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Symmetries in String Theory

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Chapter 1

Introduction

The physics of elementary particles is currently described in terms of a very successful theory called the standard model. It describes all known elementary particles and their interactions except gravitational interactions. The standard model accommodates the quarks and the leptons which are the constituents of matter, the vector particles that mediate the strong and electroweak forces, and the Higgs boson which is expected to account for the masses of the particles. So far, the standard model agrees, often to great accuracy, with all experimental findings.

The only phenomena that are known not to fit within the standard model are those involving gravity, for which there is another successful theory: Einstein's general relativity. Because of its weakness, gravity is completely negligible in all particle scattering experiments that are done to test the standard model. It is because gravity is a long-range attractive force that we can see it act between macroscopic systems.

In both general relativity and the standard model, the interactions are governed by the principle of gauge invariance. Gauge invariances are symmetries of the equations of the theory under a group of local transformations, i.e. transformations that may vary from point to point in space-time. One can think of this as the freedom to choose a reference frame (in space-time or in some internal space) independently at each space-time point. For general relativity the underlying gauge invariance is the symmetry under general space-time coordinate transformations. In the standard model, the gauge symmetry is internal, i.e. it acts on fields rather than on space-time coordinates. The gauge group of the standard model that governs the strong and electroweak forces is $SU(3)_s \times SU(2)_{ew} \times U(1)_{ew}$. It is spontaneously broken to $SU(3)_s \times U(1)_e$ at low energies by the Higgs mechanism whereby three vector bosons of the electroweak gauge group acquire a mass and the photon of the electromagnetic interaction that corresponds to the unbroken $U(1)_e$ remains massless. A consequence of gauge invariance is that certain degrees of freedom in the description are redundant, i.e. not physical. However, they are needed for a symmetric description of the theory.

An essential difference between the standard model and general relativity is that the standard model is formulated as a quantum field theory whereas Einstein's general relativity is a classical field theory. This causes conceptual difficulties in trying to combine particle physics with gravity. One might try to couple quantum matter to classical gravity, but this seems a strange thing to do and it might not even be consistent. On the other hand, attempts to formulate gravity as a quantum field theory have failed because of the nonrenormalizability of gravity. The nonrenormalizability of gravity stems from the fact that Newton's constant G_N has negative mass dimension. (In units where $\hbar = c = 1$, G_N has dimension $(\text{mass})^{-2}$, see equation (1.1) below.) In the usual perturbative approach, dimensional analysis then shows that in higher loop Feynman diagrams involving gravitons, divergences become more and more serious.

Still, one can argue that quantum gravitational effects become important at the energy scale set by the Planck mass, which is the mass constructed out of Planck's constant, the speed of light and Newton's gravitational constant,

$$M_P = \sqrt{\frac{\hbar c}{G_N}} \approx 10^{19}\,\mathrm{GeV}/c^2\,. \tag{1.1}$$

Although such energies are far beyond those that one can hope to reach in particle accelerators, theorists are still very much interested in finding a consistent theory of quantum gravity. One of the motivations is the hope that a quantum theory of gravity might be united with particle physics into a single theory of all particles and forces. For other motivations and a comprehensive review of several issues in quantum gravity, see reference [120].

Notwithstanding the great experimental successes of the standard model, there are several reasons to expect that it is not the ultimate theory for particle physics. First of all, as we discussed above, quantum effects of general relativity can no longer be ignored as we approach the Planck scale, and gravity should somehow be coupled to the rest of the theory. Secondly, there are reasons that stem from the standard model itself. The standard model depends on many parameters, such as the different particle masses, mixing angles and coupling constants, whose values have to be put in by hand to fit experimental data. This is not something expected (or at least wanted) from a fundamental theory. Instead, one might believe the standard model to be an effective low-energy theory of a more fundamental underlying theory. More fundamental means that the underlying theory has a larger domain of validity, extending to smaller distances, or equivalently, to higher energies that have not (yet) been explored by particle accelerators. An effective low-energy theory can (in principle) be obtained from its underlying theory by integrating out all fluctuations above a certain energy.

As possible underlying theories, so-called grand unified theories (GUTs) have been proposed. A grand unified theory unifies the strong, weak and electromagnetic forces in the sense that they all become part of a larger gauge group with a single coupling constant. One indication for grand unification comes from renormalization group calculations, which show that the three effective (running) coupling constants corresponding to the three factors of the standard model gauge group meet approximately at a single energy $M_{GUT} \approx 10^{15}\,\mathrm{GeV}$. Below this scale, the GUT gauge group should be broken to the standard model gauge group in much the same way that the electroweak gauge group $SU(2)_{ew} \times U(1)_{ew}$ is broken to the electromagnetic gauge group $U(1)_e$ below the weak scale $M_W \approx 100\,\mathrm{GeV}$. However, in [6] it was calculated, starting from more precise data for the coupling constants at the weak scale, that the running couplings in the standard model miss each other by several standard deviations. It was also shown that in a modified version of the standard model, the minimal supersymmetric standard model (MSSM), the three running coupling constants meet quite accurately at an energy scale of about $10^{16}\,\mathrm{GeV}$, thus supporting supersymmetric grand unification[1]. This somewhat higher energy scale is also favoured in view of the present experimental limit on the proton lifetime. Note also that it is 'only' three orders of magnitude below the Planck energy.

The minimal supersymmetric standard model has a particle content which is richer than that of the standard model. This is due to supersymmetry, which associates to any boson a fermion and to any fermion a boson. Thus, in the MSSM all particles of the standard model are accompanied by a superpartner of opposite statistics, such that the theory is invariant under their interchange. However, none of these superpartners has ever been observed. Therefore, supersymmetry, if it exists, has to be broken at low energies. The favoured energy at which this is supposed to take place is on the order of $1000\,\mathrm{GeV}$.

One of the original motivations for supersymmetry is that it softens the divergences of quantum field theory. This may make the renormalization procedure seem less artificial. For example, in the standard model the Higgs boson is expected to have a mass not too far from the weak scale, but it gets radiative corrections of leading order Λ, which is the cut-off representing the scale at which new physics appears, say M_{GUT} or the Planck scale. This is not inconsistent but it is unnatural, since renormalization requires fine tuning of parameters to obtain the correct physical value. Supersymmetry solves this unnaturalness problem, since the leading divergent contribution to the boson's mass vanishes as a consequence of cancellations between bosonic and fermionic contributions. From a technical point of view, let us mention that supersymmetry often makes a theory more tractable. Nonrenormalization theorems and other special properties induced by supersymmetry may sometimes be used to compute exact results for the quantum theory. A recent example that received a lot of attention is [178].

In a (supersymmetric) grand unified theory, the strong and electroweak gauge interactions are unified. However, they still contain many a priori undetermined parameters and moreover, gravity is still not taken into account. In summary, we are looking for a unified theory for all interactions including (quantum) gravity with as few parameters as possible that reproduces the standard model at low energies and is preferably supersymmetric. Superstring theory seems to be the best candidate for such a theory. So let us review some basic properties of string theory.

String theory is an approach to describe all phenomena in terms of strings, i.e. one-dimensional objects. This is to be contrasted with the usual picture of quantum field theory as a second quantized theory of point-particles, described as elementary excitations of the fundamental fields. String theory introduces one new parameter: the string scale M_{str}. Its inverse[2] is the typical length of an oscillating string. However,

[1] See [77] for a recent account.

[2] In units with $\hbar = c = 1$ length (or time) has dimension $(\mathrm{mass})^{-1}$.

string theory contains no other parameters like those of the standard model. Unlike a point particle, a string carries many degrees of freedom, corresponding to its possible modes of oscillation. An interesting idea of string theory is that different elementary particles are manifestations of a single string oscillating in different modes, much as different vibrations of a violin string give rise to different tones. This is a very elegant way of unifying all elementary particles. Clearly, this means that strings should be very tiny since there are no indications for any stringy extendedness from experiment. A lot of excitement was caused by the recognition [197, 170] that one of the excitations of a closed string corresponds to a massless spin-two particle with the properties of a graviton. For this particle to have also the correct coupling strength of gravity, the length scale of the string should be of order the Planck length (the inverse of the Planck mass (1.1) in $\hbar = c = 1$ units), $l_P \approx 10^{-33}\,\mathrm{cm}$. As string theory is a quantum theory, it naturally includes quantum gravity.

Strings can be either open or closed. Open strings have two ends whereas closed strings are closed loops without ends. A further differentiation among string theories is provided by the internal degrees of freedom they may carry. The simplest string theory is the bosonic string. As its name suggests, its spectrum (the Hilbert space of allowed excitations) contains only bosonic degrees of freedom. Therefore, it is impossible for this simplest string theory to accommodate a theory like the standard model. However, the bosonic string is studied extensively because it is a good starting point to learn about string theory and to investigate some of its general properties. The consistency of the bosonic string can be questioned though, since its spectrum contains a particle of negative mass squared, a tachyon.

The tachyon is eliminated by the introduction of supersymmetry in string theory. Hence, supersymmetry is probably required in a consistent string theory. The superstring also necessarily contains fermionic degrees of freedom. Moreover, certain versions of the superstring turn out to contain 'standard model like' theories as their four-dimensional low-energy field theory limits. The superstring has many attractive features. Its ultraviolet properties are better than those of most field theories. It is known that one-loop amplitudes in superstring theory are finite. For higher loops there are no complete results, but supersymmetry properties provide good hope that the superstring is finite to all orders. This is good, especially since one might fear renormalization problems in a theory that contains quantum gravity. An intuitive way of understanding the mild ultraviolet behaviour of string theory is to think of the interactions among strings to be spread out along the length of the strings. In this sense string theory has a natural short-distance cut-off built in. Instead of a perturbation theory in terms of Feynman diagrams, the perturbation series of string theory consists of a sum over topologies of two-dimensional surfaces (note that as a string moves through space-time it sweeps out a two-dimensional surface, the world-sheet). Whereas in quantum field theory the number of Feynman diagrams grows rapidly with the order in the perturbation parameter, in string theory there is only one process to consider at any order (at least for a theory containing closed strings only).

To give an honest picture of string theory, we should also mention some problems encountered in its study. The most common criticism is the lack of possible experiments to test string theory. This is a problem, since the genuinely string-like aspects of the theory are believed to appear only near the Planck scale. Nevertheless, string theory makes predictions also for the low-energy theory, for example that supersymmetry exists. It might be that supersymmetric partners will be detected by the next generation of particle accelerators (e.g. CERN's LHC starting around 2004 will look for supersymmetric partners besides the Higgs boson(s)). Other exotic particles may also be predicted by string theory. This takes us immediately to another, this time calculational problem of string theory. There are many ways of obtaining realistic low-energy effective theories from string theory. However, there is no principle within the present formulation of the theory that selects one possibility out of the many. If we knew this principle we would probably also know the precise low-energy predictions made by string theory and experiments could be planned to verify or falsify the theory. It is also not clear how supersymmetry should be broken and how the cosmological constant can be small or zero after supersymmetry breaking. As string theory contains quantum gravity and a priori makes no reference to four dimensions, it should even 'explain' why we experience the world as four-dimensional and not otherwise. All these questions cannot yet be answered in string theory.

Most of these problems are probably due to our present imperfect formulation of string theory. In the usual world-sheet approach to string theory, one describes a first-quantized string moving in a target space-time of fixed geometry. Eventually however, the geometry should be determined dynamically by the strings themselves. Perhaps these matters have to await the formulation of a string field theory. Another shortcoming of the present understanding of string theory is the fact that it is only known perturbatively in the number of quantum loops (number of handles of the world-sheet). On the other hand, the fact that the string perturbation series diverges (at least for the bosonic string) should be understood as an indication that nonperturbative effects are very important [105].

In this thesis we discuss some of the different symmetries that occur in string theory. We have seen that in the search for a possible unified theory more and more symmetry is introduced to improve certain properties of a theory or to decrease the number of parameters. String theory has many symmetries that play an important role. In the world-sheet approach one considers a two-dimensional field theory which is reparametrization invariant. This can be considered a gauge symmetry of two-dimensional coordinate transformations, i.e. the gauge symmetry of two-dimensional gravity. It is natural to ask if we can base a string theory on an even larger world-sheet gauge invariance. In this thesis we address this question for a certain class of extended world-sheet symmetries. For simplicity, we only consider bosonic string theories. These involve bosonic degrees of freedom only. String theory also has many space-time gauge symmetries. Undeniably, there is some relation between world-sheet and space-time gauge symmetries. For example, world-sheet reparametrization invariance imposes certain restrictions on the physical spectrum of excitations of space-time fields. These give the massless spin-two particle the gauge properties of a graviton. Besides gravity, there are many more space-time gauge symmetries in string theory. For example, the massless sector also includes the gauge symmetry of an antisymmetric tensor field and possible Yang-Mills fields. Recently, there has been much interest in what are believed to be discrete gauge symmetries of string theory. These so-called duality symmetries map a string theory in one background to the same string theory in another background or even one string theory to another. In the last chapter of this thesis some of these duality symmetries are discussed.

To get more grip on a complicated theory, an approach that is often useful is to reduce the number of degrees of freedom in such a way that as many as possible features of the original theory are preserved. In string theory, a way to do this is to reduce the number of dimensions of the target space-time in which the string moves. Usually, consistency demands that a string can only move in a target space-time of definite 'critical' dimension; $D = 26$ for the bosonic string or $D = 10$ for the superstring. However, it is possible to reduce the number of space-time dimensions D at the cost of introducing a new field which should be regarded as a component of the two-dimensional world-sheet metric [159]. The resulting string theories are called non-critical. So far, only results for non-critical strings in $D \leq 2$ have been obtained. In this case, even nonperturbative calculations can be performed using the discrete approach of matrix models. It would be very interesting to see if such results could be extended to higher values of D. The problems for $D > 2$ can be traced back to the representation theory of the conformal algebra or Virasoro algebra, a remnant of world-sheet reparametrization invariance. The representation theory of certain extended conformal algebras known as W-algebras suggests that the equivalent of the $D = 2$ barrier for these extended algebras takes place at higher values of D. This is one of the motivations to study string theory based on such extended conformal symmetries. In the main part of this thesis, we will look at string theories with W-symmetry ('W-strings') and compare their properties with those of the ordinary (non-critical) string that is based on conformal symmetry only. It turns out that many relations exist between strings based on different world-sheet gauge symmetries.

The W-algebras that we use in our construction are nonlinear algebras. As a consequence of these nonlinearities, explicit computations can be quite complicated. We improve on this situation by introducing transformations of the variables that simplify many calculations. These simplifications are particularly welcome in the BRST formalism, a framework for quantization of gauge theories. In string theory, the BRST formalism has proved to be very useful. We use it to compute the physical spectrum of a string based on the W_4 algebra, a typical example of a W-algebra.

The other symmetries that we consider in this thesis are the duality symmetries in string theory. In the past few years, duality transformations have been a subject of intense study, both in string theory and field theory. Some duality transformations relate strings moving in different space-time backgrounds. Such dualities are referred to as target-space dualities or T-dualities. The best known example is the duality transformation which shows the equivalence of a string moving in a background with one coordinate compactified on a circle of radius R and the same string moving in the same background but with a radius l_{str}^2/R, where l_{str} is the typical length of a string. This suggests that in string theory there is a notion of minimal distance near the Planck length and that our usual view of space-time should not be relied upon too much for distances near or below the Planck scale. We describe T-dualities in the language of canonical transformations in the two-dimensional world-sheet theory. Also, we discuss the low-energy limit of string theory (which is represented as a field theory) and review its symmetries associated to T-duality.

Many other duality symmetries have been conjectured, and among them are dualities that interchange weak and strong coupling. These are particularly interesting since they provide information on the strong coupling behaviour of certain string theories. Moreover, the strong coupling limit of one string theory turns out in many cases to be another string theory (in weak coupling) or even a field theory or something else. This has raised questions such as ‘Is string theory unique?’ or ‘Is string theory a theory of strings?’. Anyhow, it is hoped that a better understanding of string dualities gives us more control over the vast number of four-dimensional effective theories that seem to be realizable within string theory. We investigate a possible strong/weak coupling duality between a string and another extended object called a fivebrane. Strong/weak coupling dualities acting within the string theory itself are also known. These are referred to as S-dualities. In particular, we look at a possible interchange of T and S-dualities under string/fivebrane duality. Also, we use certain duality symmetries in one of the superstrings, the type IIB string, to obtain new solutions of the low-energy theory from known ones.

The organization of this thesis is as follows. First, in chapter 2, we introduce the classical bosonic string in the usual formulation of a reparametrization invariant two-dimensional action. Choosing a specific gauge leads to a conformal field theory. We describe some general properties of two-dimensional conformal field theory, which plays an important role in string theory. We also briefly review some well-known variants of string theory and start to discuss the possibility of extended world-sheet gauge symmetries, in particular those described by W-algebras. Then in chapter 3 we discuss the quantization of the string, starting with the ordinary bosonic string. We use the BRST formalism and we start this chapter by introducing this formalism. Then we apply it to W-strings. We report on the construction of the BRST operators and on realizations for W-algebras needed for quantization of the W-string. In chapter 4, the physical spectra of string theories will be investigated. We start again with the ordinary bosonic string and then proceed to a discussion of W-string spectra. At the end of this chapter, we mention some relations that have been found between string theories based on different world-sheet gauge symmetry. Chapter 5 is about duality symmetries. First we review how low-energy effective actions are obtained, and a simple compactification procedure called dimensional reduction is explained. Then we discuss T-duality from the point of view of the low-energy effective theory as well as from the world-sheet point of view. We also consider strong/weak coupling dualities, in particular one relating strings and fivebranes. At the end of this chapter, we use duality transformations in the type IIB string to obtain some new solutions. Among them are solutions that have both ‘electric’ and ‘magnetic’ charges. We conclude with a short discussion (chapter 6).

Chapter 2

Strings and conformal field theory

This chapter starts with a description of the classical bosonic string. Special attention is paid to the gauge symmetries of the classical action and to the constraints that result after gauge-fixing. References for section 2.1 and for a general introduction to string theory are [104, 145, 108]. The connection with conformal field theory, which we review in section 2.2, is emphasized. The possibility of extended conformal symmetries is also discussed. More information on conformal field theory may be found in [96, 57, 61]. In section 2.3 we briefly discuss several known string models.

2.1 Classical string action

Let us start by considering a string moving in D-dimensional Minkowski space-time labeled by coordinates X^μ. Space-time indices $\mu, \nu, ...$ are raised and lowered with the Minkowski metric $\eta = \text{diag}[-1, 1, 1, ...1]$. The string sweeps out a two-dimensional surface in space-time, the world-sheet, which is parametrized by $\sigma^a = (\tau, \sigma)$ where $\sigma^0 = \tau$ is the time-evolution parameter and $\sigma^1 = \sigma$ runs along the length of the string.

The dynamics of the string may be deduced from an action proportional to the area of the world-sheet,

$$S_{NG} = -T \int dA = -T \int d^2\sigma \sqrt{-\det(\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu})}\,, \tag{2.1}$$

where $X^\mu(\tau, \sigma)$ describes the embedding of the world-sheet into space-time. The expression in the determinant is the metric on the world-sheet induced by the space-time metric. This action, called the Nambu-Goto action, is the direct analogue of the action of a relativistic point particle, which is proportional to the length of its world-line. The constant T is called the string tension, and it has dimension one over length squared in units in which the action is dimensionless.

The action (2.1) is not always a convenient starting point for string theory due to the presence of the square root. Another action, which is classically equivalent to (2.1) and does not contain the square root, is

$$S = -\frac{T}{2}\int d^2\sigma\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X_\mu\,, \tag{2.2}$$

where $h = \det h_{ab}$. This is called the Polyakov action[1]. The world-sheet metric h_{ab} is treated here as an independent variable. Its equation of motion is $T_{ab} = 0$ where we defined the world-sheet energy-momentum tensor

$$T_{ab} \equiv -\frac{1}{T}\frac{1}{\sqrt{-h}}\frac{\delta S}{\delta h^{ab}} = \tfrac{1}{2}\partial_a X^\mu \partial_b X_\mu - \tfrac{1}{4}h_{ab}h^{cd}\partial_c X^\mu \partial_d X_\mu\,. \tag{2.3}$$

Taking the determinant of $T_{ab} = 0$, one finds $\sqrt{-h}h^{ab}\partial_a X^\mu \partial_b X_\mu = 2\sqrt{-\det \partial_a X^\mu \partial_b X_\mu}$, and substitution in the Polyakov action gives back the Nambu-Goto action.

The Polyakov action is the usual starting point for first quantization of the string where one considers a sum over two-dimensional surfaces (string trajectories) in space-time. Note that from the world-sheet point of view, however, it defines a two-dimensional field theory of scalar fields X^μ coupled to two-dimensional gravity.

Let us discuss the symmetries of the Polyakov action. First of all, it is invariant under reparametrizations of the world-sheet, as it should be. For any diffeomorphism $\sigma^a \to \sigma'^a(\sigma)$ [2] the action is invariant under

$$h'_{ab}(\sigma') = \frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}h_{cd}(\sigma)\,, \qquad X'^\mu(\sigma') = X^\mu(\sigma)\,. \tag{2.4}$$

There is an additional local invariance due to the fact that in two dimensions the combination $\sqrt{-h}h^{ab}$ is invariant under local rescalings

$$h'_{ab}(\sigma) = \Lambda(\sigma)h_{ab}(\sigma)\,. \tag{2.5}$$

This clearly is a symmetry of the Polyakov action and is called Weyl invariance. So there are three gauge invariances in total, two diffeomorphisms and one Weyl rescaling. Later we will also discuss the possibility of string models based on even more local symmetries.

The action has global symmetries as well. Because space-time is Minkowskian, the action is invariant under the D-dimensional Poincaré transformations

$$h'_{ab}(\sigma) = h_{ab}(\sigma)\,, \qquad X'^\mu(\sigma) = \Lambda^\mu{}_\nu X^\nu(\sigma) + a^\mu\,, \tag{2.6}$$

where a^μ is a constant vector and Λ is a constant $O(D-1,1)$ matrix. For a string moving in a more general background, e.g. in some curved instead of Minkowskian space-time, the global symmetry is generally smaller.

[1] This form of the world-sheet action was introduced in [51]

[2] Here and in the following, the argument σ denotes possible dependence on both $\sigma^0 = \tau$ and $\sigma^1 = \sigma$, unless otherwise stated.

Reparametrization invariance implies the existence of conserved currents $j_b^f = f^a(\sigma)T_{ab}$ for any function $f^a(\sigma)$. Note that these currents vanish on-shell. It is a consequence of Weyl invariance that the trace of the energy-momentum tensor is zero, even without using the equations of motion, since for Weyl rescalings $\delta S = \int d^2\sigma \frac{\delta S}{\delta h_{ab}(\sigma)}\lambda(\sigma)h_{ab}(\sigma) = -T\int d^2\sigma\sqrt{-h}\lambda(\sigma)T^a{}_a(\sigma)$ with λ an arbitrary (infinitesimal) function, so $T^a{}_a$=0 because $\delta S = 0$. This can be verified directly in (2.3). Invariance under global Poincaré transformations leads to the space-time energy-momentum and angular momentum currents

$$\begin{aligned} P^a_\mu &= -T\sqrt{-h}h^{ab}\partial_b X_\mu \,, \\ J^a_{\mu\nu} &= X_\mu P^a_\nu - X_\nu P^a_\mu \,. \end{aligned} \tag{2.7}$$

The conservation law $\partial_a P^a_\mu = 0$ is the equation of motion for X^μ, from which $\partial_a J^a_{\mu\nu} = 0$ also follows.

We may now use the local invariances to choose a convenient gauge in which the subsequent analysis simplifies. Since there are three gauge invariances, the two-dimensional metric can be gauged away completely, at least locally. Thus we may take $h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. This is known as the conformal gauge. In this gauge the action takes the form

$$S = -\frac{T}{2}\int d^2\sigma \eta^{ab}\partial_a X^\mu \partial_b X_\mu \,. \tag{2.8}$$

However, this gauge-fixing does not eliminate the complete gauge symmetry; there are residual symmetries. If a reparametrization $\sigma \to \sigma'(\sigma)$ is such that

$$\frac{\partial\sigma^c}{\partial\sigma'^a}\frac{\partial\sigma^d}{\partial\sigma'^b}\eta_{cd} = \Lambda(\sigma)\eta_{ab} \,, \tag{2.9}$$

we can undo this transformation by a corresponding Weyl rescaling. The conformal gauge is then preserved. The residual symmetry transformations satisfying (2.9) constitute the two-dimensional conformal group. The action (2.8) is invariant under this group[3]. Conformal invariance has proven to be a very important principle in string theory and the 'language' of conformal field theory, described in the next section, is useful for many calculations in string theory, see e.g. the review papers [90, 174].

The equation of motion for X^μ derived from (2.8) is simply the free wave equation

$$(\frac{\partial^2}{\partial\sigma^2} - \frac{\partial^2}{\partial\tau^2})X^\mu = 0 \,, \tag{2.10}$$

whose general solution is

$$X^\mu(\tau,\sigma) = X^\mu_R(\tau-\sigma) + X^\mu_L(\tau+\sigma) \,. \tag{2.11}$$

Here both X_R and X_L are arbitrary functions of their arguments; they are called right-moving and left-moving string coordinates, respectively. It is useful to introduce the

[3]Equation (2.8) represents an action for free scalar fields and therefore has even more symmetries besides conformal (and Poincaré) invariance.

so-called light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$, in which the components of the metric become $\eta_{++} = \eta_{--} = 0\,,\ \eta_{+-} = \eta_{-+} = -\frac{1}{2}$, and $\partial_{\pm} \equiv \frac{\partial}{\partial\sigma^{\pm}} = \frac{1}{2}(\partial_0 \pm \partial_1)$. The equation of motion then reads

$$\partial_+\partial_- X^\mu = 0\,. \tag{2.12}$$

We should also take care of the boundary terms that arise in the variation of the action. In the variational principle, the initial and final configurations of the string are fixed, so the fields are not varied at the spacelike boundaries $\tau = \tau_i$ and $\tau = \tau_f$. For a closed string, which has no ends, there are no other boundary terms, but we should impose periodicity $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$. The open string world-sheet has additional boundaries: the trajectories of the endpoints of the string. The corresponding boundary terms in the variation of the action are required to vanish which gives for open strings the boundary condition $\frac{\partial}{\partial\sigma}X^\mu = 0$ at both ends.

Now we can write down the general solution of (2.12) which respects the boundary conditions for the string coordinates. For a closed string the Fourier expansion is

$$X^\mu_R(\sigma^-) = \tfrac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^- + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\alpha^\mu_n e^{-in\sigma^-}\,, \tag{2.13}$$
$$X^\mu_L(\sigma^+) = \tfrac{1}{2}x^\mu + \frac{1}{4\pi T}p^\mu\sigma^+ + \frac{i}{\sqrt{4\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\bar{\alpha}^\mu_n e^{-in\sigma^+}\,.$$

The normalizations of the various modes have been chosen for later convenience. Besides the zero modes x^μ and p^μ there are also positive and negative frequency modes which describe the oscillations of the string. This is what makes strings very different from point particles, of course. The α and $\bar{\alpha}$ oscillators are referred to as right and left-moving modes, or right and left-movers, respectively. Reality of X^μ implies the relations

$$(\alpha^\mu_n)^* = \alpha^\mu_{-n} \quad \text{and} \quad (\bar{\alpha}^\mu_n)^* = \bar{\alpha}^\mu_{-n}\,. \tag{2.14}$$

For the open string, the general solution is

$$X^\mu(\sigma, \tau) = x^\mu + \frac{1}{\pi T}p^\mu\tau + \frac{i}{\sqrt{\pi T}}\sum_{n\neq 0}\tfrac{1}{n}\alpha^\mu_n e^{-in\tau}\cos n\sigma\,. \tag{2.15}$$

Here we used the convention that σ runs from 0 to π along the open string[4]. Notice that there is only one set of oscillators α, because the boundary conditions relate right and left-moving waves on the string.

This is not yet the end of the story, since we still have to impose $T_{ab} = 0$ on the solutions (2.13) and (2.15). We missed this equation of motion because the world-sheet metric h_{ab} was replaced by the fixed metric η_{ab} in our gauge-fixed action (2.8). This is something to be aware of in gauge theories: upon fixing a gauge and substituting it in the action, one may lose equations of motion which then have to be implemented separately as

[4]We use the conventions of [145]. It is often convenient to choose a system of units in which $T = \frac{1}{\pi}$ for the open string and $T = \frac{1}{4\pi}$ for the closed string. We shall do this later on.

constraints. In the conformal gauge the components of the energy-momentum tensor in the $\sigma^{\pm}$ coordinate system, $T_{\pm\pm} = \frac{\partial\sigma^a}{\partial\sigma^{\pm}}\frac{\partial\sigma^b}{\partial\sigma^{\pm}}T_{ab}$, are

$$\begin{aligned} T_{--} &= \tfrac{1}{4}(T_{00} - 2T_{01} + T_{11}) = \tfrac{1}{2}\partial_- X^\mu \partial_- X_\mu \,, \\ T_{++} &= \tfrac{1}{4}(T_{00} + 2T_{01} + T_{11}) = \tfrac{1}{2}\partial_+ X^\mu \partial_+ X_\mu \,, \\ T_{+-} &= T_{-+} = \tfrac{1}{4}(T_{00} - T_{11}) = 0\,. \end{aligned} \tag{2.16}$$

Note that $T_{+-} = T_{-+} = 0$ expresses the Weyl invariance of the original theory. Conservation of energy-momentum $\partial^a T_{ab} = 0$ now translates into $\partial_- T_{++} = 0$ and $\partial_+ T_{--} = 0$. This indeed follows immediately from (2.16) and the equations of motion (2.12). Thus the dynamics of the string is described by the solutions (2.13) or (2.15), supplemented by the constraints $T_{++} = 0$ and $T_{--} = 0$.

For later use, let us also briefly discuss the classical string in a Hamiltonian formulation. For the gauge-fixed action (2.8), the momentum conjugate to X^μ is

$$P_\mu \equiv \frac{\partial L}{\partial \dot{X}^\mu} = T\dot{X}_\mu\,. \tag{2.17}$$

We use the conventional notation $\dot{X}^\mu = \partial_0 X^\mu$ and $X'^\mu = \partial_1 X^\mu$. The equal time Poisson brackets are given by

$$\begin{aligned} \{X^\mu(\sigma), X^\nu(\sigma')\} &= \{\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\} = 0\,, \\ \{X^\mu(\sigma), \dot{X}^\nu(\sigma')\} &= \frac{1}{T}\eta^{\mu\nu}\delta(\sigma - \sigma')\,. \end{aligned} \tag{2.18}$$

The Hamiltonian for the string in the conformal gauge is

$$H = \int d\sigma(\dot{X}^\mu P_\mu - L) = \frac{T}{2}\int d\sigma(\dot{X}^2 + X'^2)\,. \tag{2.19}$$

Using the fundamental brackets (2.18), we may calculate the Poisson brackets of the energy-momentum tensor. The result is

$$\begin{aligned} \{T_{--}(\sigma), T_{--}(\sigma')\} &= -\frac{1}{2T}\left[2T_{--}(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma T_{--}(\sigma)\delta(\sigma - \sigma')\right]\,, \\ \{T_{++}(\sigma), T_{++}(\sigma')\} &= \frac{1}{2T}\left[2T_{++}(\sigma)\partial_\sigma\delta(\sigma - \sigma') + \partial_\sigma T_{++}(\sigma)\delta(\sigma - \sigma')\right]\,, \end{aligned} \tag{2.20}$$

while the other possible brackets vanish. One defines the so-called Virasoro charges as the Fourier modes of the energy-momentum tensor,

$$L_m = 2T\int d\sigma e^{-im\sigma}T_{--}\,, \qquad \bar{L}_m = 2T\int d\sigma e^{im\sigma}T_{++}\,. \tag{2.21}$$

From (2.20) it then follows that the modes satisfy the algebra

$$\begin{aligned} \{L_m, L_n\} &= -i(m-n)L_{m+n}\,, \\ \{\bar{L}_m, \bar{L}_n\} &= -i(m-n)\bar{L}_{m+n}\,, \\ \{L_m, \bar{L}_n\} &= 0\,, \end{aligned} \tag{2.22}$$

under Poisson brackets. This is the classical Virasoro algebra, and L_m and $\bar{L}_m$ are the generators of the infinite-dimensional conformal group in two dimensions. To see this, note that the conformal transformations, which satisfy (2.9), take the infinitesimal form

$$\delta\sigma^+ = \varepsilon f^+(\sigma^+)\,, \qquad \delta\sigma^- = \varepsilon f^-(\sigma^-)\,, \tag{2.23}$$

where $f^\pm$ are arbitrary functions of their argument, and ε is an infinitesimal parameter. The associated conserved currents are given by

$$j^f_+ = f^+(\sigma^+)T_{++}\,, \qquad j^f_- = f^-(\sigma^-)T_{--}\,. \tag{2.24}$$

Current conservation is expressed by $\partial_- j^f_+ + \partial_+ j^f_- = 0$. The two terms vanish separately, since $\partial_- T_{++} = \partial_+ T_{--} = 0$. Now, if we choose for the functions $f^\pm$ the complete sets $e^{im\sigma^\pm}$ satisfying the periodicity condition of the closed string, we get conserved charges $Q_m = \int d\sigma e^{im\sigma^-} T_{--}$ and $\bar{Q}_m = \int d\sigma e^{im\sigma^+} T_{++}$, closely related to the Virasoro charges L_m and $\bar{L}_m$.

The Poisson brackets (2.18) together with the expansion (2.13) can be shown to yield the algebra of oscillators

$$\{\alpha^\mu_m, \alpha^\nu_n\} = \{\bar{\alpha}^\mu_m, \bar{\alpha}^\nu_n\} = -im\delta_{m+n,0}\eta^{\mu\nu}\,. \tag{2.25}$$

All brackets between barred and non-barred oscillators vanish. The position and momentum of the string have the usual Poisson bracket $\{x^\mu, p^\nu\} = \eta^{\mu\nu}$. It is also conventional to define

$$\alpha^\mu_0 = \bar{\alpha}^\mu_0 = \frac{1}{\sqrt{4\pi T}} p^\mu\,, \tag{2.26}$$

hence (2.25) is also valid for $m = 0$ or $n = 0$. Then

$$\begin{aligned} \partial_- X^\mu_R &= \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{+\infty} \alpha^\mu_n e^{-in\sigma^-}\,, \\ \partial_+ X^\mu_L &= \frac{1}{\sqrt{4\pi T}} \sum_{n=-\infty}^{+\infty} \bar{\alpha}^\mu_n e^{-in\sigma^+}\,, \end{aligned} \tag{2.27}$$

from which we find that p^μ equals the total conjugate momentum of the string:

$$\int d\sigma P^\mu = T \int d\sigma \dot{X}^\mu(\sigma) = T \int d\sigma (\partial_- X^\mu_R + \partial_+ X^\mu_L) = p^\mu\,. \tag{2.28}$$

Expressed in terms of oscillators, the Virasoro generators are

$$L_m = \tfrac{1}{2} \sum_n \alpha_{m-n} \cdot \alpha_n\,, \qquad \bar{L}_m = \tfrac{1}{2} \sum_n \bar{\alpha}_{m-n} \cdot \bar{\alpha}_n\,, \tag{2.29}$$

and the Hamiltonian is

$$H = \tfrac{1}{2} \sum_n (\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n) = L_0 + \bar{L}_0\,. \tag{2.30}$$

The notation $A \cdot B$ is used here to mean $A^\mu B_\mu$. The constraints $T_{++} = 0$ and $T_{--} = 0$ translate into $L_m = 0$ and $\bar{L}_m = 0$. Thus we see that $H = 0$ is one of the constraints, and this gives a mass-shell condition

$$M^2 \equiv -p^\mu p_\mu = 2\pi T \sum_{n \neq 0} (\alpha_{-n} \cdot \alpha_n + \bar{\alpha}_{-n} \cdot \bar{\alpha}_n) \,. \tag{2.31}$$

Using also $L_0 - \bar{L}_0 = 0$, we see that for closed strings the right and left-moving sectors are connected through

$$M^2 = 8\pi T \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 8\pi T \sum_{n=1}^{\infty} \bar{\alpha}_{-n} \cdot \bar{\alpha}_n \,. \tag{2.32}$$

2.2 Conformal field theory and W-algebras

The formalism of conformal field theory has found many uses in physics after the pioneering work of Belavin, Polyakov and Zamolodchikov [16]. For example, in statistical mechanics, the critical behaviour of systems near a second order phase transition point can be described by conformal field theory. Scaling and conformal invariance had been used before to determine some relations between critical exponents, but in [16] it was shown that in two dimensions much more can be done. For some two-dimensional models all critical exponents can be computed exactly using conformal invariance.

Here we are interested primarily in the application of conformal field theory to string theory. We have seen in the previous section that (2.8) is the action of a two-dimensional conformal field theory because the symmetries left over from two-dimensional coordinate invariance and Weyl invariance constitute the conformal group. It turns out that, for consistency reasons, conformal invariance must be maintained upon quantization of the string moving in some background (for which the world-sheet action is not a free field action in general). This means that conformal field theory (CFT) is of great importance for the study of possible string backgrounds. We will return to this point later. Now we wish to review some general results of [16] concerning two-dimensional quantum conformal field theory. Some considerations also apply to the classical theory, though. In the next chapter we discuss the quantization of the string as a conformal field theory with constraints.

The conformal group consists of the transformations that leave the metric invariant modulo a local scale transformation as in equation (2.9). In D-dimensional Minkowski space with $D > 2$, the conformal transformations consist of Poincaré transformations, dilatations (scale transformations) and special conformal transformations, together forming a group that is locally isomorphic to $SO(D, 2)$. For more details, see e.g. [96]. It is only in two dimensions that the group of conformal transformations is infinite-dimensional. This is because conformal transformations are those transformations that leave all angles invariant, and in two dimensions these are precisely the complex analytic coordinate transformations

$$z \to f(z) \,, \qquad \bar{z} \to \bar{f}(\bar{z}) \,. \tag{2.33}$$

It is therefore convenient to formulate two-dimensional CFT on the complex plane, and in order to do so in the case of the closed string, we have to map the cylinder (world-sheet) to the complex plane. Complex coordinates on the cylinder are defined by first performing a Wick rotation to Euclidean signature, $\tau \to -i\tau$, and then defining

$$\zeta = i\sigma^- = \tau - i\sigma \,, \qquad \bar{\zeta} = i\sigma^+ = \tau + i\sigma \,. \tag{2.34}$$

Then we perform the conformal transformation

$$\zeta \to z = e^{\zeta} = e^{\tau - i\sigma} \,, \qquad \bar{\zeta} \to \bar{z} = e^{\bar{\zeta}} = e^{\tau + i\sigma} \,, \tag{2.35}$$

which maps the cylinder to the complex plane with coordinate z. Note that time evolves along the radial coordinate $|z|$; the infinite past corresponds to $z = 0$ and the infinite future to $z = \infty$. Lines of equal time are concentric circles around $z = 0$.

It is sometimes convenient to think of z and $\bar{z}$ as independent complex variables. So we write both z and $\bar{z}$ as the arguments of a general field. Holomorphic fields, depending on z only, are written as $A(z)$, and anti-holomorphic fields, depending on $\bar{z}$ only, as $\bar{A}(\bar{z})$[5]. The analytic substitutions of z and $\bar{z}$ can then be considered independent, and the conformal group is therefore a direct product of them. At the end one should of course remember that $\bar{z}$ is the complex conjugate of z.

In a two-dimensional quantum conformal field theory, the complete set of operators decomposes into representations of the conformal group, so-called conformal families $[\phi_m]$. These conformal families consist of a primary operator and an infinite number of secondary operators. Primary operators, denoted here by ϕ_m, are characterized by their special behaviour under conformal transformations. They transform covariantly (as tensors) under (2.33) with $z' = f(z)$ and $\bar{z}' = \bar{f}(\bar{z})$,

$$\phi'_m(z', \bar{z}') = \left(\frac{\partial z'}{\partial z}\right)^{-h_m} \left(\frac{\partial \bar{z}'}{\partial \bar{z}}\right)^{-\bar{h}_m} \phi_m(z, \bar{z}) \,, \tag{2.36}$$

where h_m and $\bar{h}_m$ are called the conformal weights of ϕ_m. The simplest example of a primary operator is the identity operator with $h = \bar{h} = 0$. All other operators of a conformal family are called secondary, and among them are all derivatives of any operator in $[\phi_m]$. Their transformations under (2.33) are more complicated.

As we have seen in the previous section, the modes of the energy-momentum tensor are the generators of conformal transformations. In CFT, the energy-momentum tensor has only two independent components, since the trace vanishes as a consequence of scale invariance. In complex coordinates, the two components T_{zz} and $T_{\bar{z}\bar{z}}$ (cf. T_{--} and T_{++} in light-cone coordinates) are holomorphic and anti-holomorphic[6] and are denoted simply by $T(z)$ and $\bar{T}(\bar{z})$, respectively.

The variation of a field A under the infinitesimal transformations $z \to z - \varepsilon(z)$ and $\bar{z} \to \bar{z} - \bar{\varepsilon}(\bar{z})$ is given by the 'equal-time' commutator

$$\delta_{\varepsilon,\bar{\varepsilon}} A(w, \bar{w}) = [Q_{\varepsilon,\bar{\varepsilon}}, A(w, \bar{w})] \,, \tag{2.37}$$

[5]Note that the bar on $\bar{A}$ has nothing to do with complex conjugation in general.

[6]In the classical theory, this follows from the equations of motion. In the quantum theory it is true inside correlation functions except at points where other operators are inserted.

where the charge $Q_{\varepsilon,\bar{\varepsilon}}$ is given by the spatial integral of the radial (time) component of the corresponding current, which on the complex plane means

$$Q_{\varepsilon,\bar{\varepsilon}} = \frac{1}{2\pi i} \oint \left(dz\, \varepsilon(z) T(z) + d\bar{z}\, \bar{\varepsilon}(\bar{z}) \bar{T}(\bar{z}) \right) . \tag{2.38}$$

The line integral is performed over some circle of fixed radius in the counter-clockwise sense.

Now, as is customary in quantum field theory in Euclidean space, products of operators are only defined if they are time-ordered. In the present case this means that a product $A(z)B(w)$ is only defined for $|z| > |w|$. Thus we define the radial (time) ordering operation R as

$$R(A(z)B(w)) = \begin{cases} A(z)B(w) & |z| > |w| \\ B(w)A(z) & |w| > |z| \end{cases} . \tag{2.39}$$

This allows us to evaluate the commutator in (2.37) as follows:

$$\begin{aligned} \delta_{\varepsilon,\bar{\varepsilon}} A(w,\bar{w}) &= \frac{1}{2\pi i} \left(\oint_{|z|>|w|} - \oint_{|z|<|w|} \right) \Big[dz\, \varepsilon(z) R(T(z)A(w,\bar{w})) \\ &\qquad\qquad + d\bar{z}\, \bar{\varepsilon}(\bar{z}) R(\bar{T}(\bar{z})A(w,\bar{w})) \Big] \\ &= \frac{1}{2\pi i} \oint \left(dz\, \varepsilon(z) R(T(z)A(w,\bar{w})) + d\bar{z}\, \bar{\varepsilon} R(\bar{T}(\bar{z})A(w,\bar{w})) \right) , \end{aligned} \tag{2.40}$$

where the two contours in the first line are deformed into a single contour integration of z around w in the last line.

Since everything works the same for holomorphic and anti-holomorphic parts of the theory, we will from now on concentrate on the holomorphic part only. We have learned that the variation of a field A under the infinitesimal transformation $z \to z - \varepsilon(z)$ is given by

$$\delta_\varepsilon A(w,\bar{w}) = \oint \frac{dz}{2\pi i} \varepsilon(z) T(z) A(w,\bar{w}) , \tag{2.41}$$

Here and in the following, we will assume operator products to be radially ordered, and the ordering symbol R will be omitted. Comparing (2.41) with the infinitesimal transformation of a primary field according to (2.36),

$$\delta_\varepsilon \phi_m(z,\bar{z}) = (h_m \partial\varepsilon + \varepsilon\partial)\phi_m(z,\bar{z}) , \tag{2.42}$$

we deduce, using the calculus of residues, that for a primary field ϕ_m

$$T(z)\phi_m(w,\bar{w}) = \frac{h_m \phi_m(w,\bar{w})}{(z-w)^2} + \frac{\partial\phi_m(w,\bar{w})}{z-w} + \text{regular part} . \tag{2.43}$$

We employ the usual notation $\partial = \frac{\partial}{\partial z}$ and $\bar{\partial} = \frac{\partial}{\partial \bar{z}}$. Also note that the regular part of (2.43) is not determined by the transformation law (2.42). Usually, only the singular part is of interest. It is this part which contains the information of the corresponding equal-time commutator (or Poisson bracket, classically).

Equation (2.43) is our first example of an operator product expansion. The operator product expansion (OPE) is one of the most important tools of conformal field theory. It was originally introduced by Wilson [193] in more general quantum field theories. However, in the case of CFT a lot more can be said about the general form of OPEs. It is assumed that the (infinite) set of all local operators $\{A_i(z,\bar{z})\}$ is complete in the following sense:

$$A_i(z,\bar{z})A_j(0,0) = \sum_k C_{ij}^k(z,\bar{z})A_k(0,0)\,, \tag{2.44}$$

where $C_{ij}^k(z,\bar{z})$ are single-valued functions. This is the operator product algebra of the theory, and it implies that any quantum state can be generated by the action of a linear combination of local operators on the vacuum. One should interpret (2.44) as a prescription valid inside correlation functions, usually valid in a region around $z=0$ which excludes all insertion points of other operators. From the operator product algebra one can extract all equal-time commutators between fields [193].

The functions $C_{ij}^k(z,\bar{z})$ in (2.44) are severely restricted by conformal invariance. For example, the OPE of two primary fields can be written as

$$\begin{aligned}\phi_m(z,\bar{z})\phi_n(0,0) &= \sum_p \sum_{\{k\}} \sum_{\{\bar{k}\}} C_{mn}^{p;\{k\}\{\bar{k}\}} \\ &\quad \times z^{h_p-h_m-h_n+\sum k_i}\, \bar{z}^{\bar{h}_p-\bar{h}_m-\bar{h}_n+\sum \bar{k}_i}\, \phi_p^{\{k\}\{\bar{k}\}}(0,0)\,,\end{aligned} \tag{2.45}$$

where $\phi_p^{\{k\}\{\bar{k}\}}$ are the secondary fields belonging to the conformal family $[\phi_p]$. They are defined below, in equation (2.61). The sets of positive integers k_i and $\bar{k}_i$ label the secondary fields. The primary fields in this notation are $\phi_p^{\{0\}\{0\}}$. The constants $C_{mn}^{p;\{k\}\{\bar{k}\}}$ can be represented as

$$C_{mn}^{p;\{k\}\{\bar{k}\}} = C_{mn}^p \beta_{mn}^{p;\{k\}} \bar{\beta}_{mn}^{p;\{\bar{k}\}}\,, \tag{2.46}$$

where the factors β and $\bar{\beta}$ can be calculated in terms of the conformal weights of the primary fields involved. These calculations in general get very complicated and an algorithm for their computation has been implemented in a Mathematica [196] package that computes OPEs [185], see also [186]. Thus, only the constants C_{mn}^p are not determined by conformal invariance. Their values are restricted by the requirement of associativity of the operator product algebra. However, the Jacobi identities are too complicated to solve in the general case.

From the conformal properties of the energy-momentum tensor itself (conformal weight two), one can deduce

$$\underbrace{T(z)T}(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,, \tag{2.47}$$

where $\underbrace{A(z)B}(w)$ denotes the contraction of two fields, which is defined to be the singular part of their OPE. In general, the parameter c is nonzero. Note that $T(z)$ is primary only for $c=0$. The constant c is called the central charge and depends on the particular CFT being studied.

The OPE (2.47) is the quantum version of the classical Poisson bracket algebra (2.20). It encodes the algebra of conformal transformations. Often, a nonzero central charge is a quantum effect, representing an anomaly, i.e. a breakdown of conformal symmetry due to quantization. We will also use OPEs for classical Poisson brackets. This makes calculations in the classical and quantum theory very similar. The Poisson bracket

$$\{A(z), B(w)\} = \sum_{n>0} \{AB\}_n(w) \partial_z^{n-1} \delta(z-w) \tag{2.48}$$

may equivalently be represented by the classical OPE

$$A(z)B(w) = \sum_{n>0} \frac{[AB]_n(w)}{(z-w)^n}\,, \tag{2.49}$$

with the correspondence $\{AB\}_n = \frac{1}{(n-1)!}[AB]_n$.

The modes of T and $\bar{T}$ are defined in the same way as was done in (2.21), apart from a normalization factor. If we define

$$L_n = \oint \frac{dz}{2\pi i} z^{n+1} T(z)\,, \tag{2.50}$$

then L_n here corresponds to the same L_n of (2.21). Comparing (2.50) with (2.38), we see that L_n is the generator of the conformal transformation $z \to z - \varepsilon z^{n+1}$. The inverse of (2.50) expresses T as the Laurent expansion

$$T(z) = \sum_{n=-\infty}^{+\infty} z^{-n-2} L_n\,. \tag{2.51}$$

In general, the mode expansion of an operator $A(z)$ of conformal weight h is defined by

$$A(z) = \sum_{n=-\infty}^{+\infty} z^{-n-h} A_n\,. \tag{2.52}$$

This implies that under a scaling $z \to \lambda z$ the modes scale as $A_n \to \lambda^n A_n$. An OPE can always be represented equivalently by the commutation relations of the modes involved. From (2.47) the following mode algebra is derived:

$$[L_m, L_n] = (m-n)L_{m+n} + \tfrac{c}{12} m(m^2-1)\delta_{m+n,0}\,. \tag{2.53}$$

This is the Virasoro algebra. Similar formulas exist for $\bar{L}_m$ and $\bar{T}$, but holomorphic and anti-holomorphic sectors[7] of the theory are independent, since mixed commutators vanish.

The generators $\{L_{-1}, L_0, L_1\}$ span an $sl(2)$ subalgebra of the Virasoro algebra (2.53). The operator L_{-1} simply generates translations; for $z' = z - \varepsilon$ with ε a constant, we

[7] Recall that these correspond to the right and left-moving sectors, respectively, in the string language of the previous section.

have

$$\begin{aligned}
\delta_\varepsilon A(w,\bar w) &= \oint \frac{dz}{2\pi i}\varepsilon T(z)A(w,\bar w) = \oint \frac{dz}{2\pi i}\varepsilon \sum_n (z-w)^{-n-2}L_n(w)A(w,\bar w) \\
&= \varepsilon L_{-1}(w)A(w,\bar w)\,, \qquad (2.54)
\end{aligned}$$

Here the operators $L_n(w)$ are given by the contour integrals

$$L_n(w) = \oint \frac{dx}{2\pi i}(x-w)^{n+1}T(x)\,, \qquad (2.55)$$

these being the Laurent modes of $T(z)$ around the point w. Thus, $L_{-1}(z)A(z,\bar z) = \partial A(z,\bar z)$ for any field $A(z,\bar z)$, and also $\bar L_{-1}(\bar z)A(z,\bar z) = \bar\partial A(z,\bar z)$. In the same way we have for the generators L_0 and $\bar L_0$ of dilatations $z' = z - \varepsilon z$ and $\bar z' = \bar z - \bar\varepsilon \bar z$, $L_0(z)A(z,\bar z) = hA(z,\bar z)$ and $\bar L_0(\bar z)A(z,\bar z) = \bar h A(z,\bar z)$. The sum of the conformal weights $h+\bar h$ is called the scaling dimension. The spin of a field is defined by $s = h - \bar h$, and characterizes the behaviour under rotations of the complex plane generated by $i(L_0 - \bar L_0)$. As in (2.30), the Hamiltonian is given by $H = L_0 + \bar L_0$, since this is the generator of translations in the time (radial) direction. Therefore, the scaling dimension of an operator is the energy of the state created by this operator. Finally, the operators L_1 and $\bar L_1$ generate special conformal transformations, $z' = z - \varepsilon z^2$ and $\bar z' = \bar z - \bar\varepsilon \bar z^2$, infinitesimally. It can be shown that $\{L_{-1}, L_0, L_1; \bar L_{-1}, \bar L_0, \bar L_1\}$ generate the group $SL(2,\mathbb{C})$ of global conformal transformations

$$z \to z' = \frac{az+b}{cz+d}\,, \qquad a,b,c,d \in \mathbb{C}\,, \quad ad - bc = 1\,. \qquad (2.56)$$

These are the only globally defined conformal transformations that map $\mathbb{C}$ one-to-one onto $\mathbb{C}$.

The vacuum state $|0>$ of the theory must satisfy

$$L_n|0>= 0 \quad \text{for} \;\; n \geq -1\,. \qquad (2.57)$$

Otherwise, the energy-momentum tensor would be singular at $z=0$ ($\tau = -\infty$). The L_n's with $n \geq -1$ generate conformal transformations that are regular at $z=0$, so (2.57) is a manifestation of conformal invariance of the vacuum state. To obtain a Hilbert space structure, one introduces the vacuum at $z = \infty$ ($\tau = +\infty$), $< 0|$. Hermitian conjugation then involves the map $z \to z' = 1/z$. The modes of a primary field $A(z)$ corresponding to a real (Hermitian) field on the Minkowskian cylinder can be shown to satisfy

$$A_n^\dagger = A_{-n}\,. \qquad (2.58)$$

Regularity of $T(z)$ at $z=\infty$ implies

$$< 0|L_n = 0 \quad \text{for} \;\; n \leq 1\,. \qquad (2.59)$$

Note that L_{-1}, L_0 and L_1 annihilate both 'in' and 'out' vacua. We will refer to $|0>$ as the $sl(2)$-invariant vacuum.

2.2.1 Representations of the Virasoro algebra

The Hilbert space of states must form a representation of the symmetry algebra, which in the case of a generic CFT is the Virasoro algebra. Therefore, it is important to know something about the representations of the Virasoro algebra. We mainly discuss here one particular class of representations, namely the conformal families that were already mentioned. They are representations in field space. At the level of states[8], these representations are called Verma modules.

A state created by a primary operator, $|\phi_m>\equiv \lim_{z\to 0}\phi_m(z)|0>$, has the following properties:

$$L_0|\phi_m>=h_m|\phi_m> \quad \text{and} \quad L_n|\phi_m>=0 \quad \text{for} \quad n>0\,. \tag{2.60}$$

The actions of L_{-n}'s with $n>0$ create all secondary states, which are in one-to-one correspondence with the secondary fields of the conformal family $[\phi_m]$. Let us now explain the notation for the secondary fields in (2.45):

$$\phi_m^{\{k\}\{\bar{k}\}}=L_{-k_1}...L_{-k_q}\bar{L}_{-\bar{k}_1}...\bar{L}_{-\bar{k}_r}\phi_m\,, \tag{2.61}$$

where the L_{-k} are the operators defined in equation (2.55), k_i and $\bar{k}_j$ are positive integers and their numbers q and r can be any nonnegative integer. Among the secondary fields, there are so-called quasi-primary fields which are defined by the requirement that $L_1\phi=0$. They transform like primary operators under the global conformal group. The energy-momentum tensor is an example of a quasi-primary operator. Usually, it is enough to consider only primary operators, since conformal Ward identities imply that correlation functions involving secondary fields can always be expressed in terms of correlation functions of primary fields only.

The state $|\phi_m>$ satisfying (2.60) is called a highest weight state of the Virasoro algebra. Together with all descendants (secondaries), it forms a representation of the Virasoro algebra, called a Verma module. Highest weight representations are the physically relevant ones, since they ensure that energy is bounded from below. (For example, using the Virasoro commutation relations, one easily shows that the state corresponding to the operator in (2.61) has $\sum k_i+\sum\bar{k}_j$ units of energy ($L_0+\bar{L}_0$ eigenvalue) relative to the highest weight state.)

An important question is whether or not a particular Verma module constitutes an irreducible representation of the Virasoro algebra. Let us address this question now.

A Verma module can be decomposed into sectors of definite conformal weight, i.e. sectors of states with the same L_0 eigenvalue. Because of the relation $[L_0,L_n]=-nL_n$, the action of L_n on a state increases the conformal weight by $-n$. The level of a state is defined by its conformal weight minus that of the primary from which it stems. At a certain level, there might be states with zero norm, depending on the central charge of the theory and the dimension of the highest weight state. Such zero norm states are called null vectors, and the simplest example is $L_{-1}|0>$. Its norm[9] vanishes, whatever

[8]There is a one-to-one correspondence between states and operators via the relation $|A>=A(0)|0>$.

[9]Norms are calculated using $L_n^\dagger=L_{-n}$. (Actually, (2.58) is generally true for quasi-primary fields.) The Virasoro algebra (2.53) is then used to commute lowering operators to the right to annihilate $|0>$.

the value of the central charge:

$$< 0|L_1 L_{-1}|0 >= 2 < 0|L_0|0 >= 0\,. \tag{2.62}$$

At level 2, a less trivial example is

$$\left(L_{-2} - \frac{3}{2(2h+1)} L_{-1}^2 \right) |h >\,, \tag{2.63}$$

where $|h >$ is a primary state of conformal weight h. Using the Virasoro algebra, one can show that this state is null in a theory with central charge $c = \frac{2h(5-8h)}{(2h+1)}$. A null vector can be viewed itself as a highest weight state, which means that it defines its own submodule of the Virasoro algebra. A Verma module therefore constitutes an irreducible representation only if it contains no null vectors. It is consistent to divide out a null vector simply by putting it to zero. This eliminates the complete submodule generated by the null vector.

The general formula that tells us when null vectors exist and at what level, is the expression of the Kac determinant [123] (proven in [80]),

$$\det M_N(c,h) = \alpha_N \prod_{pq \leq N} (h - h_{p,q}(c))^{P(N-pq)}\,. \tag{2.64}$$

It is the determinant of the matrix of inner products of all states at level N in the Verma module $M(c,h)$ of a primary field of weight h in a CFT with central charge c. If it is zero, there is a null vector at level N. The product in (2.64) is over all positive integers p, q whose product is less than or equal to N, and α_N is a constant independent of c and h. By $P(N-pq)$ we denote the number of partitions of $N-pq$ into positive integer parts. It represents the multiplicity of the zeroes, since a null vector at level pq gives rise to $P(N-pq)$ null states at the higher level N, these being its descendants. The values $h_{p,q}$ are given by

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}\,, \tag{2.65}$$

where m (in general complex) parametrizes the central charge as

$$c = 1 - \frac{6}{m(m+1)}\,. \tag{2.66}$$

Another important question is whether a Verma module is a unitary representation. Again, we give the result [91] without further arguments. In the region $c \geq 1$ and $h \geq 0$ the Kac determinant never changes sign at any level, so in this region the Verma modules are (or at least can be made) irreducible unitary representations. On the other hand, for $0 < c < 1$ there is only a discrete set of unitary representations, with m in (2.65) and (2.66) taking one of the values $m = 3, 4, 5, \ldots$. If $c < 0$, there will always be negative norm states in a Verma module.

For correlation functions involving primary fields with null descendants[10], one can derive linear differential equations. This provides further restrictions on such correlation functions. In particular, it is possible to derive selection rules for OPEs, known as fusion rules, which determine the possible primary fields that can appear on the RHS of OPEs involving degenerate fields. The fusion rules for the degenerate primaries $\phi_{p,q}$ of conformal weight $h_{p,q}$ may be expressed as

$$\phi_{p_1,q_1} \times \phi_{p_2,q_2} = \sum_{p_3=|p_1-p_2|+1}^{p_1+p_2-1} \sum_{q_3=|q_1-q_2|+1}^{q_1+q_2-1} \phi_{p_3,q_3} \,, \tag{2.67}$$

where p_3 (q_3) runs over even integers, provided $p_1 + p_2$ ($q_1 + q_2$) is odd and vice versa. Equation (2.67) shows that the degenerate conformal families form a closed operator algebra. For special values of the central charge in the region $0 < c < 1$, namely the series (2.66) with $m = 3, 4, 5, ...$, the bounded 'grid' of conformal families $[\phi_{p,q}]$ with $p = 1, 2, ...m - 1$ and $q = 1, 2, ...m$, forms a closed operator product algebra. This 'truncation from above' of the fusion rules is a consequence of the existence of additional null vectors at these values of c. The families $[\phi_{p,q}]$ are called completely degenerate in this case and contain two independent null vectors, due to the symmetry $(p, q) \to (m - p, m - q + 1)$ of the highest weights (2.65). Since there are only a finite number of conformal families involved in such operator algebras, the models composed of these conformal families are called minimal. In general, minimal models exist for central charges

$$c = 1 - \frac{6(r-s)^2}{rs} \,, \tag{2.68}$$

with r and $s > r$ relatively prime positive integers[11]. However, only for $s = r + 1$ these minimal models are unitary.

The case $m = 2$ in (2.66), corresponding to $c = 0$, yields a trivial minimal model. It contains only the identity operator. All its descendants are null, because $L_{-1}\mathbf{1} = 0$ and $L_{-2}\mathbf{1} = T(z)$ is null as well for $c = 0$. A number of minimal models have been shown to describe statistical models undergoing a second order phase transition. As an example, the first non-trivial member of the minimal model series ($m = 3$) has central charge $c = \frac{1}{2}$ and describes the critical point of the Ising model. In chapter 4 we will see that minimal models also arise in the study of W-string spectra.

2.2.2 The example of free scalar fields

We shall now illustrate part of the formalism introduced above by a simple application: the closed bosonic string of the previous section. The action in conformal gauge (2.8) is simply the action of D free massless bosons. From now on we take for the string tension $T = \frac{1}{4\pi}$ and write everything in $(z, \bar{z})$ coordinates. The action then becomes

$$S = \frac{1}{4\pi} \int d^2z \partial X^\mu \bar{\partial} X_\mu \,, \tag{2.69}$$

[10] Such primary fields are called degenerate.
[11] Note that this corresponds to (2.66) with rational $m = r/(s - r)$.

and the propagator of the field X^μ is calculated from

$$\frac{1}{2\pi}\partial_z\partial_{\bar z} < X^\mu(z,\bar z)X^\nu(w,\bar w) >= -\eta^{\mu\nu}\delta^2(z-w,\bar z-\bar w)\,. \tag{2.70}$$

The solution is

$$< X^\mu(z,\bar z)X^\nu(w,\bar w) >= -\eta^{\mu\nu}\log|z-w|^2\,. \tag{2.71}$$

The field $X^\mu(z,\bar z)$ is not a well-defined field due to the logarithmic infrared divergence. However, the equations of motion still imply $X^\mu(z,\bar z) = X^\mu(z) + \bar X^\mu(\bar z)$ everywhere except at coinciding points in correlation functions, and ∂X^μ and $\bar\partial \bar X^\mu$ are well-defined conformal fields with propagators[12]

$$\underbrace{\partial X^\mu(z)\partial X^\nu}_{}(w) = \frac{-\eta^{\mu\nu}}{(z-w)^2}\,, \qquad \underbrace{\bar\partial\bar X^\mu(\bar z)\bar\partial\bar X^\nu}_{}(\bar w) = \frac{-\eta^{\mu\nu}}{(\bar z-\bar w)^2}\,. \tag{2.72}$$

The holomorphic component of the energy-momentum tensor is given by

$$T(z) = -\frac{1}{2} : \partial X^\mu(z)\partial X_\mu(z) : \,. \tag{2.73}$$

Normal-ordering, denoted by : ... :, is necessary because of the singularity in the OPE (2.72). It is defined here as

$$: \partial X^\mu(w)\partial X^\nu(w) :\equiv \lim_{z\to w}\left(\partial X^\mu(z)\partial X^\nu(w) + \frac{\eta^{\mu\nu}}{(z-w)^2}\right)\,. \tag{2.74}$$

This is equivalent to taking the constant $O((z-w)^0)$ part of the OPE. Henceforth, we will use this same normal-ordering prescription for composite operators, which in the general case is expressed by [9]

$$: A(z)B(z) :\equiv \oint \frac{dw}{2\pi i}\frac{A(z)B(w)}{z-w}\,. \tag{2.75}$$

We shall usually omit the normal-ordering symbols. The normal-ordering prescription is not associative, so when a product of more than two operators at the same point is considered, the convention is to start normal-ordering from the right,

$$: A(z)B(z)C(z) :\equiv : A(z)\ : B(z)C(z) : : \,. \tag{2.76}$$

In order to calculate OPEs between composite operators, we also need the following Wick rule [9]

$$\underbrace{A(z)\ : B}_{}C : (w) = \oint \frac{dx}{2\pi i}\frac{1}{x-w}\left(\underbrace{A(z)B}_{}(x)C(w) + B(x)\underbrace{A(z)C}_{}(w)\right)\,. \tag{2.77}$$

Using the Wick rule, we calculate the OPE of $\partial X^\mu(z)$ with the energy-momentum tensor and see that it is a primary of weight $(h,\bar h) = (1,0)$:

$$\underbrace{T(z)\partial}_{}X^\mu(w) = \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w}\,. \tag{2.78}$$

[12]The propagator or two-point function is the term in the OPE which contains the unit operator.

Now the OPE of T with itself can be calculated using the Wick rule once again, and the result is

$$\underbrace{T(z)T}(w) = \frac{D/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}\,. \qquad (2.79)$$

Thus, the central charge is D and each scalar field contributes 1 to it. The fourth order pole comes from double contractions in

$$T(z)T(w) = \frac{1}{4}\ :\partial X^\mu \partial X_\mu : (z)\ :\partial X^\nu \partial X_\nu : (w)\,. \qquad (2.80)$$

There are two double contractions that can be made, resulting in $\frac{1}{2}\eta^{\mu\nu}\eta_{\mu\nu}/(z-w)^4$, which is the first term in (2.79).

We can construct other primaries from the field $X^\mu(z,\bar{z})$. The operators

$$V_p(z,\bar{z}) =: e^{ip\cdot X(z,\bar{z})} : \qquad (2.81)$$

are primary with conformal dimension $(\frac{p^2}{2}, \frac{p^2}{2})$. Classically, however, a field $e^{ip\cdot X}$ would have no conformal dimension, because X^μ is dimensionless. The conformal dimension of (2.81) is a quantum effect, and is due to double contractions in the OPE $T(z)V_p(w,\bar{w})$. The operators (2.81) are called vertex operators. In string theory their insertion on the world-sheet represents an incoming string in its ground state with momentum p_μ.

2.2.3 W-algebras

The Virasoro algebra plays an important role in string theory because it is the underlying world-sheet symmetry algebra of the bosonic string. It is interesting now, to think about the possibility of extending the world-sheet symmetry. We have seen that the Virasoro algebra is the algebra of modes of the (quasi-)primary spin-two generator $T(z)$. Extended conformal algebras are obtained by the inclusion of additional generators. Then the Virasoro algebra encoded in the OPE $T(z)T(w)$ constitutes a subalgebra. In fact, extensions by generators of spins less than two have been studied for quite a long time. Examples include the $N=1$ superconformal algebra which has an additional spin-$\frac{3}{2}$ fermionic current and the $N=2$ superconformal algebra which has two fermionic spin-$\frac{3}{2}$ currents and a spin-1 current besides the energy-momentum tensor. We will normally use the name W-algebra for an extended conformal algebra with at least one current of spin greater than two. Some other and more precise definitions of W-algebras exist in the literature, but for our purpose the description just given is sufficient. For a review of W-symmetry in conformal field theory, see [49]. There is also a recent volume of reprints on W-symmetry [50].

Zamolodchikov was the first to construct W-algebras in [198]. He argued that if among the primary fields in the theory there is a field Q_s with conformal dimensions $(h,\bar{h}) = (s,0)$ (and therefore spin s), where s is some integer or half-integer number, then there is an additional infinite symmetry in such a theory. A field with $\bar{h}=0$ is often called a chiral field; it is necessarily holomorphic, $\bar{\partial}Q_s = 0$. Therefore, an infinite number of conserved currents take the form

$$j_s^f(z) = f(z)Q_s(z)\,, \qquad \bar{\partial}j_s^f(z) = 0\,, \qquad (2.82)$$

where $f(z)$ is an arbitrary analytic function. These currents generate the additional symmetry. The set of all holomorphic fields generates what is called the chiral algebra $\mathcal{A}$ of the theory. Similarly, the anti-chiral algebra $\bar{\mathcal{A}}$ is generated by the set of anti-holomorphic fields. The full symmetry algebra is the direct product

$$\mathcal{G} = \mathcal{A} \otimes \bar{\mathcal{A}} . \tag{2.83}$$

The Hilbert space of the theory should therefore decompose into representations of $\mathcal{G}$,

$$\mathcal{H} = \bigoplus_{m,\bar{m}} [\phi_m] \otimes [\bar{\phi}_{\bar{m}}] . \tag{2.84}$$

Of special interest are the so-called rational conformal field theories (RCFTs) in which the sum in (2.84) is over a finite number of representations of $\mathcal{G}$.

We have seen that minimal models are RCFTs with the Virasoro algebra as chiral algebra. Minimal models with W-symmetry also exist. Although they usually contain an infinite number of conformal families (Virasoro representations), they decompose into a finite number of representations of the W-algebra. As usual, we shall restrict ourselves to the chiral part $\mathcal{A}$ of the symmetry algebra.

Most W-algebras are nonlinear. This means that normal-ordered products of generators appear in the commutator (OPE) of two generators. To see this, consider a W-algebra whose maximal spin generator is Q_s with spin s. The OPE $Q_s(z)Q_s(w)$ generally produces (in the second order pole) operators with spin $2s-2$, which is greater than s for $s > 2$. Such operators consist of normal-ordered products of some currents. This argument cannot be applied to W-algebras with an infinite number of currents with increasing spins. Indeed, linear W-algebras with an infinite number of higher spin currents are known [11, 163].

A characteristic example of a W-algebra is the W_3 algebra [198]. It is the algebra generated by the energy-momentum tensor $T(z)$ together with an additional spin-three current $W(z)$. In terms of OPEs, the algebra is given by

$$\begin{aligned}
\underbrace{T(z)T(w)} &= \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} , \\
\underbrace{T(z)W(w)} &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w} , \\
\underbrace{W(z)W(w)} &= \frac{c/3}{(z-w)^6} + \frac{2T(w)}{(z-w)^4} + \frac{\partial T(w)}{(z-w)^3} \\
&\quad + \frac{3}{10}\frac{\partial^2 T(w)}{(z-w)^2} + \frac{1}{15}\frac{\partial^3 T(w)}{z-w} \\
&\quad + \frac{16}{22+5c}\left(\frac{2\Lambda(w)}{(z-w)^2} + \frac{\partial\Lambda(w)}{z-w}\right) ,
\end{aligned} \tag{2.85}$$

where $\Lambda =: TT : -\frac{3}{10}\partial^2 T$. The only nonlinear term is the normal-ordered product of two energy-momentum tensors. The first equation in (2.85) gives the Virasoro algebra, while the second equation expresses the fact that W is a primary field of spin three. Also for more general W-algebras, it is customary to work in a basis in which all generators (except $T(z)$) are primary with respect to $T(z)$. The last equation of (2.85) tells us that the OPE of two spin-three generators gives rise to the conformal family of the unit operator. Up to a normalization, the coefficients in this equation are all determined by conformal invariance (cf. β-coefficients in (2.46)). The W_3 algebra exists (satisfies the Jacobi identities) for any value of the central charge c.

The W_3 algebra has been generalized to a series of W-algebras involving higher spin generators by Fateev and Lukyanov [78]. These are called W_N algebras and consist of generators of spins $2, 3, 4...N$, the spin-two generator being the energy-momentum tensor. In [78] the generators of a W_N algebra are constructed out of free scalar fields. This has the advantage that if a closed algebra is obtained (closure of W_N algebras was proven in [144]), it automatically satisfies the Jacobi identities. In the following chapters we investigate the possibility of string models based on W_N symmetries.

Although we will not use W-algebras other than the W_N series, we wish to spend a few words on some other known W-algebras, and especially their construction. There are essentially three ways by which W-algebras are constructed, see [49] for a detailed review and [50] for reprints and an extensive list of references.

The first approach is to try to write down an extended algebra by proposing a number of extra generators with given spins and closing the algebra. Usually, the extra generators are assumed to be primary and therefore the form of the OPEs is already fixed by conformal invariance as in (2.45). The difficult step in this approach is to guarantee that the algebra is associative, thereby restricting the OPE coefficients C^p_{mn}. The W_3 algebra (2.85) was in fact obtained in this way in [198]. Other W-algebras with two and three generators, among them the W_4 algebra, were constructed according to this approach in the papers [35] and [127].

A second, much more systematic approach of constructing W-algebras is the so-called Drinfeld-Sokolov reduction. This associates a W-algebra to any embedding of $sl(2)$ into a simple Lie algebra $\mathbf{g}$, see [10, 38] and references therein. This results in a large class of W-algebras. The W_N algebras are obtained if one takes the so-called principal embedding of $sl(2)$ into $sl(N)$.

A third approach is to start from a known model of a CFT and to see if there are extended symmetries in that model. Additional currents are then formed from the fields in the model. For example, in a free field theory, currents may be constructed out of the free fields. The closure of the algebra should be checked and the Jacobi identities are automatically satisfied. Another example is given by the Casimir algebras of reference [9]. In this paper it was shown that W-algebras can be obtained from an affine Lie algebra $\hat{\mathbf{g}}$ by the construction of the Casimir invariants in terms of the affine currents. The W_N algebras are the Casimir algebras of $\widehat{sl(N)}$.

Let us end this section with some remarks on representations of W-algebras. Highest weight representations of W-algebras are defined in the same way as those for the Virasoro algebra. The highest weight state is now characterized by its weights under all generators of the W-algebra. For the W_N algebra with generators $W^{(k)},\ k = 2, 3, ...N$, of spin k, one defines a highest weight state $|w> \equiv |w^{(2)}, w^{(3)}, ...w^{(N)}>$ by the requirement

$$W_0^{(k)}|w>=w^{(k)}|w>\,,$$
$$W_n^{(k)}|w>=0 \quad \text{for} \quad n>0\,. \tag{2.86}$$

The modes of the currents are defined by the usual expansion $W^{(k)}(z) = \sum_{-\infty}^{\infty} z^{-n-k} W_n^{(k)}$. The W_N representations are then given by the set of states obtained by acting with linear combinations of strings of 'raising operators' $W_{-n}^{(k)}$ $(n > 0)$ on the highest weight state.

As mentioned before, minimal models for W_N algebras also exist. Their central charges and operator contents were given in [79] for W_3 and in [78] for W_N in general. The central charges of unitary W_N minimal models are given by

$$c = (N-1)\left(1 - \frac{N(N+1)}{m(m+1)}\right)\,, \tag{2.87}$$

with $m \in \{N, N+1, N+2, ...\}$. These minimal models describe the critical behaviour of certain $\mathbb{Z}_N$ symmetric statistical systems [79, 78]. An interesting feature is that unlike Virasoro minimal models, W_N minimal models also exist for $c \geq 1$. Thus, an infinite number of Virasoro primary fields in $c \geq 1$ models may sometimes be rearranged into a finite number of W-algebra primary fields.

2.3 Some different types of string theories

In the first section of this chapter we described the classical action of the ordinary bosonic string. It is the simplest among the string theories in that it is the direct one-dimensional analogue of the relativistic particle. In the next chapter we will demonstrate the well-known result that a consistent quantization requires this string to move in a space-time of dimension $D = 26$. In order to make contact with four-dimensional physics, 22 of the 26 dimensions would somehow have to be 'compactified' such that they are invisible at low energies. Also, the bosonic string turns out to have a tachyon in its physical spectrum (see chapter 4), and no fermions. This means that the bosonic string can never yield a realistic theory of known elementary particles and their interactions. Therefore, other types of strings have been constructed. The superstring has much better physical properties; there are no tachyons and it has fermions as well as bosons in its physical spectrum. The supersymmetry of the superstring also improves the convergence of scattering amplitudes. It still needs to live in an 'unphysical' space-time dimension $D = 10$, though. However, this is not an insurmountable problem, in principle. Six of the ten dimensions may be compactified (in accordance with the equations of the theory, of course) to an internal space of incredibly small size that can't be observed by any of our instruments. Different compactification procedures give rise to different possible gauge interactions and matter content of the four-dimensional low-energy theory; some of them even contain the standard model gauge group and particles.

The classical action of the superstring is given by

$$S = -\frac{1}{8\pi} \int d^2\sigma \{\partial_a X^\mu \partial^a X_\mu - i\bar{\psi}^\mu \rho^a \partial_a \psi_\mu\}\,, \tag{2.88}$$

where ψ^μ is a Majorana spinor and ρ^a are two-dimensional Dirac matrices. From the space-time point of view, ψ^μ is a vector. This action is invariant under the supersymmetry transformations

$$\begin{aligned} \delta X^\mu &= \bar{\varepsilon}\psi^\mu\,, \\ \delta\psi^\mu &= -i\rho^a \partial_a X^\mu \varepsilon\,, \end{aligned} \tag{2.89}$$

where ε is an infinitesimal constant anti-commuting Majorana spinor. The action (2.88) should be compared to the bosonic string action in the conformal gauge (2.8). Here it is a gauge-fixed version of a locally supersymmetric action, i.e. the action of two-dimensional supergravity coupled to the 'matter' fields X and ψ. As a consequence of the gauge-fixing, the equations of motion of (2.88) have to be supplemented by two constraints:

$$J_a = 0 \quad \text{and} \quad T_{ab} = 0\,. \tag{2.90}$$

The current J_a is the conserved Noether current associated to the supersymmetry (2.89), and T_{ab} is the energy-momentum tensor. Together, J and T generate the $N = 1$ superconformal algebra. The superstring is the best known example of a string based on extended conformal symmetry.

It is known that superstring theory not only possesses world-sheet supersymmetry but that it also gives rise to a space-time supersymmetric physical spectrum (and amplitudes). This is normally an $N = 2$ supersymmetry, i.e. there are two space-time supersymmetry generators. However, space-time supersymmetry is not at all manifest in the approach based on the action (2.88). A world-sheet action in which space-time supersymmetry is manifest from the beginning was introduced by Green and Schwarz, see [104] for additional information.

We now list the well-known consistent superstring theories. They all need to live in ten-dimensional space-time.

- Type I: a theory of open[13] strings. The open string boundary conditions break one half of the supersymmetries, so the remaining space-time supersymmetry is $N = 1$. Gauge charges can be attached to the ends of an open string. The only acceptable gauge group turns out to be $SO(32)$.
- Type IIA: a theory of closed strings only, and unbroken $N = 2$ space-time supersymmetry. The two supersymmetries are of opposite chirality. There is no freedom to introduce Yang-Mills fields. They can only appear after compactification.
- Type IIB: same as type IIA, except that the two supersymmetries have the same chirality.

[13]Closed strings can be formed in the interacting theory by the joining of the end-points of an open string.

- Heterotic string: closed strings only. Here the fact that left and right-moving sectors of the theory are independent is utilized. Supersymmetry is introduced only in one of the sectors. The other sector is purely bosonic and gives rise to Yang-Mills fields. Consistency demands either $SO(32)$ or $E_8 \times E_8$ gauge symmetry. The heterotic string has $N = 1$ space-time supersymmetry.

Strings based on extended world-sheet supersymmetry ($N = 2$ and $N = 4$) have also been considered. Their physical interpretation, if any, does not seem to be very clear yet. In the previous section we discussed, among other things, different extensions of conformal symmetry, namely W-symmetry. It is expected to be possible to construct string theories based on W-algebras [33]. The full world-sheet action should then have a local W-symmetry (W-gravity). In the gauge-fixed version of this action, the vanishing of the currents of the W-algebra must be imposed as constraints. In the next two chapters we will describe some aspects of W-strings.

So far, we discussed the possibility of string theories based on different world-sheet gauge algebras. Another property that might distinguish different string theories has also been mentioned: the world-sheet topologies allowed, i.e. open or closed strings[14]. We have not yet discussed the possibility to choose different realizations of the world-sheet gauge algebras corresponding to different (extended) conformal field theories. For the Virasoro algebra we described an explicit realization in terms of the string coordinates X^μ in equation (2.73), corresponding to the action (2.69). We can also study string theory in more general backgrounds or, in other words, take a different conformal field theory as a starting point. However, it is believed that taking different CFTs corresponds to having different vacua of the same string theory.

As an example, let us consider the bosonic string again, and let it propagate in a more general background described by the nonlinear sigma model

$$S = -\frac{1}{4\pi\alpha'} \int d^2\sigma \sqrt{-h} \{ g_{\mu\nu}(X) h^{ab} \partial_a X^\mu \partial_b X^\nu + B_{\mu\nu}(X) \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)} \Phi(X) \} , \qquad (2.91)$$

where $g_{\mu\nu}$, $B_{\mu\nu}$ and Φ are the background metric, antisymmetric tensor and dilaton field. $R^{(2)}$ is the world-sheet curvature scalar and ε_{ab} is the antisymmetric symbol. The string tension has been expressed in terms of the constant $\alpha' = \frac{1}{2\pi T}$. This action is reparametrization invariant and the first two terms are also invariant under Weyl rescalings of the metric. It has been shown that for a consistent quantization the theory should be Weyl invariant at the quantum level, and this is only true if the background fields satisfy certain equations[15] [54]. In fact, any conformal field theory[16] may serve as a string background. However, the bulk of CFTs do not seem to have a space-time interpretation. Nevertheless, it is clear that a classification of CFTs is important for string theory: it amounts to a classification of all possible string vacua. The problem then arises which CFT should be regarded as the true vacuum and why.

[14] See [155] for a nice exposition.

[15] These background field equations are given, to lowest order, in section 5.1.

[16] Apart from some restrictions such as the critical value of the central charge and modular invariance.

This is an unsolved problem. However, we will see in chapter 5 that not all seemingly different string backgrounds are inequivalent. There are certain transformations called T-dualities that map one CFT to another one describing the same string dynamics.

Not only is it known that certain string backgrounds are equivalent, there are also indications that even string theories based on different world-sheet algebras can be equivalent. It was first shown by Berkovits and Vafa [29] that the $N = 1$ superstring in a special background is equivalent to the bosonic string. Later, similar relations were found between strings based on different N-extended superconformal algebras or even different W-algebras. However, the meaning of this is still not very clear, since all these relations are only valid in very special backgrounds. In section 4.4 we consider some examples.

Perhaps more promising are some of the recently conjectured strong/weak coupling dualities in string theory. They relate the weakly coupled phase of one string (or a more general extended object) to the strongly coupled phase of another. This will be further discussed in section 5.3.

Finally, as another class of strings we should mention the non-critical strings. In the case of the bosonic string it is impossible to respect both diffeomorphism and Weyl invariance in the quantum theory, unless we are in the critical dimension. However, one might choose to give up Weyl invariance, in which case the two-dimensional metric becomes a dynamical field quantum mechanically[17] [159]. In this way, it is possible to obtain strings moving in dimensions below the critical one. Such strings are called non-critical. It would for example be interesting to look at non-critical strings in four space-time dimensions. However, for the bosonic string there are serious problems for $D > 2$, related to the presence of tachyons in the spectrum. As these problems seem to be shifted to higher dimensions in the case of W-strings, this is one of the motivations for studying (non-critical) W-strings. More on this in section 3.2.1.

[17]Therefore, non-critical strings might also teach us something about two-dimensional quantum gravity.

Chapter 3

BRST quantization of strings

In the previous chapter, we described the classical string and the formalism of conformal field theory. We now turn to the quantization of the string, and at many places conformal field theory techniques will be helpful. The constraints encountered in the preceding chapter must somehow be implemented at the quantum level. We will use the BRST formalism to do this. Section 3.1 describes some aspects of the quantization of gauge theories with emphasis on the BRST formalism. In section 3.2 we discuss the BRST quantization of the bosonic string. We also mention the possibility of non-critical strings. In the last section of this chapter, the BRST quantization of W-strings is considered. The latter section contains a summary of the papers [22, 21, 20, 39].

3.1 BRST quantization of gauge theories

Classically, the presence of gauge symmetry means that general solutions of the equations of motion involve arbitrary functions of the coordinates. This indicates that not all fields in the classical action describe independent physical degrees of freedom. In a standard canonical quantization of a gauge theory, one would thus also quantize unphysical degrees of freedom. The presence of such unphysical degrees of freedom is usually reflected in the existence of negative norm states in the Hilbert space. Therefore, it is important to identify the true physical degrees of freedom in a gauge theory. One way to get rid of the unphysical degrees of freedom is to use the gauge transformations to fix a gauge at the classical level. This means that one imposes certain conditions on the fields. These conditions should be accessible: in any gauge equivalence class of field configurations there must be a representative which satisfies the conditions. Also, one would like to fix the gauge completely, such that these representatives are unique. Any non-trivial gauge transformation then leads out of the gauge. Next, one should identify a complete set of independent physical degrees of freedom. Only these physical degrees of freedom need to be quantized. However, there are some problems with this seemingly straightforward approach. First of all, it is not always easy to find good gauge conditions which eliminate all gauge symmetry. Usually, it is also a difficult task to obtain a complete set of independent physical degrees of freedom. Moreover, after imposing gauge conditions, some global symmetries (e.g. Lorentz invariance) of the theory may no longer be manifest.

It is often possible to choose gauge conditions which are manifestly symmetric under the global symmetries of the theory. However, such covariant gauge conditions usually do not fix the gauge completely. Therefore, quantization may still be problematic, in that for example negative norm states may be present in the Hilbert space. Additional constraints must be imposed to eliminate them.

A procedure which does not suffer from the problems mentioned above is BRST quantization. This way of dealing with the gauge symmetry is named after Becchi, Rouet, Stora and Tyutin [15], and is based on a global symmetry of an effective action that is obtained by adding a gauge-fixing term plus compensating ghost terms to the original gauge invariant action. The gauge invariance of the original action is replaced by this global symmetry generated by the BRST charge.

The BRST charge associated to the BRST invariance of the effective action depends on the particular gauge-fixing that is used. In the Hamiltonian formulation, however, the BRST charge does not depend on any kind of gauge-fixing. Already at the classical level the gauge symmetry may be encoded in terms of the BRST charge by enlarging the phase space to include ghost variables. One of the main advantages of the BRST approach is then that covariance is preserved because no gauge conditions need to be imposed. Instead, the unphysical degrees of freedom are eliminated by the procedure of taking the cohomology of the BRST charge. This assumes that the BRST charge is nilpotent, which reflects the closure of the gauge symmetry algebra. All degrees of freedom, including the ghosts, are to be quantized and the cohomology of the quantum BRST charge yields the physical spectrum. Below, we summarize some important ideas of the BRST approach. In particular, we discuss the construction of the classical BRST charge, which will be useful later when we consider BRST charges for gauge theories based on W-algebras. Extensive discussions of the quantization of gauge theories in the Hamiltonian formalism can be found in [110, 111].

Before turning to the BRST formalism, let us first look at a simple example of a gauge theory, Maxwell's theory, to illustrate part of the preceding discussion. Details may be found in most textbooks on quantum field theory. The action is given by

$$S = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}\,, \tag{3.1}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This action is invariant under the gauge transformations

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x)\,, \tag{3.2}$$

where Λ is an arbitrary function of the coordinates. The infinite set of Noether currents associated to the gauge symmetry (3.2) is given by

$$j^\mu_\Lambda = F^{\mu\nu}\partial_\nu\Lambda\,. \tag{3.3}$$

The momenta conjugate to A_μ are

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} = F^{\mu 0}\,, \tag{3.4}$$

and this yields the primary constraint $T_1 \equiv \Pi^0 = 0$. Using this, we find that the canonical Hamiltonian is

$$\begin{aligned} H &= H_0 - \int d^3x A_0 \partial_i \Pi^i\,, \\ H_0 &= \int d^3x \left(\tfrac{1}{2}\Pi^i\Pi_i + \tfrac{1}{4}F_{ij}F^{ij}\right)\,. \end{aligned} \tag{3.5}$$

There is another constraint which follows from the consistency condition $\dot{\Pi}^0 = 0$. Constraints which arise upon using the equations of motion are called secondary. In this case we have

$$T_2 \equiv -\frac{\delta H}{\delta A_0} = \partial_i \Pi^i = 0\,, \tag{3.6}$$

which is in fact Gauss' law $\vec{\nabla}\cdot\vec{E} = 0$. There are no further constraints. The Poisson brackets among the constraints are trivial, $\{T_a, T_b\} = 0$. Constraints that form a closed algebra under Poisson brackets are called first-class. They are the generators of gauge transformations. One might at first wonder why there are two generators (with two independent gauge parameters), but this is because (3.2) involves the time derivative of Λ which, in the Hamiltonian formulation, is treated independently from Λ and its spatial derivatives.

It is possible to fix the gauge completely by introducing two gauge conditions. For example, the conditions $A_0 = 0$ and $\partial_i A^i = 0$ eliminate all gauge invariance[1]. This selects out the physical degrees of freedom, namely the transverse polarizations of the photon. These degrees of freedom can then be quantized. Unfortunately, though, Lorentz invariance is no longer manifest since a Lorentz transformation leads out of the gauge. In principle, one should then express the Lorentz generators in terms of the physical degrees of freedom and check whether the algebra is still intact quantum-mechanically.

Another possibility is to impose a covariant gauge such as $\partial_\mu A^\mu = 0$, known as the Lorentz gauge. In the Lagrangian formalism this can be done by changing the action to

$$S = \int d^4x \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \tfrac{1}{2\alpha}(\partial_\mu A^\mu)^2\right)\,. \tag{3.7}$$

The extra term eliminates the constraints. However, in order that we are still dealing with the same theory, we have to impose $\partial_\mu A^\mu = 0$. In the quantum theory it then turns out that in the Hilbert space one has to impose

$$\partial^\mu A_\mu^{(+)}|\psi> = 0\,, \tag{3.8}$$

[1]To ensure that constraints plus gauge conditions are effectively zero inside brackets, the original Poisson bracket should be replaced by a Dirac bracket.

as a condition for physical states, where the + denotes the positive energy part. This ensures that expectation values $< \psi|\partial^\mu A_\mu|\psi >$ vanish. This is the Gupta-Bleuler quantization procedure. It eliminates all unphysical negative norm states, while Lorentz invariance is manifest. Due to the fact that the condition $\partial^\mu A_\mu = 0$ does not fix the gauge completely, there may still be states containing the scalar and longitudinal photons in a certain combination which satisfies the constraint (3.8), but these have zero norm and decouple.

3.1.1 The BRST formalism

We start with a classical description. As is well-known, and as we have seen in the example above, gauge invariance leads to constrained Hamiltonian dynamics. We assume that the constraints, denoted by T_a[2], are first-class, with $\{T_a, T_b\} = f^c_{ab}T_c$, and generate all gauge transformations. We also assume that the constraints are independent.

In the BRST approach, the gauge symmetry is replaced by a global fermionic symmetry in an extended phase space[3] consisting of the original phase space together with the ghosts c^a and their momenta b_a, with $\{b_a, c^b\} = -\delta^b_a$. The ghosts are anticommuting for bosonic constraints and commuting for fermionic constraints. The generator Q of the symmetry is called the BRST charge. The essential property of Q is its nilpotency, $\{Q, Q\} = 0$. This enables one to define a differential (BRST) complex, consisting of extended phase space functions, with a ghost number[4] grading, in which Q acts as the differential. It is further required that Q is Grassmann odd, has ghost number one, is real (Hermitian), and has the form $Q = c^a T_a + ...$, where the dots denote terms nonlinear in the ghost variables c^a. The BRST charge Q is defined such that its ghost number zero cohomology, $H^0(Q)$, coincides with the set of (equivalence classes of) gauge invariant functions on the constraint surface $T_a = 0$. In more physical terms, the physical gauge invariant functions are those that satisfy $\{Q, f\} = 0$, and two functions f_1 and f_2 are equivalent if $f_1 = f_2 + \{Q, g\}$ for some g. Effectively, what happens is that in going to the cohomology the ghost variables 'kill' the gauge degrees of freedom. All of this is carried out without loss of covariance since no gauge-fixing is applied.

Given any gauge symmetry, one can construct a classical BRST charge Q. This can be done using a procedure which we now sketch. The expansion of Q in ghost variables is

$$Q = \sum_{p=0}^{r} c^{b_{p+1}} c^{b_p} ... c^{b_1} U^{(p)\, a_1 ... a_p}_{b_1 ... b_{p+1}} b_{a_p} b_{a_{p-1}} ... b_{a_1} \, , \tag{3.9}$$

which is the most general ghost number one expression in the extended phase space. Here $U^{(p)}$ are called the structure functions of order p (they are independent of the ghost variables), and r is called the rank of the set of constraints $\{T_a\}$. The zeroth

[2]For simplicity we suppress coordinate dependence, and in the case of a field theory we assume that the index a includes the spatial coordinates.

[3]This is sometimes called the BFV extended phase space named after Batalin, Fradkin and Vilkovisky.

[4]The (total) ghost number assignments are $G(c^a) = 1$, $G(b_a) = -1$ and zero for the other fields. This can be stated equivalently in terms of Poisson brackets with the ghost number charge $b_a c^a$.

order structure functions are required to be the generators T_a so that the first term in (3.9) represents a general gauge generator with parameters replaced by ghosts. The other structure functions can be found by requiring Q to be nilpotent under the Poisson bracket. Each term in $\{Q,Q\}$ contains $n+1$ ghosts c and $n-1$ ghost-momenta b. All these terms have to vanish separately. For $n=1$ this amounts to

$$c^a c^b \left(\{T_a, T_b\} + 2U^{(1)\,c}_{ab} T_c \right) = 0\,. \tag{3.10}$$

The solution, whose existence is guaranteed by the closure of the constraint algebra, is $U^{(1)c}_{ab} = -\frac{1}{2} f^c_{ab}$. For $n=2$, nilpotency implies

$$\begin{aligned} c^c c^b c^a \left(D^{(1)\,d}_{abc} + 4U^{(2)\,de}_{abc} T_e \right) b_d &= 0\,, \\ D^{(1)\,d}_{abc} &= -f^e_{[ab} f^d_{c]e} + \{f^d_{[ab}, T_{c]}\}\,. \end{aligned} \tag{3.11}$$

For a Lie algebra, where f^c_{ab} are constants, the second term in $D^{(1)}$ of course vanishes, and the first term vanishes due to the Jacobi identities. So in that case there is no second order structure function, and a nilpotent BRST operator is given by

$$Q = c^a T_a + \tfrac{1}{2} f^c_{ab} c^a c^b b_c\,. \tag{3.12}$$

However, in general (and in particular in the case of W-algebras), the structure constants can be field-dependent. The Jacobi identity, which reads $D^{(1)\,d}_{abc} T_d = 0$, then implies that $D^{(1)\,d}_{abc} = X^{de}_{abc} T_e$ for some fully antisymmetric tensor X, and (3.11) is solved by $U^{(2)} = -\frac{1}{4} X$. This procedure continues (if necessary) for the higher order structure functions, and Jacobi identities always guarantee the existence of appropriate structure functions that constitute a nilpotent BRST charge.

Quantization changes the Poisson bracket constraint algebra into a commutator algebra

$$\{T_a, T_b\} = f^c_{ab} T_c \;\rightarrow\; [\hat{T}_a, \hat{T}_b] = i\hbar \hat{f}^c_{ab} \hat{T}_c + \hbar^2 \hat{C}_{ab}\,, \tag{3.13}$$

where the structure constants may have changed by order $\hbar$ quantum corrections, and there may be terms $\hat{C}_{ab}$ (e.g. central charges) that break the naive closure of the quantum algebra. But even if $\hat{C}_{ab} \neq 0$, BRST quantization may still be consistent. The reason is that the ghost variables may not only cancel gauge degrees of freedom, but possibly also anomalies. This happens, for example, with the central charge anomaly in the Virasoro algebra of the 26-dimensional bosonic string, as will be shown in the next section. It is another advantage of the BRST approach. The quantum BRST operator $\hat{Q}$ should involve the quantum structure constants, and is required to be nilpotent in order to be able to identify the physical states with the cohomology classes of $\hat{Q}$. However, due to problems like normal-ordering, it is usually not an easy task to obtain the BRST operator; it is not even guaranteed to exist. Fortunately though, the classical BRST charge often guides us to the corresponding quantum operator.

Before moving on to the application to string theory, we first continue our example of the free electromagnetic field to illustrate the BRST formalism. For the constraints

T_1 and T_2 we introduce corresponding anticommuting ghost fields (c_1, b_1) and (c_2, b_2), respectively, and the classical BRST charge is then simply given by

$$Q = \int d^3x \left(c_1 \Pi^0 + c_2 \partial_i \Pi^i \right) . \tag{3.14}$$

There are no higher order ghost terms since the constraint algebra is abelian. The classical cohomology can then be shown to consist only of the transverse degrees of freedom, see e.g. [111]. A BRST invariant Hamiltonian is given by H_0 in (3.5). Changing the gauge corresponds to adding an appropriate BRST exact expression $\{Q, \Psi\}$ to H_0, where Ψ is called a gauge-fixing fermion. This produces another BRST invariant Hamiltonian.

In the temporal gauge $A_0 = 0$, which eliminates the constraint $\Pi^0 = 0$, the remaining degrees of freedom are (Π^i, A_i)[5]. The residual symmetry is $A_i \to A_i + \partial_i \Lambda(x^j)$, and associated currents are $F^{\mu i}\partial_i \Lambda$. The charges are then given by

$$q_\Lambda = \int d^3x \, F^{0i} \partial_i \Lambda = \int d^3x \, \partial_i \Pi^i \Lambda \, . \tag{3.15}$$

They generate the residual symmetry transformations and may be compared with the L_n and $\bar{L}_n$ Virasoro charges that generate residual transformations in the conformal gauge of a two-dimensional generally covariant plus Weyl invariant theory. Now one only needs (c_2, b_2) to eliminate the longitudinal degree of freedom. For this minimal sector, the BRST charge is given by $Q = \int d^3x \, c_2 \partial_i \Pi^i$. In the subsequent discussion of the BRST formulation of the bosonic string in the conformal gauge, the reader may notice some similarities with the formulation of electromagnetism in the temporal gauge.

3.2 Critical and non-critical bosonic strings

We next study the quantization of the bosonic string using the BRST formalism [92, 126, 119]. Let us first note that we are dealing with first-quantized string theory. That is, in a path integral formulation one sums over all possible paths of a single string between two fixed string configurations. However, from the world-sheet point of view, the quantization of the bosonic string may be regarded as second quantization of the two-dimensional field theory of scalar fields $X^\mu(\sigma, \tau)$ and two-dimensional metric $h_{ab}(\sigma, \tau)$ (in the Polyakov formulation).

We start with a classical BRST analysis. The Polyakov action of the classical bosonic string defines a two-dimensional gauge theory as we described in section 2.1. Let us repeat here the Lagrangian

$$\mathcal{L} = -\frac{1}{8\pi} \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X_\mu \, . \tag{3.16}$$

The gauge invariances are two-dimensional general covariance and Weyl symmetry. We know then that there are constraints among the phase space variables. To begin with,

[5] The Dirac bracket is simply the Poisson bracket with (Π^0, A_0) ignored.

there is a primary constraint

$$p_{ab} \equiv \frac{\partial \mathcal{L}}{\partial \dot{h}^{ab}} = 0 \,. \tag{3.17}$$

The secondary constraint that results from the condition that (3.17) is conserved in time, is the Virasoro constraint,

$$\dot{p}_{ab} = -\frac{\partial \mathcal{H}}{\partial h^{ab}} = \frac{\partial \mathcal{L}}{\partial h^{ab}} = -\frac{1}{4\pi}\sqrt{-h}T_{ab} = 0 \,, \tag{3.18}$$

where the energy-momentum tensor is given in (2.3). To get rid of the primary constraint, we impose the gauge condition $h_{ab} = \eta_{ab}$[6], as we did in section 2.1.

Note that the imposition of the conformal gauge condition $h_{ab} = \eta_{ab}$ is similar to the temporal gauge-fixing in electromagnetism. They both eliminate the primary constraint, associated with the Lagrange multiplier fields h_{ab} and A_0 in the respective theories. Whereas in electromagnetism this breaks manifest Lorentz invariance, in the string case we do not lose space-time Lorentz invariance.

Now we are left with the constraint $T_{ab} = 0$ and, as argued in chapter two, in the present conformal gauge $h_{ab} = \eta_{ab}$ the energy-momentum tensor is composed of one holomorphic component $T(z)$ and one anti-holomorphic component $\bar{T}(\bar{z})$ on the complex plane. Alternatively, if one starts from the Nambu-Goto action (2.1), the Virasoro constraints (which are the string-equivalents of the mass-shell constraint $p^2 + m^2 = 0$ of the relativistic particle) are primary and there are no other constraints, because there is no independent metric variable. At the Lagrangian level, we already saw in section 2.1 how the Virasoro constraint comes in after substituting the conformal gauge condition into the action: the equation of motion $T_{ab} = 0$ is then lost and has to be imposed as a separate constraint.

The classical BRST charge can now be obtained following the standard procedure outlined before. Thereto we note that the algebra of first-class constraints is

$$\begin{aligned} T(z)T(w) &= \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \,, \\ \bar{T}(\bar{z})\bar{T}(\bar{w}) &= \frac{2\bar{T}(\bar{w})}{(\bar{z}-\bar{w})^2} + \frac{\bar{\partial}\bar{T}(\bar{w})}{(\bar{z}-\bar{w})} \,, \\ T(z)\bar{T}(\bar{w}) &= 0 \,, \end{aligned} \tag{3.19}$$

which is the classical Poisson bracket algebra (2.20) expressed in the language of OPEs as explained in equations (2.48) and (2.49). Next, introduce anticommuting ghost variables $(c(z), b(z))$ and $(\bar{c}(\bar{z}), \bar{b}(\bar{z}))$ with canonical Poisson brackets, in OPE language,

$$b(z)c(w) = \frac{1}{(z-w)} \,, \qquad \bar{b}(\bar{z})\bar{c}(\bar{w}) = \frac{1}{(\bar{z}-\bar{w})} \,. \tag{3.20}$$

The BRST charge can now be read off from the constraint algebra, knowing that no higher than first order structure functions appear, since the structure constants of the

[6] This gauge is accessible in a neighborhood of any point on the world-sheet. However, it is in general not accessible globally.

Virasoro algebra are really constant, i.e. not field-dependent. Thus the BRST charge is given by

$$Q = \int d\sigma \left(c(z)T(z) + c(z)\partial c(z)b(z) + \bar{c}(\bar{z})\bar{T}(\bar{z}) + \bar{c}(\bar{z})\bar{\partial}\bar{c}(\bar{z})\bar{b}(\bar{z}) \right) . \tag{3.21}$$

We again restrict ourselves to the holomorphic sector, so from now on we consider the BRST charge

$$Q = \oint \frac{dz}{2\pi i} j(z) , \tag{3.22}$$

where $j(z)$ is the BRST current

$$j(z) = c(z)T(z) + c(z)\partial c(z)b(z) . \tag{3.23}$$

The BRST charge can be shown to be nilpotent as is guaranteed by the method of construction. In the language of OPEs the nilpotency statement translates to the first order pole of the OPE $j(z)j(w)$ being a total derivative. Using (3.19) plus the canonical OPE for the ghosts, one easily finds that the first order pole of $j(z)j(w)$ vanishes.

Note that we can rewrite (3.23) as

$$j(z) = c(z) \left(T(z) + \tfrac{1}{2} T_{gh}(z) \right) , \tag{3.24}$$

where the ghost energy-momentum tensor is defined by $T_{gh} = -2b\partial c - \partial bc$. This implies that the conformal weights of c and b are -1 and 2, respectively. Their mode expansions are

$$c(z) = \sum_{n=-\infty}^{+\infty} c_n z^{-n+1} , \qquad b(z) = \sum_{n=-\infty}^{+\infty} b_n z^{-n-2} . \tag{3.25}$$

Now we are ready to quantize. In subsection 2.2.2 we already saw that because of double contractions among the free scalar fields (the string coordinates), the Virasoro algebra obtains a central charge,

$$< T(z)T(w) > = \frac{D/2}{(z-w)^4} . \tag{3.26}$$

Thus the conformal algebra does not close anymore and this is referred to as the conformal anomaly. As a consequence, under the assumption that the BRST operator is still given by the expression (3.23) now taken to be normal-ordered, it is no longer nilpotent in general,

$$Q^2 = \tfrac{1}{2}\{Q, Q\} = \oint \left(\tfrac{3}{2}\partial^2 c \partial c + (-\tfrac{2}{3} + \tfrac{D}{12})\partial^3 c c \right) . \tag{3.27}$$

However, the integrand becomes a total derivative[7] for $D = 26$ and therefore the BRST operator is nilpotent only if the number of scalar fields is 26, i.e. if the string is moving in 26-dimensional space-time. In fact, the conformal anomaly (3.26) is cancelled by the

[7] This total derivative is absent if one adds the total derivative $\frac{3}{2}\partial^2 c$ to the BRST current j. This in addition makes j a $(1,0)$ primary field [90].

ghosts precisely in $D = 26$, because then the total energy-momentum tensor of matter plus ghosts[8],

$$T_{tot} = T + T_{gh} = -\tfrac{1}{2}\partial X^\mu \partial X_\mu - 2b\partial c - \partial bc\,, \tag{3.28}$$

has zero central charge. This follows from the OPE of two ghost energy-momentum tensors

$$\underbrace{T_{gh}(z)T_{gh}}(w) = \frac{-13}{(z-w)^4} + \frac{2T_{gh}(w)}{(z-w)^2} + \frac{\partial T_{gh}(w)}{(z-w)}\,, \tag{3.29}$$

which shows that the ghosts contribute -26 to the central charge cancelling that of the matter energy-momentum tensor formed by 26 string coordinates. For $D = 26$, the quantum theory is independent of the world-sheet metric, as is also indicated by the fact that T_{tot} is BRST trivial, $T_{tot} = \{Q, b\}$.

We have witnessed here one of the virtues of BRST quantization: requiring nilpotency of the BRST operator immediately yields the condition for a consistent quantization, in this case the critical dimension $D = 26$.

The physical operators of the theory correspond to cohomology classes of the BRST operator. This will be discussed in the next chapter. Let us mention one subtlety now. The ghost zero modes c_0 and b_0 commute with the BRST invariant Hamiltonian $L_0 + L_0^{gh}$, and from the relations $c_0^2 = b_0^2 = 0$, $\{c_0, b_0\} = 1$ it then follows that the ghost sector ground state is degenerate, and consists of two states, $|\uparrow>$ and $|\downarrow>$, which satisfy

$$\begin{aligned} c_0|\downarrow> &= |\uparrow>\,, \qquad c_0|\uparrow> = 0\,, \\ b_0|\downarrow> &= 0\,, \qquad b_0|\uparrow> = |\downarrow>\ . \end{aligned} \tag{3.30}$$

In operator language, the state $|\downarrow>$ corresponds to $c(z)$ (i.e. $c(z=0)|0> = |\downarrow>$ with $|0>$ the $sl(2)$-invariant vacuum) and $|\uparrow>$ corresponds to $\partial c(z)c(z)$. These operators are indeed the ones with lowest possible conformal weight (energy) -1 in the ghost sector. If we take $|\downarrow>$ as our ghost vacuum, physical operators corresponding to string states with momentum p_μ will be of the form

$$\mathcal{O} = cV(X,p) = cP(\partial X)e^{ip\cdot X}\,, \tag{3.31}$$

where P is some polynomial in the derivatives of X^μ, and $V(X,p)$ is called a vertex operator.

There is also a simple interpretation for the operators corresponding to the other ghost vacuum, $\partial ccV(X,p)$. It is related to the anomaly in the ghost number current $J_{gh} = cb$ (which is a primary field, classically),

$$\underbrace{T_{gh}(z)J_{gh}}(w) = \frac{-3}{(z-w)^3} + \frac{J_{gh}(w)}{(z-w)^2} + \frac{\partial J_{gh}(w)}{(z-w)}\,. \tag{3.32}$$

This has the consequence that a correlation function on the sphere, say the two-point function $< \mathcal{O}_2(z)\mathcal{O}_1(w) >$, vanishes unless the total ghost number of the operators in

[8] The ghost energy-momentum tensor can also be derived from the BRST invariant gauge-fixed action involving Faddeev-Popov ghost terms, see e.g. [90].

it equals three. One way to see this is as follows. Inserting the ghost number charge $Q_{gh} = \oint \frac{dx}{2\pi i} J_{gh}(x)$ into the correlator and contracting it around z and w using OPEs gives the total ghost number times the correlator. We can also contract the integral over x to a point on the other side of the sphere, in which case we have to use the conformal transformation $x \to -\frac{1}{x}$. Then we meet no operator insertions and one would expect to get zero, indicating conservation of ghost number. However, due to the anomalous central term in (3.32), which means that J_{gh} does not transform as a tensor under (global) conformal transformations, the result will instead be three times the correlator. Thus, for a correlation function to be nonzero, the total ghost number of the operators in it must be equal to three. Now for a nonzero two-point function, if $\mathcal{O}_1$ is of the standard form (3.31) with ghost number one, $\mathcal{O}_2$ should belong to the other ghost vacuum with ghost number two.

It is not difficult to show that the condition $[Q, \mathcal{O}] = 0$ with $\mathcal{O}$ of the form (3.31) is equivalent to

$$(L_m - \delta_{m,0})|V>= 0 \quad \text{for } m \geq 0\,, \tag{3.33}$$

where L_m are the modes of the matter energy-momentum tensor and $|V>= V(0)|0>$ with V the vertex operator in (3.31). Here $|0>$ is the $sl(2)$-invariant vacuum, defined in equation (2.57).

The conditions (3.33) are nothing but the positive energy modes of the Virasoro constraint $T(z) = 0$ imposed on the Hilbert space. We see therefore that physical states are highest weight states with weight one with respect to the matter energy-momentum tensor. The vertex operators $V(X, p)$ are therefore primary operators with conformal weight one. The specific form of the L_0 constraint is due to a normal-ordering effect. In fact, L_0 is the only mode which has an ambiguous normal-ordering, which can be seen from (the quantum commutator version of) (2.25) and (2.29). In (3.33) we implicitly defined the normal-ordered operator L_0 to be

$$L_0 = \tfrac{1}{2}\alpha_0^2 + \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n\,. \tag{3.34}$$

However, an arbitrary constant $-a$ could be added because of the commutation relations $[\alpha_m^\mu, \alpha_n^\nu] = m\delta_{m+n,0}\eta^{\mu\nu}$. The correct value $a = 1$ directly follows from BRST quantization as we see in (3.33).

A number of other quantization methods have also been used to quantize the bosonic string. One of these methods is the light-cone gauge quantization. This is an example of a full gauge-fixing prior to quantization. The light-cone gauge is obtained from the conformal gauge by using the residual holomorphic and anti-holomorphic reparametrization invariance to eliminate the two light-cone coordinates $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$. The remaining physical degrees of freedom X^i, $i = 1, 2, ...D-2$, correspond to transverse excitations. These are to be quantized. Disadvantage of this method is that manifest space-time Lorentz invariance is lost. Therefore one should check Lorentz invariance and the result is that the commutators of space-time angular momentum constitute the Lorentz algebra if and only if $D = 26$ and $a = 1$ [100].

Another quantization method, which goes under the name old covariant quantization, is more like the Gupta-Bleuler quantization of electromagnetism. One adopts the conformal gauge, which is manifestly space-time Lorentz covariant, and imposes the Virasoro constraints on the physical states by requiring the positive energy modes to annihilate physical states:

$$(L_m - a\delta_{m,0})|\psi>= 0 \quad \text{for } m \geq 0 \,. \tag{3.35}$$

A careful analysis of the norms in the Hilbert space shows that for $a = 1$ and $D = 26$ there are no negative norm states.

3.2.1 Non-critical strings

In the preceding discussion of BRST quantization of the bosonic string, we selected the conformal gauge $h_{ab} = e^{\phi}\eta_{ab}$ and forgot about the conformal factor e^{ϕ} since it drops out of the classical action due to Weyl symmetry. However, it was shown by Polyakov [159] that, quantum-mechanically, the conformal factor may be ignored only if $D = 26$. In other space-time dimensions it becomes dynamical thus providing a non-trivial model for two-dimensional quantum gravity.

The approach of [159] starts with the Euclidean path integral[9]

$$Z = \int \mathcal{D}h\mathcal{D}X \exp\left[-\frac{1}{8\pi}\int d^2\sigma\sqrt{h}\, h^{ab}\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu}\right] . \tag{3.36}$$

In order to perform the path integration without overcounting, it is necessary to fix a gauge. Let us first write

$$h_{ab} = e^{\phi(\sigma)}\hat{h}_{ab} \,, \tag{3.37}$$

where $\hat{h}_{ab}$ is a fiducial metric. Since the action is also invariant under diffeomorphisms we have to integrate not over all $\hat{h}_{ab}$ but only over those that are not related by a world-sheet diffeomorphism. In fact, it can be shown that the space of metrics modulo diffeomorphisms and Weyl transformations is a finite-dimensional space (see e.g. [69]), the moduli space of Riemann surfaces. For the sphere (genus 0 Riemann surface), the moduli space consists of a single point only.

Gauge-fixing with respect to the group of diffeomorphisms leads to the introduction of the Faddeev-Popov ghosts. After factoring out the volume of the group of diffeomorphisms we are left with

$$Z = \int d\mu\, \mathcal{D}_{e^\phi \hat{h}}X\mathcal{D}_{e^\phi \hat{h}}\phi\mathcal{D}_{e^\phi \hat{h}}b\mathcal{D}_{e^\phi \hat{h}}c \exp\left\{-S_P[X,\hat{h}] - S_{gh}[b,c,\hat{h}]\right\} , \tag{3.38}$$

where $\mathcal{D}_{e^\phi \hat{h}}$ indicates that the integration measure is evaluated with respect to the original metric h_{ab}, and $d\mu$ is the measure in the space of fiducial metrics (moduli space). S_P is the Polyakov action and S_{gh} the action for the ghost fields,

$$S_{gh}[b,c,\hat{h}] = \frac{i}{2\pi}\int d^2\sigma\sqrt{\hat{h}}\hat{h}^{ab}b_{bc}\hat{\nabla}_a c^c \,. \tag{3.39}$$

[9]We follow here the discussion as given in [5, 2].

Next, one would like to write the integration measures in (3.38) with respect to the fiducial metric, in order to make the dependence on $\phi(\sigma)$ explicit. For the fields X^μ, it can be shown [2] that

$$\mathcal{D}_{e^\phi \hat{h}} X = e^{(D/48\pi) S_L[\phi,\hat{h}]} \mathcal{D}_{\hat{h}} X \,, \tag{3.40}$$

and for the ghost fields

$$\mathcal{D}_{e^\phi \hat{h}} b \mathcal{D}_{e^\phi \hat{h}} c = e^{(-26/48\pi) S_L[\phi,\hat{h}]} \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \,, \tag{3.41}$$

where $S_L[\phi, \hat{h}]$ is the Liouville action for the conformal factor,

$$S_L[\phi, \hat{h}] = \int d^2\sigma \sqrt{\hat{h}} \left(\frac{1}{2} \hat{h}^{ab} \partial_a \phi \partial_b \phi + R^{(2)} \phi + \lambda e^\phi \right) \,, \tag{3.42}$$

in which λ is the cosmological constant, needed for renormalization purposes [159]. Thus, we get

$$\begin{aligned} Z &= \int d\mu\, \mathcal{D}_{e^\phi \hat{h}} \phi \mathcal{D}_{\hat{h}} X \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \\ &\times \exp\left\{ -S_P[X, \hat{h}] - S_{gh}[b, c, \hat{h}] - \frac{26 - D}{48\pi} S_L[\phi, \hat{h}] \right\} . \end{aligned} \tag{3.43}$$

For $D \neq 26$, the conformal mode of the metric (the Liouville field) becomes a dynamical field. Hence the realization of the conformal symmetry is very different depending on whether $D = 26$ or $D \neq 26$. Non-critical strings are those with $D \neq 26$, and the Liouville field is necessary in order to satisfy the requirement of conformal invariance. For $D = 26$, corresponding to the critical string, the Liouville field decouples from the action and the integration over ϕ can be absorbed in the normalization of the path integral. The only remainder of the integration over metrics in this case is the integration over moduli $d\mu$,

$$Z = \int d\mu\, \mathcal{D}_{\hat{h}} X \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \, \exp\left\{ -S_P[X, \hat{h}] - S_{gh}[b, c, \hat{h}] \right\} . \tag{3.44}$$

Under some plausible assumptions the dependence of the measure in (3.43) on the conformal factor can be further extracted, resulting in

$$\begin{aligned} Z &= \int \mathcal{D}_{\hat{h}} \phi \mathcal{D}_{\hat{h}} X \mathcal{D}_{\hat{h}} b \mathcal{D}_{\hat{h}} c \\ &\times \exp\left\{ -S_P[X, \hat{h}] - S_{gh}[b, c, \hat{h}] - \frac{25 - D}{48\pi} S_L^{'}[\phi, \hat{h}] \right\} , \end{aligned} \tag{3.45}$$

where $S_L^{'}[\phi, \hat{h}]$ is given by

$$S_L^{'}[\phi, \hat{h}] = \int d^2\sigma \sqrt{\hat{h}} \left(\tfrac{1}{2} \hat{h}^{ab} \partial_a \phi \partial_b \phi + R^{(2)} \phi + \lambda e^{\gamma\phi} \right) \,, \tag{3.46}$$

with

$$\gamma = \frac{1}{12} \left(25 - D - \sqrt{(25 - D)(1 - D)} \right) . \tag{3.47}$$

This expression was obtained by David and Distler and Kawai [66]. A similar analysis but in the so-called chiral gauge was performed in [129]. The results obtained are reasonable only for $D \leq 1$. For $D > 1$ some of the arguments used in [129, 66] might break down. This $c = 1$ barrier (recall that $D = 1$ corresponds to a matter central charge $c = 1$) is usually viewed as a transition to a strong coupling phase of 2d-gravity for $c > 1$. This phase, which might be related to the existence of tachyonic excitations for the bosonic string in space-time dimensions $D > 2$, is still not well-understood.

If we rescale

$$\phi \to \sqrt{\frac{12}{25-D}}\phi\,, \tag{3.48}$$

the propagator of ϕ becomes the same as that of the X^μ fields. The contribution to the energy-momentum tensor of the rescaled ϕ is

$$T_\phi = -\tfrac{1}{2}\partial\phi\partial\phi + \tfrac{Q}{2}\partial^2\phi, \tag{3.49}$$

where $Q = \sqrt{\frac{25-D}{3}}$. The second term in T_ϕ comes from the term $R^{(2)}\phi$ in $S_L^{'}$ and $\frac{Q}{2}$ is called a background charge. The central charge of T_ϕ can be calculated using OPEs, and one finds $c_\phi = 1 + 3Q^2$. We now observe that the total central charge vanishes,

$$c_{tot} = c + c_\phi + c_{gh} = D + 1 + 3Q^2 - 26 = 0\,, \tag{3.50}$$

consistent with overall conformal invariance. Physical operators can now be obtained using the BRST operator (3.24) with T replaced by $T_X + T_\phi$. Operators of the matter theory thus get 'dressed' by the Liouville field in such a way that their total conformal weight equals one [129, 66].

We can interpret (3.45) in two ways. First, since the Liouville action is an effective action for the world-sheet metric, we may interpret (3.45) as a quantum theory of 2d-gravity described by the Liouville field coupled to D scalar fields. Due to the $c = 1$ barrier, we can only consider $D = 0$ or $D = 1$. However, the discussion leading to (3.45) does not depend on the particular matter CFT, but only on its central charge. Therefore it is also possible to consider $c < 1$ minimal models coupled to 2d-gravity. In the other interpretation ϕ is regarded as one of the string coordinates, hence we have a $(D+1)$-dimensional non-critical string theory. A special case is $D = 25$, where the Liouville field may be viewed as the 26th coordinate of the critical bosonic string. For more information on non-critical string theory and 2d-gravity, also from the matrix model point of view, the reader may consult [97, 1].

3.3 BRST quantization of W-strings

We now come to the study of W-string theories, i.e. string theories based on W-algebras instead of the Virasoro algebra. W-algebras were briefly discussed in subsection 2.2.3. A first question we could ask ourselves is what the analogue of the Polyakov action is. The Polyakov action describes the Weyl invariant coupling of 2d-gravity to matter, and the Virasoro algebra is the symmetry algebra that remains after fixing the conformal gauge. Generalizing the Virasoro algebra to a W-algebra, we know that the W-algebra is the symmetry algebra in a conformal type gauge, resulting from a covariant action describing the W-Weyl invariant coupling of W-gravity to matter. Although W-algebras themselves are known explicitly, it is not a simple matter to obtain such corresponding covariant actions that generalize the Polyakov action. Anyway, a consistent quantization of matter coupled to gravity leads to a string theory, critical or non-critical depending on whether or not the world-sheet gravity field(s) decouple. The next task would then be to study the physical properties of these string theories, such as the spectrum of physical states.

Gauging (the reverse of fixing a gauge) the Virasoro algebra leads to the Polyakov action. Thus one should try to gauge a W-algebra in order to obtain the W-gravity coupling. Let us quickly see how this might be done by considering a simple example of gauging a classical version of the W_3 algebra. For a review, see [115]. See also [36].

Consider a free field theory of scalar fields with action

$$S_0 = \frac{1}{4\pi} \int d^2 z \partial\phi_i \bar{\partial}\phi^i \, . \tag{3.51}$$

From the equations of motion $\bar{\partial}\partial\phi_i = 0$, it follows that the chiral part of the symmetry algebra is generated by $\{\partial\phi_i\}$ (and the anti-chiral part by $\{\bar{\partial}\phi_i\}$). In particular, the energy-momentum tensor is given by

$$T = -\tfrac{1}{2}\partial\phi_i\partial\phi^i \, . \tag{3.52}$$

As discussed in the previous chapter, it generates holomorphic coordinate transformations (sometimes referred to as semi-local or semi-rigid symmetries). Gauging this chiral copy of the Virasoro algebra is accomplished by the minimal linear coupling to a spin-two gauge field which enters the action as a Lagrange multiplier imposing the Virasoro constraint. Gauging of both chiral and anti-chiral copies of the Virasoro algebra ultimately yields the Polyakov action.

We could, however, just as well gauge a larger part of the chiral algebra. Consider another conserved current, say, the spin-three current

$$W = \tfrac{1}{3} d_{ijk} \partial\phi^i \partial\phi^j \partial\phi^k \, . \tag{3.53}$$

In order to gauge the corresponding symmetries, the algebra generated by the currents (T, W) must close. This restricts the coefficients d_{ijk} to solutions of

$$d^k_{(ij} d_{lm)k} = \kappa \delta_{(ij}\delta_{lm)} \, . \tag{3.54}$$

The classical OPE of W with itself then closes on the square of T,

$$W(z)W(w) = \frac{-4\kappa\Lambda(w)}{(z-w)^2} + \frac{-2\kappa\partial\Lambda(w)}{z-w} \, , \tag{3.55}$$

with $\Lambda = T^2$ and κ a constant. This, together with the OPEs involving T, is a classical version of Zamolodchikov's W_3 algebra (2.85), and we will refer to it as the w_3 algebra.

The infinitesimal transformation parameters associated to $(T(z), W(z))$ are arbitrary holomorphic functions $(\varepsilon(z), \lambda(z))$ and the transformations they generate are

$$\delta\phi^i(z,\bar{z}) = \oint \frac{dw}{2\pi i} \left(\varepsilon(w)T(w) + \lambda(w)W(w)\right) \phi^i(z,\bar{z}) = \varepsilon(z)\partial\phi^i - \lambda(z)d^i{}_{jk}\partial\phi^j\partial\phi^k \,. \tag{3.56}$$

To gauge this chiral algebra we have to promote this symmetry to one where (ε, λ) can be arbitrary functions (not necessarily holomorphic). This can be done by employing the Noether procedure in which the gauge fields are minimally coupled to the currents. Denoting the spin-two and spin-three gauge fields by h and B, the coupling is given by

$$S = S_0 + \frac{1}{2\pi}\int d^2z\,(hT + BW) \,. \tag{3.57}$$

With appropriate transformation rules for the gauge fields, this action is invariant under chiral w_3 gauge transformations.

It has been shown that for gauging any chiral algebra, a linear coupling to the gauge fields suffices [114]. The action (3.57) is the action of scalar matter coupled to chiral w_3 gravity, but it may also be viewed as the action of covariant w_3 gravity coupled to matter in the so-called chiral gauge. To obtain a covariant non-chiral coupling of w_3-gravity, one should also gauge the anti-holomorphic components[10] $(\bar{T}, \bar{W})$ which satisfy the same w_3 algebra. For such non-chiral gaugings it is not enough to have linear couplings to the gauge fields, and in fact one should add higher and higher order terms to the action to make it gauge invariant. In the case of ordinary gravity, we know that this process can be circumvented, since the coupling to gravity is well-known and takes the form as in the Polyakov action. But for the higher-spin gauge field an *explicit* closed form of the coupling is not known. This is related to the fact that W-geometry is not really well-understood. However, closed forms for non-chiral W-gravity coupled to matter actions are known [173]. These involve auxiliary fields which, after elimination using the equations of motion, again yield an infinite power series in the gauge fields.

Here we do not really need covariant actions of W-gravity coupled to W-matter, since our purpose is at first to obtain the spectrum of physical states of a W-string. This can be obtained by constructing the BRST operator corresponding to the W-algebra (which should be viewed as the algebra of first-class constraints that remain after going to the conformal type gauge). If a nilpotent BRST operator can be constructed, the physical spectrum can in principle be computed through the cohomology.

3.3.1 BRST analysis of W-symmetries

Let us start with the construction of the BRST operator for the W_3 string. Its classical constraint algebra w_3 (in general we write w_N for the centerless classical versions of W_N), takes the form

$$T(z)T(w) \;=\; \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \,,$$

[10] It is a consequence of generalized Weyl (W-Weyl) symmetry that the W generators are traceless symmetric tensors which in two dimensions implies that they have only two independent components.

$$
\begin{aligned}
T(z)W(w) &= \frac{3W(w)}{(z-w)^2} + \frac{\partial W(w)}{z-w}\,, \\
W(z)W(w) &= \frac{\lambda T^2(w)}{(z-w)^2} + \frac{\lambda T\partial T(w)}{z-w}\,,
\end{aligned}
\tag{3.58}
$$

where λ is a constant which depends on the normalization of W. This symmetry is realized by a set of free scalar fields, as in (3.52) and (3.53). We introduce two ghost pairs (c_2, b_2) and (c_3, b_3) corresponding to the spin-two and spin-three symmetries, respectively. They satisfy the standard Poisson bracket

$$
c_i(z)b_j(w) = \frac{\delta_{ij}}{z-w}\,. \tag{3.59}
$$

It is not difficult to write down the classical BRST current for the algebra in (3.58). Following the procedure described in the previous section, we see that in spite of the fact that the algebra is nonlinear, no second or higher order structure functions are needed. The result is

$$
\begin{aligned}
j(z) &= c_2(T + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}) + c_3 W + \tfrac{\lambda}{2}c_3\partial c_3 b_2 T\,, \\
T_{c_k,b_k} &= -k b_k \partial c_k + (1-k)\partial b_k c_k\,,
\end{aligned}
\tag{3.60}
$$

with T_{c_k,b_k} the spin-k ghosts' contribution to the total energy-momentum tensor. The BRST charge is given by $\oint \frac{dz}{2\pi i} j(z)$. Note the appearance of T in the cubic ghost term of the BRST current. Here it plays the role of structure constant (first order structure function). It originates from the nonlinearity in the OPE of W with itself: one T is interpreted as generator, the other as structure constant.

A quantum extension of (3.60) for $\lambda = 2b^2$, with $b^2 = \frac{16}{22+5c}$ as in (2.85), is obtained by normal-ordering and adding the following quantum corrections:

$$
j(z) \;\to\; j(z) + \tfrac{25\lambda}{192}c_3(9\partial c_3\partial^2 b_2 + 15\partial^2 c_3\partial b_2 + 10\partial^3 c_3 b_2)\,. \tag{3.61}
$$

The expression for the quantum BRST operator was first found by Thierry-Mieg [187]. The construction of BRST operators for more general quadratically nonlinear Lie algebras has been described in [172]. For the quantum W_3 BRST operator to be nilpotent the central charge must be fixed to $c = 100$. This is the W_3 analogue of the $c = 26$ requirement for the ordinary (Virasoro) bosonic string. The reason for this is that the ghost fields contribute -100 to the central charge (-26 from (c_2, b_2) and -74 from (c_3, b_3)) which can be checked by considering the OPEs of the ghost energy-momentum tensors. This central charge must be cancelled by that of the matter W_3 algebra so that the conformal anomaly disappears.

As in the classical case, the various terms in (3.61) can be associated to terms in the algebra, which has now become the quantum W_3 algebra. The field realizations of T and W have to be normal-ordered, and counterterms that cancel anomalies have to be introduced. These counterterms are such that the renormalized currents satisfy the W_3 algebra [160, 114].

The classical currents (T, W) of equations (3.52) and (3.53) no longer generate a closed algebra quantum-mechanically, since double and triple contractions give rise to terms not proportional[11] to T or W. Such terms can be constant (central charges), or field-dependent. The central terms are sometimes called universal anomalies. The field-dependent terms are called matter-dependent anomalies, and they are characteristic of nonlinearly realized algebras such as is the case here, see (3.56). They do not occur for the ordinary bosonic string, as only a central charge anomaly arises in that case. These anomalies have been calculated for chiral w_3 gravity where they correspond to the non-invariance of the effective action (usually called the induced action), obtained by integrating out the matter fields, under chiral w_3 gauge transformations. For a review, see [115][12]. One can add local counterterms to the action (3.57) of chiral w_3 gravity of the form

$$S_c = \frac{1}{2\pi} \int d^2z \left(a_i \partial^2 \phi^i h + (e_{ij} \partial\phi^i \partial^2 \phi^j + f_i \partial^3 \phi^i) B \right) , \tag{3.62}$$

for some constants (a_i, e_{ij}, f_i). This means that the currents are modified to

$$\begin{aligned} T &= -\tfrac{1}{2}\partial\phi_i\partial\phi^i + a_i\partial^2\phi^i , \\ W &= \tfrac{1}{3}d_{ijk}\partial\phi^i\partial\phi^j\partial\phi^k + e_{ij}\partial\phi^i\partial^2\phi^j + f_i\partial^3\phi^i . \end{aligned} \tag{3.63}$$

These appear to be the most general terms that can be added without introducing new fields or dimensionful couplings. It has been shown [164] that all anomalies (universal and matter-dependent) cancel if the coefficients (a_i, e_{ij}, f_i) are such that (3.63) generate the quantum W_3 algebra with central charge $c = 100$, and if ghost variables (c_2, b_2) and (c_3, b_3) are taken into account appropriately. These conditions are equivalent to the existence of a nilpotent BRST operator, as we have seen.

As in the case of the ordinary Virasoro string, one might also consider the non-critical string where the matter does not yield a $c = 100$ realization of the W_3 algebra. Then, not all anomalies are cancelled, and the W-gravity gauge fields become dynamical quantum-mechanically. Much work has been done on the computation of the effective action for quantum W-gravities. In particular, the relation between affine Lie algebras and W-algebras has been exploited to learn more about these effective actions. See for example [36] and references therein. For W_N algebras, much evidence has been provided for the suggestion that the effective W_N gravity actions in the conformal gauge are $sl(N)$ Toda actions, i.e. straightforward generalizations of the Liouville action. These are realized in terms of $N-1$ scalar fields which can be interpreted as coming from generalized conformal factors that become propagating fields quantum-mechanically due to anomalous W-Weyl symmetry.

Both matter and gravity sectors (in the non-critical case) separately should realize the W-symmetry. However, it is not obvious how the complete matter-coupled-to-gravity system realizes the W-symmetry. Due to the nonlinearity of the W-algebra it is impossible to obtain new realizations by just adding two commuting realizations of

[11] Terms proportional to the generators that are not present in the classical algebra do not affect the closure of the algebra but represent a quantum deformation of the classical algebra. Such 'anomalies' are cancelled simply by modifying the transformation laws of the gauge fields.

[12] More recent discussions of chiral W_3 gravity and its anomalies have been given in [148, 190]. In [148], instead of the conformal gauge a derivative gauge condition is employed which leads to a simplification of the Hamiltonian BRST formulation. In [190], the anomalies of chiral W_3 gravity are computed as an illustration of the Lagrangian Batalin-Vilkovisky quantization method. See also [67].

the currents. However, for the classical w_3 algebra we can add two independent copies of the algebra to form a new algebra in the following way. Define

$$T = T_M + T_L \quad \text{and} \quad W = W_M + iW_L\,, \tag{3.64}$$

for two commuting copies (T_M, W_M) and (T_L, W_L) of the w_3 algebra. The Poisson bracket of W with itself then reads

$$W(z)W(w) = \frac{(T_M - T_L)T}{(z-w)^2} + \frac{\frac{1}{2}\partial((T_M - T_L)T)}{(z-w)}\,, \tag{3.65}$$

with structure constant $T_M - T_L$. This algebra is called modified w_3 [26]. The corresponding BRST current is simply (3.60) with T replaced by $T_M - T_L$ in the last term. A corresponding quantum BRST operator was found in [30, 26]. Its current is

$$\begin{aligned} j &= c_2(T_M + T_L + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}) + c_3(\frac{W_M}{b_M} \pm i\frac{W_L}{b_L}) \\ &+ c_3\partial c_3 b_2(T_M - T_L) + \nu c_3\partial^2 c_3\partial b_2 + \tfrac{5}{3}\nu c_3\partial^3 c_3 b_2\,, \end{aligned} \tag{3.66}$$

where $(b_{M,L})^2 = \frac{16}{22+5c_{M,L}}$ and $\nu = \frac{c_M - 50}{32}$. It is nilpotent if and only if $c_M + c_L = 100$. We should stress that there is no closed quantum algebra that can be extracted from (3.66). This does not seem to be a problem, since it is the nilpotency of the BRST operator that allows for a consistent quantization.

The construction following (3.64) can be generalized to w_n algebras as follows. Suppose we have two commuting copies of the w_n algebra, generated by spin-k ($k = 2, 3, \ldots N$) currents v_M^k and v_L^k. Then if we define new spin-k currents

$$v^k = v_M^k + i^{k-2}v_L^k\,, \tag{3.67}$$

we obtain a new closed algebra. This is however not a w_N algebra. Instead, the algebra involves structure constants that are functions of the separate currents v_M and v_L as in (3.65). The reason that this construction gives a closed algebra can be traced back to the fact that w_N algebras only have second-order poles in their classical OPEs (plus first order poles that involve an additional derivative). Since quantum W_N algebras have higher-order poles as well, this construction will not work in that case, as observed for the modified BRST operator for W_3 [30]. We expect that nilpotent BRST operators based on these modified w_N algebras can nevertheless be constructed also for $N > 3$, although explicit expressions will be complicated. For the non-critical W_3 string we assume that (3.66) is the correct BRST operator. Evidence for this has also been given in [37].

Only a few BRST operators for nonlinear W-algebras are known explicitly. Besides the critical and non-critical BRST operators for the W_3 algebra, of the W_N series only the critical W_4 BRST operator is known explicitly [113, 200, 20]. Already for $N = 4$ does the complicated nonlinear structure of the algebra make it very elaborate to compute explicit expressions such as the BRST operator. Moreover, if these expressions have been obtained, they are usually too involved to use in further calculations.

It is expected that BRST operators for arbitrary W_N algebras exist and that they are unique up to (quantum) canonical transformations[13]. Canonical transformations may be used to simplify the form of the BRST operator and the subsequent cohomology analysis. In the case of an explicit realization of the currents, for example in terms of free bosons, a larger number of different canonical transformations can be performed, since the fields in the realization may also be used to build generating functions. In the next subsection we describe realizations of W_N algebras and after that we will see that canonical transformations involving scalar fields of the realization can indeed be used to simplify the BRST analysis of W-algebras [21, 20, 39].

Finally, we mention that also some BRST operators for other than W_N nonlinear algebras are known. These are the BRST operators for a number of $W_{2,N}$ algebras, which are nonlinear algebras generated by an energy-momentum tensor and a spin-N current. A BRST operator for the $W_{2,4}$ algebra has been given in [200], in a realization-independent basis. For $N = 4, 5, 6$ and 7, BRST operators in a realization-dependent basis have been constructed in [139, 142]. A BRST operator for a non-critical $W_{2,4}$ model has been given in [141], see also [153]. Some progress in constructing BRST charges for higher-spin strings ($W_{2,N}$ and W_N strings) has also been reported in [87].

3.3.2 Realizations of W-algebras

Since we are interested in the construction of string theories based on W-algebras, the realizations that are most relevant to us are free bosonic realizations. Fortunately, the W_N algebras, nonlinear algebras generated by currents of spins $2, 3, ...N$, were originally constructed in terms of free bosons [78]. Let us recall this construction. For more details we refer to [78, 144, 65, 136, 32].

The currents of the W_N algebra can be obtained from the following differential operator of order N,

$$(\sqrt{2}\alpha_0)^N D_N = \prod_{m=1}^{N} \left(\sqrt{2}\alpha_0 \partial - \vec{h}_m \cdot \partial\vec{\phi}(z)\right) , \tag{3.68}$$

where $\vec{\phi}$ is an $(N-1)$-dimensional vector of scalar fields, and $\vec{h}_m\,,\ m = 1, 2, ...N$ are $(N-1)$-dimensional vectors satisfying

$$\vec{h}_m \cdot \vec{h}_n = \delta_{mn} - \frac{1}{N}\,, \qquad \sum_{m=1}^{N} \vec{h}_m = 0\,. \tag{3.69}$$

Expanding this differential operator in powers of ∂, we can write

$$D_N = \sum_{k=0}^{N} (\sqrt{2}\alpha_0)^{-k} U_k(z) \partial^{N-k}\,, \tag{3.70}$$

[13]At the classical level, it is known that the BRST charge associated to a constraint surface is unique up to canonical transformations, see e.g. [111].

for currents $U_k(z)$ with spin k. As always, normal-ordering is understood. Comparing (3.70) with (3.68) we find $U_0 = 1$, $U_1 = 0$ and, after some rearrangements,

$$U_2(z) = -\tfrac{1}{2}\partial\vec{\phi}(z)\cdot\partial\vec{\phi}(z) - \sqrt{2}\alpha_0\vec{\rho}\cdot\partial^2\vec{\phi}(z)\,, \tag{3.71}$$

where $\vec{\rho} = \sum_{i=1}^{N-1}\vec{\lambda}_i$, the sum of the $sl(N)$ fundamental weights $\vec{\lambda}_i$, is the $sl(N)$ Weyl vector. The central charge of $T \equiv U_2$ is given by

$$c = N - 1 + 24(\alpha_0)^2\rho^2 = (N-1)(1 + 2N(N+1)(\alpha_0)^2)\,. \tag{3.72}$$

Equation (3.70) yields a realization of the W_N algebra for arbitrary central charge parametrized by α_0. In (3.71) we recognize an energy-momentum tensor in terms of $N-1$ scalars. It has been shown [144] that the currents $\{U_k(z)\}\,,\ k = 2, 3, ...N$ generate a closed operator product algebra which is quadratically nonlinear. The transformation from the algebra of free scalar fields to the algebra generated by $\{U_k(z)\}$ is the quantum Miura transformation. For that reason we call the resulting realizations of W_N algebras Miura realizations. We first must note, however, that the currents $\{U_k(z)\}$ are not primary with respect to $T(z) = U_2(z)$, whereas usually the W_N algebras are assumed to be generated by T plus a set of primary currents of spins $3, 4, ...N$. To obtain a primary spin-k current starting from $U_k(z)$, one has to add appropriate terms involving derivatives and composites of lower-spin currents to it. The virtue of a primary basis is that OPEs for primary fields take the relatively simple form given in equation (2.45). A less attractive consequence is that the algebra in the primary basis contains higher than quadratic nonlinearities.

For $N > 3$, the explicit realizations and OPEs become rather awkward. The OPEs of the W_4 algebra have been given explicitly in [35] and [127]. The complete W_5 algebra has been given in [112, 199].

The N weights $\vec{h}_m^{(N)}$ satisfying (3.69) may be defined recursively as [65, 136]

$$\vec{h}_m^{(N)} = \left(\vec{h}_m^{(N-1)}, \frac{1}{\sqrt{N(N-1)}}\right)\,, \tag{3.73}$$

for $1 \le m \le N-1$, starting from $N = 2$ with $\vec{h}_1^{(2)} = \left(\frac{1}{\sqrt{2}}\right)$. In other words, the first $N-2$ components of the first $N-1$ vectors are precisely the $\vec{h}_m^{(N-1)}$ vectors of W_{N-1}. The N^{th} vector $\vec{h}_N^{(N)}$ is fixed by the second condition in (3.69). This enables one to re-express the W_N currents in terms of W_{N-1} currents and the scalar field ϕ_{N-1} which is the last component of $\vec{\phi}$ in the W_N realization. This is a special property of the Miura realizations of W_N algebras and turns out to have some restrictive implications on the spectrum of physical states of W_N strings based on these realizations. Explicit expressions for W_N currents in terms of W_{N-1} currents plus a scalar field have been given in [136].

It is known that two classical limits of the Miura realization of W_N algebras can be considered, one being a truncation of the other. First, a classical limit which in general still involves central charges is obtained by letting $\hbar \to 0$[14] and $\alpha_0 \to \pm\infty$ ($c \to \infty$) such that the product $\hbar c = x$ remains finite. This results in a closed Poisson bracket algebra. A truncation to a classical limit without central charges (or background charges) is obtained by now letting $x \to 0$. In this classical limit, which gives the w_N algebras mentioned before, only terms of highest order in the scalar fields (and thus with lowest number of derivatives) survive. In [20], we considered this second classical limit of the Miura transformation. We exploited there the relation between Miura realizations of W_N and W_{N-1} to redefine the W_N currents in such a way that a nested subalgebra structure arises[15]. This structure is induced by the fact that the spin-k currents w_N^k of the redefined w_N algebra only depend on the scalar fields $\{\phi_{N-1}, \phi_{N-2}, ...\phi_{k-1}\}$. For example, the redefined spin-N current is proportional to $(\partial\phi_{N-1})^N$ and therefore automatically defines a subalgebra with field-dependent structure constant,

$$w_N^N(z) w_N^N(w) = c(N) \left(\frac{1}{(z-w)^2} + \frac{\frac{1}{2}\partial}{(z-w)} \right) (\partial\phi_{N-1})^{N-2} w_N^N \, , \tag{3.74}$$

where $c(N)$ is some constant that depends only on N. In the redefined basis, closed expressions have been obtained for all currents of the w_N algebra [20]. It is now clear that the highest-spin current acts only within the Fock space generated by $\partial\phi_{N-1}$. For w_3 this means that the Miura transformation provides reducible representations only (if we also recall that the energy-momentum tensor is diagonal in the scalar fields).

Let us now consider free boson realizations of the quantum W_3 algebra in somewhat more detail. Classically, the condition for a realization (3.53) of the w_3 algebra (3.58) is the quadratic relation for the d_{ijk} tensor given in (3.54). In a different context, an interesting relationship amongst solutions of (3.54) and Jordan algebras was shown to exist in [106]. The classification of such algebras then leads to two classes of classical w_3 realizations: the 'generic' realizations existing for any value n of scalar fields, and four 'magical' solutions with $n = 5, 8, 14$ and 26. It was shown by Romans [168] that all generic solutions can be extended by adding quantum corrections as in (3.63) to realizations of Zamolodchikov's W_3 algebra. In particular, starting from a certain ansatz for the generators, the following n-scalar realization was found:

$$\begin{aligned} T &= -\frac{1}{2} AA - \sqrt{3}\alpha_0 \partial A + T_X \, , \\ W &= \frac{1}{3} AAA + \sqrt{3}\alpha_0 A\partial A + \alpha_0^2 \partial^2 A + 2AT_X + \sqrt{3}\alpha_0 \partial T_X \, , \end{aligned} \tag{3.75}$$

where A is the derivative of a scalar field. The other $n-1$ scalars are represented by T_X which commutes with A and satisfies a Virasoro algebra with central charge $c_X = \frac{1}{4}c + \frac{1}{2}$. Note that for $c = -2$, T_X is null and can be set to zero, and this is in fact the only central charge for which a one-scalar realization of W_3 exists [43]. The background charge parameter α_0 is related to the central charge c via $c = 2(1 + 24\alpha_0^2)$. The resulting realization coincides for $n = 2$ with the Miura realization derived from (3.70) above, where A should be identified with $\partial\phi_2$. These two-scalar realizations were first obtained

[14] In the string sigma model context, the role of Planck's constant is played by $\alpha' = \frac{1}{4\pi T}$. A scalar field has the dimension of $\sqrt{\hbar}$.

[15] A similar structure is also found in certain linearizing algebras for W_N [130].

in [79]. In fact, all realizations (3.75) are essentially the Miura realization for W_3 which, as we mentioned before, can be written in terms of an explicit scalar field and an energy-momentum tensor that can be realized by the fields of any conformal field theory with central charge c_X. The four 'magical' w_3 realizations cannot be extended to realizations of the quantum W_3 algebra [168, 149, 82, 190, 83]. This is unfortunate, especially since in contrast to the generic (Miura) w_3 realizations the magical realizations are irreducible. The critical and non-critical W_3 BRST operators need realizations of the quantum W_3 algebra and therefore can't be based on the magical realizations of w_3. A more general ansatz for a nilpotent quantum extension of the classical BRST charge (3.60) has been considered in [83] and it seems that the (reducible) Romans realizations are forced upon us for the construction of a W_3 string in terms of scalar fields. In the next chapter we will see that as a consequence, the W_3 string is rather similar to an ordinary bosonic string or two[16]. Nevertheless, some interesting structures are present in the spectrum of W-strings based on Romans realizations.

In [22], we investigated the possibility of additional realizations of the W_3 algebra if a null spin-four field is allowed to occur in the OPE of W with itself. Four two-scalar realizations with nonvanishing but null spin-four fields were obtained for fixed values of the central charge; two $c = -2$ realizations and two $c = \frac{4}{5}$ realizations. One of the $c = -2$ realizations is up to a null energy-momentum tensor precisely the $c = -2$ one-scalar realization of W_3 [43]. Together with one of the $c = \frac{4}{5}$ realizations, it has the property that it can be written in terms of an energy-momentum tensor plus a single scalar field. These realizations are therefore generalizable to multi-scalar realizations. The $c = \frac{4}{5}$ realization with this property is the only two-scalar W_3 realization that has one real and one imaginary background charge or equivalently, in a real basis, has one timelike and one spacelike scalar. This realization appears in the physical state analysis of $W_{2,4}$ [138] and W_4 [40] strings.

The other $c = -2$ and $c = \frac{4}{5}$ realizations are only known as two-scalar realizations and also appear in [12] as specific truncations of a nonlinear W_∞ algebra. They can also be derived from the second realization mentioned in a footnote of [79]. The explicit form of all these solutions can be found in [22]. A family of modulo spin-four W_3 realizations for generic central charge was also found in [22]. These are extensions of the Romans solutions by a null energy-momentum tensor. We note that the modulo spin-four realizations do not satisfy in their classical limit the closure condition (3.54), and therefore, their classical limits do not seem to correspond to the w_3 algebra.

Recently, another method of finding realizations for W-algebras has been proposed in [130]. There it was shown that W-algebras can be embedded into linear conformal algebras. For example, for the W_3 algebra, the linearizing algebra W_3^{lin} (in a non-primary basis) consists of currents $(\hat{J}, \hat{T}, \hat{W})$ of spins $1, 2$ and 3, respectively. The W_3 algebra is then obtained from W_3^{lin} by an invertible redefinition of the currents, of the form $T = \hat{T}$ and $W = \hat{W} + a_1 \hat{J}\hat{T} + a_2(\hat{J})^3 + a_3 \hat{J}\partial\hat{J} + a_4 \partial\hat{T} + a_5 \partial^2 \hat{J}$ for some coefficients a_i. This implies that given a realization of the linear algebra W_3^{lin}, one obtains a realization

[16]The analysis of [83] does not completely rule out the possibility of W_3 strings based on irreducible scalar field realizations. Also, one may try to include other than free scalar fields in the construction of an 'irreducible W_3 string'.

of the W_3 algebra through this redefinition. One of the advantages of a linear algebra is that two independent realizations give rise to another realization simply by adding the currents of both realizations. Some W_3 realizations have been obtained [130] starting from realizations of the linearizing algebra. However, apart from the Miura realizations these seem to involve vertex operators (see also [169]). A number of modulo null fields realizations of W_3 has also been obtained using the linearizing method [17]. Starting from realizations of bigger algebras, at certain central charges it may happen that all currents except the spin-two and spin-three currents become null (see also [117]). This, then, gives rise to a modulo null fields realization of the W_3 algebra. In this way all previously known modulo spin-four null field realizations and some more can be obtained [17].

We have seen that multi-scalar W_3 realizations can be constructed for arbitrary values of the central charge. In particular, $c = 100$ Romans realizations exist for an arbitrary number of scalar fields. This is important for the construction of a W_3 string. However, it is not difficult to see that any of the known realizations of W_3 for $c = 100$ necessarily involves nonzero background charges. For the Virasoro algebra one simply takes 26 free scalar fields without any background charge to get the critical central charge, but for W_3 taking 100 free scalar fields without background charges does not give a realization. This is somewhat disappointing from a physical point of view, since background charges break Lorentz invariance. Also, the distinction between critical and non-critical W-strings is not as clear as it is for the ordinary bosonic string.

3.3.3 Simplifications and canonical transformations

In order to calculate BRST operators for W-algebras other than W_3 one should find some simplification method. Otherwise, the construction becomes soon too complicated because of the increasing number of nonlinearities. It turns out that the BRST analysis can be simplified by performing certain canonical transformations which in the case of explicit realizations may also involve the fields of this realization.

In [135], it was found that a particular transformation of the fields that enter the description of a W_3 string leads to a great simplification in the analysis of the physical states. Consequently, it was found in [21] that this transformation could be interpreted as a redefinition of the constraints at the classical level. Then in [20] it was shown that a similar redefinition could be used to simplify the BRST analysis of W_N algebras with $N > 3$. These redefinitions lead to a nested subalgebra structure as mentioned before.

To illustrate the idea, we consider the W_3 algebra, as usual. The classical Miura realization for the classical w_3 algebra is given by

$$\begin{aligned} T &= -\tfrac{1}{2}\partial\phi\partial\phi + T_X \,, \\ W &= \frac{1}{6}\partial\phi\partial\phi\partial\phi + \partial\phi T_X \,, \end{aligned} \tag{3.76}$$

in terms of one explicit scalar field ϕ and an arbitrary classical stress tensor T_X (with zero central charge). Now define new generators

$$\tilde{T} = T \,, \qquad \tilde{W} = W - \partial\phi T = \frac{2}{3}(\partial\phi)^3 \,. \tag{3.77}$$

Now the main idea of the redefinition is already clear: the redefined spin-three generator only depends on a single scalar field and therefore generates a subalgebra by itself. Indeed, the $W(z)W(w)$ OPE has changed to (cf. equation (3.74))

$$\tilde{W}(z)\tilde{W}(w) = \frac{-6\partial\phi\tilde{W}}{(z-w)^2} + \frac{-3\partial(\partial\phi\tilde{W})}{z-w} \,. \tag{3.78}$$

The corresponding BRST charge can be read off directly from the algebra, with the structure 'constants' appearing in the cubic ghost terms. No higher-order ghost terms are needed. This BRST charge, and also its quantum extension [135], has the nice property that it can be written as a sum of two nilpotent charges Q_1 and Q_2 given, classically, by

$$Q_1 = \oint dz c_2(T + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}) \,, \qquad Q_2 = \oint dz c_3(W - 3\partial\phi\partial c_3 b_3) \,,$$
$$\{Q_1, Q_1\} = \{Q_2, Q_2\} = \{Q_1, Q_2\} = 0 \,, \tag{3.79}$$

where we have dropped the tildes on T and W. In somewhat more physical words, (3.77) shows that whereas gravity (via the metric) couples to all fields, the spin-three gauge field does not.

Alternatively, we can describe a redefinition of the constraint algebra by a canonical transformation in the extended phase space. The generating function of the canonical transformation in the case of the redefinition in equation (3.77) turns out to be $G = -\partial\phi c_3 b_2$. Its action on an extended phase space function F is, in OPE language,

$$F(w) \;\rightarrow\; F(w) + \oint \frac{dz}{2\pi i} G(z)F(w) + \frac{1}{2!}\oint \frac{dz}{2\pi i} G(z) \oint \frac{dx}{2\pi i} G(x)F(w) + \,\ldots \,. \tag{3.80}$$

The BRST current in (3.60) transforms into the one given in (3.79) under this canonical transformation.

Similar redefinitions can be carried out for generic w_N algebras. Starting from the Miura realization in terms of $N-1$ scalar fields, we can perform a redefinition of the generators such that the highest spin current only involves one scalar field, the next highest spin current involves this scalar field plus one other scalar field, etc. Only the energy-momentum tensor then depends on all $N-1$ scalar fields. This necessarily induces a nested subalgebra structure where the k highest spin currents form a subalgebra for $k = 1, 2, \ldots N-1$. Of course, the energy-momentum tensor, which is not affected by the redefinition, always generates a subalgebra by itself. This also results in a nested structure of the BRST charge Q. After the redefinition we can write $Q = \sum_{i=2}^N Q_i$, and define nilpotent BRST charges $Q_N^k \equiv \sum_{i=k}^N Q_i$ associated to the various subalgebras in the nested basis,

$$\{Q_N^k, Q_N^k\} = 0 \;\text{ for }\; k = 2, 3, \ldots N \,, \qquad \{Q_2, Q_2\} = 0 \,, \tag{3.81}$$

where Q_i denotes the 'spin-i contribution' to the BRST charge [20]. This generalizes equation (3.79). The nested basis makes the calculation of the BRST charge simpler since it can now be done in steps, starting with the highest-spin part Q_N and going downwards. The same redefinitions can be performed in one of the sectors of the modified w_N algebras. For modified w_3 this results in the same nested structure as in (3.79). To obtain the quantum BRST operator one can parametrize all possible quantum corrections to the classical expression and then demand nilpotency. For the explicitly known cases (critical and non-critical W_3 and critical W_4), the nested basis survives quantization.

In [20], the BRST operator of the W_4 algebra has been found by making use of the nested basis. This BRST operator will be used in the next chapter to study the physical spectrum of the W_4 string. In references [113, 200] the BRST operator of the W_4 algebra was found in the original basis, where only generators appear as structure constants, but these expressions are very lengthy which makes further calculations practically impossible. One difference is that second and third-order structure functions are nonzero in the original basis, while in the nested basis there are at most terms cubic in the ghost variables (first-order structure functions). However, it should be admitted that also in the nested basis higher-order ghost terms are expected for W_N with $N > 4$.

With the BRST operator in the nested basis, it is possible to study the physical states of the W_4 string [40]. As noticed originally [135] for W_3, the physical state analysis simplifies dramatically in the new basis. Another advantage of the nested basis is that it elucidates an apparent relation of W_N strings with minimal models [65]. This will be discussed at length in sections 4.2 and 4.3.

Chapter 4

Physical spectra in string theory

In this chapter we compute cohomology classes of the BRST operator for a number of string models. Within the formalism of BRST quantization, these cohomology classes build the spectrum of physical states. As usual, we start with a discussion of the bosonic string. In section 4.2 we turn to W-string spectra. To illustrate the general structure of such spectra and some methods of computation, we consider in section 4.3 the BRST analysis of the W_4 string in detail. In the last section of this chapter we review some of the relations that exist between strings based on different world-sheet gauge symmetries.

4.1 The bosonic string

First we should stress that all considerations in this chapter are concerned with free strings only. That is, we compute physical spectra of free strings, but no correlation functions. Let us just briefly motivate the term 'vertex operator' that is used for operators which create physical states. First, Weyl transformations can be used to map any two-dimensional surface representing a string scattering process to a compact surface with the same number of handles (quantum loops), but on which the external strings are mapped to points. The quantum numbers of the external string states are then to be described by local operators of the two-dimensional quantum field theory inserted at these points. These operators, which create the external string states, are precisely the physical operators or vertex operators to be discussed in this chapter. String scattering amplitudes then involve the correlation functions of products of vertex operators in the conformal field theory. See for example [104, 90] for discussions of string scattering.

The physical spectrum of the critical bosonic string has been known since the early days of string theory. From the previous chapter, we know that the holomorphic part of the BRST operator is given by $Q = \oint \frac{dz}{2\pi i} j$ with

$$j = c(T + \tfrac{1}{2}T_{gh}) , \qquad T_{gh} = -2b\partial c - \partial bc , \tag{4.1}$$

where the matter energy-momentum tensor involves the 26 string coordinates, $T = -\frac{1}{2}\partial X_\mu \partial X^\mu$. A basis of operators on which we act with the BRST operator to determine the physical spectrum, is given by

$$F(b, c, \partial X)e^{ip\cdot X} , \tag{4.2}$$

where F is an arbitrary polynomial in the ghost variables and ∂X^μ plus their derivatives. All expressions are assumed to be normal-ordered. We will call an operator (or state) 'physical' if it belongs to the BRST cohomology, irrespective of its ghost number.

The cohomology analysis is most easily done in a level by level computation[1], where one starts from level 0 spanned by the operators $F(b, c, \partial X)$ of lowest conformal weight. As has been mentioned in the previous chapter, there are two such operators: c and $c\partial c$, having ghost numbers $G = 1$ and $G = 2$, respectively. These are the only ghost numbers that can occur at level 0. The BRST variations are[2]

$$\begin{aligned} &[Q, ce^{ip\cdot X}] = -(\tfrac{1}{2}p^2 - 1)c\partial ce^{ip\cdot X} , \\ &[Q, c\partial ce^{ip\cdot X}] = 0 , \end{aligned} \tag{4.3}$$

where we compute $[Q, A]$ using OPEs. From the relation between equal-time commutators and OPEs described in chapter 2, in particular equations (2.41) and (2.37) (where in the present case the infinitesimal transformation parameter ϵ is an anticommuting constant and we consider transformations generated by the BRST current $j(z)$ instead of conformal transformations generated by $T(z)$), we readily see that the BRST commutator is the first order pole of the OPE $j(z)A(w)$. Since no $G = 0$ operators exist at level 0, the operators $ce^{ip\cdot X}$ cannot be BRST exact. Therefore, we conclude that $ce^{ip\cdot X}$ is physical for $p^2 = 2$. These operators create tachyonic states, with $M^2 = -p^2 = -2$,

$$|0; p >\equiv \lim_{z,\bar{z}\to 0} c(z)e^{ip\cdot X(z,\bar{z})}|0 > , \tag{4.4}$$

where $|0>$ is the $sl(2)$-invariant vacuum. The operators $c\partial ce^{ip\cdot X}$ are trivially BRST invariant, because no $G = 3$ level 0 operators exist. However, as is clear from (4.3), they are only non-exact for $p^2 = 2$. These operators then generate copies of the tachyonic states (4.4) at the next ghost number. Their role has already been discussed in the previous chapter as providing states that give nonzero inner product with the standard states (4.4). We can consider the operators $ce^{ip\cdot X}$ and $c\partial ce^{ip\cdot X}$ as corresponding to the same physical operator in a different picture. The picture changing operator is given by

$$P_a = [Q, a_\mu X^\mu] = a_\mu c\partial X^\mu , \tag{4.5}$$

[1]The BRST operator respects the grading by level since it has itself zero conformal dimension and it does not change momentum p_μ.

[2]We use the notation $[,]$ for both commutators and anticommutators. As usual, when two operators of odd ghost number are considered, an anticommutator is understood.

for some polarization vector a_μ. Since P_a is BRST invariant[3], the normal-ordered product of P_a with a BRST invariant operator is also BRST invariant. We obtain the $G = 2$ tachyon operator by taking the normal-ordered product of P_a (assuming $a_\mu p^\mu \neq 0$) with the $G = 1$ tachyon operator,

$$c\partial c e^{ip\cdot X} \propto \oint \frac{dz}{2\pi i} \frac{P_a(z) c e^{ip\cdot X}(w)}{(z-w)} \,. \tag{4.6}$$

At the first excited level, where the operators $F(b, c, \partial X)$ have conformal weight zero, the lowest ghost number is $G = 0$, corresponding to the purely exponential operators $e^{ip\cdot X}$. It is easy to see that BRST invariance requires $p_\mu = 0$, which is the statement that the unit operator is physical. The corresponding state is the $sl(2)$-invariant vacuum. This is a discrete state, it is only physical for a single momentum. The $sl(2)$-invariant vacuum together with the picture changing operators[4] plus their conjugates (the conjugate of the identity operator is $\partial^2 c\partial c c$) create all discrete states of the chiral sector of the bosonic string [88, 109]. The next ghost number at the first excited level, $G = 1$, admits as the most general operator

$$V = (a_\mu c\partial X^\mu + x\partial c)e^{ip\cdot X} \,, \tag{4.7}$$

where x is a free parameter. The BRST variation is

$$[Q, V] = -(\tfrac{1}{2}p^2 a_\mu - ixp_\mu)c\partial c\partial X^\mu e^{ip\cdot X} - (-\tfrac{i}{2}a_\mu p^\mu - x)c\partial^2 c e^{ip\cdot X} \,. \tag{4.8}$$

Therefore, the BRST invariant combination has $x = -\frac{i}{2}a\cdot p$, and the polarization vector must satisfy $p^2 a_\mu = 2ixp_\mu$. However, this operator is BRST exact if a_μ is proportional to p_μ. Thus, the physical operators are given by $a_\mu c\partial X^\mu e^{ip\cdot X}$ with $p^2 = a\cdot p = 0$. Furthermore we have the equivalence relation $a_\mu \simeq a_\mu + \alpha p_\mu$ for any constant α.

Let us interpret these results. For an open string, there are no separate left and right-moving sectors, and the operator given in equation (4.1) is the full BRST operator. The level 1 physical operators (apart from the discrete ones) correspond to massless states ($p^2 = 0$) with transverse polarizations ($a\cdot p = 0$), and an equivalence relation $a_\mu \simeq a_\mu + \alpha p_\mu$. This leaves the 24 positive norm states expected for a massless vector particle in 26 dimensions.

For the closed string, we recall that there is also an anti-holomorphic sector that we usually ignore since it is treated in exactly the same way as the holomorphic sector. The BRST operator is in fact given by the sum of holomorphic and antiholomorphic BRST operators as in (3.21). The cohomology in the anti-holomorphic sector is isomorphic to that in the holomorphic sector and the total cohomology is obtained by tensoring

[3] One might think that P_a is BRST trivial as well. This is not the case though, since we do not include the non-derivative fields X^μ in the BRST complex. However, see [7] for a discussion of an extended BRST complex which does include X^μ (i.e. its center of mass operator x^μ besides the modes α_n^μ). In this extended complex, there is no doubling of operators since the $G = 2$ copies are BRST exact. Moreover, the 26 picture changing operators themselves are no longer physical.

[4] The picture changing operators are actually part of the generic $p^2 = 0$ spectrum. However, they are singular in the sense that they cannot be reached from generic light-like excitations by Lorentz transformation. They are also left out in the proof of the no-ghost theorem [109].

physical states from both sectors. Since the exponential factors of physical operators are common to both sectors, or in other words the left and right-moving momenta are assumed to be equal (see equation (2.26)), the levels of excitation in both sectors are required to be the same by the $L_0 - \bar{L}_0 = 0$ constraint. This is the only constraint that connects left and right-moving sectors. For the first excited level this yields the operators

$$e_{\mu\nu} c\bar{c}\partial X^\mu \bar{\partial} X^\nu e^{ip\cdot X} \,. \tag{4.9}$$

They correspond to massless states, $p^2 = 0$, and the polarization tensor satisfies $p^\mu e_{\mu\nu} = p^\nu e_{\mu\nu} = 0$. The BRST equivalence relation reads $e_{\mu\nu} \simeq e_{\mu\nu} + p_\mu k_\nu + k'_\mu p_\nu$ with $p \cdot k = p \cdot k' = 0$. The resulting physical states can be decomposed under the transverse rotation group $SO(24)$ into a traceless symmetric tensor, antisymmetric tensor and invariant. These correspond to the graviton, antisymmetric tensor and dilaton, respectively. Note that the free spectrum of the open string does *not* contain a massless spin-two particle. However, if interactions are taken into account, two open strings may join to form a closed string. Therefore, a theory of open strings contains closed strings as well and thus also the graviton.

It is interesting to note that for $c = 26$ all states of the form (2.63) with $h = -1$ correspond to null states of physical weight one. Together with the null states $L_{-1}|h = 0>$ they arrange for the decoupling of longitudinal excitations in the spectrum. This is one example of the interplay between space-time gauge symmetry in string theory and its underlying conformal field theory.

States at excitation levels two or higher are massive. This may be seen from the mass-shell formula which differs from the classical formula (2.32) by the normal-ordering constant in L_0 and $\bar{L}_0$,

$$M^2 = -p^2 = -2 + 2\sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n = 2(N-1) \,, \tag{4.10}$$

where α_n are the Fourier modes of the string coordinates, and N is the total energy level of these harmonic oscillators. Reintroducing $\alpha' = \frac{1}{2\pi T}$, we find that the mass-shell condition reads $M^2 = \frac{4}{\alpha'}(N-1)$ in the case of closed strings, and $M^2 = \frac{1}{\alpha'}(N-1)$ in the case of open strings. The massive physical states fill out representations of $SO(25)$, and the maximum spin at level N is N for open strings and $2N$ for closed strings. Thus we have the inequalities $J \leq \alpha' M^2 + 1$ for open strings and $J \leq \frac{1}{2}\alpha' M^2 + 2$ for closed strings, corresponding to the well-known Regge trajectories.

All physical operators, apart from the identity operator and its conjugate, take the form

$$S = cV(X,p) = cP(\partial X^\mu) e^{ip\cdot X} \,. \tag{4.11}$$

We ignore for the moment their copies at the next ghost number, and also restrict ourselves, as usual, to the holomorphic sector of the theory. Using the Wick rule (2.77), one can show that the first order pole of the OPE $j(z)S(w)$[5] is given by

$$[jS]_1 = -\partial c c V + \sum_{n\geq 1} \partial^n c c [TV]_{n+1} \,. \tag{4.12}$$

[5]We use the notation $[AB]_n$ for the operator in the n^{th} order pole of $A(z)B(w)$.

It follows that the condition of BRST invariance requires the operator $V(X,p)$ to be primary of conformal dimension one with respect to T, as claimed before in equation (3.33). The complete operator S is then a primary of the *total* energy-momentum tensor of vanishing weight. Thus the problem of constructing the complete cohomology of the bosonic string is the same as the problem of identifying all dimension one primary operators $V(X,p)$ or their corresponding highest weight states $|V>$. However, among these states there are still zero norm states that are BRST exact. All nonzero norm states can be generated by the so-called spectrum generating algebra of DDF operators [68]. This algebra is isomorphic to the algebra of transverse oscillators α_n^i. Application of the DDF operators to the tachyonic ground state results in the DDF states which have been proven to span the complete spectrum of the bosonic string in the case that $D=26$ and $a=1$. More precisely, any physical state $|\phi>$ can be uniquely decomposed as

$$|\phi>=|f>+|n>, \tag{4.13}$$

where $|f>$ is a DDF state and $|n>$ is a null state. A null state is a spurious state[6] that is physical as well. In the language of CFT, a spurious state is a Virasoro descendant and a physical state is a dimension one primary state. A state which is both a descendant and primary is null and decouples completely from the theory. In the BRST formalism, a DDF state $|f>$ corresponds to a certain representative of a cohomology class, whereas a null state $|n>$ corresponds to a BRST trivial state. From the fact that the DDF operators only generate excitations in the transverse directions, it follows that the spectrum contains no negative norm states. This is the no-ghost theorem [52, 102, 188]. For a more complete discussion, see [104]. The no-ghost theorem in the BRST quantization was established in [85, 88].

4.2 W-strings

The purpose of this and the next section is to describe the general structure of the physical spectrum of a class of W-strings. This class consists of the W_N strings where the W_N algebra is realized by a Miura realization. We mainly restrict ourselves to the critical case, where the fields of the Miura realization carry a total central charge c_{crit} given by minus the ghosts' contribution

$$c=c_{\text{crit}}=2\sum_{k=2}^{N}(6k^2-6k+1)=2(N-1)(2N^2+2N+1)\,. \tag{4.14}$$

The results for W_N strings turn out to be somewhat disappointing in the sense that their spectra appear to be quite similar to that of the ordinary bosonic string. This is a consequence of a special property of the Miura realizations, as described in subsection 3.3.2. As explained there, the currents of the Miura realization for W_N can be re-expressed in terms of W_{N-1} currents and one explicit scalar field ϕ_{N-1}. Repeating this

[6] A spurious state is a state of the form $|n>=L_{-1}|1>+L_{-2}|2>$ for some states $|1>,|2>$ and is therefore orthogonal to any physical state.

procedure, we obtain a W_N realization in terms of an energy-momentum tensor and $N-2$ scalar fields $\{\phi_2, \phi_3, ...\phi_{N-1}\}$. It turns out that the physical state conditions 'freeze' the scalar fields $\{\phi_2, \phi_3, ...\phi_{N-1}\}$, and leave a set of ordinary Virasoro type string sectors with different intercepts [164, 65, 136]. This is most easily understood in the nested basis. The BRST charge inherits the nested structure (at least classically) given in (3.81). Starting the cohomology computation with the highest-spin part Q_N^N of the BRST operator and going downwards to Q_N^3, we see that this successively fixes excitations and momenta corresponding to $\phi_{N-1}\,, \phi_{N-2}\,, ...\phi_2$. In the next section we intend to clarify this in the example of the W_4 string.

It is interesting to see what are the central charges of the embedded W_n algebras ($n \leq N$) in the Miura realization of the W_N string. The condition for criticality (4.14) fixes the free parameter of the Miura realization, α_0,

$$c = (N-1)(1+2N(N+1)(\alpha_0)^2) = c_{\text{crit}} \quad \Rightarrow \quad (\alpha_0)^2 = \frac{(2N+1)^2}{2N(N+1)}\,. \tag{4.15}$$

This then fixes the background charges of the scalar fields, and the contribution of ϕ_n to the central charge becomes

$$c_{\phi_n} = 1 + 3(2N+1)^2 \frac{n(n+1)}{N(N+1)}\,, \tag{4.16}$$

for $n = 1, 2, ...N-1$. The total central charge of the fields $\{\phi_{N-1}; c_N, b_N\}$ is now

$$c_N^N = c_{\phi_{N-1}} - 2(6N^2 - 6N + 1) = \frac{2(N-2)}{N+1} = c_{N-1,N}\,, \tag{4.17}$$

which is precisely the central charge of the first unitary W_{N-1} minimal model. We recall from equation (2.87) that the central charges of unitary W_N minimal models are given by

$$c_{N,q} = (N-1)\left(1 - \frac{N(N+1)}{q(q+1)}\right)\,, \quad q \geq N\,. \tag{4.18}$$

For $N = 3$, for example, $c_3^3 = \frac{1}{2}$ is the central charge of the Ising model, the first unitary Virasoro minimal model. More generally, the central charge of the fields $\{\phi_{n-1}, ...\phi_{N-1}; c_n, b_n, ...c_N, b_N\}$, which may be considered to correspond to the sub-sector of the $N-n+1$ highest-spin currents of the W_N algebra [20], adds up to

$$\begin{aligned} c_N^n &= \sum_{k=n-1}^{N-1} c_{\phi_k} - 2\sum_{k=n}^{N}(6k^2 - 6k + 1) \\ &= (n-2)\left(1 - \frac{n(n-1)}{N(N+1)}\right) = c_{n-1,N}\,, \end{aligned} \tag{4.19}$$

which is the central charge of the $(N, N+1)$ unitary W_{n-1} minimal model. So these central charge counting arguments suggest that a critical W_N string is related to a series of $(N, N+1)$ W_n minimal models with $n = 2, 3, ...N-1$ [136, 20]. This is indeed what is found also in cohomology computations, as we will now discuss.

For simplicity, we concentrate on the W_3 string. Explicit results for W_4 will be discussed in the next section. For $N > 4$, the explicit form of the BRST operator is not known. However, from the results for W_3 and W_4, the general pattern for critical W_N strings based on Miura realizations seems to be clear. More on W_N strings may be found in the review papers [161, 192, 115].

The $c = 100$ realization needed for a nilpotent BRST operator is given in (3.75) if we choose $\alpha_0^2 = \frac{49}{24}$. The freedom to choose the sign for α_0 corresponds to the simple OPE automorphism $A \to -A$. We write $A = \partial\phi$ and take the energy-momentum tensor T_X to be realized by D scalar fields X^μ,

$$T_X = -\frac{1}{2}\partial X_\mu \partial X^\mu + a_\mu \partial^2 X^\mu \,, \tag{4.20}$$

with central charge $c_X = D + 12 a_\mu a^\mu = \frac{1}{4}c + \frac{1}{2} = 25\frac{1}{2}$. It is clear that at least one of the scalars X^μ must have a nonzero background charge.

A background charge q in $T_\phi = -\frac{1}{2}\partial\phi\partial\phi + q\partial^2\phi$ corresponds to a coupling of ϕ to the world-sheet curvature scalar,

$$S_0 \to S_0 + \frac{q}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)}\phi \,, \tag{4.21}$$

as observed before in the case of the Liouville field of the non-critical string. If we consider a correlation function on the sphere

$$< \prod_k V_k(p_k) >= \int \mathcal{D}\phi e^{-S_0 - \frac{q}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)}\phi} \prod_k V_k(p_k) \,, \tag{4.22}$$

where the vertex operators $V_k(p_k)$ have exponential parts $e^{ip_k\phi}$, the change of variable $\phi \to \phi + a$ yields the Ward identity

$$\sum_k p_k = -2iq \,. \tag{4.23}$$

This follows from the Gauss-Bonnet theorem, $\frac{1}{4\pi}\int d^2\sigma\sqrt{h}R^{(2)} = 2(1-g)$ with g the genus of the Riemann surface that represents the world-sheet. We see then that scalars with real background charges, such as the scalars in the usual Miura realization, are supposed to have imaginary momenta. Rescaling such scalars by $\sqrt{-1}$ gives real momenta but changes the sign of the OPE and therefore suggests that they may be thought of as timelike coordinates[7]. One introduces so-called screening charges, to be inserted in correlation functions, to ensure that (4.23) is satisfied. For the two-scalar W_3 Miura realization ($D = 1$ in (4.20)), screening currents are given by

$$S_i^\pm = e^{i\alpha_\pm \vec{e}_i\cdot\vec{\phi}} \,, \tag{4.24}$$

[7]However, in the case of 'frozen' scalar fields whose allowed momenta are discrete, a space-time interpretation, if any, is not clear. Perhaps they can be viewed as corresponding to compact directions. See [162] for a discussion.

where $\vec{e}_i$, $i = 1, 2$ are the simple roots of $sl(3)$ and $\alpha_\pm$ are determined from the requirement that the currents are spin-one primaries,

$$\alpha_\pm = \tfrac{i}{\sqrt{2}}(\alpha_0 \pm \sqrt{\alpha_0^2 - 2})\,. \tag{4.25}$$

We use the basis in which $\vec{\phi} = (X, \phi)$. The screening charges commute with the generators of the W_3 algebra. It turns out that the momenta of cohomology classes of the two-scalar W_3 string are all of the form [166]

$$\vec{p} = \sum_{i=1}^{2} (n_i^+ \alpha_+ + n_i^- \alpha_-)\vec{e}_i\,, \tag{4.26}$$

where $n_i^\pm$ are integers and $\vec{p} = (p_X, p_\phi)$. This guarantees (at least for negative integers $n_i^\pm$) that with appropriate insertions of screening charges momentum conservation (4.23) can always be satisfied.

W_3 string ground states are created by

$$V = c_2 \partial c_3 c_3 e^{i\vec{p}\cdot\vec{\phi}}\,. \tag{4.27}$$

The BRST invariance condition for standard operators (operators with ghost structure as in (4.27)), is the condition that they are W_3 primary with spin-two and spin-three weights (intercepts) $(h, w) = (4, 0)$. Since this amounts to a quadratic and a cubic equation in the momenta, there are six physical values (p_X, p_ϕ). They form a multiplet under the following action of the Weyl group $\mathcal{W}$ of $sl(3)$,

$$w \cdot \vec{p} \equiv w(\vec{p} - i\sqrt{2}\alpha_0\vec{\rho}) + i\sqrt{2}\alpha_0\vec{\rho}\,, \qquad w \in \mathcal{W}\,. \tag{4.28}$$

These are the transformations which leave (h, w) invariant. The six-to-one map from momenta to weights is an artefact of the Miura transformation [78, 32].

As argued in the previous chapter, the BRST analysis is simplified considerably after an appropriate canonical transformation. We want the BRST operator in (3.60) and (3.61) to become a sum of two terms with spin-two and spin-three ghost numbers $(G_2, G_3) = (1, 0)$ and $(G_2, G_3) = (0, 1)$. To accomplish this, it turns out that we have to perform the quantum canonical transformation generated by $G = \frac{4i}{3\sqrt{29}}\partial\phi c_3 b_2 - \frac{7i}{3\sqrt{58}}\partial c_3 b_2$. The change in the fields is calculated using (3.80), where in the present case only the first three terms contribute. After another OPE-preserving rescaling of the spin-three ghost fields, the BRST operator takes the form $Q = Q_0 + Q_1$, with

$$\begin{aligned} Q_0 &= \oint \frac{dz}{2\pi i} c_2 \left(T + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}\right)\,, \\ Q_1 &= \oint \frac{dz}{2\pi i} c_3 \left((\partial\phi)^3 + 3\sqrt{3}\alpha_0 \partial^2\phi\partial\phi + \tfrac{19}{8}\partial^3\phi + \tfrac{9}{2}\partial\phi b_3 \partial c_3 + \tfrac{3\sqrt{3}}{2}\partial b_3 \partial c_3\right)\,. \end{aligned} \tag{4.29}$$

As a simple consequence of the graded structure, we note that $Q_0^2 = Q_1^2 = \{Q_0, Q_1\} = 0$. This form of the BRST operator was first given in [135].

It can be shown[8] that all physical operators with standard ghost structure are of the form

$$V = c_2 \partial c_3 c_3 e^{-\frac{1}{2\sqrt{2}} p\phi} Y_\Delta(X)\,, \tag{4.30}$$

where $Y_\Delta(X)$ is primary under T_X with weight Δ. Physical operators (4.30) belong to one of three sectors, namely

$$p = 6\,,\ \Delta = 1\,; \qquad p = 7\,,\ \Delta = \frac{15}{16}\,; \qquad p = 8\,,\ \Delta = 1\,. \tag{4.31}$$

Thus we see that the scalar ϕ is 'frozen' in the sense that the momentum in the ϕ direction is restricted to three discrete values, and no ϕ excitations are physical [162]. Since the total weight of physical operators is zero, the 'highest-spin part' $\partial c_3 c_3 e^{ip\phi}$ of (4.30) can only have weight 0 or $\frac{1}{16}$. Note that in the case of the two-scalar W_3 string, the momentum of X is also discrete.

We recall from equation (4.17) that the fields $(\phi; b_3, c_3)$ together have central charge $c_3^3 = \frac{1}{2}$, the central charge of the first (non-trivial) unitary Virasoro minimal model, the Ising model. The appearance of an Ising model structure is now becoming clear by noticing that operators of weight 0 (corresponding to the identity operator) and $\frac{1}{16}$ also appear in the Ising model. The third primary operator of the Ising model has conformal weight $h = \frac{1}{2}$ and appears in the W_3 cohomology as a non-standard operator (an operator with non-standard ghost structure), see below. The fields X^μ constitute an effective space-time sector, and we see that the standard ghost structure operators give rise to effective space-time sectors with intercepts 1 and $\frac{15}{16}$. The sector with intercept 1 is almost the same as the standard bosonic string spectrum. The difference is that here we have $c_X = 25\frac{1}{2}$ instead of 26. Note that $Y_\Delta(X)$ can be any $h = \Delta$ primary. In the ordinary critical string, physical states are built from excitations in $D-2$ transverse directions. In the effective space-time sectors of the W_3 string, however, fewer states decouple, and excitations in $D-1$ directions are physical. Unitarity requires the effective space-time intercepts to be of the form $\Delta = 1 - h_{p,q}(m = 3)$, where the conformal weights $h_{p,q}$ of unitary representations (in this case of the Ising model) are given in (2.65). For an explanation, see references [191, 192].

Besides states of standard ghost structure, one can also consider states of non-standard ghost structure. Among them are states that correspond to the $h = \frac{1}{2}$ operator of the Ising model. The simplest such operator is the level 1 operator

$$c_2 c_3 e^{-\frac{1}{2\sqrt{2}} p\phi} Y_\Delta(X)\,, \tag{4.32}$$

with $p = 4$ and $\Delta = \frac{1}{2}$. Operators of non-standard ghost structure are usually associated with vanishing null states of the W_3 algebra. In this context, let us note that precisely the W_3 modules with momenta (weights) as in (4.26) are degenerate, i.e. contain null vectors.

The operators in (4.30) and (4.32) by no means exhaust the BRST cohomology. The physical spectrum turns out to contain an infinite number of operators with different

[8] See the review papers [161, 192] and references therein.

$(\phi; b_3, c_3)$-dependence for all three effective space-time sectors. The $(\phi; b_3, c_3)$-dependent parts of physical operators can be found quite easily from the cohomology of the Q_1 operator in (4.29) alone. The Q_1 cohomology is the first term in a spectral sequence that can be associated with the double complex defined by all fields and the differential $Q = Q_0 + Q_1$. The Q_1 cohomology has been shown to provide a new realization of the Ising model [116]. Singular vectors are divided out in the sense that they are Q_1 exact. The infinite number of copies of each primary are all connected by screening operators. However, from the W_3 string point of view there is no reason to identify them. The Q_1 cohomology acting on the Fock space generated by the fields $(\phi; b_3, c_3)$ *plus* the identifications using screening operators would seem to play a role similar to the Felder reduction that provides irreducible Virasoro minimal model realizations from Fock space representations.

Note that all physical operators described up to now have the factorized form

$$c_2 U(\phi; b_3, c_3) V_\Delta(X)\,, \tag{4.33}$$

where $U(\phi; b_3, c_3)$ are Ising model operators and $V(X)$ are primaries with weights dual to those of U. The operators (4.33) create continuous momentum states of the W_3 string. 'Continuous momentum' here refers to the momenta p_μ of X^μ in the multi-scalar case $(D > 1)$. In addition to continuous momentum states, the physical spectrum also contains discrete states which are physical only for $p_\mu = 0$ or $p_\mu = -2ia_\mu$. For $p_\mu = 0$ the corresponding operators take the form [161]

$$V = c_2 U_1(\phi; b_3, c_3) + U_2(\phi; b_3, c_3)\,, \tag{4.34}$$

where U_2 is a $h = 0$ primary in the Q_1 cohomology. The identity operator is the special case with $U_1 = 0$ and $U_2 = \mathbf{1}$. Note that whereas the operators (4.33) have standard *spin-two* ghost structure, the operators corresponding to discrete states do not have this property. In [140], two invertible discrete physical operators were found. These are guaranteed to give BRST non-trivial physical states when normal-ordered with any physical operator. They have been used to compute the complete spectrum of the W_3 string for the two-scalar as well as for the multi-scalar case [140].

A generic feature of the cohomology is that ghost number G physical states have partners at ghost number $2 - G$. This corresponds to Hermitian conjugation under which the BRST operator is invariant. Conjugation pairs the G and $2 - G$ sectors in the ghosts' Fock space and changes the momenta (p_μ, p_ϕ) of scalar fields to $(-p_\mu - 2ia_\mu, -p_\phi + 2i\sqrt{3}\alpha_0)$. Furthermore, all states described thus far are so-called prime states. Acting with picture changing operators on prime states gives additional states at the next few ghost numbers [165]. For the bosonic string we have seen that the picture changing operator accounts for the doubling of the states. For the two-scalar W_3 string, two independent picture changing operators $a_X = [Q, X]$ and $a_\phi = [Q, \phi]$ generate, starting from a prime state $|P>$, quartets of physical states $\{|P>, a_X|P>, a_\phi|P>, a_X a_\phi|P>\}$ at ghost numbers $\{G, G+1, G+1, G+2\}$.

The appearance of Ising model operators in the spectrum suggests that the critical W_3 string is related to the non-critical Virasoro string with the Ising model as its matter sector. Indeed, the Q_1 cohomology represents the Ising model whose operators are then dressed by the 'Liouville' scalar X (in the two-scalar case) to operators of vanishing total conformal dimension in the Q cohomology. Moreover, in both the non-critical string spectrum [133] and the critical W_3 spectrum [140], the ghost numbers of cohomology classes range from $-\infty$ to $+\infty$. It is also known [70] that W_3 constraints appear as Dyson-Schwinger equations in the two-matrix model corresponding to the Ising model coupled to 2d-gravity. In the same way, critical W_N strings are believed to be intimately related to the $(N, N+1)$ unitary Virasoro minimal model coupled to 2d-gravity. For $N \to \infty$ one would then expect some connection between $c = 1$ matter coupled to 2d-gravity (often referred to as the two-dimensional string) and the critical W_∞ string. It is indeed known that the two-dimensional string has a W_∞ symmetry structure [8, 128, 194].

The critical W_N string based on the Miura realization is not only related to the non-critical Virasoro string with $(N, N+1)$ minimal matter, but also to a series of non-critical W_n strings for all $3 < n < N$, as argued before by counting central charges. The Q_N^{n+1} cohomology (cf. equation (3.81)) is conjectured to realize the unitary $(N, N+1)$ W_n minimal model and is coupled to W_n gravity through the transition to the total cohomology. For W_4 we give details below.

What we have been calling the critical W_N string is in fact more like the analogue of pure gravity and it is perhaps better to call it pure W_N gravity. Indeed, the W_N symmetry of the $sl(N)$ Toda action expected to describe quantum W_N gravity in a conformal gauge is realized by the $(N-1)$-scalar Miura realization. Pure gravity is described by the Liouville scalar with a background charge such that $c_L = 26$. Its cohomology classes are known to extend through all ghost numbers [133]. Similarly, the critical W_3 string described above also has cohomology classes at all ghost numbers. As mentioned before, direct W-extensions of the 26-dimensional critical bosonic string do not seem to exist since scalar field realizations of W_N algebras always involve background charges.

4.2.1 Non-critical W-strings

Non-critical W_N strings describe W_N matter coupled to W_N gravity and provide generalizations of the critical W_N string which corresponds to the special case of W_N gravity coupled to trivial ($c = 0$) matter. Results on the spectrum of non-critical W_N strings have been given in [30, 31, 46, 18, 47, 48].

The matter sector is usually taken to be a W_N minimal model. The (p, q) W_N minimal models have central charges

$$c_{p,q} = (N-1)\left(1 - N(N+1)\frac{(p-q)^2}{pq}\right) , \tag{4.35}$$

for positive integers p and q. The unitary models have $q = p + 1 > N + 1$ and their central charges were given before in (2.87). The central charge of the W_N gravity (Toda) theory is then given by $c_{p,-q}$, since

$$c_{p,q} + c_{p,-q} = c_{\text{crit}} . \tag{4.36}$$

The BRST charge for the non-critical W_3 string in its canonical form was given in equation (3.66). To compute the physical spectrum we need a realization of the matter and Liouville currents. For the Liouville sector it is natural to take the usual two-scalar Miura realization of the W_3 algebra. To facilitate computation of the cohomology, the matter minimal model is usually also represented by the two-scalar Miura realization of the appropriate central charge. However, one has to perform a further reduction in order to obtain a minimal model. This should be similar to the Felder BRST reduction of the Coulomb gas realizations for Virasoro minimal models. However, Fock space resolutions of irreducible modules for minimal models are very complex in the case of W-algebras. (In the case of W_3 such resolutions were constructed in [89, 44].) Nevertheless, a complete classification of physical states for a $W[g]$ minimal model coupled to $W[g]$ gravity has been conjectured in [46]. For $g = sl(N)$ this corresponds to W_N minimal matter coupled to W_N gravity.

The BRST analysis for the non-critical W_3 string can again be simplified using the redefinition that leads, at the classical level, to a nested subalgebra structure. In [18] the non-critical W_3 string is investigated using the redefined BRST operator of [21], which is the sum of two nilpotent operators Q_0 and Q_1. The minimal model structure of the cohomology is then elucidated. In particular, it is shown in [18] that the Q_1 cohomology is closely related to a (p, q) Virasoro minimal model if one chooses for the matter sector a (p, q) W_3 minimal model. This generalizes the connections to minimal models of critical W_N strings. One might also wonder what happens if one takes a (non-unitary) $(p, q) = (2, 3)$ W_3 model as matter sector with central charge -2. This might lead to the trivial $c = 0$ Virasoro minimal model in the Q_1 cohomology. In fact, this particular W_3 non-critical string model is used in [23] and we come back to it in section 4.4.

Another interesting class of non-critical W_N strings is obtained if the matter theory is realized by $N-1$ free scalar fields. For $N = 2$ this corresponds to the $D = 2$ string whose physical states have been calculated in [134]. Interesting algebraic structures have been found in the $D = 2$ string, as described, for example, in [194]. For $N = 3$, the $D = 4$ W_3 string has been extensively studied in [47]. In this work the algebraic structure of the cohomology is emphasized.

4.3 An example: the W_4 string

This section closely follows [40]. In order to study the physical spectrum of the W_4 string, we need the BRST operator for the W_4 algebra. It was given in [113, 200], but a more convenient form of the BRST operator was found in [20]. Let us summarize how it was constructed.

The energy-momentum tensor in the three scalar Miura realization is

$$T_M = -\tfrac{1}{2}\partial\vec{\phi}\cdot\partial\vec{\phi} - \sqrt{2}\alpha_0\vec{\rho}\cdot\partial^2\vec{\phi}, \tag{4.37}$$

where $\vec{\phi} = (\phi_1, \phi_2, \phi_3)$ and $\vec{\rho}$ is the Weyl vector of $sl(4)$. We use the representation $\vec{\rho} = \frac{1}{2}(\sqrt{2}, \sqrt{6}, \sqrt{12})$, and denote by $q_i = \sqrt{2}\alpha_0\rho_i$, $i = 1, 2, 3$, the background charges of the scalar fields. The central charge is

$$c_M = 3 + 24(\alpha_0)^2\rho^2 = 3 + 120(\alpha_0)^2\,. \tag{4.38}$$

As described in the previous chapter, in the classical limit it is possible to redefine the generators such that the algebra is brought to a special form with a nested subalgebra structure [20]. The energy-momentum tensor is not affected by the redefinition. The BRST charge associated to the resulting classical algebra inherits the same nested structure. Quantization by parametrising all possible quantum corrections and demanding nilpotency leads to a BRST operator which still has this nested structure, and therefore, as we will see, is convenient for studying the spectrum. It should also be possible to relate the operators in the original and redefined basis by a (quantum) canonical transformation, as in the case of the W_3 string. Explicitly, the BRST current is

$$\begin{aligned} j_2 &= c_4\{(\partial\phi_3)^4 + 4q_3\partial^2\phi_3(\partial\phi_3)^2 + \tfrac{41}{5}(\partial^2\phi_3)^2 + \tfrac{124}{15}\partial^3\phi_3\partial\phi_3 \\ &\quad + \tfrac{46}{135}q_3\partial^4\phi_3\} - 8(\partial\phi_3)^2 c_4\partial c_4 b_4 + \tfrac{16}{9}q_3\partial^2\phi_3 c_4\partial c_4 b_4 \\ &\quad + \tfrac{32}{9}q_3\partial\phi_3 c_4\partial^2 c_4 b_4 + \tfrac{4}{5}c_4\partial^3 c_4 b_4 - \tfrac{16}{3}c_4\partial c_4\partial^2 b_4\,, \end{aligned} \tag{4.39}$$

$$\begin{aligned} j_1 &= c_3\{(\partial\phi_2)^3 + \tfrac{3}{4}\partial\phi_2(\partial\phi_3)^2 + \tfrac{5\sqrt{2}}{8}(\partial\phi_3)^3 + 3q_2\partial\phi_2\partial^2\phi_2 \\ &\quad + \tfrac{3}{2}q_3\partial\phi_2\partial^2\phi_3 + \tfrac{9}{2}q_2\partial\phi_3\partial^2\phi_3 + \tfrac{93}{40}\partial^3\phi_2 + \tfrac{69\sqrt{2}}{10}\partial^3\phi_3\} \\ &\quad - \tfrac{9}{2}\partial\phi_2 c_3\partial c_3 b_3 + \tfrac{3}{2}q_2 c_3\partial^2 c_3 b_3 - \tfrac{243}{64}c_3\partial c_3 b_4 \\ &\quad - \tfrac{9}{2}\partial\phi_2 c_3 c_4\partial b_4 - 6\partial\phi_2 c_3\partial c_4 b_4 + \tfrac{9}{2}q_2 c_4\partial^2 c_3 b_4 \\ &\quad + \tfrac{3}{2}q_2 c_3\partial^2 c_4 b_4 - \tfrac{9\sqrt{2}}{2}\partial\phi_3 c_4\partial c_3 b_4 - 3\sqrt{2}\partial\phi_3 c_3\partial c_4 b_4 + j_2\,, \end{aligned} \tag{4.40}$$

$$j = c_2(T_M + T_{c_3,b_3} + T_{c_4,b_4} + \tfrac{1}{2}T_{c_2,b_2}) + j_1\,. \tag{4.41}$$

Here (c_k, b_k) is the conjugate ghost pair of the spin-k symmetry with conformal dimension $(1-k,k)$ with respect to the corresponding energy-momentum tensor $T_{c_k,b_k} = -kb_k\partial c_k + (1-k)\partial b_k c_k$. The total W_4 BRST operator is $Q = \oint \frac{dz}{2\pi i} j(z)$ and as the way of representing it in equations (4.39-4.41) suggests, it involves two other nilpotent BRST charges: $Q_2 = \oint \frac{dz}{2\pi i} j_2(z)$ is a BRST operator corresponding to a spin-four symmetry, and $Q_1 = \oint \frac{dz}{2\pi i} j_1(z)$ is a BRST operator corresponding to a symmetry generated by spin-three and spin-four currents. We have

$$(Q_2)^2 = (Q_1)^2 = (Q)^2 = 0\ ; \qquad (Q_{Vir})^2 = \{Q_{Vir}, Q_1\} = 0\,, \tag{4.42}$$

where we defined $Q_{Vir} = Q - Q_1$. Note that Q_{Vir} is just the usual Virasoro BRST operator when we consider the spin-(3,4) ghost systems to belong to the matter part. The spin-four part Q_2 was obtained before in [139]. We should also mention that momenta and ghost numbers of physical operators are not affected by the redefinition leading to (4.39-4.41). However, the explicit expressions for physical operators are expected to be much simpler in the new basis, as in the W_3 string case [135].

The BRST operator (4.41) is nilpotent provided the total central charge of matter plus ghosts vanishes. This requires T_M to have central charge $c_M = 246$ implying $(\alpha_0)^2 = \frac{81}{40}$. Then we obtain what we call a critical W_4 string. For a non-critical string one would expect another sector with W_4 symmetry. Unfortunately, however, the redefinition described above can only be applied to one of both sectors [20] which means that the description of the non-critical W_4 string would be much more involved.

Screening operators play an important role in the physical state analysis. The standard W_4 screening currents [78] are

$$S_i^{\pm} = e^{i\alpha_{\pm}\vec{e}_i\cdot\vec{\phi}}\,, \tag{4.43}$$

where $\vec{e}_i$ are the $sl(4)$ simple roots, and $\alpha_{\pm}$ are given in (4.25). The screening charges $\oint \frac{dz}{2\pi i} S_i^{\pm}$ commute with the W_4 generators in the Miura realization, and they will appear in modified form in the discussion of the Q-cohomology. Besides, there will be more screening operators that simplify the classification of physical states. In general, for a physical operator $\mathcal{O}$ of zero conformal weight, one can find an associated screening current $S_{\mathcal{O}}$ via the descent equation

$$[Q, S_{\mathcal{O}}(z)] = \partial\mathcal{O}(z)\,, \tag{4.44}$$

where Q is the BRST charge under consideration. The corresponding screening charge $\oint \frac{dz}{2\pi i} S_{\mathcal{O}}(z)$ will then commute with Q.

Picture changing operators are defined by

$$P_i(z) = [Q, \phi_i(z)]\,. \tag{4.45}$$

In the usual BRST complex these are physical $h = 0$ primaries. Applying a picture changing operator to a physical state, i.e. taking the normal-ordered product, either gives zero or another physical state.

As in (4.26), the momenta of physical states are multiples of the momenta of the screening currents (4.43):

$$\vec{p} = \sum_{i=1}^{3} (n_i^+\alpha_+ + n_i^-\alpha_-)\vec{e}_i\,. \tag{4.46}$$

All physical operators described below indeed have momenta on this lattice. It is then convenient to rewrite the momenta as

$$p_i = \frac{iq_i}{27}\tilde{p}_i\,, \tag{4.47}$$

because (4.46) now implies that

$$\tilde{p}_1 \in 3\mathbb{Z}\,; \quad \tilde{p}_2 \in \mathbb{Z}\,; \quad \tilde{p}_3 \in 2\mathbb{Z}\,. \tag{4.48}$$

Note that the momenta p_i are imaginary. In the following we will usually refer to $\tilde{p}_i$ as the momentum.

We now proceed to determine the cohomology of Q in steps, starting with the Q_2 cohomology which imposes the spin-four constraint.

4.3.1 The Q_2 cohomology

Since the BRST current j_2 only depends on a single scalar field ϕ_3 and the spin-four ghost variables (c_4, b_4), we only need to consider operators built from these fields. These fields together have central charge $\frac{4}{5}$, the central charge of the $(p, p') = (4, 5)$ W_3 unitary minimal model.

The Q_2 physical states at some low energy levels are given implicitly in a discussion of the $W_{2,4}$ string in [139, 138]. The extra Virasoro constraint of the $W_{2,4}$ algebra does little more than dress the primary Q_2 physical operators to operators of total spin zero. In [141], the complete cohomology of the critical $W_{2,4}$ string is given. Ignoring the Virasoro constraint, i.e. ignoring the 'Liouville dressings', this cohomology seems to be equivalent to the W_3 primary part of the Q_2 cohomology obtained below, apart from some descendants which couple to the Virasoro fields in the $W_{2,4}$ cohomology.

The ghost vacuum is given by acting on the $sl(2)$-invariant vacuum with $\partial^2 c_4 \partial c_4 c_4$. In the following, we will always write down operators that are supposed to act on the $sl(2)$-invariant vacuum. First consider operators of the form

$$V_0^0 = \partial^2 c_4 \partial c_4 c_4 e^{ip_3\phi_3} \,. \tag{4.49}$$

The lower index denotes the level. The upper index refers to the ghost number G of the state created by this operator. Thus the $sl(2)$-invariant vacuum is assigned ghost number -3 for the moment. Level 1 states at lowest ghost number ($G = -1$) are created by

$$V_1^{-1} = \partial c_4 c_4 e^{ip_3\phi_3} \,. \tag{4.50}$$

The physical states of lowest ghost number at a particular level are easy to find. Since they can't be Q_2 exact, one only has to impose the vanishing of their Q_2 variation. We restrict ourselves to operators on levels 0 and 1, since this will turn out to be enough to understand the full structure of the Q_2 cohomology. Imposing the physical state conditions on (4.49) and (4.50), we obtain the results listed in table 1.

V_0^0	$\tilde{p}_3$	h	w
	24	0	0
	30	0	0
	26	1/15	1
	28	1/15	-1

V_1^{-1}	$\tilde{p}_3$	h	w
	16	1/15	-1
	18	2/5	0
	20	2/3	-26

Table 1. *Level 0 and 1 physical states in the Q_2 cohomology. Momenta are denoted by $\tilde{p}_3$, see (4.47). The last two columns give the weights h and w with respect to the spin-two and three generators of the $c = \frac{4}{5}$ W_3 algebra.*

Note that the physical values of p_3 agree with equations (4.46-4.48) (we only consider the third component of (4.46)). The last two columns give the weights of the physical states with respect to the generators of a W_3 algebra. It turns out that the physical states in the Q_2 cohomology can be organized in representations of the $c = \frac{4}{5}$ W_3 algebra whose generators (T, W) are physical operators at levels 8 and 9 with zero momentum. The Virasoro generator T is the energy-momentum tensor built from the fields $(\phi_3; c_4, b_4)$, and the spin-three generator is given by [138]

$$
\begin{aligned}
W \quad = \quad & \sqrt{\tfrac{2}{13}}\{\tfrac{5}{3}(\partial\phi_3)^3 + 5q_3\partial^2\phi_3\partial\phi_3 + \tfrac{25}{4}\partial^3\phi_3 + 20\partial\phi_3 b_4\partial c_4 \\
& +12\partial\phi_3\partial b_4 c_4 + 12\partial^2\phi_3 b_4 c_4 + 5q_3\partial b_4\partial c_4 + 3q_3\partial^2 b_4 c_4\}\,. \qquad (4.51)
\end{aligned}
$$

They generate the $c = \frac{4}{5}$ W_3 algebra with standard normalization, up to an extra primary spin-four operator, which turns out to be a multiple of the Q_2 exact operator $V = \{Q_2, b_4\}$. It was noticed by the authors of [138] that after bosonizing the spin-four ghost pair, this realization of the W_3 algebra coincides with a special two-scalar realization found in [22]. As a side-remark, we note that this $c = \frac{4}{5}$ realization is unique in the sense that it has one real and one imaginary background charge, the latter belonging to the 'ghost scalar'.

The physical states in table 1 are all primary with respect to the W_3 algebra, with L_0 and W_0 eigenvalues h and w, respectively. For convenience, the weights w have been rescaled as in [138]. The Virasoro weights are given in terms of p_3 as

$$
h = \tfrac{1}{2}(p_3)^2 - iq_3p_3 + l - 6\,, \qquad (4.52)
$$

where l is the level. The spin-three weight is a cubic polynomial in p_3 and depends on the detailed structure of the operator.

Let us now compare the Q_2 spectrum with the spectrum of primaries in a $c = \frac{4}{5}$ W_3 minimal model. The spectrum of conformal weights in a generic (p, p') W_3 minimal model is given by (see e.g. [49])

$$
\begin{aligned}
h(r_1, r_2; s_1, s_2) \quad = \quad & -\frac{(p-p')^2}{pp'} \qquad (4.53) \\
& +\frac{1}{3pp'}\{\sum_{i\le j=1}^{2}(p'(r_i+1) - p(s_i+1))(p'(r_j+1) - p(s_j+1))\}\,,
\end{aligned}
$$

where the non-negative integers r_i, s_i run over the range

$$
0 \le r_1 + r_2 \le p - 3\,; \quad 0 \le s_1 + s_2 \le p' - 3\,. \qquad (4.54)
$$

Note that the level 0 states in table 1 correspond to the 'diagonal' entries of the $(p, p') = (4, 5)$ Kac table (4.53), since $h(0,0;0,0) = 0$ and $h(0,1;0,1) = h(1,0;1,0) = \frac{1}{15}$. The weights $\frac{2}{5}$ and $\frac{2}{3}$ of the level 1 states are also in the set (4.53). Moreover, at levels 0 and 1 together, all conformal weights of the $(4, 5)$ W_3 minimal model occur. It is also interesting to note that the maximum possible conformal dimension of an operator at a particular level, $h_{max} = \frac{1}{2}q_3^2 + l - 6$ (see (4.52)), forbids the appearance of $h = \frac{2}{5}$ and $h = \frac{2}{3}$ operators on level 0.

The spin-three weights w corresponding to the Virasoro weights $h(r_1, r_2; s_1, s_2)$ in a (p, p') W_3 minimal model are given by [49]

$$\begin{aligned} w(r_1, r_2; s_1, s_2) &= C(p,p')(p'(r_1 - r_2) - p(s_1 - s_2)) \\ \times(p'(2r_1 + r_2 + 3) - p(2s_1 + s_2 + 3))&(p'(r_1 + 2r_2 + 3) - p(s_1 + 2s_2 + 3)) \,, \end{aligned} \tag{4.55}$$

where $C(p, p')$ depends on the normalization of the spin-three current. The w-values in table 1 are indeed in agreement with the minimal model values (4.55). Under the $\mathbb{Z}_2$ transformation $(r_1, r_2; s_1, s_2) \to (r_2, r_1; s_2, s_1)$ h is invariant, while w changes sign. We observe that the level 0 states occur in such $\mathbb{Z}_2$ pairs.

From table 1 it is clear that a physical state with $(h, w) = (\frac{2}{3}, +26)$ is missing at levels 0 and 1. However, we only discussed states of lowest ghost number. In particular, any state of ghost number G has a conjugate state at ghost number $1 - G$ at the same level, and it turns out that the $(\frac{2}{3}, +26)$ state occurs at level 1, $G = 2$. It is in fact the conjugate of the $(\frac{2}{3}, -26)$ state. More generally, the $\mathbb{Z}_2$ symmetry mentioned above is part of the conjugation of a physical operator. This completes the identification of all minimal model primaries in the Q_2 cohomology, at levels 0 and 1.

For the purpose of finding physical states at higher levels and different ghost numbers we introduce the following screening operators,

$$S = b_4 e^{ip_3\phi_3} \,, \qquad \text{with } \tilde{p}_3 = -6 \,, \tag{4.56}$$

$$R = \partial c_4 c_4 e^{ip_3\phi_3} \,, \qquad \tilde{p}_3 = 30 \,, \tag{4.57}$$

$$\bar{R} = \partial c_4 c_4 e^{ip_3\phi_3} \,, \qquad \tilde{p}_3 = 24 \,. \tag{4.58}$$

They are spin-one primaries whose charges commute with Q_2. It is not difficult to see that R and $\bar{R}$ are the screening currents associated to the level 0, $h = 0$ physical operators (see table 1) via the descent equation (4.44). With these screening charges it is possible to obtain new physical states by acting on the level 0 and 1 states described above. The OPEs of T and W with the screening currents are total derivatives (in the case of R and $\bar{R}$ this is true up to Q_2 exact terms), which means that W_3 primaries are mapped to W_3 primaries of the same (h, w) under the action of the screening charges.

We follow [86], where a similar discussion for the W_3 string can be found. For the action of n screening charges on a physical state of momentum p to be well-defined, the following expression must be an integer,

$$P_n \equiv n - 1 + \sum_{i<j=1}^{n} p_{s_i} p_{s_j} + p \sum_{i=1}^{n} p_{s_i} \,, \tag{4.59}$$

with screening momenta p_{s_i}. Using this, one can show that, for example, the action of S on $V_0^0[\tilde{p}_3 = 30]$ is well-defined. However, this action[9] is trivial in the sense that it gives zero, and to obtain a new physical state we have to make use of the picture

[9] By the action of a screening operator S on a physical operator V we mean the commutator $\oint \frac{dz}{2\pi i} S(z)V(w)$, whereas the action of a picture changing operator P is the normal-ordered product $\oint \frac{dz}{2\pi i} \frac{P(z)V(w)}{z-w}$, with an integration contour around w.

changing operator $P_3(z) = [Q_2, \phi_3(z)]$. Taking the normal-ordered product of P_3 with $V_0^0[\tilde{p}_3 = 30]$ and then acting with S gives the physical state $V_0^0[\tilde{p}_3 = 24]$. Generalizing this, we can write down infinite series of operators by analogy with the W_3 case [86, 116]. If we define $V(0,0) = V_0^0[\tilde{p}_3 = 30]$, the series with $(h,w) = (0,0)$ may be written as

$$\bar{V}(0,n) = SP_3V(0,n)\,, \quad V(0,n) = (S)^4P_3\bar{V}(0,n-1)\,. \tag{4.60}$$

The other series are

$$\begin{aligned}
& V_-(\tfrac{1}{15},0) \equiv V_0^0[\tilde{p}_3 = 28]\,, && (4.61)\\
\bar{V}_-(\tfrac{1}{15},n) \;=\;& (S)^2P_3V_-(\tfrac{1}{15},n)\,, \quad V_-(\tfrac{1}{15},n) = (S)^3P_3\bar{V}_-(\tfrac{1}{15},n-1)\,; &&\\
& V_+(\tfrac{1}{15},0) \equiv V_0^0[\tilde{p}_3 = 26]\,, && (4.62)\\
\bar{V}_+(\tfrac{1}{15},n) \;=\;& (S)^3P_3V_+(\tfrac{1}{15},n)\,, \quad V_+(\tfrac{1}{15},n) = (S)^2P_3\bar{V}_+(\tfrac{1}{15},n-1)\,; &&\\
& V(\tfrac{2}{5},0) \equiv V_1^{-1}[\tilde{p}_3 = 18]\,, && (4.63)\\
\bar{V}(\tfrac{2}{5},n) \;=\;& (S)^2P_3V(\tfrac{2}{5},n)\,, \quad V(\tfrac{2}{5},n) = (S)^3P_3\bar{V}(\tfrac{2}{5},n-1)\,; &&\\
& V_-(\tfrac{2}{3},0) \equiv V_1^{-1}[\tilde{p}_3 = 20]\,, && (4.64)\\
\bar{V}_-(\tfrac{2}{3},n) \;=\;& SP_3V_-(\tfrac{2}{3},n)\,, \quad V_-(\tfrac{2}{3},n) = (S)^4P_3\bar{V}_-(\tfrac{2}{3},n-1)\,; &&\\
& V_+(\tfrac{2}{3},0) \equiv V_1^{1}[\tilde{p}_3 = 34]\,, && (4.65)\\
\bar{V}_+(\tfrac{2}{3},n) \;=\;& (S)^4P_3V_+(\tfrac{2}{3},n)\,, \quad V_+(\tfrac{2}{3},n) = SP_3\bar{V}_+(\tfrac{2}{3},n-1)\,. &&
\end{aligned}$$

The notation, not to be confused with the previous notation V_l^G with level and ghost number indices, is $V_\pm(h,n)$, where h is the spin and $\pm$ indicates the sign of the spin-three weight w (see table 1). Although the actions of the screening operator S in general do not seem to have inverses, one can act on any operator in (4.60-4.65) with R, thus extending the series to negative n,

$$\begin{aligned}
V_\pm(h,n-1) &= RP_3V_\pm(h,n)\,,\\
\bar{V}_\pm(h,n-1) &= RP_3\bar{V}_\pm(h,n)\,.
\end{aligned} \tag{4.66}$$

We observe that the operator $V(0,1)$ is the identity, and $\bar{V}(0,1)$ is another $h = 0$ operator at the same ghost number $G = -3$ relative to the tachyonic operators (4.49). To summarize, we list all these operators with their momentum, ghost number and level in table 2.

We did not prove that all these operators are BRST non-trivial (they are certainly BRST closed). The operators in table 2 are supposed to be the prime operators [165] from which new physical operators are obtained by normal-ordering with the picture changing operator P_3. Thus all operators in the Q_2 cohomology come in doublets.

operator	$\tilde{p}_3$	G	level
$V(0,n)$	$30-30n$	$-3n$	$\frac{1}{2}(3n(5n-1))$
$\bar{V}(0,n)$	$24-30n$	$-3n$	$\frac{1}{2}(3n(5n+1))$
$V_-(\frac{1}{15},n)$	$28-30n$	$-3n$	$\frac{1}{2}(n(15n-1))$
$\bar{V}_-(\frac{1}{15},n)$	$16-30n$	$-1-3n$	$\frac{1}{2}(15n^2+11n+2)$
$V_+(\frac{1}{15},n)$	$26-30n$	$-3n$	$\frac{1}{2}(n(15n+1))$
$\bar{V}_+(\frac{1}{15},n)$	$8-30n$	$-2-3n$	$\frac{1}{2}(15n^2+19n+6)$
$V(\frac{2}{5},n)$	$18-30n$	$-1-3n$	$\frac{1}{2}(15n^2+9n+2)$
$\bar{V}(\frac{2}{5},n)$	$6-30n$	$-2-3n$	$\frac{1}{2}(15n^2+21n+8)$
$V_-(\frac{2}{3},n)$	$20-30n$	$-1-3n$	$\frac{1}{2}(15n^2+7n+2)$
$\bar{V}_-(\frac{2}{3},n)$	$14-30n$	$-1-3n$	$\frac{1}{2}(15n^2+13n+4)$
$V_+(\frac{2}{3},n)$	$34-30n$	$1-3n$	$\frac{1}{2}(15n^2-7n+2)$
$\bar{V}_+(\frac{2}{3},n)$	$10-30n$	$-2-3n$	$\frac{1}{2}(15n^2+17n+6)$

Table 2. *Operators in the Q_2 cohomology.*

From equations (4.60-4.65) one observes that the action of five S screening charges (together with two picture changes) is special. It lowers $\tilde{p}_3$ by 30 and G by 3. Indeed, a screening operator exists which does the same in one go (together with one picture change), namely

$$S_x = \partial^3 b_4 \partial^2 b_4 \partial b_4 b_4 e^{ip_3\phi_3}\,, \quad \text{with } \tilde{p}_3 = -30\,. \tag{4.67}$$

This screening operator is also used in [141]. It has a well-defined action on all physical operators. The physical operator x associated with S_x through the descent equation turns out to be (up to an irrelevant constant factor)

$$x(z) = \oint \frac{dw}{2\pi i} S_x(w) P_3(z)\,, \tag{4.68}$$

which we identify as $V(0,2)$ in (4.60). We will not give x explicitly; it is a complicated expression consisting of 50 terms. We may recover S_x from x via

$$S_x(w) = (b_4)_{-1}(w)x(w) \equiv \oint \frac{dz}{2\pi i}(z-w)^2 b_4(z)x(w)\,. \tag{4.69}$$

The operator x has a physical inverse x^{-1}, such that the normal-ordered product of x with x^{-1} is a nonvanishing multiple of the identity. This inverse is precisely the physical level 0 operator $V_0^0[\tilde{p}_3 = 30]$. We may write it as

$$x^{-1} = \oint \frac{dw}{2\pi i} R(w) P_3(z)\,. \tag{4.70}$$

Thus we have identified three members of the family (4.60): $V(0,0) = x^{-1}$, $V(0,1) = \mathbf{1}$ and $V(0,2) = x$. We also see that RP_3 acts as the inverse of S_xP_3.

In [140], analogous invertible operators enabled the computation of the complete cohomology of the critical W_3 string by the observation that normal-ordered products of arbitrary powers of x or x^{-1} with a physical operator give new non-trivial physical operators. The situation here is somewhat different in that the operators in the Q_2 cohomology do not all have vanishing total conformal weight, due to the lack of a Virasoro constraint. As this is an essential argument used in [140], we have no complete proof here that the operators of table 2 plus their W_3 descendants and picture changed versions generate the full Q_2 cohomology.

Equation (4.69) seems to give the general procedure to obtain the screening current associated to a $h = 0$ physical operator in the Q_2 cohomology. For $S_\mathcal{O} \equiv (b_4)_{-1}\mathcal{O}$, with $\mathcal{O}$ an arbitrary $h = 0$ physical operator, one has

$$[Q_2, S_\mathcal{O}(w)] = V_{-1}(w)\mathcal{O}(w) = \oint \frac{dz}{2\pi i}(z-w)^2 V(z)\mathcal{O}(w)\,, \tag{4.71}$$

where, as before, V is the spin-four current $\{Q_2, b_4\}$, and the RHS is indeed a (multiple of) $\partial\mathcal{O}$ in the cases examined.

4.3.2 The Q_1 cohomology

We now take the next nilpotent BRST operator, Q_1, and study its cohomology. It is the part of the total W_4 BRST current (4.41) which does not involve the Virasoro sector. It imposes only a spin-three and a spin-four constraint. The Fock space must now be extended to include also the scalar ϕ_2 and the spin-three ghosts. Together, the fields $(\phi_2, \phi_3; c_3, b_3; c_4, b_4)$ have central charge $c = \frac{7}{10}$ which is the central charge of the $(p, p') = (4, 5)$ unitary Virasoro minimal model.

Operators in the Q_1 cohomology can be computed from operators in the Q_2 cohomology in a systematic way using a spectral sequence argument (for a review, see e.g [45]). Taking the spin-three ghost number G_3 as an extra grading on the complex of scalar plus ghost Fock spaces, one can decompose Q_1 in three parts with $G_3 = 0, 1$ and 2:

$$Q_1 = d_0 + d_1 + d_2\,, \tag{4.72}$$

where the $G_3 = 0$ part, d_0, is Q_2. There is only one term in Q_1 with $G_3 = 2$, namely $d_2 = -\frac{243}{64} c_3 \partial c_3 b_4$. This term prevents the complex from being a double complex. The remaining terms have $G_3 = 1$ and form d_1. The first term of the spectral sequence $(E_r, \delta_r)_{r=0}^{\infty}$ associated to this gradation, is the Q_2 cohomology

$$\begin{aligned} E_1 &= H(Q_2, \mathcal{F}(\phi_2, \phi_3; c_3, b_3; c_4, b_4)) \\ &= \mathcal{F}(\phi_2; c_3, b_3) \otimes H(Q_2, \mathcal{F}(\phi_3; c_4, b_4))\,, \end{aligned} \tag{4.73}$$

where the second equality follows from the fact that Q_2 acts trivially on any of the fields $(\phi_2; c_3, b_3)$. So we can start with a Q_2 physical operator and extend it (if possible) to a Q_1 physical operator by computing the next terms in the spectral sequence. The successive terms that are added to the original Q_2 physical operator have increasing spin-three ghost number G_3 (but of course the same total ghost number). At low levels the spectral sequence will collapse after a few terms due to the small range of ghost numbers available there, but at higher levels the procedure becomes increasingly laborious.

Having said this, we found it just as convenient to compute operators in the Q_1 cohomology by imposing the complete Q_1 physical condition at once. We used the Mathematica package OPEdefs [185] for computing OPEs. Still, it is useful to observe from the above-mentioned arguments that the Q_1 physical operators are extensions of Q_2 physical operators, so that only the ϕ_2 momentum and the spin-three ghost structure remain to be determined from the Q_1 physical condition.

The level 0 operators are now of the form

$$W_0^0 = \partial c_3 c_3 \partial^2 c_4 \partial c_4 c_4 e^{ip_2\phi_2 + ip_3\phi_3} . \tag{4.74}$$

The notation is the same as in (4.49) except that we use W for operators in the Q_1 cohomology. Level 1 operators that create states with the lowest ghost number $G = -1$ can now be linear combinations of two terms of different ghost structure:

$$W_1^{-1} = (x_1 c_3 \partial^2 c_4 \partial c_4 c_4 + x_2 \partial c_3 c_3 \partial c_4 c_4) e^{ip_2\phi_2 + ip_3\phi_3} . \tag{4.75}$$

Table 3 lists the momenta for which the level 0 and level 1 operators are physical.

W_0^0	$\tilde{p}_3$	$\tilde{p}_2$	h
	24	24	0
		27	3/80
		30	0
	26	22	0
		28	1/10
		31	3/80
	28	23	3/80
		26	1/10
		32	0
	30	24	0
		27	3/80
		30	0

W_1^{-1}	$\tilde{p}_3$	$\tilde{p}_2$	h
	16	23	3/80
		26	1/10
		32	0
	18	18	1/10
		27	7/16
		36	1/10
	20	19	7/16
		22	3/5
		40	0
	24	12	1/10
		15	7/16
	26	16	3/5
	28	11	3/80
	30	12	1/10
		15	7/16

Table 3. *Level 0 and 1 operators in the Q_1 cohomology.*

All cohomology classes are one-dimensional. The level 1 operators with $\tilde{p}_3$-values 16, 18 and 20 have $x_1 = 0$ in (4.75) while the other level 1 operators have a nonzero ratio $\frac{x_1}{x_2}$.

The last column in table 3 shows the total conformal weight of the physical operators with respect to the $c = \frac{7}{10}$ energy-momentum tensor, which is itself a physical operator at level 11. Thus the physical states are organized into $c = \frac{7}{10}$ Virasoro representations. Unitarity then requires that all primary physical operators have conformal dimensions of the corresponding Kac table. This appears to be the case. In particular, the level 0 physical operators correspond to the diagonal entries of the Kac table. The multiplicities of operators of fixed weight can be understood from Weyl group transformations [136] (cf. (4.28)). The presence of non-diagonal operators at level 0 is impossible because of the maximum conformal weight

$$h_{max}(l) = \tfrac{1}{2}(q_2^2 + q_3^2) + l - 9 = \tfrac{9}{80} + l\,. \tag{4.76}$$

At level 1, primary operators corresponding to the first off-diagonal in the Kac table, with $h = \frac{7}{16}$ and $h = \frac{3}{5}$, are allowed by (4.76), and they are indeed in the Q_1 cohomology as can be seen from table 3. Also observe that there is no physical operator at levels 0 and 1 corresponding to the outermost entry in the Kac table, $h = \frac{3}{2}$. From (4.76) it is clear that such an operator can exist only at levels $l \geq 2$. So it is natural to look for this missing operator at level 2. At this level, the ghost number can take values $-2 \leq G \leq 4$. To see if there is a $h = \frac{3}{2}$ physical state, it suffices to consider only $G \leq 0$, since for $G \geq 1$ the spectrum consists of conjugates of $G \leq 1$ states with the same conformal weight. The lowest ghost number operator at level 2 has the form $W_2^{-2} = c_3 \partial c_4 c_4 e^{ip_2\phi_2 + ip_3\phi_3}$ and is physical for two values of the momenta (p_2, p_3) giving rise to two $h = \frac{3}{80}$ operators. For $G = -1$ there is no $h = \frac{3}{2}$ cohomology either. However, for $G = 0$ there is a one-dimensional $h = \frac{3}{2}$ cohomology class, with momentum $(\tilde{p}_2, \tilde{p}_3) = (34, 20)$. It may be represented by $\partial^3 c_3 \partial c_3 c_3 \partial c_4 c_4 e^{ip_2\phi_2 + ip_3\phi_3}$ which is primary up to Q_1 exact terms. Also, there is a cohomology class with the conjugate momentum[10] (and thus also $h = \frac{3}{2}$), $(\tilde{p}_2, \tilde{p}_3) = (20, 34)$. One can understand the appearance of states at $G = 0$ in pairs with conjugate momenta as follows. First note that states in the Q_1 cohomology occur in quartets with ghost numbers $(G, G+1, G+1, G+2)$, where the state at lowest ghost number is called the prime state [165], and the other states are obtained by applying the picture changing operators to this prime state (remember that we have two independent picture changing operators in the Q_1 cohomology description). Besides, any state at ghost number G has a conjugate state at ghost number $2 - G$ with the conjugate momentum. Combining these observations, we see that prime states at $G = 0$ occur in pairs with conjugate momenta.

We have now identified all operators of the $c = \frac{7}{10}$ minimal model in the Q_1 cohomology at levels 0,1 and 2. The next objective is to show that all physical operators (at least the ones found so far) of the same conformal weight are related to each other through the action of screening operators and picture changes. Therefore, let us introduce a number of useful screening charges, which are now required to commute with Q_1. First of all, the operator S in equation (4.56) is still a screening current in the Q_1 cohomology, as is

[10] We recall that the momentum conjugate to $\vec{p}$ is $-\vec{p} + 2i\vec{q}$, where $\vec{q}$ is the background charge vector. In the tilded variables it means that $54 - \tilde{p}_i$ is conjugate to $\tilde{p}_i$.

S_x. The Q_2 screening operators R and $\bar{R}$ commute with Q_1 only after adding an extra term,

$$R = (\partial c_4 c_4 + \tfrac{15}{88} q_2 c_3 c_4) e^{ip_3\phi_3}\,, \quad \tilde{p}_3 = 30\,, \tag{4.77}$$

$$\bar{R} = (\partial c_4 c_4 + \tfrac{15}{56} q_2 c_3 c_4) e^{ip_3\phi_3}\,, \quad \tilde{p}_3 = 24\,. \tag{4.78}$$

New screening currents are given by (the notation will become clear in the next subsection)

$$T_3^- = (1 - \tfrac{256}{729} q_2 b_3 c_4) e^{i\vec{p}\cdot\vec{\phi}}\,, \quad \tilde{\vec{p}} \equiv (\tilde{p}_2, \tilde{p}_3) = (-8, 8)\,, \tag{4.79}$$

$$T_3^+ = (1 - \tfrac{32}{729} q_2 b_3 c_4) e^{i\vec{p}\cdot\vec{\phi}}\,, \quad \tilde{\vec{p}} = (-10, 10)\,. \tag{4.80}$$

Screening currents with positive $\tilde{p}_2$-values are

$$R' = c_3 e^{ip_2\phi_2}\,, \quad \tilde{p}_2 = 30\,, \tag{4.81}$$

$$\bar{R}' = c_3 e^{ip_2\phi_2}\,, \quad \tilde{p}_2 = 24\,. \tag{4.82}$$

Of course, many more screening currents at higher or lower ghost numbers exist, but we expect that they can be represented by composite actions of the given ones, together with P_2 and/or P_3 picture changes.

The $h = 0$ physical operators are obtained through the action of the associated screening currents on a picture changed version of the identity operator, so they can all be viewed as different screened versions of the identity. Also, operators of table 3 with the same conformal weight can be connected to each other more directly by the action of certain combinations of the screening charges given above. It is more important, however, to find operators which can normal-order with any physical operator and thereby create new physical operators. The operator x that was found in the previous section is easily extended to the Q_1 cohomology, since the associated screening current S_x is still given by (4.67). Again, it can also be expressed as

$$x(z) = \oint \frac{dw}{2\pi i} S_x(w) P_3(z)\,, \tag{4.83}$$

where now $P_3 = [Q_1, \phi_3]$ contains some additional terms compared with the P_3 operator in the Q_2 discussion. Its inverse now is the level 3 physical operator

$$\begin{aligned} x^{-1} = {} & (\partial^2 c_4 \partial c_4 c_4 + \tfrac{45}{56} \partial\phi_2 c_3 \partial c_4 c_4 - \tfrac{45\sqrt{2}}{56} \partial\phi_3 c_3 \partial c_4 c_4 - \tfrac{5}{56} q_2 c_3 \partial^2 c_4 c_4 \\ & + \tfrac{5}{28} q_2 \partial c_3 \partial c_4 c_4 + \tfrac{3645}{19712} \partial c_3 c_3 c_4) e^{ip_3\phi_3}\,, \end{aligned} \tag{4.84}$$

with $\tilde{p}_3 = 30$. This is the operator corresponding to the screening current (4.77) via the descent equation, and it also equals the commutator of R with P_3. Conversely, we get back R as $(b_3)_{-1} x^{-1}$.

There should also be similar operators y and y^{-1} with nonzero ϕ_2 momentum. We expect y to be a level 37 physical operator with momentum $(\tilde{p}_2, \tilde{p}_3) = (-40, -20)$. Such an operator can normal-order with any physical operator. This can be easily checked using (4.59), and by noting that $\tilde{p}_2 + \tilde{p}_3$ is a multiple of 3 for all physical operators (this also follows from (4.46)). We did not try to construct the operator y. However, y^{-1} should be the level 1 physical operator with $(\tilde{p}_2, \tilde{p}_3) = (40, 20)$ of table 3.

The overall picture of the Q_1 cohomology is then the following. Physical operators come in minimal model modules of the $c = \frac{7}{10}$ Virasoro algebra realized in terms of the scalar fields (ϕ_2, ϕ_3) and the ghost fields $(c_3, b_3; c_4, b_4)$. There seems to be an infinite number of representatives of each minimal model primary (but only a finite number at fixed ghost number). We expect that all primaries belonging to the Q_1 cohomology can be written as normal-ordered products of powers of the operators x, y and their inverses acting on a set of physical operators at some low-lying levels. Since the ghost numbers and momenta of the operators x, y, x^{-1}, y^{-1} are known, we can predict the ghost numbers, level numbers, and momenta of Q_1 cohomology classes as in table 2 for the Q_2 cohomology.

4.3.3 The complete cohomology

We now consider the cohomology of $Q = Q_1 + Q_{Vir}$ on the full Fock space generated by the three scalar fields and the three conjugate ghost pairs. Because of the Virasoro constraint, the Q cohomology contains only $h = 0$ primaries. This is different from the Q_2 and Q_1 cohomologies which also contain descendants of minimal model primaries.

Since Q_1 and Q_{Vir} anticommute, see (4.42), they define a double complex. Note that Q_1 does not involve the fields $(\phi_1; c_2, b_2)$, and since Q_1 physical operators have already been computed, we take a spectral sequence where the first term is the Q_1 cohomology. This spectral sequence provides a systematic procedure to obtain operators in the Q cohomology by adding to Q_1 physical operators terms with higher spin-two ghost number G_2 (but the same total ghost number). Physical operators in the total cohomology are then given by

$$\mathcal{O} = \sum_{i=k}^{\infty} \mathcal{O}_i \,, \tag{4.85}$$

where the first term in the sum, $\mathcal{O}_k$ with $G_2 = k$, is an operator in the Q_1 cohomology, and the higher G_2 terms are defined by $[Q_1, \mathcal{O}_{i+1}] = -[Q_{Vir}, \mathcal{O}_i]$. At small values of the level, the sum in (4.85) will only have a few terms (e.g. only one term at level 0 and $G = 0$).

The 'tachyon operators' (level 0, $G = 0$) take the form

$$X_0^0 = c_2 \partial c_3 c_3 \partial^2 c_4 \partial c_4 c_4 e^{ip_1\phi_1 + ip_2\phi_2 + ip_3\phi_3} \,. \tag{4.86}$$

They are physical for 24 values of the momenta which form a multiplet of the $sl(4)$ Weyl group [136]. For the explicit values of the physical momenta, we refer to [40]. The 24 physical operators X_0^0 correspond to the 12 Q_1 operators W_0^0 'dressed up' with the c_2 ghost and the ϕ_1 part of the exponential, to operators of vanishing total conformal dimension (thus giving two possible p_1-values for each Q_1 operator). The level 0 operators of the W_4 string were already constructed by Das, Dhar and Rama [65] in

1992. The W_4 BRST operator was not known at that time, and their computation was based on an assumption about the existence of the 'cosmological constant operator'.

At level 1 and lowest ghost number $G = -1$, we can form the operators

$$X_1^{-1} = (x_1 \partial c_3 c_3 \partial^2 c_4 \partial c_4 c_4 + x_2 c_2 c_3 \partial^2 c_4 \partial c_4 c_4 + x_3 c_2 \partial c_3 c_3 \partial c_4 c_4) e^{i\vec{p}\cdot\vec{\phi}} . \qquad (4.87)$$

All momenta at which such operators become physical have been listed in [40]. The set of physical level 1 operators can be divided in continuous and discrete momentum operators. The continuous momentum operators correspond to Q_1 operators W_1^{-1} that have been dressed to operators of the total cohomology. They have $x1 = 0$, thus their spin-two ghost structure is standard. The discrete momentum operators have nonzero x_1, x_2 and x_3, thus their spin-two ghost structure is non-standard. The latter operators have $p_1 = 0$. This is all in agreement with (or rather analogous to) the observations made for the W_3 string in equations (4.33) and (4.34).

Next we compute some screening charges, which commute with Q. First, we note that all Q_1 screening currents are Q screening currents as well. So S_x is still a screening current, and its associated physical operator is still given by the relation (4.83), where now $P_3 = [Q, \phi_3]$ contains two additional terms relative to $[Q_1, \phi_3]$. The physical operator x^{-1} can be found using the spectral sequence argument described at the beginning of this subsection. We find that it is given by (4.84) with the following modification:

$$x^{-1} \;\rightarrow\; x^{-1} - (\tfrac{15}{28} c_2 \partial c_4 c_4 + \tfrac{225}{2464} q_2 c_2 c_3 c_4) e^{ip_3\phi_3} , \quad \tilde{p}_3 = 30 . \qquad (4.88)$$

In the Q cohomology, this is a level 4 operator. Similar physical operators y, y^{-1}, z, z^{-1} are also expected to exist, where y is supposed to have momentum $(0, -40, -20)$ and z should have nonzero ϕ_1 momentum in order to connect states with different p_1-values.

We find four new screening currents involving b_2 and ϕ_1,

$$\begin{aligned}
T_2^- &= (1 + \tfrac{2}{3} q_2 b_2 c_3) e^{i\vec{p}\cdot\vec{\phi}} , & \tilde{\vec{p}} &\equiv (\tilde{p}_1, \tilde{p}_2, \tilde{p}_3) = (-12, 12, 0) , \\
T_2^+ &= (1 + \tfrac{5}{6} q_2 b_2 c_3) e^{i\vec{p}\cdot\vec{\phi}} , & \tilde{\vec{p}} &= (-15, 15, 0) , \\
T_1^- &= e^{ip_1\phi_1} , & \tilde{p}_1 &= 24 , \\
T_1^+ &= e^{ip_1\phi_1} , & \tilde{p}_1 &= 30 .
\end{aligned} \qquad (4.89)$$

The operators $T_i^\pm$, $i = 1, 2, 3$, (recall the Q_1 screening currents in (4.79) and (4.80)) have exactly the same momenta as the standard screening currents $S_i^\pm$ given in (4.43). In fact, $T_1^\pm = S_1^\pm$. The other screening currents have been modified by a ghost contribution. This is a consequence of the redefinition that we carried out to obtain the W_4 BRST charge (4.41), since in this redefinition the scalar fields and ghosts are mixed to some extent [136, 20]. In [65] it was noted that the tachyonic physical operators are precisely the composites that can be formed out of the screening currents $S_i^\pm$.

Now that we have included the Virasoro constraint, it is trivial to obtain screening currents associated to physical operators, since the descent equation (4.44) is solved by $S_{\mathcal{O}}(w) = \oint \frac{dz}{2\pi i} b_2(z) \mathcal{O}(w)$.

In [46] a classification of physical states for a $W[g]$ minimal model coupled to $W[g]$ gravity is given. These results have already been seen to agree with those of [140] in the case of the two-scalar W_3 string (or pure W_3 gravity), see [46, 18].

If we take $g = sl(4)$ and the trivial ($c = 0$) W_4 minimal model, we are able to compare with our results. Non-trivial cohomology classes exist, according to [46], at the following values of the momentum[11]:

$$\vec{p} = w^{-1}(\alpha_-\sigma\vec{\rho} - \alpha_+\vec{\rho}) + i\sqrt{2}\alpha_0\vec{\rho}\,, \tag{4.90}$$

where $\vec{\rho}$ and the parameters $\alpha_0, \alpha_\pm$ are as before (see (4.25)), and w is an element of the $sl(4)$ Weyl group $\mathcal{W}$ while σ can be an element of the $\widehat{sl(4)}$ affine Weyl group $\widehat{\mathcal{W}}$. The ghost number at which the state with momentum (4.90) occurs is given by $-l_w(\sigma)$, where $l_w(\sigma)$ is the twisted length of σ,

$$l_w(\sigma) = \lim_{N\to\infty}\left(l(t_{-Nw\rho}\sigma) - l(t_{-Nw\rho})\right), \tag{4.91}$$

and l is the ordinary length of an affine Weyl group element. In order to compute the twisted length, one should decompose the translation $t_{-Nw\rho}$ into the simple affine Weyl reflections $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ and then look for the cancellations that take place between $t_{-Nw\rho}$ and σ. For $\sigma \in \mathcal{W}$, (4.91) reduces to $l_w(\sigma) = l(w^{-1}\sigma) - l(w^{-1})$. The action of σ_0 on ρ should be taken here as $\sigma_0\rho = \sigma_\theta\rho + 5\theta$, where θ is the highest root of $sl(4)$. For completeness we give the decompositions of the translations associated with the simple roots:

$$\begin{aligned} t_{e_1} &= \sigma_2\sigma_3\sigma_0\sigma_3\sigma_2\sigma_1\,, \\ t_{e_2} &= \sigma_3\sigma_1\sigma_0\sigma_1\sigma_3\sigma_2\,, \\ t_{e_3} &= \sigma_2\sigma_1\sigma_0\sigma_1\sigma_2\sigma_3\,. \end{aligned} \tag{4.92}$$

For $\sigma = \mathbf{1}$, (4.90) yields all level 0 physical states corresponding to X_0^0, when w runs over the 24 elements of $\mathcal{W}$. Whereas the Weyl group action in (4.90) does not change the level, the affine Weyl group action does. If we let σ run over the simple Weyl reflections $\{\sigma_1, \sigma_2, \sigma_3\}$ and w over all elements in $\mathcal{W}$, we obtain all momenta and ghost numbers of the level 1 prime physical states of which the ones with $G = -1$ agree with our findings for the operators X_1^{-1}. Thus we find complete agreement with the results of [46] at levels 0 and 1.

The affine Weyl elements can be decomposed into ordinary Weyl transformations and translations in the co-root lattice, $\sigma = t_\beta w$. The translations associated with the simple roots (4.92) correspond to the following changes in the momenta,

$$\begin{aligned} \Delta_1\tilde{\vec{p}} &= (120, 0, 0)\,, \\ \Delta_2\tilde{\vec{p}} &= (-60, 60, 0)\,, \\ \Delta_3\tilde{\vec{p}} &= (0, -40, 40)\,. \end{aligned} \tag{4.93}$$

[11]It can be checked that they are compatible with the selection rule (4.46).

Physical operators with such momenta are supposed to be invertible and they can be used to classify the complete cohomology in terms of a set of low level physical operators. Recall that the operator x with momentum $\tilde{\vec{p}} = (0, 0, -30)$ can also be used for this purpose. Using x and the operators corresponding to the simple root translations, one finds that an alternative basis of x-like operators has momenta $(0, 0, -30)$, $(0, -40, -20)$ and $(-60, -20, -10)$.

As before, we only considered prime operators. Seven other sectors of operators can be obtained by acting with the picture changing operators P_1, P_2 and P_3.

The energy-momentum tensor for ϕ_1 can be replaced by an arbitrary effective energy-momentum tensor T_{eff} with the same central charge. Some three-scalar states will then generalize to continuous momentum multi-scalar states, some will generalize to discrete momentum multi-scalar states and some may not be generalized at all to multi-scalar states. An effective space-time exponential may be replaced by any effective space-time operator with the same OPE under T_{eff} to obtain other physical operators. See [140] for a discussion of this multi-scalar generalization in the case of W_3.

It is expected [20] that the W_N BRST charge can be decomposed in a way similar to the W_4 BRST charge. This is certainly true at the classical level. The studies of W_3 and W_4 strings make the following picture of minimal models in the W_N string very plausible. Imposing the spin-N constraint results in an $(N, N+1)$ unitary W_{N-1} minimal model. In the next step, where the spin-$(N-1)$ constraint is added, the operators are dressed to operators of the $(N, N+1)$ W_{N-2} minimal model. This goes on in the same way, resulting in an $(N, N+1)$ Virasoro minimal model in the Q_1 cohomology, and the total cohomology is obtained from the double complex with BRST charges Q_1 and Q_{Vir}. This agrees with the counting of central charges as discussed earlier. A similar discussion for *non-critical* W_N strings may be found in [18].

Some results on higher-spin strings based on $W_{2,N}$ algebras have been obtained in [139, 138, 141, 142]. A complication noted by the authors of these papers is that for $N \geq 5$, the central charge of the spin-N sector, corresponding to a W_{N-1} minimal model, becomes greater or equal to one, as can be seen in equation (4.17). Consequently, the number of effective space-time sectors which couple to the spin-N fields is no longer finite. This complication is not expected to occur for the W_N string, since there is a sequence of W_k minimal models, the last one being the $(N, N+1)$ Virasoro minimal model which of course has $c < 1$ so that there is only a finite number of effective space-time intercepts.

4.4 Relations between strings based on different gauge symmetries

In the previous sections we discussed some relations between (non-critical) W_N strings for different N. It was argued that in the cohomology of the (p, q) non-critical W_N string all (p, q) non-critical W_n strings with $2 \leq n < N$ naturally appear. This is not to say that they all are equivalent. In fact, the minimal model operators appearing in the spectra of W_N strings are repeated at an infinite number of different ghost numbers. In principle, this degeneracy could be lifted by using certain screening operators with nonzero ghost number to identify all the copies of a particular minimal model operator. However, it is not natural to do so. See [18] for a discussion.

In this section we briefly describe further relations between string theories based on different world-sheet gauge symmetries. In [29] it was shown that the bosonic string may be viewed as a special background for the $N = 1$ superstring, and that the $N = 1$ superstring may be viewed as a special background for the $N = 2$ superstring.

Let us review some of the arguments for the $N = 0 \subset N = 1$ case. Starting from a critical bosonic string with a $c_{bos} = 26$ energy-momentum tensor T_{bos}, one obtains a realization of the $N = 1$ superconformal algebra by introducing fermionic (i.e. anticommuting) fields (b_1, c_1) of spin $(3/2, -1/2)$ and defining [29]

$$\begin{aligned} T &= T_{bos} - \tfrac{3}{2} b_1 \partial c_1 - \tfrac{1}{2} \partial b_1 c_1 + \tfrac{1}{2} \partial^2 (c_1 \partial c_1) \, , \\ G &= b_1 + c_1 \left(T_{bos} + \partial c_1 b_1 \right) + \tfrac{5}{2} \partial^2 c_1 \, . \end{aligned} \tag{4.94}$$

The fields (b_1, c_1) can be viewed as twisted versions of spin-two ghost fields (b, c). Then the second term in G has the structure of the BRST current of the bosonic string. Note that this particular realization of the $N = 1$ superconformal algebra acts nonlinearly due to the first term in G.

The currents T and G generate the $N = 1$ superconformal algebra with critical central charge $c = 15$. Thus the system $\{T_{bos}, (b_1, c_1)\}$ can serve as a background for the $N = 1$ fermionic string. To impose the constraints T and G, one needs to introduce the usual anticommuting ghosts (b, c) and the commuting ghosts (β, γ), respectively. It is then shown in [29] that correlation functions of the fermionic string in this specific background reduce to corresponding bosonic string correlation functions essentially because the path integral over the (β, γ) fields cancels that over the (b_1, c_1) fields.

The arguments of [29] use the assumption that all physical operators of the bosonic string with energy-momentum tensor T_{bos} are of the standard form. However, the equivalence of the class of $N = 1$ strings based on the realizations (4.94) to the bosonic string based on T_{bos} holds also in the case where other operators of non-standard ghost structure are present, as in non-critical string theories. This follows from the work of Ishikawa and Kato [121] in which a similarity transformation is used to prove that the cohomology of the $N = 1$ BRST operator is isomorphic to the cohomology of the bosonic string BRST operator. Indeed, for a certain generating function R [121],

$$e^R Q_{N=1} e^{-R} = Q_{bos} + Q_{top} \, , \tag{4.95}$$

where

$$Q_{top} = \oint \frac{dz}{2\pi i} \left(-\tfrac{1}{2} b_1 \gamma \right) , \tag{4.96}$$

and Q_{bos} is the standard bosonic string BRST operator with energy-momentum tensor T_{bos}. The cohomology of $Q_{N=1}$ is simply the direct product of the cohomology of Q_{bos} and Q_{top}. From the form of Q_{top} and the fact that (b_1, c_1) and (β, γ) are conjugate pairs, it immediately follows that the cohomology of Q_{top} consists of only the ground state in the $\{(b_1, c_1), (\beta, \gamma)\}$ system. Decoupling of all excitations in these fields is due to the quartet mechanism of Kugo and Ojima. Moreover, (4.95) is a similarity transformation and therefore preserves the form of all OPEs. As a result, any correlation function in the $N = 1$ theory reduces to the corresponding one in the $N = 0$ theory[12].

Further embeddings of strings into string models with larger world-sheet gauge symmetry have since been found. The embedding described above has been generalized to a hierarchy of superstrings [14] where N-extended superstrings may be viewed as a special class of $(N+1)$-extended superstrings. In [28], the embeddings are generalized to non-critical superstrings and in [132] a hierarchy of w-strings[13] is obtained.

A very interesting idea behind all these embeddings is that there might exist a universal string theory which includes all the others by special choices of vacua. This universal string theory would then be the most symmetrical one, and less symmetric string theories arise by spontaneous symmetry breaking, i.e. by the choice of a certain vacuum. However, all realizations used in the embeddings are of a very special kind. In particular, the symmetries of the more symmetrical string theory are always nonlinearly realized, as in (4.94).

It is known that nonlinear realizations[14] for some symmetry algebra G may be induced from realizations of some smaller algebra H. In the case that G is a finite dimensional group and H a subgroup, the BRST charge for the nonlinear realization of G is related via a similarity transformation to the BRST charge of H [131, 146, 81]. Thus it seems that the existence of hierarchies of string theories with different world-sheet symmetries is purely a consequence of the fact that the algebras are nonlinearly realized. In other words, one can start from the bosonic string and add fields such that a nonlinear realization of some larger symmetry is obtained. Gauging the extra symmetry in effect eliminates the new degrees of freedom and therefore gives back the original theory. This mechanism of enlarging the symmetry is rather trivial and makes the significance of string embeddings unclear. See also the discussion in [155].

In the case of W_N strings we also described similarity transformations like (4.95). In particular, for the W_3 string a similarity transformation (or canonical transformation, it preserves OPEs) turns the BRST operator into a sum of two nilpotent terms $Q = Q_0 + Q_1$, as described above equation (4.29). There are some differences, however. One is that the W_3 symmetry is realized nonlinearly in another sense, namely higher than quadratically. Indeed, we know that Q_1 does not correspond to a topological sector of the theory, rather its cohomology is a realization of the Ising model. Another difference is that the critical value $c_{Vir} = 26$ does not lead, through the Miura realization, to the critical value $c_{W_3} = 100$ but rather to $c_{W_3} = 102$. The latter difference can be eliminated if we consider a non-critical W_3 string with a $c_L = -2$ Liouville sector and therefore a $c_M = 102$ matter sector [23]. Then this matter sector can be realized by the usual 26 free scalar fields of the bosonic string plus an additional scalar field. For

[12] However, there may be problems for a complete identification on higher genus surfaces where (super)moduli play a role.

[13] The w-strings considered in [132] are based on linear versions of the W_N algebras.

[14] In this section we mean by 'nonlinear' that there are transformations with terms of zeroth order in the fields. For example, G in (4.94) acts nonlinearly in this sense on c_1.

example, the matter energy-momentum tensor is given by

$$T_M = -\tfrac{1}{2}\partial X_\mu \partial X^\mu - \tfrac{1}{2}\partial\phi\partial\phi + \tfrac{5}{2}\partial^2\phi\,. \tag{4.97}$$

After performing our usual redefinition in the matter sector, the BRST current can be cast into the form $j = j_0 + j_1$ with

$$\begin{aligned} j_0 &= c_2\left(T_M + T_L + T_{c_3,b_3} + \tfrac{1}{2}T_{c_2,b_2}\right)\,, \\ j_1 &= c_3 F(T_L, W_L; \phi; c_3, b_3)\,. \end{aligned} \tag{4.98}$$

The explicit expression for F is given in [21, 23]; it is not important here. From the form of the BRST operator $Q = Q_0 + Q_1$ we can already see that this non-critical W_3 string contains the complete critical bosonic string spectrum in its cohomology. Indeed, we can rewrite $Q = Q_{Vir} + Q_R$, where Q_{Vir} is the standard BRST operator of the bosonic string. Since Q_R does not depend on X^μ and b_2, any bosonic string physical state of the form $c_2 V(X^\mu)$ is automatically Q invariant. However, the results of [18] that show how the Q_1 cohomology in the case that the Liouville sector is a (p,q) minimal model reduces to the (p,q) Virasoro minimal model, cannot be directly applied here since there does not seem to be a W_3 minimal model at $c_L = -2$. Although $c = -2$ corresponds to $(p,q) = (3,2)$ in the formula for minimal model central charges, a corresponding Kac table of minimal model primaries does not exist. One could still restrict the Liouville sector to its identity operator only[15], just to see to what model it leads in the Q_1 cohomology. It does not lead to the $(3,2)$ $(c = 0)$ trivial minimal Virasoro model. Instead, the Q_1 cohomology involves operators of dimensions $\{n, \frac{1}{8}+n, \frac{5}{8}+n\}$ for non-negative integers n. Applying naively the fusion rules of BPZ, it can be seen that dimension $0, \frac{1}{8}, \frac{5}{8}$ primaries are part of a closed fusion algebra. However, only the dimension 0 operators yield the bosonic string spectrum in the complete cohomology. It is clear that the relation described here between a special W_3 string realization and the bosonic string is by no means an equivalence. For more details we refer to [23].

Other attempts of embedding the bosonic string in a W-string, apart from the hierarchy of linearized w_N strings [132], have been described in [27, 13, 137, 143]. In [13], the critical bosonic string is realized as a particular background of a string based on the linearized W_3 algebra W_3^{lin} [130]. The nonlinear W_3 algebra obtained from this critical W_3^{lin} algebra by a redefinition has central charge 102, the same value as in the matter sector of the non-critical W_3 string discussed above. Related work in [137] shows that both the $c_M = 100$ critical and $c_M = 102$ non-critical BRST operators can be altered without losing nilpotence by adding an extra term to the Q_1 operator. They then become equivalent up to a similarity transformation. Moreover, with this extra term, which corresponds to a nonlinearly (in the sense of 0th order in the fields) realized W_3 symmetry, they also become equivalent to the bosonic string BRST operator plus a topological part. The latter part decouples a quartet of fields leaving precisely the critical bosonic string. The W_3 realizations used in [137] are obtained using the linearized approach of [130]. They involve a bosonic bc system and are, therefore, different from the usual Miura realizations.

[15] However, this is not a modular invariant restriction.

Chapter 5

Duality

In this chapter we study a different kind of symmetry in string theory which is known under the name duality symmetry. Duality transformations act on background space-time fields in which the string moves, whereas the gauge transformations discussed in chapters 2 and 3 act on the world-sheet fields. In section 5.1 we present the low-energy effective action for the background fields and discuss a simple compactification procedure that gives an effective action in lower dimensional space-time. We discuss global symmetries that appear upon reduction to lower dimensions. Discrete subgroups of these symmetry groups are believed or known to be symmetries of the full string theory, called dualities. In section 5.2 we describe, from the sigma model point of view, a subclass of possible dualities, the so-called target-space duality or T-duality. Section 5.3 is about strong/weak coupling dualities in string theory.

5.1 Introduction

We first continue our discussion of strings moving in background fields started in section 2.3. Let us repeat here the action of the nonlinear sigma model which describes the bosonic string moving in a background determined by a space-time metric, antisymmetric tensor and dilaton:

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{-h}\{g_{\mu\nu}(X)h^{ab}\partial_a X^\mu \partial_b X^\nu + B_{\mu\nu}(X)\varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu + \alpha' R^{(2)}\Phi(X)\}\,, \qquad (5.1)$$

This action does not turn into a free field action by going to the conformal gauge $h_{ab} = e^{\phi}\eta_{ab}$. Therefore, its quantization is not straightforward and one must resort to perturbation theory in the parameter α' which plays the role of the loop-counting parameter of the nonlinear sigma model viewed as a two-dimensional quantum field theory. As mentioned in section 2.3, demanding Weyl invariance of this quantum field theory results in a number of differential equations for the background fields. The Weyl anomaly is the trace of the energy-momentum tensor and is given in reference [54] (see also [84]),

$$< T^a{}_a >= \beta^\Phi \sqrt{-h} R^{(2)} + \beta^g_{\mu\nu} \sqrt{-h} h^{ab} \partial_a X^\mu \partial_b X^\nu + \beta^B_{\mu\nu} \sqrt{-h} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu \,, \quad (5.2)$$

where β^Φ , β^g and β^B are the beta-functionals associated to the coupling functions $\Phi(X)$, $g_{\mu\nu}(X)$ and $B_{\mu\nu}(X)$. They are given by

$$\begin{aligned} \beta^\Phi &\propto D - 26 + 3\alpha' \left(R + 4\nabla_\mu \Phi \nabla^\mu \Phi - 4\nabla^2 \Phi + \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) + O(\alpha'^2) \,, \\ \beta^g_{\mu\nu} &\propto R_{\mu\nu} - 2\nabla_\mu \nabla_\nu \Phi + \tfrac{9}{4} H_{\mu\rho\sigma} H_\nu{}^{\rho\sigma} + O(\alpha') \,, \\ \beta^B_{\mu\nu} &\propto \nabla_\rho H^\rho{}_{\mu\nu} - 2H^\rho{}_{\mu\nu} \nabla_\rho \Phi + O(\alpha') \,. \end{aligned} \quad (5.3)$$

The field strength $H_{\mu\nu\rho}$ of the antisymmetric tensor field is defined by

$$H_{\mu\nu\rho} = \partial_{[\mu} B_{\nu\rho]} \,, \quad (5.4)$$

using antisymmetrization conventions $A_{[\mu_1 \ldots \mu_n]} \equiv \frac{1}{n!} \sum_P (-1)^{|P|} A_{\mu_{P_1} \ldots \mu_{P_n}}$ where the sum is over all $n!$ permutations $\{P_1, ...P_n\}$ of $\{1, ...n\}$. $R_{\mu\nu}$ is the space-time Ricci tensor and $R = R_\mu{}^\mu$ is the space-time scalar curvature, not to be confused with the world-sheet scalar curvature $R^{(2)}$. The expressions (5.3) arise from one-loop calculations. They receive corrections of higher orders in α', but these will not concern us here.

From (5.2) it is clear that cancellation of the Weyl anomaly requires

$$\beta^\Phi = \beta^g = \beta^B = 0 \,. \quad (5.5)$$

Note that for vanishing dilaton and antisymmetric tensor fields the equation $\beta^g_{\mu\nu} = 0$ becomes Einstein's equation in D-dimensional empty space-time if we ignore also the α' corrections. Nonzero Φ and $B_{\mu\nu}$ provide sources for Einstein's equation.

The equations (5.5) can be viewed as equations of motion of the action

$$S = \int d^D X \sqrt{-g} e^{-2\Phi} \left(-\frac{D-26}{3} - R + 4\partial_\mu \Phi \partial^\mu \Phi - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) \,. \quad (5.6)$$

This is the low-energy effective action. Only the massless fields from the string spectrum are represented in this action. Note that $B_{\mu\nu}$ is a gauge field; the action is invariant under the gauge transformation $B_{\mu\nu} \to B_{\mu\nu} + \partial_{[\mu} \Lambda_{\nu]}$. From now on we assume that we are in the critical dimension, i.e. that the first term in (5.6) vanishes. Thus the low-energy effective action for the bosonic string is the action of 26-dimensional gravity coupled to a scalar field and an antisymmetric tensor field.

For other string theories, one can follow the same steps to arrive at a low-energy effective action. A general procedure is to consider a string moving in a background with fields corresponding to the massless states in the string spectrum. The dynamics of the string in this background is described by a nonlinear sigma model, and the condition for conformal invariance at the quantum level supplies the equations of motion for the background fields. From these equations one may construct the low-energy effective action[1].

[1]Another method that has been used to derive the low-energy effective action is to calculate scattering amplitudes of the massless string modes within string perturbation theory and construct the field theory action for the background fields which reproduces these amplitudes.

The bosonic massless spectrum of the heterotic string contains, besides the metric, antisymmetric tensor and dilaton, gauge bosons of $E_8 \times E_8$ or $SO(32)$. The bosonic part of the low-energy effective action of the heterotic string is, to lowest order in α',

$$S = \int d^{10}x \sqrt{-g} e^{-2\Phi} \left(-R + 4\partial_\mu \Phi \partial^\mu \Phi - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \right) , \tag{5.7}$$

which is, apart from the dimension, identical to (5.6). When fermion fields are included, the low-energy effective action to lowest order in α' is the $D = 10$ $N = 1$ supergravity action [58]. First order α' corrections involve Yang-Mills fields

$$S^{(\alpha')} \sim \alpha' \int d^{10}x \sqrt{-g} e^{-2\Phi} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) , \tag{5.8}$$

as well as Chern-Simons terms in the antisymmetric tensor field strength. Including fermions, one obtains supersymmetric Yang-Mills coupled to $N = 1$ supergravity [59, 25, 60].

It should be kept in mind that we use only lowest order approximations to the full effective actions. The full effective action is approximated by an expansion in two independent parameters. One of them is α' which is the loop-counting parameter from the sigma model point of view, as mentioned before. It defines a length scale $\sqrt{\alpha'}$ which is assumed to be small, probably not too far from the Planck length, and the background fields are assumed to vary slowly in space-time relative to this scale. The other parameter is the string loop parameter which is the expectation value of $e^{2\Phi}$. It counts the number of loops in a string scattering process, i.e. the genus of the world-sheet. This can be understood by noting that in the nonlinear sigma model the constant mode of the dilaton field multiplies the Euler characteristic

$$\chi(\Sigma) = \frac{1}{4\pi} \int_\Sigma d^2\sigma \sqrt{h} R^{(2)} = 2 - 2g , \tag{5.9}$$

where g is the genus of the world-sheet Σ. Thus we see that in the (Euclidean) path integral genus g amplitudes are weighted by a factor $e^{(2g-2)\Phi} = (e^{2\Phi})^{g-1}$. Since each handle corresponds to two string 'vertices', the string coupling constant should be identified with e^{Φ}. As the dilaton exponential in (5.6), (5.7) and (5.8) indicates, these effective actions correspond to tree level string theory, i.e. to the classical zero-loop scatterings of massless string modes.

Higher order kinetic terms for the gravitational and Yang-Mills fields appear either as α' corrections or as string loop corrections or as both. For the gravitational field the first few terms in the expansion schematically look like

$$e^{-2\Phi} \left(R + \alpha' R^2 + (\alpha')^3 R^4 + e^{2\Phi} (\alpha')^3 R^4 + ... \right) . \tag{5.10}$$

In particular, this implies that string theory predicts corrections with terms of higher order in derivatives to standard gravity or supergravity. Further information, in particular on aspects of supersymmetry concerning higher order corrections may be found in [183] and references therein.

5.1.1 Compactification and dimensional reduction

In order to eventually make contact with the world as we know it, it is necessary to 'compactify' the superstring theory from ten to four dimensions. The six internal dimensions should be hidden from our view, so these dimensions should form a compact space of tiny proportions, presumably of order the Planck length [124]. This means that we have to look for a solution of the background field equations (5.5) with the property that the ten-dimensional space-time M is a product of four-dimensional space-time M^4 and a six-dimensional compact internal manifold K, $M = M^4 \times K$. From now on we use hatted indices $\hat{\mu}, \hat{\nu}, \ldots$ for ten-dimensional space-time coordinates, unhatted $\mu, \nu, \ldots$ for four-dimensional space-time coordinates and $m, n, \ldots$ for six-dimensional internal coordinates, and we write $x^{\hat{\mu}} = (x^\mu, y^m)$. Moreover, ten-dimensional fields will be hatted and four-dimensional fields unhatted in our notation.

The 10d fields can be reinterpreted as 4d fields as follows. In general, a 10d spin-s field will decompose into several fields with spins $\leq s$ from the 4d point of view. This takes place according to the decomposition of the representations of the appropriate Lorentz groups, i.e. of $SO(1,9)$ representations into $SO(1,3) \times SO(6)$ representations. For example, the 10d metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, which transforms in the 55-dimensional representation of $SO(1,9)$ decomposes under $SO(1,3) \times SO(6)$ as $\mathbf{55} = \mathbf{10} \times \mathbf{1} + \mathbf{4} \times \mathbf{6} + \mathbf{1} \times \mathbf{21}$, corresponding to the components $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu m}$ and $\hat{g}_{mn}$, respectively. Thus in four dimensions these fields correspond to a metric, 6 vector fields and 21 scalar fields, respectively.

In general, however, the fields depend on all ten coordinates x^μ and y^m. This means that the 10d fields give rise to an infinite number of 4d fields, essentially because in four dimensions the y-dependences are reinterpreted as internal degrees of freedom. However, only a finite number of these fields correspond to massless particles in four dimensions, the precise number depending on the topology of K, see e.g. [104]. All the other fields correspond to massive particles, presumably with masses of order M_{Planck}. Since we are (at first) interested in the low-energy effective theory, these massive particles may be ignored as an approximation.

We only consider the simplest way of compactification, namely toroidal compactification. Then the compact internal manifold is assumed to be a six-dimensional torus, $K = T^6 = S^1 \times \ldots \times S^1$. From the phenomenological point of view this may not be the most interesting way of compactifying but, as we will see, it gives rise to a number of interesting symmetry structures some of which are also expected (or known) to be present in more general compactifications. Let us just mention that other, more realistic compactification procedures, for example the ones introduced in [56], may produce four-dimensional models with $N = 1$ supersymmetry, chiral fermions, gauge groups which contain the standard model group $SU(3) \times SU(2) \times U(1)$, etc.. For a review of various compactification methods and corresponding descriptions of the internal degrees of freedom by conformal field theories, see [174].

In fact, we will consider only a special case of toroidal compactification; we require that all fields be independent of the compact coordinates y^m. This is called dimensional reduction. Such a y-independent field configuration should be a solution of the string equations of motion, and for this to be consistent, K must be flat (the metric on K is independent of the coordinates on K). A torus is flat, thus dimensional reduction is consistent with torus compactification. We now discuss how a four-dimensional action is obtained by dimensional reduction. Explicit formulas for dimensional reduction have been given in [171, 62, 147].

First of all, it should be noted that the gauge symmetries of the 10d action are necessarily broken because we require that the fields remain y-independent after a gauge transformation. Let us illustrate this with a vector field $\hat{A}_{\hat{\mu}}$. Under a 10d local coordinate transformation $\delta x^{\hat{\mu}} = -\xi^{\hat{\mu}}(x^{\hat{\nu}})$, it transforms according to

$$\delta \hat{A}_{\hat{\mu}} = \xi^{\hat{\nu}} \partial_{\hat{\nu}} \hat{A}_{\hat{\mu}} + \partial_{\hat{\mu}} \xi^{\hat{\nu}} \hat{A}_{\hat{\nu}} = \xi^{\nu} \partial_{\nu} \hat{A}_{\hat{\mu}} + \partial_{\hat{\mu}} \xi^{\nu} \hat{A}_{\nu} + \partial_{\hat{\mu}} \xi^{n} \hat{A}_{n} \tag{5.11}$$

where we used that $\partial_n \hat{A}_{\hat{\mu}} = 0$. Requiring $\partial_m \delta \hat{A}_{\hat{\mu}} = 0$ implies

$$\partial_m \xi^{\mu} = 0 \quad \text{and} \quad \partial_m \partial_{\hat{\mu}} \xi^n = 0\,, \tag{5.12}$$

which has the solution

$$\xi^{\mu} = \xi^{\mu}(x^{\nu}) \quad \text{and} \quad \xi^m = a^m{}_n y^n + \xi^m(x^{\nu})\,, \tag{5.13}$$

where $a^m{}_n$ are constants. Thus the remaining transformations originating from 10d general coordinate invariance are

$$\begin{aligned} \delta x^{\mu} &= -\xi^{\mu}(x^{\nu}) && \text{4d general coordinate invariance}\,, \\ \delta y^m &= -\xi^m(x^{\nu}) && \mathrm{U}(1)^6 \text{ 4d gauge invariance}\,, \\ \delta y^m &= -a^m{}_n y^n && \mathrm{GL}(6) \text{ global invariance}\,, \end{aligned} \tag{5.14}$$

where it should be noted that the second line really defines 4d gauge transformations, since the gauge parameters are arbitrary functions of the 4d coordinates. Below we will see how these transformations act on the 4d fields. A similar analysis of the 10d gauge transformations $\delta \hat{B}_{\hat{\mu}\hat{\nu}} = \partial_{[\hat{\mu}} \Lambda_{\hat{\nu}]}$ shows that the remaining 4d transformations, consistent with dimensional reduction, are

$$\begin{aligned} \delta \hat{B}_{\mu\nu} &= \partial_{[\mu} \lambda_{\nu]}\,, \\ \delta \hat{B}_{\mu n} &= \partial_{\mu} \lambda_n + b_{\mu n}\,, \\ \delta \hat{B}_{mn} &= b_{mn}\,, \end{aligned} \tag{5.15}$$

where λ_{ν} and λ_n are functions of x^{μ} only and $b_{\mu n}$ and b_{mn} are constants. The first equation shows the usual gauge invariance for a 4d antisymmetric tensor field. The second equation shows that there are six $U(1)$ gauge invariances and some constant shifts. The last equation expresses invariance under constant shifts of the 4d scalars $\hat{B}_{mn}$. Note that we have not yet defined the 4d fields. The fields in the above equations are components of 10d fields and the 4d fields should be defined in terms of them. However, they should be defined in a sensible way such that they transform in the usual way under 4d (gauge) transformations.

It is convenient to use the vielbein formulation in which the metric is written as $\hat{g}_{\hat{\mu}\hat{\nu}} = \hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} \hat{e}_{\hat{\nu}}{}^{\hat{\beta}} \eta_{\hat{\alpha}\hat{\beta}}$, where $\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}}$ is the ten-dimensional vielbein and $\hat{\alpha}, \hat{\beta}, \ldots$ denote flat ten-dimensional indices. The metric is invariant under local Lorentz transformations $\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} \to \Lambda(x)_{\hat{\beta}}{}^{\hat{\alpha}} \hat{e}_{\hat{\mu}}{}^{\hat{\beta}}$. This allows us to choose the gauge

$$\hat{e}_{\hat{\mu}}{}^{\hat{\alpha}} = \begin{pmatrix} e_\mu{}^\alpha & A_\mu^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix} , \tag{5.16}$$

where $\alpha = 0,1,2,3$ and $a = 4,5,...9$ are flat indices. This parametrization has been chosen such that

- $e_\mu{}^\alpha$ is the 4d vierbein, and $g_{\mu\nu} = e_\mu{}^\alpha e_\nu{}^\beta \eta_{\alpha\beta}$ is the 4d metric

- $e_m{}^a$ produces 4d scalars $G_{mn} = e_m{}^a e_n{}^b \delta_{ab}$

- A_μ^m are six 4d abelian vector fields.

This can be checked using (5.14). For example, under the residual 10d general coordinate transformations (5.14) with parameters $\xi^m(x^\nu)$, we have

$$\delta A_\mu^m = \partial_\mu \xi^m , \tag{5.17}$$

which shows that A_μ^m are the gauge fields belonging to the residual $U(1)^6$.

Other four-dimensional fields can also be defined conveniently using vielbeins. We give one example. Define the four-dimensional vector fields corresponding to the gauge transformation (5.15) by

$$B_{\mu m} = e_\mu{}^\alpha e_m{}^a \hat{e}_\alpha{}^{\hat{\mu}} \hat{e}_a{}^{\hat{\nu}} \hat{B}_{\hat{\mu}\hat{\nu}} , \tag{5.18}$$

which is equivalent to saying that $B_{\alpha a} = \hat{B}_{\alpha a}$. This ensures that B_μ^m transforms as a 4d vector under 10d general coordinate transformations. In particular, it guarantees that B_μ^m is invariant under the $U(1)$ transformations in (5.14). Using (the inverse of) (5.16), we find

$$B_{\mu m} = \hat{B}_{\mu m} - A_\mu^n B_{mn} , \tag{5.19}$$

where we defined $B_{mn} = \hat{B}_{mn}$, or equivalently, $B_{ab} = \hat{B}_{ab}$.

The vector fields A_μ^m are usually called Kaluza Klein vector fields, since Kaluza and Klein put forward the idea that gravity in higher dimensions gives rise to gravity plus gauge symmetry in lower dimensions. The fields $B_{\mu m}$ are usually called winding vector fields since they can be shown to couple to the states corresponding to strings that wind around the periodic y^m direction.

Let us write down the low-energy effective action (5.7) of the heterotic string, dimensionally reduced to four dimensions[2] [147],

$$\begin{aligned} S = \int d^4x \sqrt{-g} e^{-2\Phi} [& -R_g + 4\partial_\mu \Phi \partial^\mu \Phi - \tfrac{3}{4} H_{\mu\nu\rho} H^{\mu\nu\rho} \\ & -\tfrac{1}{4} \mathcal{F}_{\mu\nu}^T LML \mathcal{F}^{\mu\nu} + \tfrac{1}{8} \mathrm{Tr}(\partial_\mu ML \partial^\mu ML)] . \end{aligned} \tag{5.20}$$

The various fields in this action are given by the definitions above and

$$\mathcal{F}_{\mu\nu}(\mathcal{A}) \quad = \quad \begin{pmatrix} F_{\mu\nu}^m(A) \\ F_{\mu\nu\, m}(B) \end{pmatrix} = \begin{pmatrix} \partial_\mu A_\nu^m - \partial_\nu A_\mu^m \\ \partial_\mu B_{\nu m} - \partial_\nu B_{\mu m} \end{pmatrix} ,$$

[2] We take $\int d^6 y = 1$ for the volume of the six-dimensional torus.

$$
\begin{aligned}
H_{\mu\nu\rho} &= \partial_{[\mu}B_{\nu\rho]} + \tfrac{1}{2}\mathcal{A}^T_{[\mu}L\mathcal{F}_{\nu\rho]}\,, \\
B_{\mu\nu} &= \hat{B}_{\mu\nu} + \tfrac{1}{2}A^m_\mu B_{\nu m} - \tfrac{1}{2}A^m_\nu B_{\mu m} - B_{mn}A^m_\mu A^n_\nu\,, \\
\Phi &= \hat{\Phi} - \tfrac{1}{4}\log\det G_{mn}\,.
\end{aligned}
\tag{5.21}
$$

Furthermore, we defined the 12×12 matrix

$$
L = \begin{pmatrix} 0 & I_6 \\ I_6 & 0 \end{pmatrix}\,, \tag{5.22}
$$

where I_6 is the 6×6 identity matrix. Transformations that preserve L, $\Omega^T L \Omega = L$, form the group $O(6,6)$. The last term in (5.20) represents the kinetic terms of all scalars G_{mn} and B_{mn}, rearranged in the matrix

$$
M = \begin{pmatrix} G^{-1} & G^{-1}B \\ -BG^{-1} & G - BG^{-1}B \end{pmatrix}\,, \tag{5.23}
$$

where we use the matrix notation $(G)_{mn} = G_{mn}$ and $(B)_{mn} = B_{mn}$. This scalar matrix is in fact an $O(6,6)$ matrix, since $M^T L M = L$. It is also symmetric, $M^T = M$. Together, these facts imply that M parametrizes the coset $\frac{O(6,6)}{O(6)\times O(6)}$. The scalar fields G_{mn} and B_{mn} characterize the torus of compactification, and the coset they span is also called the moduli space of toroidal compactifications.

The way in which the terms in the action (5.20) have been arranged shows that it is $O(6,6)$ symmetric. For $\Omega \in O(6,6)$, the following transformations are symmetries of the action:

$$
M \to \Omega M \Omega^T\,, \qquad \mathcal{F}_{\mu\nu} \to \Omega \mathcal{F}_{\mu\nu}\,. \tag{5.24}
$$

We also see that Kaluza Klein and winding vector fields are transformed into each other under (5.24).

If fermions are included, (5.20) is the action of $D=4$ $N=4$ supergravity coupled to six abelian vector fields[3]. Toroidal compactification does not break any supersymmetries and therefore we get four (Majorana) supersymmetry charges in $D=4$ from one (Majorana-Weyl) supersymmetry charge in $D=10$. The non-compact global $O(6,6)$ symmetry of (5.20) is one example of a generic phenomenon in extended supergravity theories. In such theories there is a non-compact global symmetry group G with a maximal compact subgroup H. The scalar fields of the theory are associated with the coset G/H. This implies, in particular, that their number is $\dim(G) - \dim(H)$. In our case $G = O(6,6)$ and $H = O(6)\times O(6)$ and therefore $\dim(G) - \dim(H) = 66 - 2.15 = 36$ which corresponds to the number of scalars G_{mn} (21) and B_{mn} (15). The action (5.20) contains yet another scalar, the dilaton. We will see in section 3 of this chapter that it also belongs to a coset associated to an additional non-compact global symmetry.

The presence of an $O(6,6)$ symmetry of the dimensionally reduced action can partly be ascribed to residual 10d gauge transformations. A $GL(6)$ subgroup of $O(6,6)$, acting on indices $m,n,...$, arises from the 10d coordinate transformations $\delta y^m = -a^m{}_n y^n$. Also, the shifts $B_{mn} \to B_{mn} + b_{mn}$, accompanied by $B_{\mu m} \to B_{\mu m} - b_{mn}A^n_\mu$, are relics of

[3]The supergravity multiplet itself contains the other six vector fields.

gauge transformations of the 10d antisymmetric tensor field, see equations (5.15) and (5.19). Together, they correspond to the subgroup

$$\begin{pmatrix} a & 0 \\ -(a^T)^{-1}b & (a^T)^{-1} \end{pmatrix} \subset O(6,6)\,, \tag{5.25}$$

where $a \in GL(6)$ and b is an antisymmetric 6×6 matrix. Whereas this subgroup acts linearly (in the sense of no quadratic or higher nonlinearities) on the scalars, the other $O(6,6)$ transformations do not.

The $O(6,6)$ transformations which do not correspond to 10d gauge transformations can be used as solution generating transformations[4]. It was shown by Sen [179] that these transformations form an $\frac{O(6)\times O(6)}{O(6)}$ coset, with transformation matrices

$$\Omega = \frac{1}{2}\begin{pmatrix} R+S & R-S \\ R-S & R+S \end{pmatrix}\,, \tag{5.26}$$

with $R, S \in O(6)$. The diagonal subgroup with $R = S$ is divided out since it is part of the $GL(6)$ transformations in (5.25).

This analysis can be generalized to include the vector fields (5.8) of the heterotic string. It is known that the requirement of anomaly cancellation restricts the gauge group to either $E_8 \times E_8$ or $SO(32)$ [103]. Toroidal compactification generically gives masses to all charged vector fields so that only the vector fields associated to the Cartan subalgebra remain massless. This gives 16 abelian vector fields in both cases. These vector fields can be grouped together with the 12 abelian vector fields coming from the metric and the antisymmetric tensor field to form an $O(6,22)$ vector multiplet. The scalars (including those that come from the additional 10d vector fields) now form a coset $\frac{O(6,22)}{O(6)\times O(22)}$. The dimensionally reduced action then has a global $O(6,22)$ symmetry.

Another straightforward generalization is to dimensionally reduce with respect to the d coordinates of a torus T^d. The resulting $10-d$ dimensional effective action will then have a global symmetry group $O(d,d)$, or $O(d,d+16)$[5] if additional vector fields are present.

5.2 T-duality in toroidal compactifications

It is well-known that whereas the low-energy effective action of the heterotic string has an $O(6,22)$ global symmetry, this can not be true for the full string theory. It turns out that only a discrete subgroup may survive as a symmetry of the full string theory. The reason for this is that in the full string theory the transformations $\Omega \in O(6,22)$ must

[4]These transformations can be used to obtain possibly new solutions starting from a known solution. All solutions must have (in this case) six isometries as they must be compatible with dimensional reduction.

[5]Larger symmetry groups, containing $O(d,d+16)$ as a subgroup, arise for $d \geq 6$, i.e. for $(D \leq 4)$-dimensional reduced actions, see e.g. [118] and references therein.

preserve a 28-dimensional even self-dual lattice of charges, with an $O(6,22)$ metric

$$L = \begin{pmatrix} 0 & I_6 & 0 \\ I_6 & 0 & 0 \\ 0 & 0 & -I_{16} \end{pmatrix} . \tag{5.27}$$

This lattice should be thought of as the lattice of allowed [151] charges of string states with respect to the 28 massless abelian vector fields that appear upon toroidal compactification of the heterotic string to four dimensions. The lattice is spanned by vectors $(a_1, ...a_6, b_1, ...b_6, \vec{k})$ where a_i and b_i are integers and the vectors $\vec{k}$ belong to the root lattice of $E_8 \times E_8$ or $SO(32)$. The integers a_i and b_i can be interpreted as Kaluza Klein momenta (momenta along the internal directions) and winding numbers (number of times a closed string wraps around a compact direction), respectively. The transformations that preserve the lattice form a discrete group $O(6,22;\mathbb{Z})$. This is expected to be a symmetry of the full string theory, called target-space duality or T-duality in short. In fact, T-duality is known to hold in each order of string perturbation theory. Moreover, T-duality transformations have been shown to correspond to discrete gauge symmetries in the space of string backgrounds [71]. We refer to [98] for an extensive review.

Before, we called the coset $\frac{O(6,22)}{O(6)\times O(22)}$ the moduli space of toroidal compactification of the heteroric string to four dimensions. We now see, however, that the moduli space of *inequivalent* toroidal compactifications is this coset modulo $O(6,22;\mathbb{Z})$ T-duality transformations.

Next, we describe T-duality at the level of the sigma model, and we will see how it relates equivalent conformal field theories underlying the toroidal compactifications. Again we refer to review articles [4, 98] for additional information and references.

It was shown by Buscher [53] that given a sigma model in a background (g, B, Φ)[6] with an abelian isometry, there exists an equivalent sigma model with background fields $(\tilde{g}, \tilde{B}, \tilde{\Phi})$ related to the former by a duality transformation. If we choose an adapted coordinate system $(x^0, x^\mu) = (y, x^\mu)$, where the isometry acts by translations of y, i.e. the background fields are independent of y, the backgrounds of two equivalent sigma models are related by

$$\begin{aligned}
&\tilde{g}_{00} = 1/g_{00} \,, \qquad \tilde{g}_{0\mu} = -B_{0\mu}/g_{00} \,, \\
&\tilde{g}_{\mu\nu} = g_{\mu\nu} - (g_{0\mu}g_{0\nu} - B_{0\mu}B_{0\nu})/g_{00} \,, \\
&\tilde{B}_{0\mu} = -g_{0\mu}/G_{00} \,, \\
&\tilde{B}_{\mu\nu} = B_{\mu\nu} - (g_{0\mu}B_{0\nu} - g_{0\nu}B_{0\mu})/g_{00} \,, \\
&\tilde{\Phi} = \Phi - \tfrac{1}{2}\log|g_{00}| \,.
\end{aligned} \tag{5.28}$$

Note that these equations describe a $\mathbb{Z}_2$ transformation. For string theory this means that two string backgrounds (conformally invariant sigma models) with an abelian isometry in the y direction and related by (5.28) yield the same string dynamics.

[6] In the following discussion all fields considered are conventional sigma model background fields, and we omit hats.

As a simple example, let us take a diagonal metric g, and $B=0$. If the coordinate $x^0=y$ is compactified on a circle, and g and Φ are independent of y, the above transformations give $\tilde{g}_{00}=1/g_{00}$ (and $\tilde{\Phi}=\Phi-\frac{1}{2}\log|g_{00}|$)[7]. This corresponds to an inversion of the radius of the circle, $R\to\frac{1}{R}$, or $R\to\frac{\alpha'}{2R}$ if we restore the unit of length. This suggests that string geometry near the Planck scale (more precisely, the string scale) may be quite different from ordinary geometry.

5.2.1 Canonical approach to T-duality

The Buscher duality transformation relates two equivalent conformal field theories. One would expect that such an equivalence should be obtainable in the Hamiltonian formalism as a canonical transformation. This is indeed the case as has been shown in [99, 3]. In this subsection, we generalize the canonical approach to arbitrary $O(d,d;\mathbb{Z})$ T-duality transformations.

We start with the nonlinear sigma model with Lagrangian

$$\mathcal{L}=\tfrac{1}{2}g_{\hat{\mu}\hat{\nu}}(X)\partial_a X^{\hat{\mu}}\partial^a X^{\hat{\nu}}+\tfrac{1}{2}\varepsilon^{ab}B_{\hat{\mu}\hat{\nu}}(X)\partial_a X^{\hat{\mu}}\partial_b X^{\hat{\nu}}\,, \tag{5.29}$$

where we chose the conformal gauge and a convenient normalization. The results will be for the bosonic string, but can also be applied to the (bosonic sector of the) heterotic string. We let $\hat{\mu},\hat{\nu},...\in\{0,1,...\bar{D}-1\}$, $\mu,\nu,...\in\{0,1,...D-1\}$ and $m,n,...\in\{D,D+1,...\bar{D}-1\}$, and we assume that the background fields $g_{\hat{\mu}\hat{\nu}}$ and $B_{\hat{\mu}\hat{\nu}}$ are independent of the coordinates X^m. Thus we have $d\equiv\bar{D}-D$ abelian isometries whose orbits we assume to be compact. This means we are considering toroidal compactification of a $\bar{D}$-dimensional sigma model background on a d-dimensional torus. For the periodicities of the compact coordinates we may take without loss of generality[8]

$$X^m\simeq X^m+2\pi\,. \tag{5.30}$$

Closed strings[9] can wrap around the circles,

$$X^m(\sigma+2\pi)=X^m(\sigma)+2\pi b^m\,, \tag{5.31}$$

where the integers b^m are the winding numbers. We will demonstrate how the T-duality group $O(d,d;\mathbb{Z})$ emerges from the action of canonical transformations. We follow the same steps as in [3, 4] where a single abelian isometry is considered.

The momentum canonically conjugate to $X^{\hat{\mu}}$ is

$$P_{\hat{\mu}}\equiv\frac{\partial\mathcal{L}}{\partial\dot{X}^{\hat{\mu}}}=g_{\hat{\mu}\hat{\nu}}\dot{X}^{\hat{\nu}}+B_{\hat{\mu}\hat{\nu}}X'^{\hat{\nu}}\,, \tag{5.32}$$

[7] The shift of the dilaton ensures that the sigma model remains conformally invariant, at least to one-loop order in α' [53].

[8] We assume the information on the radii of the torus to be contained in the metric.

[9] We restrict ourselves to closed strings here. For open strings, T-duality also has interesting consequences. It interchanges Neumann and Dirichlet boundary conditions, and leads to the emergence of extended objects called D-branes. For a review, see [158].

and the Hamiltonian is given by

$$\begin{aligned}\mathcal{H} &\equiv P_{\hat\mu}\dot X^{\hat\mu} - \mathcal{L} = \tfrac{1}{2}g_{\hat\mu\hat\nu}\dot X^{\hat\mu}\dot X^{\hat\nu} + \tfrac{1}{2}g_{\hat\mu\hat\nu}X'^{\hat\mu}X'^{\hat\nu} \\ &= \tfrac{1}{2}g^{\hat\mu\hat\nu}(P_{\hat\mu} - B_{\hat\mu\hat\rho}X'^{\hat\rho})(P_{\hat\nu} - B_{\hat\nu\hat\sigma}X'^{\hat\sigma}) + \tfrac{1}{2}g_{\hat\mu\hat\nu}X'^{\hat\mu}X'^{\hat\nu}\,.\end{aligned} \tag{5.33}$$

Let us now rewrite $\mathcal{L}$ as follows

$$\begin{aligned}\mathcal{L} &= \tfrac{1}{2}g_{mn}(X)\left(\dot X^m\dot X^n - X'^m X'^n\right) + B_{mn}\dot X^m X'^n \\ &\quad +(\dot X^m + X'^m)J_{m-} + (\dot X^m - X'^m)J_{m+} + V\,,\end{aligned} \tag{5.34}$$

where

$$\begin{aligned}&J_{m-} = \tfrac{1}{2}g_{m\mu}(\dot X^\mu - X'^\mu) + \tfrac{1}{2}B_{m\mu}(X'^\mu - \dot X^\mu) = \tfrac{1}{2}(g_{m\mu} - B_{m\mu})\partial_- X^\mu\,,\\ &J_{m+} = \tfrac{1}{2}g_{m\mu}(\dot X^\mu + X'^\mu) + \tfrac{1}{2}B_{m\mu}(X'^\mu + \dot X^\mu) = \tfrac{1}{2}(g_{m\mu} + B_{m\mu})\partial_+ X^\mu\,,\\ &V = \tfrac{1}{2}g_{\mu\nu}(\dot X^\mu\dot X^\nu - X'^\mu X'^\nu) + B_{\mu\nu}\dot X^\mu X'^\nu = \tfrac{1}{2}(g_{\mu\nu} - B_{\mu\nu})\partial_+ X^\mu\partial_- X^\nu\,.\end{aligned} \tag{5.35}$$

To obtain a dual sigma model using canonical transformations, we need to apply the Legendre transformation only to the internal coordinates $(X^m, \dot X^m)$. Then the momenta are[10]

$$P_m = G_{mn}\dot X^n + B_{mn}X'^n + J_{m-} + J_{m+}\,, \tag{5.36}$$

and the 'Hamiltonian' corresponding to this incomplete Legendre transformation is

$$\begin{aligned}\mathcal{H} &= P_m\dot X^m - \mathcal{L} = \tfrac{1}{2}G^{mn}P_mP_n - G^{mn}B_{nk}P_mX'^k + \tfrac{1}{2}(G - BG^{-1}B)_{mn}X'^mX'^n \\ &\quad +G^{mn}B_{mk}(J_{n+} + J_{n-})X'^k - G^{mn}(J_{n+} + J_{n-})P_m \\ &\quad +X'^m(J_{m+} - J_{m-}) + \tfrac{1}{2}G^{mn}(J_{m+} + J_{m-})(J_{n+} + J_{n-}) - V\,,\end{aligned} \tag{5.37}$$

where we eliminated $\dot X^m$ using (5.36). We may now perform canonical transformations using a generating function that depends on X^m and the new coordinates $\tilde X^m$. (A nice explanation of generating functions for canonical transformations can be found in [167].) We take as the generating function

$$F(X, \tilde X) = \int_0^{2\pi} d\sigma\left(a_{mn}X^m\tilde X'^n + \tfrac{1}{2}b_{mn}\tilde X^m\tilde X'^n\right)\,, \tag{5.38}$$

where a is a non-singular $d \times d$ matrix and b is an antisymmetric $d \times d$ matrix (the symmetric part of b would contribute a total derivative to the integrand in (5.38)). The momenta are given by

$$\begin{aligned}P_m &= \frac{\delta F}{\delta X^m} = a_{mn}\tilde X'^n\,,\\ \tilde P_m &= -\frac{\delta F}{\delta \tilde X^m} = a_{nm}X'^n + b_{nm}\tilde X'^n\,.\end{aligned} \tag{5.39}$$

The zero modes of the momenta P_m are integers. This follows from requiring single-valuedness of the wave functions associated to string states under $X^m \to X^m + 2\pi$.

[10] To stick with our notation in the previous section, we denote the 'internal' components of the metric by G_{mn}.

Moreover, the zero mode of X'^m is the winding number b^m. In order that the zero modes remain integer under duality transformations, we should restrict the coefficients a_{mn} and b_{mn} to integers. Thus we observe also here (and in fact for the same reason that the string spectrum is required to be duality-invariant) that the T-duality group is a discrete group.

To obtain the new Hamiltonian $\tilde{\mathcal{H}}$ we first substitute P_m of (5.39) in (5.37) to obtain an expression in terms of X' and $\tilde{X}'$, and then use the second line in (5.39) to eliminate X'. Note that because $\mathcal{H}$ has no explicit X^m dependence this leads to a local expression. The result is, after some simplifications,

$$\begin{aligned}\tilde{\mathcal{H}} \;=\; & \tfrac{1}{2}\tilde{X}'a^TG^{-1}a\tilde{X}' - \tilde{X}'a^TG^{-1}B(a^T)^{-1}(\tilde{P}+b\tilde{X}') \\ & +\tfrac{1}{2}(\tilde{P}-\tilde{X}'b)a^{-1}(G-BG^{-1}B)(a^T)^{-1}(\tilde{P}+b\tilde{X}') \\ & +(J_++J_-)G^{-1}B(a^T)^{-1}(\tilde{P}+b\tilde{X}') - (J_++J_-)G^{-1}a\tilde{X}' \\ & +(J_+-J_-)(a^T)^{-1}(\tilde{P}+b\tilde{X}') + \tfrac{1}{2}(J_++J_-)G^{-1}(J_++J_-) - V\,. \end{aligned} \tag{5.40}$$

We suppress all indices by using matrix notation. In order to get the new sigma model Lagrangian we perform the inverse Legendre transformation from $(\tilde{X},\tilde{P})$ to $(\tilde{X},\dot{\tilde{X}} = \frac{\partial\tilde{\mathcal{H}}}{\partial\tilde{P}})$. We get, after some rearrangements,

$$\begin{aligned}\tilde{\mathcal{L}} \;=\; & \tfrac{1}{2}\dot{\tilde{X}}a^TWa\dot{\tilde{X}} - \tfrac{1}{2}\tilde{X}'a^TWa\tilde{X}' - \dot{\tilde{X}}(a^TWBG^{-1}a+b)\tilde{X}' \\ & -J_+(Wa+WBG^{-1}a)(\dot{\tilde{X}}-\tilde{X}') + J_-(Wa-WBG^{-1}a)(\dot{\tilde{X}}+\tilde{X}') \\ & -2J_+(W+WBG^{-1})J_- + V\,. \end{aligned} \tag{5.41}$$

For notational convenience we defined the matrix

$$W = (G-BG^{-1}B)^{-1}\,, \tag{5.42}$$

and we used $G^{-1}BW = WBG^{-1}$.

This is the dual sigma model Lagrangian and we can read off the new background fields by comparing with (5.34). Thus we see that

$$\begin{aligned}\tilde{G}_{mn} &= (a^TWa)_{mn} = (a^T(G-BG^{-1}B)^{-1}a)_{mn}\,, \\ \tilde{B}_{mn} &= -(a^TWBG^{-1}a)_{mn} - b_{mn}\,. \end{aligned} \tag{5.43}$$

Furthermore,

$$\begin{aligned}\tilde{J}_+ &= -(a^TW - a^TG^{-1}BW)J_+\,, \\ \tilde{J}_- &= \;(a^TW + a^TG^{-1}BW)J_-\,, \end{aligned} \tag{5.44}$$

from which we derive, using the definitions of $J_\pm$ given in (5.35),

$$\begin{aligned}\tilde{g}_{m\mu} &= (a^TG^{-1}BW)_m{}^n g_{n\mu} - (a^TW)_m{}^n B_{n\mu}\,, \\ \tilde{B}_{m\mu} &= -(a^TW)_m{}^n g_{n\mu} + (a^TG^{-1}BW)_m{}^n B_{n\mu}\,. \end{aligned} \tag{5.45}$$

Finally, we get from $\tilde{V} = V - 2J_+(W + WBG^{-1})J_-$,

$$\begin{aligned} \tilde{g}_{\mu\nu} &= g_{\mu\nu} - g_{m\mu} W^{mn} g_{n\nu} \\ &\quad + 2g_{m(\mu}(WBG^{-1})^{mn} B_{n\nu)} + B_{m\mu} W^{mn} B_{n\nu} \,, \\ \tilde{B}_{\mu\nu} &= B_{\mu\nu} - 2g_{m[\mu} W^{mn} B_{n\nu]} \\ &\quad + g_{m\mu}(WBG^{-1})^{mn} g_{n\nu} - B_{m\mu}(WBG^{-1})^{mn} B_{n\nu} \,. \end{aligned} \tag{5.46}$$

The transformations (5.43), (5.45) and (5.46) generalize the $\mathbb{Z}_2$ duality rules of equation (5.28).

We note that for $a_{mn} = \delta_{mn}$ and $b_{mn} = 0$, the transformation in (5.43) can be written as $E \to 1/E$, where $E_{mn} \equiv (G + B)_{mn}$. This generalizes the $R \to 1/R$ circle duality to a d-torus.

We have not yet obtained all $O(d, d; \mathbb{Z})$ transformations, because transformations connected to the identity (in the continuous group $O(d, d; \mathbb{R})$, that is) cannot be obtained using the generating function F in (5.38). To obtain the latter transformations, we have to use a generating function that depends on the pair $(X^m, \tilde{P}_m)$. Using

$$F(X, \tilde{P}) = \int_0^{2\pi} d\sigma \left(a_m{}^n X^m \tilde{P}_n + \tfrac{1}{2} b_{mn} X^m X'^n \right) \,, \tag{5.47}$$

together with

$$\begin{aligned} P_m &= \frac{\delta F}{\delta X^m} = a_m{}^n \tilde{P}_n + b_{mn} X'^n \,, \\ \tilde{X}^m &= \frac{\delta F}{\delta \tilde{P}_m} = X^n a_n{}^m \,, \end{aligned} \tag{5.48}$$

we obtain all transformations of the form (5.25), i.e. general linear transformations acting on the indices $m, n, ...$ and constant shifts of B_{mn}, where now we have integer transformation parameters because of the periodicities of the internal coordinates.

We can also apply the Legendre transformation to a single coordinate X^m only, and use a generating function

$$F^m(X, \tilde{X}) = \int_0^{2\pi} d\sigma X^m \tilde{X}'^m \,, \tag{5.49}$$

with no summation over m understood. It generates the $R \to 1/R$ duality for the single compact coordinate X^m. This completes our identification of all $O(d, d; \mathbb{Z})$ transformations with canonical transformations. Further details on different classes of $O(d, d; \mathbb{Z})$ transformations may be found in [98].

One thing to note is that a transformation rule for the dilaton like the one that is part of the Buscher rules does not follow from this canonical approach. This comes as no surprise because we started with a world-sheet action in the conformal gauge and without the dilaton. There is however more than one way to see that the dilaton has to transform in order to maintain conformal invariance of the sigma model. One way is to look at the vanishing of the β-functions (5.5), or equivalently to look at the nilpotency condition $Q^2_{BRST} = 0$. In order for these to be satisfied, the dilaton has to transform as

$$\tilde{\Phi} = \Phi - \tfrac{1}{2} \log |\det G_{mn}| \,. \tag{5.50}$$

As argued in [4], one of the advantages of the canonical approach is that the dual fields can be obtained in a covariant way for dualities with respect to arbitrary abelian isometries. This makes it easier to obtain global information on the dual manifold. The implementation of the canonical transformation in the path integral is described in [94, 4].

5.3 Strong/weak coupling dualities

In this final section we describe another type of duality symmetry in string theory, namely dualities that act nonlinearly on the string coupling constant. We will be rather brief. The following text is essentially that of [41]. It is based on the papers [42, 19].

At the moment, the interest in strong/weak coupling dualities in field theory and string theory is rapidly growing. Such dualities map the weak coupling region of a theory into the strong coupling region of the same or another theory. Strong/weak coupling dualities may prove to be very important since they provide a way to go beyond perturbation theory. We discuss, at the level of low-energy effective actions, an early example of a strong/weak coupling duality in the context of string theory, namely string/fivebrane duality. This duality was first proposed by Strominger in [182] where a fivebrane soliton solution of the heterotic string effective theory was found. Further support for string/fivebrane duality has been presented in [74, 75].

Whereas a particle couples naturally to a one-form gauge field, a generic p-brane, an extended object that sweeps out a $(p+1)$-dimensional world-volume in space-time, couples to a $(p+1)$-form gauge field. For a string this is the well-known coupling to the two-form field in the sigma model, as in (5.1). In D space-time dimensions, a $(p+1)$-form gauge field is related by Poincaré duality to a $(D-p-3)$-form gauge field carrying the same degrees of freedom. This $(D-p-3)$-form gauge field couples naturally to a $(D-p-4)$-brane and thus we see that the dual of a p-brane is a $(D-p-4)$-brane. The duality transformation interchanges electric and magnetic components of the field strengths. Because of a generalized Dirac quantization argument [152], electric-magnetic dualities are accompanied by a strong/weak coupling interchange.

Apart from string/fivebrane duality in $D = 10$, some other strong/weak coupling dualities have been conjectured. The first of these was the Montonen-Olive particle/particle duality [150, 101] in $D = 4$ supersymmetric field theory. More recently, string/string duality in $D = 6$ has received a lot of attention, see [72, 118, 195, 181, 107]. Many other dualities between different string theories in several dimensions have now been proposed.

Besides strong/weak coupling dualities between different string (or p-brane) theories, some other relations that usually act within one particular string (or p-brane) theory are also known under the name dualities. Target-space duality or T-duality (see [98] for a review) is a symmetry of string theory that relates different background field configurations in which the string dynamics is exactly the same. It is discussed in the previous section. Target-space duality is known to hold in each order of string perturbation theory. On the other hand, S-duality (see [180] and references therein) is a strong/weak coupling duality and therefore can only be true nonperturbatively in the string coupling.

An interesting observation was made in references [176, 34]: S-duality and T-duality seem to get interchanged under string/fivebrane duality. The roles of S-moduli and some of the T-moduli are interchanged in going from the string to the fivebrane formulation. This is also consistent with an earlier observation [74] that string/fivebrane duality interchanges world-sheet and space-time loop expansion parameters.

We first discuss to what extent T and S-dualities are interchanged in the case of heterotic string/fivebrane duality. A more extensive exposition may be found in [42]. Consequently we turn our attention to solutions of the low-energy field theory, and describe how dyonic solutions can be obtained using electric/magnetic duality rotations that are symmetries of the equations of motion in a particular dimensionally reduced effective action [19].

5.3.1 $SL(2,\mathbb{R})$ symmetry of strings and fivebranes

Unfortunately, it is unknown how to quantize the fivebrane starting from the sigma model, and therefore we have no means to derive the fivebrane effective action. What is known however, is that since the fivebrane couples to a six-form, this effective action must involve a six-form gauge field. A natural candidate for the fivebrane effective action then comes to mind: $D = 10$ $N = 1$ supergravity in the dual formulation [59] which contains a six-form gauge field $A^{(6)}$ instead of the two-form gauge field B. This seems to be the best guess for the fivebrane effective action. Another problem is that it is unknown how heterotic fivebranes couple to vector fields. However, we will again adopt the point of view that the low-energy effective action is obtained by applying the standard dualization procedure to the two-form field of $D = 10$ $N = 1$ supergravity coupled to Yang-Mills fields A. Let us briefly explain this dualization procedure. First, the 3-form field strength $H = dB + A \wedge F$ is regarded as an independent unconstrained field in the usual $D = 10$ $N = 1$ action, thus one ignores its expression in terms of B and the Yang-Mills Chern-Simons term. Instead, the Bianchi identity of H is imposed by adding a Lagrange multiplier term to the Lagrangian:

$$\mathcal{L}(B, ...) \rightarrow \mathcal{L}(H, ...) + A^{(6)} \wedge (dH - F \wedge F)\,, \tag{5.51}$$

where $A^{(6)}$ is the six-form Lagrange multiplier and F is the Yang-Mills field strength. Using the equation of motion for $A^{(6)}$, we recover the Bianchi identity for H and get back to the original formulation. However, if we eliminate H by its equation of motion, we obtain an action in terms of the six-form gauge potential $A^{(6)}$ instead of B. As we see from (5.51), the Chern-Simons term in the original formulation becomes a topological term in the dual formulation.

The heterotic string dimensionally reduced to $D=4$ has an $SL(2,\mathbb{R})$ invariance of the equations of motion, though not of the action. This invariance is related to S-duality, which consists of transformations belonging to the discrete subgroup $SL(2,\mathbb{Z})$ believed to be a symmetry of the full string theory[11]. In order to see the $SL(2,\mathbb{R})$ S-duality invariance for the fivebrane, we dimensionally reduced (the bosonic part of) the dual action to four dimensions. This reduction was first done by Chamseddine in [59] where the emphasis was on properties of the resulting scalar potential, in view of possible supersymmetry breaking. In [176], the reduction of the bosonic part of the dual ten-dimensional action without vector fields was performed, and an $SL(2,\mathbb{R})$ symmetry of the *action* was recognized. We now describe the results of [42], in which we performed a dimensional reduction including vector fields and investigated the resulting symmetries in four dimensions.

Let us compare the dimensional reductions of the two-form (string) and six-form (fivebrane) formulations. In the table below we compare the four-dimensional fields[12] coming from the two-form and six-form gauge fields. We use $\mu, \nu, ...$ as $D=4$ space-time indices and $m, n, ...$ as indices belonging to the six internal coordinates.

two-form version	six-form version
$B_{\mu\nu}$ dualized to λ_1	$\lambda_1 \equiv \varepsilon^{m_1 \dots m_6} A^{(6)}_{m_1 \dots m_6}$
B_{mn}	$A_{\mu\nu mn} \equiv \varepsilon_{mn}{}^{m_1 \dots m_4} A^{(6)}_{\mu\nu m_1 \dots m_4}$ dualized to B_{mn}
B^m_μ	$B^m_\mu \equiv \varepsilon^{m m_1 \dots m_5} A^{(6)}_{\mu m_1 \dots m_5}$

Table 4. *Four-dimensional fields arising from the ten-dimensional antisymmetric tensor fields in the two and six-form formulations.*

In the table it is indicated that the two-forms in four dimensions have been dualized to scalars. In the two-form formulation this makes the $SL(2,\mathbb{R})$ symmetry of the equations of motion manifest. The pseudo-scalar λ_1 is usually referred to as the axion. As could be expected from the dualizations in the table above compared to the dualization in $D=10$, the net difference between the two resulting four-dimensional actions boils down to a dualization of the six winding vector fields B^m_μ.

Let us compare the symmetries of both actions. We will not write down the full actions here, they can be found for example in [180] (string action) and [42] (fivebrane action including Yang-Mills vector fields). The four-dimensional string action is also given in equation (5.20) where the two-form field is not dualized. Let us first assume that there are no vector fields in $D=10$.

In the string action, the kinetic terms for the vector fields that come from the dimen-

[11] For references, see [180].

[12] More precise formulas for the four-dimensional fields that ensure correct $D=4$ transformation properties can be found in [176]. See also the discussion on dimensional reduction in section 2.1.1.

sional reduction take the form[13] (schematically)

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\text{string}} \sim \lambda_2 \mathcal{F}^T LMLF + \lambda_1 \mathcal{F}^T L\ {}^*\mathcal{F}\,, \tag{5.52}$$

where

$$\mathcal{F} = \begin{pmatrix} F^m(A) \\ F^m(B) \end{pmatrix} \tag{5.53}$$

is the multiplet of twelve vector fields, six fields A_μ^m arising from the 10d metric (Kaluza Klein vector fields), and six fields B_μ^m arising from the 10d antisymmetric tensor (winding vector fields). The Hodge star acts as ${}^*F_{\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$. As in equation (5.23), M is the symmetric $O(6,6)$ matrix that parametrizes the coset $\frac{O(6,6)}{O(6)\times O(6)}$ of scalar fields G_{mn} and B_{mn} coming from the 10d metric and 10d antisymmetric tensor fields, and the constant 12×12 matrix L is the $O(6,6)$ metric as given in (5.22). The scalars λ_2 and λ_1 parametrize the coset $SL(2,\mathbb{R})/U(1)$ and are usually combined into the axion/dilaton field $S = \lambda_1 + i\lambda_2$ where λ_1 is the axion and $\lambda_2 = e^{-\phi}$ is the inverse string coupling as ϕ is the dilaton. This action is invariant under $O(6,6)$ transformations that act on the vector and scalar fields as

$$\mathcal{F} \to \Omega\mathcal{F}\,, \qquad M \to \Omega M \Omega^T\,. \tag{5.54}$$

We see that Kaluza-Klein and winding electric charges are transformed into one another by T-duality. The axion and dilaton fields are invariant under these transformations. There is, however, an additional symmetry of the *equations of motion*. This is an $SL(2,\mathbb{R})$ invariance, related to S-duality, that acts on the S field as

$$S \to \frac{dS+c}{bS+a}\,, \qquad ad - bc = 1\,. \tag{5.55}$$

Alternatively, in terms of the symmetric $SL(2,\mathbb{R})$ matrix

$$\mathcal{M} = \frac{1}{\lambda_2}\begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & \lambda_1^2+\lambda_2^2 \end{pmatrix}\,, \tag{5.56}$$

the transformation law is

$$\mathcal{M} \to \omega\mathcal{M}\omega^T\,, \qquad \omega = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,. \tag{5.57}$$

Simultaneously, $SL(2,\mathbb{R})$ acts on the vector fields by duality rotations[14],

$$\begin{pmatrix} {}^*\mathcal{F} \\ \mathcal{G} \end{pmatrix} \to \omega \begin{pmatrix} {}^*\mathcal{F} \\ \mathcal{G} \end{pmatrix}\,, \tag{5.58}$$

where $\mathcal{G}$ is the equation of motion tensor $\mathcal{G}^a_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}L^a{}_b\frac{\partial\mathcal{L}}{\partial\mathcal{F}_b^{\mu\nu}}$. Indices a, b label the twelve vector fields. Bianchi identities and equations of motion are rotated into each

[13] We work in the Einstein frame, in which the Einstein-Hilbert action takes its canonical form.
[14] For a general treatment of duality rotations of vector fields, see [93]

other which implies that it is only an on-shell symmetry, not a symmetry of the action. The canonical (Einstein) metric is invariant under $O(6,6) \times SL(2,\mathbb{R})$. We see that S-duality transforms Kaluza-Klein electric (magnetic) charges and winding magnetic (electric) charges into each other.

In the $D=4$ fivebrane action, the relevant terms are

$$\frac{1}{\sqrt{-g}}\mathcal{L}_{\text{fivebrane}} \sim G_{mn}\mathcal{F}^{m\,T}\mathcal{L}\mathcal{M}\mathcal{L}\mathcal{F}^n + B_{mn}\mathcal{F}^{m\,T}\mathcal{L}\,{}^*\mathcal{F}^n\,, \tag{5.59}$$

where the constant 2×2 matrix $\mathcal{L}$ is the $SL(2,\mathbb{R})$ invariant metric, $\omega^T\mathcal{L}\omega = \mathcal{L}$. We clearly see that the roles of T-moduli (G_{mn}, B_{mn}) and S-moduli (λ_2, λ_1) are interchanged as compared to the string case. This also means that now $SL(2,\mathbb{R})$ is a symmetry of the action, acting as

$$\mathcal{F} \to \omega\mathcal{F}\,, \qquad \mathcal{M} \to \omega\mathcal{M}\omega^T\,. \tag{5.60}$$

The $O(6,6)$ transformations now act through duality rotations and therefore constitute a symmetry of the equations of motion only. We see that now $SL(2,\mathbb{R})$ transforms Kaluza-Klein electric fields and winding electric fields into each other. Thus there are strong indications for an interchange of T and S-dualities [176].

Now we include the vector fields in the ten-dimensional effective actions. We assume that n additional abelian vector fields are present after dimensional reduction to $D=4$. That makes the total number of vector fields $12+n$. Then it is known for the string action that the $O(6,6)$ symmetry of the action can be extended to an $O(6,6+n)$ symmetry of the action. $SL(2,\mathbb{R})$ remains a symmetry of the equations of motion which now also involves duality rotations of the additional vector fields.

For the fivebrane action, however, things change, as was noted before in [176]. The $SL(2,\mathbb{R})$ invariance is no longer a symmetry of the action but remains a symmetry of the equations of motion only. This is because of its action on the vector fields:

$$\begin{pmatrix} F(A) \\ F(B) \end{pmatrix} \to \omega \begin{pmatrix} F(A) \\ F(B) \end{pmatrix}\,, \qquad \begin{pmatrix} {}^*F(V) \\ G(V) \end{pmatrix} \to \omega \begin{pmatrix} {}^*F(V) \\ G(V) \end{pmatrix}\,, \tag{5.61}$$

where $F(V)$ are the field strength tensors of the additional vector fields and $G(V) = -\frac{2}{\sqrt{-g}}\frac{\partial\mathcal{L}}{\partial F(V)}$. The additional vector fields transform by duality rotations under $SL(2,\mathbb{R})$. Therefore, it can only be an equation of motion symmetry. Thus, $SL(2,\mathbb{R})$ no longer acts as a pure T-duality symmetry transforming electric Kaluza Klein, winding and gauge fields into each other. Rather, it acts on the gauge vector fields as electric-magnetic duality rotations. It can no longer be interpreted as a T-duality symmetry. The other symmetry, $O(6,6+n)$, remains an invariance of the equations of motion.

One may try to dualize some of the vector fields into their duals in order to obtain a manifestly $SL(2,\mathbb{R})$ invariant action. However, this is impossible since there always remain vector fields on which $SL(2,\mathbb{R})$ must be realized by duality rotations. What we conclude is that $SL(2,\mathbb{Z})$ cannot be an ordinary T-duality symmetry of the fivebrane since it involves electric-magnetic duality rotations. However, we see no reason to regard this as evidence against string/fivebrane duality.

Suggestions for making an $SL(2,\mathbb{R})$ invariant $D=4$ action including vector fields have been presented in [176]. The price to pay is that one has to give up manifest general covariance. More recently, a construction involving more auxiliary fields has been shown to yield an $SL(2,\mathbb{R}) \times O(6,22)$ manifestly general covariant action [154].

It has been shown in [72] that the interchange of T and S-dualities is more clearly realized in $D=6$ string/string duality. It seems that at least until more knowledge about quantization of fivebranes is available, a study of string/string duality will be more fruitful. However, $D=10$ string/fivebrane duality and $D=6$ string/string duality might be the same thing, the latter being a six-dimensional version of the former.

5.3.2 String and fivebrane solutions of low-energy effective theories

Classical solutions of the low-energy effective theories may provide further clues for duality. In the context of string/fivebrane duality, it is known that a solitonic fivebrane solves the heterotic string equations of motion [182]. Also, the dual fivebrane (six-form) theory admits a solitonic string [73] as a classical solution. For a recent review on string solitons, see [76].

It is known that upon reducing the heterotic string effective action to six dimensions, ignoring all vector fields, there is a $\mathbb{Z}_2$ symmetry of the equations of motion. This is the string/string duality transformation [72, 118, 195, 181, 107]

$$\phi' = -\phi\,, \qquad H' = e^{-2\phi}\,{}^*H\,, \tag{5.62}$$

in the Einstein frame. It is a duality rotation of the antisymmetric tensor field, where the Bianchi identity and the equation of motion are interchanged. This maps the fundamental string solution [64] into the solitonic string which is the ten-dimensional fivebrane solution with four spatial directions wrapped around T^4. When vector fields are included, the transformation (5.62) maps the heterotic string on T^4 to the type IIA string on K3 [118, 195, 181, 107].

Type II string effective actions dimensionally reduced to D dimensions have a larger non-compact global symmetry group[15] than type I actions. For example, in six dimensions the U-duality group is $SO(5,5)$ (see [184]). In particular, these transformations involve duality rotations of the two-form fields (of which there are five in $D=6$). It is then interesting to see whether we can use this to generate dyonic solutions of the equations of motion. This can indeed be done, and we will see that in the type II theory one can interpolate between the electrically charged fundamental string and the magnetically charged solitonic string. We will now show this using a simple $D=6$ model derived from the type IIB string. This was described in [19].

It is known that the field equations of $D=10$ type IIB supergravity[16] cannot be derived from a covariant action. This is due to the presence of the self-dual four-

[15] The corresponding conjectured discrete symmetry of the full string theory is named U-duality [118].

[16] We use formulas given in [24] in which also the relation between type IIA and type IIB fields in the presence of one isometry is given.

form field $\hat{D}$[17]. It is possible though to write down an action which reproduces the type IIB equations (with all fermionic fields put to zero) except for the self-duality condition $\hat{F}(\hat{D}) = {}^*\hat{F}(\hat{D})$. The equation of motion for $\hat{D}$ derived from this action has to be identical to the Bianchi identity, in order to be consistent with self-duality. Then putting $\hat{F}(\hat{D}) = {}^*\hat{F}(\hat{D})$ will yield the correct type IIB equations. The action we are looking for is given here in the Einstein frame in which the $SL(2,\mathbb{R})$ symmetry of the IIB theory is manifest:

$$\begin{aligned} S_{\rm IIB} &= \int d^{10}x\,\sqrt{-\hat{g}}\,\Big\{-\hat{R} + \tfrac{1}{4}\mathrm{Tr}\left(\partial\mathcal{M}\partial\mathcal{M}^{-1}\right) + \tfrac{3}{4}\hat{\mathcal{H}}^T\mathcal{L}\mathcal{M}\mathcal{L}\hat{\mathcal{H}} \\ &\qquad -\tfrac{5}{6}(\hat{F}(\hat{D}))^2 - \tfrac{1}{96}\frac{\varepsilon}{\sqrt{-\hat{g}}}\hat{D}\hat{\mathcal{H}}^T\mathcal{L}\hat{\mathcal{H}}\Big\}\,, \end{aligned} \tag{5.63}$$

where $\mathcal{H}^T = d\mathcal{B}^T = (H^{(1)}, H^{(2)}) = (dB^{(1)}, dB^{(2)})$ with $B^{(1)}$ and $B^{(2)}$ the Neveu-Schwarz Neveu-Schwarz (NSNS) and Ramond Ramond (RR) antisymmetric tensor fields, respectively. The scalar matrix $\mathcal{M}$ is again given by (5.56) but here the axion is replaced by the RR scalar $\hat{\ell}$: $\lambda = \lambda_1 + i\lambda_2 = \hat{\ell} + ie^{-\hat{\phi}}$. The field strength of the four-form field is given by

$$\hat{F}_{\hat{\lambda}\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}}(\hat{D}) = \partial_{[\hat{\lambda}}\hat{D}_{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}]} + \tfrac{3}{4}\hat{\mathcal{B}}^T_{[\hat{\lambda}\hat{\mu}}\mathcal{L}\hat{\mathcal{H}}_{\hat{\nu}\hat{\rho}\hat{\sigma}]}\,. \tag{5.64}$$

The $SL(2,\mathbb{R})$ symmetry is easily recognized:

$$\mathcal{H} \to \omega\mathcal{H}\,, \qquad \mathcal{M} \to \omega\mathcal{M}\omega^T\,. \tag{5.65}$$

From now on we will refer to this symmetry as $SL(2,\mathbb{R})_{\rm IIB}$. Note that these transformations act nonlinearly on the string coupling constant. It was conjectured in [118] that an $SL(2,\mathbb{Z})$ subgroup is an exact symmetry of the full type IIB string. It involves a $\mathbb{Z}_2$ transformation that inverts the string coupling constant and interchanges NSNS and RR two-form fields.

We now use a simple ansatz for dimensional reduction to $D = 6$. Our goal is to extend the $\mathbb{Z}_2$ symmetry (5.62) to an $SL(2,\mathbb{R})$ symmetry and we will accomplish this by taking into account two additional scalar fields coming from the internal metric and the four-form field $\hat{D}$. Our ansatz for the ten-dimensional fields is

$$\begin{aligned} \hat{g}_{\mu\nu} = e^{-G}g_{\mu\nu}\,, &\qquad \hat{g}_{mn} = -\delta_{mn}e^G\,, \\ \hat{B}^{(i)}_{\mu\nu} = B^{(i)}_{\mu\nu}\,, &\qquad \hat{\ell} = \ell\,, \qquad \hat{\phi} = \phi\,, \\ \hat{D}_{\mu\nu\rho\sigma} = D_{\mu\nu\rho\sigma}\,, &\qquad \hat{D}_{mnpq} = D_{mnpq}\,, \end{aligned} \tag{5.66}$$

with all other components of ten-dimensional fields vanishing. Unhatted fields are six-dimensional. The four-form field $D_{\mu\nu\rho\sigma}$ is dual to a scalar in $D = 6$. This scalar is related by the self-duality constraint to $D \equiv \varepsilon^{mnpq}D_{mnpq}$. Thus we obtain one scalar from the nonzero components of $\hat{D}$ in the ansatz (5.66).

[17] We put hats again on ten-dimensional fields.

We obtain the following six-dimensional action:

$$
\begin{aligned}
S &= \int d^6x\sqrt{-g}\Big\{-R + \frac{2\partial\lambda\partial\bar{\lambda}}{(\lambda-\bar{\lambda})^2} + \frac{2\partial\kappa\partial\bar{\kappa}}{(\kappa-\bar{\kappa})^2} \\
&\quad + \kappa_2\mathcal{H}^T\mathcal{L}\mathcal{M}\mathcal{L}\mathcal{H} + \kappa_1\mathcal{H}^T\mathcal{L}\,{}^*\mathcal{H}\Big\}\,,
\end{aligned} \tag{5.67}
$$

where

$$
\kappa = \kappa_1 + i\kappa_2 = \tfrac{1}{8}D + \tfrac{3}{4}ie^{2G}\,. \tag{5.68}
$$

Now we observe that besides $SL(2,\mathbb{R})_{IIB}$, there is another $SL(2,\mathbb{R})$ invariance, namely the one that acts on the $SL(2,\mathbb{R})/U(1)$ coset described by κ. This $SL(2,\mathbb{R})$ is different in character from $SL(2,\mathbb{R})_{IIB}$ because it acts through duality rotations on the two-form fields:

$$
\kappa \to \frac{p\kappa+q}{r\kappa+s}\,, \qquad \mathcal{H} \to (r\kappa_1+s)\mathcal{H} + r\kappa_2\mathcal{M}\mathcal{L}\,{}^*\mathcal{H}\,. \tag{5.69}
$$

This symmetry is very similar to S-duality in the $D=4$ heterotic string. We will refer to this symmetry group as $SL(2,\mathbb{R})_{EM}$ to distinguish it from $SL(2,\mathbb{R})_{IIB}$ that was already present in ten dimensions.

We may now use $SL(2,\mathbb{R})_{IIB}\times SL(2,\mathbb{R})_{EM}$ as solution generating transformations. To this end we apply the most general $SL(2,\mathbb{R})_{\rm IIB}\times SL(2,\mathbb{R})_{EM}$ transformation, with parameters a,b,c,d ($ad-bc=1$) and p,q,r,s ($ps-qr=1$), respectively, to the neutral fivebrane solution [74, 55], reinterpreted as a $D=6$ solitonic string,

$$
\begin{aligned}
&ds^2 = (dx^0)^2 - (dx^1)^2 - e^{2\phi}\delta_{ab}dx^a dx^b\,, \\
&H^{(1)}_{abc} = \tfrac{2}{3}\varepsilon_{abcd}\partial^d\phi\,,
\end{aligned} \tag{5.70}
$$

where $a,b,\ldots$ are indices for the four transverse coordinates. The result is, written in the string frame,

$$
\begin{aligned}
ds^2 &= A\left[(dx^0)^2 - (dx^1)^2\right] - Ae^{2C}\delta_{ab}dx^a dx^b\,, \\
\begin{pmatrix} H^{(1)} \\ H^{(2)} \end{pmatrix} &= \begin{pmatrix} asH - \frac{3}{4}bre^{-2C}\ {}^*H \\ csH - \frac{3}{4}dre^{-2C}\ {}^*H \end{pmatrix}\,, \\
\lambda &= \ell + ie^{-\phi} = \frac{ac+bde^{-2C}}{a^2+b^2e^{-2C}} + i\frac{e^{-C}}{a^2+b^2e^{-2C}}\,, \\
\kappa &= \tfrac{1}{8}D + \tfrac{3}{4}ie^{2G} = \frac{qs+\frac{9}{16}pre^{-2C}}{s^2+\frac{9}{16}r^2e^{-2C}} + \tfrac{3}{4}i\frac{e^{-C}}{s^2+\frac{9}{16}r^2e^{-2C}}\,,
\end{aligned} \tag{5.71}
$$

where A and H_{abc} are functions of C,

$$
\begin{aligned}
A &= \sqrt{(a^2+b^2e^{-2C})(s^2+\tfrac{9}{16}r^2e^{-2C})}\,, \\
H_{abc} &= \tfrac{2}{3}\varepsilon_{abcd}\partial^d C\,,
\end{aligned} \tag{5.72}
$$

and C depends only on the transverse coordinates x^a and satisfies $\Box e^{2C}=0$.

A characteristic feature of the above dyonic solutions is that nonzero RR fields are needed in order for the solution to carry electric as well as magnetic charges with respect to the two-form fields. See [19] for more details and for a discussion on how these solutions may be interpreted in ten dimensions. Let us conclude by identifying a few well-known purely electric or magnetic solutions:

- We start from the ten-dimensional neutral fivebrane solution (5.70), corresponding to the identity transformation of $SL(2,\mathbb{R})_{IIB} \times SL(2,\mathbb{R})_{EM}$, $a = d = p = s = 1$, $b = c = q = r = 0$.
- Applying the non-diagonal $SL(2,\mathbb{R})_{IIB}$ transformation ($b = -c = 1$, $a = d = 0$) one obtains (reinterpreted in $D = 10$) a fivebrane solution with a non-trivial RR two-form field. The string coupling is mapped to its inverse.
- The $\mathbb{Z}_2$ string/string duality (5.62) corresponds to choosing $b = -c = -\frac{4}{3}q = \frac{3}{4}r = 1$ and $a = d = p = s = 0$. This yields the fundamental string solution [64] which has no RR fields.
- Applying the non-diagonal $SL(2,\mathbb{R})_{EM}$ transformation ($-\frac{4}{3}q = \frac{3}{4}r = 1$, $p = s = 0$), one obtains a solution with non-trivial RR two-form field. This is the $D = 6$ version of the other type IIB string solution which may also be regarded as a solution of the type I string in the context of a heterotic/type I duality, see [63] for recent discussions.

We have not investigated any further properties of the solutions (5.71). However, we know that they break half of the type II supersymmetries and therefore saturate a Bogomolnyi bound because it is known that the two string and fivebrane solutions just described do, and both $SL(2,\mathbb{R})$ transformations are consistent with the full set of type II supersymmetries.

It would be interesting to study the properties of the dyonic solutions described above in more detail. For example, the fact that both electrically and magnetically charged solutions exist, implies that only discrete subgroups of the $SL(2,\mathbb{R})$ transformations give rise to possible (dyonic) strings. In [175], electrically charged string solutions of the $D = 10$ type IIB theory have been considered in detail. Here the $SL(2,\mathbb{R})_{IIB}$ symmetry of the effective action is used together with a charge quantization rule (due to the existence of magnetic fivebrane solutions) to obtain an $SL(2,\mathbb{Z})_{IIB}$ multiplet of $D = 10$ string solutions with NSNS and/or RR charges. Other work on dyonic p-branes has appeared in [122], in which dyonic membranes are constructed. Membranes may carry both electric and magnetic charges in $D = 8$. A general discussion of p-brane solutions and their charges is given in [189].

At the moment of writing, the field of strong/weak coupling dualities is developing very fast. Let me conclude this chapter by mentioning one of the recent developments which also has some connection with the work presented in this subsection.

The solutions with nonzero RR charges in type II theories have always been something of a mystery. On the one hand, these solutions are singular which means that they probably need a sigma model source term. On the other hand, however, it is expected that the string sigma model only couples to solutions with NSNS charges, related to the fact that the perturbative string spectrum contains NSNS charged states but no states with RR charges. A solution to this problem is described in [156]. There it is shown that Dirichlet-branes (D-branes), extended objects to which the endpoints of open strings with mixed Dirichlet and Neumann boundary conditions are confined, carry a complete set of electric and magnetic RR charges. It is then argued that D-branes are intrinsic to type II string theory and are the RR sources required by string duality.

Chapter 6

Discussion

The main part of this thesis, and indeed the main part of my research in the last four years, has been concerned with the investigation of W-strings. Whereas the world-sheet description of ordinary strings is based on conformal invariance, W-strings are based on higher-spin extended conformal invariance. We only considered a special class of W-strings: those based on free scalar realizations of W_N algebras. One of our main results is that we found redefinitions of the classical constraint algebras associated with W_N gauge symmetry that simplify the BRST analysis. These redefinitions can also be interpreted as canonical transformations in the extended phase space including the ghosts. In the case of the critical W_3 string [135], the non-critical W_3 string [21, 18] and the critical W_4 string [20, 40] the simplifications have also been found to apply at the quantum level. In particular, this elucidated a similarity of critical W_N strings to (non-critical) Virasoro strings. From this and especially from a lot of work that had been done before on the W_3 string, the overall picture of W-strings, at least concerning a number of issues such as the spectrum, seems to be clear. Summarizing, we can say the following about the status of these W-strings:

- The only known realizations with critical value of the central charge in terms of scalar fields only, are the Miura realizations or multi-scalar generalizations thereof. In particular, it is impossible to build a genuine *critical* W-string based on string coordinates without background charges.

- It is not clear whether W-strings can be interesting from a phenomenological point of view. For example, space-time interpretations are not straightforward like those for the bosonic and superstrings. However, W-strings and especially non-critical W-strings may allow one to study two-dimensional (W-)gravity coupled to matter with central charge greater than one, or non-critical strings in $D > 2$. An example is the $D = 4$ W_3 string discussed extensively in [47].

- The spectra of critical W-strings show a remarkable resemblance to Virasoro string spectra. This is mostly due to the special structure of the Miura realizations of

W-algebras. Nevertheless, the spectra have quite a rich structure. The spectrum of a critical W_N string is closely connected to the spectra of non-critical W_n strings for all $n = 2, 3, ...N$ where the matter sector is the $(N, N+1)$ unitary W_n minimal model.

- It would be interesting to find different realizations of W-algebras such that the corresponding string spectra do not reduce to several effective Virasoro string sectors. However, this seems impossible to realize with just scalar fields [83]. Other types of realization are then needed. Perhaps such new realizations can be found using the linearizing conformal algebras of [130].

The relations that have been found between W_N strings for different N are in some sense similar to the Berkovits-Vafa construction and its generalizations to hierarchies of superstrings and strings based on linear W-algebras. In both cases, certain strings in special realizations are shown to be related to other strings based on different world-sheet gauge symmetries. In the Berkovits-Vafa construction the relation has been shown to be an equivalence, at least concerning the physical spectrum [121]. In this light, some recent observations in supersymmetric field theory are also worth mentioning. It has been argued for certain $N = 1$ supersymmetric field theories that different dual descriptions of the same theory can have a different gauge symmetry [177]. All this reminds us of the fact that gauge symmetry is not a physical symmetry but rather a symmetry of the description. For example, the Hilbert space of physical states of a theory comes in representations of the global symmetry group but not in representations of any gauge group. Indeed, the gauge symmetry is divided out to obtain the physical spectrum.

Canonical or similarity transformations have proved to be a useful tool to show relations among string theories. Also in the context of duality symmetries canonical transformations may elucidate certain equivalences in string theory (see e.g. [4, 125, 95]). For example, in section 5.2 we described all $O(d, d; \mathbb{Z})$ T-duality transformations for a toroidally compactified string in the language of canonical transformations.

However, canonical transformations in the two-dimensional sigma model description of strings can probably not say much about possible nonperturbative duality transformations. Nowadays there are many examples of nonperturbative strong/weak coupling dualities, and they have passed many non-trivial tests. In section 5.3 we discussed two examples of strong/weak coupling duality. In the first example [42] we studied, at the level of low-energy effective actions, the relation between string S-duality and fivebrane T-duality and vice versa. It turns out that vector fields in the ten-dimensional effective action destroy this relation. See also [176, 180]. In the second example [24] we studied a six-dimensional reduction of the type IIB effective theory. We used an $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ symmetry of this model to generate dyonic string solutions with nonzero Ramond Ramond charges. These solutions interpolate between fundamental string and solitonic string (compactified fivebrane) solutions.

New results on dualities are appearing at a fast rate and now the view is favoured that the known superstring theories correspond to different perturbative descriptions of the same nonperturbative 'superstring' theory. Interestingly, other extended objects and eleven-dimensional supergravity seem to play prominent roles in this picture. For a relatively recent review on string dualities, we refer to [157]. It seems that these developments are already leading to a better understanding of (nonperturbative) string theory.

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Stellingen

bij het proefschrift
Symmetries in String Theory
Harm Jan Boonstra

De suggestie van S.R. Das et al.[1] dat voor W_N strings het equivalent van de $c = 1$ barrière een $c = \frac{6}{N(N+1)}$ barrière is, gaat alleen op voor de speciale manier van materie-koppeling zoals die hier aangenomen wordt. Volgens de nu meer gebruikelijke opvatting van materie-koppeling[2] lijkt het er eerder op dat een dergelijke barrière bij materie centrale lading $c = N$ optreedt.

1. S.R. Das, A. Dhar, S.K. Rama, Int. J. Mod. Phys. A7 (1992) 2295
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Kanonieke of equivalentie-transformaties, en in het bijzonder transformaties waarbij coördinaten en kanoniek geconjugeerde impulsen worden verwisseld, leiden nog vaak tot verrassende vereenvoudigingen of tot onverwachte equivalenties van theorieën. (Zie bijvoorbeeld hoofdstukken 3,4 en 5 van dit proefschrift)

Uit het feit dat er binnen het vakgebied van stringtheorie de laatste tijd geregeld vragen als 'Wat is stringtheorie?' en 'Is stringtheorie een theorie van strings?' opduiken, blijkt dat er hier nog genoeg werk te doen is.

Het valt te betwijfelen of onderstaand citaat op een goede wetenschappelijke instelling duidt.

> What is matter? Never mind.
> What is mind? No matter.
> *Punch* 29 (1855) 19

De vooruitgang van de wetenschap is meestal niet gebaat bij concurrentie die men van elkaar ondervindt of meent te ondervinden.

Vaak is het zo dat hoe zekerder iets wordt uitgedrukt in een stelling hoe minder waarheidsgehalte deze bezit. Het omgekeerde geldt nog vaker.

Fietsen in de bergen weerspiegelt het leven met zijn ups en downs.

Geen stelling is opgewassen tegen "Wat gij niet wilt dat u geschiedt - doe dat ook een ander niet".

Jongeren moeten niet meer gestimuleerd worden om het rijbewijs te halen.

Samenvatting

De elementaire deeltjesfysica heeft als doel de verscheidenheid aan elementaire deeltjes en hun onderlinge wisselwerkingen beter te begrijpen. Elementaire deeltjes zijn de bouwstenen van materie. Vooral elektronen en quarks zijn typische bouwstenen omdat, zoals we weten, atomen uit een door elektronen omringde kern van protonen en neutronen bestaan, waarbij deze laatsten op hun beurt opgebouwd zijn uit quarks. We noemen deeltjes elementair als ze, voor zover bekend, niet opgebouwd zijn uit nog kleinere deeltjes. In deze zin zijn protonen en neutronen dus niet elementair, maar quarks en elektronen wel.

Voor het opstellen en testen van theorieën over het gedrag van elementaire deeltjes zijn experimenten waarbij verschillende deeltjes met elkaar in botsing worden gebracht van groot belang. Tijdens de botsing vinden interacties plaats die tot resultaat kunnen hebben dat andere deeltjes verschijnen en waargenomen worden. Een algemene regel is dat hoe kleiner de structuur is die onderzocht wordt, hoe hoger de energieën van de botsende deeltjes moeten zijn (ze dringen dan dieper in elkaar door). De elementaire deeltjesfysica wordt daarom ook wel hoge-energie fysica genoemd.

De huidige fundamentele theorie van elementaire deeltjes is 'het standaardmodel'. Deze is ontstaan na jaren van speurwerk naar orde en structuur in de experimentele gegevens en na uitgebreide selectie van theoretische modellen. Om het standaardmodel te testen wordt het gebruikt voor het voorspellen van uitkomsten van nieuwe experimenten. Tot nu toe heeft het standaardmodel deze beproevingen met succes doorstaan, al zijn er naar het zich laat aanzien recentelijk enige afwijkingen gevonden.

Het standaardmodel beschrijft alle bekende elementaire deeltjes, waaronder quarks en elektronen, en hun onderlinge wisselwerking ten gevolge van drie fundamentele interacties of krachten. Dit zijn de elektromagnetische, zwakke en sterke interacties. Deze laatste bijvoorbeeld, is verantwoordelijk voor het bijeenhouden van de deeltjes in een atoomkern. In het standaardmodel zijn de drie krachten zeer nauw gerelateerd aan symmetrieën, via het zogenaamde ijkprincipe. Een symmetrie houdt in dat de vergelijkingen van de theorie dezelfde vorm behouden na bepaalde wiskundige bewerkingen (symmetrietransformaties). Een eenvoudig voorbeeld is translatiesymmetrie, waarbij de vergelijkingen dezelfde vorm behouden na willekeurige verschuivingen van het gebruikte assenstelsel. Dit correspondeert met de observatie dat de natuurwetten overal in het heelal hetzelfde zijn. Een ander voorbeeld is rotatiesymmetrie, waarbij de vergelijkingen dezelfde vorm behouden na willekeurige rotaties van het assenstelsel. Bij de ijksymmetrieën van het standaardmodel werken de symmetrietransformaties niet op het coördinatenstelsel van de ruimte, maar zijn het een soort rotaties in een interne ruimte van variabelen die de verschillende elementaire deeltjes representeren. We kunnen een symmetrietransformatie meestal beschouwen als een verandering van referentiesysteem waarbij de theorie onveranderd blijft. Het kenmerkende van ijksymmetrie is dat die verandering van referentiesysteem in elk punt van de ruimte en op elk tijdstip onafhankelijk gekozen mag worden. Het ijkprincipe houdt nu in dat het aantal mogelijke interacties tussen de verschillende deeltjes beperkt is omdat deze allemaal aan de symmetrie moeten gehoorzamen. Aangezien deze beperkingen experimenteel worden bevestigd, is ijksymmetrie een belangrijke leidraad gebleken op weg naar betere theorieën in de elementaire deeltjesfysica.

Het standaardmodel is een quantumtheorie. Sinds het begin van deze eeuw is bekend dat de natuurkunde op microscopische schaal, en dus zeker de elementaire deeltjesfysica, bijzondere eigenschappen bezit die we normaal gesproken niet zien op voor ons meer natuurlijke lengteschalen. Dit zijn quantummechanische eigenschappen. Op zeer kleine schaal wordt onder meer duidelijk dat de golven in een elekromagnetisch veld gequantiseerd zijn in pakketjes met bepaalde energieën. Deze kunnen we interpreteren als deeltjes. Voor het elektromagnetische veld worden deze deeltjes fotonen genoemd. In het standaardmodel worden alle interacties verklaard door de uitwisseling van dit soort elementaire deeltjes, ook wel ijkbosonen genoemd.

Het is bekend dat het standaardmodel geen volledige theorie kan zijn. Het bevat namelijk geen enkele referentie aan de zwaartekracht, terwijl een volledige theorie alle elementaire deeltjes inclusief alle mogelijke wisselwerkingen zou moeten beschrijven. Nu is het zo dat bij botsingsexperimenten met elementaire deeltjes helemaal niets van de zwaartekracht te merken is; deze is gewoon veel te zwak en valt in het niet bij de andere krachten. Voor de zwaartekracht beschikken we over een afzonderlijke theorie: Einstein's algemene relativiteitstheorie. Deze verklaart gravitatie in termen van de geometrie van de vier-dimensionale ruimte-tijd. Zo geeft het een zeer nauwkeurige beschrijving van de bewegingen van hemellichamen ten opzichte van elkaar, waarvoor overigens de veel oudere theorie van Newton ook meestal een zeer goede benadering is. Dat op grote schaal (tussen grote objecten en over grote afstanden) de zwaartekracht het kennelijk wint van de andere krachten kan begrepen worden uit het feit dat de zwaartekracht uitsluitend aantrekkend is (en niet zoals bijvoorbeeld de elektromagnetische kracht soms afstotend en soms aantrekkend afhankelijk van de ladingen) en zijn sterkte maar langzaam afneemt met de afstand.

Een van de grote vraagstellingen binnen de theoretische hoge-energie fysica is hoe deeltjesfysica (het standaardmodel) en gravitatie (de algemene relativiteitstheorie (ART)) verenigd kunnen worden in één theorie. Dat ze niet zomaar te verenigen zijn blijkt als we bedenken dat de ART een klassieke, dat wil zeggen niet-quantummechanische theorie is. Als je echter elementaire deeltjes zou kunnen versnellen tot extreem hoge energieën, dan zou de zwaartekracht op een gegeven moment significant moeten worden, want volgens de ART koppelt de zwaartekracht net zo goed aan energie als aan massa. Ook verwacht men dat quantummechanische effecten van de zwaartekracht dan een belangrijke rol gaan spelen. Dergelijke extreme energieën (in de buurt van de zogenaamde Planck energie of Planck massa) zijn weliswaar niet te realiseren in deeltjesversnellers, maar waren hoogstwaarschijnlijk bij het ontstaan van het heelal alomtegenwoordig. Natuurlijk heeft men geprobeerd, en probeert men nog steeds, een quantummechanische versie van de ART te construeren, zij het zonder enige experimentele aanwijzing. Vanwege grote wiskundige problemen is dit tot nu toe niet gelukt.

Een andere aanwijzing voor het niet compleet zijn van het standaardmodel is de aanwezigheid van vele parameters in de theorie. Deze parameters, bijvoorbeeld de massa's van de elementaire deeltjes, moeten 'met de hand in de theorie worden gestopt', zodanig dat er overeenstemming is met experimentele bevindingen. Men kan zich voorstellen dat hoe meer van dit soort parameters een theorie heeft, des te kleiner haar voorspellende kracht is. Het zou fraaier zijn als uiteindelijk deze parameters hun waarde uit de theorie zelf krijgen.

Op het ogenblik wordt er veel theoretisch onderzoek gedaan naar wat waarschijnlijk de beste kandidaat is voor een volledige theorie die de zwaartekracht met de deeltjesfysica combineert: stringtheorie. Wellicht het belangrijkste idee van stringtheorie is dat het concept van puntvormig deeltje (zoals we ons een elementair deeltje meestal voorstellen) wordt vervangen door dat van een string of snaar, d.w.z. een één-dimensionaal object. In tegenstelling tot een puntdeeltje heeft een snaar inwendige vrijheidsgraden, omdat hij op tal van manieren kan trillen. De verschillende trillingstoestanden van een snaar kunnen dan verschillende elementaire deeltjes representeren, zoals verschillende trillingen van een vioolsnaar verschillende tonen produceren. Het zal duidelijk zijn dat een snaartje dat alle elementaire deeltjes verenigt minuscuul klein moet zijn, ook al omdat het nog nooit gedetecteerd is.

Door bestudering van de mogelijke trillingstoestanden van een string kan uitgerekend worden welke deeltjes met welke eigenschappen door stringtheorie beschreven worden. De verzameling van deze deeltjes wordt het spectrum genoemd. Een van de grote verrassingen van stringtheorie is dat het spectrum een deeltje bevat dat precies de eigenschappen heeft van het graviton, het quantum (gequantiseerd energiepakketje of deeltje) van het zwaartekrachtsveld. Bovendien lijkt stringtheorie gevrijwaard te zijn van de moeilijkheden die optreden bij pogingen om rechtstreeks een quantummechanische versie van de theorie van de zwaartekracht te construeren. Naast het graviton bevat het spectrum nog een oneindig aantal andere deeltjes. Slechts een eindig aantal hiervan is massaloos, en daaronder bevinden zich ook ijkbosonen, de deeltjes die geassocieerd worden met krachten, zoals die van het standaardmodel.

Stringtheorie zou dus in principe een volledige theorie voor alle elementaire deeltjes en alle krachten kunnen zijn. Een duidelijk probleem is echter, zoals hiervoor al gebleken is, dat de meeste voorspellingen van stringtheorie (nog) niet experimenteel getest kunnen worden. Daarvoor zijn de energieën waarbij karakteristieke string effecten of quantumgravitatie effecten optreden veel te hoog. Het is wèl mogelijk om een benadering van stringtheorie te bestuderen die alleen geldig is bij lage energieën. Deze zou voorspellingen kunnen opleveren die ook experimenteel getest kunnen worden. Voordat het zover is moeten er nog wel enige technische (wiskundige) moeilijkheden overwonnen worden.

Stringtheorie heeft in tegenstelling tot het standaardmodel geen vrije parameters. Dat betekent dat alle waarden van bijvoorbeeld massa's van deeltjes, maar zelfs het aantal dimensies van de ruimte waarin we leven, zouden moeten volgen uit de vergelijkingen van de theorie zelf. Dat is op zich een gewenste eigenschap, maar het wijst ons wel op tekortkomingen van de huidige formulering van stringtheorie. Het standaardmodel met al z'n parameters zou op de een of andere manier als vier-dimensionale lage-energielimiet van stringtheorie tevoorschijn moeten komen. Nu blijkt inderdaad dat stringtheorie oplossingen heeft die veel op het standaardmodel lijken. Het probleem is dat er nog veel en veel meer oplossingen zijn en dat voor stringtheorie al deze oplossingen even goed of even slecht zijn. De huidige formulering van stringtheorie lijkt dan ook niet compleet te zijn. Bovendien is deze formulering gebaseerd op storingsrekening. Processen waarbij strings wisselwerken kunnen daarin bij benadering berekend worden. Voor elke verbetering in de benadering moet een extra term uitgerekend worden, corresponderend met de uitwisseling van een extra string tijdens het proces. Het is een onmogelijke opgave om al deze correctietermen te berekenen (er zijn er in principe oneindig veel), en in de praktijk wordt dan ook meestal alleen naar de laagste orde benadering gekeken. Deze benadering is beter naarmate de koppeling tussen strings (de sterkte van hun interactie) kleiner is. Maar ongetwijfeld zijn er ook belangrijke effecten die zowiezo niet met behulp van storingsrekening kunnen worden berekend, zogenaamde niet-storingseffecten. Als we bijvoorbeeld in detail willen weten welke voorspellingen stringtheorie doet bij lage energieën, of hoe eventueel het standaardmodel uiteindelijk tevoorschijn komt, dan zullen we eerst een beter begrip van niet-storingseffecten moeten hebben.

Tot zover de inleiding. Nu zal ik vertellen waar dit proefschrift nou eigenlijk over gaat. Symmetrieën spelen een belangrijke rol in het standaardmodel en in de algemene relativiteitstheorie. Stringtheorie combineert symmetrieën van standaardmodel en ART in een nog veel grotere groep van symmetrieën. De verwachting is dat een beter begrip van deze symmetrieën kan bijdragen tot een meer volledige formulering van stringtheorie. In dit proefschrift worden enkele van de symmetrieën in stringtheorie onderzocht. Het grootste deel van het proefschrift gaat over generalizaties van stringtheorie gebaseerd op uitbreidingen van een belangrijke onderliggende symmetrie. Stringtheorie is namelijk niet uniek, zo lijkt het. Het meest bekend zijn de superstring modellen. Dit zijn de beste kandidaten voor een volledige elementaire deeltjestheorie. Maar er zijn meer mogelijkheden.

Zoals het traject van een deeltje dat zich in een bepaalde ruimte voortbeweegt een lijn is, is het traject van een string een twee-dimensionaal oppervlak. Stringtheorie kan dan geformuleerd worden op zo'n twee-dimensionaal oppervlak. Dit wordt uitgebreid beschreven in hoofdstuk 2 van dit proefschrift. Nu is het zo dat de parametrisatie van het twee-dimensionale oppervlak er niet toe mag doen; de theorie moet invariant zijn onder 'herparametrisaties' van dit oppervlak; dit is derhalve een symmetrie (en wel een ijksymmetrie) van stringtheorie. Het blijkt mogelijk te zijn om deze symmetrie uit te breiden. De uitbreidingen zoals beschreven in dit proefschrift worden W-symmetrieën genoemd en de stringtheorie hierop gebaseerd heet W-string. Symmetrieën worden, wiskundig gezien, beschreven door groepen of algebra's. In het geval van de W-symmetrieën zijn deze algebra's (W-algebra's) niet-lineair en dat maakt de analyse ervan nogal ingewikkeld. Om deze situatie te verbeteren hebben we vereenvoudigingen ontwikkeld in de formulering van W-strings. Deze vereenvoudigingen ontstaan na speciale transformaties van de variabelen en hebben ons in staat gesteld meer te weten te komen over onder andere het spectrum van W-strings. Gedetailleerde berekeningen zijn gedaan voor een string gebaseerd op de W_4 algebra. Het berekenen van het spectrum gebeurt in het zogenaamde BRST formalisme. Dit is een formalisme dat geschikt is voor de quantisatie (de constructie van een quantummechanische versie) van theorieën met ijksymmetrie. Het is gebleken dat deze W-strings weliswaar niet qua spectrum beter zijn dan de originele string (ze lijken er zelfs erg op), maar dat ze belangrijk kunnen zijn in verband met de niet-kritische string, een stringmodel met minder vrijheidsgraden waarvoor ook nietstoringsberekeningen gedaan kunnen worden. Hoofdstukken 3 en 4 geven een overzicht van het onderzoek naar W-strings.

Tenslotte handelt hoofdstuk 5 over het tweede deel van mijn onderzoek dat zich toegespitst heeft op een ander soort symmetrie van stringtheorie: de zogenaamde dualiteitssymmetrieën. Het belangrijkste idee van dualiteitssymmetrieën is dat ze twee (of meer) verschillende beschrijvingen van dezelfde theorie aan elkaar relateren. We onderscheiden verschillende soorten dualiteitssymmetrieën. Sommige relateren strings bewegend in verschillende ruimten. Deze T-dualiteitstransformaties relateren onder meer grote en kleine lengteschalen in stringtheorie. In de eerste helft van hoofdstuk 5 leggen we uit hoe T-dualiteit werkt, zowel in de lage-energielimiet van stringtheorie als in de formulering op het twee-dimensionale oppervlak.

Naast T-dualiteit bestaat er een andere groep van dualiteitstransformaties die waarschijnlijk zeer belangrijk is voor een beter begrip van niet-storingseffecten in stringtheorie. Deze noemen we sterk/zwak koppelingsdualiteiten. Ze relateren een stringtheorie met zwakke koppeling (waarvoor storingsrekening goede resultaten geeft) aan een (mogelijk andere) stringtheorie met een sterke koppeling. Dat betekent dat storingsrekening in de ene theorie informatie geeft over sterke koppelingseffecten (niet-storingseffecten) van de andere theorie. Een van de sterk/zwak koppelingsdualiteiten is S-dualiteit. Deze relateert verschillende formuleringen van dezelfde (lage-energielimiet van) stringtheorie waarbij, naast een transformatie van de koppeling, elektrische en magnetische velden worden verwisseld.

De laatste jaren is ook duidelijk geworden dat strings waarschijnlijk niet de enige fundamentele objecten zijn binnen stringtheorie. Er zijn namelijk oplossingen van de vergelijkingen van stringtheorie die objecten voorstellen van andere dimensies, bijvoorbeeld puntdeeltjes (of zwarte gaten) en ook hoger-dimensionale objecten. Deze worden p-branen genoemd, waarbij p het aantal dimensies van het object voorstelt. Voor $p = 0$ is dit een deeltje, voor $p = 1$ een string, voor $p = 2$ een membraan, enzovoort. Een mogelijke sterk/zwak koppelingsdualiteit is string/5-braan dualiteit. Hierbij zou een 5-braan theorie equivalent zijn aan een stringtheorie, maar dan met sterke en zwakke koppeling verwisseld. Wij onderzochten de rol van S-dualiteit en T-dualiteit, onder string/5-braan dualiteit. Er zijn aanwijzingen dat deze dualiteiten verwisseld worden onder string/5-braan dualiteit, maar we hebben gevonden dat dit alleen het geval kan zijn als bepaalde deeltjes uit het spectrum worden weggelaten. Daarnaast hebben we speciale oplossingen van één van de bekende superstringtheorieën (type IIB) onderzocht. Deze oplossingen kunnen zowel elektrische als magnetische ladingen hebben en worden verbonden door verschillende dualiteitssymmetrieën. Dit soort oplossingen kan verdere verbanden tussen de verschillende superstringtheorieën verduidelijken alsook informatie over het niet-storingsspectrum van stringtheorie verschaffen.

Nieuwe ontwikkelingen op het gebied van dualiteit in stringtheorie verschijnen tegenwoordig in een snel tempo. Zo wordt er nu veel onderzoek gedaan naar een mogelijke onderliggende theorie waarvan verschillende ‘reducties’ duaal zijn aan de bekende superstringtheorieën. Er is in ieder geval genoeg reden om optimistisch te zijn over verdere ontwikkelingen op dit gebied. Die zouden kunnen leiden tot een meer volledige formulering van stringtheorie waarin ook niet-storingseffecten beter begrepen kunnen worden.

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