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A retrospective review of von Neumann's analysis of hidden variables in quantum mechanics

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Abstract

This article reviews the history of J. von Neumann's analysis of hidden variables in quantum mechanics and the subsequent analysis by others. In his book *The Mathematical Foundations of Quantum Mechanics*, published in 1932, von Neumann performed an analysis of the consequences of introducing hidden parameters (hidden variables) into quantum mechanics. He arrived at two principal conclusions: first, hidden variables cannot be incorporated into the existing theory of quantum mechanics without major modifications, and second, if they did exist, the theory would have already failed in situations where it has been successfully applied. This analysis has been taken as an "incorrect proof" against the existence of hidden variables, possibly due to a mistranslation of the German word *prufen*. von Neumann's so-called proof isn't even wrong as such a proof does not exist, but it is an examination of the limitations imposed by internal consistency of the Hilbert space formulation of the theory. One of the earliest attempts to eliminate uncertainty, by D. Bohm, requires a major modification of quantum mechanics (observables are not represented by Hermitian operators), which supports von Neumann's first principal conclusion. However, testing the Bohm theory requires constructing a physically impossible initial state. As such, the theory has no experimental consequences, so W. Pauli referred to it as an "uncashable check". As there are no observable consequences, the Bohm theory is possibly a counterexample to von Neumann's second conclusion that hidden variables in particular would have already led to a failure of the theory.

Keywords: *hidden variables, quantum statistics, density matrix, history*

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1. Background

The Mathematical Foundations of Quantum Mechanics (MFQM) by J. von Neumman, published in 1932, is a masterpiece of theoretical physics [1–3]. (We refer to the 1955 English translation except as noted.) In MFQM, von Neumann provided a complete exposition of the fundamentals of quantum mechanics as a linear operator theory, through Hermitean operators and Hilbert spaces. von Neumann further applied mathematical analysis to the problems of quantum theory, such as quantum statistical mechanics and the measurement processes, that continue to serve as the basis of the theory.

One of von Neumann's analyses in MFQM that has remained of central interest for nearly a century is the consequences of introducing hidden parameters (hidden variables) into quantum mechanics to eliminate dispersion in simultaneous measurements of non-commuting observables. He arrives at two principal conclusions: first, hidden variables (or equivalently, something to eliminate dispersion) cannot be incorporated into the existing theory of quantum mechanics without major modifications to the fundamental theory, and second, if they did exist, quantum mechanics would have already failed in situations where it has been successfully applied.

One of the earliest attempts to eliminate dispersion, by D. Bohm [4], represents a major modification of quantum mechanics, in that it expands beyond Hilbert space and observables are no longer the results of Hermitian operators. Bohm's theory is compatible with von Neumann's first principal conclusion. However, testing Bohm's theory requires constructing an initial state that is physically impossible, as it would involve violating the quantum mechanical uncertainty principle, and therefore, it has no testable consequences. This is possibly a counterexample to von Neumann's second principal conclusion, which states that hidden variables would lead to observable deviations from the usual form of quantum mechanics. Since the Bohm theory has no experimental consequences, W. Pauli referred to it as an "uncashable check".

To our knowledge, a successful hidden variable extension to quantum mechanics or other means to eliminate dispersion with testable consequences has not yet been produced, leading us to conclude that von Neumann's analysis is worthy of rehabilitation.

Our analysis follows a different tack than the recent work by Acuña [5] who gives a detailed overview of the intense debate concerned with the interpretation and validity of von Neumann's work, which has unfolded over several decades (see, e.g., [6, 7]). (Our

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discussion partly overlaps that of Jeffrey Bub [7].) He points out that the Bohm theory abandons the notion that physical observables are the result of Hermitian operators and therefore expands quantum theory beyond its Hilbert space formulation. That formulation is the foundational basis of von Neumann's analysis and he did not consider the possibility of further expansion which, as asserted, would be a major modification of the theory. As such, von Neumann does not provide proof, but he provides an investigation of the possible outcomes due to the introduction of hidden parameters and other modifications within the Hilbert space formulation of quantum mechanics, as we will discuss.

Dieks [6] writes, "According to what has become a standard history of quantum mechanics, in 1932 von Neumann persuaded the physics community that hidden variables are impossible as a matter of principle, after which leading proponents of the Copenhagen interpretation put the situation to good use by arguing that the completeness of quantum mechanics was undeniable. This state of affairs lasted, so the story continues until Bell in 1966 exposed von Neumann's proof as obviously wrong". Much earlier, Grete Hermann criticized von Neumann's analysis and raised objections similar to Bell's, although her work was only later fully appreciated [8].

In fact, in MFQM, von Neumann showed that adding hidden parameters to the existing theory leads to a logical contradiction. His analysis is essentially a *reductio ad absurdum* argument: if hidden variables exist, it would be possible to construct dispersion-free coordinate states, after showing that such states are not possible within the existing Hilbert space framework of quantum mechanics. He concluded that the present quantum theory would have already given false predictions if hidden variables existed. However, he left open the possibility of hidden parameters while recognizing that they would require a vastly modified theory.

In this article, we summarize von Neumann's discussion and suggest some new ways of looking at the situation. It is noted that the presentations in MFQM are sometimes rather opaque.

We have retained the page numbers of MFQM throughout the article. A mistranslation of the German original occurs on p. 210 of MFQM, with the original being, "Bis uns eine genauere Analyse der Aussagen der Quantenmechanik nicht in die Lage versetzen wird, die Möglichkeit der Einfürung verborgener Parameter objektiv zu prüfen, was am oben erfolgen wird, wollen wir auf diese Erklärungsmöglichkeit verzichten," for which the relevant part was translated to English as "prove objectively the possibility of the introduction of hidden parameters" (see p. 109 of the 1932 edition [1–3]). Prüfen does not mean prove, but it should be translated as "examine" or "test", which is precisely what von Neumann does in MFQM. (It should be noted that Hermann read the original German, so it is difficult to assess the overall consequences of the mistranslations.)

2. von Neumann's analysis of hidden parameters

2.1. Overview

von Neumann's analysis progresses in several steps and is centered on the density matrix. In MFQM, he placed quantum mechanics on a firm mathematical footing based on operators and vectors in Hilbert space and showed that it did not need to rely on the then mathematically dubious delta-function introduced by Dirac. He also put considerable effort into the discussion of the possibility that the statistical behavior associated with the quantum states might be due to fluctuations in some unknown parameters (hidden variables), whose variations would lead to random behavior, just as averaging over the positions and velocities of the individual molecules leads to the statistical behavior in classical statistical mechanics.

von Neumann considers quantum mechanics to be characterized by the relation for the expectation value of a physical variable represented by the Hermitian operator $\mathbf{R}$, in the state represented by the Hilbert space vector $|\phi\rangle$:

$$\langle R\rangle = \langle \phi|\mathbf{R}|\phi\rangle . \tag{1}$$

This is the inner product of the state $|\phi\rangle$ with the state $\mathbf{R}|\phi\rangle$.

We expand $|\phi\rangle$ in a complete orthonormal set $|\psi_j\rangle$:

$$|\phi\rangle = \sum_j |\psi_j\rangle \langle\psi_j|\ \phi\rangle \ , \tag{2}$$

so that

$$\begin{aligned} \langle \phi|\mathbf{R}|\ \phi\rangle &= \sum_{j,k} \langle\phi|\ \psi_j\rangle \langle\psi_j|\mathbf{R}|\psi_k\rangle \langle\psi_k|\ \phi\rangle & (3)\\ &= \sum_{j,k} \langle\psi_k|\ \phi\rangle \langle\phi|\ \psi_j\rangle \langle\psi_j|\mathbf{R}|\psi_k\rangle & (4)\\ &= \sum_{j,k} \langle\psi_k|\rho|\psi_j\rangle \langle\psi_j|\mathbf{R}|\psi_k\rangle = \sum_{j,k}\rho_{kj}R_{jk} & (5)\\ &= \mathrm{Tr}\,(\rho\mathbf{R}) = \langle R\rangle\,, & (6)\end{aligned}$$

where ρ is the projection operator onto the state $|\phi\rangle$ given by

$$\rho = |\phi\rangle\langle\phi| . \tag{7}$$

The operator ρ is called the density matrix or statistical operator and represents all the information available about the quantum state, equivalent to the wave function. It allows the introduction of statistical mixtures of different states.

Equation (7) holds when the system is in a single quantum state, called a pure state. In the case when the system is described by a statistical ensemble consisting of systems in various states $|\phi_i\rangle$ with probabilities w_i, what is called a mixed state, the density operator is given by

$$\rho = \sum_i w_i |\phi_i\rangle\langle\phi_i| \ . \tag{8}$$

If Eq. (1) is considered an assumption it can never be disproved, so von Neumann took one step back and replaced Eq. (1) by a set of assumptions, which can lead to Eq. (6). Without specifying a definite method for calculating expectation values, he listed several properties he would expect to be features of such a method. Among these are (where $\langle\ldots\rangle$ represents the expectation value)

$$\langle\alpha\mathbf{R}\rangle = \alpha\langle\mathbf{R}\rangle\,, \tag{9}$$

where α is a number and

$$\langle\mathbf{R}+\mathbf{S}+\mathbf{T}+\cdots\rangle = \langle\mathbf{R}\rangle + \langle\mathbf{S}\rangle + \langle\mathbf{T}\rangle + \cdots\,, \tag{10}$$

von Neumann was well aware of the problem connected with the addition of operators that do not commute (see, e.g., MFQM

[1–3], p. 298; we refer to the English translation). Equation (10) is certainly valid in quantum mechanics since it follows from Eq. (6), but to cover the more general case, von Neumann defined the sum of the operators $\mathbf{R} + \mathbf{S} + \mathbf{T} + \ldots$ to be that operator, which had the expectation value given by the right-hand side (RHS) of Eq. (10) (i.e., an implicit definition). Indeed, he considered explicitly the case of the Hamiltonian for the electron in a hydrogen atom, in which

$$\begin{aligned} E &= \langle \mathbf{H} \rangle = \left\langle \frac{p^2}{2m} - \frac{e^2}{r} \right\rangle \\ &= \left\langle \frac{(p_x^2 + p_y^2 + p_z^2)}{2m} \right\rangle - \left\langle \frac{e^2}{\sqrt{x^2+y^2+z^2}} \right\rangle . \end{aligned} \quad (11)$$

Measurement of the first term on the RHS of Eq. (11) requires a momentum measurement, while that of the second term is a coordinate measurement. The sum is measured in an entirely different way. Each measurement requires an entirely different apparatus. The variables p_i and x_i (and hence functions of them) cannot be simultaneously and precisely determined due to their noncommutativity. Nonetheless, as in the present case of the hydrogen atom, the sum of $p^2/2m$ and $-1/r$ is defined and has a precise value, which is the energy. The role of a hidden variable theory is to allow the determination of the values of p and r without any statistical fluctuations, as can be done in classical mechanics, and provide an internal framework that corresponds to classical expectations. In such a theory, quantum mechanical uncertainty would arise simply due to our inability to see these hidden parameters or variables.

von Neumann then proceeded to derive Eq. (6) from Eqs. (9), (10), and several other assumptions as follows.

Obviously Eq. (10) follows from the trace rule (Eq. 6) so that in any case where Eq. (10) fails, Eq. (6) and hence quantum mechanics will also not be valid.

In a simplified approach, we start with the representation of an operator by its matrix elements:

$$\mathbf{R} = \sum_{mn} |m\rangle \langle m| \mathbf{R} |n\rangle \langle n| = \sum_{mn} |m\rangle \langle n| R_{mn} , \quad (12)$$

von Neumann takes the expectation value of both sides (note that this applies to any method of taking expectation values that satisfies Eqs. (9) and (10)):

$$\langle \mathbf{R} \rangle = \left\langle \sum_{mn} |m\rangle \langle n| R_{mn} \right\rangle = \sum_{mn} \langle |m\rangle \langle n| \rangle R_{mn} \quad (13)$$

$$= \sum_{mn} \rho_{nm} R_{mn} = \mathrm{Tr}\,(\rho \mathbf{R}) , \quad (14)$$

where we defined $\rho_{nm} = \langle |m\rangle \langle n| \rangle$ as the physical quantity represented by that operator $|m\rangle \langle n|$.

2.2. Implications of hidden variables

If the statistical variations in experimental results were due to averaging over unknown "hidden variables", the ensembles described by quantum states would consist of separate subensembles, each with some exact value of all physical variables. These values would have to be eigenvalues of the corresponding operators for the results to agree with observations, which are found to always yield eigenvalues, in agreement with the current version of quantum mechanics. In addition, it would be possible, at least in principle, to separate these subensembles, so that each of the separate ensembles would have the property that all variables had exact values and there would be no scatter in the measured values of observables. von Neumann called such states *dispersion-free states*, he called the resulting subensembles *homogeneous ensembles*. A "homogeneous ensemble" is an ensemble that can only be divided into subensembles that are identical to the original ensemble. If the statistical behavior associated with the quantum states was due to hidden variables, these quantum states would represent "inhomogeneous ensembles".

2.3. "Dispersion-free" states and homogeneous ensembles in quantum mechanics

Starting with Eq. (6), von Neumann wrote the dispersion (mean square fluctuation; cast in modern notation by James Albertson [9]) of the variable represented by the operator $\mathbf{R}$ as

$$\sigma^2 = \mathrm{Tr}\left[\rho(\mathbf{R} - \langle \mathbf{R} \rangle)^2\right] = \mathrm{Tr}\left[\rho(\mathbf{R}^2 - \langle \mathbf{R} \rangle^2)\right] = \langle \mathbf{R}^2 \rangle - \langle \mathbf{R} \rangle^2 \quad (15)$$

because $\langle \mathbf{R} \rangle$ is a number. σ^2 is, in general, non-zero. It is zero if $\mathbf{R}|\phi\rangle = R|\phi\rangle$, which means that $|\phi\rangle$ is an eigenstate of $\mathbf{R}$, and the system is in that eigenstate.

We call a state "dispersion-free" when from Eq. (15)

$$\mathrm{Tr}\left[\rho \mathbf{R}^2\right] = (\mathrm{Tr}\left[\rho \mathbf{R}\right])^2 \quad (16)$$

for all Hermitian measurement operators $\mathbf{R}$.

von Neumann then considered $\mathbf{R}$ to be the projection operator onto a state $|\phi\rangle$, so that $\mathbf{R} = \mathbf{P}_\phi = |\phi\rangle\langle\phi|$. When $\mathbf{P}_\phi$ operates on *any* state $|\Psi\rangle$ that is formed from a complete set of eigenfunctions that include $|\phi\rangle$,

$$(\mathbf{P}_\phi)^N |\Psi\rangle = \mathbf{P}_\phi |\Psi\rangle = c_\phi |\phi\rangle , \quad (17)$$

which means that the same result is obtained when $\mathbf{P}_\phi$ is applied multiple times.

With $\mathbf{R} = \mathbf{P}_\phi$ in Eq. (16) (note: $\mathrm{Tr}\left[\rho \mathbf{P}_{[\phi]}\right] = \langle\phi|\rho|\phi\rangle$),

$$\begin{aligned} \mathrm{Tr}\left[\rho \mathbf{R}^2\right] &= (\mathrm{Tr}\left[\rho \mathbf{R}\right])^2 = \mathrm{Tr}\left[\rho \mathbf{P}_{[\phi]}^2\right] \\ &= \mathrm{Tr}\left[\rho \mathbf{P}_{[\phi]}\right] = \left(\mathrm{Tr}\left[\rho \mathbf{P}_{[\phi]}\right]\right)^2 \end{aligned} \quad (18)$$

$$\langle\phi|\rho|\phi\rangle = \langle\phi|\rho|\phi\rangle^2 \quad (19)$$

for all $|\phi\rangle$. Therefore,

$$\langle\phi|\rho|\phi\rangle = 0 \;\text{ or }\; \langle\phi|\rho|\phi\rangle = 1 . \quad (20)$$

This should hold for any normalized state $|\phi\rangle$. Taking $\phi = c_1|\phi_1\rangle + c_2|\phi_2\rangle$, we vary c_1 and c_2 in a continuous manner, so that $|\phi\rangle$ starts at $|\phi_1\rangle$ and ends at $|\phi_2\rangle$ (MFQM, pp. 320–321). During this process, the relation $\langle\phi|\rho|\phi\rangle = 0$ or 1 must hold over the entire variation, and we therefore conclude that

$$\rho = 0 \;\text{ or }\; \rho = 1 \quad (21)$$

for a dispersion-free ensemble. This means that ρ is a diagonal matrix with elements all 0 or all 1. The case of all zero diagonal elements is trivial because in this case $\langle \mathbf{R} \rangle = 0$ always, which is not possible in a realistic system.

The obvious dispersion-free case of ρ being in a pure state of the $\mathbf{R}$ basis is possible for one or even for several measurement operators **R**, but it cannot simultaneously be true for *all* measurement operators since some of them do not commute. Thus, such a state would not be dispersion-free because it would still show dispersion in the measurements of some observables.

In the case of $\rho = 1$, von Neumann tells us (top of MFQM, p. 321) that there is a normalization problem, in that

$$\mathrm{Tr}\,(\rho) = N \to \infty, \tag{22}$$

where N is the number of elements (dimension of the state space). However, this sum should be unity if ρ represents the system average density matrix. This is because the diagonal matrix elements are the probabilities of being in each eigenstate, and

$$\sum_n w_n = 1 = \mathrm{Tr}\,(\rho)\,. \tag{23}$$

This would seem to preclude the possibility of constructing a general density matrix with $\rho = 1$ because it obviously cannot be normalized. However, we read (MFQM, p. 310, where Exp[**R**] is the expectation value):

> ... we shall admit not only $\langle \mathbf{R} \rangle = \mathrm{Exp}(\mathbf{R})$ functions representing expectation values, but also functions corresponding to relative values – i.e. we allow the normalization condition ($\mathrm{Exp}(1) = 1$) to be dropped. ....($\mathrm{Exp}(1) = \infty$) is an entirely different matter and it is actually for this sake that we want this extension...

and (MFQM, p. 320, and $U = \rho$ in our notation) that "It is for an infinite $\mathrm{Tr}(U)$ only that we have essentially relative probabilities and expectation values." So it seems infinite traces are not *a priori* forbidden. This together with the fact that $\rho = 1$ is not dispersion-free, while it is supposed to be a solution for zero dispersion, raises some questions about the relevance of this solution. If instead of considering the dispersion for an observable represented by a projection operator, we focus on a general operator with $\mathbf{R}^2 \neq \mathbf{R}$, the only solution to Eq. (16) valid for all **R** is $\rho = 0$.

Thus, von Neumann concluded that if the assumptions Eqs. (9) and (10) hold, dispersion-free states do not exist. The impossibility of dispersion-free states is related to the Heisenberg uncertainty principle. Non-commuting observables cannot be dispersion-free in the same state and the commutation relations prescribe a limit to the possible accuracy of their measurement.

2.4. No hidden variables "theorem"

After investigating the possibility of dispersion-free states, von Neumann studied uniform or homogeneous ensembles, that is, ensembles that can be divided into subensembles, which would all give the same expectation value for every physical quantity. In other words, all physical quantities would have the same probability distribution in every subensemble. He then shows that such ensembles exist and they are pure quantum states and hence not dispersion-free. As this argument does not seem to have been challenged, we have placed it in an the Supplementary materials. If the dispersion shown in the homogeneous ensembles (quantum states) was due to some hidden variables with different values, it would be possible to separate them into subensembles according to the different values of these hidden variables in contradiction to their property of being homogeneous. Either of the two results would rule out hidden variables coexisting with quantum mechanics. Dispersion-free states are impossible and the quantum states are homogeneous ensembles so it is impossible, according to quantum mechanics, to break up a quantum mechanical ensemble into subensembles with different physical properties. According to von Neumann:

> But this [the existence of hidden variables] is impossible for two reasons: First, because then the homogeneous ensemble in question could be represented as a mixture of two different ensembles, contrary to its definition. Second, because the dispersion-free ensembles, which would have to correspond to the "actual" states (i.e., which consist only of systems in their own "actual" states), do not exist. (MFQM, p. 324)

von Neumann summarizes his argument:

> It should be noted that we need not go any further into the mechanism of the "hidden parameters" since we now know that the established results of quantum mechanics can never be rederived with their help. In fact, we have even ascertained that it is impossible that the same physical quantities exist with the same functional connections, if other variables (i.e., "hidden parameters") should exist in addition to the wave functions. Nor would it help if there existed other, as yet undiscovered, physical quantities, in addition to those represented by the operators in quantum mechanics, because the relations assumed by quantum mechanics (i.e., **I.**, **II.**) would have to fail already for the currently known quantities, those that we discussed above. It is therefore not, as is often assumed, a question of a re-interpretation of quantum mechanics,—the present system of quantum mechanics would have to be objectively false, in order that another description of the elementary processes than the statistical one be possible. [MFQM (1), p. 324. Relations **I.** and **II.** are found on pp. 313–314 of MFQM (1) and correspond to our Eqs. (9) and (10). It is clear that von Neumann implies that he is employing all previously used conditions, **A'.**, **B'.**, α), β) defined on pp. 311–312 of MFQM(1), in addition to **I.** and **II.**, as stated on p. 323.]

3. Reactions to von Neumann's discussion of hidden variables

von Neumann's discussion of hidden variables unleashed a vigorous debate that has lasted almost a hundred years. A reasonably comprehensive summary has been given by Acuña [5].

A year after the publication of MFQM, Grete Hermann, a philosophy student who was defending the philosophical tradition that causality was a necessary constituent for any scientific view of the world, produced a criticism of von Neumann's argument [8]; however, the current popular notion that the development of quantum mechanics would have followed a different course had her works been appreciated is not supported by further development of her views. She'd had extensive discussions with Heisenberg, and with von Weizsäcker (for details see [6]; she eventually came to accept the statistical nature of the theory). Hermann called attention to the fact that the assumption

$$\text{Exp}[a\mathbf{R} + b\mathbf{S}] = a\text{Exp}[\mathbf{R}] + b\text{Exp}[\mathbf{S}]$$

fails for quantities that are not simultaneously measurable (their operators do not commute). She then noted that von Neumann had recognized this fact and implicitly defined the operator for the sum of physical quantities as that operator whose expectation value was the sum of the expectation values on the RHS. She deduces from this that von Neumann needs another "proof for quantum mechanics". Then she claimed that he finds this in the use of

$$\langle \mathbf{R} \rangle = \text{Exp}[\mathbf{R}] = \langle \phi | \mathbf{R} | \phi \rangle$$

for the expectation value of **R** in the state $|\phi\rangle$. For this definition, the expectation value of a sum is given by the sum of the expectation values. However, this only holds for those ensembles whose definition depends only on those physical quantities that are considered in the present quantum theory, those quantities that determine the state $|\phi\rangle$. She states that von Neumann's argument for the necessity of dispersion applies only to such ensembles. It has not been shown that the expectation value has the above form for ensembles that agree not only in the state $|\phi\rangle$ but also in possibly yet to be discovered, presently unknown, quantities (hidden variables). A physicist who only knows a given system by its Schrödinger wave function is bound to find themself limited by the uncertainty principle. For the von Neumann proof to hold, we must assume that $\langle \phi | \mathbf{R} | \phi \rangle$ represents the average value of measurements in any ensemble whose elements agree with each other not only with respect to $|\phi\rangle$ but also with respect to arbitrary as yet undiscovered quantities. Hermann stated,

> That all these ensembles have the same average value is an assumption justified neither by previous experience nor by the hitherto confirmed theory of quantum mechanics. Without it, the proof of indeterminism collapses.

This argument was repeated in a slightly different form in her "Natural-Philosophical Foundations of Quantum Mechanics" whose English translation has been published in Chapter 15 of [8]. Later, Hermann came to accept that indeterminism as a necessary consequence of quantum mechanics, and that the statistical distribution can be precisely calculated. It is now recognized that the topic of Hermann's work that has drawn the most popular attention (the critique of von Neumann's no hidden variable proof) was not actually a primary motivation, nor a major conclusion [10].

This work remained largely unknown and the von Neumann proof was generally accepted as providing strong support for the Copenhagen interpretation of quantum mechanics until in 1952 David Bohm [4] reinvented a theory that had been proposed by de Broglie in 1927. For details of these theories, see [11, 12]. The Bohm theory treats the position of a particle as exactly knowable, but determining this with more accuracy than allowed by quantum mechanics requires an initial state that violates the uncertainty principle. Because of this, Pauli called it a "check that can't be cashed" ([13], Pauli to Bohm, Dec. 3, 1953, p. 436, no. 13).

Bohm's theory led to an outbreak of papers analyzing and criticizing von Neumann's "proof." The most influential of these was a series of papers by John Bell. His reaction to reading Bohm's paper was as follows:

> But in 1952 I saw the impossible done. It was in papers by David Bohm. Bohm showed explicitly how parameters could indeed be introduced into nonrelativistic wave mechanics, with the help of which the indeterministic description could be transformed into a deterministic one. More importantly, in my opinion, the subjectivity of the orthodox version, the necessary reference to the "observer" could be eliminated [14]. This [assumption (10)] is true for quantum mechanical states, *it is required by von Neumann of the hypothetical dispersion free states also* [emphasis added]. At first sight, the required additivity of expectation values seems very reasonable, and it is the non-additivity of allowed values (eigenvalues) which requires explanation. Of course, the explanation is well known: A measurement of a sum of noncommuting variables cannot be made by combining trivially the results of separate operations on the two terms—it requires a quite distinct experiment. For example the measurement of σ_x for a magnetic particle might be made with a suitably oriented Stern-Gerlach magnet. The measurement of σ_y would require a different orientation and the measurement of $(\sigma_x + \sigma_y)$ a third and different orientation. But this explanation of the non-additivity of allowed values also established the non-triviality of the additivity of expectation values".

In another paper [15], Bell writes,

> The latter [our Eq. (10)] is a quite peculiar property of quantum mechanical states, not to be expected *a priori*. There is no reason to demand it individually of the hypothetical dispersion-free states, whose function it is to reproduce the *measurable* properties of quantum mechanics when *averaged over*. [emphasis added]

David Mermin [16] characterized the assumption (Eq. 10) as "silly" and quoted Bell in a published interview:

> Yet the von Neumann proof, if you actually come to grips with it, falls apart in your hands! There is *nothing* to it. It's not just flawed, it's *silly*! . . . When you translate [his assumptions] into terms of physical disposition, they're nonsense. The proof of von Neumann is not merely false but *foolish*! [17]

Bell's point, quoted above with emphasis, is correct: In dispersion-free states, the expectation value of a variable would have to be one of the eigenvalues of the corresponding operator, and so Eq. (10) could not apply. But contrary to Bell's point above, von Neumann's argument does not require Eq. (10) to be true for dispersion-free states.

To reiterate, von Neumann's logical argument, which has the appearance of circularity (sort of a *reductio ad absurdum*), is based on the following three propositions:

- **A.** The sum of expectation values assumption, Eq. (10).
- **B.** $\langle \mathbf{R} \rangle = \text{Tr}(\rho\mathbf{R})$, Eq. (14) that is, the trace of ρ times an operator gives the expectation value of that operator.
- **C.** Dispersion-free states do NOT exist.

von Neumann has shown that $\mathbf{A} \Rightarrow \mathbf{B} \Rightarrow \mathbf{C}$. If this conclusion is taken as the argument against hidden variables, then Bell, Hermann, and others are correct because **A** does not hold for dispersion-free states, and it is, therefore, a circular, silly argument [6]. However, let us look at the logic. It is obvious that $\mathbf{B} \Rightarrow \mathbf{A}$ so that if **A** were false (which is the case for a dispersion-free state), **B** would have to be false, that is, the predictions of quantum mechanics would have to fail for already known quantities as von Neumann stated. Moreover, assumption **A** is only necessary for deriving **B**. But nobody would deny that **B** represents the essence of quantum mechanics. Since $\mathbf{B} \Rightarrow \mathbf{C}$, if **C** were false, that is, there were dispersion-free states, **B** would have to be false and again the conclusion is that quantum mechanics would have to fail with the existence of hidden parameters.

4. Conclusions

We have shown that examination of the logic of von Neumann's argument leads to the conclusion that the existence of hidden variables capable of allowing the exact prediction of all physical quantities would mean that quantum mechanics in its present form would have to be false, that is, the existence of hidden variables would contradict quantum mechanics, and their inclusion requires a vastly modified theory. Of course, this follows already from the fact that physical quantities represented by non-commuting operators must satisfy an uncertainty relation.

Another powerful argument against hidden variables has been presented by Pauli. In a letter to Fierz he wrote ([18, 19] Pauli to Fierz, Jan. 6, 1952, p. 499, no 1337):

> I want to call special attention to the thermodynamics of ensembles, consisting of the same type of subensembles (Einstein-Bose or Fermi-Dirac statistics). What is important to me is not the energy values but the statistical weights, further the indifference of the thermodynamic-statistical reasoning to the "wave-particle" alternative and Gibbs' point that *identical* or only similar states behave qualitatively differently. If hidden parameters exist, not only on paper, but determine a really different behavior of different single systems (e.g.particles)—according to their "real" values—so must—completely independent of the question of the technical measurability of the parameters—the Einstein-Bose or Fermi-Dirac statistics be completely disrupted. Since there is no basis to assume that the thermodynamic weights should be determined by only half (or a part of) "reality". Either two states are identical or not (there is no "similar") and if the ψ function is not a complete description of single systems, states with the same ψ function will not be identical. Every argument with the goal of saving the Einstein-Bose and Fermi-Dirac statistics from the causal parameter mythology must fail because it - taking into account the usual theory in which the ψ function is a complete description of a state—declares the other half of reality to be unreal.

In the case of Bohm's theory, each particle follows its own trajectory, and the wavefunction is in $3N$ dimensional space and is even or odd under particle exchange. The hidden variable is the initial position of each particle, which is not an additional degree of freedom because in ordinary quantum mechanics the particle positions are already considered the degrees of freedom. For identical particles, when the initial wavefunction is specified, the even or odd superposition results from the creation of a multiparticle state at each initial position. Overall, there is no surprise that the system evolution is indistinguishable from the usual formulation of quantum mechanics.

In the Supplementary materials, we review von Neumann's second attempt to eliminate dispersion by decomposition of the density matrix into sub-assemblies. However, this approach also fails because the homogeneous ensembles are considered quantum states and, therefore, are not dispersion-free.

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Conflict of interest

The authors declare no conflict of interest.

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