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A Fast Algorithmic Way to Calculate the Degree Growth of Birational Mappings

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Abstract: We present an algorithmic method for the calculation of the degrees of the iterates of birational mappings based on Halburd's method for obtaining the degrees from the singularity structure of the mapping. The method uses only integer arithmetic with additions and, in some cases, multiplications by small integers. It is therefore extremely fast. Several examples of integrable and non-integrable mappings are presented. In the latter case, the dynamical degree we obtain from our method is always in agreement with that calculated by previously known methods.

Keywords: mappings; integrability; singularities; degree growth; dynamical degree

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1. Introduction

Integrability is a rare phenomenon. This fact, linked to the nice properties it is generally associated with, led at the beginning of the 20th century to a profusion of studies [1–5] dealing almost exclusively with ordinary differential equations. Subsequently however, the interest in integrability waned considerably, and its revival only came with the discovery of the soliton and of integrable partial differential equations in the 1960s [6–9]. The main bulk of the ensuing studies again concerned differential systems, but notable exceptions included the works of Hirota [10,11], who singlehandedly derived discrete analogues of all the classical integrable evolution equations. His studies remained mainly ignored till the 1990s, when the interest of the integrable community shifted towards discrete systems, giving rise to an explosion of results in the domain.

Put in very vague terms, integrability is associated with the existence of a 'sufficient number' of conserved quantities. (The authors are of course aware that for the Painlevé equations, to which they have devoted several studies [12], the relation of integrability to conserved quantities is rather tenuous). While the existence of conserved quantities does (again with the appropriate caveats) guarantee integrability, it is not of great use when one is facing the question whether a given system is integrable or not. What is really needed is an easily implementable integrability criterion.

By analogy to the continuous differential systems, where singularities play a capital role in integrability [13], it was considered that the structure of singularities of a discrete system could provide useful information on its integrable or non-integrable character. All the more so, it was observed that several systems, all of them integrable through spectral methods, had a very special singularity structure: a singularity that appeared at some iteration step, due to a special choice of initial conditions, disappeared again after a few more iterations. This property was dubbed singularity confinement [14] and has been extensively used as a discrete integrability criterion ever since. For example, the derivation of the discrete analogues of the Painlevé equations [15] largely proceeded based on it. However, it turned out that the singularity confinement property is not a sufficient condition [16] for integrability. In fact, infinitely many non-integrable discrete systems exist, having confined singularities. What was therefore needed was a more refined way of analysing the singularity confinement constraints. This was done in the approach we dubbed full-deautonomisation [17], which allows one to distinguish between integrable and non-integrable discrete systems with confined singularities. A rigorous, algebro-geometric justification of the full-deautonomisation method was given in [18].

Another approach to the detection of discrete integrability is based on the formal identification of discrete systems and delay equations. The studies of Ablowitz and collaborators [19] have shown that at the continuum limit, when a delay equation goes over to a differential one, the singularities of the latter that are at finite distance can be associated with the behaviour of the solutions of the discrete system at infinity. What ensued was an approach aiming at studying this behaviour using the arsenal of the Nevanlinna theory [20]. We will not discuss further the success of the Nevanlinna approach for the characterisation of discrete integrability. The point we shall retain from these studies is that integrability is incompatible with too fast an asymptotic growth of the solutions of the discrete system.

Veselov [21] put this in a more general setting by stating that '[discrete] integrability has an essential correlation with the weak growth of certain characteristics' based on a statement by Arnold [22] introducing the notion of complexity for mappings on the plane. The latter is defined as the number of intersection points of a fixed curve with the images of a second curve under the n-th iteration of the mapping. In fact, when it comes to mappings on the plane, Diller and Favre [23] have presented a classification of the possible degree growths (something equivalent to Arnold's complexity) of the iterates of (autonomous) second-order birational autonomous mappings. Three integrable cases can be distinguished:

– The degree growth is bounded. In this case, the mapping is either periodic or can be transformed into a projective mapping after some birational transformations.
– The degree grows linearly with n. In this case, the mapping is linearisable, as it preserves a rational fibration. Moreover, such a mapping has at least one non-confined singularity.
– The degree grows quadratically with n. In this case, the mapping is considered to be integrable, as it always preserves an elliptic fibration, and moreover, it always enjoys the singularity confinement property.

Non-integrable mappings, on the other hand, have exponential degree growth and have no typical preserved structures, but as mentioned above, they may or may not possess the singularity confinement property. (A classification, for confining non-autonomous mappings, similar to the above, is given in [24]).

A simple way to encode these different behaviours is through the dynamical degree. If d_n represents the degree of the nth iterate of a rational mapping, then the dynamical degree is defined as

$$\lambda = \lim_{n\to\infty} d_n^{1/n}. \tag{1}$$

Integrable mappings have a dynamical degree equal to 1, while a dynamical degree larger than 1 signals non-integrability.

In a nutshell, if one wishes to make predictions concerning the integrable character of a discrete system, it suffices to examine its complexity properties. For a rational mapping, this is rather straightforward: introduce appropriate initial conditions, iterate them, and compute the degree of a suitably chosen variable in the numerator and denominator of the rational expression obtained. This paper is devoted to the calculation of the successive degrees obtained when iterating a birational mapping. A fast algorithmic approach is presented based on the theory proposed by R. Halburd in [25]. In what follows, we start with a short refresher on how the degrees are usually calculated, and then we proceed to the presentation of our method.

2. A Refresher

Suppose one is given a birational mapping and wishes to make a prediction concerning its integrability. The simplest way to do this is by computing its dynamical degree. At this point, it should be stressed that there is no theorem like the one of Diller and Favre concerning mappings of orders higher than two, so concluding that the mapping is not integrable once the dynamical degree turns out to be larger than 1 may sound somewhat arbitrary. Still, we believe this assumption to be a reasonable one.

By far the simplest way to compute the dynamical degree of a mapping is by using the Diophantine method introduced by Halburd [26]. Here is how the method works. One starts from initial conditions that are simple rational numbers and then computes the successive fractions obtained by iterating the mapping. The calculations are simple, all the more so since one does not have to manipulate any symbolic variables. From the successive iterates of the mapping, obtained as fractions of the form N_n/D_n, one computes the successive approximations of the dynamical degree as $\log N_n/\log N_{n-1}$ (or equivalently $\log D_n/\log D_{n-1}$). If the mapping is integrable, the dynamical degree converges (in general, very slowly) to 1. One typically needs hundreds of additional iterations in order to gain an order of magnitude in accuracy, but the calculations in this case are rather fast (since the size of the fractions involved grows slowly). In the case of non-integrable mappings, the convergence is much faster, but, since the growth of the size of the fractions involved is exponential, the calculations can easily become prohibitive. Let us illustrate these points with two tangible examples.

We start with a well-known integrable mapping:

$$x_{n+1}x_{n-1} = 1 + \frac{1}{x_n}. \tag{2}$$

We choose some integer values for x_0 and x_1 and compute the dynamical degree using the Diophantine prescription. After 500 iterations, we find $\lambda \approx 1.004$ which becomes 1.002 after 1000 iterations, with the calculation being 30 times longer this time. However it is clear that the value of the dynamical degree converges to 1 in agreement with the integrable character of the mapping.

Next, we examine the non-integrable mapping [27]:

$$x_{n+1}x_{n-1} = x_n + \frac{1}{x_n}. \tag{3}$$

In [17], we obtained the exact expression of its dynamical degree $(1+\sqrt{17})/4 + \sqrt{(1+\sqrt{17})/8}$, with an approximate value of 2.081019. Using the Diophantine method, we find that after 10 iterations, only the first decimal after 2, i.e., 2.0, has converged. Pushing the calculation to 20 iterations gives a value where four decimals have converged, i.e., 2.0810, but the time necessary for this calculation is a few thousand times longer. Of course, with four converged decimals, one does not really need to push the calculation further.

Similar behaviour is observed in every case, be it of order two or higher, that we have studied. The Diophantine method is a reliable tool for the obtention of the dynamical degree. However, it does not provide access to the degrees themselves and can lead to prohibitively long calculations in the case of non-integrable mappings. Finding the exact degree growth is in fact quite simple. We shall illustrate this using the two mappings introduced above while also pointing out the relation of the growth to the singularity structure of the mapping.

We start from (2) and take initial conditions $x_0 = 1$ and $x_1 = z$. We then iterate the mapping and find the following sequence of iterates (where $P_n(z)$ represents a polynomial of degree n in z):

$$x_2 = \frac{z+1}{z}$$

$$x_3 = \frac{2z+1}{z(z+1)}$$

$$x_4 = \frac{zP_2(z)}{(z+1)(2z+1)}$$

$$x_5 = \frac{(z+1)P_3(z)}{(2z+1)P_2(z)}$$

$$x_6 = \frac{(z+1)(2z+1)P_4(z)}{z(z+1)P_2(z)P_3(z)}$$

$$x_7 = \frac{(z+1)^2(2z+1)P_2(z)P_5(z)}{(z+1)^2(2z+1)P_3(z)P_4(z)}$$

$$x_8 = \frac{z(z+1)^3(2z+1)^2P_2(z)P_3(z)P_6(z)}{(z+1)^3(2z+1)^2P_2(z)P_4(z)P_5(z)},$$

where we have yet to implement all possible simplifications between the numerators and denominators in the above rational functions. The degrees computed as the maximum of the degrees in z of these numerators and denominators, before simplification, are 0, 1, 1, 2, 3, 4, 7, 10, and 17. Once the simplifications are carried out however, the degrees become 0, 1, 1, 2, 3, 4, 6, 7, and 10. In order to understand how these simplifications come about, we must discuss the singularities and indeterminacies of the mapping. Suppose that z takes the value -1. Then, x_2 vanishes and the successive values of x_n are $-1, 0, \infty, \infty, 0, -1$, and the subsequent values are regular. Thus, the value -1 for x_1 introduces a singularity, but the latter is confined, disappearing after five steps. We remark that this is only possible if at that iterate the mapping becomes indeterminate, which is the same as saying that the numerator and denominator at that iterate have a certain number of common factors. Hence, simplifications start appearing in the successive iterates once the singularity is confined, and they have as a result that the growth, which would have been exponential (with a dynamical degree equal to $(1+\sqrt{5})/2)$ in their absence, becomes polynomial, leading to a dynamical degree equal to 1. (Note that, contrary to the above, if we assume now that z vanishes, we obtain the following pattern $0, \infty, \infty, 0, \bullet, \infty, \bullet, 0$, where $\bullet$ stands for a finite value. In fact, pursuing the iterations, we find that the pattern $\{\bullet, \infty, \bullet, 0, \infty, \infty, 0\}$ repeats indefinitely; it is what we call a cyclic pattern).

While obtaining the degrees directly from the iterates of the mapping is straightforward, it becomes easily unmanageable since one has to perform simplifications of polynomials of growing degrees. The situation becomes still worse in the case of non-integrable mappings. We show this in the case of mapping (3). With the same initial conditions 1 and z, we find the following succession of x_n (where, again, Q_n is a polynomial of degree n in z):

$$x_2 = \frac{z^2+1}{z}$$

$$x_3 = \frac{Q_4(z)}{z^2(z^2+1)}$$

$$x_4 = \frac{Q_8(z)}{z(z^2+1)^2 Q_4(z)}$$

$$x_5 = \frac{zQ_{18}(z)}{(z^2+1)Q_4(z)Q_8(z)}$$

$$x_6 = \frac{(z^2+1)^2 Q_4(z)Q_{38}(z)}{(z^2+1)Q_4^2(z)Q_8^2(z)Q_{18}(z)}$$

$$x_7 = \frac{(z^2+1)Q_4^2(z)Q_8(z)Q_{80}(z)}{z(z^2+1)Q_4(z)Q_8^2(z)Q_{18}(z)Q_{38}^2(z)}$$

$$x_8 = \frac{(z^2+1)^2 Q_4^2(z)Q_8^2(z)Q_{18}(z)Q_{166}(z)}{z(z^2+1)^2 Q_4^2(z)Q_8(z)Q_{18}^2(z)Q_{38}^2(z)Q_{80}(z)}$$

The corresponding degrees are 0, 1, 2, 4, 9, 19, 46, 98, and 213, without taking into account any possible simplifications. After simplification, we find the sequence 0, 1, 2, 4, 9, 19, 40, 84, and 175. As in the previous case, a singularity appears if z is equal to $\pm i$. Then, x_2 vanishes, and the successive values of x_n are $\pm i, 0, \infty, \infty^2, \infty, 0, \mp i$ (The symbol ∞^2 is introduced here, since the term z^2+1 vanishing in the denominator of x_4 has a multiplicity of 2). This is again a confined singularity. A cyclic one does also exist: assuming that z vanishes, we obtain the sequence $\bullet, 0, \infty, \infty^2, \infty, 0, \bullet, \infty, \infty$, and in fact, this pattern repeats indefinitely. Note that while some simplifications do appear, due to the presence of a confined singularity, they do not suffice to curb the exponential growth, and the ratio of the degrees of x_8 to x_7 is already 2.08, i.e., very close to the dynamical degree.

Thus, computing the degree growth of a given birational mapping by iterating some initial condition and obtaining the successive rational expressions is not a very efficient method, due to the simplifications involved. But it is precisely these simplifications which condition the integrability of the mapping, so they cannot be ignored. Fortunately, a much more efficient method for the obtention of the degrees does exist.

3. Obtaining the Degrees with the Help of Singularities

In the previous section, we have presented an explicit calculation of the degrees of the successive iterates of two second-order birational mappings, starting from initial conditions $x_0 = 1$ and $x_1 = z$. The choice of the value 1 for x_0 is not constraining, meaning that the value 1 is not special. In fact, the idea behind the method of Halburd, which we shall present in what follows, is that x_0 must be generic (in the sense that it does not satisfy any special relation). On the other hand, the value z of x_1 can take values freely in the closure of $\mathbb{C}$. Starting from such initial conditions and iterating up to $x_n(z)$, one can obtain the degree d_n as the number of preimages of $x_n(z) = w$ for some w. Halburd's method is based on the observation that if one chooses for w a value that appears in the singularity pattern, the computation of the degree is greatly simplified.

Let us illustrate this in the case of (2). The confined singularity pattern is $\{-1, 0, \infty, \infty, 0, -1\}$. Asking what are the contributions to the number of preimages of -1, we see that either a -1 appears spontaneously at some iteration, "opening" the singularity pattern, or it appears five iterations later, "closing" the pattern. If we denote the number of spontaneous occurrences of the value -1 at step n by M_n, we have for the degree

$$d \equiv d_n(-1) = M_n + M_{n-5}. \tag{4}$$

But the degree must be the same when one considers the occurrences, i.e., the number of preimages, of 0 or ∞. We remark that a 0 appears if it is preceded by a -1 one or four steps earlier. Similarly, the occurence of an ∞ is linked to the existence of -1 two or three steps earlier. We find thus for the degrees the relations

$$d_n(0) = M_{n-1} + M_{n-4}, \tag{5}$$

$$d_n(\infty) = M_{n-2} + M_{n-3}, \tag{6}$$

which, however, are not complete. In fact, 0 and ∞ also appear in the cyclic pattern $\{\infty, \bullet, 0, \infty, \infty, 0, \bullet\}$, which has period 7, and thus, the right-hand side of (5) and (6) must be complemented by period-7 functions, accounting for the presence of 0 and ∞ in the cyclic pattern, and obtained by the repetition of the strings $[0,1,0,1,1,0,1]$ and $[0,1,1,1,1,1,1]$ (where the first position corresponds to $n = 0 \bmod 7$). Equating the three expressions for the degree (4), (5), and (6), one obtains linear equations for M_n. For instance, subtracting (5) from (6), we find $M_{n+3} - M_{n+2} - M_{n+1} + M_n = [0,0,-1,0,0,-1,0]$. The corresponding characteristic polynomial is just $(k+1)(k-1)^2$, and the solution one finds for M_n is $M = n^2/14 + [-2,2,-2,0,1,1,0]/7$. The degree of the successive iterates grows quadratically with n as

$$d_n = \frac{n^2 + \psi_7(n)}{7}, \tag{7}$$

where $\psi_7(n)$ is a period-7 function obtained by the periodic repetition of the string $[0,6,3,5,5,3,6]$.

Before proceeding to the presentation of the algorithmic method of computing the degrees that is the subject of our paper, let us deal with the non-integrable mapping (3). As we saw, the confined singularity pattern is $\{\pm i, 0, \infty, \infty^2, \infty, 0, \mp i\}$. Here, there are two ways to enter the singularity: either by a $+i$ or by a $-i$. We denote by I_n and M_n the spontaneous occurrences of $+i$ and $-i$, respectively, at iteration n. We remark that a $-i$ may appear either spontaneously or due to the presence of a $+i$ six iterations earlier and similarly for the appearance of $+i$. We thus have the following for the degrees:

$$d_n(i) = I_n + M_{n-6}, \tag{8a}$$

$$d_n(-i) = M_n + I_{n-6}. \tag{8b}$$

The occurence of 0 is linked to the appearance of either a $+i$ or a $-i$ one or five steps earlier, while infinity appears due to the presence of $\pm i$ two, three (with multiplicity of 2), or four steps earlier. We thus obtain the expressions for the degrees:

$$d_n(0) = I_{n-1} + I_{n-5} + M_{n-1} + M_{n-5}, \tag{9a}$$

$$d_n(\infty) = I_{n-2} + 2I_{n-3} + I_{n-4} + M_{n-2} + 2M_{n-3} + M_{n-4}. \tag{9b}$$

As in the case of (2), one must add to the right-hand sides of (9) periodic terms, with period 9, to account for the appearance of 0 and ∞ in the cyclic pattern $\{0, \infty, \infty^2, \infty, 0, \bullet, \infty, \infty, \bullet, \cdots\}$. However, since we are only interested in this non-integrable case in obtaining the dynamical degree, we can apply our "express" version [28] of Halburd's method, which does away with these periodic contributions. Equating the expressions for the degrees, we find a linear equation for the quantity $I_n + M_n$, the characteristic polynomial of which is $k^6 - k^5 - k^4 - 2k^3 - k^2 - k + 1$, which factorises into $(k^2+1)(k^4 - k^3 - 2k^2 - k + 1)$. The largest root of the latter is $\mu + \sqrt{\mu^2 - 1}$, with $\mu = (1 + \sqrt{17})/4$, which is precisely the dynamical degree that we had obtained in [17] using the full-deautonomisation approach.

The method of Halburd, as we have presented it, leads, at least in the case of an integrable mapping, to a closed expression for the degrees of the iterates. This requires the solution of some linear system of equations, a task which, admittedly, does not present particular difficulties. In the case of non-integrable mappings however, the same approach usually does not yield a useful handle for the computation of the degrees, and the fact that the degrees grow exponentially precludes the use of the direct, brute force approach based on the simplifications of rational expressions. Fortunately, Halburd's method of working with the values that appear in a singularity pattern can be cast in a way that is both convenient for calculations and can be described as an algorithm allowing the computation of the degrees to very high orders. In [29], we outlined this method but without insisting on the fact that it can be cast into a simple algorithm. In what follows, we start by explaining how the method works for the first few iterations and then proceed to its algorithmic description. We start with mapping (2) and remark that the string $\{\infty, \bullet, 0, \infty, \infty, 0, \bullet\}$ introduces in fact two possible ways to enter the cyclic pattern: $\{\bullet, \infty, \bullet, 0, \infty, \infty, 0\}$ and $\{\bullet, 0, \infty, \infty, 0, \bullet, \infty\}$. Since the degree is the same whether we compute it from the occurrences (i.e., the number of preimages) of 0, ∞ or -1, we must balance the 0 and ∞ of the cyclic patterns by a -1 for a confined pattern. This means that at the next iteration step, an ∞ (from the cyclic pattern) and a 0 (from the confined pattern) are present. Again, they must be balanced by a -1 for a new confined pattern. So at the next iteration, we have two 0 values and two ∞ values from the cyclic and confined patterns. They must be balanced by two -1 values for two new confined patterns, and so on. This leads to the table below. We have adopted the following convention for the setup of the table. The abscissa corresponds to the iterations, while on the ordinate, we represent the values that appear in the singularity pattern. In each cell, we give the singularities in the upper right corner, while their multiplicity is given in the lower left corner. The degree for each iteration is obtained by adding the multiplicities corresponding to one of the values appearing in the singularity pattern (Since the singularities are balanced any one of those leads to the same result). We thus obtain Figure 1.

d_n: 0	1	1	2	3	4	6	7	10	12	15	…
$\bullet$	∞ (1)	$\bullet$	0 (1)	∞ (1)	∞ (1)	0 (1)	$\bullet$	∞ (1)	$\bullet$	0 (1)	…
$\bullet$	0 (1)	∞ (1)	∞ (1)	0 (1)	$\bullet$	∞ (1)	$\bullet$	0 (1)	∞ (1)	∞ (1)	…
	-1 (1)	0 (1)	∞ (1)	∞ (1)	0 (1)	-1 (1)					
		-1 (1)	0 (1)	∞ (1)	∞ (1)	0 (1)	-1 (1)				
			-1 (2)	0 (2)	∞ (2)	∞ (2)	0 (2)	-1 (2)			
				-1 (3)	0 (3)	∞ (3)	∞ (3)	0 (3)	-1 (3)		
					-1 (4)	0 (4)	∞ (4)	∞ (4)	0 (4)	-1 (4)	
						-1 (5)	0 (5)	∞ (5)	∞ (5)	0 (5)	-1 (5)
							-1 (6)	0 (6)	∞ (6)	∞ (6)	0 (6)
								-1 (8)	0 (8)	∞ (8)	∞ (8)

Figure 1. Balancing of the singularities for the mapping (2). The degrees are given just above the table. The first two lines in the table correspond to the cyclic pattern (the dots on the left side represent the finite values preceding the two possible entries). In each cell, we give the singularities in the upper right corner, while their multiplicity is given in the lower left corner.

We turn now to the non-integrable mapping (3). The string $\{0, \infty, \infty^2, \infty, 0, \bullet, \infty, \infty, \bullet\}$ again leads to two entry possibilities for the cyclic pattern: $\{\bullet, \infty, \infty, \bullet, 0, \infty, \infty^2, \infty, 0\}$ and $\{\bullet, 0, \infty, \infty^2, \infty, 0, \bullet, \infty, \infty\}$. There are also two confined singularity patterns corresponding to the sign choice in $\{\pm i, 0, \infty, \infty^2, \infty, 0, \mp i\}$. Just as in the previous case, we set up a table by balancing the singularities at each iteration. Since the quantity ∞^2 appears in the singularity and in the cyclic patterns, it must be counted twice when it comes to obtaining the multiplicities of each singularity. We thus obtain Figure 2:

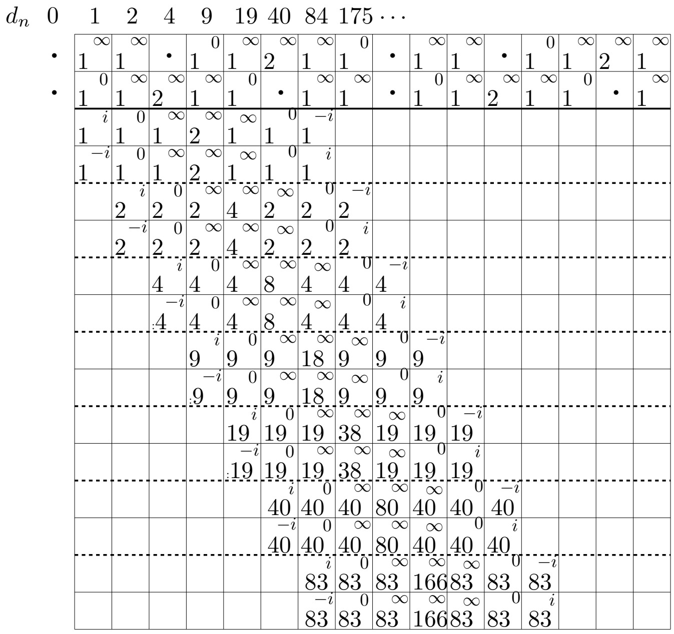

Figure 2. Balancing the singularities for the mapping (3). The degrees are given just above the table. The first two lines in the table correspond to the cyclic pattern (the dots on the left side represent the finite values preceding the two possible entries).

Balancing the singularities, as was done in the two tables above, can be cast into an algorithm, with the calculations being performed automatically. We start by introducing variables for the preimages of the various quantities that appear in the (confined) singularity pattern. In the case of mapping (2), we have M_n (already introduced in the previous paragraphs) as well as Z_n and F_n for zero and infinity, respectively. Here, we define objects with two indices $M_{n,m}$, $Z_{n,m}$, $F_{n,m}$ corresponding to the two dimensions of the table, where n refers to the column index which starts at 1.

To start, we set their values equal to 0 for all m and n. Then, we define the contribution coming from the cyclic pattern by assigning values to F, Z, and M. If more than one cyclic patterns exists, this must be done for each of them. However, when the cyclic patterns have the same periodicity, it is simpler to merge them into a single one combining the contributions. In the case of mapping (2) for instance, we have $F_{1,0} = F_{2,0} = F_{3,0} = F_{4,0} = F_{5,0} = F_{6,0} = 1$, $F_{7,0} = 0$, $Z_{1,0} = Z_{3,0} = Z_{4,0} = Z_{6,0} = 1$, and $Z_{2,0} = Z_{5,0} = Z_{7,0} = 0$, combined with the periodicity condition $F_{n+7,0} = F_{n,0}$, $Z_{n+7,0} = Z_{n,0}$. And since -1 does not appear in the cyclic pattern, we have $M_{n,0} = 0$ for all n.

Next, we introduce the initial condition for the confined singularity pattern. We have, always for mapping (2), $M_{1,1} = M_{6,1} = 1$, $Z_{2,1} = Z_{5,1} = 1$, and $F_{3,1} = F_{4,1} = 1$. The condition for balancing the singularities at any iteration is $M_{n,n} = \sum_{k=0}^{n-1} Z_{n,k} - M_{n,n-5}$, which is valid for $n > 5$ (but it suffices to set $M_{n,k} = 0$ for $k < 0$ for the relation to be valid everywhere). The $M_{n,n-5}$ term corresponds to the value -1 that "closes" the singularity pattern and must be subtracted, lest we double-count the contribution of the singularity -1.

It goes without saying that the sum entering the expression for $M_{n,n}$ could have run over the F instead of the Z. Once M is obtained, we compute the remaining quantities on the n-th line as $F_{n+3,n} = F_{n+4,n} = M_{n,n}$, $Z_{n+2,n} = Z_{n+5,n} = M_{n,n}$, and $M_{n+6,n} = M_{n,n}$. We iterate this process up to some arbitrarily high N and compute the degree as $d_n = \sum_{k=0}^{N} M_{n,k}$.

The computations based on the algorithm just described do not necessitate computer algebra: they can be carried out in any programming language using integer arithmetic. As a consequence, the calculations are incredibly fast and allow one to obtain the succession of the degrees up to very high orders. We illustrate this by presenting the first 100 degrees for the mapping (2):

0, 1, 1, 2, 3, 4, 6, 7, 10, 12, 15, 18, 21, 25, 28, 33, 37, 42, 47, 52, 58, 63, 70, 76, 83, 90, 97, 105, 112, 121, 129, 138, 147, 156, 166, 175, 186, 196, 207, 218, 229, 241, 252, 265, 277, 290, 303, 316, 330, 343, 358, 372, 387, 402, 417, 433, 448, 465, 481, 498, 515, 532, 550, 567, 586, 604, 623, 642, 661, 681, 700, 721, 741, 762, 783, 804, 826, 847, 870, 892, 915, 938, 961, 985, 1008, 1033, 1057, 1082, 1107, 1132, 1158, 1183, 1210, 1236, 1263, 1290, 1317, 1345, 1372, 1401.

The ease and power of our method become even more apparent in the case of the non-integrable mapping (3). The computation of the degrees up to numbers with 32 digits takes up a negligible amount of time resulting in the following:

0, 1, 2, 4, 9, 19, 40, 84, 175, 364, 759, 1580, 3288, 6843, 14241, 29636, 61674, 128345, 26708
8, 555817, 1156666, 2407044, 5009105, 10424043, 21692632, 45142780, 93942983, 195497132, 4
06833247, 846627716, 1761848360, 3666439907, 7629931097, 15878031556, 33042485298, 6876
2039601, 143095110656, 297783643601, 619693419218, 1289593777476, 2683669148857, 55847
66479427, 11622005135400, 24185613465636, 50330721067007, 104739186654252, 2179642371
18503, 453587718028380, 943924657852632, 1964325144373643, 4087797940988785, 85067851
69560324, 17702781538058906, 36839824673794697, 76664374978484048, 15954002069457202
5, 332005813787275914, 690910405481109316, 1437797678771749121, 2992084282826671643,
6226584232064003288, 12957640071007986380, 26965095139190915479, 561148752304442198
84, 116776121347770033935, 243013326876841402804, 505715349663634775080, 10524032495
34643394739, 2190071154390984313769, 4557579676247064475524, 9484409884900041722722,
19737237142250511592801, 41073565433906675200000, 85474869927060675632801, 17787482
8052224495902882, 370160896198002010775684, 770311856795605003014249, 1603033607316
772844835699,3335943389053760365737000, 6942161564284909047646964, 1444677009291359
7620942415,30064003003220403237137644, 62563761364278747160932439, 1301963758993482
42208503180, 270941131538212542146563288, 563833641697967370487364443, 117334851930
9461949828061761, 2441760558344260990740850756, 5081350107122939718737775434, 10574
38610142295627956017426 5, 22005498354703634757948514128, 45793860106328226045065787
337, 95297892810035512121785214426, 198316725275972642690305128964, 41270087264766
8258220992831025, 858838355903320829678322516043, 1787258933664594476688828092632,
3719319792842931751576160826780, 7739975143427773276411146697063, 1610703530687491
0426573973927212, 33519046452908794254283600055487.

Having presented the detail of our algorithmic method for the computation of the degree growth, we turn now to its application to some illustrative examples.

4. A Collection of Examples

In this section, we are going to present the application of the algorithm we introduced in the previous section to the computation of the degree growth of second- or higher-order mappings, integrable or non-integrable, and possessing confined or unconfined singularities to show that our approach can be applied in all cases studied and indeed leads to the correct value of the dynamical degree.

4.1. An Integrable Mapping (Autonomous Limit of Discrete Painlevé I)

We start with the mapping

$$x_{n+1} + x_n + x_{n-1} = \frac{a}{x_n} + 1, \tag{10}$$

depending on a non-zero constant a. It possesses a confined singularity pattern $\{0, \infty, \infty, 0\}$ as well as a cyclic one with the string $\{\bullet, \infty, \infty\}$ repeated indefinitely. The equations for the degree can be obtained in a straightforward way. Introducing the quantity Z_n for the number of spontaneous occurences of 0 at step n, we have for the degree $d_n(0) = Z_n + Z_{n-3}$. Similarly linking the degree to the occurences of infinity, we find $d_n(\infty) = Z_{n-1} + Z_{n-2}$ to which one must add a function of period 3 to account for the presence of infinity in the cyclic pattern. We could proceed to integrate the equation for Z_n obtained by equating the two expressions for the degree, but instead we go on directly to apply the algorithm introduced in the previous section. Starting from initial conditions, a generic value for x_0 and $x_1 = z$, we compute the degrees in z of the iterates using our algorithm. We obtain the following sequence for the first 100 terms:

0, 1, 2, 3, 6, 9, 12, 17, 22, 27, 34, 41, 48, 57, 66, 75, 86, 97, 108, 121, 134, 147, 162, 177, 192, 209, 226, 243, 262, 281, 300, 321, 342, 363, 386, 409, 432, 457, 482, 507, 534, 561, 588, 617, 646, 675, 706, 737, 768, 801, 834, 867, 902, 937, 972, 1009, 1046, 1083, 1122, 1161, 1200, 1241, 1282, 1323, 1366, 1409, 1452, 1497, 1542, 1587, 1634, 1681, 1728, 1777, 1826, 1875, 1926, 1977, 2028, 2081, 2134, 2187, 2242, 2297, 2352, 2409, 2466, 2523, 2582, 2641, 2700, 2761, 2822, 2883, 2946, 3009, 3072, 3137, 3202, 3267, 3334, 3401, 3468, 3537, 3606, 3675, 3746, 3817, 3888, 3961, 4034, 4107, 4182, 4257, 4332, 4409, 4486, 4563, 4642, 4721, 4800, 4881, 4962, 5043, 5126, 5209, 5292, 5377, 5462, 5547, 5634, 5721, 5808, 5897, 5986, 6075, 6166, 6257, 6348, 6441, 6534, 6627, 6722, 6817, 6912, 7009, 7106, 7203, 7302, 7401, 7500, 7601, 7702, 7803, 7906, 8009, 8112, 8217, 8322, 8427, 8534, 8641, 8748, 8857, 8966, 9075, 9186, 9297, 9408, 9521, 9634, 9747, 9862, 9977.

In this case, it is easy to show that the sequence above can be represented by the expression

$$d_n = \frac{n^2 + \psi_3(n)}{3}, \tag{11}$$

where $\psi_3(n)$ is a periodic function obtained by the repetition of the string $[0, 2, 2]$.

4.2. A Non-Integrable Mapping with Confined Singularities

The mapping

$$x_{n+1} + x_{n-1} = x_n + \frac{a}{1 - x_n^2}, \tag{12}$$

with constant a, has two confined singularity patterns $\{\pm 1, \infty, \infty, \mp 1\}$ and a cyclic one, which are obtained from the repetitions of the string $\{\bullet, \infty, \infty\}$. Here, we introduce the quantities U_n and V_n for the number of spontaneous occurences of 1 and -1 at step n and find for the degree $d_n(1) = U_n + V_{n-3}$, $d_n(-1) = V_n + U_{n-3}$ and $d_n(\infty) = U_{n-1} + U_{n-2} + V_{n-1} + V_{n-2}$ (to which one must add a function of period 3 to account for the presence of infinity in the cyclic pattern). Again, we introduce initial conditions $x_0, x_1 = z$ and compute the degree in z based on the algorithm we described earlier. We obtain the following sequence:

0, 1, 3, 8, 23, 61, 160, 421, 1103, 2888, 7563, 19801, 51840, 135721, 355323, 930248, 243542
3, 6376021, 16692640, 43701901, 114413063, 299537288, 784198803, 2053059121, 5374978560,
14071876561, 36840651123, 96450076808, 252509579303, 661078661101, 1730726404000, 4531
100550901, 11862575248703, 31056625195208, 81307300336923, 212865275815561, 557288527
109760, 1459000305513721, 3819712389431403, 10000136862780488, 26180698198910063, 685
41957733949701, 179445175002939040, 469793567274867421, 1229935526821663223, 3220013
013190122248, 8430103512748703523, 22070297525055988321, 57780789062419261440, 15127

2069662201796001, 3960354199241861265563, 1036834190110356583688, 271446715040688362
4503, 7106567261110294289821, 18605234632923999244960, 48709136637661703445061, 1275
2217528006111090223, 333857389202521629825608, 874049992327503778386603, 228829258
7779989705334201, 5990827771012465337616000, 15684190725257406307513801, 4106174440
4759753584925403, 107501042489021854447262408, 281441383062305809756861823, 7368231
06697895574823323061, 1929027937031380914713107360, 5050260704396247169315999021, 1
3221754176157360593234889703, 34615001824075834610388670088, 906232512960701432379
31120563, 237254752064134595103404691601, 621141004896333642072282954240, 162616826
2624866331113444171121, 4257363782978265351268049559123, 1114592308630992972269070
4506248, 29180405475951523816804063959623, 76395293341544641727721487372621,
where we have eliminated from the list all degrees longer than 32 digits. Computing the dynamical degree from the values for the degrees of the list, we obtain of 2.618034, which is in perfect agreement with the exact value $(3+\sqrt{5})/2$ of the dynamical degree of (12).

4.3. A Mapping Non-Integrable Due to Bad Deautonomisation

We examine the mapping

$$x_{n+1} + x_{n-1} = 1 + \frac{a_n}{x_n}, \tag{13}$$

where a_n is a function of n. When a_n is linear in n, the equation is a well-known integrable one: it is in fact a discrete analogue of Painlevé I [30]. But suppose we opt for a different n-dependence, taking for instance an a_n quadratic in n. In this case, the singularity confinement conditions are not satisfied, and the singularity pattern which was confined as $\{0, \infty, \bullet, \infty, 0\}$ in the autonomous or a_n linear in n case now becomes unconfined $\{0, \infty, \bullet, \infty, 0, \infty, \bullet, \infty, 0, \infty, \bullet, \infty, 0, \cdots\}$ with the bloc $\{\infty, \bullet, \infty, 0, \}$ repeating indefinitely. The cyclic pattern $\{\bullet, \infty, \cdots\}$, present in the integrable case, still exists. Let us show first how to deal with the unconfined pattern when it comes to balancing the singularities. Since the unconfined pattern is preceded by a semi-infinite sequence of regular values, just like a confined pattern, the way to deal with it is exactly the same as for the latter. The only difference is that at every iteration step, we introduce a semi-infinite sequence of singularities that must be balanced. Figure 3 below shows the first few iterations.

Already inspecting the degrees presented in Figure 3, we see that the dynamical degree must be larger than 1, with the ratio of 189 to 129 being equal to 1.465. In fact, it is possible to implement the Diophantine approximation and obtain a better estimate of the dynamical degree, $\lambda = 1.4655$ (where we are giving the converged decimals). Using Halburd's method, the equations for the degree can be obtained formally if one introduces Z_n for the number of spontaneous occurences of 0 at step n. We find that $d_n(0) = Z_n + Z_{n-4} + Z_{n-8} + \cdots$, $d_n(\infty) = Z_{n-1} + Z_{n-3} + Z_{n-5} + Z_{n-7} + \cdots$, and $(1 + (-1)^n)/2$ must be added to the latter to account for the presence of infinity in the cyclic pattern. Given that we must now deal with infinite sums of possible contributions, it is far simpler (and faster) to implement our algorithm and obtain the sequence of degrees. We find the following:

0, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, 41, 60, 88, 129, 189, 277, 406, 595, 872, 1278, 1873, 2745, 402
3, 5896, 8641, 12664, 18560, 27201, 39865, 58425, 85626, 125491, 183916, 269542, 395033, 5789
49, 848491, 1243524, 1822473, 2670964, 3914488, 5736961, 8407925, 12322413, 18059374, 2646
7299, 38789712, 56849086, 83316385, 122106097, 178955183, 262271568, 384377665, 56333284
8, 825604416, 1209982081, 1773314929, 2598919345, 3808901426, 5582216355, 8181135700, 11
990037126, 17572253481, 25753389181, 37743426307, 55315679788, 81069068969, 1188124952
76, 174128175064, 255197244033, 374009739309, 548137914373, 803335158406, 117734489771
5, 1725482812088, 2528817970494, 3706162868209, 5431645680297, 7960463650791, 11666626
519000, 17098272199297, 25058735850088, 36725362369088, 53823634568385, 7888237041847
3, 115607732787561, 169431367355946, 248313737774419, 363921470561980, 53335283791792

6, 781666575692345, 1145588046254325, 1678940884172251, 2460607459864596, 3606195506118921, 5285136390291172, 7745743850155768, 11351939356274689, 16637075746565861, 24382819596721629, 35734758952996318,
and by computing the dynamical degree from the ratio of the two last degrees, we find the value $\lambda = 1.465571232$.

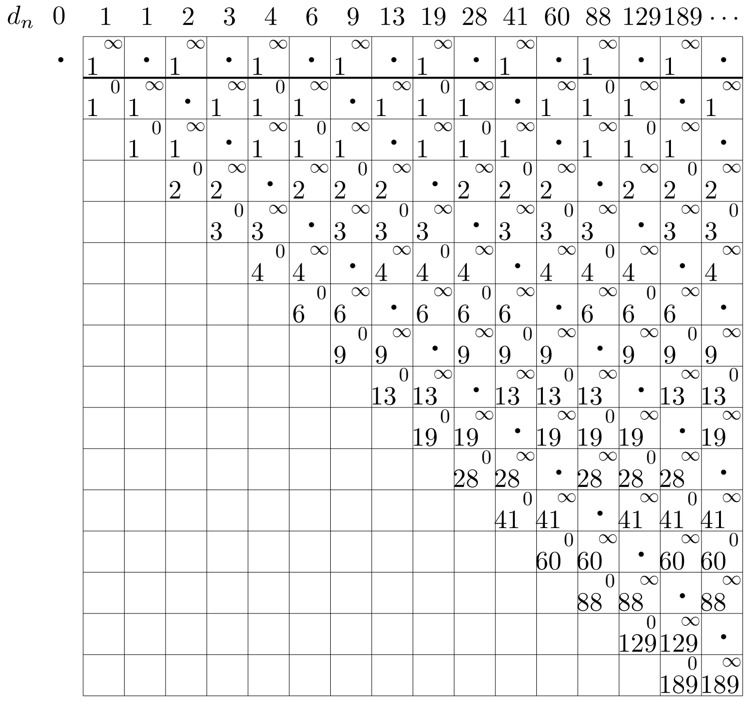

Figure 3. Balancing the singularities for the mapping (13). The degrees are given just above the table. The first line in the table corresponds to the cyclic pattern (the dot on the left side represents the finite value preceding the entry).

It turns out that, thanks to the methods we developed in [31], it is possible to provide an exact expression for the dynamical degree in a very simple way. The unconfined pattern can be considered as resulting from a confinement condition that has been infinitely delayed. Thus, the characteristic equation which was $1 - \lambda - \lambda^3 + \lambda^4 = 0$ in the integrable case now becomes $1 - \lambda - \lambda^3 + \lambda^4 - \lambda^5 - \lambda^7 + \lambda^8 - \lambda^9 - \lambda^{11} + \lambda^{12} - \cdots = 0$. We remark that the geometric series that appears can be easily summed up to some order m. We thus find for the characteristic equation

$$1 - \left(\frac{\lambda^{4m} - 1}{\lambda^4 - 1}\right)\lambda(1 + \lambda^2 - \lambda^3) = 0 \tag{14}$$

and since λ is larger than 1, at the limit when m goes to infinity (i.e., an infinitely delayed confinement) the dynamical degree is given by the (supergolden ratio) equation $\lambda^3 - \lambda^2 - 1 = 0$. The latter has one real root, with its value coinciding with the dynamical degree previously computed.

Having shown how our method works in the case of second-order mappings, integrable or not, we now proceed to apply it to higher-order ones. A caveat is in the order at this point. As stressed in a recent work of ours [29], the way to proceed to the calculation of the degree growth and the dynamical degree follows the spirit of that employed for second-order mappings. Namely, for a mapping of order N, we start with an initial condition where $x_0, \cdots, x_{N-2}$ are generic, and x_{N-1} is free to take *any* value including special ones. ('Special' in this context is to be understood not only as a value appearing in a singularity pattern but, in fact, any value which, combined with the previous generic ones, may lead to a singularity at some later iteration). And we proceed to compute the degree of the successive iterates in x_{N-1}. While there is no guarantee that this suffices (as it did for second order mappings), in order to obtain the right value of the dynamical degree, it turned out that, in all cases examined, the value of the dynamical degree obtained was the expected one. The examples that we shall present in the following confirm this result, further justifying our approach.

4.4. A Coupling of Two Linearisable Mappings

In [32], we presented the couplings of various mappings with linear or linearisable ones. In this subsection, we consider one of those mappings, more precisely the coupling of what we dubbed a "third-kind" linearisable mapping [33]

$$x_{n+1}x_{n-1} = x_n^2 - 1, \tag{15a}$$

to a "Gambier"-type mapping [34], through

$$x_n = z_{n+1} - z_n + \frac{1}{z_n} - \frac{1}{z_{n-1}}. \tag{15b}$$

The resulting fourth-order mapping has two singularity patterns: a confining one $\{0, \infty\}$ and an anticonfining one $\{\cdots, 0^3, 0^2, 0, \bullet, \bullet, \bullet, \bullet, \bullet, \infty, \infty^2, \infty^3, \infty^4 \cdots\}$. The degree growth was obtained in [32] by direct computation, leading to the sequence d_n= 0, 0, 0, 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157$\cdots$, which, for $n > 2$, can be represented by the expression $d_n = (n-2)(n-3) + 1$. What is interesting in the present case is the anticonfined pattern. As we have explained in [35], the presence of an anticonfined pattern is not incompatible with integrability unless the growth of the multiplicities of the values that appear in this pattern is exponential. This is not the case here, and thus we expect the coupled system to be integrable. In order to compute its degree growth, we proceed, as in the previous cases, to the balancing of singularities. However, the situation is now complicated by the fact that instead of just three free values in the anticonfined pattern (as expected given its order), here we apparently have five. And since one can put the value ∞ to any position compatible with the number of consecutive 'free' values, one would have to balance the following singularity patterns (in the positive n direction): $\{\bullet, \bullet, \bullet, \infty, \infty^2, \infty^3, \cdots\}$, $\{\bullet, \bullet, \bullet, \bullet, \infty, \infty^2, \infty^3, \cdots\}$, and $\{\bullet, \bullet, \bullet, \bullet, \bullet, \infty, \infty^2, \infty^3, \cdots\}$. However, it turns out that the pattern of the middle is not acceptable: it exists only under a constraint between z_0, z_1, and z_2. We are thus left with the remaining two patterns and balancing the singularities for the first few iterations, which leads to Figure 4 below.

Introducing the quantity Z_n for the number of spontaneous occurences of 0 at step n, we have for the degrees $d_n(0) = Z_n$ and $d_n(\infty) = Z_{n-1} + 2n - 6$ the latter term coming for the contributions $n-2$ and $n-4$ of the anticonfined pattern. Implementing our algorithm for the computation of the degrees is elementary, resulting to the following sequence for the first hundred degrees:

0, 0, 0, 1, 3, 7, 13, 21, 31, 43, 57, 73, 91, 111, 133, 157, 183, 211, 241, 273, 307, 343, 381, 421, 463, 507, 553, 601, 651, 703, 757, 813, 871, 931, 993, 1057, 1123, 1191, 1261, 1333, 1407, 1483, 1561, 1641, 1723, 1807, 1893, 1981, 2071, 2163, 2257, 2353, 2451, 2551, 2653, 2757, 2863, 2971, 3081, 3193, 3307, 3423, 3541, 3661, 3783, 3907, 4033, 4161, 4291, 4423, 4557, 4693, 4831, 4971, 5113, 5257, 5403, 5551, 5701, 5853, 6007, 6163, 6321, 6481, 6643, 6807, 6973, 7141, 7311, 7483, 7657, 7833, 8011, 8191, 8373, 8557, 8743, 8931, 9121, 9313, 9507, 9703, 9901.

d_n 0 0 0	1	3	7	13	21	31	43	57	73	$\cdots$
• • •	1^{∞}	2^{∞}	3^{∞}	4^{∞}	5^{∞}	6^{∞}	7^{∞}	8^{∞}	9^{∞}	10^{∞}
• • •	•	•	1^{∞}	2^{∞}	3^{∞}	4^{∞}	5^{∞}	6^{∞}	7^{∞}	8^{∞}
	1^{0}	1^{∞}								
		3^{0}	3^{∞}							
			7^{0}	7^{∞}						
				13^{0}	13^{∞}					
					21^{0}	21^{∞}				
						31^{0}	31^{∞}			
							43^{0}	43^{∞}		
								57^{0}	57^{∞}	
									73^{0}	73^{∞}

Figure 4. Balancing the singularities for the mapping (15). The degrees are given just above the table. The first two lines in the table correspond to the anticonfined pattern (the dots on the left side represent the finite values preceding the two acceptable entries).

4.5. Reductions of the Bogoyavlensky Lattice

The discrete form of the Bogoyavlensky system was obtained by Suris [36] and independently by Papageorgiou and Nijhoff [37]. One of the authors gave in [38] the "travelling wave" reductions of these lattices. In what follows, we shall work with one of the latter, namely,

$$\frac{1-\prod_{k=0}^{N} x_{n+1+k}}{1-\prod_{k=0}^{N} x_{n-k}} = \frac{x_{n+1}}{x_n}. \tag{16}$$

Equation (14) can be integrated once, leading to

$$x_n + \sum_{m=1}^{N} \prod_{k=0}^{N+1} x_{n+m-k} = C. \tag{17}$$

The $N = 1$ case gives, up to a rescaling, the mapping (2). For $N = 2, 3, 4$ we find the mappings

$$x_{n+1}x_{n-1}(x_{n+2} + x_{n-2}) = \frac{C}{x_n} - 1, \tag{18a}$$

$$x_{n+1}x_{n-1}(x_{n+3}x_{n+2} + x_{n+2}x_{n-2} + x_{n-2}x_{n-3}) = \frac{C}{x_n} - 1. \tag{18b}$$

$$x_{n+1}x_{n-1}\big(x_{n-2}x_{n-3}(x_{n+2} + x_{n-4}) + x_{n+2}x_{n+3}(x_{n+4} + x_{n-2})\big) = \frac{C}{x_n} - 1. \tag{18c}$$

The integrated form (17) has, for $N > 1$, a confined singularity pattern $\{0, \infty, (N-1 \text{ finite values } \bullet), \infty, 0\}$. A cyclic pattern also exists, but its details depend on N. For $N = 2$, the cyclic pattern is obtained from the repetition of the string $\{\infty, \bullet, \bullet, \bullet\}$. For $N = 3$, the basic string is $\{\infty, \bullet, 0, \infty, \bullet, \bullet, \infty, 0, \bullet, \infty, \bullet, \bullet, \bullet, \bullet, \bullet\}$ of length 15, while for $N = 4$, we have a string of length 12, $\{\infty, \bullet, 0, \bullet, \infty, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet, \bullet\}$. In fact, the length of the basic string of the cyclic patterns is equal to $N(N+2)$ for odd $N > 1$ and $N(N+2)/2$ for even N. We focus now on the case $N = 4$, which is a mapping of order 8. Introducing Z_n for the number of spontaneous occurrences of the value 0 at the step, we find that the contribution of the preimages of 0 to the degree from the confined pattern is

$$d_n(0) = Z_n + Z_{n-6} \tag{19a}$$

to which we must add a period-12 contribution coming form the cyclic pattern. Similarly, for the contribution of the preimages of infinity, we have

$$d_n(\infty) = Z_{n-1} + Z_{n-5} \tag{19b}$$

and again, there is a (different) period-12 contribution due to the cyclic pattern. Equating the two expressions for the degree, we find the characteristic polynomial $(\lambda^5 - 1)(\lambda - 1)$, which signals the presence of a period-5 term in the expression of d_n. And since there is a period-12 contribution coming from the cyclic pattern, we expect the degree to include a period-60 term. Performing the calculation of the degrees in the standard way, through the simplification of the successive rational expressions, is prohibitively long. But this is not a real obstacle. Using the algorithm we introduced in the previous section, we compute a large number of degrees. A hundred of them are displayed below (but the calculation of thousands is literally instantaneous):

0, 0, 0, 0, 0, 0, 0, 1, 1, 1, 0, 1, 2, 3, 2, 1, 2, 4, 5, 5, 4, 5, 6, 8, 8, 8, 8, 9, 11, 12, 12, 13, 14, 16, 16, 17, 18, 20, 21, 21, 22, 24, 26, 28, 28, 29, 30, 33, 35, 36, 36, 37, 40, 43, 44, 45, 46, 49, 51, 53, 54, 56, 58, 60, 62, 64, 66, 69, 71, 73, 74, 77, 80, 83, 84, 85, 88, 92, 95, 97, 98, 101, 104, 108, 110, 112, 114, 117, 121, 124, 126, 129, 132, 136, 138, 141, 144, 148, 151, 153, 156, 160, 164, 168, 170, 173, 176, ...

Once the degrees are obtained, we proceed to fit them with a quadratic-plus-periodic term. The result is

$$d_n = \frac{n^2 - 8n + 1 - \psi_{60}(n)}{60}$$

which reproduces perfectly the degrees for $n > 0$ and where $\psi_{60}(n)$ is a period-60 function obtained by the repetitions of the string [0, 53, 48, 45, 44, 45, 48, 53, 0, 9, 20, 93, 48, 5, −36, 45, 128, 93, 0, −31, 0, 93, 68, 45, −36, 5, 48, 93, 80, 9, 0, 53, 48, 45, −16, 45, 48, 53, 0, 9, 80, 93, 48, 5, −36, 45, 68, 93, 0, −31, 0, 93, 128, 45, -36, 5, 48, 93, 20, 9].

4.6. A Mapping Proposed by Viallet

In [39], Viallet introduced the fourth-order mapping

$$x_{n+2}x_{n-2} = \frac{x_{n+1}x_{n-1}}{x_n(1 - x_n)} \tag{20}$$

and studied its properties. The mapping is non-integrable but, as we showed in [29], it has confined singularities with pattern

$\{1, \bullet, \infty, \infty, 0, 0^2, 0, 0, 0, \infty, \infty^2, \bullet, 0^2, 0, 0, 0^2, \bullet, \infty^2, \infty, 0, 0, 0, 0^2, 0, \infty, \infty, \bullet, 1\}$.

A cyclic pattern does also exist,

$\{\bullet, \bullet, \bullet, 0, 0, \bullet, \bullet, \bullet, \infty, \infty, 0, 0^2, 0, 0, 0, \infty, \infty^2, \bullet, 0^2, 0, 0, 0^2, \bullet, \infty^2, \infty, 0, 0, 0, 0^2, 0, \infty, \infty, \cdots\}$.

Viallet calculated the dynamical degree of the mapping and obtained a value of $\lambda = 1.400618098$. In [29], we confirmed his result, first by performing a calculation using the Diophantine method, secondly by using the express method for the calculation of the dynamical degree, and thirdly by applying the full-deautonomisation approach.

We are not going to give any details concerning those calculations here: they can be found in the aforementioned reference. Instead, we shall show how the algorithmic method developed in this paper allows us to calculate the degree of the successive iterates of the mapping up to a very high order, giving their exact numerical values. Given the length of the singularity patterns, we cannot present a table as for the other cases, but it suffices to say that because the two strings $[\bullet, \bullet, \bullet, 0]$ and $[\bullet, \bullet, \bullet, \infty]$ appear in the cyclic pattern, there are two entry possibilities, and thus, in order to establish the balance of singularities, we must consider the cyclic pattern twice, once for each distinct entry. Calculating the degrees of the successive iterates up to numbers with 32 digits, we find the following sequence:

0, 0, 0, 1, 1, 1, 2, 2, 4, 5, 7, 11, 13, 21, 27, 39, 56, 74, 110, 148, 211, 296, 409, 583, 805, 1137,
1589, 2220, 3125, 4355, 6122, 8561, 11988, 16812, 23510, 32973, 46147, 64646, 90567, 126792, 1
77668, 248766, 348474, 488089, 683547, 957525, 1340969, 1878312, 2630758, 3684603, 5160931,
7228205, 10124236, 14180020, 19860756, 27817615, 38961372, 54570561, 76432069, 107052279,
149939657, 210007460, 294141267, 411978590, 577025261, 808192154, 1131967568, 158545596
8, 2220616352, 3110237027, 4356253777, 6101446832, 8545799460, 11969397802, 16764558686,
23480742219, 32887552057, 46062904113, 64516531156, 90363028131, 126564087071, 1772679
52700, 248284706512, 347752044329, 487067819683, 682195991057, 955496059093, 133828507
4905, 1874426284426, 2625355399233, 3677120262371, 5150241209020, 7213521042247, 10103
388111876, 14150988276811, 19820130246014, 27760433178461, 38881765102819, 5445850389
4687, 76275566209288, 106832938422914, 149632147145442, 209577493323421, 293538030185
392, 411134677688492, 575842670331675, 806535666047600, 1129648450708241, 15822060649
69660, 2216066449995075, 3103862776940992, 4347326380558854, 6088944007905416, 852828
5177786624, 11944870568143731, 16730201900482150, 23432623571899094, 328201566669588
92, 45968505420209104, 64384320647061992, 90177844750615639, 126304721431861067, 1769
04678748378423, 247775894745546074, 347039402524002567, 486069668029109121, 68079797
4119525108, 953537963896201536, 1335542529734901571, 1870585038311532699, 2619975259
233867999, 3669584765411676264, 5139686836002157877, 7198738402553836452, 1008268329
2172993102, 14121988699333644965, 19779512957514883590, 27703543825590987029, 388020
84871609573672, 54346902326352142398, 76119254989614038154, 106614006174251491940, 1
49325506589293239414, 209148007079909727440, 292936483957867009444, 410292141109513
412654, 574662598464288635564, 804882835875694547385, 1127333467009268246062, 157896
3656811522231183, 2211525074426955280631, 3097502044280978247274, 43384174230134159
95101, 6076465961033910313538, 8510808199261987730729, 11920391995794362546576, 1669
5916769187586022690, 23384603196101909963173, 32752898460079439467899, 458743023578
47838179820, 64252378133319645626005, 89993043677989226761213, 12604588570440298686
0053, 176542148744917218261287, 247268128660416604553159, 346328216156283003233114,
485073567530880410120385, 679402817733327888901663, 951583882613022567489241, 13328
05608122009518427854, 1866751656368554103347084, 2614606155105353528248079, 3662064
700996912975961005, 5129154097683684709104535, 7183986058636136905461982, 100620208
92292003555832968, 14093048568101168311547683, 19738978885941926307047355, 27646771
071346539539732415, 38722567924514505893561355, 54235529451129113043360373, 7596326
4124900443401373673, 106395522545954499195852156, 149019494465299846283504013, 2087
19400960711821725749125, 292336170470250123359369254, 409451331174037516919182256,
573484944851367455523562861, 803233392911801831007474170, 112502322734177457337432
4509, 1575727893321634484608393739, 2206993005520891540594895571, 30911543465480281
14128212281, 4329526722685536031350771505, 6064013485247335678192491024, 8493367036
305597799567522659, 11895963587300670007217826259, 16661701898136554995376614133, 2
3336681228473790510107047926, 32685778084909149303229895079, 457802923456137696942
16481407, 64120706008754881456103081748, 89808621317270719787789006990.

It goes without saying that the ratio of d_n/d_{n-1} reproduces the value of the dynamical degree once n is sufficiently large.

4.7. A Rational But Not Birational Mapping

In [40], some of the present authors introduced, among others, the higher-order non-integrable mapping

$$x_{n+N} + x_{n-1} = \frac{1}{x_{n+N-1}^{2m}} + \frac{1}{x_n^{2m}} \tag{21}$$

where $N > 1$, and $m > 1$. As shown in [29], (21) has a confined singularity pattern $\{0, \infty^{2m}, (N-2 \text{ finite values } \bullet), \infty^{2m}, 0\}$, as well as a cyclic one $\{(N-1 \text{ finite values } \bullet), \infty)\}$. Here, we shall study a variant of that mapping

$$x_{n+N} + x_{n+1} = \frac{1}{x_{n+N-1}^m} + \frac{1}{x_n^m}, \tag{22}$$

where $N > 1$, and $m > 1$. This is *not* a birational mapping: only the forward evolution is described in a rational way. It is elementary to verify the existence of a confined singularity $\{0, \infty^m, (N-2 \text{ finite values})\}$. The appearance of the finite values in the confined singularity pattern may appear strange at first sight. However, as we have regularly explained, a singularity is confined when all indeterminacies are lifted and one recovers all the degrees of freedom of the mapping. In the present case, one needs these extra $N-2$ steps in order to recover all the missing degrees of freedom.

But another singularity does also exist. Starting with $N-1$ finite values and assuming that the Nth one is infinity, one finds that the infinity does not disappear in the subsequent iterations. However, this singularity is not unconfined but rather an anticonfined one.

Let us start with the $N = 2$ case with finite x_0 and assume that x_1 behaves like $1/\epsilon$ (and thus $x_1 \to \infty$ when $\epsilon \to 0$). From the expression (22) of the mapping, we have $x_1 + x_0 - x_0^{-m} - x_{-1}^{-m} = 0$. This means that if x_1 and x_{-1} are related through $x_1 x_{-1}^m - 1 = 0$ (up to subdominant terms), it is possible to have the $1/\epsilon$ behaviour we assume for x_1. Using the instances of (22) for negative indices, we find that the x_{-n} are related to x_1 by $x_1 x_{-n}^m + (-1)^n = 0$ (always up to subdominant terms). On the other hand, iterating (22) forwards, we find that if x_1 diverges, then all the subsequent values of x for positive indices are infinite. This solution is not what we call unconfined: the values for negative n are not regular either. In fact, they all vanish like $\epsilon^{1/m}$. Only x_0 is finite. This solution is anticonfined and of the form

$$\{\ldots, 0^{1/m}, 0^{1/m}, 0^{1/m}, x_0, \infty, \infty, \infty, \ldots\},$$

where x_0 is the only free value.

In the case $N = 3$, we start with x_0, x_1 finite and assume that x_2 goes to infinity like $1/\epsilon$ when $\epsilon \to 0$, which we find by iterating a sequence of infinities alternating with finite values. The appearance of ∞ with period 2 leads to a periodic term in the expression for the degree. From (22), we have $x_2 + x_0 - x_1^{-m} - x_{-1}^{-m} = 0$, and thus, x_1 and x_{-1} are related through $x_2 x_{-1}^m - 1 = 0$ (up to subdominant terms), i.e., a situation similar to that encountered in the $N = 2$ case. Similarly, we find that x_{-2} obeys the relation $x_1 + x_{-1} - x_0^{-m} - x_{-2}^{-m} = 0$, and since x_0, x_1 are finite and x_{-1} is not divergent, x_{-2} has a finite value. Going one step backwards, we find for x_{-3} the relation $x_2 - x_{-2} - x_1^{-m} + x_{-3}^{-m} = 0$, and, neglecting subdominant terms, we find that x_2 and x_{-3} must satisfy the relation $x_2 x_{-3}^m + 1 = 0$. The pattern now becomes clear: the x values with negative odd indices are such that they must compensate the divergence of x_2 through a relation $x_2 x_{1-2k}^m + (-1)^k = 0$ for $k > 0$. Again, an anticonfined pattern is present here of the form

$$\{\ldots, 0^{1/m}, \bullet, 0^{1/m}, \bullet, 0^{1/m}, x_0, x_1, \infty, \bullet, \infty, \bullet, \infty, \ldots\}.$$

We have checked several values for N, and we expect the general singularity pattern to be

$$\{\cdots, 0^{1/m}, (N-2 \text{ finite values } \bullet), 0^{1/m}, (N-2 \text{ finite values } \bullet), x_0, x_1, \cdots, x_{N-2}, \infty, (N-2 \text{ finite values } \bullet), \infty \cdots\}.$$

We can now apply Halburd's method used in the previous examples to compute the degree growth of the mapping. For positive n, let us call Z_n the number of occurences of the value 0 at step $n > 0$. The only appearance of a zero value is in the confined singularity pattern, and the degree $d_n(0)$ is equal to Z_n. For $n > 1$, the infinite values for x_n come either from the anticonfined singularity, which provides one such value for $n = 1 \bmod (N-1)$ and m values for each instance of the confined singularity, starting at $n-1$, namely, Z_{n-1}. We thus have $d_n(\infty) = mZ_{n-1} + \psi_{N-1}(n)$, where $\psi_{N-1}(n)$ is a period-$(N-1)$ function. The periodic function can be written in terms of the $(N-1)$th root of unity ω ($\omega^{N-1} = 1$, with $\omega \neq 1$) as $\psi_{N-1}(n) = \sum_{k=0}^{N-2} \omega^{k(n-1)}/(N-1)$ (The case $N = 2$ is special, since there is an entry for infinity at every n, and thus $d_n(\infty) = mZ_{n-1} + 1$). Finally, we have for the degree the recursion $d_n = md_{n-1} + \psi_{N-1}(n)$ leading to a dynamical degree equal to m (plus a subdominant term coming from the periodic function).

What is interesting with mapping (22) is that it contains a family of integrable subcases. Indeed, for $m = 1$, the mapping

$$x_{n+N} + x_{n+1} = \frac{1}{x_{n+N-1}} + \frac{1}{x_n}, \tag{23}$$

belongs to the Gambier family of linearisable mappings. It possesses a confined singularity $\{0, \infty, (N-2 \text{ finite values})\}$. An anticonfined singularity is also present. It has the form

$$\{\cdots, 0, (N-2 \text{ finite values } \bullet), 0, (N-2 \text{ finite values } \bullet), x_0, x_1, \cdots, x_{N-2}, \infty, (N-2 \text{ finite values } \bullet), \infty, \cdots\}$$

which has the same structure as the singularity pattern found in the non-integrable, $m > 1$ case. And, as expected, the degree of the iterates is given by the recursion $d_n = d_{n-1} + \psi_{N-1}(n)$, with the periodic function ψ defined in the previous paragraphs.

Before concluding this section, it is interesting to give the integration of (23). We start by rewriting it as $x_{n+N} - 1/x_{n+N-1} = -(x_{n+1} - 1/x_n)$ and introduce $y_n = x_{n+1} - 1/x_n$. The mapping now simply becomes $y_{n+N-1} + y_n = 0$, i.e., an equation linear in y of order $N-1$. Solving it introduces $N-1$ integration constants, and one needs only to integrate the remaining homographic equation for x.

5. Conclusions

Integrability of discrete systems is related to low-growth properties, in particular for birational systems, to the degree growth of the iterates of some initial condition. This property is encapsulated in the dynamical degree, with a value larger than 1 for the latter signalling non-integrability. The practical way to compute the degree of the solution is usually done by obtaining the successive rational expressions and simplifying out the greatest common divisor. This approach, though straightforward, is time-consuming and becomes easily prohibitive, in particular for non-integrable systems.

Halburd has provided two different approaches aiming at simplifying this task. The first consists of starting from initial conditions that are pure rational numbers, in which case the greatest common divisor simplifications are computationally less heavy (Even thus, the calculation of the dynamical degree for non-integrable mappings may turn out to be computationally demanding). The other approach is based on singularities which, as we have explained, are at the origin of the simplifications of the rational expressions. This approach can give the degree of the solution of an integrable mapping. In the case of a non-integrable one, the dynamical degree can be obtained simply using the "express" method [28], but the degrees of the solution cannot be obtained in the closed form. In this work, we showed that by applying Halburd's method of 'balancing the singularities' (which is a way of saying that the calculation of the degree growth based on any of the values appearing in the singularity pattern should lead to the same result), the calculation of the degrees can be described algorithmically. Moreover, the computations use integer arithmetic with only addition, subtraction and, in some cases, multiplication by small integers and are thus lightning fast. We demonstrated the efficiency of our method by computing the explicit degrees for various systems, both integrable and not.

In the case of mappings of orders higher than two, we applied the prescription introduced in [29], whereupon the first initial values of the variables of the mapping are generic, and the degree is calculated only in terms of the last one. (For a mapping of order N, this means generic values for indices 0 to $N-2$ and choosing the variable with index $N-1$ as special). There is no guarantee that this approach would lead to the correct dynamical degree. However, in a slew of cases we studied, some of which were presented here, it turned out that the value of the dynamical degree we obtained coincided with the one given by the Diophantine approach.

Among the examples presented in this paper, one mapping was not of the birational type. Although the techniques we use have been applied for birational mappings, it turned out that the study of singularities can also be constructively applied to this non-birational case. Whether this can be generalised to other non-birational or even non-algebraic mappings [41] is an open question.

Another intriguing question is that of mappings which, although birational, do not possess rational invariants. The examples of what was called "solvable chaos" [42] spring to mind. Consider for instance the linear equation $\omega_{n+2}+\omega_{n-2}=2(\omega_{n+1}+\omega_{n-1})$ and introduce a variable x through $x=\tan\omega$. Using the addition formulae for the tangent, we obtain the mapping [43]

$$2\frac{1-x_{n+2}x_{n-2}}{x_{n+2}+x_{n-2}}=\frac{1-x_{n+1}x_{n-1}}{x_{n+1}+x_{n-1}}-\frac{x_{n+1}+x_{n-1}}{1-x_{n+1}x_{n-1}}. \tag{24}$$

The dynamical degree is $(1+\sqrt{3})/2+\sqrt{\sqrt{3}/2}\approx 2.29663$, and the sequence of degrees, 0, 0, 0, 1, 2, 4, 10, 23, 52, 120, 276, 633, 1454,... conforms with this value. The only singularities of the mapping are the two anticonfined ones $\{\cdots,\mp i^{\{2\}},\mp i,\bullet,\bullet,\bullet,\pm i,\pm i^{\{2\}},\pm i^{\{4\}},\pm i^{\{10\}},\cdots\}$, where the upper index (between braces) corresponds to the multiplicity of the root and directly gives the degree growth.

As can be judged from the analysis at the end of Section 4, already the analysis of a single concrete non-birational mapping can be extremely challenging, and the tools for tackling non-algebraic mappings are unfortunately still largely lacking. Hence, we will refrain from making any further statements concerning the general applicability of our present results to such very general settings. However, a domain where the fast algorithm may turn out to be useful is that of lattice systems. Thanks to recent results [44] by T. Mase, it is now clear how one can extend Halburd's singularity-based method to the case of partial difference equations and obtain the degree growth. We hope to address some of the open problems hinted at in this section in some future work of ours.

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