(s^-, t^-) = &\left(1 - \frac{\beta_{a,b}(-s^-)}{\gamma_{a,b}(-s^-)}\right) \cdot \mathrm{av}_{a,b}(\xi^+, \xi^-) \\ &+ \frac{\beta_{a,b}(-s^-)(1-\beta_{a,b}(-s^-))}{\gamma_{a,b}(-s^-)} \cdot \xi^+(s^- + R_{a,b}, t^- + \vartheta_{a,b}) + \frac{\beta_{a,b}(-s^-)^2}{\gamma_{a,b}(-s^-)} \xi^-(s^-, t^-) \end{aligned} \tag{4.2}$$

for $(s^-, t^-) \in \mathbb{R}^- \times S^1$.

Proposition 4.3.1. *The map*

$$\pi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times E \to E, \quad (a, b, (\xi^+, \xi^-)) \mapsto \pi_{a,b}(\xi^+, \xi^-)$$

is sc-smooth.

Proof. The argument follows precisely the same reasoning as in the proof that the projection π_a defined via the usual gluing and anti-gluing is sc-smooth, namely by checking differentiability of individual terms in the explicit formula for $\pi_{a,b}$ and applying the chain rule. Full details must follow the lengthy arguments given in [19, Section 2.4].

If we write $\xi^\pm = c + r^\pm$, where c is the common asymptotic constant, then the formula for η^+ takes the form

$$\begin{aligned} \eta^+(s,t) = c + &\frac{1}{2}\left(1 - \frac{\beta_{a,b}}{\gamma_{a,b}}(s)\right) \cdot ([r^+]_{R_{a,b}} + [r^-]_{-R_{a,b}}) \\ &+ \frac{\beta_{a,b}^2}{\gamma_{a,b}}(s) \cdot r^+(s,t) + \frac{\beta_{a,b}(1-\beta_{a,b})}{\gamma_{a,b}}(s) \cdot r^-(s - R_{a,b}, t - \vartheta_{a,b}) \end{aligned} \tag{4.3}$$

We can decompose this expression into a combination of the following maps.

M1. The map

$$H^{3,\delta_0}_c(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N, \quad \xi^+ \mapsto c$$

which associates with ξ^+ its asymptotic constant c.

M2. The map

$$B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N, \quad (a, b, r^+) \mapsto [r^+]_{R_{a,b}}.$$

M3. The map

$$B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N), \quad (a, b, r^+) \mapsto \frac{\beta_{a,b}}{\gamma_{a,b}} \cdot [r^+]_{R_{a,b}}.$$

M4. The map

$$B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N), \quad (a, b, r^+) \mapsto \frac{\beta^2_{a,b}}{\gamma_{a,b}} \cdot r^+.$$

M5. The map

$$B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N),$$
$$(a, b, r^-) \mapsto \frac{\beta_{a,b}(1-\beta_{a,b})}{\gamma_{a,b}} r^-(\cdot - R_{a,b}, \cdot - \vartheta_{a,b}).$$

Proposition 4.3.2. *The maps **M1-M5** listed above are sc-smooth in a neighborhood of $a = b = 0$.*

The proof of the proposition follows from a sequence of lemmas.

Lemma 4.3.3. *[19, Lemma 2.18] The map $H^{3,\delta_0}_c(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N, \quad \xi^+ \mapsto c$, which associates with ξ^+ its asymptotic constant c is sc-smooth.*

Lemma 4.3.4. *[19, Lemma 2.12] Consider the map $R : B_{\frac{1}{2}} \setminus \{0\} \to [0, \infty)$ defined via the exponential gluing profile by*

$$R_a = e^{\frac{1}{|a|}} - e.$$

For every multi-index $\alpha = (\alpha_1, \alpha_2)$, there exists a constant C such that

$$|DR_a| \leq C \cdot R_a \cdot [\ln(R_a)]^{2|\alpha|}$$

for $0 < |a| < \frac{1}{2}$.

Lemma 4.3.5. *(Analog of [19, Lemma 2.19]) The map* $\Phi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to \mathbb{R}^N$ *, defined by* $\Phi(a, b, h) = 0$ *for* $a = 0$ *or* $b = 0$ *and*

$$\Phi(a, b, h) = [h]_{R_{a,b}} = \int_{S^1} h\left(\frac{R_{a,b}}{2}, t\right)\, dt$$

for $a \neq 0, b \neq 0$, *is sc-smooth.*

Proof. We follow the argument of [19, Lemma 2.19]. We abbreviate the sc-Banach space by $F = H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$.

Using the Sobolev embedding theorem for bounded domains into continuously differentiable functions we see that the map

$$(0, \infty) \times (0, \infty) \times F_m \to C^0(S^1, \mathbb{R}^N), \quad (R_a, R_b, h) \mapsto h\left(\frac{R_a + R_b}{2}, \cdot\right)$$

is of class C^{m+1} for every $m \geq 0$. By [19, Corollary 2.5] this implies that the map

$$\hat{\Phi} : (0, \infty) \times (0, \infty) \times F \to \mathbb{R}^N, \quad (R_a, R_b, h) \mapsto [h]_{R_{a,b}}$$

is sc-smooth. The maps $a \mapsto R_a := \varphi(|a|)$ and $b \mapsto R_b := \varphi(|b|)$ are diffeomorphisms. Using the chain rule for sc-smooth maps, it follows that the map

$$\Phi : (B_{\frac{1}{2}} \setminus \{0\}) \times (B_{\frac{1}{2}} \setminus \{0\}) \times F \to \mathbb{R}^N, \quad (a, b, h) \mapsto [h]_{R_{a,b}}$$

is sc-smooth.

We claim that the map Φ is sc^0 at every point $(a, 0, h)$, $(0, b, h)$, and $(0, 0, h) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F$. Indeed, suppose that $(a_k, b_k h_k) \in (B_{\frac{1}{2}} \setminus \{0\}) \times (B_{\frac{1}{2}} \setminus \{0\}) \times F_m$ is a sequence converging to $(a, 0, h)$, $(0, b, h)$ or $(0, 0, h)$. We show that $|\Phi(a_k, b_k, h_k)| = |[h_k]_{R_{a_k,b_k}}| \to 0$.

Abbreviate $\Sigma_k = (\frac{R_k}{2} - 1, \frac{R_k}{2} + 1) \times S^1$ where $R_k := R_{a_k,b_k} = \varphi(|a_k|) + \varphi(|b_k|)$. By the Sobolev embedding theorem on bounded domains and using the bound $|h_k|_m \leq C'$, we estimate

$$|e^{\delta_m \cdot} \cdot h_k|_{C^0(\Sigma_k)} \leq C|e^{\delta_m \cdot} \cdot h_k|_{H^{m+3}(\Sigma_k)} \leq C''.$$

This implies

$$|[h_k]_{R_k}| \leq C'' \cdot e^{-\delta_m R_k/2} \tag{4.4}$$

and hence Φ is sc^0 as claimed.

At this point we know that

1. $\Phi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F \to \mathbb{R}^N$ is sc^0,
2. the restriction to $(B_{\frac{1}{2}} \setminus \{0\}) \times (B_{\frac{1}{2}} \setminus \{0\}) \times F$ is sc^∞.

We shall denote points in $T^k(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F)$ by (a, b, H). We shall prove inductively the following.

($\mathbf{S}_k$). *The map* $\Phi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F \to \mathbb{R}^N$ *is of class* sc^k *and* $T^k\Phi(a,0,H) = T^k\Phi(0,b,H) = T^k\Phi(0,0,H) = 0$ *for every* $(a,0,H)$, $(0,b,H)$, *and* $(0,0,H)$ *in* $T^k(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F)$. *Moreover, if* $\pi : T^k(\mathbb{R}^N) \to \mathbb{R}^N$ *is the projection onto a factor of* $T^k\mathbb{R}^N$, *then the composition* $\pi \circ T^k\Phi$ *is a linear combination of maps* Γ *of the following types,*

$$\begin{aligned}&\Gamma : B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F_m \to \mathbb{R}^N,\\ &(a,\alpha,b,\beta,v) \mapsto \boldsymbol{R}(a)(\alpha_1,\ldots,\alpha_{n_a}) \cdot \boldsymbol{R}(b)(\beta_1,\ldots,\beta_{n_b}) \cdot [\partial_s^j v]_{R_{a,b}}\end{aligned}$$

for $a \neq 0, b \neq 0$ *and* $\Gamma(a,\alpha,0,\beta,v) = \Gamma(0,\alpha,b,\beta,v) = \Gamma(0,\alpha,0,\beta,v) = 0$. *Here* $j \leq m$, *and* $n_a + n_b \leq k$. *Moreover,* $\boldsymbol{R}(a)$ *is the product of derivatives of the function* $R(a) = e^{\frac{1}{|a|}} - e$ *of the form*

$$\boldsymbol{R}(a)(\alpha_1,\ldots,\alpha_{n_a}) = D^{n_1}R(a)(\alpha_1,\ldots,\alpha_{n_1}) \cdot \ldots \cdot D^{n_l}R(a)(\alpha_{n_1+\ldots+n_{l-1}+1},\ldots,\alpha_{n_a}),$$

where the integer $n_a = n_1 + \ldots + n_l$ *is called the order of* $\boldsymbol{R}(a)$. *We set* $\boldsymbol{R}(a) = 1$ *if* $n_a = 0$. *We define* $\boldsymbol{R}(b)$ *similarly.*

We begin by verifying that ($\mathbf{S}_0$) holds. In this case, the projection $\pi : T^0\mathbb{R}^N = \mathbb{R}^N \to \mathbb{R}^N$ is the identity map, the indices j,k,m and n_a, n_b are equal to 0, and the composition $\pi \circ T^0\Phi$ is just the map $\Phi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F \to \mathbb{R}^N$ given by

$$(a,b,v) \to [v]_{R_{a,b}}.$$

The map has the required form with $\mathbf{R}(a) = \mathbf{R}(b) = 1$ of order 0. With $\Phi(a,0,v) = \Phi(0,b,v) = \Phi(0,0,v) = 0$, we already know that Φ is sc^0. Therefore, the base case ($\mathbf{S}_0$) holds.

Assuming that ($\mathbf{S}_k$) holds, we show that ($\mathbf{S}_{k+1}$) also holds. By induction hypothesis, the map Φ is sc^k, so that $T^k\Phi$ is sc^0. Moreover, $T^k\Phi(a,0,H) = T^k\Phi(0,b,H) = T^k\Phi(0,0,H) = 0$, $T^{k+1}\Phi$ is sc-smooth at points (a,b,H) with $a \neq 0, b \neq 0$, and $\pi \circ T^k\Phi$ can be written as a linear combination of maps of a certain form.

Setting $DT^k\Phi(a,0,H) = DT^k\Phi(0,b,H) = DT^k\Phi(0,0,H) = 0$, we will approximate $T^k\Phi$ at the points $(a,0,H),(0,b,H),(0,0,H) \in (T^k(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F)^1$ as follows. Recalling that $T^k\Phi(a,0,H) = T^k\Phi(0,b,H) = T^k\Phi(0,0,H) = 0$, for $(a,b,H) = (a,0,H),(0,b,H)$, or $(0,0,H)$, we will show that

$$\frac{1}{\|(\delta a,\delta b,\delta H)\|_1}|T^k\Phi(a+\delta a,b+\delta b,H+\delta H)|_0 \to 0 \quad \text{as } \|(\delta a,\delta b,\delta H)\|_1 \to 0. \tag{4.5}$$

where the subscripts 0 and 1 refer to the levels of the iterated tangents. By the inductive assumption ($\mathbf{S}_k$), we know that the compositions $\pi \circ T^k\Phi$ with projections π on different factors of $T^k\mathbb{R}^N$ are linear combinations of maps Γ described in ($\mathbf{S}_k$). The proof of (4.5) amounts to showing that at the points $(0,\alpha,b,\beta,v),(a,\alpha,0,\beta,v),(0,\alpha,0,\beta,v)\in B_{\frac12}\times\mathbb{C}^{n_a}\times B_{\frac12}\times\mathbb{C}^{n_b}\times F^{m+1}$ we have

$$\frac{1}{|\delta a|+|\delta\alpha|+|\delta b|+|\delta\beta|+|\delta v|_{m+1}}|\Gamma(a+\delta a,\alpha+\delta\alpha,b+\delta b,\beta+\delta\beta,v+\delta v)|\to 0 \tag{4.6}$$

as $|\delta a|+|\delta\alpha|+|\delta b|+|\delta\beta|+|\delta v|_{m+1}\to 0$ for the maps

$$\begin{aligned}&\Gamma: B_{\frac12}\times\mathbb{C}^{n_a}\times B_{\frac12}\times\mathbb{C}^{n_b}\times F^m\to\mathbb{R}^N,\\ &(a,\alpha,b,\beta,v)\mapsto \mathbf{R}(a)(\alpha_1,\dots,\alpha_{n_a})\cdot\mathbf{R}(b)(\beta_1,\dots,\beta_{n_b})\cdot[\partial_s^j v]_{R_{a,b}}\end{aligned}$$

defined in $(\mathbf{S}_k)$.

Using as above the Sobolev estimate on the bounded domain $\Sigma_{R_{a,b}}=\left(\frac{R_{a,b}}{2}-1,\frac{R_{a,b}}{2}+1\right)\times S^1$, we obtain

$$|e^{\delta_{m+1}\cdot}\partial_s^j(v+\delta v)|_{C^0(\Sigma_{R_{a,b}})}\le C|e^{\delta_{m+1}\cdot}\partial_s^j(v+\delta v)|_{H^{m+3}(\Sigma_{R_{a,b}})},$$

where $j\le m$, and estimate

$$[\partial_s^j(v+\delta v)]_R\le Ce^{-\delta_{m+1}\frac{R_{a,b}}{2}}|v+\delta v|_{m+1}.$$

Utilizing Lemma 4.3.4, we obtain the estimates

$$\begin{aligned}&|\Gamma(a+\delta a,\alpha+\delta\alpha,b+\delta b,\beta+\delta\beta,v+\delta v)|\\ &\qquad\le C^2\cdot e^{-\delta_{m+1}\frac{R_{a,b}}{2}}\cdot|R_{a+\delta a}|^{3n_a}\cdot|\alpha+\delta\alpha|^{n_a}\cdot|R_{b+\delta b}|^{3n_b}\cdot|\beta+\delta\beta|^{n_b}\cdot|v+\delta v|_{m+1}\end{aligned}$$

for $a+\delta a\neq 0$ and $b+\delta b\neq 0$. Consequently,

$$\begin{aligned}&\frac{|\Gamma(a+\delta a,\alpha+\delta\alpha,b+\delta b,\beta+\delta\beta,v+\delta v)|}{|\delta a|+|\delta\alpha|+|\delta b|+|\delta\beta|+|\delta v|_{m+1}}\\ &\le\frac{C^2\cdot e^{-\delta_{m+1}\frac{R_{a+\delta a,b+\delta b}}{2}}\cdot|R_{a+\delta a}|^{3n_a}\cdot|R_{b+\delta b}|^{3n_b}}{|\delta a|+|\delta\alpha|+|\delta b|+|\delta\beta|+|\delta v|_{m+1}}\cdot|\alpha+\delta\alpha|^{n_a}\cdot|\beta+\delta\beta|^{n_b}\cdot|v+\delta v|_{m+1}.\end{aligned}\tag{4.7}$$

In the case $(a,\alpha,0,\beta,v)$, for δb small we have $2R_{\delta b}\ge 2\ln R_{\delta b}\ge\frac{1}{|\delta b|}$, we can bound the right-hand side of (4.7) by

$$C^2\cdot e^{-\delta_{m+1}\frac{R_{a+\delta a,\delta b}}{2}}\cdot|R_{a+\delta a}|^{3n_a}\cdot|R_{\delta b}|^{4n_b}\cdot|\alpha+\delta\alpha|^{n_a}\cdot|\beta+\delta\beta|^{n_b}\cdot|v+\delta v|_{m+1}$$

In the case $(0,\alpha,b,\beta,v)$, for δa small we have $2R_{\delta a}\ge 2\ln R_{\delta a}\ge\frac{1}{|\delta a|}$, we can bound the right-hand side of (4.7) by

$$C^2\cdot e^{-\delta_{m+1}\frac{R_{\delta a,b+\delta b}}{2}}\cdot|R_{\delta a}|^{4n_a}\cdot|R_{b+\delta b}|^{3n_b}\cdot|\alpha+\delta\alpha|^{n_a}\cdot|\beta+\delta\beta|^{n_b}\cdot|v+\delta v|_{m+1}$$

In the case $(0, \alpha, 0, \beta, v)$, for δa small we have $2R_{\delta a} \geq 2 \ln R_{\delta a} \geq \frac{1}{|\delta a|}$, we can bound the right-hand side of (4.7) by

$$C^2 \cdot e^{-\delta_{m+1} \frac{R_{\delta a, \delta b}}{2}} \cdot |R_{\delta a}|^{4n_a} \cdot |R_{\delta b}|^{3n_b} \cdot |\alpha + \delta\alpha|^{n_a} \cdot |\beta + \delta\beta|^{n_b} \cdot |v + \delta v|_{m+1}$$

Each of these expressions converges to 0 as $(\delta a, \delta\alpha, \delta b, \delta\beta, \delta v) \to (0, 0, 0, 0, 0)$ in $\mathbb{C} \times \mathbb{C}^{n_a} \times \mathbb{C} \times \mathbb{C}^{n_b} \times F^{m+1}$. Summing up our discussion so far, we have shown the map $T^k\Phi$ satisfies

$$DT^k\Phi(a, 0, H) = DT^k\Phi(0, b, H) = DT^k\Phi(0, 0, H) = 0$$

for all $(a, 0, H), (0, b, H), (0, 0, H) \in \left(T^k(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F)\right)_1$. To complete the proof, it remains to show that $T^{k+1}\Phi$ is of class sc^0 (which will imply that Φ is of class sc^k) and to show that the compositions $\pi \circ T^{k+1}\Phi$ have the required form.

We now consider $\pi \circ T^{k+1}\Phi(a, 0, H)$, $\pi \circ T^{k+1}\Phi(0, b, H)$, and $\pi \circ T^{k+1}\Phi(0, 0, H)$. If π is the projection onto one of the first 2^k factors, then $\pi \circ T^{k+1}\Phi$ has the form of the map Γ in $(\mathbf{S}_k)$. The only thing is that the indices are raised by 1. Denoting the new indices by j', m', and n'_a, n'_b, we have $j' = j$, $m' = m + 1$, and $n'_a = n_a, n'_b = n_b$ which obviously satisfy $j' \leq m' \leq k + 1$, and $n'_a + n'_b \leq k + 1$. If π is the projection onto one of the remaining 2^k factors, then $\pi \circ T^{k+1}\Phi$ is equal to the sum of derivatives of maps in the induction hypothesis $(\mathbf{S}_k)$. So, if the map

$$\begin{aligned}\Gamma &: B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F_m \to \mathbb{R}^N, \\ &(a, \alpha, b, \beta, v) \mapsto \mathbf{R}(a)(\alpha_1, \ldots, \alpha_{n_a}) \cdot \mathbf{R}(b)(\beta_1, \ldots, \beta_{n_b}) \cdot [\partial_s^j v]_{R_{a,b}}\end{aligned}$$

for $a \neq 0, b \neq 0$ and $\Gamma(a, \alpha, 0, \beta, v) = \Gamma(0, \alpha, b, \beta, v) = \Gamma(0, \alpha, 0, \beta, v) = 0$, is one of the maps from $(\mathbf{S}_k)$ and if we take the sc-derivative of Γ (which we have shown exists at every point), we obtain a linear combination of maps of the following types:

1. $B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F^{m+1} \to \mathbb{R}^N$ defined by

   $$(a, \alpha_1, \ldots \delta\alpha_i, \ldots, \alpha_{n_a}, b, \beta, v) \mapsto \mathbf{R}(a)(\alpha_1, \ldots, \delta\alpha_i \ldots, \alpha_{n_a}) \cdot \mathbf{R}(b)(\beta_1, \ldots, \beta_{n_b}) \cdot [\partial_s^j v]_{R_{a,b}}$$

   for every $1 \leq i \leq n_a$.

2. $B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F^{m+1} \to \mathbb{R}^N$ defined by

   $$(a, \alpha, b, \beta_1, \ldots, \delta b_i, \ldots, \beta_{n_b}, v) \mapsto \mathbf{R}(a)(\alpha_1, \ldots, \alpha_{n_a}) \cdot \mathbf{R}(b)(\beta_1, \ldots, \delta\beta_i \ldots, \beta_{n_b}) \cdot [\partial_s^j v]_{R_{a,b}}$$

   for every $1 \leq i \leq n_b$.

3. $B_{\frac{1}{2}} \times \mathbb{C}^{n_a+1} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F^{m+1} \to \mathbb{R}^N$ defined by

   $$(a, (\delta a, \alpha), b, \beta, v) \mapsto \mathbf{R}'(a)(\delta a, \alpha) \cdot \mathbf{R}(b)(\beta) \cdot [\partial_s^j v]_{R_{a,b}}$$

   and obtained by differentiation of $\mathbf{R}(a)$ with respect to a.

4. $B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b+1} \times F^{m+1} \to \mathbb{R}^N$ defined by

$$(a, \alpha, b, (\delta b, \beta), v) \mapsto \mathbf{R}(a)(\alpha) \cdot \mathbf{R}'(b)(\delta b, \beta) \cdot [\partial_s^j v]_{R_{a,b}}$$

and obtained by differentiation of $\mathbf{R}(b)$ with respect to b.

5. $B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F^{m} \to \mathbb{R}^N$ defined by

$$(a, \alpha, b, \beta, \delta v) \to \mathbf{R}(a)(\alpha) \cdot \mathbf{R}(b)(\beta) \cdot [\partial_s^j \delta v]_{R_{a,b}}$$

and obtained by differentiating with respect to v.

6. $B_{\frac{1}{2}} \times \mathbb{C}^{n_a+1} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F^{m+1} \to \mathbb{R}^N$ defined by

$$(a, (\delta a, \alpha), b, \beta, v) \to \mathbf{R}_1(a)(\delta a, \alpha) \cdot \mathbf{R}(b)(\beta) \cdot [\partial_s^{j+1} v]_{R_{a,b}}$$

which is obtained by differentiating R_a in the term $[\partial_s^j v]_{R_{a,b}}$ with respect to a. Hence $\mathbf{R}_1(a)(\delta a, \alpha) = (DR_a \delta a) \cdot \mathbf{R}(a)(\alpha)$.

7. $B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b+1} \times F^{m+1} \to \mathbb{R}^N$ defined by

$$(a, \alpha, b, (\delta b, \beta), v) \to \mathbf{R}(a)(\alpha) \cdot \mathbf{R}_1(b)(\delta b, \beta) \cdot [\partial_s^{j+1} v]_{R_{a,b}}$$

which is obtained by differentiating R_b in the term $[\partial_s^j v]_{R_{a,b}}$ with respect to b. Hence $\mathbf{R}_1(b)(\delta b, \beta) = (DR_b \delta b) \cdot \mathbf{R}(b)(\beta)$.

Note that in all of the above cases the new indices j', m' and n'_a, n'_b stay the same or are raised by 1 so that we have $j' \leq m' \leq k+1$ and $n'_a + n'_b \leq k+1$. We have verified that the statement $(\mathbf{S}_{k+1})$ holds true. This completes the proof of Lemma 4.3.5. □

Lemma 4.3.6. *(Analog of [19, Lemma 2.20]) The map* $\Psi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times (H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \times H^{3,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^N)) \to H_c^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$*, defined by* $\Psi(a, b, r^+, r^-) = 0$ *for* $a = 0$ *or* $b = 0$ *and*

$$\Psi(a, b, r^+, r^-) = \left(1 - \frac{\beta_{a,b}}{\gamma_{a,b}}\right) \cdot \Big([r^+]_{R_{a,b}} + [r^-]_{-R_{a,b}}\Big),$$

for $a \neq 0, b \neq 0$*, is sc-smooth.*

Proof. We follow the argument of [19, Lemma 2.20]. We abbreviate by G the sc-Banach space $H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \times H^{3,\delta_0}(\mathbb{R}^- \times S^1, \mathbb{R}^N)$ and by F the sc-Banach space $H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$. We already know that the maps

$$\begin{aligned} &B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times G \to \mathbb{R}^N, \\ &(a, b, r^+, r^-) \mapsto [r^+]_{R_{a,b}},\ [r^-]_{-R_{a,b}} \end{aligned}$$

are sc-smooth. We will now consider the map

$$\Phi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F \to F,$$
$$(a,b,r) \mapsto \frac{\beta_{a,b}}{\gamma_{a,b}} \cdot [r]_{R_{a,b}}$$

if $a \neq 0, b \neq 0$ and $\Phi(a,0,r) = \Phi(0,b,r) = \Phi(0,0,r) = 0$. The similar map for (a,b,r^-) can be dealt with the same way. The map Φ is sc-smooth on the set $(B_{\frac{1}{2}} \setminus \{0\}) \times (B_{\frac{1}{2}} \setminus \{0\}) \times F$ and we shall prove the sc-smoothness at the points $(a,0,r), (0,b,r), (0,0,r) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F$. We set $\sigma_{a,b} := \frac{\beta_{a,b}}{\gamma_{a,b}}$. We shall prove the following statements $(\mathbf{S}_k)$ by induction:

> $(\mathbf{S}_k)$. *The map Φ is of class sc^k and $T^k\Phi(a,0,H) = T^k\Phi(0,b,H) = T^k\Phi(0,0,H) = 0$. Moreover, if $\pi : T^k(B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F) \to \mathbb{R}^N$ is a projection onto a factor of T^kF, then the composition $\pi \circ T^k\Phi$ is the linear combination of maps of the following type,*
>
> $$A : B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F^m \to F^j$$
> $$(a,\alpha,b,\beta,v) \to \boldsymbol{R}(a)(\alpha) \cdot \boldsymbol{R}(b)(\beta) \cdot \sigma^{(p)}_{a,b} \cdot [\partial^q_s v]_{R_{a,b}}$$
>
> *for $a \neq 0, b \neq 0$ and $A(0,\alpha,b,\beta,v) = A(a,\alpha,0,\beta,v) = A(0,\alpha,0,\beta,v) = 0$. In addition, the indices satisfy $p+q = n_a + n_b$, $m \leq k$ and $l \leq m - j$.*

We begin by verifying that $(\mathbf{S}_0)$ holds. In this case there is only one projection $\pi : T^0F = F \to F$, namely, $\pi = \mathrm{id}$. Clearly $\pi \circ \Phi = \Phi$ has the required form with $\mathbf{R}(a) = \mathbf{R}(b) = 1$ of order 0 and all indices j, l, p, and q equal to 0. Hence we only need to show that the map Φ has sc^0-property. This is clearly true at points $(a,b,v) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F^m$ for $a \neq 0, b \neq 0$. We carry out the proof of the sc^0-property for the map Φ at $(a,0,v), (0,b,v), (0,0,v)$. Consider a sequence (a_k, b_k, v_k) converging to one of the following points: $(a,0,v), (0,b,v)$ or $(0,0,v) \in B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times F_m$ and we claim that $\Phi(a_k,b_k,v_k) \to 0$ in F_m. Observe that $\sigma_{a,b}$ vanishes on $[\frac{R_{a,b}}{2} + 1, \infty)$. Hence we can estimate

$$|\Phi(a_k,b_k,v_k)|_m = |\sigma_{a_k,b_k} \cdot [v_k]_{R_{a_k,b_k}}|^2_m = \sum_{|\alpha| \leq m} |[v_k]_{R_{a_k,b_k}}|^2 \int_{\mathbb{R}^+ \times S^1} |D^\alpha \sigma_{a_k,b_k}(s)|^2 e^{2\delta_m s}\, ds dt$$
$$\leq \sum_{|\alpha| \leq m} C_\alpha |[v_k]_{R_{a_k,b_k}}|^2 e^{2\delta_m (\frac{R_{a_k,b_k}}{2} + 1)}$$

with constants C_α depending only on the cutoff β, the multi-index α, and m. So, to prove them claim we have to show that $[v_k]_{R_{a_k,b_k}} \to 0$ in $\mathbb{R}^N$. Abbreviate $\Sigma_k = [\frac{R_{a_k,b_k}}{2} - 1, \frac{R_{a_k,b_k}}{2} + 1] \times S^1$. By the Sobolev embedding theorem on bounded domains,

$$|e^{\delta_m \cdot} v|_{C^0(\Sigma_k)} \leq C |e^{\delta_m \cdot} v|_{H^m(\Sigma_k)} =: \varepsilon_k$$

with the constant C independent of v and k. This shows that

$$|[v]_{R_{a_k,b_k}}| \leq \varepsilon_k \cdot e^{-\delta_m R_{a_k,b_k}}. \tag{4.8}$$

Also note that since v belongs to E_m, the sequence ε_k converges to 0. Similarly, we have

$$|e^{\delta_m \cdot}(v_k - v)|_{C^0(\Sigma_k)} \leq C|e^{\delta_m \cdot}(v_k - v)|_{H^m(\Sigma_k)} \leq C|v_k - v|_m =: \varepsilon'_k$$

which implies that

$$|[v_k]_{R_{a_k,b_k}} - [v]_{R_{a_k,b_k}}| = |[v_k - v]_{R_{a_k,b_k}}| \leq \varepsilon'_k \cdot e^{-\delta_m R_{a_k,b_k}}.$$

By assumption, $|v - v_k|_m = \varepsilon_k \to 0$. Consequently,

$$|[v_k]_{R_{a_k,b_k}}|e^{\delta_m R_{a_k,b_k}} \leq |[v_k]_{R_{a_k,b_k}} - [v]_{R_{a_k,b_k}}| \cdot e^{\delta_m R_{a_k,b_k}} + |[v]_{R_{a_k,b_k}}| \cdot e^{\delta_m R_{a_k,b_k}} \leq \varepsilon'_k + \varepsilon_k \to 0$$

which proves our claim. Therefore, the base case ($\mathbf{S}_0$) holds.

Assuming that ($\mathbf{S}_k$) holds, we show that ($\mathbf{S}_{k+1}$) also holds. By induction hypothesis, the map Φ is of class sc^k, so that $T^k\Phi$ is sc^0, and $T^k\Phi(a, 0, H) = T^k\Phi(0, b, H) = T^k\Phi(0, 0, H) = 0$. Moreover, $\pi \circ T^k\Phi$ can be written as a linear combination of maps of a certain form. We also know that $T^{k+1}\Phi$ is sc-smooth at points (a, b, H) with $a \neq 0, b \neq 0$.

We will approximate $T^k\Phi$ at the points $(a, 0, H), (0, b, H), (0, 0, H)$ as follows. Consider the maps A described in ($\mathbf{S}_k$) of the form

$$\begin{aligned}&A : B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times F^m \to \mathbb{R}^N\\ &(a, \alpha, b, \beta, v) \to \mathbf{R}(a)(\alpha) \cdot \mathbf{R}(b)(\beta) \cdot \sigma^{(p)}_{a,b} \cdot [\partial^q_s v]_{R_{a,b}}\end{aligned}$$

for $a \neq 0, b \neq 0$ and $A(a, \alpha, 0, \beta, v) = A(0, \alpha, b, \beta, v) = A(0, \alpha, 0, \beta, v) = 0$. We will show that if $(a, \alpha, b, \beta, v) \in B_{\frac{1}{2}} \times \mathbb{C}^{n_a} \times B_{\frac{1}{2}} \times \mathbb{C}^{n_b} \times E^{m+1}$ is of the form $(a, \alpha, 0, \beta, v), (0, \alpha, b, \beta, v)$, or $(0, \alpha, 0, \beta, v)$, then

$$\frac{1}{|\delta a| + |\delta\alpha| + |\delta b| + |\delta\beta| + |\delta v|_{m+1}}|A(a + \delta a, \alpha + \delta\alpha, b + \delta b, \beta + \delta\beta, v + \delta v)|_j \to 0$$

as $|\delta a| + |\delta\alpha| + |\delta b| + |\delta\beta| + |\delta v|_{m+1} \to 0$ which will prove that A has the approximation property at $(a, \alpha, b\beta, v)$ with respect to the linearized map $DA(a, \alpha, b, \beta, v) = 0$.

Exactly as in the proof of Lemma 4.3.5, we can obtain the estimate

$$\begin{aligned}&\frac{|A(a + \delta a, \alpha + \delta\alpha, b + \delta b, \beta + \delta\beta, v + \delta v)|}{|\delta a| + |\delta\alpha| + |\delta b| + |\delta\beta| + |\delta v|_{m+1}}\\ &\leq \frac{C^2 \cdot e^{-\delta_{m+1}\frac{R_{a+\delta a, b+\delta b}}{2}} \cdot |R_{a+\delta a}|^{3n_a} \cdot |R_{b+\delta b}|^{3n_b}}{|\delta a| + |\delta\alpha| + |\delta b| + |\delta\beta| + |\delta v|_{m+1}} \cdot |\alpha + \delta\alpha|^{n_a} \cdot |\beta + \delta\beta|^{n_b} \cdot |v + \delta v|_{m+1}.\end{aligned}$$

Again, one can show this converges to 0 as $(\delta a, \delta\alpha, \delta b, \delta\beta, \delta v) \to (0,0,0,0,0)$ in $\mathbb{C} \times \mathbb{C}^{n_a} \times \mathbb{C} \times \mathbb{C}^{n_b} \times F^{m+1}$.

Finally, we need to show that $\pi \circ T^{k+1}\Phi$ is a linear combination of the maps of the required form and which have the required continuity properties at points with vanishing a. The terms making up $\pi \circ T^{k+1}\Phi$ are the terms guaranteed by $(\mathbf{S}_k)$ provided π is the projection onto one of the first 2^k factors. In this case the indices m and j are raised by one. If π is the projection onto one of the last 2^k factors, then $\pi \circ T^{k+1}\Phi$ is a linear combination of derivatives of maps guaranteed by $(\mathbf{S}_k)$. We once again refer to the case by case study of Lemma 4.3.5. □

Lemma 4.3.7. *(Analog of [19, Lemma 2.21]) The map $\Phi : B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N)$, defined by $\Phi(a,b,r) = r$ if $a = 0$ or $b = 0$, and by*

$$\Phi(a,b,r) = \frac{\beta_{a,b}^2}{\gamma_{a,b}} \cdot r \tag{4.9}$$

if $a \neq 0, b \neq 0$, is sc-smooth.

Proof. We follow the argument of [19, Lemma 2.21]. We need only prove the sc-smoothness at points $(a,0,r), (0,b,r)$, and $(0,0,r)$ in $B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1)$. We may assume that a and b are small. Choose a smooth function $\chi_1 : \mathbb{R}^+ \to [0,1]$ satisfying $\chi_1(s) = 1$ for $s \in [0,1]$ and $\chi_1(s) = 0$ for $s \geq 2$, and set $\chi_2 = 1 - \chi_1$. Then the map

$$(a,b,r) \mapsto \frac{\beta_{a,b}^2}{\gamma_{a,b}} \cdot \chi_1 \cdot r$$

is obviously sc-smooth since for $|a|$ and $|b|$ small it is equal to the map

$$(a,b,r) \to \chi_1 \cdot r$$

which is independent of a and b. It remains to deal with the map

$$(a,b,r) \mapsto \frac{\beta_{a,b}^2}{\gamma_{a,b}} \cdot \chi_2 \cdot r.$$

This map can be factored as follows. First, we apply the map

$$B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^N) \to B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N), \quad (a,b,r) \mapsto (a,b,\chi_2 r)$$

which is an sc-operator and hence sc-smooth. Then compose with the map

$$B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N) \to H^{3,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^N), \quad (a,b,u) \to \frac{\beta_{a,b}^2}{\gamma_{a,b}} \cdot u.$$

We claim that this map is sc-smooth; a full proof of this fact would require an analog of [19, Proposition 2.8].

Finally, compose with the restriction map

$$H^{3,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^N)\to H^{3,\delta_0}(\mathbb{R}^+\times S^1,\mathbb{R}^N).$$

which as an sc-operator is also sc-smooth. Hence we can write (4.9) as a composition of sc-smooth maps, and the proof is complete. ☐

Lemma 4.3.8. *(Analog of [19, Lemma 2.22]) The map* $\Phi : B_{\frac{1}{2}}\times B_{\frac{1}{2}}\times H^{3,\delta_0}(\mathbb{R}^-\times S^1,\mathbb{R}^N)\to H^{3,\delta_0}(\mathbb{R}^+\times S^1,\mathbb{R}^N)$, *defined by* $\Phi(a,b,r)=0$ *if* $a=0$ *or* $b=0$, *and by*

$$\Phi(a,b,r)=\frac{\beta_{a,b}(1-\beta_{a,b})}{\gamma_{a,b}}\cdot r(\cdot - R_{a,b},\cdot-\vartheta_{a,b})$$

if $a\neq 0, b\neq 0$, *is sc-smooth.*

Proof. We follow the argument of [19, Lemma 2.22]. We study the map for a and b small. Choose a smooth map $\chi:\mathbb{R}^-\to[0,1]$ satisfying $\chi(s)=1$ for $s\leq -1$ and $\chi(s)=0$ for $s\in[-\frac{1}{2},0]$. If $|a|$ and $|b|$ is small, the map Φ is the composition of the following three maps. The first map is defined by

$$B_{\frac{1}{2}}\times B_{\frac{1}{2}}\times H^{3,\delta_0}(\mathbb{R}^-\times S^1,\mathbb{R}^N)\to B_{\frac{1}{2}}\times B_{\frac{1}{2}}\times H^{3,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^N),\quad (a,b,u)\mapsto(a,b,\chi\cdot u).$$

It is an sc-operator and hence sc-smooth. The second map is defined by

$$B_{\frac{1}{2}}\times B_{\frac{1}{2}}\times H^{3,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^N)\to H^{3,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^N)$$
$$(a,b,u)\mapsto \frac{\beta_{a,b}(1-\beta_{a,b})}{\gamma_{a,b}}\cdot u(\cdot-R_{a,b},\cdot-\vartheta_{a,b}).$$

We again claim this map is sc-smooth, which requires an analog of [19, Proposition 2.8]. The last map is the restriction map $H^{3,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^N)\to H^{3,\delta_0}(\mathbb{R}^+\times S^1,\mathbb{R}^N)$ which is an sc-operator. This completes the proof.. ☐

This complete the proof of Proposition 4.3.1.

Local M-Polyfold Models for the New Polyfold

Following our construction of the sc-smooth splicing $\pi_{a,b}$, the construction of good uniformizing families of stable maps follows the same recipe as in the construction of the original GW-polyfolds. In particular, for a stable map $\alpha=(\Sigma,j,M,D,u)$ with constant destabilizing ghost components C_k the construction of good data and a stabilization of the underlying noded, not necessarily stable, Riemann surface is identical. Note that the construction of a stabilization does not require the addition of any marked points to a ghost component C_k, as such a component will already satisfy the DM-stability condition.

Thus, assume for α we have chosen good data with an associated a stabilization S and linear constraints $H_{u(z_s)} \subset T_{u(z_s)}M$ for $z_s \in S$. As before, define an sc-Banach space by

$$E_u = \{\eta \in H_c^{3,\delta_0}(u^*TM) \mid \eta(z_s) \in H_{u(z_s)} \text{ for } z_s \in S\}$$

which consists of sections along the smooth map $u : \Sigma \to M$ which satisfy the linear constraint $H_{u(z_s)}$ at the points $z_s \in \Sigma$. There exists a good uniformizing family

$$(\underline{a}, v) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \cup S)_{\underline{a}}, D_{\underline{a}}), \quad (\underline{a}, v) \in O$$

of stabilized noded Riemann surfaces.

The parameters $(\underline{a}, v)$ give splicing parameters, and together with the gluing construction allow us to define a splicing core

$$K = \{(\underline{a}, v, \eta) \mid (\underline{a}, v) \in O, \eta \in E_u, \quad \pi_a(\eta) = \eta, \pi_{a,b}(\eta) = \eta\}.$$

where the projections are understood as being defined on the appropriate regions of nodes on the underlying Riemann surface.

Definition 4.3.9. Assume that we have good data and a stabilization centered at a stable map $\alpha = (\Sigma, j, M, D, u)$ with constant destabilizing ghost component C_k, and with isotropy group $G(\alpha)$. A **good uniformizing family of stable maps** centered at $\alpha = (\Sigma, j, M, D, u)$ is a family of stable maps

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}}^{ft} \exp_u(\eta)) \quad \text{where} \quad (\underline{a}, v, \eta) \in \mathcal{O}.$$

Here $\mathcal{O}$ is an open subset of the splicing core K.

The coordinates $(\underline{a}, v, \eta) \in \mathcal{O}$ are therefore local sc-coordinates of the M-polyfold models on which we define the polyfold $\mathcal{Z}_{A,g,k}^{ft}$. The analogs of [22, Proposition 3.10, Proposition 3.12], concerning the action of the group $G(\alpha)$ on the local sc-coordinates carries over verbatim to this context.

sc-Smooth Compatibility of Good Uniformizers

At this point, in addition to the old local M-polyfold models which are defined at points which do not contain constant ghost components C_k, we now have defined new local M-polyfold models at points which contain constant ghost components C_k. Our next task is therefore to demonstrate the sc-smooth compatibility of all good uniformizing families of stable maps, both the original ones and the new ones we have defined with the new splicing construction. This will also allow us to establish the ètale condition for the ep-groupoid structure we wish to construct. Thus we need to verify the following version of [22, Theorem 3.13] in our present context.

Proposition 4.3.10. *We consider two good uniformizing families of stable maps* $(\underline{a}, v, \eta) \mapsto \alpha_{(\underline{a},v,\eta)}$ *parametrized by* $(\underline{a}, v, \eta) \in \mathcal{O}$ *and centered at a stable map* $\alpha = (\Sigma, j, M, D, u)$*; and* $(\underline{a}', v', \eta') \mapsto \alpha'_{(\underline{a}',v',\eta')}$ *parametrized by* $(\underline{a}', v', \eta') \in \mathcal{O}'$ *and centered at* $\alpha' = (\Sigma', j', M', D', u')$*. Here we understand that 'good uniformizing family' may be either of the following types*

- *the usual good uniformizing families centered at stable maps which do not contain a destabilizing ghost component*
- *the new good uniformizing families centered at stable maps which do contain a destabilizing ghost component, families which are defined using the new splicing defined using the projection* $\pi_{a,b}$*.*

We assume that for two points $(\underline{a}_0, v_0, \eta_0) \in \mathcal{O}$ *and* $(\underline{a}'_0, v'_0, \eta'_0) \in \mathcal{O}'$ *there exists an isomorphism*

$$\phi_0 : \alpha_{(\underline{a}_0,v_0,\eta_0)} \to \alpha'_{(\underline{a}'_0,v'_0,\eta'_0)}$$

between the associated stable maps. Then there exist a unique local germ of an sc-diffeomorphism

$$f : (\mathcal{O}, (\underline{a}_0, v_0, \eta_0)) \to (\mathcal{O}', (\underline{a}'_0, v'_0, \eta'_0)), \quad (\underline{a}, v, \eta) \mapsto (\underline{a}', v', \eta') = f((\underline{a}, v, \eta))$$

between the parameter spaces satisfying $f(\underline{a}_0, v_0, \eta_0) = (\underline{a}'_0, v'_0, \eta'_0)$ *and there exists a core-smooth germ of a family* $(\underline{a}, v, \eta) \mapsto \phi_{(\underline{a},v,\eta)}$ *of isomorphisms*

$$\phi_{(\underline{a},v,\eta)} : \alpha_{(\underline{a},v,\eta)} \to \alpha'_{f(\underline{a},v,\eta)}$$

satisfying $\phi_{(\underline{a}_0,v_0,\eta_0)} = \phi_0$*.*

Proof. The proof once again will follow nearly verbatim the reasoning and arguments given in [22]. We indicate precisely where to expect modifications to the argument.

To begin, we reiterate that the Deligne–Mumford construction remains the same, as do the constructions of stabilizations and associated linear constraints, and so we may observe that the results of Lemmas 3.14, 3.15, and 3.16 will hold. In particular, there exists an sc-smooth germ near $(\underline{a}_0, v_0, \eta_0)$:

$$(\underline{a}, v, \eta) \mapsto (\underline{a}'(\underline{a}, v, \eta), v'(\underline{a}, v, \eta))$$

and a family of isomorphisms between the underlying (unstable) noded Riemann surfaces:

$$\varphi_{(\underline{a},v,\eta)} : (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}) \to (\Sigma'_{\underline{a}'(\underline{a},v,\eta)}, j'(\underline{a}'(\underline{a}, v, \eta), v'(\underline{a}, v, \eta), M'_{\underline{a}'(\underline{a},v,\eta)}, D'_{\underline{a}'(\underline{a},v,\eta)}).$$

From there, the argument requires us to compare the glued maps on the glued Riemann surfaces in order to get a map $(\underline{a}, v, \eta) \to \eta'$; this map will be uniquely determined by expressions of the form:

$$\oplus \exp'_{u'}(\eta') = \oplus \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \text{ and } \ominus \eta' = 0$$

where the gluing and antigluings may be of either kind we have defined. Such an expression may be broken down into the following possible local expressions:

1.
$$\eta' = (\exp'_{u'})^{-1} \circ \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \tag{4.10}$$

2.
$$\eta' = (\exp'_{u'})^{-1} \circ \oplus_a \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \tag{4.11}$$

3.
$$\oplus_{a'(a,v,\eta)} \exp'_{u'}(\eta') = \oplus_a \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \text{ and } \ominus_{a'(a,v,\eta)} \exp'_{u'}(\eta') = 0 \tag{4.12}$$

4.
$$\eta' = (\exp'_{u'})^{-1} \circ \oplus^{ft}_{a,b} \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \tag{4.13}$$

5.
$$\oplus^{ft}_{a'(a,v,\eta),b'(a,v,\eta)} \exp'_{u'}(\eta') = \oplus^{ft}_{a,b} \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \text{ and } \ominus^{ft}_{a'(a,v,\eta),b'(a,v,\eta)} \exp'_{u'}(\eta') = 0 \tag{4.14}$$

6.
$$\oplus^{ft}_{a'(a,v,\eta),b'(a,v,\eta)} \exp'_{u'}(\eta') = \oplus_a \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \text{ and } \ominus^{ft}_{a'(a,v,\eta),b'(a,v,\eta)} \exp'_{u'}(\eta') = 0 \tag{4.15}$$

7.
$$\oplus_{a'(a,v,\eta)} \exp'_{u'}(\eta') = \oplus^{ft}_{a,b} \exp_u(\eta) \circ \varphi^{-1}_{(\underline{a},v,\eta)} \text{ and } \ominus_{a'(a,v,\eta)} \exp'_{u'}(\eta') = 0. \tag{4.16}$$

The first three expressions (4.10), (4.11), (4.12) arise in the construction of the GW polyfolds in [22], and hence are sc-smooth. The proof that (4.13) is sc-smooth is identical to the proof that (4.11) is sc-smooth. The proof that (4.14), (4.15), and (4.16) are sc-smooth is discussed below. □

The proof that (4.14), (4.15), and (4.16) are sc-smooth will follow the same argument given that 3 is sc-smooth, which follows from [22, Theorem 2.57], which is proved in [22, Appendix 5.1]. The proof follows from writing down explicit formulas for the above expressions.

As an example, we discuss the localized setup for (4.16). This situation arises if we try to check the compatibility of a source good uniformizing family centered at a stable map which has an destabilizing ghost component C_k of type 1, and hence two nodal points $\{x_a, y_a\}, \{x_b, y_b\}$; and target good uniformizing family with only one node $\{x_{a'}, y_{a'}\}$. We may assume the node $\{x_a, y_a\}$ will correspond to the node $\{x_{a'}, y_{a'}\}$; there will be no morphisms for points in the overlap of the two good uniformizing families when $b = 0$.

This discussion is localized to a region of the destabilizing ghost component C_k. Thus, we have parameters on the source $(a, b) \in B_\epsilon \times D \subset \mathbb{C}^2$ and $w \in W$, where $D \subset \mathbb{C}$ is an open neighborhood such that $\mathrm{cl}(D)$ does not contain 0, and W is an open neighborhood of 0 in some $\mathbb{C}^N$. We have parameters on the target $a' \in B_\epsilon \subset \mathbb{C}$.

We assume D_{x_a}, D_{y_a}, D_{x_b}, and D_{y_b} are the small-disk structures associated to the gluing parameters a and b, while $D'_{x_{a'}}$ and $D'_{y_{a'}}$ are associated to the parameter a'. Using the exponential gluing profile, we have constructed glued surfaces $Z_a \sqcup Z_b$ (which we may identify with $Z_{a,b}$) on which we have defined the dual glued maps $\oplus_{a,b}$, and $Z'_{a'}$ on which we have the usual gluing $\oplus_{a'}$.

For $h > 0$ and a (or a') small enough one can define the sub-cylinders $Z_a(-h)$ and $Z'_{a'}(-h)$ as in [22, p24-25] and recall that $Z_0(-h) = D_x(-h) \cup D_y(-h)$ if $a = 0$, and similarly for Z'_0.

We assume that the following data are given:

(1) A smooth map $(a, w) \mapsto a'(a, w) \in \mathbb{C}$ defined for $w \in W$ and $a \in B_\epsilon$ with ϵ sufficiently small, and satisfying $a'(0, w) = 0$ for all $w \in W$. In this situation, variation of the gluing parameter b only does not correspond to another gluing parameter.

(2) For sufficiently large $h > 0$, a core-smooth family of holomorphic embeddings

$$\phi_{(a,w)} : Z_a(-h) \to Z'_{a'(a,w)}$$

parameterized by $(a, w) \in B_\epsilon \oplus W$.

(3) There exists $H > 0$ such that

$$Z'_{a'(a,w)}(-H) \subset \phi_{(a,w)}(Z_a(-h))$$

for all $(a, w) \in B_\epsilon \oplus W$ where ϵ is sufficiently small.

Here ϕ differs from the map φ above; it only depends on finite-dimensional parameters. In fact, φ factors through ϕ by a map $(a, b, v, \eta) \mapsto w$ determined via pullback of the transversal constraint; see the argument of [22, Theorem 3.13].

Consider the sc-Banach space E of maps $\eta : D_{x_a}(-h) \sqcup D_{y_b}(-h) \to \mathbb{R}^{2n}$ of class $(3, \delta_0)$ satisfying $\eta(x_a) = \eta(y_b)$. Similarly, consider the sc-Banach space E' of maps $\eta' : D'_{x_{a'}}(-H) \sqcup D'_{y_{a'}}(-H)$ of class $(3, \delta_0)$ satisfying $\eta'(x_{a'}) = \eta'(y_{a'})$.

With these spaces we define the map

$$\Phi : B_\epsilon \oplus D \oplus W \oplus E \to E', \quad (a, w, \eta) \mapsto \eta'$$

where $\eta' \in E'$ is defined as the unique solution of the two equations

$$\oplus_{a'(a,w)}(\eta') = \oplus^{ft}_{a,b}(\eta) \circ \phi^{-1}_{(a,w)} \text{ and } \ominus_{a'(a,w)}(\eta') = 0.$$

Proposition 4.3.11. *The map Φ, which corresponds to the local expression* (4.16)*, is sc-smooth.*

To check sc-smoothness, we write down an explicit expression for $\eta'^{\pm}$ as follows. Writing $\eta^{\pm} = r^{\pm} + c$, where c is a common asymptotic constant, abbreviating a' for $a'(a, w)$, $\beta_{a'} =$

$\beta(\cdot - R_{a'}/2)$, and $\gamma_{a'} = \beta_{a'}^2 + (1-\beta_{a'})^2$ we have:

$$\begin{aligned}\eta'^{+} = c &+ \left(1 - \frac{\beta_{a'}}{\gamma_{a'}}\right) \cdot \int_{S^1} \oplus_{a,b}(r) \left(\phi^{-1}_{(a,w)}\left(\tfrac{R_{a,b}}{2}, t\right)\right) dt \\ &+ \frac{\beta_{a'}}{\gamma_{a'}} \cdot \beta_{a,b}(\phi^{-1}_{(a,w)}) \cdot r^{+}(\phi^{-1}_{(a,w)}) \\ &+ \frac{\beta_{a'}}{\gamma_{a'}} \cdot \left(1 - \beta_{a,b}(\phi^{-1}_{(a,w)})\right) \cdot r^{-}\left(\phi^{-1}_{(a,w)} - (R_{a,b}, \vartheta_{a,b})\right).\end{aligned}$$

Three is an analogous formula for the component η'^{-}. As in [22, Appendix 5.1], it is possible to check sc-smoothness of this expression through further decomposition into smaller terms, and application of the chain rule. The process for checking the sc-smoothness of the expressions (4.14) and (4.15) follows the same procedure.

Remark 4.3.12. An important reality check is the following natural question: What would happen if we tried to compare two good uniformizing families centered at a stable map α with constant destabilizing ghost component, one which uses the old gluing, and one which uses the new gluing?

There is a map defined in one direction (from the new gluing to the old), which one can check is sc-smooth; this in fact is also a local expression of the well-defined inclusion $\iota : \mathcal{Z}^{ft}_{A,g,k} \hookrightarrow \mathcal{Z}_{A,g,k}$.

In the other direction, there is only a partially defined map. At points where it is defined and where the gluing parameters surrounding the ghost component do not vanish, this map is sc-smooth; this is not a surprise, given that such points must also be sc-smooth compatible with other local sc-coordinates. At other points it will necessarily not be defined or be sc-smooth.

Compactness Properties for Morphisms

We need to establish a version of [22, Proposition 3.22], which concerns the compactness properties for the space of morphisms between two good uniformizing families. This result should be thought of as an important initial step in showing the properness condition of our eventual ep-groupoid structures.

We take two good uniformizing families of stable maps

$$(\underline{a}, v, \eta) \mapsto \alpha_{(\underline{a},v,\eta)}, \quad \text{where} \quad (\underline{a}, v, \eta) \in \mathcal{O}$$

and

$$(\underline{a}', v', \eta') \mapsto \alpha'_{(\underline{a}',v',\eta')}, \quad \text{where} \quad (\underline{a}', v', \eta') \in \mathcal{O}'$$

centered at smooth stable maps α and α'. The set of **morphisms** between these M-polyfolds is given by:

$$\begin{aligned}M(\mathcal{O}, \mathcal{O}') = \{\phi : \alpha_{(\underline{a},v,\eta)} \to \alpha'_{(\underline{a}',v',\eta')} \,|\, &(\underline{a}, v, \eta) \in \mathcal{O}, (\underline{a}', v', \eta') \in \mathcal{O}', \\ &\phi \text{ is an isomorphism of stable maps}\}. \qquad (4.17)\end{aligned}$$

For a morphism $\phi : \alpha_{(\underline{a},v,\eta)} \to \alpha'_{(\underline{a}',v',\eta')}$ we can define the source map $s : M(\mathcal{O},\mathcal{O}') \to \mathcal{O}$ by $s(\phi) = (\underline{a}, v, \eta)$ and the target map $t : M(\mathcal{O},\mathcal{O}') \to \mathcal{O}'$ by $t(\phi) = (\underline{a}', v', \eta')$. Using Proposition 4.3.10, we can give $M(\mathcal{O},\mathcal{O}')$ a topology, and moreover, an M-polyfold structure.

Proposition 4.3.13. *For every point $(\underline{a}, v, \eta) \in \mathcal{O}$ there exists an open neighborhood $U \subset \mathcal{O}$ which has the following property. Consider a sequence $\phi_k \in M(\mathcal{O},\mathcal{O}')$ of morphisms where $s(\phi_k) \in U$ and such that $t(\phi_k)$ converges to a point $(\underline{a}', v', \eta') \in \mathcal{O}'$. Then there exists a subsequence of ϕ_k which converges in $M(\mathcal{O},\mathcal{O}')$.*

The crucial ingredient in the proof of this proposition is one final technical result - an analogue of [22, Proposition 2.59]. As in the discussion above, there are versions of the proposition corresponding to the local expressions (4.14), (4.15), and (4.16). Here, we will only discuss the analog of (4.14).

We assume that $((D_{x_a}, D_{y_a}), (x_a, y_a))$, $((D_{x_b}, D_{y_b}), (x_b, y_b))$ as well as $((D'_{x_{a'}}, D'_{y_{a'}}), (x_{a'}, y_{a'}))$, $((D'_{x_{b'}}, D'_{y_{b'}}), (x_{b'}, y_{b'}))$ are disk pairs so that we obtain via gluing the associated cylinders $Z_{a,b}$ and $Z'_{a',b'}$. We assume that E is the sc-Banach space of maps $\eta : D_{x_a}(-h) \sqcup D_{y_b}(-h) \to \mathbb{R}^{2n}$ of class $(3, \delta_0)$ satisfying $\eta(x_a) = \eta(y_b)$; similarly, we assume E' is the sc-Banach space of maps $\eta' : D'_{x_{a'}}(-H) \sqcup D'_{y_{a'}}(-H)$ of class $(3, \delta_0)$ satisfying $\eta'(x_{a'}) = \eta'(y_{a'})$.

We assume that (a_k, b_k) and (a'_k, b'_k) are small gluing parameters converging to $(0, 0)$ and

$$\phi_k : Z_{a_k,b_k} \to Z'_{a'_k,b'_k}$$

is a sequence of holomorphic embeddings having the following properties. The restrictions of the maps to a fixed annulus-type neighborhood of the boundary of the discs D converge, viewed as the maps into the discs D', in the C^∞-sense. Then we consider a sequence (η'_k) in E' converging on level 0 to some $\eta'_0 = (\eta'^+_0, \eta'^-_0) \in E'$ and define the maps $\eta_k \in E$ as the unique solutions of the two equations

$$\oplus^{ft}_{a_k,b_k}(\eta_k) = (\oplus^{ft}_{a'_k,b'_k}(\eta'_k)) \circ \phi_k \quad \text{and} \quad \ominus^{ft}_{a_k,b_k}(\eta_k) = 0.$$

Proposition 4.3.14. *Under the assumptions stated above the sequence (ϕ_k) converges in C^∞_{loc} in finite distance to the left and the right boundary to holomorphic embeddings $\phi^+ : D_{x_a} \to D'_{x_{a'}}$ and $\phi^- : D_{y_b} \to D'_{y_{b'}}$ satisfying $\phi^+(x_a) = x_{a'}$ and $\phi^-(y_b) = y_{b'}$. The sequence $(\eta_k) \subset E$ converges on level 0 to the map $\eta_0 = (\eta^+_0, \eta^-_0) \in E$ given by*

$$\eta^+_0 = \eta'^+_0 \circ \phi^+ \quad \textit{and} \quad \eta^-_0 = \eta'^-_0 \circ \phi^-.$$

The convergence of the sequence (ϕ_k) is a fact about Deligne–Mumford spaces. The remainder of the statement follow the arguments of [22]. There are similar propositions for the local expressions (4.15) and (4.16).

4.4 A Natural Polyfold Structure on the Gromov–Witten Polyfold of Stable Curves with Constant Destabilizing Ghost Components

In this section, we discuss how all the puzzle pieces fit together to give a natural polyfold structure on $\mathcal{Z}^{ft}_{A,g,k}$. The crucial ingredients are Proposition 4.3.10 and Proposition 4.3.13, which are can be viewed as precursors of the properties of being étale and proper, respectively.

Before we can use these to try to define an ep-groupoid structure, we must first establish necessary facts about the underlying topology of the space $\mathcal{Z}^{ft}_{A,g,k}$. These facts are derived essentially through a combination of knowledge about the topology of our local M-polyfold models, a decomposition of $\mathcal{Z}^{ft}_{A,g,k}$ into components which are covered by Banach manifolds, and judicious use of Proposition 4.3.10 and Proposition 4.3.13. It is only once we have established these facts that we can give $\mathcal{Z}^{ft}_{A,g,k}$ a polyfold structure.

The Topology on $\mathcal{Z}^{ft}_{A,g,k}$

A topology on the new polyfold $\mathcal{Z}^{ft}_{A,g,k}$ is induced from the topologies of the local M-polyfold models in the following way. Consider a good uniformizer of stable maps $(\underline{a}, v, \eta) \to \alpha_{(\underline{a},v,\eta)}$ defined on an M-polyfold $\mathcal{O}$ and centered at a stable map $\alpha = (\Sigma, j, M, D, u)$. Denote the automorphism group of α by $G(\alpha)$. The M-polyfold $\mathcal{O}$ comes with a finite group action by sc-diffeomorphisms

$$G(\alpha) \times \mathcal{O} \to \mathcal{O}, \quad (g, (\underline{a}, v, \eta)) \mapsto g * (\underline{a}, v, \eta).$$

The map defined on the quotient space

$$G(\alpha)\backslash\mathcal{O} \to \mathcal{Z}^{ft}_{A,g,k}, \quad [\underline{a}, v, \eta] \mapsto [\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}}(\exp_u(\eta))]$$

into the space of stable curves is injective. We may therefore define a topology on $\mathcal{Z}^{ft}_{A,g,k}$ through the requirement that all such maps will be homeomorphisms onto open sets; showing this is a well-defined topology requires Proposition 4.3.10.

M-polyfold models have topology given by the subspace topology of the 0th level of a sc-Banach space (which itself is a Banach space). Hence M-polyfolds models (and the quotient space of an M-polyfold models) will have a topology which is second countable, paracompact, and Hausdorff.

From this we may prove that at a point in $\mathcal{Z}^{ft}_{A,g,k}$ there exists a countable basis of open neighborhoods. That $\mathcal{Z}^{ft}_{A,g,k}$ is a Hausdorff topological space will then follow from the fact that our M-polyfolds are Hausdorff in addition to Proposition 4.3.13. We could also use the fact that the inclusion map $\iota : \mathcal{Z}^{ft}_{A,g,k} \hookrightarrow \mathcal{Z}_{A,g,k}$ is a continuous injection on the underlying topological spaces, and that $\mathcal{Z}_{A,g,k}$ is a Hausdorff topological space. Finally, the existence of a countable basis at a point and Proposition 4.3.13 can be used to show $\mathcal{Z}^{ft}_{A,g,k}$ is completely regular. The proof that there exists a countable basis for $\mathcal{Z}^{ft}_{A,g,k}$ uses a countable decomposition of $\mathcal{Z}^{ft}_{A,g,k}$ into diffeomorphism types, which are then proven to be separable.

Having shown $\mathcal{Z}^{ft}_{A,g,k}$ is Hausdorff, completely regular, and second countable, we may use Urysohn's metrization theorem to conclude it is also metrizable and consequently, paracompact.

Proposition 4.4.1. *The topology on $\mathcal{Z}^{ft}_{A,g,k}$ uniquely determined by the local M-polyfold models is a Hausdorff, second countable, paracompact topology.*

Polyfold Structure on $\mathcal{Z}^{ft}_{A,g,k}$

We may now define a polyfold structure on $\mathcal{Z}^{ft}_{A,g,k}$ as follows. Using Proposition 4.4.1, we may obtain a countable collection of good uniformizing families of stable maps $\mathcal{O}_\lambda$, indexed by $\lambda \in \Lambda$, whose images $|\mathcal{O}_\lambda|$ form an locally finite cover of the topological space $\mathcal{Z}^{ft}_{A,g,k}$. We thus define the object set as

$$\coprod_{\lambda\in\Lambda} \mathcal{O}_\lambda.$$

We may then define the morphism set as

$$\coprod_{\lambda,\lambda'\in\Lambda} M(\mathcal{O}_\lambda, \mathcal{O}_{\lambda'})$$

where $M(\mathcal{O}_\lambda, \mathcal{O}_{\lambda'})$ is defined by (4.17). The definitions of the additional structure maps of multiplication, unit, and inverse are all the obvious ones, and allow us to define a groupoid structure.

This groupoid structure is étale, as follows immediately from Proposition 4.3.10. The fact that it is also proper follows from both Proposition 4.3.13 and the fact that the object set we have defined is a locally finite cover of $\mathcal{Z}^{ft}_{A,g,k}$. Hence, we have a polyfold structure on $\mathcal{Z}^{ft}_{A,g,k}$.

As in the usual case, this polyfold structure is natural in the sense that, having fixed a strictly increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$, if we construct another polyfold structure with this recipe they both will possess a common refinement, and hence are Morita equivalent.

Theorem 4.4.2. *Having defined the new gluing procedure, for a fixed strictly increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$, the second countable paracompact topological space $\mathcal{Z}^{ft}_{A,g,k}$ of stable curves with constant destabilizing ghost components possesses a natural equivalence class of polyfold structures.*

4.5 The Polyfold Fredholm Problem for the New Gromov–Witten Polyfold

We have already described the underlying set of the new strong polyfold bundle $\mathcal{W}^{ft}_{A,g,k}$ as a set of equivalence classes of tuples $(\Sigma, j, M, D, u, \xi)$ in Definition 4.1.3.

New Strong Polyfold Bundle Structure

To give $\mathcal{W}^{ft}_{A,g,k}$ a strong polyfold bundle structure we must define lifted versions of the good uniformizing families, following the procedure of [22, p116-119].

Let

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta)) \quad \text{where} \quad (\underline{a}, v, \eta) \in \mathcal{O}.$$

be a good uniformizing family of stable maps associated to the new gluing as in Definition 4.2.9.

Let $\delta(\underline{a}, v) : (T\Sigma_{\underline{a}}, j(\underline{a}, v)) \to (T\Sigma_{\underline{a}}, j(\underline{a}, 0))$ be the complex linear map given by

$$\delta(\underline{a}, v)h = \tfrac{1}{2}(\mathrm{id} - j(\underline{a}, 0) \circ j(\underline{a}, v))h.$$

Let Γ be defined via parallel transport of a complex connection, as follows. Fix a complex connection on the almost complex vector bundle $(TQ, J) \to Q$, i.e. if ∇_X is the covariant derivative on Q belonging to the Riemaniann metric $\omega \circ (\mathrm{id} \oplus J)$, the connection $\tilde{\nabla}_X$, defined by $\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J(\nabla_X J)Y$, defines a complex connection, in the sense that it satisfies $\tilde{\nabla}_X(JY) = J(\tilde{\nabla}_X Y)$. If $\eta \in T_pQ$ is a tangent vector, the parallel transport of a complex connection along the path $t \mapsto \exp_p(t\eta)$ for $t \in [0, 1]$, defines the linear map

$$\Gamma(\exp_p(\eta), p) : (T_pQ, J(p)) \to (T_{\exp_p(\eta)}Q, J(\exp_p(\eta)))$$

which is complex linear, hence $\Gamma(\exp_p(\eta), p) \circ J(p) = J(\exp_p(\eta)) \circ \Gamma(\exp_p(\eta), p)$.

The component ξ is a complex anti-linear map

$$\xi(z) : (T_z\Sigma, j) \to (T_{u(z)}Q, J(u(z)).$$

As we have already remarked, we can define new versions of the hat gluings. By implanting the new hat gluing $\hat{\oplus}^{ft}_{\underline{a}}$, we have defined the new glued section

$$\hat{\oplus}^{ft}_{\underline{a}}(\xi)(z) : (T_z\Sigma_{\underline{a}}, j(\underline{a}, 0)) \to (T_{\oplus^{ft}_{\underline{a}} u(z)}Q, J(\oplus^{ft}_{\underline{a}} u(z)))$$

along the glued base map $\oplus^{ft}_{\underline{a}} u$ into Q. The composition $\xi \circ \delta(v)$ is a complex anti-linear map

$$(T_z\Sigma, j(v)) \to (T_{u(z)}Q, J(u(z)))$$

which satisfies the identity

$$\hat{\oplus}^{ft}_{\underline{a}}(\xi \circ \delta(v)) = \hat{\oplus}^{ft}_{\underline{a}}(\xi) \circ \delta(\underline{a}, v).$$

Given a point $(\underline{a}, v, \eta, \xi) \in K^{\mathcal{R}}$ we define the complex anti-linear map

$$\Xi(\underline{a}, v, \eta, \xi)(z) : (T_z\Sigma_{\underline{a}}, j(\underline{a}, v)) \to (T_{\oplus^{ft}_{\underline{a}} \exp_u(\eta)(z)} Q, J(\oplus^{ft}_{\underline{a}} \exp_u(\eta)(z)))$$

at the points $z \in \Sigma_{\underline{a}} \setminus \{\text{not-glued nodal points}\}$ by

$$\Xi(\underline{a}, v, \eta, \xi)(z) = \Gamma[\oplus^{ft}_{\underline{a}} \exp_u(\eta)(z), \oplus^{ft}_{\underline{a}} u(z)] \circ \hat{\oplus}^{ft}_{\underline{a}}(\xi)(z) \circ \delta(\underline{a}, v)(z).$$

Definition 4.5.1. Given a good uniformizing family associated to the new gluing construction,

$$(\underline{a}, v, \eta) \to (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta))$$

we can associate the **lifted family** defined by

$$(\underline{a}, v, \eta, \xi) \to (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta), \Xi(\underline{a}, v, \eta, \xi))$$

where $(\underline{a}, v, \eta, \xi) \in K^{\mathcal{R}}$.

Following the arguments in [22, p119-124], using these lifted families we can define a strong polyfold bundle structure $W^{ft}_{A,g,k}$ on the set $\mathcal{W}^{ft}_{A,g,k}$.

The Cauchy–Riemann section

Consider the Cauchy–Riemann section $\overline{\partial}_J : \mathcal{Z}^{ft}_{A,g,k} \to \mathcal{W}^{ft}_{A,g,k}$. This is defined on the underlying sets of the polyfold $\mathcal{Z}^{ft}_{A,g,k}$ and strong polyfold bundle $\mathcal{W}^{ft}_{A,g,k}$ by the equation

$$[\Sigma, j, M, D, u] \mapsto [\Sigma, j, M, D, u, \tfrac{1}{2}(du + J(u) \circ du \circ j))].$$

In a local sc-trivialization $K \to O$ it has the following local expression

$$(\underline{a}, v, \eta) \mapsto (\underline{a}, v, \eta, \overline{\xi})$$

where $\overline{\xi}$ is the unique solution of the equations

$$\Gamma(\oplus^{ft}_{\underline{a}} \exp_u \eta, \oplus^{ft}_{\underline{a}} u) \cdot \hat{\oplus}^{ft}_{\underline{a}} \overline{\xi} \circ \delta(\underline{a}, v) = \overline{\partial}_{J, j(\underline{a}, v)}(\oplus^{ft}_{\underline{a}} \exp_u \eta),$$
$$\hat{\ominus}^{ft}_{\underline{a}} \overline{\xi} \cdot \tfrac{\partial}{\partial_s} = 0.$$

To show that $\overline{\partial}_J$ is proper we can observe that the unperturbed solution set $\mathcal{S}(\overline{\partial}_J)$ with the topology given by the stable map compactification of [26] is a compact topological space. We can then show that this topology is equivalent to the the subspace topology induced by the underlying topological space of the Gromov–Witten polyfold $\mathcal{Z}^{ft}_{A,g,k}$.

To show that $\overline{\partial}_J$ is regularizing consider the following commutative diagram of maps defined between the underlying sets.

$$\begin{array}{ccc} \mathcal{W}^{ft}_{A,g,k} & \xhookrightarrow[\iota_{\mathcal{W}}]{} & \mathcal{W}_{A,g,k} \\ \overline{\partial}_J \Big\uparrow\Big\downarrow & & \Big\downarrow\Big\uparrow \overline{\partial}_J \\ \mathcal{Z}^{ft}_{A,g,k} & \xhookrightarrow[\iota_{\mathcal{Z}}]{} & \mathcal{Z}_{A,g,k} \end{array}$$

We then only need to observe that these maps preserve the induced grading of the sc-structures.

We can define a filled section for $\overline{\partial}_J$ associated to the new gluing constructions following [22, p129-130]. Moreover, we may write down a local expression for the filled section in a region of a destabilizing ghost component of type I as an operator:

$$\mathfrak{f} : O \to \mathcal{F}, \qquad (a, v, \eta) \mapsto \xi$$

where ξ is the unique solution to the equations.

$$\begin{aligned} \Gamma(\oplus^{ft}_{a,b}(u^+ + \eta^+, u^- + \eta^-), \oplus^{ft}_{a,b}(u^+, u^-)) \circ \hat{\oplus}^{ft}_{a,b}(\xi) &= \overline{\partial}_J(\oplus^{ft}_{a,b}(u^+ + \eta^+, u^- + \eta^-)) \\ \hat{\ominus}^{ft}_{a,b}(\xi) \cdot \frac{\partial}{\partial s} &= \overline{\partial}_{J(0)}(\ominus^{ft}_{a,b}(\eta^+, \eta^-)). \end{aligned}$$

Here O is an open neighborhood of $(0, 0, 0)$ in $B_{\frac{1}{2}} \times B_{\frac{1}{2}} \times E \times H^{3+m,\delta_m}_c(\mathbb{R}^\pm \times S^1, \mathbb{R}^{2n})$ and $\mathcal{F} = H^{2+m,\delta_m}(\mathbb{R}^\pm \times S^1, \mathbb{R}^{2n})$. We may check sc-smoothness of this expression associated to the new gluing using local expressions as in Proposition 4.3.1. To check that this expression can be conjugated to an sc^0-contraction germ requires mimicking the proof of [22, Proposition 4.8]

Lastly, calculation of the Fredholm index for the case of a punctured Riemann surface is classical, the technology can be found in [29] [32].

Chapter 5

Pulling Back Perturbations via the kth-Marked Point Forgetting Map

We have defined a new Gromov–Witten polyfold of stable curves with constant destabilizing ghost components, on which we can consider a well-defined kth-marked point forgetting map

$$ft_k : \mathcal{Z}^{ft}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}.$$

In this chapter, we show how to pullback perturbations via this map. This map is sc^0, and fails to be sc^1 at stable curves which contain a destabilizing ghost component of type I. As a consequence, pulling back perturbations is not quite as automatic as in the case of the permutation maps in Corollary 3.1.5, and will require a hands-on approach.

Throughout this chapter, we will assume that $(A, g, k) \neq (0, 0, 2), (0, 0, 1), (0, 0, 0), (0, 1, 0)$ and $(A, g, k) \neq (0, 0, 3), (0, 1, 1)$ as these cases have already been dealt with, see Remark 2.3.4 and Remark 2.6.2, respectively.

5.1 The kth-Marked Point Forgetting Map Redux

We require the stable curves in $\mathcal{Z}^{ft}_{A,g,k}$ are of class $(3, \delta_0)$ at all marked points $\{z_1, \ldots z_{k-1}\}$, and of class H^3_{loc} at the marked point z_k. In order to get a well-defined map, we will require that the stable curves in $\mathcal{Z}_{A,g,k-1}$ are of class $(3, \delta_0)$ at all marked points $\{z_1, \ldots z_{k-1}\}$

Definition 5.1.1. We define the kth-marked point

$$ft_k : \mathcal{Z}^{ft}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}$$

on the underlying sets of the GW-polyfolds as follows. Let $[\Sigma, j, M, D, u]$ be a stable curve; to define ft_k we distinguish three cases for the component C_k which contains the kth-marked point, $z_k \in C_k$.

- The component $C_k \setminus \{z_k\}$ is stable, i.e.

$$2 \cdot g(C_k) + \sharp((M \setminus \{z_k\}) \cup |D|)_{C_k} \geq 3 \quad \text{or} \quad \int_{C_k} u^*\omega > 0.$$

  We therefore define

$$ft_k([\Sigma, j, M, D, u]) = [\Sigma, j, M \setminus \{z_k\}, D, u].$$

- The component C_k is a destabilizing ghost component of type I. Then $ft_k([\Sigma, j, M, D, u])$ is give by the stable curve obtained as follows. Delete z_k, delete the component C_k, and delete the two nodal pairs. We add a new nodal pair $\{x_a, y_b\}$ given by two points of the former nodal pairs.

- The component C_k is a destabilizing ghost component of type II. Then $ft_k([\Sigma, j, M, D, u])$ is give by the stable curve obtained as follows. Delete z_k, delete the component C_k, and delete the nodal pair. We add a new marked point z_i, given by the former nodal point which did not lie on C_k.

Theorem 5.1.2 (Pulling Back Perturbations via the kth-Marked Point Forgetting Map)**.** *We can construct a regular perturbation which pulls back to a regular perturbation via the kth-marked point forgetting map*

$$ft_k : \mathcal{Z}^{ft}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}.$$

Proof. Our strategy is to use the methods of Chapter 3, and attempt to pullback a suitable perturbation. This approach is complicated by the fact that ft_k is sc^0, and fails to be sc^1 at stable curves which contain a destabilizing ghost component of type I. This is an unavoidable consequence of the fact that our construction of the GW-polyfolds uses as a base the exponential Deligne–Mumford spaces, and we have shown in Proposition 2.6.5 that $ft_k : \overline{\mathcal{M}}^{\exp}_{g,k} \to \overline{\mathcal{M}}^{\exp}_{g,k-1}$ fails to be C^1 at precisely the components of type I.

We may still consider the pullback via ft_k of the polyfold Fredholm problem given by the strong polyfold bundle $\mathcal{W}_{A,g,k-1} \to \mathcal{Z}_{A,g,k-1}$ and the Cauchy–Riemann seciton $\overline{\partial}_J$, as illustrated in the below commutative diagram.

$$\begin{array}{ccc} ft_k^*\mathcal{W}_{A,g,k-1} & \xrightarrow[\text{proj}_2]{} & \mathcal{W}_{A,g,k-1} \\ {\scriptstyle ft_k^*\overline{\partial}_J} \uparrow\downarrow & & \downarrow\uparrow {\scriptstyle \overline{\partial}_J} \\ \mathcal{Z}^{ft}_{A,g,k} & \xrightarrow[ft_k]{} & \mathcal{Z}_{A,g,k-1} \end{array}$$

However, the pullback $ft_k^*\mathcal{W}_{A,g,k-1}$ does not carry an sc-smooth structure. We may replace the étale condition with an sc^0-étale condition (where the source and target maps are required to be surjective local homeomorphisms) and hence we may consider $ft_k^*\mathcal{W}_{A,g,k-1}$ as carrying a *topological* polyfold structure.

The strong polyfold bundle $\mathcal{W}^{ft}_{A,g,k} \to \mathcal{Z}^{ft}_{A,g,k}$ carries an sc-smooth structure. We observe there is a natural sc^0-homeomorphism $\mathcal{W}^{ft}_{A,g,k} \simeq ft_k^*\mathcal{W}_{A,g,k-1}$. We may therefore consider the pullback of a parametrized sc$^+$-multisection $\Lambda^t : \mathcal{W}_{A,g,k-1} \to \mathbb{Q}^+$ as defining a parametrized sc^0-multisection $\text{proj}_2^*\Lambda^t : \mathcal{W}^{ft}_{A,g,k} \to \mathbb{Q}^+$. This is defined identically as in Definition 3.5.3; except here the local section structures $ft_k^*s_1^t, \ldots, ft_k^*s_j^t : O_x \to K[1]$ can only be assumed to be sc^0.

A multisection which is sc^0 is unsuitable for running a transversality argument. However, if we are careful in our construction of the sc$^+$-multisection Λ we can actually ensure that the pullback local section structures $ft_k^*s_1^t, \ldots, ft_k^*s_j^t$ will be sc-smooth. The main idea is the following: in the local expressions we can pinpoint exactly where the failure of differentiability occurs; the map between the gluing parameters $(a,b) \mapsto a *_{\exp} b$ fails to be C^1 whenever $a *_{\exp} b = 0$. We can define a cutoff function $\beta : B_{\frac{1}{2}} \subset \mathbb{C} \to [0,1]$ to be constant on a small neighborhood of the gluing parameter $c = 0$. Hence, while the expression $a *_{\exp} b$ is not C^1, the composition $\beta(a *_{\exp} b)$ is smooth. The construction of the sc$^+$-multisection Λ with sc-smooth pullback is made explicit in the subsequent two sections. In Section 5.3, we examine the local expressions for ft_k in the sc-coordinates of M-polyfold charts. In Section 5.4, we make explicit the construction of the parametrized sc$^+$-multisection Λ^t.

Following this observation, it is easy to follow the procedure of Theorem 3.1.1 to achieve simultaneous transversality. In order to achieve simultaneous compactness, notice that auxiliary norms are only assumed to be sc^0, and hence the pullback of an auxiliary norm by ft_k gives a well-defined auxiliary norm on the strong polyfold bundle $\mathcal{W}^{ft}_{A,g,k} \to \mathcal{Z}^{ft}_{A,g,k}$. It is then a topological exercise to show that the map ft_k satisfies the topological pullback condition 3.1.2, and hence we may apply Theorem 3.1.3. □

5.2 Structure of the kth-Marked Point Forgetting Map on the Perturbed Gromov–Witten Solution Spaces

Consider the kth-marked point forgetting map $ft_k : \mathcal{Z}^{ft}_{A,g,k} \to \mathcal{Z}_{A,g,k-1}$ defined in Definition 5.1.1. By Theorem 5.1.2 we can construct a regular perturbation which pulls back to a regular perturbation via ft_k. Hence the kth-marked point forgetting map persists after perturbation, giving a well-defined map between compact oriented weighted branched orbifolds,

$$ft_k|_{\mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J,\text{proj}_2^*\Lambda)} : \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \text{proj}_2^*\Lambda) \to \mathcal{S}_{A,g,k-1}(\overline{\partial}_J, \Lambda).$$

Considered on the underlying topological spaces, this map is continuous. Considered on the branched ep-subgroupoid structures, the associated functor

$$\hat{ft}_k|_{S^{ft}_{A,g,k}(\hat{\overline{\partial}}_J,\text{proj}_2^*\hat{\Lambda})} : S^{ft}_{A,g,k}(\hat{\overline{\partial}}_J, \text{proj}_2^*\hat{\Lambda}) \to S_{A,g,k-1}(\hat{\overline{\partial}}_J, \hat{\Lambda})$$

is continuous at points which contain a destabilizing ghost component of type I, and smooth at all other points. The restricted kth-marked point forgetting map ft_k and its associated functor $\hat{ft}_k$ are both weight preserving.

The preimage of ft_k at any un-noded stable curve $[\Sigma, j, M, \emptyset, u] \in \mathcal{S}_{A,g,k-1}(\overline{\partial}_J, \Lambda)$ can be identified with the Riemann surface Σ, i.e.

$$ft_k^{-1}([\Sigma, j, M, \emptyset, u]) \simeq \Sigma.$$

At any un-noded stable map $(\Sigma, j, M, \emptyset, u) \in S_{A,g,k-1}(\hat{\overline{\partial}}_J, \hat{\Lambda})$ there exists a local branching structure $(M_i)_{i\in I}$ such that any stable map in a given local branch is also un-noded. In a neighborhood consisting of un-noded local branches, $\hat{ft}_k$ has the structure of a fiber bundle with fiber the un-noded Riemann surface Σ, i.e.

$$\hat{ft}_k^{-1}(M_i) \simeq \Sigma \times M_i.$$

By pulling back a perturbation via the map ft_k, the perturbed Cauchy–Riemann section is independent of the placement of the kth-marked point; hence given a perturbed solution we may move the kth-marked point freely and still have a perturbed solution.

Remark 5.2.1. In [43] we will show that the expected behavior for the orientation is given as follows. The orientation at any stable map in $\hat{ft}_k^{-1}(M_i) \simeq \Sigma \times M_i$ is given by the orientation defined by the complex orientation of Σ and the orientation of M_i.

5.3 Local Expressions for the kth-Marked Point Forgetting Map

Having defined ft_k on the underlying sets of our GW-polyfolds, we now write local expressions for ft_k in the coordinates given by the (alternative) good uniformizing families of stable maps. Our exposition will be quite similar to the local expressions for the kth-marked point forgetting map considered on the DM-spaces $ft_k : \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k-1}$ which we have derived in Section 2.5.

Let $[\alpha] = [\Sigma, j, M, D, u] \in \mathcal{Z}^{ft}_{A,g,k}$ be a stable curve, and let $\alpha = (\Sigma, j, M, D, u) \in Z^{ft}_{A,g,k}$ be a stable map representative. There are four cases to consider.

Suppose first that the component $C_k \setminus \{z_k\}$ is stable, i.e.

$$2 \cdot g(C_k) + \sharp((M \setminus \{z_k\}) \cup |D|)_{C_k} \geq 3 \quad \text{or} \quad \int_{C_k} u^*\omega > 0.$$

Hence

$$ft_k(\Sigma, j, M, D, u) = (\Sigma, j, M \setminus \{z_k\}, D, u).$$

There are now two possibilities.

1. The component $C_k \setminus \{z_k\}$ satisfies the DM-stability condition i.e.

$$2 \cdot g(C_k) + \sharp((M \setminus \{z_k\}) \cup |D|)_{C_k} \geq 3.$$

2. The component $C_k \setminus \{z_k\}$ does not satisfy the DM-stability condition i.e.

$$2 \cdot g(C_k) + \sharp((M \setminus \{z_k\}) \cup |D|)_{C_k} < 3.$$

However,

$$\int_{C_k} u^* \omega > 0.$$

Case 1—The component $C_k \setminus \{z_k\}$ is stable and satisfies the DM-stability condition

Suppose that the component $C_k \setminus \{z_k\}$ is stable and satisfies the DM-stability condition i.e.

$$2 \cdot g(C_k) + \sharp((M \setminus \{z_k\}) \cup |D|)_{C_k} \geq 3.$$

As in Section 2.3, we may choose good data and a stabilization S for α. This determines an alternative good uniformizing family of stable maps centered at α, given by

$$(\underline{a}, v, z, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), ((M \setminus \{z_k\}) \cup \{\phi(z)\} \cup S)_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta)).$$

We may use the same exact stabilization S for $\hat{ft}_k(\alpha)$, and (nearly) identical good data. This determines a good uniformizing family of stable maps centered at $\hat{ft}_k(\alpha)$, given by

$$(\underline{a}', v', \eta') \mapsto (\Sigma_{\underline{a}'}, j(\underline{a}', v'), (M \cup S)_{\underline{a}'}, D_{\underline{a}'}, \oplus_{\underline{a}'} \exp_u(\eta')).$$

Note that the gluing constructions are the same, due to the absence of any destabilizing ghost components.

Comparing the good uniformizing families of stable maps, we deduce

$$\begin{aligned}&\hat{ft}_k(\Sigma_{\underline{a}}, j(\underline{a}, v), ((M \setminus \{z_k\}) \cup \{\phi(z)\} \cup S)_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta)) \\ &= (\Sigma_{\underline{a}'}, j(\underline{a}', v'), (M \cup S)_{\underline{a}'}, D_{\underline{a}'}, \oplus_{\underline{a}'} \exp_u(\eta'))\end{aligned}$$

precisely when $\underline{a}' = \underline{a}$, $v' = v$, and $\eta' = \eta$. Hence considered as a map between the sc-coordinates given by these good uniformizing families of stable maps, a local expression for $\hat{ft}_k$ is given by

$$\hat{ft}_k : (\underline{a}, v, z, \eta) \mapsto (\underline{a}, v, \eta).$$

Case 2—The component $C_k \setminus \{z_k\}$ is stable but does not satisfy the DM-stability condition

Suppose the component $C_k \setminus \{z_k\}$ is stable but does not satisfy the DM-stability condition i.e.

$$2 \cdot g(C_k) + \sharp((M \setminus \{z_k\}) \cup |D|)_{C_k} < 3.$$

However,

$$\int_{C_k} u^*\omega > 0.$$

This is nearly identical to the above case. However, note that a stabilization for α is not necessarily a stabilization for $ft_k(\alpha)$; the component C_k already satisfies the DM-stability condition (in the case of the former), but once we forget the kth-marked point, it is no longer stable and needs to be stabilized (in the case of the latter). To address this, simply choose a stabilization and good data for α that also gives a stabilization and good data for $ft_k(\alpha)$. There is no issue with choosing a stabilization and good data that adds points to a component which already satisfies the DM-stability condition.

The process is now identical. As in Section 2.3, we may choose good data and a stabilization S for α. This determines an alternative good uniformizing family of stable maps centered at α, given by

$$(\underline{a}, v, z, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), ((M \setminus \{z_k\}) \cup \{\phi(z)\} \cup S)_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta)).$$

We may use the same exact stabilization S for $\hat{ft}_k(\alpha)$, and (nearly) identical good data. This determines a good uniformizing family of stable maps centered at $\hat{ft}_k(\alpha)$, given by

$$(\underline{a}', v', \eta') \mapsto (\Sigma_{\underline{a}'}, j(\underline{a}', v'), (M \cup S)_{\underline{a}'}, D_{\underline{a}'}, \oplus_{\underline{a}'} \exp_u(\eta')).$$

Again, the gluing constructions are the same, due to the absence of any destabilizing ghost components.

Comparing the good uniformizing families of stable maps, we deduce

$$\begin{aligned} &\hat{ft}_k(\Sigma_{\underline{a}}, j(\underline{a}, v), ((M \setminus \{z_k\}) \cup \{\phi(z)\} \cup S)_{\underline{a}}, D_{\underline{a}}, \oplus^{ft}_{\underline{a}} \exp_u(\eta)) \\ &= (\Sigma_{\underline{a}'}, j(\underline{a}', v'), (M \cup S)_{\underline{a}'}, D_{\underline{a}'}, \oplus_{\underline{a}'} \exp_u(\eta')) \end{aligned}$$

precisely when $\underline{a}' = \underline{a}$, $v' = v$, and $\eta' = \eta$. Hence considered as a map between the sc-coordinates given by these good uniformizing families of stable maps, a local expression for $\hat{ft}_k$ is given by

$$\hat{ft}_k : (\underline{a}, v, z, \eta) \mapsto (\underline{a}, v, \eta).$$

Case 3—The component C_k is a destabilizing ghost component of type I

Suppose the stable map $\alpha = (\Sigma, j, \{z_1, \ldots, z_k\}, D, u)$ has a destabilizing ghost component of type I. Hence the component C_k contains the marked point z_k, and there exist two nodal pairs $\{x_a, y_a\}$ and $\{x_b, y_b\}$ such that $y_a, x_b \in C_k$. Then $ft_k(\alpha)$ is given by the stable map determined from the following procedure. We delete z_k, delete the component C_k, and delete the two nodal pairs. We add a new nodal pair $\{x_a, y_b\}$ given by two points of the former nodal pairs.

For simplicity, let us assume that $\{x_a, y_a\}$ and $\{x_b, y_b\}$ are the only nodal pairs on $[\Sigma, j, M, D]$ and hence $\{x_c, y_c\}$ is the only nodal pair on $ft_k([\Sigma, j, M, D])$. We make this assumption entirely on the basis of notational convenience and clarity. The general case easily follows from this description. Hence we assume $\alpha = (\Sigma, j, M, \{\{x_a, y_a\}, \{x_b, y_b\}\}, u)$ and that $\hat{ft}_k(\alpha) = (\Sigma \setminus C_k, j, M \setminus \{z_k\}, \{\{x_c, y_c\}\}, u)$.

There is a unique biholomorphism between $C_k \setminus \{y_a, x_b\}$ and $\mathbb{R} \times S^1$ which sends the marked point z_k to the point $(0, 0)$, the puncture y_a to $-\infty$, and the puncture x_b to $+\infty$. We may choose the small disk structure at y_a such that there is a biholomorphism between $D_{y_a} \setminus \{y_a\}$ and $\mathbb{R}^- \times S^1 \subset \mathbb{R} \times S^1$. Likewise, we choose the small disk structure at x_b such that there is a biholomorphism between $D_{x_b} \setminus \{x_b\}$ and $\mathbb{R}^+ \times S^1 \subset \mathbb{R} \times S^1$.

Let $(a, b, v, \eta) \mapsto (\Sigma_{a,b}, j(a, b, v), (M \cup S)_{a,b}, \{\{x_a, y_a\}, \{x_b, y_b\}\}_{a,b}, \oplus^{ft}_{a,b} \exp_u(\eta))$ be a good uniformizing family of stable maps centered at $(\Sigma, j, M, \{\{x_a, y_a\}, \{x_b, y_b\}, u)$. Explicitly, this family is given by the following.

- The glued Riemann surface $\Sigma_{a,b}$ is given by

$$\Sigma_{a,b} = (\Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b})) \sqcup Z_a \sqcup Z_b.$$

- The good complex deformation is given by

$$j(a, b, v) = \begin{cases} j(v) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b}) \\ i & \text{on } Z_a \sqcup Z_b. \end{cases}$$

- The marked points $z_1, \ldots, z_{k-1} \in M_{a,b}$ are given by the former marked points $\{z_1, \ldots, z_{k-1}\} \subset \Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b})$. The marked point $z_k \in M_{a,b}$ is given by

$$z_k = \begin{cases} (0, 0) \in \mathbb{R} \times S^1 & \text{when } a = 0 \text{ and } b = 0 \\ (\varphi(r_a), \theta_a) \in Z_a & \text{when } a \neq 0 \text{ and } b = 0 \\ (0, 0) \in Z_b & \text{when } a = 0 \text{ and } b \neq 0 \\ (\varphi(r_a), \theta_a) \in Z_a (\text{equivalently, } (0, 0) \in Z_b) & \text{when } a \neq 0 \text{ and } b \neq 0. \end{cases}$$

- The set of nodal pairs $\{\{x_a, y_a\}, \{x_b, y_b\}_{a,b}$ is given by

$$\{\{x_a, y_a\}, \{x_b, y_b\}\}_{a,b} = \begin{cases} \{\{x_a, y_a\}, \{x_b, y_b\}\} & \text{when } a = 0 \text{ and } b = 0 \\ \{\{x_b, y_b\}\} & \text{when } a \neq 0 \text{ and } b = 0 \\ \{\{x_a, y_a\}\} & \text{when } a = 0 \text{ and } b \neq 0 \\ \emptyset & \text{when } a \neq 0 \text{ and } b \neq 0. \end{cases}$$

- The map $\oplus^{ft}_{a,b} \exp_u(\eta) : \Sigma_{a,b} \to Q$ defined by the new gluing construction 4.2.4 is given by

$$\oplus^{ft}_{a,b} \exp_u(\eta) = \begin{cases} \exp_u(\eta) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k \sqcup D_{y_b}) \\ \oplus^{ft}_{a,b} \exp_u(\eta) & \text{on } Z_a \sqcup Z_b \end{cases}$$

Let $(c, v', \eta') \mapsto ((\Sigma \setminus C_k)_c, j(c, v'), (M \setminus \{z_k\})_c \cup S_c, \{\{x_c, y_c\}\}_c, \oplus_c \exp_u(\eta'))$ be a good uniformizing family centered at $\hat{ft}_k(\alpha) = (\Sigma \setminus C_k, j, M \setminus \{z_k\}, \{\{x_c, y_c\}\}, u)$. Explicitly, this family is given by the following.

- The glued Riemann surface $(\Sigma \setminus C_k)_c$ is given by

$$(\Sigma \setminus C_k)_c = (\Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c})) \sqcup Z_c$$

- The good complex deformation is the same as above on $\Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c})$, and is given by

$$j(c, v') = \begin{cases} j(v') & \text{on } \Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c}) \\ i & \text{on } Z_c \end{cases}$$

- The marked points $z_1, \ldots, z_{k-1} \in M_{a,b}$ are given by the former marked points $\{z_1, \ldots, z_{k-1}\} \subset \Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c})$.

- The set of nodal pairs $\{\{x_c, y_c\}\}_c$ is given by $\{\{x_c, y_c\}\}$ when $c = 0$ and $\emptyset$ when $c \neq 0$.

- The map $\oplus_c \exp_u(\eta') : \Sigma_{a,b} \to Q$ is given by

$$\oplus_c \exp_u(\eta') = \begin{cases} \exp_u(\eta') & \text{on } \Sigma \setminus (D_{x_c} \sqcup C_k \sqcup D_{y_c}) \\ \oplus_c \exp_u(\eta') & \text{on } Z_c \end{cases}$$

We now compare the good uniformizing families of stable maps, and deduce

$$\begin{aligned} &\hat{ft}_k(\Sigma_{a,b}, j(a, b, v), M_{a,b} \cup S_{a,b}, \{\{x_a, y_a\}, \{x_b, y_b\}\}_{a,b}, \oplus^{ft}_{a,b} \exp_u(\eta)) \\ &= (\Sigma'_c, j(c, v'), (M \setminus \{z_k\})_c \cup S_c, \{x_c, y_c\}_c, \oplus_c \exp_u(\eta')) \end{aligned}$$

precisely when $v' = v$ and when $c = a *_{\exp} b$, as can be seen from the DM case in Section 2.5. We claim that $\eta' = \eta$. To see this, observe that $Z_a \sqcup Z_b$ are $Z_{a *_{\exp} b}$ biholomorphic. By the construction of the new gluing, we may also observe that the interpolations for the new gluing at the double parameter (a, b) and the gluing for the single parameter $a *_{\exp} b$ are the same. Explicitly, for a pair of reference curves (u^+, u^-) with $u^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^{2n}$ we have

$$\oplus^{ft}_{a,b}(u^+, u^-) = \oplus_{a *_{\exp} b}(u^+, u^-),$$

considered as maps $Z_a \sqcup Z_b \to \mathbb{R}^{2n}$. For a pair of vector fields (h^+, h^-) along these reference curves given by $h^\pm : \mathbb{R}^\pm \times S^1 \to \mathbb{R}^{2n}$ we also have

$$\oplus^{ft}_{a,b}(h^+, h^-) = \oplus_{a *_{\exp} b}(h^+, h^-),$$
$$\ominus^{ft}_{a,b}(h^+, h^-) = \ominus_{a *_{\exp} b}(h^+, h^-).$$

It therefore follows that $\eta' = \eta$ uniquely solves the equations

$$\oplus^{ft}_{a,b} \exp_u(\eta) = \oplus_{a *_{\exp} b} \exp_u(\eta'),$$
$$\ominus^{ft}_{a,b} \exp_u(\eta) = 0, \qquad \ominus_{a *_{\exp} b} \exp_u(\eta') = 0.$$

Therefore, considered as a map between the sc-coordinates given by these good uniformizing families of stable maps, a local expression for $\hat{ft}_k$ is given by

$$\hat{ft}_k : (a, b, v, \eta) \mapsto (a *_{\exp} b, v, \eta).$$

Case 4—The component C_k is a destabilizing ghost component of type II

Suppose the stable map $\alpha = (\Sigma, j, \{z_1, \ldots, z_k\}, D, u)$ has a destabilizing ghost component of type II. Hence the component C_k contains the marked point z_k together with another marked point z_i, and there exist a nodal pairs $\{x_a, y_a\}$ such that $y_a \in C_k$. Then $ft_k(\alpha)$ is given by the stable map determined from the following procedure. We delete z_k, delete the component C_k, and delete the nodal pair. We add a new marked point z_i, given by the former nodal point which did not lie on C_k, i.e. $\{x_a\}$.

Once again, for simplicity, let us assume that $\{x_a, y_a\}$ is the only nodal pair on (Σ, j, M, D) and hence $ft_k(\Sigma, j, M, D, u)$ contains no nodal pairs. We make this assumption entirely on the basis of notational convenience and clarity. The general case easily follows from this description. Hence we assume $\alpha = (\Sigma, j, M, \{\{x_a, y_a\}\}, u)$ and that $ft_k(\alpha) = (\Sigma \setminus C_k, j, M \setminus \{z_k\}, \emptyset, u)$.

There is a unique biholomorphism between $C_k \setminus \{y_a, z_i\}$ and $\mathbb{R} \times S^1$ which sends the marked point z_k to the point $(0, 0)$, the puncture y_a to $-\infty$, and the puncture at the marked point z_i to $+\infty$. We may choose the small disk structure at y_a such that there is a biholomorphism between $D_{y_a} \setminus \{y_a\}$ and $\mathbb{R}^- \times S^1 \subset \mathbb{R} \times S^1$.

Let $(a, v, \eta) \mapsto (\Sigma_a, j(a, v), M_a \cup S_a, \{\{x_a, y_a\}\}_a, \oplus^{ft}_a \exp_u(\eta)$ be a good uniformizing family centered at $(\Sigma, j, M, \{\{x_a, y_a\}\}, u)$. Explicitly, this family is given by the following.

- The glued Riemann surface Σ_a is given by

$$\Sigma_a = (\Sigma \setminus (D_{x_a} \sqcup C_k)) \sqcup (\mathbb{R}^+ \times S^1) \sqcup \{+\infty\}.$$

- The good complex deformation is given by

$$j(a, v) = \begin{cases} j(v) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k) \\ i & \text{on } \mathbb{R}^+ \times S^1. \end{cases}$$

- The marked points $z_1, \dots, \hat{z}_i, \dots, z_{k-1} \in M_a$ (with z_i omitted) are given by the former marked points $\{z_1, \dots, \hat{z}_i, \dots, z_{k-1}\} \subset \Sigma \setminus (D_{x_a} \sqcup C_k)$. The marked point $z_k \in M_a$ is given by

$$z_k = \begin{cases} (0,0) \in \mathbb{R} \times S^1 & \text{when } a = 0 \\ (\varphi(r_a), \theta_a) \in \mathbb{R}^+ \times S^1 & \text{when } a \neq 0. \end{cases}$$

  Meanwhile, the marked point $z_i \in M_a$ is given by the point $\{+\infty\}$

- The set of nodal pairs $\{\{x_a, y_a\}\}_a$ is given by $\{\{x_a, y_a\}\}$ when $a = 0$ and $\emptyset$ when $a \neq 0$.

- The map $\oplus_a^{ft} \exp_u(\eta) : \Sigma_a \to Q$ defined by the new gluing construction 4.2.7 is given by

$$\oplus_a^{ft} \exp_u(\eta) = \begin{cases} \exp_u(\eta) & \text{on } \Sigma \setminus (D_{x_a} \sqcup C_k) \\ \oplus_a^{ft} \exp_u(\eta) (= \exp_u(\eta), \text{ by definition}) & \text{on } \mathbb{R}^+ \times S^1 \end{cases}$$

Let $(v', \eta') \mapsto (\Sigma \setminus C_k, j(v'), (M \setminus \{z_k\}) \cup S, \emptyset, \exp_u(\eta'))$ be a good uniformizing family centered at $\hat{ft}_k(\Sigma, j, M, \{\{x_a, y_a\}\}, u) = (\Sigma \setminus C_k, j, M \setminus \{z_k\}, \emptyset, u)$. Explicitly, this family is given by the following.

- The Riemann surface is $\Sigma \setminus C_k$. We may identify the neighborhoods $D_{z_i} \setminus \{z_i\} \simeq D_{x_a} \setminus \{z_i\} \simeq \mathbb{R}^+ \times S^1$; this sends the marked point z_i to $+\infty$.

- The good complex deformation can be taken the same as above, i.e.

$$j(v') = \begin{cases} j(v') & \text{on } \Sigma \setminus D_{z_i} \\ i & \text{on } \mathbb{R}^+ \times S^1 \end{cases}$$

  using the identification of $D_{z_i} \setminus \{z_i\}$ with $\mathbb{R}^+ \times S^1$.

- The marked points are given by $z_1, \dots, \hat{z}_i, \dots, z_{k-1} \in M$, and as mentioned, the marked point z_i is be identified with the former nodal point x_a.

- The set of nodal pairs is the empty set.

- The map $\exp_u(\eta') : \Sigma \setminus C_k \to Q$.

Comparing the good uniformizing families of stable maps, we see

$$\hat{ft}_k(\Sigma_a, j(a, v), M_a \cup S_a, \{\{x_a, y_a\}\}_a, \oplus_a^{ft} \exp_u(\eta)) = (\Sigma \setminus C_k, j(v'), (M \setminus \{z_k\}) \cup S, \emptyset, \exp_u(\eta'))$$

precisely when $v' = v$ and $\eta' = \eta$. Therefore, considered as a map between the sc-coordinates given by these good uniformizing families of stable maps, a local expression for $\hat{ft}_k$ is given by

$$\hat{ft}_k : (a, v, \eta) \mapsto (v, \eta).$$

5.4 Pulling Back Perturbations via the kth-Marked Point Forgetting Map

We now give an explicit construction of an sc$^+$-multisection $\Lambda^t : \mathcal{W}_{A,g,k-1} \to \mathbb{Q}^+$ such that the pullback sc$^+$-multisection $\text{proj}_2^*\Lambda^t : \mathcal{W}^{ft}_{A,g,k} \to \mathbb{Q}^+$ (which, a priori, is only sc^0) can be checked to be sc-smooth, as required in the proof of Theorem 5.1.2.

Pulling Back an sc-Smooth Bump Function via an sc^0-Map

A crucial ingredient in the construction of an sc$^+$-multisection are the sc-smooth bump functions. We begin by recalling the discussion from [18, p174]. Given an sc-Hilbert space H with associated norm $\|\cdot\|$, the map $x \mapsto \|x\|^2$ is sc-smooth. Then, choosing a smooth function $\beta : \mathbb{R} \to [0,1]$ with compact support and $\beta(0) = 1$, the function $\beta(\|x\|^2)$, $x \in H$ defines an sc-smooth bump function.

We construct an sc-smooth bump function at a point $[\Sigma, j, M, D, u] \in \mathcal{Z}_{A,g,k-1}$ according to the following procedure. Let $(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), (M \cup S)_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}} \exp_u(\eta)$ be a good uniformizing family of stable maps centered at the stable map representative $(\Sigma, j, M, D, u) \in Z_{A,g,k-1}$. The sc-coordinates $(\underline{a}, v, \eta)$ lie in the sc-retract O which is a subset of the sc-Banach space $(B_{\frac{1}{2}})^{\sharp D} \times E \times H^{3,\delta_0}_{c,S}(u^*TQ)$.

Choose a smooth cutoff function $\beta : \mathbb{R} \to [0,1]$ such that $\operatorname{supp} \beta \subset (-1,1)$ and $\beta \equiv 1$ on a small neighborhood of $0 \in \mathbb{R}$. For $\epsilon > 0$ we define $\beta_\epsilon : O \to [0,1]$ by

$$\beta_\epsilon : (\underline{a}, v, \eta) \mapsto \left(\prod_{a \in \underline{a}} \beta(\tfrac{1}{\epsilon}|a|^2) \right) \cdot \beta(\tfrac{1}{\epsilon}\|v\|^2) \cdot \beta(\tfrac{1}{\epsilon}\|\eta\|^2).$$

This is an sc-smooth bump function with support contained in an arbitrarily small neighborhood of $(0,0,0) \in O$ (recalling that O is equipped with the subspace topology induced from the sc-Banach space $(B_{\frac{1}{2}})^{\sharp D} \times E \times H^{3,\delta_0}_{c,S}(u^*TQ)$).

Then the pullback of this sc-smooth bump function via the sc^0 map $\hat{ft}_k$ (considered as a local expression on local M-polyfold models) is sc-smooth. We see this explicitly by considering the above four possible cases for the local form of $\hat{ft}_k$. The crucial observation from these local expressions is the fact that while the function $a *_{\exp} b$ is not C^1, the pullback term $\beta(\frac{1}{\epsilon}|a *_{\exp} b|^2)$ is smooth, by our requirement that $\beta \equiv 1$ on a small neighborhood of $0 \in \mathbb{R}$.

Construction of the Perturbation

We now construct the parametrized sc$^+$-multisection Λ^t for the strong polyfold bundle $\mathcal{W}_{A,g,k-1} \to \mathcal{Z}_{A,g,k-1}$. The construction follows the same exact procedure as in [18, Theorem 15.4].

Local Construction: Let $[\alpha] = [\Sigma, j, M, D, u] \in \mathcal{Z}_{A,g,k-1}$ be a stable curve, and let $\alpha = (\Sigma, j, M, D, u)$ be a stable map representative. Let O_α be an M-polyfold chart centered at α, and invariant under the induced action by $\boldsymbol{G}(\alpha)$. Thus we have good uniformizing family

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}} \exp_u(\eta)), \qquad (\underline{a}, v, \eta) \in O_\alpha,$$

and let $K \to O_\alpha$ be a local strong bundle model, with sc-coordinates given by $(\underline{a}, v, \eta, \xi)$. Choose vectors $w_1, \ldots, w_k$ such that $w_1, \ldots, w_k$ together with the linearization of $\hat{\overline{\partial}}_J$ at α together span $H^{2,\delta_0}(\Sigma, \Lambda^{0,1} \otimes_J u^*TQ)$. Let $\beta : O_\alpha \to [0,1]$ be the sc-smooth bump function constructed above.

Define a parametrized sc$^+$-section $O_\alpha \to K$, with parameters $t \in \mathbb{R}^k$ by

$$s^t : O_\alpha \to K, \qquad (\underline{a}, v, \eta) \mapsto (\underline{a}, v, \eta, \beta(\underline{a}, v, \eta) \cdot (\Sigma_{i=1}^k t_i \rho_{\underline{a}}(w_i))),$$

where $\rho_{\underline{a}}$ is the strong bundle projection defined using the hat gluings, see [18, p117] and [22, p65-67]. Using the natural action of isotropy group $\boldsymbol{G}(\alpha)$, we can define a collection of sc$^+$-sections $\{s_g^t\}_{g \in \boldsymbol{G}(\alpha)}$, each with weight $\frac{1}{\sharp \boldsymbol{G}(\alpha)}$, which together give a local section structure for a well-defined sc$^+$-multisection $\lambda^t : \mathcal{W}_{A,g,k-1} \to \mathbb{Q}^+$.

By our construction, the pullback sc$^+$-multisection $\text{proj}_2^* \lambda^t : \mathcal{W}_{A,g,k}^{ft} \to \mathbb{Q}^+$ is sc-smooth. First, the pullbacks via ft_k of the sc-smooth bump functions will be sc-smooth. Second, the pullback terms $\rho_{\underline{a}}(w_i)$ are sc-smooth given that the hat gluing constructions are identical for $\mathcal{W}_{A,g,k}^{ft}$ and $\mathcal{W}_{A,g,k-1}$.

Local to Global: Taking a finite sum of sc$^+$-multisections as constructed above gives the desired perturbation, $\Lambda^t := \oplus \lambda^t$. We can use the approach of Theorem 3.1.1 to achieve simultaneous transversality. By requiring that the locally constructed λ^t satisfy $N_2[\lambda^t] \leq 1$ and dom-supp$(\lambda^t) \subset \mathcal{U}_2$ we can ensure the sum Λ^t is controlled by a pair $(N_2, \mathcal{U}_2)$ for parameters $t \in B_\epsilon \subset \mathbb{R}^l$. The fact that $\text{proj}_2^* \Lambda^t$ is controlled by a pair now follows from Theorem 3.1.3 and the fact that ft_k satisfies the topological pullback condition of Definition 3.1.2.

Chapter 6

Naturality of the Polyfold Gromov–Witten Invariants

Small modifications to the construction of a polyfold yield invariants which, a priori, we can not assume are equivalent. In this chapter we give a framework for proving stability of polyfold invariants under modifications to the polyfold construction. We use this to establish equality of the Gromov–Witten invariants defined by the polyfold of stable curves with constant destabilizing ghost components $\mathcal{Z}^{ft}_{A,g,k}$ constructed in Chapter 4, and the polyfold $\mathcal{Z}^{\text{HWZ}}_{A,g,k}$ constructed in [22] (Theorem 4.1.5). We also use it to establish equality of the Gromov–Witten invariants associated to polyfolds which have different choices of punctures at the marked points (Theorem 6.1.2). Finally, we use this to establish equality of the Gromov–Witten invariants for polyfolds constructed with different strictly increasing sequences $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$ (Theorem 6.1.4).

6.1 Gromov–Witten Polyfold Variations

We begin by introducing three pairs of GW-polyfolds, each with a slight modification to the polyfold construction.

Polyfold of Stable Curves with Constant Destabilizing Ghost Components and Polyfold as Constructed in [22]

In Chapter 4 we have constructed a new polyfold $\mathcal{Z}^{ft}_{A,g,k}$ of stable curves with constant destabilizing ghost components. We would like to compare the GW-invariants defined by this new polyfold with the GW-invariants defined by the GW-polyfold $\mathcal{Z}^{\text{HWZ}}_{A,g,k}$ as constructed in [22].

Theorem 6.1.1. *The polyfold Gromov–Witten invariants associated to the new polyfold $\mathcal{Z}^{ft}_{A,g,k}$ and the Gromov–Witten invariants associated to the polyfold $\mathcal{Z}^{HWZ}_{A,g,k}$ constructed in [22] are equal.*

To begin, consider the inclusion map

$$\mathcal{Z}^{ft}_{A,g,k} \hookrightarrow \mathcal{Z}^{\mathrm{HWZ}}_{A,g,k}$$

where we require all stable map representatives are of class $(3, \delta_0)$ at the marked points $z_1, \ldots, z_{k-1}$ for both polyfolds. The sequence of points in $\mathcal{Z}_{A,g,k}$ defined in Proposition 2.6.5 can be used to show that the above inclusion map is not proper, considered as a continuous map on the underlying topological spaces of the polyfolds. Furthermore, the pullback strong polyfold bundle is not the same as the new strong polyfold bundle on $\mathcal{Z}^{ft}_{A,g,k}$. Hence, the methods of pulling back perturbations used to construct compatible perturbations in Chapter 3 do not apply.

Gromov–Witten Polyfolds and Punctures at the Marked Points

As elaborated in Remark 2.3.5, situations naturally arise where we need to require that the stable curves of the GW-polyfold satisfy exponential decay estimates on neighborhoods of punctures at some fixed subset of the marked points. These situations include the GW-polyfolds considered in the previous subsection, as well as the GW-polyfolds used to define the joining and self-joining maps.

Theorem 6.1.2. *The Gromov–Witten invariants defined by the Gromov–Witten polyfolds do not depend on the choices of punctures at the marked points.*

Consider a GW-polyfold $\mathcal{Z}'_{A,g,k}$ where we require all stable map representatives are of class H^{3,δ_0} at the marked points z_i for all $i \in \{i_1, \ldots, i_l\} \subset \{1, \ldots, k\}$ and of class H^3_{loc} at the remaining marked points.

Additionally, consider the GW-polyfold $\mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}]$ where we require all stable map representatives are of class H^{3,δ_0} and of class H^3_{loc} at all the marked points. For any such GW-polyfolds we can consider the inclusion map

$$\mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}] \hookrightarrow \mathcal{Z}'_{A,g,k}.$$

There exist sequences of maps which converge in H^3_{loc} but do not converge in H^{3,δ_0}. Consequently, in general the above inclusion map is not proper. Furthermore, the pullback strong polyfold bundle is not the same as the standard strong polyfold bundle on $\mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}]$. Once more, the methods of pulling back perturbations used to construct compatible perturbations in Chapter 3 do not apply.

The Gromov–Witten Polyfolds with Sequences (δ_i)

At the beginning of the construction of the GW-polyfolds, one chooses a strictly increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$, i.e.

$$0 < \delta_0 < \delta_1 < \cdots < 2\pi.$$

This sequence is used to define sc-Banach spaces which are then used to define the M-polyfold models of the GW-polyfold $\mathcal{Z}^{3,\delta_0}_{A,g,k}$ (see Section 2.4 of [22]). The following result implies this is one of the few choices which might lead to GW-polyfolds which are not Morita-equivalent.

Theorem 6.1.3. *[22, Theorem 3.37] Having fixed the exponential gluing profile and a strictly increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$, the underlying topological space $\mathcal{Z}^{3,\delta_0}_{A,g,k}$ possesses a natural equivalence class of polyfold structures.*

The construction of a polyfold structure requires choices, such as the choice of a cut-off function in the gluing constructions, choices of good uniformizing families of stable maps, choice of a locally finite refinement of a cover of M-polyfold charts, as well as the exponential gluing profile and the choice of the strictly increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$. The above theorem states that, having fixed the exponential gluing profile and a strictly increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$, different choices lead to Morita-equivalent polyfold structures.

We will further extend this theorem with the following result.

Theorem 6.1.4. *[Naturality of the Polyfold Gromov–Witten Invariants] The Gromov–Witten invariants defined by the Gromov–Witten polyfolds do not depend on the choice of an increasing sequence $(\delta_i)_{i\geq 0} \subset (0, 2\pi)$.*

Given two sequences $(\delta_i) \subset (0, 2\pi)$ and $(\delta'_i) \subset (0, 2\pi)$ we can always find a third sequence $(\delta''_i) \subset (0, 2\pi)$ which satisfies

$$\delta_i \leq \delta''_i, \qquad \delta'_i \leq \delta''_i$$

for all i. The GW-polyfold associated to the sequence (δ''_i) can be seen to give a refinement of the GW-polyfolds associated to (δ_i) and (δ'_i), in the sense that there are inclusion maps

$$\mathcal{Z}^{3,\delta'_0}_{A,g,k} \hookleftarrow \mathcal{Z}^{3,\delta''_0}_{A,g,k} \hookrightarrow \mathcal{Z}^{3,\delta_0}_{A,g,k}.$$

It is therefore sufficient to consider inclusion maps of the form

$$\mathcal{Z}^{3,\delta'_0}_{A,g,k} \hookrightarrow \mathcal{Z}^{3,\delta_0}_{A,g,k}$$

with $\delta_i \leq \delta'_i$ for all i and demonstrate that the associated GW-invariants are equal. Note that if $\delta_0 < \delta'_0$ this inclusion map is not proper; to see this, exploit the difference in exponential weights. Furthermore, the pullback strong polyfold bundle is not the same as the standard strong polyfold bundle on $\mathcal{Z}^{3,\delta'_0}_{A,g,k}$. So once again, the methods of pulling back perturbations used to construct compatible perturbations in Chapter 3 do not apply.

6.2 Equality of Polyfold Gromov–Witten Invariants

We have given three different inclusion maps between three different sets of GW-polyfolds, each with some slight modification to the polyfold construction. We can consider the following three commutative diagrams of maps between polyfolds.

GW-Polyfold of Constant Destabilizing Ghost Components and HWZ GW-Polyfold:

$$\begin{array}{ccc} \mathcal{W}^{ft}_{A,g,k} & \xhookrightarrow[\iota_{\mathcal{W}}]{} & \mathcal{W}^{\mathrm{HWZ}}_{A,g,k} \\ \overline{\partial}_1 \uparrow\downarrow & & \downarrow\uparrow \overline{\partial}_2 \\ \mathcal{Z}^{ft}_{A,g,k} & \xhookrightarrow[\iota_{\mathcal{Z}}]{} & \mathcal{Z}^{\mathrm{HWZ}}_{A,g,k} \end{array}$$

GW-Polyfolds and Punctures at the Marked Points:

$$\begin{array}{ccc} \mathcal{W}_{A,g,k}[H^2 \cap H^{2,\delta_0}] & \xhookrightarrow[\iota_{\mathcal{W}}]{} & \mathcal{W}'_{A,g,k} \\ \overline{\partial}_1 \uparrow\downarrow & & \downarrow\uparrow \overline{\partial}_2 \\ \mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}] & \xhookrightarrow[\iota_{\mathcal{Z}}]{} & \mathcal{Z}'_{A,g,k} \end{array}$$

GW-Polyfolds with Different Strictly Increasing Sequences $(\delta_i)_{i\geq 0}$:

$$\begin{array}{ccc} \mathcal{W}^{2,\delta'_0}_{A,g,k} & \xhookrightarrow[\iota_{\mathcal{W}}]{} & \mathcal{W}^{2,\delta_0}_{A,g,k} \\ \overline{\partial}_1 \uparrow\downarrow & & \downarrow\uparrow \overline{\partial}_2 \\ \mathcal{Z}^{3,\delta'_0}_{A,g,k} & \xhookrightarrow[\iota_{\mathcal{Z}}]{} & \mathcal{Z}^{3,\delta_0}_{A,g,k} \end{array}$$

In each of these cases, we have sc-smooth proper Fredholm sections of the same index, $\overline{\partial}_1$ and $\overline{\partial}_2$, and we have sc-smooth inclusion maps between polyfolds $\iota_{\mathcal{W}}$ and $\iota_{\mathcal{Z}}$. The associated functors between polyfold structures $\hat{\iota}_{\mathcal{Z}}$ are fully faithful, and are also injections on both the object and the morphism sets. Likewise the associated functors between polyfold strong bundle structures $\hat{\iota}_{\mathcal{W}}$ are also fully faithful, are an injection on both the object and the morphism sets, and moreover are bundle maps (i.e. restrict to a linear map on the fibers). Finally, in each case $\mathcal{S}(\overline{\partial}_2) \subset \mathrm{Im}(\iota_{\mathcal{Z}})$.

It cannot be overstated that the differences between each set of GW-polyfolds are subtle; despite this, there are enough similarities in the setups that for the time being we may consider a single, general setup described as follows. Consider a commutative diagram of polyfolds as follows.

$$\begin{array}{ccc} \mathcal{W}_1 & \xhookrightarrow[\iota_{\mathcal{W}}]{} & \mathcal{W}_2 \\ \overline{\partial}_1 \uparrow\downarrow & & \downarrow\uparrow \overline{\partial}_2 \\ \mathcal{Z}_1 & \xhookrightarrow[\iota_{\mathcal{Z}}]{} & \mathcal{Z}_2 \end{array}$$

Suppose $\mathcal{W}_1 \to \mathcal{Z}_1$ is a strong polyfold bundle. (Later we will further specify that $\mathcal{Z}_1$ is one of the three polyfolds considered above, i.e. $\mathcal{Z}^{ft}_{A,g,k}$, $\mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}]$, or $\mathcal{Z}^{3,\delta'_0}_{A,g,k}$. Likewise, we will further specify that $\mathcal{W}_1$ is one of the three strong polyfold bundles $\mathcal{W}^{ft}_{A,g,k}$, $\mathcal{W}_{A,g,k}[H^{2,\delta_0}\cap H^2]$, or $\mathcal{W}^{2,\delta_0'}_{A,g,k}$.)

Suppose $\mathcal{W}_2 \to \mathcal{Z}_2$ is a strong polyfold bundle. (Later we will further specify that $\mathcal{Z}_2$ is one of the three polyfolds considered above, i.e. $\mathcal{Z}^{\text{HWZ}}_{A,g,k}$, $\mathcal{Z}'_{A,g,k}$, or $\mathcal{Z}^{3,\delta_0}_{A,g,k}$. Likewise we will further specify that $\mathcal{W}_2$ is one of the three strong polyfold bundles $\mathcal{W}^{\text{HWZ}}_{A,g,k}$, $\mathcal{W}'_{A,g,k}$, or $\mathcal{W}^{2,\delta_0}_{A,g,k}$.)

We thus have strong polyfold bundles $\mathcal{W}_1 \to \mathcal{Z}_1$ and $\mathcal{W}_2 \to \mathcal{Z}_2$. Suppose that $\overline{\partial}_1$ and $\overline{\partial}_2$ are sc-smooth proper Fredholm sections of the same index, and suppose we have sc-smooth inclusion maps between polyfolds $\iota_{\mathcal{W}} : \mathcal{W}_1 \hookrightarrow \mathcal{W}_2$ and $\iota_{\mathcal{Z}} : \mathcal{Z}_1 \hookrightarrow \mathcal{Z}_2$. Suppose that the associated functor between polyfold structures $\hat{\iota}_{\mathcal{Z}} : (Z_1, \boldsymbol{Z}_1) \hookrightarrow (Z_2, \boldsymbol{Z}_2)$ is fully faithful, and is also an injection on both the object and the morphism sets. Suppose that the associated functor between polyfold strong bundle structures $\hat{\iota}_{\mathcal{W}} : (W_1, \boldsymbol{W}_1) \hookrightarrow (W_2, \boldsymbol{W}_2)$ is also fully faithful, is an injection on both the object and the morphism sets, and moreover is a bundle map (i.e. restricts to a linear map on the fibers). Finally, suppose that $\mathcal{S}(\overline{\partial}_2) \subset \operatorname{Im}(\iota_{\mathcal{Z}})$.

It follows from commutativity that $\iota_{\mathcal{Z}}$ restricts to a continuous bijection between the unperturbed solution sets i.e. $\iota_{\mathcal{Z}}|_{\mathcal{S}(\overline{\partial}_1)} : \mathcal{S}(\overline{\partial}_1) \to \mathcal{S}(\overline{\partial}_2)$. In fact, this map is a homeomorphism as can be shown from some point-set topology, noting that $\mathcal{S}(\overline{\partial}_1)$ is compact and $\mathcal{S}(\overline{\partial}_2)$ is Hausdorff (see [35, Remark 3.1.15]).

Perturbations which Restrict

While we cannot pullback perturbations, in concrete examples we may be able to construct perturbations which have well-behaved restrictions. We highlight the conditions we would like such a pair of parametrized sc$^+$-multisections to possess by introducing the following definition.

Definition 6.2.1. We say the parametrized sc$^+$-multisection $\Lambda_2^t : \mathcal{W}_2 \to \mathbb{Q}^+$ is a **regular perturbation with a compatible restriction** if there exists a parametrized sc$^+$-multisection $\Lambda_1^t : \mathcal{W}_1 \to \mathbb{Q}^+$ with common parameters $t \in B^l_\epsilon \subset \mathbb{R}^l$ such that the following conditions are satisfied.

Restriction condition: We require that the following diagram of maps between polyfolds commutes.

$$\begin{array}{ccccc} & & & & \mathbb{Q}^+ \\ & & \nearrow^{\Lambda_1^t} & \nearrow_{\Lambda_2^t} & \\ \mathcal{W}_1 & \xhookrightarrow[\iota_{\mathcal{W}}]{} & \mathcal{W}_2 & & \end{array} \tag{6.1}$$

We require the following diagram between local section structures commutes for every $1 \leq i \leq k$, and we require that the weights are equal, $\sigma_i = \sigma_i'$.

$$\begin{array}{ccc} K_1 & \xhookrightarrow{\hat{\iota}_{\mathcal{W}}} & K_2 \\ s_i^t \uparrow\downarrow & & \downarrow\uparrow s_i'^t \\ O_x & \xhookrightarrow{\hat{\iota}_{\mathcal{Z}}} & O_y \end{array} \tag{6.2}$$

Here $[x] \in \mathcal{Z}_1$ maps to $[y] := \iota_{\mathcal{Z}}([x]) \in \mathcal{Z}_2$, and we have local section structures at $[x]$ and $[y]$ given by the following data:

- a local sc-trivialization $K_1 \to O_x$ at a representative x, and parametrized local sections (s_i^t) with weights σ_i for $i \in I$,
- a local sc-trivialization $K_2 \to O_y$ at a representative y and parametrized local sections $(s_i'^t)$ with weights σ_i' for $i \in I$.

Transversality condition:

- $(\overline{\partial}_1, \Lambda_1^t)$ is a transversal pair for the strong polyfold bundle $B_\epsilon^l \times \mathcal{W}_1 \to B_\epsilon^l \times \mathcal{Z}_1$
- $(\overline{\partial}_2, \Lambda_2^t)$ is a transversal pair for the strong polyfold bundle $B_\epsilon^l \times \mathcal{W}_2 \to B_\epsilon^l \times \mathcal{Z}_2$

Compactness condition: We require that Λ_2^t is controlled by a pair $(N_2, \mathcal{U}_2)$ which controls the compactness of $\overline{\partial}_2$, i.e.

- $N_2[\Lambda_2^t] \leq 1$
- dom-supp$(\Lambda_2^t) \subset \mathcal{U}_2$.

Anti push off condition: We require that for every $t_0 \in B_\epsilon^l$,

$$\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}) \subset \iota_{\mathcal{Z}}(\mathcal{Z}_1). \tag{6.3}$$

Proposition 6.2.2. *Consider the three inclusion maps outlined in Section 6.1,*

- $\mathcal{Z}_{A,g,k}^{ft} \hookrightarrow \mathcal{Z}_{A,g,k}^{HWZ}$,
- $\mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}] \hookrightarrow \mathcal{Z}'_{A,g,k}$,
- $\mathcal{Z}_{A,g,k}^{3,\delta_0'} \hookrightarrow \mathcal{Z}_{A,g,k}^{3,\delta_0}$.

There exist regular perturbations with compatible restrictions for these maps.

We prove this proposition in Section 6.3. Given a pair of such perturbations, the transversality condition implies we can find a common regular value $t_0 \in B_{\epsilon/2}^l$, and hence by Theorem 3.4.4 the perturbed solution sets $\mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0})$ and $\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$ will have the structure of weighted branched orbifolds. By the commutativity of the diagram, we can also conclude that there is a well-defined continuous map between the perturbed solution sets,

$$\iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0}) \hookrightarrow \mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}).$$

By the compactness condition, we know that $\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}) \subset \mathcal{Z}_2$ is compact.

Observe the following:

- dom-supp$(\Lambda_1^{t_0}) = \iota_{\mathcal{Z}}^{-1}\left(\text{dom-supp}(\Lambda_2^{t_0})\right)$

- the topology on $\mathcal{Z}_1$ will in general be strictly finer than the subspace topology of the embedded image $\iota_{\mathcal{Z}}(\mathcal{Z}_1) \subset \mathcal{Z}_2$.

It follows from these two observations that for an arbitrary open neighborhood $\overline{\partial}_1^{-1}(0) \subset \mathcal{U}_1 \subset \mathcal{Z}_1$ we cannot expect dom-supp$(\Lambda_1^{t_0}) \subset \mathcal{U}_1$. Hence we cannot in general expect that $\Lambda_1^{t_0}$ will be controlled by a pair. And moreover, we cannot assume that $\mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0})$ is compact.

Proposition 6.2.3 (Branched Suborbifolds and Invariance of Domain)**.** *The restriction of the inclusion map to the perturbed solution sets*

$$\iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0}) \to \mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$$

is a homeomorphism between the underlying topological spaces. Moreover, the associated functor

$$\hat{\iota}_{\mathcal{Z}} : S(\hat{\overline{\partial}}_1, \hat{\Lambda}_1^{t_0}) \to S(\hat{\overline{\partial}}_2, \hat{\Lambda}_2^{t_0})$$

considered as a continuous map on object and morphism sets, is a homeomorphism onto its image. Furthermore, (6.1) *implies the inclusion map is weight preserving, i.e.* $(\Lambda_2^{t_0} \circ \overline{\partial}_2) \circ \iota_{\mathcal{Z}} = \Lambda_1^{t_0} \circ \overline{\partial}_1$ *and* $(\hat{\Lambda}_2^{t_0} \circ \hat{\overline{\partial}}_2) \circ \hat{\iota}_{\mathcal{Z}} = \hat{\Lambda}_1^{t_0} \circ \hat{\overline{\partial}}_1$.

We prove this proposition in Section 6.4. By Theorem 3.4.4 the transversality condition implies that the thickened solution sets given by

- $\mathcal{S}(\overline{\partial}_1, \Lambda_1^t; t \in B_\epsilon^l) := \{(t, [z]) \in B_\epsilon^l \times \mathcal{Z}_1 \mid \Lambda_1^t(\overline{\partial}_1([z])) > 0\} \subset B_\epsilon^l \times \mathcal{Z}_1$,
- $\mathcal{S}(\overline{\partial}_2, \Lambda_2^t; t \in B_\epsilon^l) := \{(t, [z]) \in B_\epsilon^l \times \mathcal{Z}_2 \mid \Lambda_2^t(\overline{\partial}_2([z])) > 0\} \subset B_\epsilon^l \times \mathcal{Z}_2$.

will also have the structure of weighted branched orbifolds. The compactness condition implies that the set

$$\mathcal{S}(\overline{\partial}_2, \Lambda_2^t; t \in \mathrm{cl}(B_{\epsilon/2}^l)) := \{(t, [z]) \in \mathrm{cl}(B_{\epsilon/2}^l) \times \mathcal{Z}_2 \mid \Lambda_2^t(\overline{\partial}_2([z])) > 0\}$$

is compact. And by the commutativity of the diagram, we can also conclude that there is a well-defined continuous map between the thickened solution sets,

$$\mathrm{id} \times \iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^t; t \in B_\epsilon^l) \hookrightarrow \mathcal{S}(\overline{\partial}_2, \Lambda_2^t; t \in B_\epsilon^l), \qquad (t, [z]) \mapsto (t, \iota_{\mathcal{Z}}([z])).$$

Proposition 6.2.4 (Thickened Solutions Sets and Invariance of Domain)**.** *The restriction of the inclusion map to the thickened solution sets*

$$id \times \iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^t; t \in B_\epsilon^l) \to \mathcal{S}(\overline{\partial}_2, \Lambda_2^t; t \in B_\epsilon^l), \qquad (t, [z]) \mapsto (t, \iota_{\mathcal{Z}}([z]))$$

is a homeomorphism between the underlying topological spaces. The associated functor

$$id \times \hat{\iota}_{\mathcal{Z}} : S(\hat{\overline{\partial}}_1, \hat{\Lambda}_1^t; t \in B_\epsilon^l) \to S(\hat{\overline{\partial}}_2, \hat{\Lambda}_2^t; t \in B_\epsilon^l), \qquad (t, z) \mapsto (t, \hat{\iota}_{\mathcal{Z}}(z))$$

considered as a continuous map on object and morphism sets, is a homeomorphism onto its image.

We prove this proposition in Section 6.4. Since $\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$ is compact, we may use Lemma 6.4.9 to conclude that $\mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0})$ is also compact, and therefore has the structure of a *compact* weighted branched suborbifold. We may now use Definition 7.2.1 to get a well-defined branched integral

$$\int_{(\mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0}), \Lambda_1^{t_0} \circ \overline{\partial}_1)} \omega$$

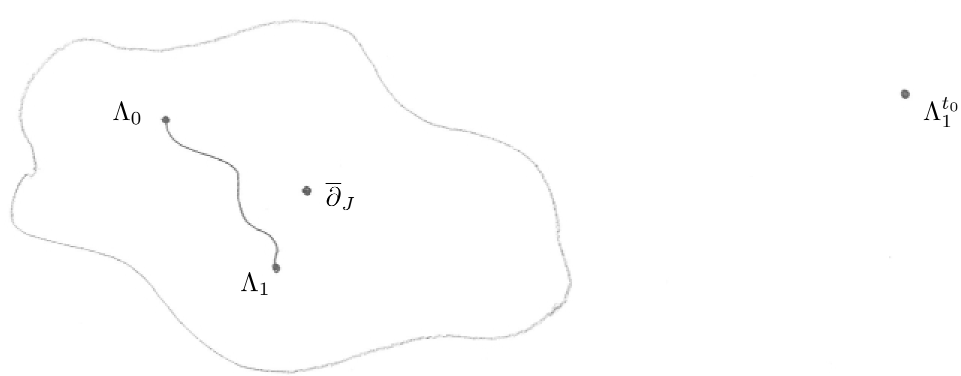
for a given sc-smooth differential form $\omega \in \Omega^*_\infty(\mathcal{Z}_1)$.

Furthermore, the above Proposition 6.2.3 implies the inclusion map between the perturbed solution sets satisfies the necessary hypothesis for the change of variable theorem 7.2.2, and therefore for a given sc-smooth differential form $\omega \in \Omega^*_\infty(\mathcal{Z}_2)$ we have

$$\int_{(\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}), \Lambda_2^{t_0} \circ \overline{\partial}_2)} \omega = \int_{(\mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0}), \Lambda_1^{t_0} \circ \overline{\partial}_1)} \iota_{\mathcal{Z}}^* \omega.$$

However, we cannot assume that the perturbation $\Lambda_1^{t_0}$ is a regular perturbation (see Definition 3.4.9). This is problematic since Proposition 3.4.11 only implies the existence of a compact cobordism between the perturbed solution spaces of two perturbations which are both assumed to be regular perturbations.

Figure 6.1: Compact Cobordism Between Regular Perturbations



Fredholm Multisections

In order to strengthen the results of Proposition 3.4.11, and therefore obtain a compact cobordism between the perturbed solution spaces of the perturbation $\Lambda_1^{t_0}$ and a regular perturbation, we introduce in this subsection the notion of a Fredholm multisection.

Definition 6.2.5. Let $\mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle. We define an **sc-smooth proper Fredholm multisection** as a pair $(\overline{\partial}, \Lambda)$ consisting of an sc-smooth Fredholm section $\overline{\partial} : \mathcal{Z} \to \mathcal{W}$ and an sc$^+$-multisection $\Lambda : \mathcal{W} \to \mathbb{Q}^+$ such that the solution set

$$\mathcal{S}(\overline{\partial}, \Lambda) = \{[z] \in \mathcal{Z} \mid \Lambda \circ \overline{\partial}([z]) > 0\} \subset \mathcal{Z}$$

is a compact set.

As it turns out Fredholm multisections carry the full strength of the abstract perturbation theory which we recalled in Chapter 3. In particular, we have the following analog of Proposition 3.4.11.

Proposition 6.2.6. *Let $\mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, let $\overline{\partial}$ be an sc-smooth proper Fredholm section.*

Let $[0,1] \times \mathcal{W} \to [0,1] \times \mathcal{Z}$ be a strong polyfold bundle, and let $(\tilde{\overline{\partial}}, \Lambda)$ be an sc-smooth Fredholm multisection, consisting of the sc-smooth Fredholm section $\tilde{\overline{\partial}}$ defined by $(s,[z]) \mapsto (s, \overline{\partial}([z]))$ and an sc$^+$-multisection $\Lambda : [0,1] \times \mathcal{W} \to \mathbb{Q}^+$.

Suppose that $(\tilde{\overline{\partial}}, \Lambda)$ is proper, i.e.

$$\mathcal{S}(\tilde{\overline{\partial}}, \Lambda) = \{(s,[z]) \in [0,1] \times \mathcal{Z} \mid \Lambda(s, \overline{\partial}([z])) > 0\}$$

is a compact subset of the underlying topological space $[0,1] \times \mathcal{Z}$.

Then there exists a pair $(N, \mathcal{U})$ which controls the compactness of $(\tilde{\overline{\partial}}, \Lambda)$. Furthermore, there exists a regular perturbation Γ of $(\tilde{\overline{\partial}}, \Lambda)$ with respect to the pair $(N, \mathcal{U})$, i.e. there exists an sc$^+$-multisection $\Gamma : [0,1] \times \mathcal{W} \to \mathbb{Q}^+$ which satisfies the following:

- *$(\tilde{\overline{\partial}}, \Lambda \oplus \Gamma)$ is a transversal pair of the strong polyfold bundle $[0,1] \times \mathcal{W} \to [0,1] \times \mathcal{Z}$,*
- *$N[\Gamma] \leq 1$ and dom-supp$(\Gamma) \subset \mathcal{U}$.*

Consider the strong polyfold bundle $[0,1] \times \mathcal{W}_1 \to [0,1] \times \mathcal{Z}_1$, and consider the sc-smooth Fredholm section $\tilde{\overline{\partial}}_1$ defined by $(s,[z]) \mapsto (s, \overline{\partial}_1([z]))$. We can consider an sc$^+$-multisection $\Lambda_1^{s \cdot t_0} : [0,1] \times \mathcal{W}_1 \to \mathbb{Q}^+$ defined by $(s,[w]) \mapsto \Lambda_1^{s \cdot t_0}([w])$. Notice that we can identify the restriction $\Lambda_1^{s \cdot t_0}|_{\{1\} \times \mathcal{W}_1}$ with the sc$^+$ multisection $\Lambda_1^{t_0}$.

Consider the solution set

$$\mathcal{S}(\tilde{\overline{\partial}}, \Lambda_1^{s \cdot t_0}) := \{(s,[z]) \in [0,1] \times \mathcal{Z}_1 \mid \Lambda_1^{s \cdot t_0}(\overline{\partial}_1([z])) > 0\}.$$

By Proposition 6.2.3, the thickened solution sets $\mathcal{S}(\overline{\partial}_1, \Lambda_1^t; t \in B_\epsilon^l)$ and $\mathcal{S}(\overline{\partial}_2, \Lambda_2^t; t \in B_\epsilon^l)$ are homeomorphic, which implies that the subset $\mathcal{S}(\overline{\partial}_1, \Lambda_1^t; t \in \text{cl}(B_{\epsilon/2}^l))$ is homeomorphic to the compact set $\mathcal{S}(\overline{\partial}_2, \Lambda_2^t; t \in \text{cl}(B_{\epsilon/2}^l))$. Hence the solution set $\mathcal{S}(\tilde{\overline{\partial}}, \Lambda_1^{s \cdot t_0})$ is compact.

We can now interpret the pair $(\tilde{\overline{\partial}}_1, \Lambda_1^{s \cdot t_0})$ as an sc-smooth Fredholm multisection of the strong polyfold bundle $[0,1] \times \mathcal{W}_1 \to [0,1] \times \mathcal{Z}_1$. By the observation that the solution set $\mathcal{S}(\tilde{\overline{\partial}}, \Lambda_1^{s \cdot t_0})$ is compact, this Fredholm multisection is proper.

Proposition 6.2.7. *Consider the strong polyfold bundle $\mathcal{W}_1 \to \mathcal{Z}_1$, and consider the sc-smooth proper Fredholm section $\overline{\partial}_1$. Consider the strong polyfold bundle $[0,1] \times \mathcal{W}_1 \to [0,1] \times \mathcal{Z}_1$, and consider the sc-smooth proper Fredholm multisection $(\tilde{\overline{\partial}}_1, \Lambda_1^{s\cdot t_0})$.*

There exists a pair $(N,\mathcal{U})$, consisting of an auxiliary norm N for $[0,1] \times \mathcal{W}$ and a neighborhood $\mathcal{U} \subset [0,1] \times \mathcal{Z}_1$ of the solution set $\mathcal{S}(\tilde{\overline{\partial}}, \Lambda_1^{s\cdot t_0})$, which controls the compactness of $(\tilde{\overline{\partial}}_1, \Lambda_1^{s\cdot t_0})$, such that the following holds.

The pair $(N_0,\mathcal{U}_0)$ controls the compactness of $\overline{\partial}_1$, where

- *N_0 is given by the restriction of N to $\{0\} \times \mathcal{W}_1$,*
- *$\mathcal{U}_0 = \mathcal{U} \cap (\{0\} \times \mathcal{Z}_1)$.*

Moreover, there exists a regular perturbation Γ of $(\tilde{\overline{\partial}}_1, \Lambda_1^{s\cdot t_0})$ with respect to the pair $(N,\mathcal{U})$ such that

- *the restriction $\Gamma|_{\{0\}\times\mathcal{W}_1}$ can be identified with a regular perturbation Γ_0 of $\overline{\partial}_1$ with respect to the pair $(N_0,\mathcal{U}_0)$,*
- *in a neighborhood of $\{1\} \times \mathcal{Z}_1$, the regular perturbation Γ may be chosen so that it is the 0 perturbation.*

Proof. Consider the inclusion map

$$\{0\} \times \mathrm{id} : \mathcal{Z}_1 \hookrightarrow [0,1] \times \mathcal{Z}_1, \qquad [z] \mapsto (0,[z]).$$

Observe that this map satisfies the topological pullback condition 3.1.2. We may identify the strong polyfold bundle $\mathcal{W}_1$ and the pullback strong polyfold bundle $(\{0\} \times \mathrm{id})^*([0,1] \times \mathcal{W}_1)$. We may also identify the sc-smooth proper Fredholm section $\overline{\partial}_1$ with the pullback section $(\{0\} \times \mathrm{id})^*(\tilde{\overline{\partial}}_1, \Lambda_1^{s\cdot t_0})$. We may consider the following commutative diagram of maps between polyfolds.

$$\begin{array}{ccc} \mathcal{W}_1 & \xrightarrow{\{0\}\times\mathrm{id}} & [0,1] \times \mathcal{W}_1 \\ {\scriptstyle\overline{\partial}_1} \uparrow\downarrow & & \downarrow\uparrow {\scriptstyle(\tilde{\overline{\partial}}_1, \Lambda_1^{s\cdot t_0})} \\ \mathcal{Z}_1 & \xrightarrow[\{0\}\times\mathrm{id}]{} & [0,1] \times \mathcal{Z}_1 \end{array}$$

Then the existence of the pair $(N,\mathcal{U})$ follows from Theorem 3.1.3 and the fact that $\{0\} \times \mathrm{id}$ satisfies the topological pullback condition 3.1.2. The auxiliary norm N_0 is given by the pullback $(\{0\} \times \mathrm{id})^*N$, while $\mathcal{U}_0 = (\{0\} \times \mathrm{id})^{-1}(\mathcal{U})$.

Furthermore, we may construct the desired regular perturbation Γ of $(\tilde{\overline{\partial}}_1, \Lambda_1^{s\cdot t_0})$ with respect to the pair $(N,\mathcal{U})$ using the reasoning of Theorem 3.1.1. We do not need to perturb in a neighborhood of $\{1\} \times \mathcal{Z}_1$, as by assumption $(\overline{\partial}_1, \Lambda_1^{t_0})$ is a transversal pair. □

All the pieces are now in place.

Theorem 6.2.8 (Proof of Theorems 6.1.1, 6.1.2, 6.1.4)**.** *Consider the GW-invariants of Definition 7.2.6. The GW-invariants associated to each of the pairs of GW-polyfolds we have described are equal, i.e.*

- $GW^Q_{A,g,k}[\mathcal{Z}^{ft}_{A,g,k}](\alpha_1,\ldots,\alpha_k;\beta) = GW^Q_{A,g,k}[\mathcal{Z}^{HWZ}_{A,g,k}](\alpha_1,\ldots,\alpha_k;\beta),$
- $GW^Q_{A,g,k}[\mathcal{Z}_{A,g,k}[H^3\cap H^{3,\delta_0}]](\alpha_1,\ldots,\alpha_k;\beta) = GW^Q_{A,g,k}[\mathcal{Z}'_{A,g,k}](\alpha_1,\ldots,\alpha_k;\beta),$
- $GW^Q_{A,g,k}[\mathcal{Z}^{3,\delta_0'}_{A,g,k}](\alpha_1,\ldots,\alpha_k;\beta) = GW^Q_{A,g,k}[\mathcal{Z}^{3,\delta_0}_{A,g,k}](\alpha_1,\ldots,\alpha_k;\beta).$

where $\alpha_1,\ldots,\alpha_k\in H_*(Q;\mathbb{R})$*, and* $\beta\in H_*(\overline{\mathcal{M}}^{log}_{g,k};\mathbb{R})$*. (Here we use "*$GW^Q_{A,g,k}[\mathcal{Z}]$*" to denote the GW-invariant considered with respect to a specific GW-polyfold* $\mathcal{Z}$*.)*

Proof. Suppose that the inclusion map between polyfolds $\iota_{\mathcal{Z}}:\mathcal{Z}_1\hookrightarrow\mathcal{Z}_2$ is given by one of the pairs of GW-polyfolds we described at the start, i.e.

- $\mathcal{Z}^{ft}_{A,g,k}\hookrightarrow\mathcal{Z}^{\text{HWZ}}_{A,g,k},$
- $\mathcal{Z}_{A,g,k}[H^3\cap H^{3,\delta_0}]\hookrightarrow\mathcal{Z}'_{A,g,k},$
- $\mathcal{Z}^{3,\delta_0'}_{A,g,k}\hookrightarrow\mathcal{Z}^{3,\delta_0}_{A,g,k}.$

Then we have a commutative diagram of polyfolds

$$\begin{array}{ccc} & & Q^k\times\overline{\mathcal{M}}^{\log}_{g,k} \\ & \nearrow^{\Pi^k_{i=1}ev_i\times\pi} & \nwarrow_{\Pi^k_i ev'_i\times\pi'} \\ \mathcal{Z}_1 & \xhookrightarrow[\iota_{\mathcal{Z}}]{} & \mathcal{Z}_2 \end{array} \tag{6.4}$$

as described in Section 7.1, where we consider the orbifold $Q^k\times\overline{\mathcal{M}}^{\log}_{g,k}$ as a finite-dimensional polyfold.

We now review the argument up to this point. Consider the following commutative diagram of polyfolds, which moreover satisfy the hypothesis outlined at the start of this section.

$$\begin{array}{ccc} \mathcal{W}_1 & \xhookrightarrow[\iota_{\mathcal{W}}]{} & \mathcal{W}_2 \\ \overline{\partial}_1\uparrow\downarrow & & \downarrow\uparrow\overline{\partial}_2 \\ \mathcal{Z}_1 & \xhookrightarrow[\iota_{\mathcal{Z}}]{} & \mathcal{Z}_2 \end{array}$$

By Proposition 6.2.2, there exists two parametrized sc$^+$-multisections $\Lambda^t_1:\mathcal{W}_1\to\mathbb{Q}^+$ and $\Lambda^t_2:\mathcal{W}_2\to\mathbb{Q}^+$ which are transversally compatible perturbations, as described in Definition 6.2.1.

Let $t_0 \in B^l_{\epsilon/2}$ be a common regular value of the two transversally compatible perturbations. Then Proposition 6.2.3 implies that restriction of the inclusion map to the perturbed solution sets

$$\iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0}) \to \mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$$

is a homeomorphism between compact weighted branched suborbifolds, and moreover satisfies the hypothesis of the change of variables theorem 7.2.2. Therefore for a given sc-smooth differential form $\omega \in \Omega^*_\infty(\mathcal{Z}_2)$ we have

$$\int_{(\mathcal{S}(\overline{\partial}_2,\Lambda_2^{t_0}),\Lambda_2^{t_0}\circ\overline{\partial}_2)} \omega = \int_{(\mathcal{S}(\overline{\partial}_1,\Lambda_1^{t_0}),\Lambda_1^{t_0}\circ\overline{\partial}_1)} \iota_{\mathcal{Z}}^*\omega. \tag{6.5}$$

Finally, Proposition 6.2.7 together with Stokes' theorem 7.2.3 implies that there exists a regular perturbation $\Gamma_0 : \mathcal{W}_1 \to \mathbb{Q}^+$ such that

$$\int_{(\mathcal{S}(\overline{\partial}_1,\Lambda_1^{t_0}),\Lambda_1^{t_0}\circ\overline{\partial}_1)} \iota_{\mathcal{Z}}^*\omega = \int_{(\mathcal{S}(\overline{\partial}_1,\Gamma_0),\Gamma_0\circ\overline{\partial}_1)} \iota_{\mathcal{Z}}^*\omega. \tag{6.6}$$

Hence considering the GW-invariants as branched integrals via Definition 7.2.6, we conclude

$$\begin{aligned}
\mathrm{GW}^Q_{A,g,k}[\mathcal{Z}_2](\alpha_1,\ldots,\alpha_k;\beta) &= \int_{(\mathcal{S}(\overline{\partial}_2,\Lambda_2^{t_0}),\Lambda_2^{t_0}\circ\overline{\partial}_2)} \bigwedge_{i=1}^k ev_i'^*\mathrm{PD}(\alpha_i)\wedge\pi'^*\mathrm{PD}(\beta) \\
&= \int_{(\mathcal{S}(\overline{\partial}_1,\Lambda_1^{t_0}),\Lambda_1^{t_0}\circ\overline{\partial}_1)} \iota_{\mathcal{Z}}^*\left(\bigwedge_{i=1}^k ev_i'^*\mathrm{PD}(\alpha_i)\wedge\pi'^*\mathrm{PD}(\beta)\right) \\
&= \int_{(\mathcal{S}(\overline{\partial}_1,\Gamma_0),\Gamma_0\circ\overline{\partial}_1)} \iota_{\mathcal{Z}}^*\left(\bigwedge_{i=1}^k ev_i'^*\mathrm{PD}(\alpha_i)\wedge\pi'^*\mathrm{PD}(\beta)\right) \\
&= \int_{(\mathcal{S}(\overline{\partial}_1,\Gamma_0),\Gamma_0\circ\overline{\partial}_1)} \bigwedge_{i=1}^k ev_i^*\mathrm{PD}(\alpha_i)\wedge\pi^*\mathrm{PD}(\beta) \\
&= \mathrm{GW}^Q_{A,g,k}[\mathcal{Z}_1](\alpha_1,\ldots,\alpha_k;\beta)
\end{aligned}$$

where in the second equality we used (6.5), in the third equality we used (6.6), and in the fourth equality we used $\iota_{\mathcal{Z}}^* ev_i'^* = ev_i^*$ and $\iota_{\mathcal{Z}}^*\pi'^* = \pi^*$ by commutativity of (6.4). □

6.3 Existence of Perturbations which Restrict

In this section we prove Proposition 6.2.2.

Local Surjectivity of the Linearized Cauchy–Riemann Operator

We begin with some basic facts about the standard, linear Cauchy–Riemann operator.

Proposition 6.3.1. *[22, Proposition 4.15] Let $H^{3,\delta_0}_c(\mathbb{R}\times S^1,\mathbb{R}^{2n})$ be the usual sc-Hilbert space with antipodal asymptotic constants, where the level m is given by regularity $(m+3,\delta_m)$. Let $J(0)$ be a constant almost complex structure on $\mathbb{R}^{2n}$. The Cauchy–Riemann operator*

$$\partial_s + J(0)\partial_t : H^{3,\delta_0}_c(\mathbb{R}\times S^1,\mathbb{R}^{2n}) \to H^{2,\delta_0}(\mathbb{R}\times S^1,\mathbb{R}^{2n})$$

is an sc-isomorphism.

This expression might look familiar, it gives the filled version of the Cauchy–Riemann operator, (see [22, p129-130]). We have a similar result for disks.

Proposition 6.3.2. *Let $H^3_{loc}(\mathbb{D},\mathbb{R}^{2n})$ be an sc-Hilbert space. Let $J(0)$ be a constant almost complex structure on $\mathbb{R}^{2n}$. The Cauchy–Riemann operator*

$$\partial_s + J(0)\partial_t : H^3_{loc}(\mathbb{D},\mathbb{R}^{2n}) \to H^2_{loc}(\mathbb{D},\mathbb{R}^{2n})$$

is surjective.

Proof. This follows from [32, Exercise B.3.3], where solutions can be constructed using the existence of solutions to the Laplacian. We recall the argument therein for the convenience of the reader. Identify $(\mathbb{R}^{2n},J(0))$ with $(\mathbb{C}^n,i)$. In order to solve

$$(\partial_s + i\partial_t)w = h$$

write $w=u+iv$ and $h=f+ig$ and solve the second order equations

$$\Delta u_j = \partial_s f_j + \partial_t g_j, \qquad \Delta v_j = \partial_s g_j - \partial_t f_j.$$

□

Consider now the Cauchy–Riemann section, defined on the underlying sets of the polyfold $\mathcal{Z}^{3,\delta_0}_{A,g,k}$ and strong polyfold bundle $\mathcal{W}^{2,\delta_0}_{A,g,k}$ by the equation

$$[\Sigma,j,M,D,u] \mapsto [\Sigma,j,M,D,u,\tfrac{1}{2}(du + J(u)\circ du\circ j))].$$

In a local sc-trivialization $K\to O$ it has the following local expression

$$(a,v,\eta)\mapsto (a,v,\eta,\overline{\xi})$$

where $\overline{\xi}$ is the unique solution of the equations[1]

$$\Gamma(\oplus_a \exp_u \eta, \oplus_a u) \cdot \hat{\oplus}_a \overline{\xi} \circ \delta(a,v) = \overline{\partial}_{J,j(a,v)}(\oplus_a \exp_u \eta),$$
$$\hat{\ominus}_a \overline{\xi} \cdot \tfrac{\partial}{\partial_s} = 0. \tag{6.7}$$

We can simplify this expression considerably by fixing the coordinates $a = 0$, $v = 0$. Moreover, we may identify a neighborhood of a point $q \in Q$ with a neighborhood of $0 \in \mathbb{R}^{2n}$ under which the Euclidean metric pulls back to the Riemannian metric on Q. The expression (6.7) now becomes

$$\overline{\xi} = \Gamma(u+\eta, u)^{-1} \cdot \left(\partial_s(u+\eta) + J(u+\eta)\partial_t(u+\eta)\right).$$

Consider the linearization at a solution $\overline{\partial}_{J,j} u = 0$ with respect to the coordinate η. It now has the following local form

$$\eta \mapsto \tfrac{1}{2}\left(\partial_s \eta + J(u)\partial_t \eta + \partial_\eta J(u) \partial_t u\right).$$

Lemma 6.3.3 (Local Surjectivity of the Linearized Cauchy–Riemann Operator)**.** *Let (Σ, j, M, D, u) be a stable map which is a solution to the Cauchy–Riemann operator $\hat{\overline{\partial}}$. Let*

$$D_u \hat{\overline{\partial}} : H^{3,\delta_0}(\Sigma, u^*TQ) \to H^{2,\delta_0}(\Sigma, \Lambda^{0,1} \otimes_J u^*TQ)$$

be the linearization of $\hat{\overline{\partial}}$ at (Σ, j, M, D, u), considered as an sc-Fredholm operator between sc-Banach spaces, where in addition we are leaving the complex structure on (Σ, j) and the gluing parameters fixed. Then there exist open subsets of the Riemann surface Σ

- *a disk-like neighborhood D_{z_i} at every marked point $z_i \in M$ (regardless of whether we require u is of class H^{3,δ_0} or of class H^3_{loc} at z_i)*
- *disk-like neighborhoods $D_{x_a} \sqcup D_{y_a}$ at every nodal pair $\{x_a, y_a\} \in D$*
- *(if it exists) a component $S^2 \subset \Sigma$ with two punctures, on which u is constant*

such that the restriction of $D_u \hat{\overline{\partial}}$ to each of these regions is a surjective operator.

[1] Furthermore, here $\delta(a,v) : (T\Sigma_a, j(a,v)) \to (T\Sigma_a, j(a,0))$ is the complex linear map given by $\delta(a,v)h = \frac{1}{2}(id - j(a,0) \circ j(a,v))h$. Here Γ is defined via parallel transport of a complex connection, as follows. Fix a complex connection on the almost complex vector bundle $(TQ, J) \to Q$, i.e. if ∇_X is the covariant derivative on Q belonging to the Riemaniann metric $\omega \circ (\mathrm{id} \oplus J)$, the connection $\tilde{\nabla}_X$, defined by $\tilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2} J (\nabla_X J) Y$, defines a complex connection, in the sense that it satisfies $\tilde{\nabla}_X (JY) = J(\tilde{\nabla}_X Y)$. If $\eta \in T_p Q$ is a tangent vector, the parallel transport of a complex connection along the path $t \mapsto \exp_p(t\eta)$ for $t \in [0,1]$, defines the linear map

$$\Gamma(\exp_p(\eta), p) : (T_p Q, J(p)) \to (T_{\exp_p(\eta)} Q, J(\exp_p(\eta)))$$

which is complex linear, hence $\Gamma(\exp_p(\eta), p) \circ J(p) = J(\exp_p(\eta)) \circ \Gamma(\exp_p(\eta), p)$. A fuller explanation of these details can be found in [22, p118, p126].

Proof. We prove the existence of the first neighborhood, assuming u is of class H^3_{loc}. Consider the operator

$$D_u\hat{\bar{\partial}} : H^3_{\text{loc}}(\mathbb{D}, \mathbb{R}^{2n}) \to H^2_{\text{loc}}(\mathbb{D}, \mathbb{R}^{2n})$$

which is defined by the local expression we have just discussed, i.e.

$$D_u\hat{\bar{\partial}}\eta = \tfrac{1}{2}\left(\partial_s\eta + J(u)\partial_t\eta + \partial_\eta J(u)\partial_t u\right).$$

Moreover, assume we have identified a neighborhood of Q with a neighborhood of $\mathbb{R}^{2n}$ such that $u(0) = 0$.

For every $\epsilon > 0$ there exists a $\delta > 0$ such that we have the following estimates for the ball $B_\delta(0) \subset \mathbb{D}$:

$$\|(J(u) - J(0)\partial_t\eta\|_{H^2_{\text{loc}}(B_\delta, \mathbb{R}^{2n})} \leq \tfrac{\epsilon}{2} \cdot \|\eta\|_{H^3_{\text{loc}}(B_\delta, \mathbb{R}^{2n})}$$

and

$$\|\partial_\eta J(u)\partial_t u\|_{H^2_{\text{loc}}(B_\delta, \mathbb{R}^{2n})} \leq \tfrac{\epsilon}{2} \cdot \|\eta\|_{H^3_{\text{loc}}(B_\delta, \mathbb{R}^{2n})}.$$

We consider the restriction of $D_u\hat{\bar{\partial}}$ to $H^3_{\text{loc}}(B_\delta, \mathbb{R}^{2n})$; we may write

$$D_u\hat{\bar{\partial}}\eta = \tfrac{1}{2}\left(\partial_s\eta + J(0)\partial_t\eta\right) + \tfrac{1}{2}\left((J(u) - J(0))\partial_t\eta\right) + \tfrac{1}{2}\left(\partial_\eta J(u)\partial_t u\right).$$

From Proposition 6.3.2 the first term on the right is surjective, while we can bound the other two terms on the right in the operator norm by ϵ. From classical functional analysis the space of surjective operators is open. Hence there exists some δ such that

$$D_u\hat{\bar{\partial}} : H^3_{\text{loc}}(B_\delta, \mathbb{R}^{2n}) \to H^2_{\text{loc}}(B_\delta, \mathbb{R}^{2n})$$

is surjective.

We prove the existence of first neighborhood, assuming u is of class H^{3,δ_0} By symmetry, this will also show the existence of the second neighborhood. Consider the operator

$$D_u\hat{\bar{\partial}} : H^{3,\delta_0}_c(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n}) \to H^{2,\delta_0}(\mathbb{R}^+ \times S^1, \mathbb{R}^{2n})$$

which is defined by the same expression as before. Moreover, assume we have identified a neighborhood of Q with a neighborhood of $\mathbb{R}^{2n}$ such that $\lim_{s\to\infty} u(s) = 0$. We proceed the same as before. By [22, Lemma 4.19] there exists $R \geq 0$ such that we have the following estimate for the region $[R, \infty) \times S^1 \subset \mathbb{R}^+ \times S^1$,

$$\|(J(u) - J(0)\partial_t\eta\|_{H^{2,\delta_0}([R,\infty)\times S^1, \mathbb{R}^{2n})} \leq \tfrac{\epsilon}{2} \cdot \|\eta\|_{H^{3,\delta_0}_c([R,\infty)\times S^1, \mathbb{R}^{2n})}.$$

The same argument shows that there exists $R \geq 0$ such that

$$\|\partial_\eta J(u)\partial_t u\|_{H^{2,\delta_0}([R,\infty)\times S^1, \mathbb{R}^{2n})} \leq \tfrac{\epsilon}{2} \cdot \|\eta\|_{H^{3,\delta_0}_c([R,\infty)\times S^1, \mathbb{R}^{2n})}.$$

One should be careful to note the presence of the exponential weights in the above norms. Using Proposition 6.3.1, we may use the same argument to conclude that there exists some $R \geq 0$ such that

$$D_u \hat{\overline{\partial}} : H^{3,\delta_0}_c([R,\infty) \times S^1, \mathbb{R}^{2n}) \to H^{2,\delta_0}([R,\infty) \times S^1, \mathbb{R}^{2n})$$

is surjective.

We prove the existence of the third neighborhood. Noting that u is constant on the component S^2, and assuming in our chart $u(S^2) = 0$ the local expression for the linearized Cauchy–Riemann operator

$$D_u \hat{\overline{\partial}} : H^{3,\delta_0}_{a,b}(\mathbb{R} \times S^1, \mathbb{R}^{2n}) \to H^{2,\delta_0}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$$

is given by

$$\eta \mapsto \partial_s + J(0)\partial_t \eta$$

where $H^{3,\delta_0}_{a,b}(\mathbb{R} \times S^1, \mathbb{R}^{2n})$ is the sc-Hilbert space of maps with asymptotic constant a as $s \to -\infty$ and asymptotic constant b as $s \to +\infty$. By Proposition 6.3.1, we can observe that this is a surjective Fredholm operator, with kernel the constant maps. □

As a consequence of the above lemma, we obtain the following.

Corollary 6.3.4. *Shrink the above small disk-like neighborhoods slightly. If necessary, shrink further in order to assume the regions are all disjoint. Then there exist vectors* $w_1, \ldots, w_k \in H^{2,\delta_0}(\Sigma, \Lambda^{0,1} \otimes_J u^*TQ)$ *such that*

- $w_1, \ldots, w_k$ *together with Im*$(D_u \hat{\overline{\partial}})$ *span* $H^{2,\delta_0}(\Sigma, \Lambda^{0,1} \otimes_J u^*TQ)$
- $w_1, \ldots, w_k$ *vanish on the above regions of* Σ*.*

Remark 6.3.5. By our simplifying assumptions, this is not the full linearization of the sc-smooth proper Fredholm operator $\overline{\partial} : \mathcal{Z} \to \mathcal{W}$, rather the linearization restricted to the subset $a = 0, v = 0$. However, the image of the full linearization contains $\operatorname{Im}(D_u \hat{\overline{\partial}})$, so all this implies is that the number of vectors in the above corollary will be greater or equal to the dimension of the cokernel of the full linearization.

Construction of Perturbations which Restrict

We now construct the parametrized sc$^+$-multisection Λ^t for the strong polyfold bundle $\mathcal{W}^{2,\delta_0}_{A,g,k} \to \mathcal{Z}^{3,\delta_0}_{A,g,k}$. The construction follows the same exact procedure as in [18, Theorem 15.4], where use Corollary 6.3.4 to ensure the vectors we choose vanish on disk-like regions of the nodal and marked points, as well as on any destabilizing ghost components.

In the following exposition, we make this construction explicit, and thus make it easy to check that such a parametrized sc$^+$-multisection (and its appropriate restriction) can be used to satisfy the requirements of Definition 6.2.1.

Local Construction: Let $[\alpha] = [\Sigma, j, M, D, u] \in \mathcal{Z}^{3,\delta_0}_{A,g,k}$ be a stable curve, and let $\alpha = (\Sigma, j, M, D, u)$ be a stable map representative. Let O_α be an M-polyfold chart centered at α, and invariant under the induced action by $\boldsymbol{G}(\alpha)$. Thus we have good uniformizing family

$$(\underline{a}, v, \eta) \mapsto (\Sigma_{\underline{a}}, j(\underline{a}, v), M_{\underline{a}}, D_{\underline{a}}, \oplus_{\underline{a}} \exp_u(\eta)), \qquad (\underline{a}, v, \eta) \in O_\alpha,$$

and let $K \to O_\alpha$ be a local strong bundle model, with sc-coordinates given by $(\underline{a}, v, \eta, \xi)$ where necessarily $\xi \in H^{2,\delta_0}(\Sigma, \Lambda^{0,1} \otimes_J u^*TQ)$. Use Corollary 6.3.4 to choose vectors $w_1, \ldots, w_k$ which vanish on disk-like regions of the nodal and marked points, and any destabilizing ghost components, and such that $w_1, \ldots, w_k$ together with the linearization of $\hat{\bar{\partial}}$ at α together span $H^{2,\delta_0}(\Sigma, \Lambda^{0,1} \otimes_J u^*TQ)$. Let $\beta : O_\alpha \to [0, 1]$ be an sc-smooth cutoff function with $\beta \equiv 1$ near $(0, 0, 0)$.

Define a parametrized sc$^+$-section $O_\alpha \to K$, with parameters $t \in \mathbb{R}^k$ by

$$s^t : O_\alpha \to K, \qquad (\underline{a}, v, \eta) \mapsto (\underline{a}, v, \eta, \beta(\underline{a}, v, \eta) \cdot (\Sigma^k_{i=1} t_i \rho_{\underline{a}}(w_i))),$$

where $\rho_{\underline{a}}$ is the strong bundle projection defined using the hat gluings, see [18, p117] and [22, p65-67]. Using the natural action of isotropy group $\boldsymbol{G}(\alpha)$, we can define a collection of sc$^+$-sections $\{s^t_g\}_{g\in\boldsymbol{G}(\alpha)}$, each with weight $\frac{1}{\sharp\boldsymbol{G}(\alpha)}$, which together give a local section structure for a well-defined sc$^+$-multisection $\lambda^t : \mathcal{W}^{2,\delta_0}_{A,g,k} \to \mathbb{Q}^+$.

At this stage, it is appropriate to verify that the restriction condition of Definition 6.2.1 is satisfied for each of the three inclusion maps we are considering. In each case, the choice of vectors $w_1, \ldots, w_k$ confirm that the parametrized sc$^+$-sections $\{s^t_g\}_{g\in\boldsymbol{G}(\alpha)}$ have a well-defined restriction to $\text{Im}(W_1) \to \text{Im}(Z_1)$. It remains to check the sc$^+$-smoothness of these restrictions, considered with regards to the smooth structures of $\mathcal{Z}_1$ and $\mathcal{W}_1$. First, note that the inclusion maps are sc-smooth, and hence the pullbacks of the cutoff functions will be sc-smooth. Second, the restriction of the terms $\rho_{\underline{a}}(w_i)$ are sc-smooth in the cases of $\mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}] \hookrightarrow \mathcal{Z}'_{A,g,k}$ and $\mathcal{Z}^{3,\delta'_0}_{A,g,k} \hookrightarrow \mathcal{Z}^{3,\delta_0}_{A,g,k}$, as the hat gluing constructions are identical.

In the case of $\mathcal{Z}^{ft}_{A,g,k} \hookrightarrow \mathcal{Z}^{\text{HWZ}}_{A,g,k}$ observe that sc-smoothness at points without a degenerating ghost component is automatic, since at such points the inclusion map is an sc-diffeomorphism. At points with a degenerating ghost component, the gluings differ, as do the lifted good uniformizing families which define the sc-trivializations of the strong polyfold bundle. However, this is irrelevant as by construction $\rho_{\underline{a}}(w_i) \equiv 0$ over the glued cylinders Z_a, and for zero vectors on such regions the gluings and lifted good uniformizing families coincide.

Local to Global: Taking a finite sum of sc$^+$-multisections as constructed above gives the desired perturbation, $\Lambda^t := \oplus\lambda^t$. In particular, the restriction condition holds since it holds for each term λ^t. We can satisfy the transversality condition using the fact that $\text{Im}(W_{1,x}) \subset W_{2,x}$, and the reasoning of Theorem 3.1.1. By requiring that the locally constructed λ^t satisfy $N_2[\lambda^t] \leq 1$ and dom-supp$(\lambda^t) \subset \mathcal{U}_2$ we can ensure the sum Λ^t is controlled by a pair $(N_2, \mathcal{U}_2)$ for parameters $t \in B_\epsilon \subset \mathbb{R}^l$, and hence satisfies the compactness condition.

Verifying the anti push off condition (6.3)

The final step is to demonstrate

$$\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}) \subset \operatorname{Im}(\mathcal{Z}_1)$$

for all $t \in B_\epsilon^l$, and for $\mathcal{Z}_1 = \mathcal{Z}_{A,g,k}^{ft}, \mathcal{Z}_{A,g,k}[H^3 \cap H^{3,\delta_0}], \mathcal{Z}_{A,g,k}^{3,\delta_0'}$, and hence show that the anti-push off condition is satisfied. A priori, observe that $\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}) \subset (\mathcal{Z}_2)_\infty$. This is insufficient to prove the anti push off condition; the ∞-level of $\mathcal{Z}_{A,g,k}$ contains nonconstant ghost components, as well as smooth maps with exponential decay only up to a constant $c < \lim_{i\to\infty} \delta_i \leq 2\pi$.

However, we have constructed the perturbation Λ^t from terms λ^t whose local section structures vanish on disk-like regions of the nodal and marked points, as well as on any destabilizing ghost components. It follows that $\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}) \subset \operatorname{Im}(\mathcal{Z}_{A,g,k}^{ft})$. Indeed, let $[\alpha] \in \mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$. Consider a stable map representative $\alpha = (\Sigma, j, M, D, u)$ and let (s_i) be a local section structure for $\Lambda_2^{t_0}$ at α. By construction, for any destabilizing ghost component $S^2 \subset \Sigma$ we have $s_i|_{S^2} \equiv 0$ and hence $\overline{\partial}_J u|_{S^2} \equiv 0$. Therefore $[\alpha]$ may only contain constant degenerating ghost components, and so $[\alpha] \in \operatorname{Im}(\mathcal{Z}_{A,g,k}^{ft})$ as desired.

For the other two situations, we need a small lemma.

Lemma 6.3.6. *Let Λ^t be the parametrized sc$^+$-multisection constructed as above, i.e. as a finite sum of parametrized sc$^+$-multisections which vanish on disk-like regions of the nodal and marked points, as well as on any destabilizing ghost components. Suppose that $[\Sigma, j, M, D, u] \in \mathcal{S}(\overline{\partial}, \Lambda^t)$, and let (Σ, j, M, D, u) be a stable map representative.*

Then for every nodal pair $\{x_a, y_a\} \in D$ there exists disk-like neighborhoods $D_{x_a} \sqcup D_{y_a}$, and for every marked point $z_i \in M$ there exists a disk-like neighborhood D_{z_i} such that

$$\overline{\partial} u|_{D_{x_a} \sqcup D_{y_a}} \equiv 0, \qquad \overline{\partial} u|_{D_{z_i}} \equiv 0.$$

Proof. Let $\alpha = (\Sigma, j, M, D, u)$ be the stable map representative of $[\Sigma, j, M, D, u]$.

Consider one of the sc$^+$-multisections λ^t constructed above. Hence let $\{s_g^t\}_{g\in \boldsymbol{G}(\alpha')}$ be a local section structure for λ^t, defined on an M-polyfold chart $O_{\alpha'}$ with sc-coordinates (a', v', η'), centered at a stable map $\alpha' = (\Sigma', j', M', D', u')$.

Consider a point $(a', v', \eta') \in O_{\alpha'}$, which maps to the stable map $(\Sigma'_{a'}, j'(a', v'), M'_{a'}, D'_{a'}, \oplus_{a'} \exp_{u'} \eta')$. We consider the value of the local section structure s_g^t at the point (a', v', η') as defining a vector in $H^{2,\delta_0}(\Sigma'_{a'}, \Lambda^{0,1} \otimes_J u'^* TQ)$. By construction, $s_g^t(a', v', \eta') \equiv 0$ on the regions

- $D'_{x_{a'}} \sqcup D'_{y_{a'}} \subset \Sigma'_{a'}$ for $a' = 0$,
- $D'_{z_i'} \subset \Sigma'_{a'}$ for $z_i' \in M'_{a'}$.

Suppose there exists a morphism $\phi : \alpha'_{(a_0', v_0', \eta_0')} \to \alpha$; i.e. a biholomorphism

$$\phi : (\Sigma'_{a_0'}, j'(a_0', v_0')) \to (\Sigma, j)$$

which moreover defines bijections between the marked points $\phi : M'_{a'_0} \to M$, and between the nodal pairs $\phi : D'_{a'_0} \to D$. Consider a nodal pair $\{x_a, y_a\} \in D$ and let $D_{x_a} \sqcup D_{y_a}$ be a small disk neighborhood at this pair. We can shrink these disks so small that $D_{x_a} \sqcup D_{y_a} \subset \phi(D'_{\phi^{-1}(x_a)}) \sqcup \phi(D'_{\phi^{-1}(y_a)})$. Likewise, consider a marked point $z_i \in M$ and let D_{z_i} be a small disk neighborhood at z_i; again we can shrink this disk so that $D_{z_i} \subset \phi(D'_{z'_i})$.

This shows that the local section structure for λ^t at α, considered as vectors in $H^{2,\delta_0}(\Sigma, \Lambda^{0,1} \otimes_J u^*TQ)$ vanish on disk-like neighborhoods of the nodal pairs and marked points of α. We can repeat this process for each of the finitely many λ^t in the sum $\Lambda^t = \oplus \lambda^t$, and hence conclude the local section structure for Λ^t vanishes on the finite intersection of these disk-like neighborhoods. The conclusion of the lemma necessarily follows. □

The consequence of the lemma is the following. Any perturbed solution $[\alpha] \in \mathcal{S}(\overline{\partial}, \Lambda^t)$ is an honest solution to the $\overline{\partial}$ equation restricted to disk-like neighborhoods of the marked and nodal points. It is immediate that on these regions the perturbed solution possesses the same properties as solutions to the $\overline{\partial}$ equation, and hence we can conclude for any stable map representative $\alpha = (\Sigma, j, M, D, u)$ that $u \in C^\infty_{\text{loc}}$ and that at any puncture we have $u \in H^{3,\delta_0}$ for all $\delta_0 < 2\pi$.

This demonstrates

- $\mathcal{S}(\overline{\partial}, \Lambda^t) \subset \text{Im}(\mathcal{Z}_{A,g,k}[H^{3,\delta_0} \cap H^3])$
- $\mathcal{S}(\overline{\partial}, \Lambda^t) \subset \text{Im}(\mathcal{Z}^{3,\delta'_0}_{A,g,k})$

as desired, confirming the anti push off condition.

6.4 Branched Suborbifolds and Invariance of Domain

In this section, we prove Proposition 6.2.3. We begin by recalling some essential facts about weighted branched suborbifolds.

Subgroupoids of Ep-Groupoids

An sc-smooth Fredholm section $\overline{\partial}$ and a sc$^+$-multisection Λ together define a perturbed solution subgroupoid with object set

$$S(\hat{\overline{\partial}}, \hat{\Lambda}) := \{x \in Z \mid \hat{\Lambda}(\hat{\overline{\partial}}(x)) > 0\} \subset Z$$

and with morphisms given from the restriction of $\boldsymbol{Z}$ to this set. We pause to recall some essential facts about subgroupoids.

Definition 6.4.1. Let $(Z, \boldsymbol{Z})$ be an ep-groupoid. Let $\pi : Z \to |Z|$ denote the quotient map to the orbit space. We say that a subset of the object set, $S \subset Z$, is **saturated** if $S = \pi^{-1}(\pi(S))$. We define a **subgroupoid** as the full subcategory $(S, \boldsymbol{S})$ associated to a saturated subset of the object set.

A subgroupoid comes equipped with the subspace topology induced from the ep-groupoid $(Z, \boldsymbol{Z})$, in addition to the induced grading. It does not come with a sc-smooth structure in general, so the étale condition no longer makes sense. However, one may observe it inherits the following directly analagous properties.

- **(sc^0-étale).** The source and target maps are surjective local sc^0-homeomorphisms.
- **(proper).** For every point $x \in S$, there exists an open neighborhood $V(x)$ so that the map $t : s^{-1}(\overline{V(x)}) \to S$ is a proper mapping.[2]

Proposition 6.4.2. *Consider the orbit space of a subgroupoid, $|S|$. There are two topologies on this space we may consider:*

- *the subspace topology τ_s, induced from the inclusion $|S| \subset |Z|$,*
- *the quotient topology τ_q, induced from the map $S \to |S|$.*

These two topologies are equivalent.

Proof. • $\tau_s \subset \tau_q$

Suppose $U \subset |S|$ and $U \in \tau_s$. Then $U = V \cap |S|$ for $V \subset |Z|$ open. By definition, $\pi^{-1}(V) \subset Z$ is open. Moreover, $\pi^{-1}(U) = \pi^{-1}(V) \cap \pi^{-1}(S) = \pi^{-1}(V) \cap S$. Hence $\pi^{-1}(U)$ is open in S. It follows from the definition of the quotient topology that $U \in \tau_q$.

- $\tau_q \subset \tau_s$

Suppose $U \subset |S|$ and $U \in \tau_q$. We will show for every $[x] \in U$ there exists a subset $B \subset |S|$ such that $B \in \tau_s$ and $[x] \in B \subset U$. It will then follow that $U \in \tau_s$, as desired.

Let $x \in \pi^{-1}(U)$ be a representative of $[x]$. There exists an open neighborhood $V(x) \subset Z$ equipped with the natural action by $\boldsymbol{G}(x)$ and such that $V(x) \cap S \subset \pi^{-1}(U)$. Observe that $|V(x) \cap S| = |V(x)| \cap |S|$; this follows since S is saturated.

Let $B := |V(x)| \cap |S| \subset U$. Then observe that $|V(x)| \subset |Z|$ is open, since by [18, Proposition 7.1], the quotient map by $\pi : Z \to |Z|$ is an open map. hence $B := |V(x)| \cap |S| \subset |S|$ is open in the subspace topology. It follows that $B \in \tau_s$ and $[x] \in B \subset U$, as desired. □

The following proposition is a direct analog of Proposition 2.1.7 for subgroupoids.

Proposition 6.4.3. *[Natural Representation of $\boldsymbol{G}(x)$ for a Subgroupoid] Let $(S, \boldsymbol{S})$ be a subgroupoid of an ep-groupoid $(Z, \boldsymbol{Z})$. Let $x \in S$ with isotropy group $\boldsymbol{G}(x)$. Then for every open neighborhood V of x there exists an open neighborhood $U \subset V$ of x, a group homomorphism $\Phi : \boldsymbol{G}(x) \to Homeo_{sc^0}(U)$, $g \mapsto \Phi(g)$, and a sc^0-map $\Gamma : \boldsymbol{G}(x) \times U \to \boldsymbol{S}$ such that the following holds.*

1. *$\Gamma(g, x) = g$.*

[2]This can be shown from the definitions, using in addition that if $f : X \to Y$ is proper, then for any subset $V \subset Y$ the restriction $f|_{f^{-1}(V)} : f^{-1}(V) \to V$ is proper.

2. *$s(\Gamma(g,y)) = y$ and $t(\Gamma(g,y)) = \Phi(g)(y)$ for all $y \in U$ and $g \in \boldsymbol{G}(x)$.*

3. *If $h : y \to z$ is a morphism between points in U, then there exists a unique element $g \in \boldsymbol{G}(x)$ satisfying $\Gamma(g,y) = h$, i.e.,*

$$\Gamma : \boldsymbol{G}(x) \times U \to \{\phi \in \boldsymbol{Z} \mid s(\phi) \text{ and } t(\phi) \in U\}$$

is a bijection.

Moreover, consider the following topological spaces:

- *$\boldsymbol{G}(x)\backslash U$, equipped with quotient topology via the projection $U \to \boldsymbol{G}(x)\backslash U$,*
- *$U/\sim$, where $x \sim x'$ for $x, x' \in U$ if there exists a morphism $\phi : x \to x'$, equipped with the quotient topology via the projection $U \to U/\sim$,*
- *$|U|$, the image of U under the map $S \to |S|$, equipped with the subspace topology.*

Then these spaces are all naturally homeomorphic.

Weighted Branched Suborbifolds

When an sc-smooth Fredholm section $\overline{\partial}$ and a sc$^+$-multisection Λ are transversal (recall Definition 3.4.3) then by Theorem 3.4.4 the perturbed solution set $\mathcal{S}(\overline{\partial}, \Lambda)$ and the perturbed solution subgroupoid $S(\hat{\overline{\partial}}, \hat{\Lambda})$ carry the additional structure of a weighted branched suborbifold and a weighted branched ep-subgroupoid, respectively. This is defined as follows.

Definition 6.4.4. [18, Definition 9.1] Consider a subset $\mathcal{S}$ of a polyfold $\mathcal{Z}$, and let θ be a weight function on $\mathcal{S}$, i.e. a function $\theta : \mathcal{S} \to \mathbb{Q}^+$. Suppose that Z is an associated polyfold structure, S an associated subgroupoid, and $\hat{\theta} : S \to \mathbb{Q}^+$ an associated weight functor. We say that $(\mathcal{S}, \theta)$ is a **weighted branched suborbifold of** $\mathcal{Z}$ and that $(S, \hat{\theta})$ is a **weighted branched ep-subgroupoid of** $(Z, \boldsymbol{Z})$ if the following properties are satisfied.

1. $\mathcal{S} \subset \mathcal{Z}_\infty$, equivalently $S \subset Z_\infty$.

2. Every point $x \in S$ is contained in an open neighborhood $U \subset Z$ such that
$$S \cap U = \bigcup_{i \in I} M_i,$$
where I is a finite index set and where the sets M_i are finite dimensional submanifolds of Z (in the sense of [24, Definition 4.19]) all having the same dimension. The submanifolds M_i are called **local branches** in U.

3. There exist positive rational numbers σ_i, $i \in I$, (called **weights**) such that if $y \in S \cap U$, then
$$\hat{\theta}(y) = \sum_{\{i \in I \mid y \in M_i\}} \sigma_i.$$

4. The inclusion maps $\phi_i : M_i \to U$ are proper.

We call $(M_i)_{i\in I}$ and $(\sigma_i)_{i\in I}$ a **local branching structure**.

By shrinking the open set U we may assume that the local branches M_i (equipped with the subspace topology induced from U) are homeomorphic to open subsets of $\mathbb{R}^n$. Hence we may assume that a local branch is given by a subset $M_i \subset \mathbb{R}^n$ and an inclusion map $\phi_i : M_i \to U$ where ϕ_i is a homeomorphism onto its image, and ϕ_i is proper.

Remark 6.4.5 (Relationship between Local Section Structures and Local Branching Structures). Consider a branched ep-subgroupoid defined by an sc-smooth Fredholm section $\hat{\bar{\partial}}$ and an sc$^+$-multisection $\hat{\Lambda}$. There is a direct relationship between the local section structure (see Definition 3.4.2) for $\hat{\Lambda}$ and the local branching structure at a point $x \in S$ as follows. Consider a local sc-trivialization at x consisting of an M-polyfold chart O_x, a strong local bundle model $K \to O_x$, and a local section structure for $\hat{\Lambda}$ consisting of sc$^+$-sections $s_i : O_x \to K$ and weights σ_i for $i \in I$. The implicit function theorem for M-polyfolds [18, Theorem 3.13] implies that the sets

$$M_i = (\hat{\bar{\partial}} - s_i)^{-1}(0)$$

define finite dimensional submanifolds which together with the weights σ_i give a local branching structure in O_x.

Remark 6.4.6. Let Y be a topological space. Let X be a subset of Y, and equip X with the subspace topology. Consider an open subset $A \subset X$, hence $A = X \cap U$ for some open subset U of Y.

We may consider two topologies on the set A:

- (A, τ_X), where τ_X is the subspace topology induced from the inclusion $A \hookrightarrow X$,
- (A, τ_Y), where τ_Y is the subspace topology induced from the inclusion $A \hookrightarrow Y$.

Then these two topologies are equivalent. Moreover, $A \hookrightarrow X$ is a homeomorphism onto its image.

The point of this digression is the following. By definition, $\cup_{i\in I} M_i$ is an open subset of S. We can equip $\cup_{i\in I} M_i$ with the subspace topology induced from the inclusion $\cup_{i\in I} M_i \hookrightarrow S$ or the subspace topology induced from the inclusion $\cup_{i\in I} M_i \hookrightarrow Z$; these two topologies are equivalent. And moreover, $\cup_{i\in I} M_i \hookrightarrow S$ is a local homeomorphism.

Furthermore, the quotient map $S \mapsto |S| \simeq \mathcal{S}$ is an open map, and therefore $|\cup_{i\in I} M_i|$ is an open subset of $\mathcal{S}$. In addition to the topologies described in Proposition 6.4.3, we can also equip $|\cup_{i\in I} M_i|$ with the subspace topology induced from the inclusion $|\cup_{i\in I} M_i| \hookrightarrow \mathcal{Z}$; these topologies are all equivalent. And moreover, $|\cup_{i\in I} M_i| \hookrightarrow \mathcal{S}$ is a local homeomorphism.

Lemma 6.4.7 (Topology of Local Branching Structures)**.** *Consider the local branching structure $\bigcup_{i\in I} M_i \subset U \subset Z$ at a point in S. There are two topologies we may consider on this set:*

- $(\cup_{i\in I} M_i, \tau_s)$, *where* τ_s *is the subspace topology induced from* U
- $(\cup_{i\in I} M_i, \tau_q)$, *where* τ_q *is the quotient topology induced by the map* $q : \bigsqcup_{i\in I} M_i \to \bigcup_{i\in I} M_i$.

These two topologies are equivalent.

Proof.
- $\tau_s \subset \tau_q$

Consider the following commutative diagram where q is the quotient map, ϕ_i are the continuous inclusion maps $\phi_i : M_i \to U$, and i is the induced inclusion map.

$$\begin{array}{ccc} \bigsqcup_{i\in I} M_i & \xrightarrow{\sqcup_{i\in I}\phi_i} & U \\ {\scriptstyle q}\downarrow & \overset{i}{\nearrow} & \uparrow{\scriptstyle i} \\ (\bigcup_{i\in I} M_i, \tau_q) & \xrightarrow{\text{id}} & (\bigcup_{i\in I} M_i, \tau_s) \end{array}$$

Then by the characteristic property of the quotient topology, $\sqcup_{i\in I}\phi_i$ continuous implies $i : (\cup_{i\in I} M_i, \tau_q) \to U$ continuous, and by the characteristic property of the subspace topology, $i : (\cup_{i\in I} M_i, \tau_q) \to U$ continuous implies $i : (\cup_{i\in I} M_i, \tau_s) \to U$ is continuous.

- $\tau_q \subset \tau_s$

We begin by making a few observations. First, U is homeomorphic to an sc-retract; consequently, it is a metric space, as well as a regular space. Second, the inclusion maps $\phi_i : M_i \to U$ are homeomorphisms onto their images, and by assumption are proper. This moreover implies the images $\phi_i(M_i) \subset U$ are closed in the subspace topology; to see this note that in metric spaces, sequential compactness is equivalent to compactness, and then use properness.

Suppose $V \subset \bigcup_{i\in I} M_i$ and $V \in \tau_q$. We will show for every $x \in V$ there exists a subset $B \subset \bigcup_{i\in I} M_i$ such that $B \in \tau_s$ and $x \in B \subset V$. It will then follow that $V \in \tau_{subspace}$, as desired.

The set $q^{-1}(V) \subset \bigsqcup_{i\in I} M_i$ is open and hence $q^{-1}(V) \cap M_i$ is open in the topology on M_i. Considering x as a point in U via the set inclusion $\bigcup_{i\in I} M_i \subset U$, use the fact that $\phi_i : M_i \to U$ is an injection to observe that $q^{-1}(x) = \{x_{i_1}, \ldots, x_{i_k}\}$ where $x_{i_l} \in M_{i_l}$ for a nonempty subset $\{i_1, \ldots, i_k\} \subset I$.

Now, consider an ϵ-ball around x, that is $x \in B_\epsilon(x) \subset U$. Take ϵ small enough that $\phi_{i_l}^{-1}(B_\epsilon(x)) \subset q^{-1}(V) \cap M_{i_l}$ for all $i_l \in \{i_1, \ldots, i_k\}$, using the fact that ϕ_{i_l} is a homeomorphism onto its image, and hence $\phi_{i_l}^{-1}(B_\epsilon(x))$ give a neighborhood basis for M_{i_l} at the point x_{i_l}. Furthermore, take ϵ small enough that $\phi_j^{-1}(B_\epsilon(x)) = \emptyset$ for all $j \in I \setminus \{i_1, \ldots, i_k\}$, using the fact that x and $\phi_j(M_j)$ are closed subsets of U, and U is regular.

Thus, $B := B_\epsilon(x) \cap \bigcup_{i\in I} M_i$ is an open set in the subspace topology on $\bigcup_{i\in I} M_i$, and $x \in B \subset V$. To see $B \subset V$, let $y \in B$, then there exists $\hat{y} \in \bigsqcup_{i\in I} M_i$ such that $q(\hat{y}) = y$ and $\hat{y} \in \phi_{i_l}^{-1}(B_\epsilon(x)) \subset q^{-1}(V) \cap M_{i_l}$, hence $y \in V$.

□

Invariance of Domain

The following is a classical result due to Brouwer, published in 1911.

Theorem 6.4.8 (Invariance of Domain)**.** *[3] Let U be an open subset of $\mathbb{R}^n$, and let $f : U \to \mathbb{R}^n$ be an injective continuous map. Then f is a homeomorphism between U and $f(U)$.*

This result can immediately be generalized to manifolds; let M and N be an n-dimensional manifolds and let $f : M \to N$ be an injective continuous map. Then f is a homeomorphism onto its image. Moreover, if f is bijective, it is a homeomorphism. We further generalize this result to the branched suborbifolds of our current situation.

Lemma 6.4.9. *Let $\mathcal{Z}_1$ and $\mathcal{Z}_2$ be two polyfolds, and let $(Z_1, \boldsymbol{Z}_1)$ and $(Z_2, \boldsymbol{Z}_2)$ be associated polyfold structures. Let $\mathcal{S}_1 \subset \mathcal{Z}_1$ and $\mathcal{S}_2 \subset \mathcal{Z}_2$ be two n-dimensional branched suborbifolds, and let $(S_1, \boldsymbol{S}_1)$ and $(S_2, \boldsymbol{S}_2)$ be associated branched ep-subgroupoid structures. Consider a map*

$$f : \mathcal{S}_1 \to \mathcal{S}_2$$

which is continuous with respect to the underlying topological spaces, and which has an associated functor

$$\hat{f} : (S_1, \boldsymbol{S}_1) \to (S_2, \boldsymbol{S}_2).$$

which is continuous on the object and morphism sets.

Suppose the following conditions are satisfied.

- *The functor $\hat{f}$ is an injection considered on the object and morphism sets. Moreover, it is fully faithful.*
- *Consider a point $x \in S_1$ which maps to $\hat{f}(x) \in S_2$. Then there exists a local branching structure $(M_i)_{i\in I}$ at x, and there exists a local branching structure $(M'_j)_{j\in J}$ at $\hat{f}(x)$ such that the following holds. The index sets are the same, $I = J$, and $\hat{f}$ restricts to an injective continuous map between each branch, i.e.*

$$\hat{f}|_{M_i} : M_i \to M'_i.$$

- *The map f is a bijection.*

Then the functor $\hat{f}$ is a local homeomorphism. Furthermore, the map f is a homeomorphism.

Proof. We prove the first claim, i.e. the functor $\hat{f}$ is a local homeomorphism.

Let $x \in S_1$ which maps to $\hat{f}(x) \in S_2$. By assumption, there exists a local branching structure $(M_i)_{i\in I}$ in a neighborhood O_x of x, and there exists a local branching structure $(M'_j)_{j\in J}$ in a neighborhood $O_{\hat{f}(x)}$ of $\hat{f}(x)$ such that the index sets are the same, $I = J$, and $\hat{f}$ restricts to a injective continuous map between each branch, i.e.

$$\hat{f}|_{M_i} : M_i \to M'_i.$$

We may invoke invariance of domain 6.4.8 to see that the restricted maps $\hat{f}|_{M_i}$ are homeomorphisms onto their images. Observe that the open balls $B_\epsilon(\hat{f}(x)) \subset O_{\hat{f}(x)}$ give a neighborhood basis for M'_i at $\hat{f}(x)$ for all $i \in I$. It follows that $\hat{f}^{-1}(B_\epsilon(\hat{f}(x))) \subset O_x$ give a neighborhood basis for M_i at x for all $i \in I$. For ϵ small enough, the restricted maps

$$\hat{f}|_{M_i \cap \hat{f}^{-1}(B_\epsilon(\hat{f}(x)))} : M_i \cap \hat{f}^{-1}(B_\epsilon(\hat{f}(x))) \to M'_i \cap B_\epsilon(\hat{f}(x)) \tag{6.8}$$

are homeomorphisms for all $i \in I$.

Define a neighborhood of x by $U_x = \hat{f}^{-1}(B_\epsilon(\hat{f}(x)))$; then $N_i := M_i \cap \hat{f}^{-1}(B_\epsilon(\hat{f}(x)))$ give local branches in U_x. Define a neighborhood of $\hat{f}(x)$ by $U_{\hat{f}(x)} := B_\epsilon(\hat{f}(x))$; then $N'_i := M'_i \cap B_\epsilon(\hat{f}(x))$ give local branches in $U_{\hat{f}(x)}$. We can now rewrite (6.8) more simply as

$$\hat{f}|_{N_i} : N_i \to N'_i.$$

The maps $\hat{f}|_{N_i}$ are homeomorphisms for all $i \in I$. It follows that the map

$$\sqcup_{i\in I}(\hat{f}|_{N_i}) : \bigsqcup_{i\in I} N_i \to \bigsqcup_{i\in I} N'_i$$

is a homeomorphism.

Consider the following commutative diagram of maps.

$$\begin{array}{ccc} \bigsqcup_{i\in I} N_i & \xrightarrow{\sqcup(\hat{f}|_{N_i})} & \bigsqcup_{i\in I} N'_i \\ \downarrow q & & \downarrow q' \\ (\bigcup_{i\in I} N_i, \tau_q) & \xhookrightarrow[\hat{f}|_{\cup N_i}]{} & (\bigcup_{i\in I} N'_i, \tau_{q'}) \end{array}$$

We assert that the map $\hat{f}|_{\cup N_i} : (\cup_{i\in I} N_i, \tau_q) \hookrightarrow (\cup_{i\in I} N'_i, \tau_{q'})$ is a homeomorphism. Indeed, by assumption $\hat{f}|_{\cup N_i}$ is injective. We can use the fact that $\sqcup(\hat{f}|_{N_i})$ is a bijection to see that $\hat{f}|_{\cup N_i}$ must also be surjective. It is easy to check that $\hat{f}|_{\cup N_i}$ is continuous with respect to the quotient topologies τ_q and $\tau_{q'}$. Furthermore, $\hat{f}|_{\cup N_i}$ is an open map. To see this, let $U \subset (\cup_{i\in I} N_i, \tau_q)$ be an open set. Then $q^{-1}(U) \subset \sqcup_{i\in I} N_i$ is open by the definition of the quotient topology. Since $\sqcup(\hat{f}|_{N_i})$ is a homeomorphism, $(\sqcup(\hat{f}|_{N_i}))(q^{-1}(U))$ is open. Commutativity of the diagram and the fact that both $\sqcup(\hat{f}|_{N_i})$ and $\hat{f}|_{\cup N_i}$ are bijections implies that $(\sqcup \hat{f}|_{N_i})(q^{-1}(U)) = q'^{-1}(\hat{f}|_{\cup N_i}(U))$. It therefore follows that $\hat{f}|_{\cup N_I}(U)$ is open by the definition of the quotient topology.

By Lemma 6.4.7, the fact that $\hat{f}|_{\cup N_i} : (\cup_{i\in I} N_i, \tau_q) \hookrightarrow (\cup_{i\in I} N'_i, \tau_{q'})$ is a homeomorphism implies that $\hat{f}|_{\cup N_i} : (\cup_{i\in I} N_i, \tau_s) \hookrightarrow (\cup_{i\in I} N'_i, \tau_s)$ is a homeomorphism. By Remark 6.4.6, the inclusion maps $(\cup_{i\in I} N_i, \tau_s) \hookrightarrow S_1$ and $(\cup_{i\in I} N'_i, \tau_s) \hookrightarrow S_2$ are both local homeomorphisms. We now see that the map $\hat{f} : S_1 \to S_2$ is a local homeomorphism on an open neighborhood of the point $x \in S_1$. Since $x \in S_1$ was arbitrary, and since $\hat{f}$ is injective, we can conclude $\hat{f}$, considered on the object sets, is a local homeomorphism. It then follows from the étale property that $\hat{f}$, considered on the morphism sets, is a local homeomorphism. This proves the claim.

We prove the second claim, i.e. the map f is a homeomorphism.

Let $[x] \in \mathcal{S}_1$ and let $f([x]) \in \mathcal{S}_2$. Let x be a representative of $[x]$, then $\hat{f}(x)$ is a representative of $f([x])$. We may take the same local branching structure $(N_i)_{i\in I}$ at x, and the same local branching structure $(N'_i)_{i\in I}$ at $\hat{f}(x)$ as above. In particular, $\hat{f}|_{\cup N_i} : (\cup_{i\in I} N_i, \tau_s) \hookrightarrow (\cup_{i\in I} N'_i, \tau_s)$ is a homeomorphism.

The proof is now almost completely identical to the argument above. Consider the following commutative diagram of maps.

$$\begin{array}{ccc} \bigcup_{i\in I} N_i & \xrightarrow{\hat{f}|_{\cup N_i}} & \bigcup_{i\in I} N'_i \\ \big\downarrow{\scriptstyle q} & & \big\downarrow{\scriptstyle q'} \\ |\cup_{i\in I} N_i| & \underset{f|_{|\cup N_i|}}{\hookrightarrow} & |\cup_{i\in I} N'_i| \end{array}$$

We assert that the map $f|_{|\cup N_i|}$ is a homeomorphism. Indeed, by assumption $f|_{|\cup N_i|}$ is injective. We can use the fact that $\hat{f}|_{\cup N_i}$ is a bijection to see that $f|_{|\cup N_i|}$ must also be surjective. By assumption, f is continuous and therefore the restriction $f|_{|\cup N_i|}$ is continuous. Furthermore, $f|_{|\cup N_i|}$ is an open map. To see this, let $U \subset |\cup_{i\in I} N_i|$ be an open set. Then $q^{-1}(U) \subset \cup_{i\in I} N_i$ is open by the definition of the quotient topology. Since $\hat{f}|_{\cup N_i}$ is a homeomorphism, $(\hat{f}|_{\cup N_i})(q^{-1}(U)) \subset \cup_{i\in I} N'_i$ is open. Commutativity of the diagram and the fact that both $\hat{f}|_{\cup N_i}$ and $f|_{|\cup N_i|}$ are bijections implies that $(\hat{f}|_{\cup N_i})(q^{-1}(U)) = q'^{-1}(f|_{|\cup N_i|}(U))$. It therefore follows that $f|_{|\cup N_i|}(U)$ is open by the definition of the quotient topology.

Once again, by Remark 6.4.6, the inclusion maps $|\cup_{i\in I} N_i| \hookrightarrow \mathcal{S}_1$ and $|\cup_{i\in I} N'_i| \hookrightarrow \mathcal{S}_2$ are both local homeomorphisms. We now see that the map $f : \mathcal{S}_1 \to \mathcal{S}_2$ is a local homeomorphism on an open neighborhood of the point $[x] \in \mathcal{S}_1$. Since $[x] \in \mathcal{S}_1$ was arbitrary, and since f is bijective, we can conclude that the map f is a homeomorphism. This proves the claim. □

Proof of Proposition 6.2.3

Proof. Consider the restriction of the inclusion map to the perturbed solution sets

$$\iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0}) \to \mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$$

and consider the associated functor

$$\hat{\iota}_{\mathcal{Z}} : S(\hat{\overline{\partial}}_1, \hat{\Lambda}_1^{t_0}) \to S(\hat{\overline{\partial}}_2, \hat{\Lambda}_2^{t_0}).$$

We show that these maps satisfy the hypothesis of Lemma 6.4.9.

- The functor $\hat{\iota}_{\mathcal{Z}} : S(\hat{\overline{\partial}}_1, \hat{\Lambda}_1^{t_0}) \to S(\hat{\overline{\partial}}_2, \hat{\Lambda}_2^{t_0})$ is an injection considered on the object and morphism sets. Moreover, it is fully faithful.

This is automatically satisfied by our assumptions on $\hat{\iota}_{\mathcal{Z}}$ at the start of Section 6.2.

- Consider a point $x \in S(\hat{\overline{\partial}}_1, \hat{\Lambda}_1^{t_0})$ which maps to $y := \hat{\iota}_{\mathcal{Z}}(x) \in S(\hat{\overline{\partial}}_2, \hat{\Lambda}_2^{t_0})$. Then there exists a local branching structure $(M_i)_{i\in I}$ at x, and there exists a local branching structure $(M'_j)_{j\in J}$ at y such that the following holds. The index sets are the same, $I = J$, and $\hat{\iota}_{\mathcal{Z}}$ restricts to an injective continuous map on each branch, i.e.

$$\hat{\iota}_{\mathcal{Z}}|_{M_i} : M_i \to M'_i.$$

This follows from the restriction condition between local section structures (6.2) together with Remark 6.4.5 which describes the relationship between the local section structures and the local branching structures.

Explicitly, by the restriction condition between local section structures, there exists a local section structure $(s_i^{t_0})$ for $\hat{\Lambda}_1^{t_0}$ at x and a local section structure $(s_i'^{t_0})$ for $\hat{\Lambda}_2^{t_0}$ at y, such that the following diagram commutes, for every $i \in I$

$$\begin{array}{ccc} K_1 & \xhookrightarrow[\hat{\iota}_{\mathcal{W}}]{} & K_2 \\ {\scriptstyle \hat{\overline{\partial}}_1 - s_i^{t_0}} \uparrow\downarrow & & \downarrow\uparrow {\scriptstyle \hat{\overline{\partial}}_2 - s_i'^{t_0}} \\ O_x & \xhookrightarrow[\hat{\iota}_{\mathcal{Z}}]{} & O_y \end{array}$$

It therefore follows that $M_i = (\hat{\overline{\partial}}_1 - s_i^{t_0})^{-1}(0)$ for $i \in I$ give a local branching structure at x, and $M'_i = (\hat{\overline{\partial}}_2 - s_i'^{t_0})^{-1}(0)$ for $i \in I$ give a local branching structure at y such that we have a well-defined injective continuous restriction

$$\hat{\iota}_{\mathcal{Z}}|_{M_i} : M_i \to M'_i.$$

This proves the claim.

- The map $\iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0}) \to \mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$ is a bijection.

The map $\iota_{\mathcal{Z}} : \mathcal{Z}_1 \hookrightarrow \mathcal{Z}_2$ is an injection by assumption, and hence so is the restriction $\iota_{\mathcal{Z}}|_{\mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0})}$. To show surjectivity, let $[y] \in \mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0})$. By the anti push off condition (6.3),

$$\mathcal{S}(\overline{\partial}_2, \Lambda_2^{t_0}) \subset \iota_{\mathcal{Z}}(\mathcal{Z}_1),$$

so we can find $[x] \in \mathcal{Z}_1$ such that $\iota_{\mathcal{Z}}([x]) = [y]$. Then by commutativity,

$$\Lambda_1^{t_0} \circ \overline{\partial}_1([x]) = \Lambda_2^{t_0} \circ \overline{\partial}_2(\iota_{\mathcal{Z}}([x])) = \Lambda_2^{t_0} \circ \overline{\partial}_2([y]) > 0$$

hence $[x] \in \mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0})$, and so $\iota_{\mathcal{Z}}|_{\mathcal{S}(\overline{\partial}_1, \Lambda_1^{t_0})}$ is surjective. This proves the claim. □

Proof of Proposition 6.2.4

Proof. The same exact reasoning can be used to show that the restriction of the inclusion map to the thickened solution sets

$$\mathrm{id} \times \iota_{\mathcal{Z}} : \mathcal{S}(\overline{\partial}_1, \Lambda_1^t; t \in B_\epsilon^l) \to \mathcal{S}(\overline{\partial}_2, \Lambda_2^t; t \in B_\epsilon^l), \qquad (t, [z]) \mapsto (t, \iota_{\mathcal{Z}}([z]))$$

and the associated functor

$$\mathrm{id} \times \hat{\iota}_{\mathcal{Z}} : S(\hat{\overline{\partial}}_1, \hat{\Lambda}_1^t; t \in B_\epsilon^l) \to S(\hat{\overline{\partial}}_2, \hat{\Lambda}_2^t; t \in B_\epsilon^l), \qquad (t, z) \mapsto (t, \hat{\iota}_{\mathcal{Z}}(z))$$

satisfy the hypothesis of Lemma 6.4.9.

□

Chapter 7

Proofs of the Gromov–Witten Axioms

In this chapter we prove Theorem 1.2.2.

7.1 The Polyfold Gromov–Witten Invariants

Let (Q, ω) be a closed symplectic manifold of dimension $2n$. For a given homology class $A \in H_2(Q;\mathbb{Z})$ and for integers $g, k \geq 0$, and for a regular perturbation Λ, the perturbed solution space $\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)$ has the structure of a $2c_1(A) + (2n-6)(1-g) + 2k$ dimensional compact oriented weighted branched orbifold. Consider the following diagram

$$\begin{array}{ccc} \mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda) & \xrightarrow{\prod_{i=1}^k ev_i} & Q^k \\ \big\downarrow \pi & & \\ \overline{\mathcal{M}}_{g,k}^{log} & & \end{array}$$

where Q^k is the k-fold product manifold, $\overline{\mathcal{M}}_{g,k}^{\log}$ is the logarithmic Deligne–Mumford orbifold, ev_i is the evaluation map at the ith marked point, and π is the projection map which forgets the perturbed solution and stabilizes the domain. These maps are smooth when considered with respect to a weighted branched ep-subgroupoid structure on $\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)$.

Definition 7.1.1 (The Polyfold Gromov–Witten Invariant)**.** Suppose that $2g + k \geq 3$, so that $\overline{\mathcal{M}}_{g,k} \neq \emptyset$. Associated to the above diagram, the **Gromov–Witten invariant** is a multilinear map

$$\mathrm{GW}_{A,g,k}^{Q} : H_*(Q;\mathbb{R})^{\otimes k} \otimes H_*(\overline{\mathcal{M}}_{g,k};\mathbb{R}) \to \mathbb{R}$$

We will consider the following two ways to define this map.

1. **Branched Integral**

$$\mathrm{GW}_{\mathrm{Br.Int.}}(\alpha_1, \ldots, \alpha_k; \beta) = \int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\Lambda),\Lambda\circ\overline{\partial}_J)} \bigwedge_{i=1}^{k} ev_i^* \mathrm{PD}(\alpha_i) \wedge \pi^* \mathrm{PD}(\beta)$$

2. **Intersection Number**

$$\mathrm{GW}_{\text{Int.Num.}}(\alpha_1, \ldots, \alpha_k; \beta) = \left(\Pi_{i=1}^k ev_i \times \pi\right)(\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)) \cdot \left(\Pi_{i=1}^k X_i \times B\right)$$

We discuss these definitions in the following sections.

Remark 7.1.2. The above formulation of a GW-invariant as a multilinear map is due to [41] [40]. In [27], a GW-invariant is formulated as a multilinear map

$$I^Q_{A,g,k} : H^*(Q; \mathbb{R})^{\otimes k} \to H^*(\overline{\mathcal{M}}_{g,k}; \mathbb{R}).$$

It is related to our formulation of a GW-invariant by the following equation

$$\mathrm{GW}^Q_{A,g,k}(\alpha_1, \ldots, \alpha_k; \beta) = \langle I^Q_{A,g,k}(\mathrm{PD}(\alpha_1), \ldots, \mathrm{PD}(\alpha_k)), \beta\rangle.$$

In [32], a GW-invariant is formulated as a multilinear map defined on cohomology classes of Q (as opposed to homology classes of Q)

$$\mathcal{GW}^Q_{A,g,k} : H^*(Q; \mathbb{R})^{\otimes k} \otimes H_*(\overline{\mathcal{M}}_{g,k}; \mathbb{R}) \to \mathbb{R}.$$

It is related to our formulation of a GW-invariant by the following equation

$$\mathrm{GW}^Q_{A,g,k}(\alpha_1, \ldots, \alpha_k; \beta) = \mathcal{GW}^Q_{A,g,k}(\mathrm{PD}(\alpha_1), \ldots, \mathrm{PD}(\alpha_k); \beta).$$

7.2 The Gromov–Witten Invariants as Branched Integrals

In this section we recall the branched integration theory on compact oriented weighted branched suborbifolds, as originally developed in [23].

Let $\mathcal{S}$ be an n-dimensional, compact oriented weighted branched suborbifold with weight function $\vartheta : \mathcal{S} \to \mathbb{Q}^+$ of a polyfold $\mathcal{Z}$. There exists a map

$$\Phi_{\mathcal{S}} : \Omega^n_\infty(\mathcal{Z}) \to \mathcal{M}(\mathcal{S}, \mathcal{L}(\mathcal{S})), \qquad \omega \mapsto \mu_\omega$$

which associates to an sc-differential n-form $\omega \in \Omega^n_\infty(\mathcal{Z})$, a signed finite measure μ_ω on a canonical measure space $(\mathcal{S}, \mathcal{L}(\mathcal{S}))$ [18, Theorem 9.2].

Definition 7.2.1. The **branched integral** of a given sc-smooth differential form $\omega \in \Omega^n_\infty(\mathcal{Z})$ is defined by evaluating the measure μ_ω on the measurable set $\mathcal{S} \in \mathcal{L}(\mathcal{S})$; this is denoted by

$$\int_{(\mathcal{S},\vartheta)} \omega := \mu_\omega(\mathcal{S}).$$

The next theorem follows the same reasoning used to prove [18, Theorem 11.8].

Theorem 7.2.2 (Change of Variables)**.** *Let $f : \mathcal{Z}_1 \to \mathcal{Z}_2$ be an sc-smooth map between polyfolds. Let $\mathcal{S}_1 \subset \mathcal{Z}_1$ and $\mathcal{S}_2 \subset \mathcal{Z}_2$ be n-dimensional compact weighted branched suborbifolds with weight functions $\vartheta_1 : \mathcal{S}_1 \to \mathbb{Q}^+$ and $\vartheta_2 : \mathcal{S}_2 \to \mathbb{Q}^+$. Let S_1 and S_2 be associated oriented branched ep-subgroupoids with associated weight functors $\hat{\vartheta}_1 : S_1 \to \mathbb{Q}^+$ and $\hat{\vartheta}_2 : S_2 \to \mathbb{Q}^+$.*

Suppose that $\hat{f}$ restricts to an sc-smooth functor $\hat{f} : S_1 \to S_2$, and suppose that

- *$f : \mathcal{S}_1 \to \mathcal{S}_2$ is a homeomorphism between the underlying topological spaces,*
- *$\hat{f} : S_1 \to S_2$ is injective and a local homeomorphism,*
- *f is weight preserving, i.e. $\vartheta_2 \circ f = \vartheta_1$ and $\hat{\vartheta}_2 \circ \hat{f} = \hat{\vartheta}_1$.*

Then given a sc-smooth differential form ω,

$$\int_{(\mathcal{S}_2,\vartheta_2)} \omega = \int_{(\mathcal{S}_1,\vartheta_1)} f^*\omega.$$

Theorem 7.2.3 (Stokes' Theorem)**.** *[18, Theorem 9.4] Let $\mathcal{Z}$ be a polyfold with polyfold structure Z which admits sc-smooth partitions of unity. Let $\mathcal{S}$ be an n-dimensional, compact oriented branched suborbifold with weight function $\vartheta : \mathcal{S} \to \mathbb{Q}^+$.*

Let $\partial\mathcal{S}$ be its oriented boundary and let ω be an sc-differential form of degree $n-1$. Then

$$\int_{(\mathcal{S},\vartheta)} d\omega = \int_{(\partial\mathcal{S},\vartheta)} \omega.$$

Remark 7.2.4 (Computation of the Branched Integral). The proof of Stokes' theorem shows how to compute the branched integral using the same approach as in classical differential geometry.

Cover the underlying compact topological space $\mathcal{S}$ with finitely many open sets $|U_{x_k}| \subset \mathcal{Z}$, $k \in \{1,\dots,l\}$, where $U_{x_k} \subset Z$ are open sets admitting the natural $\boldsymbol{G}(x_k)$ action. Moreover, suppose that $S \cap U_{x_k} = \cup_{i\in I_k} M_i$ where (M_i, o_i) are oriented local branches with associated weights σ_i.

Denote by $U^*_{x_k}$ the saturations of the sets U_{x_k} in Z, i.e. $U^*_{x_k} = \pi^{-1}(\pi(U_{x_k}))$. Add another saturated open set U^*_0 so that the sets U^*_k cover Z and the sets $|U^*_{x_k} \setminus \overline{U^*_0}|$ still cover S.

By assumption Z admits sc-smooth partitions of unity, hence there exist sc-smooth functors $\beta_k : Z \to [0,1]$ which satisfy $\operatorname{supp} \beta_k \subset U^*_{x_k}$. Moreover, $\sum_{k=1}^{l} \beta_k = 1$ on S.

We may now write

$$\int_{(\mathcal{S},\vartheta)} \omega = \sum_{k=0}^{l} \int_{(\mathcal{S},\vartheta)} \beta_k \cdot \omega = \sum_{k=1}^{l} \frac{1}{\sharp \boldsymbol{G}^{\text{eff}}(x_k)} \cdot \sum_{i\in I_k} \sigma_i \cdot \int_{(M_i,o_i)} \beta_k \cdot \omega, \tag{7.1}$$

(we drop the term $k=0$ in the second equality since $\operatorname{supp} \beta_0 \cap S = \emptyset$).

Corollary 7.2.5 (Invariance of the Branched Integral)**.** *[21, Theorem 4.23] Let $\mathcal{W} \to \mathcal{Z}$ be a strong polyfold bundle, let $\overline{\partial}$ be an sc-smooth proper oriented Fredholm section, and* *let* $(N_0,\mathcal{U}_0)$, $(N_1,\mathcal{U}_1)$ *be two pairs which control the compactness of* $\overline{\partial}$*. Suppose that* Λ_0 *is a regular perturbation of* $\overline{\partial}$ *with respect to the pair* $(N_0,\mathcal{U}_0)$*, and that* Λ_1 *is a regular perturbation of* $\overline{\partial}$ *with respect to the pair* $(N_1,\mathcal{U}_1)$*.*

Let ω *be a closed sc-smooth differential form which represents a class in* $H^*_{dR}(\mathcal{Z})$*. Then*

$$\int_{(\mathcal{S}_{A,g,k}(\overline{\partial},\Lambda_0),\Lambda_0\circ\overline{\partial})} \omega = \int_{(\mathcal{S}_{A,g,k}(\overline{\partial},\Lambda_1),\Lambda_1\circ\overline{\partial})} \omega.$$

Definition 7.2.6 (Gromov–Witten Invariant as a Branched Integral)**.** [22, Theorem 1.12] Let $\alpha_1,\ldots,\alpha_k \in H_*(Q;\mathbb{R})$, and let $\beta \in H_*(\overline{\mathcal{M}}^{\log}_{g,k};\mathbb{R})$. Suppose that Λ is a regular perturbation of the Cauchy–Riemann section $\overline{\partial}_J$ with respect to a pair $(N,\mathcal{U})$. Hence the perturbed solution space $\mathcal{S}_{A,g,k}(\overline{\partial}_J,\Lambda)$ has the structure of a weighted branched suborbifold with weight function $\Lambda\circ\overline{\partial}_J$.

We can represent the Poincaré duals of the homology classes α_i and β by closed differential forms in the de Rahm cohomology groups, $\mathrm{PD}(\alpha_i) \in H^*_{dR}(Q;\mathbb{R})$ and $\mathrm{PD}(\beta) \in H^*_{dR}(\overline{\mathcal{M}}^{\log}_{g,k};\mathbb{R})$. By pulling back via the evaluation and projection maps, we obtain a closed sc-smooth differential form $\wedge_{i=1}^k ev_i^*\mathrm{PD}(\alpha_i) \wedge \pi^*\mathrm{PD}(\beta) \in H^*_{dR}(\mathcal{Z}_{A,g,k};\mathbb{R})$.

We define the GW-invariant via the branched integration of this differential form on the perturbed solution space, i.e.

$$\mathrm{GW}_{\mathrm{Br.Int.}}(\alpha_1,\ldots,\alpha_k;\beta) := \int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\Lambda),\Lambda\circ\overline{\partial}_J)} \bigwedge_{i=1}^{k} ev_i^*\mathrm{PD}(\alpha_i)\wedge \pi^*\mathrm{PD}(\beta)$$

Remark 7.2.7. As a result of Corollary 7.2.5, the branched integral does not depend on the choice of a pair which controls compactness, or on the choice of a regular perturbation with respect to this pair. As a result of Stokes' theorem 7.2.3, the branched integral does not depend on choice of closed sc-smooth differential form which represents a class in $H^*_{dR}(\mathcal{Z})$. Hence this definition gives a well-defined invariant.

7.3 The Gromov–Witten Invariants as Intersection Numbers

Compact oriented branched weighted orbifolds are generalizations of smooth finite-dimensional spaces; as such, we can generalize appropriate notions of transversality and intersection number. In this section, we show how to define the Gromov–Witten invariants as an intersection number.

Let us first consider the general situation. Consider a compact oriented orbifold $\mathcal{M}$ and a closed oriented suborbifold $\iota : \mathcal{B} \hookrightarrow \mathcal{M}$. Consider a compact oriented weighted branched suborbifold $\mathcal{S}$ (see Definition 6.4.4), and consider a smooth map $f : \mathcal{S} \to \mathcal{M}$.

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\ f\ } & \mathcal{M} \\ & & \uparrow{\scriptstyle \iota} \\ & & \mathcal{B} \end{array}$$

Definition 7.3.1. [1, Definition 2.3] Let $\iota : \mathcal{B} \to \mathcal{M}$ be a smooth map between oriented orbifolds. Let $(B, \boldsymbol{B})$ and $(M, \boldsymbol{M})$ be orbifold structures, which we moreover assume are effective. We say that $\mathcal{B}$ is a **suborbifold** of $\mathcal{M}$ if the following conditions are satisfied.

- The functor $\hat{\iota} : B \to M$ is an immersion with respect to the object / morphism sets.
- The image of the object set of $\mathcal{B}$, i.e. $\hat{\iota}(B) \subset M$, is saturated.
- Let $y \in \hat{\iota}(B) \subset M$. Then the morphism set $\boldsymbol{B}$ acts transitively on $\hat{\iota}^{-1}(y)$. Moreover, let $x_0 \in \hat{\iota}^{-1}(y)$, we require
  $$\sharp \hat{\iota}^{-1}(y) = \tfrac{\sharp \boldsymbol{G}(y)}{\sharp \boldsymbol{G}(x_0)}.$$
  where the (effective) isotropy subgroups are denoted by $\boldsymbol{G}(y)$ and $\boldsymbol{G}(x_0)$.
- Considered on the underlying topological spaces, $\iota : \mathcal{B} \to \mathcal{M}$ is proper. (It follows from the above condition that it is also injective.)

Remark 7.3.2. As a consequence of the above definition, we have the following local description. Consider $y \in \hat{\iota}(B)$, and suppose $x_0 \in \hat{\iota}^{-1}(y)$. Then there exists a $\boldsymbol{G}(y)$-invariant open neighborhood U_y, and there exist $\boldsymbol{G}(x)$-invariant open neighborhoods V_x at each of the $\frac{\sharp \boldsymbol{G}(y)}{\sharp \boldsymbol{G}(x_0)}$ many preimages $x \in \hat{\iota}^{-1}(y)$ such that

$$U_y \cap \operatorname{Im}(\hat{\iota}) = \bigcup_{g \in \boldsymbol{G}(y)} g \cdot \hat{\iota}(V_{x_0}) = \bigcup_{x \in \hat{\iota}^{-1}(y)} \hat{\iota}(V_x)$$

(see [1, Remark 2.4]).

The following proposition generalizes some results about manifolds to the current situation of orbifolds. For the relevant constructions of classical differential geometry, see [2, Chapter 1, §6].

Proposition 7.3.3. *Let $\mathcal{B}$ be a closed oriented suborbifold of an oriented orbifold $\mathcal{M}$. There exists a normal bundle $\pi : N \to \mathcal{B}$ of rank $n := \dim \mathcal{M} - \dim \mathcal{B}$. Associated to the normal bundle there exists a* ***Thom class*** *$\tau \in H^*_{dR,cv}(N; \mathbb{R})$ (i.e. a de Rahm form with compact vertical support) which is characterized by the following properties.*

- *For every $x \in B$ the restriction of τ to $H^n_{dR,c}(N_x; \mathbb{R})$ is the generator determined by the orientation. Here $H^*_{dR,c}$ denotes the space of de Rahm forms with compact support.*
- *The map*
  $$\begin{aligned} H^*_{dR}(\mathcal{B}; \mathbb{R}) &\to H^{*+n}_{dR,cv}(N; \mathbb{R}) \\ \omega &\mapsto \pi^*\omega \wedge \tau \end{aligned}$$
  is an isomorphism.

There exists a neighborhood $\mathcal{U}$ of the zero section N_0 such that there exists a well-defined map between orbifolds $j : \mathcal{U} \to \mathcal{M}$ which is a local diffeomorphism. There exists a well-defined pushforward $j_* : H^*_{dR,cv}(\mathcal{U};\mathbb{R}) \to H^*_{dR}(\mathcal{M};\mathbb{R})$. At points $y \in \hat{j}(U)$, it is defined by

$$(\hat{j}_*\omega)_y := \sum_{x \in \hat{j}^{-1}(y)} \hat{j}_*\omega_x.$$

At points $y \notin \hat{j}(U)$, it is defined by 0. This map has the following significance.

- *The Poincaré dual of $\mathcal{B}$ can be represented by the pushforward of the Thom class by this map, i.e. $PD(\mathcal{B}) = j_*\tau$.*
- *The support of the Poincaré dual of $\mathcal{B}$ can be shrunk into any given neighborhood of $\mathcal{B} \subset \mathcal{M}$.*

Example 7.3.4. Consider the example of a suborbifold defined by a point, defined on the underlying topological spaces by the inclusion map

$$\iota_x : \{x\} \to \mathcal{M}.$$

If ι_x maps $\{x\}$ to a point of $\mathcal{M}$ with non-trivial isotropy group $\boldsymbol{G}(\hat{\iota}_x(\{x\}))$, $d := \sharp\boldsymbol{G}(\hat{\iota}_x(\{x\}))$, we have the following consequences.

- The orbifold structure of $\{x\}$ has object set $\{x_1, \ldots, x_d\}$, i.e. d distinct points.
- The normal bundle structure consists of d distinct tangent planes, $N_{x_1}, \ldots, N_{x_d}$.
- The Poincaré dual is represented as the sum of the Thom classes on the d distinct tangent planes, i.e. is given by a $\boldsymbol{G}(\hat{\iota}_x(\{x\}))$-invariant volume form ω with total weight d.

When one remembers the definition of the integral on an orbifold this makes sense. Letting $U_{\hat{\iota}_x\{x\}}$ be a $\boldsymbol{G}(\hat{\iota}_x(\{x\}))$-invariant neighborhood of $\hat{\iota}_x(\{x\})$ we see that

$$\int_{\mathcal{M}} \mathrm{PD}(\{x\}) = \tfrac{1}{\sharp\boldsymbol{G}(\hat{\iota}_x(\{x\}))} \int_{U_{\hat{\iota}_x\{x\}}} \omega = 1.$$

Consider a compact oriented weighted branched suborbifold $\mathcal{S}$ (see Definition 6.4.4). At every point x in an associated oriented weighted branched ep-subgroupoid S an open neighborhood of x is given by a local branching structure $x \in \cup_{i\in I} M_i \subset S$ where I is a finite index set and where the M_i are finite dimensional submanifolds.

Definition 7.3.5. Let $\mathcal{S}$ be a compact oriented weighted branched suborbifold. Let $\mathcal{M}$ an oriented orbifold, and let $\mathcal{B}$ be an oriented suborbifold with inclusion map $\iota : \mathcal{B} \to \mathcal{M}$. Consider a smooth map $f : \mathcal{S} \to \mathcal{M}$. We say that f is **transverse** to $\mathcal{B}$, written symbolically as $f \pitchfork \mathcal{B}$, if for all $[x] \in \mathcal{S}, [x'] \in \mathcal{B}$ which satisfy $f([x]) = \iota([x']) \in \mathcal{M}$,

$$d\hat{f}_x(T_x M_i) \oplus d\hat{\iota}_{x'}(T_{x'} V_{x'}) = T_y U_y. \tag{7.2}$$

where

- $U_y \subset M$ is an open neighborhood of a representative y of $[y] := f([x]) = \iota([x'])$, where M is the object set of an orbifold structure on $\mathcal{M}$.
- $\cup_{i \in I} M_i \subset S$ is an open neighborhood of a representative x of $[x]$, where S is the object set of an weighted branched ep-subgroupoid structure on $\mathcal{S}$. Moreover, $\hat{f}(x) = y$.
- $V_{x'} \subset B$ is an open neighborhood of a representative x' of $[x']$, where B is the object set of an orbifold structure on $\mathcal{B}$. Moreover, $\hat{\iota}(x') = y$.

Moreover, we require (7.2) for all such representatives of $[y]$, $[x]$, and $[x']$.

The underlying topological space $\mathcal{S}$ is compact and hence we can cover it with a finite number of open sets of the form $|\cup_{i \in I_k} M_i|$, $k \in \{1, \ldots, l\}$ (see Remark 6.4.6). When f and $\mathcal{B}$ are transverse and $\dim \mathcal{S} + \dim \mathcal{B} = \dim \mathcal{M}$, the intersection points will be isolated. One can then show that the subset $f^{-1}(\mathcal{B}) \subset \mathcal{S}$ consists of a finite number of points.

Definition 7.3.6. Suppose that $\dim \mathcal{S} + \dim \mathcal{B} = \dim \mathcal{M}$. We define the **intersection number** by

$$f(\mathcal{S}) \cdot \mathcal{B} := \sum_{[x] \in f^{-1}(\mathcal{B})} \delta([x])$$

where $\delta([x])$ must account for orientation, weights, and istropy as determined by the following equation:

$$\delta([x]) := \left(\sum \pm \sigma_i \right) \cdot \left(\frac{1}{\sharp \boldsymbol{G}(x)} \right) \cdot \left(\frac{\sharp \boldsymbol{G}(y)}{\sharp \boldsymbol{G}(x')} \right)$$

where σ_i is the weight of the branch M_i, and where the sign is positive if the orientation of both sides of equation (7.2) are the same, and negative if the the orientations are opposite.

Theorem 7.3.7. *The branched integral and the intersection number are related by the following equation.*

$$\int_{(\mathcal{S}, \vartheta)} f^* PD(\mathcal{B}) = f(\mathcal{S}) \cdot \mathcal{B}.$$

Proof. Let $[x_1], \ldots, [x_l] \in f^{-1}(\mathcal{B})$ be the finite points of intersection. By Proposition 7.3.3 we can represent $\mathrm{PD}(\mathcal{B})$ by the pushforward of the Thom class $j_* \tau$, with support in an arbitrarily small neighborhood of $\mathcal{B} \subset \mathcal{M}$. Hence, we can assume $f^* \tau$ is supported in arbitrarily small disjoint neighborhoods of the points $[x_k]$, i.e.

$$\operatorname{supp} f^* \tau \subset \sqcup_k |\cup_{i \in I_k} M_i| \subset \mathcal{S}.$$

where $\cup_{i \in I_k} M_i \subset S$ are disjoint $\boldsymbol{G}(x_k)$-invariant open sets for representatives x_k of $[x_k]$.

We assert that

$$\begin{aligned}
\int_{(\mathcal{S},\vartheta)} f^*\mathrm{PD}(\mathcal{B}) &= \sum_{k=1}^{l} \int_{(|\cup_{i\in I_k} M_i|,\vartheta)} \hat{f}^*\hat{j}_*\tau \\
&= \sum_{k=1}^{l} \frac{1}{\sharp \boldsymbol{G}(x_k)} \cdot \sum_{i\in I_k} \sigma_i \cdot \int_{(M_i,o_i)} \hat{f}^*\hat{j}_*\tau \\
&= \sum_{k=1}^{l} \frac{1}{\sharp \boldsymbol{G}(x_k)} \cdot \sum_{i\in I_k} \sigma_i \cdot \left(\pm \frac{\sharp \boldsymbol{G}(y)}{\sharp \boldsymbol{G}(x_0')} \right) \\
&= \sum_{k=1}^{l} \delta([x_k]) \\
&= f(\mathcal{S}) \cdot \mathcal{B}.
\end{aligned}$$

This is clear, except for the third equality. To see this equality, consider the point $x_k \in M_i$ which intersects a point $x_0' \in B$, i.e. $y := \hat{f}(x_k) = \hat{\imath}(x_0') \in M$. By the definition of suborbifold 7.3.1, there are precisely $\frac{\sharp \boldsymbol{G}(y)}{\sharp \boldsymbol{G}(x_0')}$ many points in $\hat{\imath}^{-1}(y) \subset B$. By the transversality assumption, $d\hat{f}_{x_k}(T_{x_k}M_i) \oplus d\hat{\imath}_{x'}(T_{x'}V_{x'}) = T_yU_y$ at each of the points $x' \in \hat{\imath}^{-1}(y)$.

By the definition of the pushforward, and the definition of the Thom class 7.3.3, it follows that the restriction of $\hat{j}_*\tau$ to $\hat{f}(M_i)$ is a volume form of weight $\frac{\sharp \boldsymbol{G}(y)}{\sharp \boldsymbol{G}(x_0')}$, with sign $\pm$ depending on the orientation of the intersection. This justifies the equality

$$\int_{(M_i,o_i)} \hat{f}^*\hat{j}_*\tau = \pm \frac{\sharp \boldsymbol{G}(y)}{\sharp \boldsymbol{G}(x_0')}.$$

□

We now consider the setup for defining the Gromov–Witten invariants as an intersection number. If Λ is a regular perturbation, then by Proposition 3.4.10 the perturbed GW solution space $\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)$ has the structure of a compact oriented weighted branched suborbifold. These solution spaces can be given additional structure as proposed by the following remark.

Remark 7.3.8. [43] There exist regular perturbations Λ such that the perturbed GW solution space $\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)$ has the structure of a stratified space, i.e. there exists a filtration[1]

$$(\mathcal{S}_{A,g,k})_0 \subset (\mathcal{S}_{A,g,k})_2 \subset \cdots \subset (\mathcal{S}_{A,g,k})_{2c_1(A)+(2n-6)(1-g)+2k-2} \subset (\mathcal{S}_{A,g,k})_{2c_1(A)+(2n-6)(1-g)+2k}$$

where the codimension $2i$ stratum appear as the subset of $\mathcal{S}_{A,g,k}$ which consists of ith-noded stable curves.

Let $\alpha_1, \ldots, \alpha_k \in H_*(Q;\mathbb{Q})$, and let $\beta \in H_*(\overline{\mathcal{M}}_{g,k}^{\log};\mathbb{Q})$. Suppose that there exist closed submanifolds $X_i \subset Q$ and a closed suborbifold $B \subset \overline{\mathcal{M}}_{g,k}^{\log}$ such that

[1] This should not be confused with the filtration given by the sc-structure on the polyfold $\mathcal{Z}_{A,g,k}$. We may note that the induced filtration from this sc-structure on $\mathcal{S}_{A,g,k}$ is trivial since $\mathcal{S}_{A,g,k} \subset (\mathcal{Z}_{A,g,k})_\infty$.

- $[X_i] = \alpha_i$ and $[B] = \beta$,
- $(\Pi_{i=1}^k \hat{ev}_i \times \hat{\pi}) \pitchfork (\Pi_{i=1}^k X_i \times B)$.

We consider $\Pi_{i=1}^k X_i \times B$ as a suborbifold of $Q^k \times \overline{\mathcal{M}}_{g,k}^{\log}$. Thus we have the following diagram.

$$\begin{array}{ccc} \mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda) & \xrightarrow{\Pi_{i=1}^k \hat{ev}_i \times \hat{\pi}} & Q^k \times \overline{\mathcal{M}}_{g,k}^{\log} \\ & & \iota \uparrow \\ & & \Pi_{i=1}^k X_i \times B \end{array}$$

Definition 7.3.9 (Gromov–Witten Invariant as an Intersection Number)**.** Consider the above setup, and suppose that

$$\sum_{i=1}^{k}(2n - \dim(X_i)) + (6g - 6 + 2k - \dim(\mathcal{B})) = 2c_1(A) + (2n - 6)(1 - g) + 2k.$$

We may define the Gromov–Witten invariant $\mathrm{GW}_{\mathrm{Int.Num.}}(\alpha_1, \ldots, \alpha_k; \beta)$ as the **intersection number**

$$(\Pi_{i=1}^k ev_i \times \pi)(\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)) \cdot (\Pi_{i=1}^k X_i \times B) = \sum_{[x] \in \left(\Pi_{i=1}^k ev_i \times \pi\right)^{-1}(\Pi_{i=1}^k X_i \times \mathcal{B})} \delta([x])$$

Corollary 7.3.10. *The definition of GW-invariant as a branched integral is equal to the definition of GW-invariant as an intersection number, i.e.*

$$\int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda), \Lambda \circ \overline{\partial}_J)} \bigwedge_{i=1}^{k} ev_i^* PD(\alpha_i) \wedge \pi^* PD(\beta) = (\Pi_{i=1}^k ev_i \times \pi)(\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)) \cdot (\Pi_{i=1}^k X_i \times B).$$

It therefore follows that the above definition of intersection number is a well-defined invariant, and moreover does not depend on our choices of submanifolds / suborbifolds.

Proof. This follows immediately from Theorem 7.3.7. □

We now discuss the assumptions of Definition 7.3.9 that we represent homology classes as submanifolds / suborbifolds, and that these submanifolds / suborbifolds were transverse to the map $\Pi_{i=1}^k ev_i \times \pi$. The rational homology groups of any orientable manifold Q have a basis consisting of elements that are represented by smooth closed submanifolds of Q [45, Corollary II.30]. Hence for each homology class $\alpha_i \in H_*(Q; \mathbb{Q})$ there exists a submanifold $X_i \subset Q$ such that $k_i[X_i] = \alpha_i$ for some $k_i \in \mathbb{Q}$. When genus $g = 0$, the Deligne–Mumford space $\overline{\mathcal{M}}_{0,k}^{\log}$ is a manifold, and hence for any homology class $\beta \in H_*(\overline{\mathcal{M}}_{0,k}^{\log}; \mathbb{Q})$ there exists a submanifold $B \subset \overline{\mathcal{M}}_{0,k}^{\log}$ such that $k'[B] = \beta$ for some $k' \in \mathbb{Q}$. There does not yet exist an analogous theorem for smooth orbifolds.

The underlying topological space $\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)$ is compact and hence we can cover it with a finite number of open sets of the form $|\cup_{i\in I_k} M_i|_k$ (see Remark 6.4.6). When $g = 0$ we may perturb the submanifold $\Pi_{i=1}^k X_i \times B$ to be transverse to the finite number of smooth maps defined on the finite collection of branches M_i

$$\Pi_{i=1}^k \hat{ev}_i \times \hat{\pi} : M_i \to Q^k \times \overline{\mathcal{M}}_{0,k}^{\log}.$$

Moreover, by the argument in [32, Exercise 7.1.2], we may assume the perturbed submanifold is still a product submanifold.

However, if genus $g \neq 0$ then we cannot in general perturb the suborbifold $B \subset \overline{\mathcal{M}}_{g,k}^{\log}$ near orbifold singularities, and so we introduce the following condition.

Definition 7.3.11. We say that the homology class $\beta \in H_*(\overline{\mathcal{M}}_{g,k}^{\log}; \mathbb{Q})$ satisfies the **suborbifold representability condition** if the following requirements are satisfied.

- There exists a suborbifold $B \subset \overline{\mathcal{M}}_{g,k}^{\log}$ such that $[B] = \beta$.
- The projection map $\pi : \mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda) \to \overline{\mathcal{M}}_{g,k}^{\log}$ is transverse to the suborbifold B.
- The restriction of the projection map to the codimension $2i > 0$ stratum of $\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)$, i.e. $\pi : (\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda))_{2c_1(A)+(2n-6)(1-g)+2k-2i} \to \overline{\mathcal{M}}_{g,k}^{\log}$, is transverse to the suborbifold B.

These requirements are automatically satisfied in the following situations.

- When the genus is zero, i.e. for all $\beta \in H_*(\overline{\mathcal{M}}_{0,k}^{\log}; \mathbb{Q})$. In this case by [45, Corollary II.30], β can be represented by a submanifold. We can perturb this submanifold to be transverse to the finite collection of branches.
- When $\beta = [\overline{\mathcal{M}}_{g,k}^{\log}]$ and $B = \overline{\mathcal{M}}_{g,k}^{\log}$.

Now consider the finite number of smooth maps defined on the finite collection of branches M_i

$$\Pi_{i=1}^k \hat{ev}_i \times \hat{\pi} : M_i \to Q^k \times \overline{\mathcal{M}}_{g,k}^{\log}.$$

Assuming that $\beta \in H_*(\overline{\mathcal{M}}_{g,k}^{\log}; \mathbb{Q})$ satisfies the above suborbifold representability condition we may perturb the submanifold $\Pi_{i=1}^k X_i \subset Q^k$ to be transverse to the finite collection of restricted maps $\Pi_{i=1}^k \hat{ev}_i|_{\hat{\pi}^{-1}(B)}$. Denoting the perturbed submanifolds by the same letter, it follows that $\Pi_{i=1}^k \hat{ev}_i \times \hat{\pi}$ and $\Pi_{i=1}^k X_i \times B$ are transverse.

Remark 7.3.12 (Gromov–Witten Invariants as a Distinguished Homology Class). Another possible approach to defining the GW-invariants is to ignore the additional structure of a compact oriented weighted branched orbifold and consider only the underlying compact topological space. By considering the weight function $\vartheta : \mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda) \to \mathbb{Q}$ as well as the orientation, one can hope to associate to the compact topological space $\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)$ a *distinguished homology class*

$$[\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda)] \in H_{2c_1(A)+(2n-6)(1-g)+2k}(\mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda); \mathbb{Q}).$$

The existence of such a rational homology class is speculated in [18, Remark 15.8], who suggest using a triangulation of the underlying topological space to construct it. Another possible approach to the construction of such a rational homology class is given in [30], which also uses a triangulation of the underlying topological space in the proof of [30, Proposition 3.25].

The underlying compact topological space does not depend on the weight function, and hence neither does the homology of this space. In order to 'distinguish' a homology class, one uses the weight function and the orientation to make a choice of a top dimensional homology class.

We are careful in our choice of words to not call this a 'fundamental class', as in classical algebraic topology this has a very specific meaning. In the cases of finite-dimensional orientable manifolds and finite-dimensional orientable orbifolds, the top dimensional homology group has rank one, and a fundamental class by definition is a choice of generator of this top homology group.

7.4 The Gromov–Witten Axioms

We prove the Effective, Grading, Homology, Zero, Symmetry, Fundamental Class, and Divisor axioms for curves of arbitrary genus, and for all closed symplectic manifolds. Our approach is essentially the same as the approach in [32], used to prove the Gromov–Witten axioms for semipositive symplectic manifolds in the genus $g = 0$ case. The main difference is that they interpret the Gromov–Witten invariants as an intersection number, where the Gromov–Witten solution spaces have the structure of pseudocycles. In our case, we also interpret the Gromov–Witten invariants as an intersection number, except our perturbed Gromov–Witten solution spaces have the structure of compact oriented weighted branched orbifolds.

Effective Axiom

If $\omega(A) < 0$ *then* $GW^Q_{A,g,k} = 0$.

Proof. The energy of a smooth map $u : \Sigma \to Q$ is defined as

$$E(u) := \frac{1}{2}\int_\Sigma |du|^2_j \mathrm{dvol}_\Sigma.$$

By the energy identity, a J-holomorphic map must satisfy $\omega(A) = E(u) \geq 0$ (for example, see [32, Lemma 2.2.1]). Hence, the unperturbed solution space of J-holomorphic maps is the empty set, i.e. $\mathcal{S}_{A,g,k}(\overline{\partial}_J) = \emptyset$, and is trivially transverse without perturbation. Therefore,

$$\mathrm{GW}^Q_{A,g,k}(\alpha_1, \ldots, \alpha_k; \beta) = \int_\emptyset \bigwedge_{i=1}^{k} ev_i^* \mathrm{PD}(\alpha_i) \wedge \pi^* \mathrm{PD}(\beta) = 0.$$

□

Grading Axiom

If $GW^Q_{A,g,k}(\alpha_1, \ldots, \alpha_k; \beta) \neq 0$ *then*

$$\sum_{i=1}^{k} (2n - \deg(\alpha_i)) + (6g - 6 + 2k - \deg(\beta)) = 2c_1(A) + (2n - 6)(1 - g) + 2k.$$

Proof. The left hand side is the codegree of $\alpha_1 \times \cdots \alpha_k \times \beta$ in the product $Q \times \cdots \times Q \times \overline{\mathcal{M}}^{\log}_{g,k}$, while the right hand side is the dimension of the perturbed solution space. Hence, this follows directly from the definition of the GW-invariants. □

Homology Axiom

There exists a homology class

$$\sigma_{A,g,k} \in H_{2c_1(A)+(2n-6)(1-g)+2k}(Q^k \times \overline{\mathcal{M}}_{g,k}; \mathbb{R})$$

such that

$$GW^Q_{A,g,k}(\alpha_1, \ldots, \alpha_k; \beta) = \langle p_1^* PD(\alpha_1) \smile \cdots \smile p_k^* PD(\alpha_k) \smile p_0^* PD(\beta), \sigma_{A,g,k} \rangle$$

where $p_i : Q^k \times \overline{\mathcal{M}}_{g,k} \to Q$ *denotes the projection onto the ith factor and the map* $p_0 : Q^k \times \overline{\mathcal{M}}_{g,k} \to \overline{\mathcal{M}}_{g,k}$ *denotes the projection onto the last factor.*

Proof. The GW-invariants define homomorphisms $\mathrm{GW}^Q_{A,g,k} : H_*(Q; \mathbb{R})^{\otimes k} \otimes H_*(\overline{\mathcal{M}}_{g,k}; \mathbb{R}) \to \mathbb{R}$. This homomorphism defines a cohomology class in $H^*(Q^k \times \overline{\mathcal{M}}_{g,k}; \mathbb{R})$, for $* = \sum_{i=1}^k (2n - \deg(\alpha_i)) + (6g - 6 + 2k - \deg(\beta))$. The Poincaré dual of this cohomology class is the required homology class $\sigma_{A,g,k} \in H_{2c_1(A)+(2n-6)(1-g)+2k}(Q^k \times \overline{\mathcal{M}}_{g,k}; \mathbb{R})$.

□

Zero Axiom

If $A = 0, g = 0$ *then* $GW^Q_{0,0,k}(\alpha_1, \ldots, \alpha_k; \beta) = 0$ *whenever* $\deg(\beta) > 0$, *and*

$$GW^Q_{0,0,k}(\alpha_1, \ldots, \alpha_k; [pt]) = \int_Q PD(\alpha_1) \wedge \cdots \wedge PD(\alpha_k)$$

Proof. Any map $u : \Sigma \to Q$ with $\overline{\partial}_J u = 0$ and $u_*[\Sigma] = 0$ must be constant. Moreover, at all constant, genus 0 stable curves the linearization of the Cauchy–Riemann operator is surjective (see e.g. [32, Lemma 6.7.6] or Lemma 6.3.3). It therefore follows that the unperturbed solution space $\mathcal{S}_{0,0,k}(\overline{\partial}_J) = \{[\Sigma, j, M, D, u] \mid \overline{\partial}_J u = 0\} \subset \mathcal{Z}_{0,0,k}$ is transversally cut out. The map $ev_i \times \pi : \mathcal{S}_{0,0,k}(\overline{\partial}_J) \to Q \times \overline{\mathcal{M}}^{\text{exp}}_{0,k}$ for any $i = 1, \dots, k$ defines a diffeomorphism. If we use this diffeomorphism as an identification $\mathcal{S}_{0,0,k}(\overline{\partial}_J) \simeq Q \times \overline{\mathcal{M}}^{\text{exp}}_{0,k}$ then the map $\Pi^k_{i=1} ev_i \times \pi : \mathcal{S}_{0,0,k}(\overline{\partial}_J) \to Q^k \times \overline{\mathcal{M}}^{\text{log}}_{0,k}$ can be identified with the map $\text{id}^k_Q \times \text{id}_{\overline{\mathcal{M}}_{0,k}} : Q \times \overline{\mathcal{M}}^{\text{exp}}_{0,k} \to Q^k \times \overline{\mathcal{M}}^{\text{log}}_{0,k}$.

Consider the definition of GW-invariant as a branched integral, we see that

$$\begin{aligned}
\text{GW}^Q_{0,0,k}(\alpha_1, \dots, \alpha_k; [\text{pt}]) &= \int_{\mathcal{S}_{0,0,k}(\overline{\partial}_J)} ev_1^*\text{PD}(\alpha_1) \wedge \dots \wedge ev_k^*\text{PD}(\alpha_k) \wedge \pi^*\text{PD}(\beta) \\
&= \int_{Q \times \overline{\mathcal{M}}^{\text{exp}}_{0,k}} \text{id}_Q^*\text{PD}(\alpha_1) \wedge \dots \wedge \text{id}_Q^*\text{PD}(\alpha_k) \wedge \text{id}^*_{\overline{\mathcal{M}}_{0,k}}\text{PD}(\beta) \\
&= \int_Q \text{PD}(\alpha_1) \wedge \dots \wedge \text{PD}(\alpha_k) \cdot \int_{\overline{\mathcal{M}}^{\text{exp}}_{0,k}} \text{PD}(\beta).
\end{aligned}$$

If $\deg(\beta) > 0$ then $\deg(\text{PD}(\beta)) < \dim(\overline{\mathcal{M}}^{\text{exp}}_{0,k})$ and hence $\int_{\overline{\mathcal{M}}^{\text{exp}}_{0,k}} \text{PD}(\beta) = 0$. On the other hand, if $\beta = [\text{pt}]$, then $\int_{\overline{\mathcal{M}}^{\text{exp}}_{0,k}} \text{PD}[\text{pt}] = 1$. □

Symmetry Axiom

Fix a permutation $\sigma : \{1, \dots, k\} \to \{1, \dots, k\}$*. Consider the permutation map* $\sigma : \overline{\mathcal{M}}^{log}_{g,k} \to \overline{\mathcal{M}}^{log}_{g,k}$, $[\Sigma, j, M, D] \mapsto [\Sigma, j, M^\sigma, D]$ *where* $M = \{z_1, \dots, z_k\}$ *and where* $M^\sigma := \{z'_1, \dots, z'_k\}$, $z'_i := z_{\sigma(i)}$*. Then*

$$GW^Q_{A,g,k}(\alpha_{\sigma(1)}, \dots, \alpha_{\sigma(k)}; \sigma_*\beta) = (-1)^{N(\sigma;\alpha_i)} GW^Q_{A,g,k}(\alpha_1, \dots, \alpha_k; \beta)$$

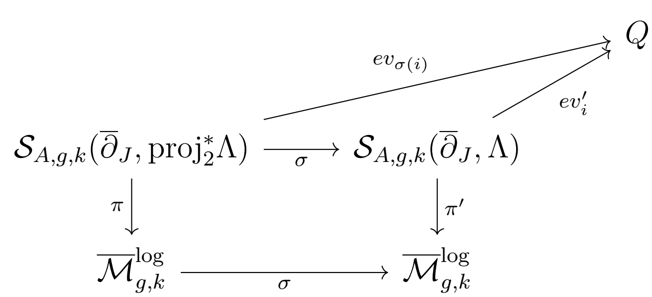
where $N(\sigma; \alpha_i) := \sharp\{i < j \mid \sigma(i) > \sigma(j), \deg(\alpha_i)\deg(\alpha_j) \in 2\mathbb{Z} + 1\}$.

Proof. In Corollary 3.1.5 we have shown we may pullback perturbations via the permutation map, yielding a persistent map between the perturbed GW solution spaces,

$$\sigma : \mathcal{S}_{A,g,k}(\overline{\partial}_J, \text{proj}_2^*\Lambda) \to \mathcal{S}_{A,g,k}(\overline{\partial}_J, \Lambda).$$

In Section 3.7 we moreover discussed the structure of this map.

To prove the symmetry axiom, consider the following commutative diagram of maps.

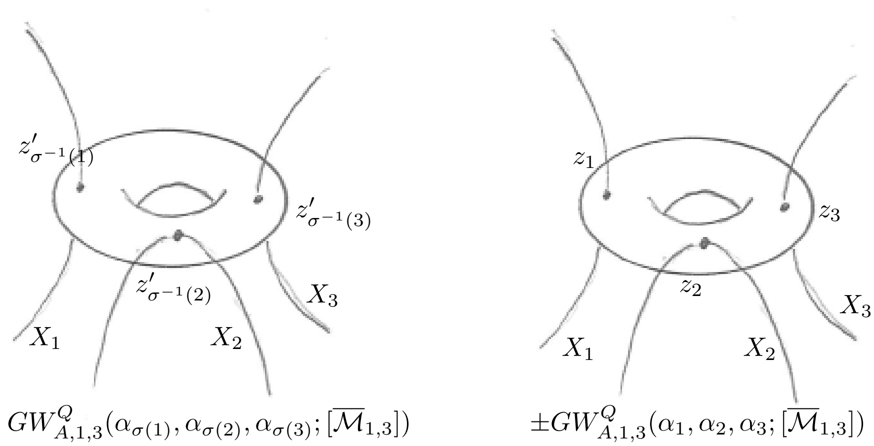


We may then compute the GW-invariants using the branched integral.

$$
\begin{aligned}
&\mathrm{GW}^Q_{A,g,k}(\alpha_{\sigma(1)},\dots,\alpha_{\sigma(k)};\sigma_*\beta)\\
&\quad= \int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\Lambda),\Lambda\circ\overline{\partial}_J)} ev_1'^*\mathrm{PD}(\alpha_{\sigma(1)})\wedge\dots\wedge ev_k'^*\mathrm{PD}(\alpha_{\sigma(k)})\wedge \pi'^*\mathrm{PD}(\sigma_*\beta)\\
&\quad= \int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\mathrm{proj}_2^*\Lambda),\mathrm{proj}_2^*\Lambda\circ\overline{\partial}_J)} \sigma^*\left(ev_1'^*\mathrm{PD}(\alpha_{\sigma(1)})\wedge\dots\wedge ev_k'^*\mathrm{PD}(\alpha_{\sigma(k)})\wedge \pi'^*\mathrm{PD}(\sigma_*\beta)\right)\\
&\quad= \int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\mathrm{proj}_2^*\Lambda),\mathrm{proj}_2^*\Lambda\circ\overline{\partial}_J)} \sigma^*ev_1'^*\mathrm{PD}(\alpha_{\sigma(1)})\wedge\dots\wedge \sigma^*ev_k'^*\mathrm{PD}(\alpha_{\sigma(k)})\wedge \pi^*\sigma^*\mathrm{PD}(\sigma_*\beta)\\
&\quad= \int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\mathrm{proj}_2^*\Lambda),\mathrm{proj}_2^*\Lambda\circ\overline{\partial}_J)} ev_{\sigma(1)}^*\mathrm{PD}(\alpha_{\sigma(1)})\wedge\dots\wedge ev_{\sigma(k)}^*\mathrm{PD}(\alpha_{\sigma(k)})\wedge \pi^*\mathrm{PD}(\beta)\\
&\quad= (-1)^{N(\sigma;\alpha_i)}\int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\mathrm{proj}_2^*\Lambda),\mathrm{proj}_2^*\Lambda\circ\overline{\partial}_J)} ev_1^*\mathrm{PD}(\alpha_1))\wedge\dots\wedge ev_k^*\mathrm{PD}(\alpha_k)\wedge \pi^*\mathrm{PD}(\beta)\\
&\quad= (-1)^{N(\sigma;\alpha_i)}\mathrm{GW}^Q_{A,g,k}(\alpha_1,\dots,\alpha_k;\beta)
\end{aligned}
$$

In the second equality, we have $\int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\Lambda),\Lambda\circ\overline{\partial}_J)}\omega = \int_{(\mathcal{S}_{A,g,k}(\overline{\partial}_J,\mathrm{proj}_2^*\Lambda),\mathrm{proj}_2^*\Lambda\circ\overline{\partial}_J)}\sigma^*\omega$ by the change of variables theorem 7.2.2. In the third equality, we use commutativity of the diagram to see $\sigma^*\pi'^*\mathrm{PD}\sigma_*(\beta) = \pi^*\sigma^*\mathrm{PD}\sigma_*(\beta)$. In the fourth equality, observe $ev_i'\circ\sigma = ev_{\sigma(i)}$ hence $\sigma^*ev_i'^* = ev_{\sigma(i)}^*$; we also observe the map $\sigma:\overline{\mathcal{M}}^{\log}_{g,k}\to\overline{\mathcal{M}}^{\log}_{g,k}$ is a diffeomorphism, and so $\sigma^*\mathrm{PD}\sigma_*(\beta) = \mathrm{PD}(\beta)$ for all $\beta\in H_*(\overline{\mathcal{M}}^{\log}_{g,k};\mathbb{Q})$. In the fifth equality, the sign $(-1)^{N(\sigma;\alpha_i)}$ is introduced by the permutation of the differential forms. □

Figure 7.1: Symmetry Axiom



Fundamental Class Axiom

Consider the fundamental classes $[Q] \in H_{2n}(Q;\mathbb{Q})$ *and* $[\overline{\mathcal{M}}^{log}_{g,k}] \in H_{6g-6+2k}(\overline{\mathcal{M}}^{log}_{g,k};\mathbb{Q})$. *Suppose that* $A \neq 0$ *and that* (A,g,k) *is not basic (Definition 1.2.1). Then*

$$GW^Q_{A,g,k}(\alpha_1,\ldots,\alpha_{k-1},[Q];[\overline{\mathcal{M}}^{log}_{g,k}]) = 0.$$

Suppose that $\beta \in H_*(\overline{\mathcal{M}}^{log}_{g,k-1};\mathbb{Q})$ *satisfies the suborbifold representability condition (Definition 7.3.11). Consider the* l*th-marked point doubling map* $s_l : \overline{\mathcal{M}}^{log}_{g,k-1} \to \overline{\mathcal{M}}^{log}_{g,k}$ *(see Definition 2.5.9). Then*

$$GW^Q_{A,g,k}(\alpha_1,\ldots,\alpha_{k-1},[Q];s_{l*}\beta) = GW^Q_{A,g,k-1}(\alpha_1,\ldots,\alpha_{k-1};\beta).$$

Proof. In Theorem 5.1.2 we have shown we construct a regular perturbation which pulls back to a regular perturbation via the kth-marked point forgetting map, yielding a persistent map between the perturbed GW solution spaces,

$$ft_k : \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda) \to \mathcal{S}_{A,g,k-1}(\overline{\partial}_J,\Lambda).$$

In Section 5.2 we moreover discussed the structure of this map.

We prove the first assertion. Since we have constructed the perturbed GW solution space $\mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda)$ with a pullback perturbation via ft_k, it follows that given a perturbed solution we may move the kth-marked freely and still have a perturbed solution.

Now, interpret the GW-invariant as an intersection number via Definition 7.3.9, and consider the finite set of points $(\Pi_{i=1}^k ev_i \times \pi)^{-1}(\Pi_{i=1}^{k-1} X_i \times Q \times \overline{\mathcal{M}}^{\log}_{g,k}) \subset \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda)$.

This set must be empty, as the last marked point is unconstrained and so any intersection point can never be isolated. Explicitly,

$$(\Pi_{i=1}^k ev_i \times \pi)^{-1}(\Pi_{i=1}^{k-1} X_i \times Q \times \overline{\mathcal{M}}^{\log}_{g,k}) = (\Pi_{i=1}^{k-1} ev_i)^{-1}(\Pi_{i=1}^{k-1} X_i) \subset \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda),$$

and $(\Pi_{i=1}^{k-1} ev_i)^{-1}(\Pi_{i=1}^{k-1} X_i)$ has no constraint on the kth-marked point.

We prove the second assertion. Consider the following commutative diagram of maps.

$$\begin{array}{ccccc} & & & & Q \\ & & {}^{ev_i}\nearrow & & \nearrow{}^{ev_i} \\ \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda) & & \xrightarrow{ft_k} & & \mathcal{S}_{A,g,k-1}(\overline{\partial}_J,\Lambda) \\ \downarrow \pi & & & & \downarrow \pi \\ \overline{\mathcal{M}}^{\log}_{g,k} & & \xrightarrow{ft_k} & & \overline{\mathcal{M}}^{\log}_{g,k-1} \end{array}$$

Again we consider the GW-invariant as an intersection number via Definition 7.3.9. By our conditions, we may assume that

- $\Pi_{i=1}^{k-1}\hat{ev}_i \times \hat{\pi} : \mathcal{S}_{A,g,k-1}(\overline{\partial}_J, \Lambda) \to Q^{k-1} \times \overline{\mathcal{M}}_{g,k-1}^{\log}$ is transverse to $\Pi_{i=1}^{k-1}X_i \times B$,
- $\Pi_{i=1}^{k}\hat{ev}_i \times \hat{\pi} : \mathcal{S}_{A,g,k}^{ft}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda) \to Q^k \times \overline{\mathcal{M}}_{g,k}^{\log}$ is transverse to $\Pi_{i=1}^{k-1}X_i \times Q \times s_l(B)$.[2]

Hence both $(\Pi_{i=1}^{k-1}ev_i \times \pi)^{-1}(\Pi_{i=1}^{k-1}X_i \times B) \subset \mathcal{S}_{A,g,k-1}(\overline{\partial}_J, \Lambda)$ and $(\Pi_{i=1}^{k}ev_i \times \pi)^{-1}(\Pi_{i=1}^{k-1}X_i \times Q \times s_l(B)) \subset \mathcal{S}_{A,g,k}^{ft}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda)$ consist of a finite set of points. The map ft_k gives a bijection between these sets of points. We assert that, taking into account orientation, weights, and isotropy, the intersection numbers will be the same, i.e.

$$(\Pi_{i=1}^{k}ev_i \times \pi)(\mathcal{S}_{A,g,k}^{ft}) \cdot (\Pi_{i=1}^{k-1}X_i \times Q \times s_l(B)) = (\Pi_{i=1}^{k-1}ev_i \times \pi)(\mathcal{S}_{A,g,k-1}) \cdot (\Pi_{i=1}^{k-1}X_i \times B)$$

and therefore

$$\mathrm{GW}_{A,g,k}^{Q}(\alpha_1, \ldots, \alpha_{k-1}, [Q]; s_{l*}\beta) = \mathrm{GW}_{A,g,k-1}^{Q}(\alpha_1, \ldots, \alpha_{k-1}; \beta).$$

□

Divisor Axiom

Suppose (A, g, k) is not basic (Definition 1.2.1) and suppose that $\beta \in H_*(\overline{\mathcal{M}}_{g,k-1}^{log}; \mathbb{Q})$ satisfies the suborbifold representability condition (Definition 7.3.11). If $\deg(\alpha_k) = 2n-2$ *then*

$$GW_{A,g,k}^{Q}(\alpha_1, \ldots, \alpha_k; PD(ft_k^* PD(\beta))) = (A \cdot \alpha_k)\ GW_{A,g,k-1}^{Q}(\alpha_1, \ldots, \alpha_{k-1}; \beta),$$

where $A \cdot \alpha_k$ is given by the homological intersection product.

Proof. We interpret the GW-invariant as an intersection number via Definition 7.3.9. Consider the commutative diagram of maps given in the Fundamental Class Axiom.

By our conditions, we may assume that $\Pi_{i=1}^{k-1}\hat{ev}_i \times \hat{\pi} : \mathcal{S}_{A,g,k-1}(\overline{\partial}_J, \Lambda) \to Q^{k-1} \times \overline{\mathcal{M}}_{g,k-1}^{\log}$ is transverse to $\Pi_{i=1}^{k-1}X_i \times B$, and hence

$$(\Pi_{i=1}^{k-1}ev_i \times \pi)^{-1}(\Pi_{i=1}^{k-1}X_i \times B) \subset \mathcal{S}_{A,g,k-1}(\overline{\partial}_J, \Lambda)$$

consists of a finite set of points. By the requirement that the restriction of $\Pi_{i=1}^{k-1}\hat{ev}_i \times \hat{\pi}$ to the codimension $2i > 0$ stratum of $\mathcal{S}_{A,g,k-1}$ is transverse to $\Pi_{i=1}^{k-1}X_i \times B$, we may observe these points consist of *un-noded stable curves*, i.e.

$$(\Pi_{i=1}^{k-1}ev_i \times \pi)^{-1}(\Pi_{i=1}^{k-1}X_i \times B) = \{[\Sigma_m, j_m, M_m, \emptyset, u_m]\}_{m\in I}.$$

[2] By assumption, $\beta \in H_*(\overline{\mathcal{M}}_{g,k}^{\log}; \mathbb{Q})$ can be represented by a suborbifold $B \subset \overline{\mathcal{M}}_{g,k-1}^{\log}$ such that $\pi : \mathcal{S}_{A,g,k-1}(\overline{\partial}_J, \Lambda) \to \overline{\mathcal{M}}_{g,k-1}^{\log}$ is transverse to B. The lth-marked point doubling map is a smooth embedding of $\overline{\mathcal{M}}_{g,k-1}^{\log}$ into $\overline{\mathcal{M}}_{g,k}^{\log}$. It follows that the homology class $s_{l*}\beta \in H_*(\overline{\mathcal{M}}_{g,k}^{\log}; \mathbb{Q})$ can be represented by the suborbifold $s_l(B) \subset \overline{\mathcal{M}}_{g,k}^{\log}$, and moreover $\pi : \mathcal{S}_{A,g,k}^{ft}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda) \to \overline{\mathcal{M}}_{g,k}^{\log}$ is transverse to B.

The map $ft_k : \overline{\mathcal{M}}^{\log}_{g,k} \to \overline{\mathcal{M}}^{\log}_{g,k-1}$ is a submersion, and hence $ft_k^{-1}(B) \subset \overline{\mathcal{M}}^{\log}_{g,k}$ is a sub-orbifold. Moreover, by the argument in [32, Lemma 7.5.5], $[ft_k^{-1}(B)] = \mathrm{PD}(ft_k^*\mathrm{PD}(\beta))$. We now observe that $\Pi_{i=1}^{k-1}\hat{ev}_i \times \hat{\pi} : \mathcal{S}^{ft}_{A,g,k} \to Q^{k-1} \times \overline{\mathcal{M}}^{\log}_{g,k}$ is transverse to $\Pi_{i=1}^{k-1} X_i \times ft_k^{-1}(B)$, and moreover

$$(\Pi_{i=1}^{k-1} ev_i \times \pi)^{-1}(\Pi_{i=1}^{k-1} X_i \times ft_k^{-1}(B)) = \{[\Sigma_m, j_m, M_m \cup \{z_k\}, \emptyset, u_m]\}_m \subset \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda).$$

Now consider the restriction of the evaluation map $ev_k : \mathcal{S}^{ft}_{A,g,k}(\overline{\partial}_J, \mathrm{proj}_2^*\Lambda) \to Q$ to this set, i.e. $ev_k : \{[\Sigma_m, j_m, M_m \cup \{z_k\}, \emptyset, u_m]\}_m \to Q$. We can identify this restriction of ev_k with the map

$$\sqcup_m u_m : \sqcup_m \Sigma_m \to Q.$$

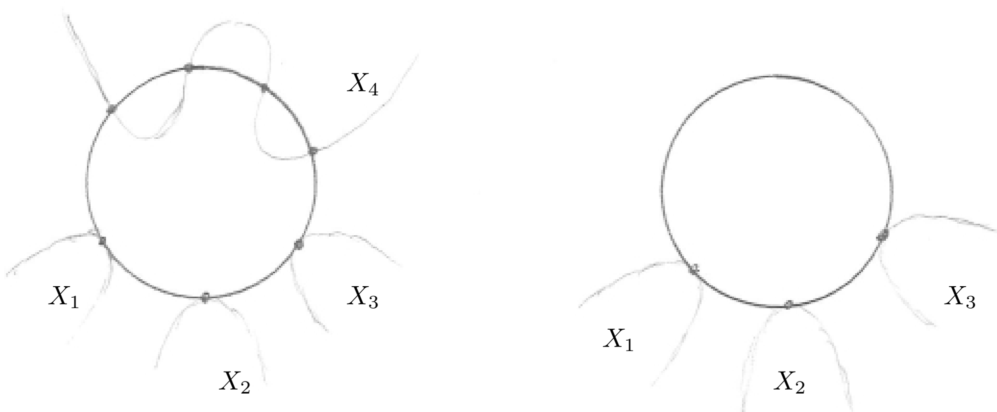
Choose a closed submanifold $X_k \subset Q$ such that $[X_k] = \alpha_k$ and such that X_k is transverse to the finitely many maps $u_m : \Sigma_m \to Q$. By the assumption $\deg(\alpha_k) = 2n-2$, these intersect in a finite set of points. Counted with sign, the intersection number $u_m(\Sigma_m) \cdot X_k$ is equal to the homological intersection product $A \cdot \alpha_k$.

It follows that $\Pi_{i=1}^{k} ev_i \times \pi : \mathcal{S}^{ft}_{A,g,k} \to Q^k \times \overline{\mathcal{M}}^{\log}_{g,k}$ is transverse to $\Pi_{i=1}^{k} X_i \times ft_k^{-1}(B)$. The intersection of X_k with the fiber $[\Sigma_m, j_m, M_m \cup \{z_k\}, \emptyset, u_m]$ contributes the additional factor $A \cdot \alpha_k$ to the intersection number over the point $[\Sigma_m, j_m, M_m, \emptyset, u_m]$. Therefore,

$$\mathrm{GW}^Q_{A,g,k}(\alpha_1, \dots, \alpha_k; \mathrm{PD}(ft_k^*\mathrm{PD}(\beta))) = (A \cdot \alpha_k)\ \mathrm{GW}^Q_{A,g,k-1}(\alpha_1, \dots, \alpha_{k-1}; \beta).$$

□

Figure 7.2: Divisor Axiom



$GW^Q_{A,0,4}(\alpha_1, \alpha_2, \alpha_3, \alpha_4; [\overline{\mathcal{M}}_{0,4}])$ $\quad$ $GW^Q_{A,0,3}(\alpha_1, \alpha_2, \alpha_3; [\overline{\mathcal{M}}_{0,3}])$

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