BLACK HOLE INFORMATION IN STRING THEORY:
NON-BPS MICROSTATES AND SUPERSTRATA

by

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To my wife, who has gracefully tolerated the increase in entropy
around the house during the writing of this work.

"What does the Questing Beast look like?"

"Ah, we call it the Beast Glatisant, you know," replied the monarch, assuming a learned air and beginning to speak quite volubly. "Now the Beast Glatisant, or, as we say in English, the Questing Beast—you may call it either," he added graciously—"this Beast has the head of a serpent, ah, and the body of a libbard, the haunches of a lion, and he is footed like a hart. Wherever this beast goes he makes a noise in his belly as it had been the noise of thirty couple of hounds questing.

"Except when he is drinking, of course," added the King.

—T.H. White, *The Once and Future King*

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Contents

List of Figures

List of Tables

Abstract

In this thesis we explore two novel directions in the quest for 3-charge "black hole microstate geometries", which are smooth, horizon-free supergravity solutions in 5 or more dimensions that correspond to the microstates of black holes.

First we find two infinite families of smooth non-BPS microstates using the "floating brane ansatz" method in 5 dimensions, based on a class of Kähler metrics studied by LeBrun. The first set of solutions is based on the LeBrun-Burns subclass, which turn out to have a trivial flux, leading to trivial bubble equations. The second set of solutions is based on the more general LeBrun metrics, which have non-trivial flux, and we find non-trivial bubble equations. In both cases, solutions are asymptotic to warped, rotating $AdS_2 \times S^3$.

Second, we realize two important steps toward the construction of *superstrata*, which are 3-charge, 2-dipole-charge smooth supergravity solutions that fluctuate as an arbitrary function of two variables. In one case, we find solutions that depend on functions of two variables; however they lack the necessary KKM charge to make them smooth. In the second case, we construct *smooth* solutions with KKM charge turned on, but in a restricted class that allows them only to depend on arbitrary functions of one variable. Nevertheless, we show that this one variable can be oriented in an arbitrary way inside a 2-torus, and many sources with different orientations inside the T^2 can be combined via superposition.

Chapter 1

Introduction

Black holes have been an object of wonder and mystery ever since Schwarzschild's[1] original 1916 solution [1] to the vacuum Einstein equations, for a point source of mass M:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r}\right)^{-1}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \tag{1.1}$$

here written in "natural units" where $c \equiv G_N \equiv \hbar \equiv k_B \equiv 1$. It took a long time to understand the key features of this solution that are taught to undergraduates today: that the singularity at $r = 2M$ is merely a coordinate singularity, and actually represents the location of the event horizon; that the event horizon is totally smooth, but is still a "surface of no return" because lightcones there "tip over" too far[2]; and that the true curvature singularity lies at $r = 0$, which is in the *future* of all observers sitting inside $r < 2M$ (the coordinate r being timelike in this region).

For some time it was thought that black holes were mere mathematical curiosities, unlikely to occur in reality. The analytical black hole solutions known for most of the 20th century had a high degree of symmetry (the Schwarzschild one, for example, has

[1]The notion of a dark star with gravity too great for even light to escape was known as early as the 18th century [2], but this was before it was understood that the velocity of light is a) absolute, and therefore cannot be slowed by the gravitational pull of any object, and b) a limit velocity that cannot be reached or exceeded by any massive observer. It is with these two additional facts that the mystery of such a "dark star" becomes so profound.

[2]Or in the Penrose-diagram understanding where a conformal map keeps the lightcones upright, the event horizon, being a null surface, is locally "moving outward at the speed of light", and thus cannot be locally outrun.

exact spherical symmetry). It was thought that the singularity was simply an artifact of this symmetry, and would be absent if the symmetry were spoiled [3]. However, the famous Hawking and Penrose singularity theorems ([4, 5], see also [6, 7]) reveal otherwise: that the development of singularities is actually a generic feature of the theory; that black holes are the most *typical* objects to exist, given enough time for matter to accumulate in one place; and that any asymmetries will be washed out in the process, radiated away as gravitational waves.

This last point seems to imply that all physical black holes, regardless of their original structure, tend toward the same final state as asymmetrical lumps are compressed and their gravitational signatures radiated away. It turns out that this can be made into a mathematical theorem, given certain general assumptions about the matter content: the black hole uniqueness, or "no hair" theorem (see [8] for a review). This theorem states that all *stationary* collapsed configurations (i.e. in equilibrium) in 4-dimensional asymptotically-flat spacetime are subsumed under the usual four black hole solutions: Schwarzschild (electrically neutral, non-spinning), Kerr (spinning), Reissner-Nordström (electrically charged), and Kerr-Newman (electrically charged and spinning). Stated another way, this means that a black hole in 4 dimensions is completely described by its mass M, charge Q, and angular momentum J, and that a unique solution to GR (or technically, GR and the Maxwell equations) results once those quantities are specified.

Black hole thermodynamics

However, this is not the full story. Several famous results of Bekenstein [9], Hawking [10], Bardeen, and Carter [11] indicate that black holes have additional properties. First Bekenstein showed in [9] that one can derive from Einstein's equations a "first law of thermodynamics":

$$\mathrm{d}M = \frac{\kappa}{8\pi}\,\mathrm{d}A_H + \Omega\,\mathrm{d}J + \Phi\,\mathrm{d}Q, \tag{1.2}$$

where the surface gravity κ and the horizon area A_H play the roles of temperature and entropy, respectively (in the work terms, Ω is angular velocity, J is angular momentum, Φ is electric potential, and Q is electric charge). Then in [11] it was shown in fact that these black hole quantities obey laws completely analogous to the usual four laws of thermodynamics, in particular showing that the horizon area A_H (representing the entropy) can never decrease. Then Hawking showed in [10] that using a semiclassical approach that the spacetime surrounding a black hole is filled with thermal radiation of a temperature (measured at infinity)

$$T_{\text{Hawk}} = \frac{\kappa}{2\pi} = \frac{1}{8\pi M}, \tag{1.3}$$

and hence according to Bekenstein's first law of black hole thermodynamics, one can identify the entropy

$$S_{\text{Bek}} = \frac{A_H}{4}. \tag{1.4}$$

But now this is rather mysterious, because the "no hair" theorems tell us that black holes are unique! Yet here the entropy of a black hole clearly exists and is non-zero (in fact it is astronomically large, A_H being measured in square Planck-lengths ℓ_P^2).

Another consequence of the Hawking temperature is that a black hole in otherwise-empty spacetime must, over time, evaporate[3]. This means that everything that falls into a black hole must somehow be radiated out again. This of course contradicts the notion of an event horizon, and we will see that black hole radiation, and eventual evaporation, bring very deep issues to light.

Classically the laws of thermodynamics can be thought of as arising from a statistical ensemble over a more fundamental theory (such as classical mechanics). In

[3]For astrophysical black holes in a spacetime filled with ambient matter and radiation such as our universe, the Hawking temperature is so tiny that a black hole is a net absorber and hence will not evaporate.

statistical mechanics, the entropy S arises out of a degeneracy $W = e^S$ of microstates whose coarse-grained properties (such as temperature, pressure, volume) correspond to the same macroscopic state of the system. For example, a balloon full of air can be described by all the microscopic data such as the positions and momenta of every individual molecule; or these data can be averaged over and we can describe the system in terms of its temperature, pressure, and volume. In the microcanonical ensemble, we can express the entropy as a function of the energy, volume, and charges, and compute other properties of the macrostate in terms of its derivatives.

By analogy, we expect to be able to define a microcanonical ensemble for black holes. The Bekenstein entropy should correspond to some degeneracy of microstates

$$W = e^{S_{\text{Bek}}}, \qquad S_{\text{Bek}} = S_{\text{Bek}}(M, Q, J), \tag{1.5}$$

where the entropy can be expressed as a state function of the extensive variables M, Q, J. But now we have a problem, because black hole uniqueness tells us that for a given M, Q, J, there is only one solution to GR, and hence the entropy must be zero. Moreover, the Hawking radiation that could in principle allow information to escape the black hole has a purely thermal spectrum, so in fact contains *no* information. This is known as the *information paradox* for black holes[4].

Another way to see the paradox is through the quantum-mechanical principle of *unitarity*. Unitary evolution is rather fundamental to quantum theory, as it guarantees the "conservation of probability". For example, in a two-state system evolving over time, unitarity guarantees that at any moment in time, probabilities of the system being in state A or state B add up to 1. This is reasonable, because there are no other states

[4]There is of course no paradox in the idea that information is simply hidden in an inaccessible place; however, since the black hole must eventually evaporate via Hawking radiation, the absence of information within this radiation is a serious problem.

available; if somehow the sum of these probabilities became less than 1, the system would have some finite probability of suddenly ceasing to exist. Unitary evolution can also be thought of as conserving *information*. But unitary evolution is at odds with the classical notion of black hole horizons. Once information crosses the horizon, it can never escape again. If one throws our two-state system into a black hole, it is forever erased from the universe. The black hole gets slightly larger, but all we know about it are its M, Q, J; the Hawking radiation it emits is purely thermal and no information can be extracted from it. So if the classical GR picture of black holes is correct, then black holes violate unitarity, and dramatically so.

However, it seems more likely that black holes really *do* have entropy, and that in a successful theory of quantum gravity their time evolution will be unitary. We expect a quantum theory of gravity to give us a resolution to this apparent paradox. We expect to be able to see the microstates whose counting gives the appropriate entropy. And the same theory should also answer further mysteries such as how the Hawking radiation is generated, how the information stored in the black hole escapes, and how the classical picture of the black hole arises in the first place, if it is so far from the truth. One candidate theory of quantum gravity is string theory, which has had many successes in tackling aspects of this problem, and this is where we will focus our efforts.

The Fuzzball Proposal

We should stress that the problem starts at the black hole *horizon*, and is not just an effect of the central singularity. Even though the horizon is just a smooth bit of spacetime, and can be made arbitrarily flat for large enough M, it is the *horizon* which traps information and effectively erases it by preventing access from the external universe. Furthermore, it is the horizon area which gives the entropy, and it is due to horizon effects that the thermal radiation can be derived.

Hence if string theory is going to resolve the information paradox, then it will not be enough to make Planck-scale modifications near the singularity, since such modifications are behind the horizon and not classically observable. In fact, Mathur has shown [12] that to extract information out of the black hole requires $\mathcal{O}(1)$ corrections *at the horizon* (a series of papers on "firewalls" arrive at a similar conclusion [13, 14], although the $\mathcal{O}(1)$ corrections they propose are radically different). This is a rather dramatic departure from the classical picture, because classically speaking, the event horizon is a smooth piece of spacetime with arbitrarily-small curvature (for large enough M), and not locally observable. However, string theory is a theory of extended objects rather than point particles, so perhaps it is sensitive to geometrical features that are nonlocal from a point-particle perspective.

This line of thinking essentially leads to the "fuzzball proposal" of Mathur [15]. To resolve the paradox we must conclude that the event horizon and the entire region within it (i.e. $r \leq 2M$ in the Schwarzschild solution) is a classical fiction. New physics should take over at the horizon scale, and instead of a black hole interior there should be a stringy "fuzzball" of astronomical size, extending throughout the classical interior region as in Figure 1.1. Infalling matter then interacts with this "fuzzball" in a complicated, but unitary, way. Information can be trapped within the fuzzball for arbitrarily long times; this gives the *appearance* to distant observers that information has been lost. Thus the classical horizon comes about as an emergent property of the ensemble of fuzzball states.

Microstate geometries

While a generic fuzzball state is in principle any sort of string theory state, it is natural to ask if we can see these states in the low-energy supergravity limits of IIA and IIB string theory. In essence, can we find *classical* supergravity solutions that correspond to

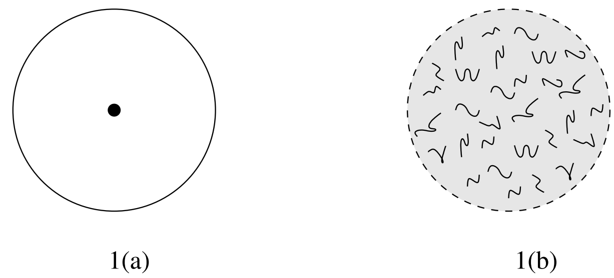

Figure 1.1: *(a) Classical picture of a black hole. (b) Stringy, "fuzzball" picture—macroscopically-extended strings reach all the way out to the classical horizon. Image from [15].*

fuzzball-type microstates? And if so, can we find enough of them to account for all of the entropy?

This question might sound absurd, because as we have already mentioned, the horizon is the feature we see at the classical level, and it is exactly the thing we were trying to get rid of. However, the uniqueness theorem that applies to black holes in 4 dimensions does not apply in higher dimensions. IIA and IIB string theory are 10-dimensional, and M theory (the strong-coupling limit of IIA) is 11-dimensional, and they contain additional massless fields (importantly, with Chern-Simons terms [16]). It turns out that in as few as 5 dimensions one can find not only many, but *infinitely many* supergravity solutions that correspond to the same M, Q, J measured at infinity. So to keep things as simple as possible, we can compactify the higher-dimensional supergravities down to 5 or 6 dimensions (and let the remaining n dimensions be wrapped up on an n-torus).

We are then interested in finding what we call "black hole microstate geometries". These are solutions of supergravity which are:

- Smooth everywhere (i.e. have no singularities),
- Have no horizons, and

- Have the same asymptotic charges as a black hole.

Such supergravity solutions do not present any information paradox individually, because they have no horizons for information to fall behind. It is then thought that taking a statistical ensemble of all such solutions having the same asymptotic charges will give something resembling a classical black hole with those charges[5].

One example of such a microstate geometry is that of the supertube [17, 18]. A supertube, as originally conceived, is a cylindrical D2-brane smeared with D0-brane and F1-string charges; hence it has two electric charges D0-F1, and one magnetic dipole D2 charge. This configuration is supersymmetric (specifically $\frac{1}{4}$-BPS), and hence is a stable bound state. The D2 cylinder can have an arbitrary cross-sectional shape while retaining supersymmetry, which means that the supertube is characterized generically by an arbitrary 1-dimensional closed curve. One can find supergravity solutions corresponding to arbitrary supertubes [19, 20, 18], and they are completely smooth and horizon-free in 6 dimensions (where they are recast, via T-duality, as D1-D5-kkm[6] bound states in the IIB theory). It furthermore turns out that supertubes are sufficient to account for (a significant, finite fraction[7] of) the entropy of the 2-charge black hole [15, 21, 22, 23, 24], which is an exciting success for the program. Unfortunately, the 2-charge black hole in 5 or 6 dimensions is, classically, a naked singularity with no horizon; it has entropy solely due to higher-order string theory corrections that give it a Planck-scale horizon.

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[5]Of course, in the full quantum theory of gravity, one must also include quantum fluctuations of the constituent strings. The question we are concerned with is how far, exactly, the classical microstate geometry picture can be pushed.

[6]Here lowercase letters indicate dipole charges. Although “KKM” means “Kaluza-Klein monopole”, it is in this case a dipole of KKM charge because it is sourced along a closed, contractible loop, and opposite ends of the loop source oppositely-oriented KKM charges.

[7]By this, we mean it grows with the appropriate power of the charges.

Since we are interested in sorting out problems with classically-large horizons, we must look to more complicated solutions in 5 or 6 dimensions with 3 charges[8].

The object of this work

In the 3-charge case necessary for classically-sized horizons in 5 or 6 dimensions, there has been a decent amount of success finding supersymmetric black hole microstate geometries (see review in [25]). However, the program is still lacking in two important ways:

First, one would like to know about non-BPS, non-extremal black holes. A BPS, extremal black hole has electric charge Q equal to its mass M, which is very unrealistic for any astrophysical black hole. A charged object tends to attract other oppositely-charged objects, and so a positively-charged black hole sitting in a universe filled with negatively-charged stuff will tend to attract that stuff preferentially, thereby decreasing its charge until it reaches zero. Therefore to understand realistic black holes, we will need to know something about non-BPS, non-extremal microstates.

Non-extremal microstates are unfortunately difficult to find [26, 27, 28, 29, 30]. However, we will show that one can find infinite families of non-BPS *extremal* microstate geometries. That is, microstates whose electric charge is still equal to their mass, but they do not have any supersymmetries. Such microstates are still not astrophysically realistic, but going beyond supersymmetry is an important first step. It turns out that in 5 dimensions, one can use a method called the "floating brane ansatz" [31] to find non-BPS extremal solutions by solving a system of linear PDEs.

Second, we would like to find enough microstates to count the entropy of the 3-charge black hole. In this case, we will stick to BPS microstates, which are easier to

[8]The number of charges required to yield a classically-macroscopic horizon varies with the dimension of the spacetime. In 4 dimensions, for example, one requires 4 charges.

analyze. It is conjectured that a 3-charge, 2-dipole-charge, $\frac{1}{8}$-BPS object exists, dubbed the "superstratum", which can take an arbitrary *2-dimensional* shape while retaining its supersymmetry [32]. This object is a supersymmetric bound state which should be a smooth geometry in the IIB frame reduced to 6 dimensions, where it has D1-D5-P electric charges and d1-d5-kkm dipole charges. Due to their arbitrary 2-dimensional shape, these objects are expected to give the correct microstate counting for the 3-charge black hole, analogously to supertubes in the 2-charge black hole. However, it is much harder to find the supergravity solution for the superstratum due to certain technicalities.

During my PhD I have written published papers with collaborators Nick Warner, Nikolay Bobev, and Orestis Vasilakis on both of these topics of research. In this thesis I will present our results, both on non-BPS microstate geometries and on the ongoing quest for the superstratum. These results are the following:

1. An infinite family of 5d non-BPS microstates based on the "floating brane ansatz" method using the LeBrun-Burns metrics;

2. A lift of these non-BPS solutions to 6d, where they are actually BPS, thus realizing explicitly a mechanism by which supersymmetry can be lost on dimensional reduction;

3. A more general family of 5d non-BPS microstates based on the more general LeBrun metrics;

4. A family of 6d BPS solutions called "supersheets" which are objects of arbitrary 2-dimensional shape, but lack the KKM dipole charge needed to make them into smooth superstrata; and

5. A family of 6d BPS solutions with KKM dipole charge, with hints on how to obtain superstrata.

This thesis is organized as follows: In Chapter 2, I will review the literature on some key topics needed in order to understand the context of this work, such as branes, BPS-ness, etc. In Chapter 3, I will discuss the specific mathematical background on which this work is based, and set up the supergravity problems to be solved. Then in Chapters 4–7 I will present my own work:

Chapter 4 will discuss the 5d non-BPS solutions of [33, 34], including the lift to 6 dimensions, and various detailed properties of these solutions. Then Chapter 5 will discuss the more general 5d non-BPS solutions of [35] and their analysis, including detailed discussion of the new base spaces on which the solutions are constructed. Then Chapter 6 will present the “superthreads” and “supersheets” of [36] which are 2-variable-arbitrary but lack KKM charge and are singular. Chapter 7 will then discuss the solutions of [37] which have KKM charge and are smooth, but only 1-variable-arbitrary (although with hints on how to obtain 2-variable arbitrary solutions).

Finally in Chapter 8 I will give an overall discussion of the results and open problems.

Chapter 2

Literature Review

All of the supergravity solutions discussed in this thesis will be sourced by the various charged objects of string theory or M theory (namely F1-strings, NS5-branes, D-branes, and M-branes), and the solutions will be *extremal*, which means they have an electric charge equal to their mass. Some solutions will be *supersymmetric*, or BPS, while other solutions will not. In this chapter, we will give a brief exposition of what these various terms mean.

2.1 Charged black holes

The prototypical example of an electrically-charged black hole in classical 4d GR is the Reissner-Nordström solution [38, 39] given by the metric

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)\mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}\mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \quad (2.1)$$

and the electromagnetic field

$$A = -\frac{Q}{r}\,\mathrm{d}t, \qquad F \equiv \mathrm{d}A = -\frac{Q}{r^2}\,\mathrm{d}t \wedge \mathrm{d}r. \quad (2.2)$$

Here M is the mass of the black hole, and Q is its electric charge. This black hole solves the Einstein-Maxwell equations

$$R_{\mu\nu} = \frac{1}{2}\Big(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}\,g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\Big), \quad (2.3)$$

where $R_{\mu\nu}$ is the Ricci tensor and $F_{\mu\nu}$ are the components of F given by

$$F \equiv \frac{1}{2} F_{\mu\nu}\, \mathrm{d}x^{\mu} \wedge \mathrm{d}x^{\nu}. \tag{2.4}$$

The metric in (2.1) has two horizons where g_{tt} vanishes, at the locations given by

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \tag{2.5}$$

One can show that these are mere coordinate singularities, and that spacetime is smooth at the horizons. However, there is a true curvature singularity at $r = 0$.

If $M^2 < Q^2$, then the $r_{\pm}$ are not real, and there are no horizons at all. Then the singularity at $r = 0$ is naked, which is probably an unphysical situation. This suggests a bound

$$M \geq |Q|, \tag{2.6}$$

relating the mass and the charge of a physically-reasonable solution. We will see that this becomes a recurring theme.

The outer horizon at r_+ is the event horizon, and the various quantities of black hole thermodynamics are defined there. The Hawking temperature is given by

$$T_{\text{Hawk}} = \frac{r_+ - r_-}{4\pi r_+^2} = \frac{\sqrt{M^2 - Q^2}}{2\pi\big(M + \sqrt{M^2 - Q^2}\big)^2}, \tag{2.7}$$

and the entropy is

$$S_{\text{Bek}} = \pi r_+^2 = \pi\big(M + \sqrt{M^2 - Q^2}\big)^2. \tag{2.8}$$

2.1.1 Extremal charged black holes

In the case that the bound (2.6) is *saturated*, i.e. $M = |Q|$, then the metric (2.1) describes an *extremal* black hole. In this case one finds that the temperature vanishes, and the entropy scales as some power of the charge:

$$T_{\text{Hawk}} = 0, \qquad S_{\text{Bek}} = \pi Q^2. \tag{2.9}$$

One can then write the metric as

$$\mathrm{d}s^2 = -\left(1 - \frac{Q}{r}\right)^2 \mathrm{d}t^2 + \left(1 - \frac{Q}{r}\right)^{-2} \mathrm{d}r^2 + r^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \tag{2.10}$$

which looks a bit simpler than (2.1). However, a coordinate change to $\rho = r - Q$ gives an even simpler expression:

$$\mathrm{d}s^2 = -\left(1 + \frac{Q}{\rho}\right)^{-2} \mathrm{d}t^2 + \left(1 + \frac{Q}{\rho}\right)^2 \big(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\theta^2 + \rho^2\sin^2\theta\,\mathrm{d}\phi^2\big), \tag{2.11}$$

where now in the second term the warp factor multiplies the entire *flat* metric of $\mathbb{R}^3$ (note also that the powers of $+2, -2$ have switched places). In these coordinates, the horizon sits at $\rho = 0$, and the coordinate patch covers only the area outside the horizon. The electromagnetic vector potential under this coordinate change becomes

$$A = -\left(1 + \frac{Q}{\rho}\right)^{-1} \mathrm{d}t. \tag{2.12}$$

2.1.2 The near-horizon limit

If one looks near the horizon in (2.11), that is taking ρ to be small, one finds the solution approaches the metric

$$\mathrm{d}s^2 = -\frac{\rho^2}{Q^2}\,\mathrm{d}t^2 + \frac{Q^2}{\rho^2}\,\mathrm{d}\rho^2 + Q^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \tag{2.13}$$

where now the spherical part has constant radius Q. Making another coordinate change $\rho = Q^2/z$ gives the metric

$$\mathrm{d}s^2 = Q^2\left(\frac{-\mathrm{d}t^2 + \mathrm{d}z^2}{z^2}\right) + Q^2\big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\big), \tag{2.14}$$

which is the metric for the product space $AdS_2 \times S^2$, also known as the Robinson-Bertotti solution [40, 41], where the AdS_2 factor and the S^2 factor each have the same radius Q.

It turns out that in higher-dimensional supergravity theories, this same basic feature continues to hold: The near-horizon regions of extremal black holes are often $AdS \times S$-like [1].

2.1.3 Multiple charged black holes

The coordinates of (2.11) are called "isotropic", because of the appearance of the *flat* metric on $\mathbb{R}^3$ multiplying the warp factor in the second term (here in spherical polar coordinates). We notice that the function $1 + Q/\rho$ is the electric potential of a point

[1] Although with some complications that depend on the dimension of spacetime and the degree of the p-form flux under which the black hole is charged. See also [42].

charge Q in flat $\mathbb{R}^3$ as well[2]. It turns out this is not accidental. If one writes the metric and electromagnetic vector potential as

$$\mathrm{d}s^2 = -H^{-2}\,\mathrm{d}t^2 + H^2\,(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{2.15}$$

$$A = -H^{-1}\,\mathrm{d}t, \tag{2.16}$$

then one can show that the Einstein-Maxwell equations (2.3) reduce to precisely Laplace's equation,

$$\nabla^2 H = \partial_x^2 H + \partial_y^2 H + \partial_z^2 H = 0. \tag{2.17}$$

So the metric (2.11) is nothing more than what results from taking H to be the potential of a single point charge. But there is nothing stopping us from writing down an H that is the potential of several point charges,

$$H = 1 + \sum_{i=1}^{N} \frac{Q_i}{|\vec{x} - \vec{a}_i|}, \tag{2.18}$$

each with an arbitrary charge $Q_i > 0$ and located at an arbitirary point $\vec{a}_i \in \mathbb{R}^3$. This gives the Majumdar-Papapetrou solution [43, 44], which corresponds to a collection of any number of extremally-charged black holes sitting in any arrangement. They remain in equilibrium because their electrostatic repulsion balances their gravitational attraction.

It is worth pointing out that although the general equations of GR, including the Einstein-Maxwell equations (2.3), are rather famously nonlinear, the equation satisfied for the "potential" of an extremal charged spacetime (2.17) is a *linear* equation. This is because the "force balancing" between the charged points allows them to be treated

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[2]Although with boundary conditions that it asymptote to a nonzero constant.

independently and therefore one can simply superpose their solutions to obtain a combined solution. This "extremal black hole superposition principle" will turn out to be very important in constructing black hole microstate solutions of higher-dimensional supergravity theories.

2.2 BPS bounds

The term "BPS" stands for Bogomol'nyi, Prasad, and Sommerfield, who first derived an energy bound that is saturated by classical solitonic solutions of Yang-Mills theory [45, 46]. It was later shown that this bound can be seen as coming from a supersymmetry algebra, and therefore it holds at the quantum level as well [47]. Einstein-Maxwell theory can be seen as the bosonic content of minimal $\mathcal{N} = 2$ supergravity in 4 dimensions, and it is interesting to see how the supersymmetry algebra relates to the extremal Reissner-Nordström solution.

The $\mathcal{N}$-extended supersymmetry algebra in 4 dimensions includes fermionic generators whose anticommutation relations are (in a Weyl basis in the rest frame):

$$\left\{Q^I_\alpha,\, \bar{Q}_{\dot{\beta}J}\right\} = 2M\delta_{\alpha\dot{\beta}}\delta^I_J, \qquad \left\{Q^I_\alpha,\, Q^J_\beta\right\} = 2Z^{IJ}\varepsilon_{\alpha\beta}, \tag{2.19}$$

where $I, J \in 1, \ldots, \mathcal{N}$ are R-symmetry indices and α, β are spinor indices. M is the mass of the system and Z^{IJ} is a matrix of central charges, which must be antisymmetric (hence the presence of Z^{IJ} requires at least $\mathcal{N} = 2$ supersymmetry).

Specializing to $\mathcal{N} = 2$, we can write $Z^{IJ} \equiv Z\varepsilon^{IJ}$. One can then show, using certain linear combinations of the Q^I_α, that

$$M - |Z|, \qquad M + |Z|, \tag{2.20}$$

are the eigenvalues of a positive semidefinite matrix. Hence it must be true that

$$M \geq |Z|. \tag{2.21}$$

This is the BPS bound, derived as a consequence of the supersymmetry algebra. So then this bound must be obeyed even at the quantum level. States that saturate the BPS bound are called BPS states. BPS states are always stable, since there are no lower-mass states available to decay to[3].

The central charge Z is related to conserved charges of the theory, such as electric charge. In fact, in $\mathcal{N}=2$ supergravity in 4 dimensions, one can show that

$$Z = Q + iP, \tag{2.22}$$

where Q is the total electric charge of a spacetime, and P is its total magnetic charge ([49], see also [50]). Hence for the Reissner-Nordström solution, we can see that the BPS bound (2.21) is really the same thing as the extremality bound (2.6).

A black hole with both electric and magnetic charges is called *dyonic*, which we will discuss in Section 2.3.

2.2.1 Residual supersymmetry

There is another consequence of saturating the BPS bound, which is that the state is annihilated by half the supersymmetries; specifically, the linear combination of them that gives the $M - |Z|$ eigenvalue in (2.20). The other half of the SUSY generators act in the usual way on the Clifford vacuum, yielding a "short multiplet" with half as many states as one would normally have, if the BPS bound were not saturated. Since the

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[3]Modulo certain caveats about wall-crossings [48].

states of this short multiplet are invariant under half the supersymmetries, we call them $\frac{1}{2}$-BPS, or more generally "supersymmetric".

For $\mathcal{N} > 2$, it is possible that Z^{IJ} has more than one distinct eigenvalue. The maximal $\mathcal{N}$ allowed in supergravity is $\mathcal{N} = 8$, and hence Z^{IJ} can have up to 4 distinct eigenvalues (they come in conjugate pairs, and only $|Z_i|$ appears in the BPS bound). The number of leftover supersymmetry generators depends on how many of the $|Z_i|$ are saturated, but since M is bounded by the highest $|Z_i|$, this in turn depends on how many of the $|Z_i|$ are the same. The highest amount of residual supersymmetry is half (or $\frac{1}{2}$-BPS), and this is when all the $|Z_i|$ are saturated:

$$M = |Z_1| = |Z_2| = |Z_3| = |Z_4|. \tag{2.23}$$

For each $|Z_i|$ that drops below M, one loses half the supersymmetry again; hence $\frac{1}{4}$-BPS, $\frac{1}{8}$-BPS, down to $\frac{1}{16}$-BPS [51].

In supergravity, the global supersymmetries generated by the Q^I_α are promoted to *local* supersymmetries, and to maintain super-gauge-invariance one must introduce $\mathcal{N}$ spin-$\frac{3}{2}$ fields or *gravitini* $\psi^I_{\mu\alpha}$. Each of these is gravitini is the state that results by lowering the spin-2 graviton state via one of the Q^I_α (and hence belongs to the supergravity multiplet). In classical supergravity solutions (such as the ones we are interested in), the fermions must always be zero, and hence in particular the gravitino variation must be zero,

$$\delta\psi_\mu = \nabla_\mu \epsilon + (\ldots), \tag{2.24}$$

where ϵ is a spinor parameter, and we omit the $\mathcal{N}$ index I and the spinor index α. The extra terms $(\ldots)$ represent various other fields in the theory, such as p-form gauge fields, etc. (which we will discuss in Section 2.3.1). Equation (2.24) is called a "SUSY variation" and one will have SUSY variations for every fermion in the theory, which must always be zero on classical solutions.

A *BPS* solution occurs precisely when the SUSY variations (2.24) admit a solution with nonzero ϵ. Then the residual supersymmetries, due to saturating the BPS bound, manifest as *spacetime* supersymmetries. The prototypical example of a spacetime supersymmetry is a "Killing spinor", defined as a spinor ϵ that solves

$$\nabla_\mu \epsilon = 0, \quad \text{where} \quad \nabla_\mu \epsilon \equiv \Big(\partial_\mu + \frac{1}{4} \omega_{\mu a b} \gamma^{ab} \Big) \epsilon, \tag{2.25}$$

so a Killing spinor is also called "covariantly constant". Acting with another ∇_ν and antisymmetrizing gives

$$\big[\nabla_\mu, \, \nabla_\nu \big] \epsilon = \frac{1}{4} R_{\mu\nu ab} \gamma^{ab} \epsilon = 0, \tag{2.26}$$

which means that the existence of a Killing spinor puts constraints on the allowed holonomy group of a spacetime solution. This can often have deep implications.

In higher-dimensional supergravity theories, these same concepts hold in more or less the same way [52]. BPS states saturate a mass-minus-charge bound which makes them *extremal* solutions. In addition, the residual supersymmetries imply that these solutions have Killing spinors.

We should note that, although we have shown that the extremal Reissner-Nordström solution is supersymmetric in 4d $\mathcal{N}=2$ supergravity, it is possible in other supergravity theories to have solutions which are extremal, yet not supersymmetric[4].

[4]One might wonder, in light of the previous discussion, how exactly a solution can saturate the $M \geq |Z|$ bound and yet not be BPS, and frankly the explanation is not clear. Nevertheless, there exist many examples of supergravity solutions (many of which we discuss in this thesis) which saturate the bound and yet have no Killing spinors.

2.3 Black holes in string theory

The familiar Maxwell's equations in 4 dimensions can be written

$$\mathrm{d}F = \underset{4}{\star} J_M, \qquad \mathrm{d} \underset{4}{\star} F = \underset{4}{\star} J_E, \tag{2.27}$$

where F is the electromagnetic field strength 2-form, J_E is the electric current 1-form, which measures the flow of charge along worldlines, and J_M is the magnetic current 1-form, if we wish to consider the existence of magnetic monopoles. To measure the amount of charge within a given region of spacetime, we integrate over some Gaussian surface Σ (which is topologically a 2-sphere) that "links" the worldlines of the charges we want to measure[5]. Then the electric charge Q and magnetic charge P linked by Σ are given by

$$Q = \frac{1}{4\pi} \oint_\Sigma \underset{4}{\star} F, \qquad P = \frac{1}{4\pi} \oint_\Sigma F. \tag{2.28}$$

It turns out that in 4 dimensions, the source term on the right-hand side of the Einstein-Maxwell equations (2.3) is invariant under the "duality rotations" that exchange F with $\star_4 F$ and hence exchange electric charge with magnetic (monopole) charge. Using this fact, one can get a "dyonic" black hole for free, with electric charge Q and magnetic charge P:

$$\begin{aligned} \mathrm{d}s^2 = &-\left(1 - \frac{2M}{r} + \frac{P^2+Q^2}{r^2}\right) \mathrm{d}t^2 + \left(1 - \frac{2M}{r} + \frac{P^2+Q^2}{r^2}\right)^{-1} \mathrm{d}r^2 \\ &+ r^2 \big(\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\phi^2\big), \end{aligned} \tag{2.29}$$

[5]In a 3-dimensional spatial slice, this is the same as the usual "enclosed charge", where the integrals are over $\vec{E} \cdot \mathrm{d}\vec{a}$ and $\vec{B} \cdot \mathrm{d}\vec{a}$ respectively. However, the "linking" picture is more universal, and is analogous to Ampère's law for electric currents.

with electromagnetic field given by

$$A = -\frac{Q}{r}\,\mathrm{d}t - P\cos\theta\,\mathrm{d}\phi, \qquad F \equiv \mathrm{d}A = -\frac{Q}{r^2}\,\mathrm{d}t\wedge\mathrm{d}r + P\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi. \quad (2.30)$$

In this case, all of the discussion in Section 2.1 carries over analogously if we replace $Q^2 \to Q^2 + P^2$. In particular, the extremality bound is again the same as the BPS bound:

$$M \geq \sqrt{Q^2+P^2}, \quad \text{or} \quad M \geq |Z|, \quad \text{where} \quad Z = Q + iP. \quad (2.31)$$

2.3.1 Branes, charges, and fluxes

The four-dimensional case is special, because in 4 dimensions, F and $\star_4 F$ are each 2-forms. Therefore their sources J_E and J_M each describe point particles, or 0-dimensional objects. This allows the same type of object to carry *both* electric and magnetic charges.

In higher dimensions, this is no longer true. A p-dimensional object (called a p-brane by analogy with "membrane" for a 2-dimensional object) will correspond to a $(p+1)$-form electric current density J^E_{p+1}, which describes the flow of p-brane charge density along its $(p+1)$-dimensional worldvolume. This electric current density J^E_{p+1} will then act as a source for a $(p+2)$-form field strength F_{p+2} via

$$\mathrm{d} \underset{n}{\star} F_{p+2} = \underset{n}{\star} J^E_{p+1}, \quad (2.32)$$

where we are now in n-dimensional spacetime. The magnetic dual object should come in as a source on the right-hand side of

$$\mathrm{d}F_{p+2} = \underset{n}{\star} J^M_q, \quad \text{for some } q. \quad (2.33)$$

In n-dimensional spacetime, the Hodge dual of a q-form is an $(n-q)$-form. On the left-hand side is a $(p+3)$-form, so then the right-hand side must be the Hodge dual of a $(n-p-3)$-form, and hence

$$\mathrm{d}F_{p+2} = \underset{n}{\star} J^M_{n-p-3}. \tag{2.34}$$

Therefore we see that the magnetic dual of a p-brane is an $(n-p-4)$-brane, which is (usually) a completely different object. The $(p+2)$-form field strength F_{p+2} is often called a "flux", because it can be used to compute the charged "linked" by the appropriate $(n-p-2)$- or $(p+2)$-dimensional surface Σ in flux integrals analogous to (2.28).

Strings, D-branes, and M-branes

This is all very general, but in string theory we have specific objects to deal with. First, the IIA and IIB superstring theories naturally live in 10 dimensions, and M theory (the strong-coupling limit of IIA) lives in 11 dimensions. So we will take n to be 10 or 11. In string theory, the most obvious object is of course the fundamental string, F1. As expected, the F1-string acts as a source for a 3-form field strength, but we will see there are additional objects as well.

The worldsheet theory of the closed superstring contains several massless bosonic excitations (see [53, 54, 42]). From the NS-NS (Neveu-Schwarz) sector come the dilaton ϕ, the spacetime metric tensor $g_{\mu\nu}$, the Kalb-Ramond field or 2-form potential $B_{\mu\nu}$, and from the R-R (Ramond-Ramond) sector come various p-form field strengths $F^{(p)}_{\mu_1\ldots\mu_p}$, where the IIA theory has p even, and the IIB theory has p odd. The B_2 field couples electrically to fundamental strings F1, by which we mean that its 3-form field strength $H_3 \equiv \mathrm{d}B_2$ has Maxwell-like equations

$$\mathrm{d}H_3 = 0, \qquad \mathrm{d} \underset{10}{\star} H_3 = \underset{10}{\star} J_{F1}, \tag{2.35}$$

One can conclude from nonperturbative effects that there is also an object to couple magnetically to H_3, the NS5-brane. So one really has

$$\mathrm{d}H_3 = \underset{10}{\star} J_{NS5}, \qquad \mathrm{d} \underset{10}{\star} H_3 = \underset{10}{\star} J_{F1}, \tag{2.36}$$

where the 6-form J_{NS5} gives the flow of NS5-brane charge.

Similarly, there are nonperturbative objects that couple electrically and magnetically to the Ramond-Ramond field strengths $F^{(p)}_{\mu_1 \cdots \mu_p}$. They are called D-branes, or "Dirichlet branes", because open strings may end on them, resulting in Dirichlet boundary conditions for the transverse coordinates of the open-string endpoints. In the presence of D-branes, we have

$$\mathrm{d}F^{(p+2)} = \underset{10}{\star} J_{D(6-p)}, \qquad \mathrm{d} \underset{10}{\star} F^{(p+2)} = \underset{10}{\star} J_{Dp}, \tag{2.37}$$

showing that a Dp-brane is a source of $F^{(p+2)}$, and the magnetic dual of a Dp-brane is a D$(6-p)$-brane. In the IIA theory one has even p, and hence D0-, D2-, D4-, D6-, and D8-branes; while in the IIB theory one has odd p, giving D1-, D3-, D5-, and D7-branes (and one can also consider D(-1)-branes, or D-instantons).

In M theory, the fundamental objects are M2-branes, which couple electrically to a 4-form field strength F_4.

$$\mathrm{d} \underset{11}{\star} F_4 = \underset{11}{\star} J_{M2}, \qquad \mathrm{d}F_4 = \underset{11}{\star} J_{M5}. \tag{2.38}$$

Their magnetic duals are M5-branes.

The low-energy limits of the IIA and IIB string theories and M theory are supergravity theories which are dominated by the effects of these p-brane objects. From M theory one gets 11-dimensional supergravity, which is unique; from the IIA and IIB string theories one gets the two 10-dimensional $\mathcal{N} = 2$ supergravities, also called IIA and IIB[6]. Each of these theories resembles Einstein-Maxwell theory in that its bosonic content comprises various kinds of p-form field strengths whose sources are $(p-2)$-dimensional objects[7].

Building black holes

In effect, then, F1-strings, NS5-branes, and Dp-branes (and M-branes) are the simplest objects in string theory (and M theory), and as such make good tools for building models over which we have some calculational control. They are especially useful in approaching the information problem. One can model a black hole by laying out various collections of strings and branes. At low string coupling, this is just a pile of objects in 10d (or 11d) Minkowski space (possibly with some directions periodically identified). At high coupling, the gravitational field turns on, and the (massive) objects become a black hole with various properties such as NS-NS or R-R charges, different horizon topologies, etc. If the configuration of strings/branes preserves supersymmetry, then one expects certain data (such as ground state degeneracy) to be protected as the coupling is increased; thus one can make certain calculations in the "stack of branes" régime that are expected to apply in the supergravity régime.

Then the task is to find what configurations of branes give rise to nice models of black holes. Individual p-branes are BPS objects; they have a mass equal to their charge, much like the extremal Reissner-Nordström black hole (or more precisely, their mass-per-unit-volume and charge-per-unit-volume are the same) [52]. When p-branes are overlapped or given funny shapes, their supersymmetries are decreased, although

[6]M theory, IIA strings, and IIB strings are also interrelated by various dualities such as T-duality.

[7]However, they also have ingredients not found in Einstein-Maxwell theory, such as Chern-Simons terms and a dilaton.

we will see that some special combinations retain more supersymmetry than naïvely expected. This will turn out to be very useful in constructing black holes.

2.3.2 2-charge black holes

A 2-charge black hole is a black hole that has two conserved charges at infinity. These can be charges sourced by p-brane objects, or they can be momentum around a compact circle (since winding number is dual to momentum).

Here we will consider a particular black hole construction in IIB theory, made of D1 and D5 branes. We are interested in a 5-dimensional black hole, since 5 is the smallest number of dimensions in which black hole uniqueness is violated (hence, we ought to be able to find microstate geometries for this black hole). To get to 5 dimensions, we compactify IIB on $S^1 \times T^4$. The D5-branes are extended along this internal $S^1 \times T^4$, and the D1-branes are extended along the S^1 (thus lying on top of the D5-branes). In the remaining 5 dimensions, both types of branes appear as point particles.

In the supergravity régime, this configuration can be described by the metric

$$\mathrm{d}s_5^2 = -(Z_1 Z_2)^{-2/3}\,\mathrm{d}t^2 + (Z_1 Z_2)^{1/3}\left(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega_3^2\right), \tag{2.39}$$

$$Z_1 = 1 + \frac{Q_1}{\rho^2}, \qquad Z_2 = 1 + \frac{Q_5}{\rho^2}, \tag{2.40}$$

where $\mathrm{d}\Omega_3^2$ is the metric on a unit 3-sphere. There are two Maxwell fields given by

$$A^1 = -Z_1^{-1}\,\mathrm{d}t, \qquad A^2 = -Z_2^{-1}\,\mathrm{d}t, \qquad F^I \equiv \mathrm{d}A^I. \tag{2.41}$$

This solution is very reminiscent of the 4-dimensional Reissner-Nordström solution in isotropic coordinates (2.11) and (2.12). The charges Q_1, Q_5 are proportional to (but not equal to) the numbers n_1, n_5 of microscopic D1 and D5 branes used to construct the solution.

The horizon of (2.39) sits at $\rho = 0$. The induced metric on a sphere at $t, \rho = \text{const}$ for small ρ is given by

$$\mathrm{d}s_5^2\bigg|_{t,\rho=\text{const}} \sim \left(Q_1 Q_5\right)^{1/3} \rho^{2/3}\, \mathrm{d}\Omega_3^2, \tag{2.42}$$

so in particular, in the $\rho \to 0$ limit we see that this black hole has *zero* horizon area! There is something pathological about 2-charge black holes in 5 dimensions—in the supergravity limit, they are (almost) naked singularities[8].

However, this black hole does have a finite entropy, which can be seen in the microscopic régime, by looking at the system as a stack of D1 and D5 branes. Considering the CFT of the D1-D5 system along the S^1 where they overlap, one can count left- and right-moving degrees of freedom on the open strings stretching between the D1 and D5 in the large-Q limit [24, 58, 15, 19, 59]. The entropy given by the Cardy formula [60] in this case is

$$S = 2\pi\sqrt{2}\sqrt{n_1 n_5}, \tag{2.43}$$

where n_1, n_5 are the numbers of D1 and D5 branes.

Here we have an apparent inconsistency. However, since the horizon of (2.39) is essentially on top of the singularity, it is in a region of very high curvature, and hence the supergravity approximation is no good. One can get away from the supergravity limit by putting in α' corrections from string theory. The resulting action has higher-derivative

[8]We have said that black holes in 4 dimensions require 4 charges to avoid this exact same pathology. However, the Reissner-Nordström black hole in (2.1) appears to have only one charge. In the string theory context, however, a 4d Reissner-Nordström black hole in fact has 4 charges, all of which happen to be equal [55, 56, 57].

terms coming from $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, etc., and the spacetime solutions are different. In particular, the 2-charge black hole acquires a "microscopic" horizon leading to the correct entropy scaling

$$S \sim \sqrt{n_1 n_5}, \tag{2.44}$$

although one cannot obtain the precise constant of $2\pi\sqrt{2}$ [61, 62].

2.3.3 Supertubes

It turns out that one can give a somewhat better semiclassical account of the entropy (2.43) in terms of microstate geometries [63, 19, 61, 15, 23, 24, 22]. These microstate geometries correspond to *supertubes*, which are particular BPS bound states of D-brane charges [17].

As originally discovered, the supertube is a configuration in IIA theory consisting of F1 and D0 electric charges smeared out over a D2-brane that has been wrapped into a cylinder [17]. The cylinder can have an arbitrary cross-sectional profile given by some closed curve $\vec{F}(\sigma)$ as in Figure 2.1, and the entire configuration is $\frac{1}{4}$-BPS for any profile $\vec{F}(\sigma)$.

Because the curve $\vec{F}(\sigma)$ is *contractible* (i.e., it's just a closed curve sitting in ordinary space; it does not wrap around any piece of topology), the D2 charge of the configuration is a dipole charge. This is analagous to a current loop in ordinary electromagnetism: an Ampèrian loop that links the current loop will measure a current I, but an Ampèrian loop at infinity, which does not link the current loop, will measure zero net current, because the local currents on each "side" of the current loop move in opposite directions. In a similar fashion, a Gaussian surface that links the D2 cylinder will measure a local D2 charge; but a Gaussian surface at infinity does not link the D2 cylinder and measures zero net charge.

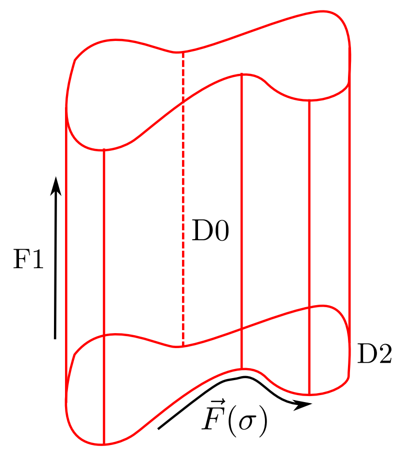

Figure 2.1: *Supertube. A D2-brane cylinder has an arbitrary cross-sectional profile along the closed curve $\vec{F}(\sigma)$. F1 charge runs vertically along the cylinder, and is smeared along $\vec{F}(\sigma)$. D0 charge is smeared over the whole surface.*

Under T-duality, the specific charges that compose a supertube get shuffled around, however the basic construction remains: a supertube is a $\frac{1}{4}$-BPS bound state of 2 electric charges and 1 magnetic dipole charge. A particular sequence of T-dualities can map the D0-F1-d2 charges[9] into D1-D5-kkm charges[10]. Therefore we can use supertubes to analyze the D1-D5 black hole.

It turns out that the singularity of (2.39) can be resolved if: 1) We lift into 6 dimensions, including the S^1 in our solution along which the D1-branes are wrapped; 2) We we smear the location $\rho = 0$ of the charges into an arbitrary closed curve $\vec{F}(\sigma)$ in $\mathbb{R}^4$; and 3) We add KKM dipole charge along $\vec{F}(\sigma)$ (which, being a dipole charge, is not measurable at infinity). This gives us the supergravity solution corresponding to a supertube, which one can show is totally smooth and horizon-free (see Section 3.3.3). Therefore one has as many microstate geometries as there are arbitrary functions of one variable, $\vec{F}(\sigma)$.

[9]Here we use lowercase letters to indicate dipole charges

[10]"KKM" means "Kaluza-Klein monopole"; however it is in this case a dipole of KKM charge because it is sourced along a closed, contractible loop.

In principle, this is now an infinite-dimensional space of smooth microstate solutions; however, one needs to quantize this space. In turns out that this process reproduces not quite the entropy (2.43), but instead a finite fraction of it [23]:

$$S = 2\pi\sqrt{\frac{c}{6}Q_1 Q_5}, \qquad c = 4, \tag{2.45}$$

where one would need the central charge $c = 12$ to get (2.43) exactly. However, the 6-dimensional picture does not capture all of the fluctuation modes of the supertube. When they are all counted, the entropy (2.43) is obtained [22, 24, 15].

2.3.4 3-charge black holes

While supertubes are successful at describing a finite portion of the entropy of the 2-charge black hole, we recall that the 2-charge black hole has a classical horizon of zero area, as in (2.42). We are interested in describing the microstates of black holes with classical horizons, and in 5 dimensions, this requires three charges.

We will use the same IIB brane configuration: A D5-brane wrapped on a compact $S^1 \times T^4$, and a D1-brane wrapped along the S^1. To obtain a third charge, we will add momentum along this S^1, giving the D1-D5-P system. The supergravity solution is again simple:

$$\mathrm{d}s_5^2 = -(Z_1 Z_2 Z_3)^{-2/3}\,\mathrm{d}t^2 + (Z_1 Z_2 Z_3)^{1/3}\left(\mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega_3^2\right), \tag{2.46}$$

$$Z_1 = 1 + \frac{Q_1}{\rho^2}, \qquad Z_2 = 1 + \frac{Q_5}{\rho^2}, \qquad Z_3 = 1 + \frac{Q_p}{\rho^2}. \tag{2.47}$$

There are now three Maxwell fields given by

$$A^I = -Z_I^{-1}\,\mathrm{d}t, \qquad F^I \equiv \mathrm{d}A^I, \qquad I \in \{1,2,3\}. \tag{2.48}$$

This time, the induced metric on a sphere at $t, \rho = \text{const}$ for small ρ is given by

$$\mathrm{d}s_5^2\Big|_{t,\rho=\text{const}} \sim \left(Q_1 Q_5 Q_p\right)^{1/3} \mathrm{d}\Omega_3^2. \tag{2.49}$$

We see that the factors of ρ cancel out and we get a horizon area of

$$A = 2\pi^2 \sqrt{Q_1 Q_5 Q_p}. \tag{2.50}$$

After relating the Q_I to the microscopic charges n_I (see [64, 15]), we obtain the entropy

$$S = 2\pi \sqrt{n_1 n_5 n_p}. \tag{2.51}$$

As we did with the 2-charge black hole, we can also approach this calculation from the microscopic perspective, using the D1-D5-P CFT along the S^1 where the branes overlap. This is the famous Strominger-Vafa state counting [65, 64], and it gives exactly the result expected:

$$S = 2\pi \sqrt{n_1 n_5 n_p}. \tag{2.52}$$

We stress that the D1-D5-P CFT calculation is done at zero string coupling $g_s = 0$, and thus effectively with gravity "turned off". The fact that it matches the calculation from the horizon area (2.50), where gravity is certainly "on", gives an important check of the theory.

2.3.5 Superstrata

The Strominger-Vafa entropy calculation counts states at zero string coupling, and it is a natural question to ask what are the microstates that give rise to the entropy (2.52) at *finite* string coupling, and can they be seen from supergravity? In fact one can find many microstate geometries for the 3-charge black hole [25]; however, one still does not have enough of them to reproduce even a finite fraction of the entropy (2.52)[11].

What one needs is something like supertubes, but with three electric charges. Supersymmetry arguments show that such an object should exist, called the *superstratum* [32]. A superstratum is an object carrying 3 electric charges and 2 independent magnetic dipole charges, and is able to take an arbitrary *2-dimensional* shape while remaining $\frac{1}{8}$-BPS and smooth. For example, in the IIB frame we have been discussing, a superstratum would carry D1-D5-P electric charges, and two dipole charges which are a combination of d1-d5-kkm[12].

It is argued in [32] that smooth superstratum solutions should exist in IIB reduced to 6 dimensions; however, due to some difficulties solving the equations, such solutions have yet to be found. We will discuss this further in Section 3.3 and Chapters 6 and 7.

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[11]However, semi-classical calculations [66, 67] indicate that quantum fluctuations on top of known solutions should come closer.

[12]All three dipole charges will be present, but a constraint reduces this to two independent charges.

Chapter 3

Background

3.1 BPS solutions in 5d

The background in this section is mostly taken from the review in [25]. Although our goal is to construct non-BPS solutions in 5 dimensions, it is helpful to expand on the BPS case in detail. Many features of the non-BPS solutions are analogous to those in the BPS case.

To discuss BPS bubbling geometries in 5 dimensions it is easiest to start from the M-theory picture. We consider M-theory on $\mathcal{M}_{4,1} \times T^6$ where the compact directions form a flat torus (with coordinates y_i) and the leftover bit $\mathcal{M}_{4,1}$ will become our 5-dimensional geometry. The 11-dimensional metric and 3-form potential are given by

$$\begin{aligned} \mathrm{d}s_{11}^2 = -Z^{-2}\,(\mathrm{d}t + k)^2 + Z\,\mathrm{d}s_4^2(\mathcal{B}) \\ + X^1\,(\mathrm{d}y_1^2 + \mathrm{d}y_2^2) + X^2\,(\mathrm{d}y_3^2 + \mathrm{d}y_4^2) + X^3\,(\mathrm{d}y_5^2 + \mathrm{d}y_6^2), \end{aligned} \tag{3.1}$$

$$C^{(3)} = A^1 \wedge \mathrm{d}y_1 \wedge \mathrm{d}y_2 + A^2 \wedge \mathrm{d}y_3 \wedge \mathrm{d}y_4 + A^3 \wedge \mathrm{d}y_5 \wedge \mathrm{d}y_6. \tag{3.2}$$

The first line of (3.1) is the metric of $\mathcal{M}_{4,1}$, which we take to be stationary, and hence can be written as a time fiber over some 4-dimensional base $\mathcal{B}$ (the powers of the function Z are chosen for convenience, as will be apparent later). The X^I, $I \in \{1, 2, 3\}$ control the sizes of three T^2's inside the T^6, and the A^I are three 1-forms having legs in $\mathcal{M}_{4,1}$. All fields are assumed independent of t and y_i.

	$\mathcal{M}_{4,1}$					T^6					
Brane	0	1	2	3	4	5	6	7	8	9	10
M2	$\updownarrow$		$\sim$			$\updownarrow$	$\updownarrow$	$\smile$	$\smile$	$\smile$	$\smile$
M2	$\updownarrow$		$\sim$			$\smile$	$\smile$	$\updownarrow$	$\updownarrow$	$\smile$	$\smile$
M2	$\updownarrow$		$\sim$			$\smile$	$\smile$	$\smile$	$\smile$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$\vec{x}(\lambda)$			$\smile$	$\smile$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$\vec{x}(\lambda)$			$\updownarrow$	$\updownarrow$	$\smile$	$\smile$	$\updownarrow$	$\updownarrow$
M5	$\updownarrow$		$\vec{x}(\lambda)$			$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\smile$	$\smile$

Table 3.1: *M-theory brane configuration. A brane is extended along "$\updownarrow$", and smeared along "$\smile$". $\vec{x}(\lambda)$ is a closed curve in the 4d base space of $\mathcal{M}_{4,1}$. "$\sim$" means a brane is smeared along the profile $\vec{x}(\lambda)$ (and not extended transverse to this profile).*

We have the M-theory brane configuration as in Table 3.1. The pairs {5,6}, {7,8}, and {9,10} are compactified, each on a square torus (but of different relative sizes). The M2-branes give electric charges, while the M5-branes give magnetic dipole charges. Hence taking a pair of M2-branes wrapping some $T^2 \times T^2$, together with the M5-brane that wraps the same $T^2 \times T^2$, gives a supertube along the profile $\vec{x}(\lambda)$ in the base space.

Reducing this configuration on the T^6 leads to $\mathcal{N} = 2$ ungauged supergravity in 5 dimensions coupled to two vector multiplets. The bosonic content of this theory is: the gravity multiplet consisting of the graviton $g_{\mu\nu}$ and graviphoton A^3_μ; and two vector multiplets consisting of two vector fields A^1_μ, A^2_μ (hence three total vectors) and two scalars φ^1, φ^2, which we will find convenient to represent via three scalars X^1, X^2, X^3 and a constraint

$$X^1 X^2 X^3 = 1. \tag{3.3}$$

Each of the X^I is the volume of one of the $T^2 \subset T^6$, and hence the constraint (3.3) states that the volume of the T^6 is fixed.

The bosonic part of the 5-dimensional action is

$$S = \frac{1}{2\kappa_5}\int \Big(\underset{5}{\star}\mathcal{R} - Q_{IJ}\,\mathrm{d}X^I \wedge \underset{5}{\star}\mathrm{d}X^J \\ - Q_{IJ}\,F^I \wedge \underset{5}{\star} F^J - \frac{1}{6}\,C_{IJK}\,F^I \wedge F^J \wedge A^K \Big), \qquad (3.4)$$

where $\mathcal{R}$ is the 5d Ricci scalar, $X^I,\ I \in \{1,2,3\}$ are scalar fields, $F^I \equiv dA^I$ are three Maxwell fields, the constants $C_{IJK} = |\varepsilon_{IJK}|$, and the kinetic terms are coupled via the matrix

$$Q_{IJ} \equiv \frac{1}{2}\,\mathrm{diag}\left((X^1)^{-2}, (X^2)^{-2}, (X^3)^{-2}\right). \qquad (3.5)$$

We parametrize the constraint (3.3) in terms of a new set of scalars Z_I:

$$X^1 = \left(\frac{Z_2\,Z_3}{Z_1^2}\right)^{1/3}, \quad X^2 = \left(\frac{Z_1\,Z_3}{Z_2^2}\right)^{1/3}, \quad X^3 = \left(\frac{Z_1\,Z_2}{Z_3^2}\right)^{1/3}. \qquad (3.6)$$

The requirement of supersymmetry further constrains the ansatz. First, the metric and the vector potentials must be related:

$$\mathrm{d}s_5^2 = -Z^{-2}\,(\mathrm{d}t + k)^2 + Z\,\mathrm{d}s_4^2, \qquad Z \equiv (Z_1 Z_2 Z_3)^{1/3}, \qquad (3.7)$$

$$A^I \equiv -Z_I^{-1}\,(\mathrm{d}t + k) + B^I. \qquad (3.8)$$

This results in a zero-force condition common to all extremal black holes. Second, there must exist a Killing spinor. In Table 3.1 there are three "flavors" of branes, and each cuts the supersymmetry in half. The surviving supersymmetry ϵ must be annihilated by each of the three projectors [25]

$$\frac{1}{2}(\mathbb{1} - \Gamma^{056})\epsilon = \frac{1}{2}(\mathbb{1} - \Gamma^{078})\epsilon = \frac{1}{2}(\mathbb{1} - \Gamma^{09(10)})\epsilon = 0, \qquad (3.9)$$

and since in the $(10,1)$-dimensional Clifford algebra we have $\Gamma^{0123\ldots(10)} \equiv \mathbb{1}$, the above conditions imply[1]

$$\frac{1}{2}(\mathbb{1} - \Gamma^{1234})\epsilon = 0. \tag{3.10}$$

In order for a Killing spinor satisfying this projection to exist, the curvature of the base $\mathcal{B}$ must be self-dual, or equivalently its holonomy must be in $SU(2)$; hence the base $\mathcal{B}$ must be hyper-Kähler.

It is convenient to introduce the "magnetic" 2-forms given by

$$\Theta^{(I)} \equiv \mathrm{d}B^I. \tag{3.11}$$

One then finds that setting the SUSY variations to zero leads to a linear system of equations, called the "BPS equations" [25]:

$$\Theta^I - \underset{4}{\star}\Theta^I = 0, \tag{3.12}$$

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_I = \frac{1}{2}C_{IJK}\,\Theta^J \wedge \Theta^K, \tag{3.13}$$

$$\mathrm{d}k + \underset{4}{\star}\mathrm{d}k = Z_I\,\Theta^I, \tag{3.14}$$

where $\star_4$ is taken with respect to the metric on $\mathcal{B}$. The solutions to these equations then determine $\mathrm{d}s_5^2$ and the three $F^I \equiv \mathrm{d}A^I$.

3.1.1 Gibbons-Hawking metrics

Supersymmetry requires that the base space $\mathcal{B}$ be hyper-Kähler. To obtain solutions that look like a black hole in $\mathbb{R}^4$, we need the base space to look like $\mathbb{R}^4$ at infinity. There

[1] This can be seen by taking the product of the three projectors in (3.9) and acting from the left with Γ^{1234}.

is a theorem that the only hyper-Kähler manifold asymptotic to $\mathbb{R}^4$ is $\mathbb{R}^4$ itself [68]; we will find a way around this theorem, however.

For now, we will choose a fairly simple family of hyper-Kähler metrics: the Gibbons-Hawking or Taub-NUT metrics[2]. These are the most general hyper-Kähler metrics with a tri-holomorphic $U(1)$ isometry (that is, a $U(1)$ isometry under which all three complex structures are invariant). They take the form [69],

$$\mathrm{d}s_4^2 = \frac{1}{V}\,(\mathrm{d}\psi + A)^2 + V\,(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \tag{3.15}$$

where the function V and 1-form A depend on x, y, z only, and

$$\mathrm{d} \underset{3}{\star} \mathrm{d}V = 0, \qquad \mathrm{d}A = \underset{3}{\star} \mathrm{d}V. \tag{3.16}$$

The three Kähler forms of (3.15) are anti-self-dual and given by

$$\Omega_-^{(1)} = (\mathrm{d}\psi + A) \wedge \mathrm{d}x - V\,\mathrm{d}y \wedge \mathrm{d}z, \tag{3.17}$$

$$\Omega_-^{(2)} = (\mathrm{d}\psi + A) \wedge \mathrm{d}y - V\,\mathrm{d}z \wedge \mathrm{d}x, \tag{3.18}$$

$$\Omega_-^{(3)} = (\mathrm{d}\psi + A) \wedge \mathrm{d}z - V\,\mathrm{d}x \wedge \mathrm{d}y. \tag{3.19}$$

These are each closed, by virtue of (3.16). The Ricci tensor of (3.15) vanishes.

The function V solves the Laplace equation in $\mathbb{R}^3$, and can be thought of as the electric potential for a number of point charges[3] located at some points $\vec{a}_i$. Then A is the

[2]Taub-NUT metrics go to $\mathbb{R}^3 \times S^1$ at infinity, and are useful for relating the 5-dimensional solutions here to the 4-dimensional black holes of our universe.

[3]One could also consider more general sources for the harmonic function V, such as dipoles, line charges, etc., but one can always obtain these solutions by combining many point charges. It turns out that only solutions corresponding to isolated point charges yield nonsingular metrics.

corresponding vector potential to give magnetic monopoles at the same points. Hence we can write

$$V = \varepsilon_0 + \sum_{i=1}^{N} \frac{q_i}{|\vec{x} - \vec{a}_i|}, \qquad A = \sum_{i=1}^{N} q_i \cos\theta_i \, \mathrm{d}\phi_i, \tag{3.20}$$

where $\vec{x} \equiv (x, y, z)$, and (θ_i, ϕ_i) are spherical polar coordinates centered around $\vec{a}_i$. In order to prevent the signature of the metric from flipping $(++++)$ to $(----)$, all the q_i must be positive (however, this assumption can be relaxed in the full 5-dimensional context due to the warp factor Z).

Near the points $\vec{a}_i$, we have $V \sim q_i/r_i$, which means that the ψ circle is shrinking to zero size. The metric can be locally written (dropping the i index)

$$\mathrm{d}s_4^2 = \frac{r}{q} \left(\mathrm{d}\psi + q \cos\theta \, \mathrm{d}\phi\right)^2 + \frac{q}{r} \left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right), \tag{3.21}$$

and making the coordinate change $r = \frac{1}{4}\rho^2$ one obtains the standard flat metric

$$\mathrm{d}s_4^2 \sim \mathrm{d}\rho^2 + \rho^2 \, \mathrm{d}\Omega_3^2, \tag{3.22}$$

where $\mathrm{d}\Omega_3^2$ is the round metric on $S^3/\mathbb{Z}_{|q|}$. Hence if each $q_i = \pm 1$, then the fiber pinches off smoothly at each $\vec{a}_i$, giving a metric that is free of singularities. For generic integers q_i, the metric locally approaches $\mathbb{R}^4/\mathbb{Z}_{|q_i|}$, and since such orbifolds are benign in string theory, we can also count such points as regular, as backgrounds for string theory.

The ψ fiber therefore pinches off smoothly at each $\vec{a}_i$, creating a network of homology 2-spheres, or "bubbles", as in Figure 3.1. On these homology 2-cycles we can construct dual cohomological fluxes of the form

$$\Theta = -\partial_a \left(\frac{H}{V}\right) \Omega_+^{(a)}, \qquad \Omega_+^{(a)} \equiv (\mathrm{d}\psi + A) \wedge \mathrm{d}x^a + V \underset{3}{\star} \mathrm{d}x^a, \tag{3.23}$$

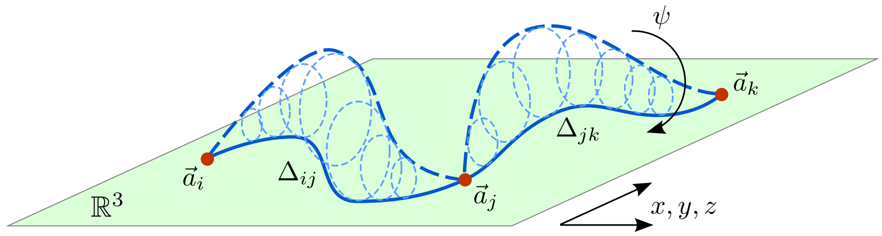

Figure 3.1: *Homological 2-cycles in the Gibbons-Hawking metric. The ψ fiber pinches off at the points $\vec{a}_i$. Sweeping the fiber along a path between any two points forms a homological 2-sphere. Two intersecting 2-cycles are shown.*

where $x^a \in \{x, y, z\}$ for $a \in \{1, 2, 3\}$. The 2-form Θ is manifestly self-dual, and it is harmonic when H solves the Laplace equation in $\mathbb{R}^3$. Writing

$$H = h_0 + \sum_{i=1}^{N} \frac{h_i}{|\vec{x} - \vec{b}_i|}, \tag{3.24}$$

we see that choosing the $\vec{b}_i$ to coincide with the $\vec{a}_i$ results in Θ being everywhere smooth. With the given fall-off behavior at infinity, such a Θ is also square-integrable, so it is a member of the 2nd cohomology group $H^2(\mathcal{B}, \mathbb{R})$. Such fluxes are "threaded" on the bubbles swept out by the ψ fiber; they have no singular sources, and yet have nonzero integrals over a Gaussian surface that links a given bubble.

At infinity, the function V in (3.15) approaches

$$V \sim \varepsilon_0 + \frac{1}{r}\sum_{i=1}^{N} q_i + \mathcal{O}\left(\frac{1}{r^2}\right). \tag{3.25}$$

Defining $q_0 \equiv \sum_i q_i$, then if $\varepsilon_0 = 0$, the geometry approaches $\mathbb{R}^4/\mathbb{Z}_{|q_0|}$. If $|q_0| \neq 1$, this space is called "asymptotically *locally* Euclidean" or ALE. As indicated by the previously-mentioned theorem [68], one can only get *global* $\mathbb{R}^4$ at infinity if one has strictly $\mathbb{R}^4$ throughout, and hence we must in fact have only one $q_1 = 1$, since any other combination of points will result in an orbifold of $\mathbb{R}^4$ at infinity. We will show how to get around this restriction in the following section.

Alternatively, if $\varepsilon_0 \neq 0$, then the metric is called Taub-NUT. At infinity, V goes to a constant and the metric approaches $\mathbb{R}^3 \times S^1$. Taking the time fiber into account, the whole 5d geometry approaches $\mathbb{R}^{3,1} \times S^1$, so the Taub-NUT metric is useful for relating 5d physics to 4d black holes.

3.1.2 Solutions on a Gibbons-Hawking base

To write the solutions to equations (3.12)–(3.14), we make the ansätze

$$\Theta^{(I)} = -\partial_a \left(\frac{K^I}{V}\right) \Omega_+^{(a)}, \tag{3.26}$$

$$Z_I = \frac{1}{2} C_{IJK} \frac{K^J K^K}{V} + L_I, \tag{3.27}$$

$$k = \mu \left(\mathrm{d}\psi + A\right) + \omega, \tag{3.28}$$

$$\mu = \frac{1}{6} C_{IJK} \frac{K^I K^J K^K}{V^2} + \frac{1}{2V} \sum_I K^I L_I + M, \tag{3.29}$$

where now the solution is expressed in terms of eight functions K^I, L_I, V, M and one 1-form ω on the $\mathbb{R}^3$ base. The BPS equations then imply that each of the functions K^I, L_I, V, M is *harmonic*:

$$K^I = k_0^I + \sum_{i=1}^N \frac{k_i^I}{|\vec{x} - \vec{a}_i|}, \qquad L_I = \ell_I^0 + \sum_{i=1}^N \frac{\ell_I^i}{|\vec{x} - \vec{a}_i|}, \tag{3.30}$$

$$V = \varepsilon_0 + \sum_{i=1}^N \frac{q_i}{|\vec{x} - \vec{a}_i|}, \qquad M = m_0 + \sum_{i=1}^N \frac{m_i}{|\vec{x} - \vec{a}_i|}, \tag{3.31}$$

and ω satisfies

$$\vec{\nabla} \times \vec{\omega} = V \vec{\nabla} M - M \vec{\nabla} V + \frac{1}{2} \left(K^I \vec{\nabla} L_I - L_I \vec{\nabla} K^I\right). \tag{3.32}$$

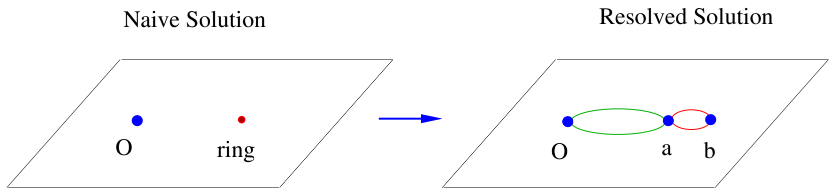

Figure 3.2: *Geometric transition of black ring: The naïve singular black ring is replaced by the resolved geometry. Two GH centers a and b are "pair created" and replace the ring with a bubble containing an equivalent amount of flux. Image from [25].*

We have hinted that it is possible to circumvent the theorem that $\mathbb{R}^4$ is the unique hyper-Kähler manifold asymptotic to $\mathbb{R}^4$. To see this, we look at the full 5-dimensional metric,

$$\begin{aligned} \mathrm{d}s_5^2 = &-(Z_1 Z_2 Z_3)^{-2/3}\,(\mathrm{d}t + k)^2 \\ &+ (Z_1 Z_2 Z_3)^{1/3}\left[\frac{1}{V}\,(\mathrm{d}\psi + A)^2 + V\,(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)\right]. \end{aligned} \tag{3.33}$$

In this context we see that what is really important is that the 5d metric keep the signature $(-++++)$ consistently. Therefore the function V is allowed to change sign, so long as each of the Z_1, Z_2, Z_3 change sign at the same place. Hence the hyper-Kähler base $\mathcal{B}$ is allowed to be "ambipolar", meaning its signature can change from regions of $(++++)$ to regions of $(----)$. This in turn allows us to choose q_i both positive and negative, allowing for an infinite variety of metrics even subject to the constraint $\sum_i q_i = 1$. One can show that the 5d metric is smooth across the $V = 0$ critical surfaces [25].

By choosing different combinations of parameters in (3.30) and (3.31), we can create a variety of supersymmetric solutions. Generically, solutions will be singular at the $\vec{a}_i$, which correspond to the locations of 3-charge black rings. However, for certain combinations of parameters, the singularity will be resolved into a bubble, or homology 2-sphere, as in Figure 3.2. The fluxes $\Theta^{(I)}$ threaded on this 2-sphere can be integrated over a Gaussian surface to reveal charges, although there is no singular charged object sourcing them. The amount of charge trapped on the bubble can be obtained by finding the periods of the $\Theta^{(I)}$ on the cycles Δ_{ij}:

$$\Pi^{(I)}_{ij} \equiv \frac{1}{4\pi}\int_{\Delta_{ij}} \Theta^{(I)} = \left(\frac{k^I_j}{q_j} - \frac{k^I_i}{q_i}\right). \tag{3.34}$$

These are *magnetic dipole* charges, and the Gaussian surfaces are 2-spheres that, in the naïve black ring solution, would link the ring. The *electric* charges of the solution are found by integrating over the 3-sphere at infinity, which cannot measure the magnetic flux trapped on the bubbles. However, due to the Chern-Simons terms in the action (3.4), the electric potentials Z_I have a quadratic source $\frac{1}{2}C_{IJK}\Theta^{(J)} \wedge \Theta^{(K)}$ in (3.13). Hence the dipole charges trapped on the bubbles can source electric charges indirectly. Thus the original charges of the naïve black ring geometry have become "dissolved in fluxes".

One is interested in smooth resolutions of the black ring geometry, and to create smooth bubbles, one must choose the parameters such that the functions Z_I, μ remain finite at each of the GH centers $\vec{a}_i$. This condition amounts to setting

$$\boxed{\ell^i_I = -\frac{1}{2}C_{IJK}\frac{k^J_i k^K_i}{q_i}, \qquad m_i = \frac{1}{12}C_{IJK}\frac{k^I_i k^J_i k^K_i}{q_i^2},} \tag{3.35}$$

at each GH center.

Supertube boundary conditions

Alternatively, one can create a *supertube* by following the usual prescription of two electric charges and one magnetic dipole charge. This requires, e.g. Z_1 and Z_2 to be singular with $1/r$ behavior, while Z_3 and V remain finite (so the ψ fiber does not pinch off). This can be accomplished by turning on ℓ^i_1, ℓ^i_2 and k^3_i at some point (where $q_i = 0$). The supertube follows a circular profile wrapping around the ψ fiber. This configuration will be singular in 5 dimensions; however, it can be lifted to 6d and made smooth, as we will see in Section 3.3.3.

3.1.3 The Bubble Equations

In order to make physical sense, the 5d solutions we have obtained must be free of causal pathologies; i.e. closed timelike curves (CTCs). Looking at a $t = \text{const}$ slice, we can re-arrange the metric as follows:

$$\mathrm{d}s_5^2\bigg|_{t=\text{const}} = \frac{\mathcal{Q}}{V^2 Z^2}\left(\mathrm{d}\psi + A - \frac{V^2\mu}{\mathcal{Q}}\,\omega\right)^2 + ZV\left(r^2 \sin^2\theta\,\mathrm{d}\phi^2 - \frac{\omega^2}{\mathcal{Q}}\right) + ZV\,(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2), \tag{3.36}$$

where

$$\mathcal{Q} \equiv Z_1 Z_2 Z_3 V - V^2\mu^2, \qquad Z \equiv (Z_1 Z_2 Z_3)^{1/3}, \tag{3.37}$$

$$k \equiv \mu\,(\mathrm{d}\psi + A) + \omega. \tag{3.38}$$

To avoid CTCs, this slice metric must be everywhere positive definite. This requires

$$\mathcal{Q} \geq 0, \qquad ZV \geq 0, \qquad r^2\sin^2\theta\,\mathrm{d}\phi^2 \geq \frac{\omega^2}{\mathcal{Q}}, \tag{3.39}$$

everywhere[4].In general, it is hard to guarantee this simply by choosing the local parameters in the solution (3.30), (3.31). One must numerically explore the solution and check these conditions. Near the GH centers, however, one can write down a condition that

[4]One might also argue that a solution should be *stably causal* rather than merely being free of CTC's. Stably causal means that there is a finite lower bound on the size of a perturbation needed to produce CTC's, and this is equivalent to having a global time function [7]. In the above parameters, this becomes $\mathcal{Q} - \omega^2 > 0$, which is a stronger condition than (3.39). See also [25].

must be *locally* satisfied. Assuming a regular bubbling solution according to (3.35), one finds that the sufficient local condition is that the function $\mu \to 0$ at every $\vec{a}_i$. This results in the “bubble equations”, which can be written (near each $\vec{a}_i$)

$$\boxed{\sum_{\substack{j=1 \\ j\neq i}}^{N} \Pi_{ij}^{(1)}\Pi_{ij}^{(2)}\Pi_{ij}^{(3)}\frac{q_i q_j}{r_{ij}} = -2\left(m_0\, q_i + \frac{1}{2}\sum_{I=1}^{3} k_i^I\right),} \tag{3.40}$$

where $\Pi_{ij}^{(I)}$ are the fluxes as in (3.34), and $r_{ij} \equiv |\vec{a}_i - \vec{a}_j|$ is the distance (in the $\mathbb{R}^3$ base) between $\vec{a}_i$ and $\vec{a}_j$. This formula contains the most interesting part of the physics of bubbling solutions: the fluxes $\Pi_{ij}^{(I)}$ on each bubble determine the *size* of that bubble by the constraint on r_{ij}. Essentially the fluxes threaded on a bubble hold that bubble open against gravitational collapse. We also see from the product of three $\Pi_{ij}^{(I)}$ that all three types of flux are necessary for this physical effect; if one type of flux is missing, then the left-hand side is zero and the system becomes degenerate.

Taking the sum of all the bubble equations, the left-hand side vanishes identically, and the resulting condition is simply that $\mu \to 0$ at infinity. So there are really $N-1$ independent bubble equations, which are exactly enough to determine the $N-1$ independent bubble diameters.

3.2 Non-BPS solutions in 5d from Floating Branes

The material in this section is taken mostly from [31]. We will keep the discussion brief and only point out the essential differences from the BPS case in Section 3.1.

To get away from BPS, we start again from the action (3.4) but relax the requirement of supersymmetry. This means tackling the full Einstein equations rather than the SUSY variations, which can be quite tedious. Thankfully, the hard work has been done in [31], starting again from the ansätze

$$\boxed{\begin{aligned} &\mathrm{d}s_5^2 = -Z^{-2}\,(\mathrm{d}t + k)^2 + Z\,\mathrm{d}s_4^2(\mathcal{B}), \qquad Z \equiv (Z_1 Z_2 Z_3)^{1/3}, && (3.41)\\ &A^I \equiv -Z_I^{-1}\,(\mathrm{d}t + k) + B^I. && (3.42)\end{aligned}}$$

As before, the warp factors Z_I appear in both the metric and the Maxwell potentials, so we are assuming a zero-force condition where the branes "float"; hence this is called the "floating brane" ansatz. However, we do not impose the condition that the base space $\mathcal{B}$ must have a covariantly constant spinor.

To continue, one defines the frames

$$e^0 \equiv -Z^{-1}\,(\mathrm{d}t + k), \qquad e^a \equiv Z^{1/2}\,\hat{e}^a, \tag{3.43}$$

where $\hat{e}^a$, $a \in \{1,2,3,4\}$ are frames on $\mathrm{d}s_4^2$. Then one can define

$$\Theta^{(I)} \equiv \mathrm{d}B^{(I)} = \frac{1}{2}\Theta^{(I)}_{ab}\,\hat{e}^a \wedge \hat{e}^b, \qquad K \equiv \mathrm{d}k = \frac{1}{2}K_{ab}\,\hat{e}^a \wedge \hat{e}^b, \tag{3.44}$$

where the indices a, b refer to the components in the *hatted* frames $\hat{e}^a$. It will also help to define 2-forms $\omega_-^{(I)}$ using the anti-self-dual parts of the $\Theta^{(I)}$ via

$$\frac{1}{2}\big(\Theta^{(I)} - \underset{4}{\star}\,\Theta^{(I)}\big) = C_{IJK}\,Z_J\,\omega_-^{(K)}. \tag{3.45}$$

The (00) components of the Einstein equations are then written

$$\sum_I Z_I^{-1}\hat{\nabla}^2 Z_I = -\frac{1}{4}\,Z^{-3}\sum_I Z_I\,\Theta^{(I)}_{ab}\,\big(Z_I\,\Theta^{(I)}_{ab} - 2K_{ab}\big), \tag{3.46}$$

where $\hat{\nabla}^2$ is the Laplacian on $\mathrm{d}s_4^2$. The off-diagonal $(0a)$ components of the Einstein equations give

$$\mathrm{d} \underset{4}{\star} K = \sum_I \mathrm{d}Z_I \wedge \underset{4}{\star} \Theta^{(I)}. \tag{3.47}$$

To write the remaining equations it is helpful to define some 2-forms on $\mathcal{B}$:

$$\mathcal{R}_\pm^{(I)} \equiv \frac{1}{2} Z_I \left(\Theta^{(I)} \pm \underset{4}{\star} \Theta^{(I)}\right), \qquad \mathcal{P}_\pm \equiv \frac{1}{2}\left(K \pm \underset{4}{\star} K\right) - \frac{1}{2} \sum_{J=1}^{3} \mathcal{R}_\pm^{(J)}, \tag{3.48}$$

(no sum on I). Then define a bilinear form $\mathcal{T}_{ab}(X, Y)$ of two 2-forms X, Y:

$$\mathcal{T}_{ab}(X, Y) \equiv \frac{1}{2}\left(X_{ac} Y_{bc} + X_{bc} Y_{ac}\right) - \frac{1}{4} \delta_{ab} X_{cd} Y_{cd}. \tag{3.49}$$

This definition is motivated by the fact that $\mathcal{T}_{ab}(X, X)$ is just the stress-energy tensor for the electromagnetic field strength X:

$$\mathcal{T}_{ab}(X, X) = X_{ac} X_{bc} - \frac{1}{4} \delta_{ab} X_{cd} X_{cd}. \tag{3.50}$$

Using $\mathcal{T}_{ab}(X, Y)$, the Einstein equations on the 4d base $\mathcal{B}$ can be written

$$\hat{R}_{ab} - \frac{1}{2} \hat{R}\, \delta_{ab} = 2Z^{-3}\, \mathcal{T}_{ab}(\mathcal{P}_+, \mathcal{P}_-) - \sum_{I=1}^{3} \mathcal{T}_{ab}\left(\frac{1}{2}\left(\Theta^{(I)} + \underset{4}{\star} \Theta^{(I)}\right), \omega_-^{(I)}\right), \tag{3.51}$$

where we note that in fact the 4d Ricci scalar $\hat{R} = 0$ because the right-hand side is traceless.

Next we have the equation of motion for the scalars Z_I:

$$Z_1^{-1} \hat{\nabla}^2 Z_1 \; - \; Z_3^{-1} \hat{\nabla}^2 Z_3 = \frac{1}{2} Z^{-3} \Big[Z_1\, \Theta_{ab}^{(1)} \left(Z_1\, \Theta_{ab}^{(1)} - 2K_{ab}\right) - Z_3\, \Theta_{ab}^{(3)} \left(Z_3\, \Theta_{ab}^{(3)} - 2K_{ab}\right) \Big], \tag{3.52}$$

and similarly for cyclic permutations of $\{1,2,3\}$. Finally, the Maxwell equations,

$$\mathrm{d}\underset{5}{\star}\big(Q_{IJ}\,F^J\big) = \frac{1}{4}C_{IJK}\,F^J\wedge F^K, \tag{3.53}$$

reduce to the two equations

$$\begin{aligned}\mathrm{d}\underset{4}{\star}\mathrm{d}Z_I &= \frac{1}{2}C_{IJK}\,\Theta^{(J)}\wedge\Theta^{(K)}\\ &\qquad + Z^{-3}Z_I\,K\wedge\Big(K+\underset{4}{\star}K+2\,\mathcal{R}_-^{(I)}-\sum_{J=1}^{3}Z_J\Theta^{(J)}\Big),\end{aligned} \tag{3.54}$$

and

$$\mathrm{d}\bigg(Z^{-3}Z_I\Big(K+\underset{4}{\star}K+2\,\mathcal{R}_-^{(I)}-\sum_{J=1}^{3}Z_J\Theta^{(J)}\Big)\bigg)=0, \tag{3.55}$$

where there is no sum on I and $\star_4\,\mathrm{d}\star_4\,\mathrm{d}Z_I \equiv \hat{\nabla}^2 Z_I$. Combining (3.46), (3.52) and (3.54), one obtains three algebraic constraints

$$\begin{aligned}&\mathcal{P}_+\wedge\mathcal{P}_+ + \mathcal{P}_+\wedge\mathcal{R}_+^{(1)}\\ &\qquad + \frac{1}{4}\big(\mathcal{R}_-^{(1)}-\mathcal{R}_-^{(2)}+\mathcal{R}_-^{(3)}\big)\wedge\big(\mathcal{R}_-^{(1)}+\mathcal{R}_-^{(2)}-\mathcal{R}_-^{(3)}\big)=0,\end{aligned} \tag{3.56}$$

along with cyclic permutations of $\{1,2,3\}$.

3.2.1 The floating brane equations

To reach a tractable system of equations, the next step in [31] is to make the simplifications

$$\mathcal{P}_+ = 0, \qquad \omega_-^{(1)} = \omega_-^{(2)} = 0, \qquad \omega_-^{(3)} \neq 0. \tag{3.57}$$

Then the above equations can be reduced to the following: First, the base space $\mathcal{B}$ is required to be a Euclidean-signature Einstein-Maxwell solution,

$$\hat{R}_{\mu\nu} = \frac{1}{2}\Big(\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho} - \frac{1}{4}\, g_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\Big), \tag{3.58}$$

where $\mathcal{F}$ is a Maxwell 2-form determined by the base geometry, and unrelated to the F^I. We decompose $\mathcal{F}$ as

$$\mathcal{F} \equiv \Theta^{(3)} - \omega_-^{(3)}, \tag{3.59}$$

where $\Theta^{(3)}$ is self-dual, and $\omega_-^{(3)}$ is anti-self-dual. The Maxwell equations $\mathrm{d}\mathcal{F} = \mathrm{d}\star_4\mathcal{F} = 0$ imply that $\Theta^{(3)}$ and $\omega_-^{(3)}$ are harmonic. As the notation implies, this defines the magnetic 2-form field strength $\Theta^{(3)}$. Once the base geometry is determined, the remaining equations reduce to a linear system which we call the "floating brane equations":

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_1 = \Theta^{(2)} \wedge \Theta^{(3)}, \qquad \Theta^{(2)} - \underset{4}{\star}\Theta^{(2)} = 2\, Z_1\, \omega_-^{(3)}, \tag{3.60}$$

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_2 = \Theta^{(1)} \wedge \Theta^{(3)}, \qquad \Theta^{(1)} - \underset{4}{\star}\Theta^{(1)} = 2\, Z_2\, \omega_-^{(3)}, \tag{3.61}$$

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_3 = \Theta^{(1)} \wedge \Theta^{(2)} - \omega_-^{(3)} \wedge (\mathrm{d}k - \underset{4}{\star}\mathrm{d}k), \tag{3.62}$$

$$\mathrm{d}k + \underset{4}{\star}\mathrm{d}k = \frac{1}{2}\sum_I Z_I\, (\Theta^{(I)} + \underset{4}{\star}\Theta^{(I)}). \tag{3.63}$$

This system is somewhat reminiscent of the BPS equations (3.12)–(3.14), but with an extra complication due to the anti-self-dual parts of $\Theta^{(1)}, \Theta^{(2)}$.

Therefore the equations of motion of the 5d $\mathcal{N} = 2$ theory on the floating brane ansatz can be solved by the following steps: First, find a Euclidean-Einstein-Maxwell base. The Maxwell 2-form defines the 2-forms $\Theta^{(3)}$ and $\omega_-^{(3)}$ via (3.59). We then solve the first layer of coupled linear equations (3.60) and (3.61) for $Z_1, Z_2, \Theta^{(1)}$, and $\Theta^{(2)}$. These enter as sources in the second layer of coupled linear equations (3.62) and (3.63), which we solve finally for Z_3 and k.

3.3 BPS solutions in 6d

The material in this section comes largely from three papers, [70, 71, 72].

In 6 dimensions we consider $\mathcal{N}=1$ supergravity coupled to one anti-self-dual tensor multiplet. The minimal 6d $\mathcal{N}=1$ supergravity contains a graviton $g_{\mu\nu}$, a left-handed gravitino ψ_μ^A, and a 2-form potential $B_{\mu\nu}^+$ with self-dual 3-form field strength $G_+^{(3)}$. Due do the self-duality condition on $G_+^{(3)}$, this theory does not have a covariant Lagrangian formulation. To this theory we add an anti-self-dual tensor multiplet consisting of a 2-form potential $B_{\mu\nu}^-$ with anti-self-dual field strength, a right-handed fermion χ_A, and a scalar ϕ. The combined 3-form field strength $G^{(3)}$ now is *general* with both self-dual and anti-self-dual parts. Hence this new theory does have a covariant Lagrangian formulation; however, we will not need to write it down.

The purpose of adding the extra tensor multiplet is to give a theory that yields, upon dimensional reduction on a circle, the previously-discussed 5d $\mathcal{N}=2$ theory with two vector multiplets. That is, the 6d $\mathcal{N}=1$ theory with one anti-self-dual tensor is the simplest place to look for 3-charge microstate geometries in 6 dimensions.

Since we are again looking for BPS solutions, we can take advantage of supersymmetry. Thus rather than solving the full Einstein equations, we can look at the SUSY variation $\delta\psi_\mu^A = 0$. This gives the equation

$$\nabla_\mu \epsilon - \frac{1}{4} G_{\mu\rho\lambda}\, \gamma^{\rho\lambda}\, \epsilon = 0, \tag{3.64}$$

which indicates the existence of a "twisted" Killing spinor ϵ. One can then choose a frame in which $G^{(3)}$ "cancels" the spin connection in this equation such that ϵ satisfies the simpler

$$\partial_\mu \epsilon = 0. \tag{3.65}$$

Out of ϵ one can form various spinor bilinears, and from those conclude the general structure of supersymmetric solutions. This ultimately leads to the metric ansatz [70]

$$\boxed{\mathrm{d}s_6^2 = -2H^{-1}\,(\mathrm{d}v + \beta)\big(\mathrm{d}u + \omega + \frac{1}{2}\mathcal{F}(\mathrm{d}v + \beta)\big) + H\,\mathrm{d}s_4^2(\mathcal{B}),} \tag{3.66}$$

where the functions $H, \mathcal{F}$ and the 1-forms β, ω are assumed independent of u such that $\partial/\partial u$ is a null Killing vector. The coordinate v is periodically identified. The metric on the 4d base $\mathcal{B}$ is given by

$$\mathrm{d}s_4^2(\mathcal{B}) = h_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \tag{3.67}$$

where the h_{ij} may also be functions of v, so that $\mathrm{d}s_4^2$ is really a 1-parameter family of 4-metrics. The 1-forms β, ω have legs only along the 4d base $\mathcal{B}$. We can write the metric (3.66) using a null-orthonormal frame

$$\mathrm{d}s_6^2 = -2e^+\,e^- + \delta_{ij}\,e^i\,e^j, \tag{3.68}$$

by defining

$$e^+ \equiv H^{-1}\big(\mathrm{d}v + \beta\big), \qquad e^- \equiv \mathrm{d}u + \omega + \frac{1}{2}\mathcal{F}H\,e^+, \qquad e^i = H^{1/2}\,\tilde{e}^i, \tag{3.69}$$

where $\tilde{e}^i$, $i \in 1, 2, 3, 4$ are an orthonormal frame on $\mathcal{B}$. In contrast to the conventions of [70], we will work using the more standard definition of the Hodge dual,

$$\underset{n}{\star}\,(e^{i_1} \wedge \ldots \wedge e^{i_p}) = \frac{1}{(n-p)!}\,\varepsilon^{i_1 \ldots i_p}{}_{j_1 \ldots j_{n-p}}\,(e^{j_1} \wedge \ldots \wedge e^{j_{n-p}}). \tag{3.70}$$

We will find it helpful to define a "restricted" exterior derivative $\tilde{\mathrm{d}}$ on $\mathcal{B}$ that acts only on the x^i and treats v as a constant:

$$\tilde{\mathrm{d}}(\varphi_I\,\mathrm{d}x^I) = \left(\frac{\partial \varphi_I}{\partial x^i}\right)_v \mathrm{d}x^i \wedge \mathrm{d}x^I, \tag{3.71}$$

where I is a multi-index and $\varphi_I\,\mathrm{d}x^I$ is a generic p-form. Using this, we define a "covariant" exterior derivative $\mathcal{D}$ via

$$\mathcal{D}\varphi \equiv \tilde{\mathrm{d}}\varphi - \beta \wedge \partial_v \varphi, \tag{3.72}$$

such that the total exterior derivative d (on u-independent fields) can be written

$$\mathrm{d}\varphi = (\mathrm{d}v + \beta) \wedge \dot{\varphi} + \mathcal{D}\varphi, \tag{3.73}$$

where we have used a dot to denote the v-derivative $\dot{\phi} \equiv \partial_v \phi$.

Next, supersymmetry implies the existence of 3 almost complex structures (i.e. linear operators) $\hat{J}^{(A)}$ on $\mathcal{B}$ that satisfy the quaternion algebra

$$\boxed{\hat{J}^{(A)}\hat{J}^{(B)} = -\delta^{AB} + \varepsilon^{ABC}\hat{J}^{(C)}.} \tag{3.74}$$

A note regarding notation: With hats, $\hat{J}^{(A)} : T\mathcal{B} \to T\mathcal{B}$ are linear operators on the tangent space of a point in $\mathcal{B}$ that satisfy the properties of an almost complex structure. Without hats, $J^{(A)} : T\mathcal{B} \times T\mathcal{B} \to \mathbb{R}$ are the 2-forms (i.e. Kähler 2-forms) associated to these almost complex structures. They are related via

$$\hat{J}^{(A)}(X) = \delta^{ab} J^{(A)}(\tilde{e}_a, X)\,\tilde{e}_b, \tag{3.75}$$

where $X \in T\mathcal{B}$ is any vector, and $\tilde{e}_a \in T\mathcal{B}$ are an orthonormal frame of $\mathcal{B}$. Supersymmetry further implies that the 2-forms $J^{(A)}$ and the 1-form β must satisfy

$$\boxed{\begin{aligned} \tilde{\mathrm{d}} J^{(A)} &= \partial_v(\beta \wedge J^{(A)}), \\ \mathcal{D}\beta &= \underset{4}{\star}\mathcal{D}\beta. \end{aligned}} \qquad \begin{aligned} &(3.76) \\ &(3.77) \end{aligned}$$

These equations together define the structure of the 4d "base space" $\mathcal{B}$ (really a 1-parameter family of base spaces parametrized by the coordinate v).

We should point out here that the main source of difficulty with 6-dimensional solutions is the β equation (3.77). From the definition of $\mathcal{D}$ (3.72) we see that this is a *non-linear* equation. We should not be surprised that non-linear equations turn up in supergravity; however, this is in contrast to both of the 5-dimensional systems previously discussed, where it was possible to reduce things to linear systems. Expanding β in Fourier modes around the v circle,

$$\beta \equiv \sum_{m=-\infty}^{\infty} e^{imv}\, \beta_{(m)}, \tag{3.78}$$

where the 1-forms $\beta_{(m)}$ are independent of v, one can re-write (3.77) for each mode as

$$\big(1 - \underset{4}{\star}\big)\bigg(\tilde{\mathrm{d}}\beta_{(\ell)} + \frac{i}{2}\sum_{m,n}(m-n)\,\delta_{\ell,m+n}\,\beta_{(m)} \wedge \beta_{(n)}\bigg) = 0, \tag{3.79}$$

which are the self-dual Yang-Mills equations based on the Witt algebra (or classical Virasoro algebra, without central extension):

$$[L_m, L_n] = (m-n)L_{m+n}, \qquad m, n \in \mathbb{Z}. \tag{3.80}$$

The Witt algebra generates the (identity component of) the diffeomorphism group of the circle. Then (3.77) resembles the equation for a $\mathrm{Diff}(S^1)$-instanton on the family of base spaces $\mathcal{B}$. However, $\pi_3(S^1) = 0$, so there are no topological solutions in the usual sense of instantons.

3.3.1 The equations of motion

In order to write the rest of the equations of motion, we first define the anti-self-dual 2-forms ψ and $\hat{\psi}$

$$\psi \equiv H\hat{\psi} \equiv \frac{1}{8} H\, \varepsilon_{ABC}\, \langle J^{(A)}, \partial_v J^{(B)} \rangle_{\mathcal{B}}\, J^{(C)}, \tag{3.81}$$

where $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ is the contraction on the base space $\mathcal{B}$, defined on p-forms via

$$\langle a, b \rangle_{\mathcal{B}} = \frac{1}{p!} a_{i_1..i_p} b^{i_1..i_p}. \tag{3.82}$$

where indices are raised with the metric on $\mathcal{B}$. Then one can make the following ansatz for the 3-form field strength G:

$$\begin{aligned} e^{\sqrt{2}\phi} G = &-\tfrac{1}{2} \underset{4}{\star} (\mathcal{D}H + H\dot{\beta} - \sqrt{2} H\, \mathcal{D}\phi) \\ &- \tfrac{1}{2}\, e^+ \wedge e^- \wedge (H^{-1}\mathcal{D}H + \dot{\beta} + \sqrt{2}\mathcal{D}\phi) \\ &- e^+ \wedge (-H\psi + \tfrac{1}{2}(\mathcal{D}\omega)^- - \mathcal{K}) + \tfrac{1}{2} H^{-1} e^- \wedge \mathcal{D}\beta, \end{aligned} \tag{3.83}$$

where $\mathcal{K}$ is a self-dual 2-form (yet to be determined) on $\mathcal{B}$, and

$$(\mathcal{D}\omega)^{\pm} \equiv \tfrac{1}{2} (\mathcal{D}\omega \pm \underset{4}{\star} \mathcal{D}\omega). \tag{3.84}$$

The equations of motion to be satisfied by G are

$$\mathrm{d}G = 0, \qquad\qquad \mathrm{d}(e^{2\sqrt{2}\phi} \underset{6}{\star} G) = 0, \tag{3.85}$$

which under the above ansatz become (using (3.77))

$$\mathcal{D}\Big(e^{\sqrt{2}\phi}H^{-1}(\mathcal{K}-H\mathcal{G}-H\psi)\Big)-\tfrac{1}{2}\partial_v \underset{4}{\star}\Big(\mathcal{D}(e^{\sqrt{2}\phi}H)+e^{\sqrt{2}\phi}H\dot{\beta}\Big) \\ -e^{\sqrt{2}\phi}H^{-1}\dot{\beta}\wedge\big(\mathcal{K}-H\mathcal{G}-H\psi\big)=0, \tag{3.86}$$

$$\mathcal{D}\Big(e^{-\sqrt{2}\phi}H^{-1}(\mathcal{K}+H\mathcal{G}+H\psi)\Big)+\tfrac{1}{2}\partial_v \underset{4}{\star}\Big(\mathcal{D}(e^{-\sqrt{2}\phi}H)+e^{-\sqrt{2}\phi}H\dot{\beta}\Big) \\ -e^{-\sqrt{2}\phi}H^{-1}\dot{\beta}\wedge\big(\mathcal{K}+H\mathcal{G}+H\psi\big)=0, \tag{3.87}$$

and

$$\mathcal{D}\underset{4}{\star}\Big(\mathcal{D}(e^{\sqrt{2}\phi}H)+e^{\sqrt{2}\phi}H\dot{\beta}\Big)=-2e^{\sqrt{2}\phi}(\mathcal{K}-H\mathcal{G})\wedge\mathcal{D}\beta, \tag{3.88}$$

$$\mathcal{D}\underset{4}{\star}\Big(\mathcal{D}(e^{-\sqrt{2}\phi}H)+e^{-\sqrt{2}\phi}H\dot{\beta}\Big)=2e^{-\sqrt{2}\phi}(\mathcal{K}+H\mathcal{G})\wedge\mathcal{D}\beta, \tag{3.89}$$

where the 2-form $\mathcal{G}$ is defined as

$$\mathcal{G}\equiv\frac{1}{2H}\big[(\mathcal{D}\omega)^{+}+\tfrac{1}{2}\mathcal{F}\,\mathcal{D}\beta\big]. \tag{3.90}$$

There is also a portion of the Einstein equations that is not automatically solved under the SUSY conditions (3.74), (3.76) and (3.77). This gives one final equation of motion. In order to write it down, first define the 1-form

$$\mathcal{L}\equiv\dot{\omega}+\tfrac{1}{2}\mathcal{F}\,\dot{\beta}-\tfrac{1}{2}\mathcal{D}\mathcal{F}. \tag{3.91}$$

Then the final equation is written

$$\underset{4}{\star}\mathcal{D}\underset{4}{\star}\mathcal{L}-2\langle\dot{\beta},\mathcal{L}\rangle_{\mathcal{B}}=-\tfrac{1}{2}Hh^{ij}\partial_v^2(Hh_{ij})-\tfrac{1}{4}\partial_v(Hh^{ij})\partial_v(Hh_{ij}) \\ -2H^2\dot{\phi}^2+\tfrac{1}{2}H^{-2}\big\|\mathcal{D}\omega+\tfrac{1}{2}\mathcal{F}\,\mathcal{D}\beta\big\|_{\mathcal{B}}^2 \\ -2H^{-2}\big\|\mathcal{K}-H\psi+\tfrac{1}{2}(\mathcal{D}\omega)^{-}\big\|_{\mathcal{B}}^2, \tag{3.92}$$

where on p-forms,

$$\|a\|^2_{\mathcal{B}} = \langle a, a\rangle_{\mathcal{B}} = \frac{1}{p!} a_{i_1..i_p} a^{i_1..i_p}. \tag{3.93}$$

3.3.2 A linear system

The above equations look rather daunting. However, it was found in [72] that there is yet another *linear* system of equations hiding within them. First one defines the following suggestively-named quantities,

$$\Theta^{(1)} \equiv e^{-\sqrt{2}\phi} H^{-1}(\mathcal{K} + H\mathcal{G} + H\psi), \qquad \Theta^{(2)} \equiv e^{\sqrt{2}\phi} H^{-1}(-\mathcal{K} + H\mathcal{G} + H\psi), \tag{3.94}$$

$$Z_1 \equiv e^{\sqrt{2}\phi} H, \qquad Z_2 \equiv e^{-\sqrt{2}\phi} H. \tag{3.95}$$

These relations can be easily inverted, which we write down here for completeness:

$$H^2 = Z_1 Z_2, \qquad e^{2\sqrt{2}\phi} = Z_1 Z_2^{-1}, \tag{3.96}$$

$$2\mathcal{K} = \tfrac{1}{2} Z_1 (\Theta^{(1)} + \underset{4}{\star}\Theta^{(1)}) - \tfrac{1}{2} Z_2 (\Theta^{(2)} + \underset{4}{\star}\Theta^{(2)}), \tag{3.97}$$

$$2H\mathcal{G} = \tfrac{1}{2} Z_1 (\Theta^{(1)} + \underset{4}{\star}\Theta^{(1)}) + \tfrac{1}{2} Z_2 (\Theta^{(2)} + \underset{4}{\star}\Theta^{(2)}), \tag{3.98}$$

$$\tfrac{1}{2}(\Theta^{(1)} - \underset{4}{\star}\Theta^{(1)}) = e^{-\sqrt{2}\phi}\psi = Z_2 \hat{\psi}, \tag{3.99}$$

$$\tfrac{1}{2}(\Theta^{(2)} - \underset{4}{\star}\Theta^{(2)}) = e^{\sqrt{2}\phi}\psi = Z_1 \hat{\psi}. \tag{3.100}$$

Using the functions Z_1, Z_2 and the 2-forms $\Theta^{(1)}, \Theta^{(2)}$, it was shown in [72] that the equations of the previous section can be rearranged into the following system:

$$\boxed{\tfrac{1}{2}(\Theta^{(1)} - \underset{4}{\star}\Theta^{(1)}) = Z_2 \hat{\psi}, \qquad \tfrac{1}{2}(\Theta^{(2)} - \underset{4}{\star}\Theta^{(2)}) = Z_1 \hat{\psi}} \tag{3.101}$$

and

$$\boxed{\begin{aligned} &\mathcal{D}\Theta^{(2)} - \dot\beta \wedge \Theta^{(2)} = -\tfrac{1}{2}\partial_v\big(\underset{4}{\star}(\mathcal{D}Z_1 + Z_1\dot\beta)\big), && (3.102)\\ &\mathcal{D}\Theta^{(1)} - \dot\beta \wedge \Theta^{(1)} = -\tfrac{1}{2}\partial_v\big(\underset{4}{\star}(\mathcal{D}Z_2 + Z_2\dot\beta)\big), && (3.103)\\ &\mathcal{D}\underset{4}{\star}\big(\mathcal{D}Z_1 + Z_1\dot\beta\big) = 2\Theta^{(2)} \wedge \mathcal{D}\beta, && (3.104)\\ &\mathcal{D}\underset{4}{\star}\big(\mathcal{D}Z_2 + Z_2\dot\beta\big) = 2\Theta^{(1)} \wedge \mathcal{D}\beta, && (3.105)\end{aligned}}$$

and

$$\boxed{\begin{aligned} \underset{4}{\star}\mathcal{D}\underset{4}{\star}\mathcal{L} - 2\langle\dot\beta, \mathcal{L}\rangle_{\mathcal{B}} = &-\tfrac{1}{2}\sqrt{Z_1Z_2}\,h^{ij}\partial_v^2\big(\sqrt{Z_1Z_2}\,h_{ij}\big)\\ &-\tfrac{1}{4}\partial_v\big(\sqrt{Z_1Z_2}\,h^{ij}\big)\,\partial_v\big(\sqrt{Z_1Z_2}\,h_{ij}\big)\\ &-2Z_1Z_2\big(\partial_v\phi\big)^2 + 2\underset{4}{\star}\Big(\Theta^{(1)} \wedge \Theta^{(2)} - \hat\psi \wedge \mathcal{D}\omega\Big),\end{aligned}} \tag{3.106}$$

and finally

$$\boxed{\mathcal{D}\omega + \underset{4}{\star}\mathcal{D}\omega = 2Z_1\Theta^{(1)} + 2Z_2\Theta^{(2)} - \mathcal{F}\,\mathcal{D}\beta - 4Z_1Z_2\hat\psi,} \tag{3.107}$$

where

$$\mathcal{L} \equiv \dot\omega + \tfrac{1}{2}\mathcal{F}\,\dot\beta - \tfrac{1}{2}\mathcal{D}\mathcal{F}. \tag{3.108}$$

We can see that this new system is linear, and vaguely reminiscent of the "floating brane" system (3.60)–(3.63), but with new complications due to v-dependence. We will see in Chapter 4 that this resemblance is not accidental.

So then the process of finding 6d BPS solutions is as follows: First determine a v-family of base spaces $\mathcal{B}$ with almost-complex structures $J^{(A)}$ that satisfy the relations (3.76), (3.77):

$$\tilde{\mathrm{d}}J^{(A)} = \partial_v(\beta \wedge J^{(A)}), \tag{3.109}$$

$$\mathcal{D}\beta = \underset{4}{\star}\mathcal{D}\beta. \tag{3.110}$$

The $J^{(A)}$ then determine the anti-self-dual 2-form $\hat{\psi}$ via

$$\hat{\psi} \equiv \frac{1}{8}\,\varepsilon_{ABC}\,\langle J^{(A)}, \partial_v J^{(B)}\rangle_{\mathcal{B}}\,J^{(C)}. \tag{3.111}$$

Next using the definitions (3.101), solve the coupled linear equations (3.102)–(3.105) for $Z_1, Z_2, \Theta^{(1)}, \Theta^{(2)}$. Finally, solve the equations (3.106) and (3.107) to obtain ω and $\mathcal{F}$.

3.3.3 The v-independent case: Supertubes

If we turn off the v dependence in equations (3.76), (3.77) and (3.101)–(3.107), they simplify immensely. First we see from (3.76) that the base $\mathcal{B}$ must be hyper-Kähler with anti-self-dual Kähler forms $J^{(A)}$. The nonlinear β equation (3.77) simplifies to

$$\tilde{\mathrm{d}}\beta = \underset{4}{\star}\tilde{\mathrm{d}}\beta, \tag{3.112}$$

and setting $\Theta^{(3)} \equiv 2\,\tilde{\mathrm{d}}\beta$ defines a 2-form $\Theta^{(3)}$ that is self-dual and harmonic. Next we see that $\hat{\psi} = 0$, and hence (3.101)–(3.103) tell us that $\Theta^{(1)}, \Theta^{(2)}$ are also self-dual and harmonic. The rest of the equations simplify to

$$\Theta^{(I)} = \underset{4}{\star}\Theta^{(I)}, \qquad \mathrm{d}\Theta^{(I)} = 0, \tag{3.113}$$

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_1 = \Theta^{(2)} \wedge \Theta^{(3)}, \tag{3.114}$$

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_2 = \Theta^{(1)} \wedge \Theta^{(3)}, \tag{3.115}$$

$$-\mathrm{d}\underset{4}{\star}\mathrm{d}\mathcal{F} = 4\,\Theta^{(1)} \wedge \Theta^{(2)}, \tag{3.116}$$

$$\mathrm{d}\omega + \underset{4}{\star}\mathrm{d}\omega = 2Z_1\Theta^{(1)} + 2Z_2\Theta^{(2)} - \frac{1}{2}\mathcal{F}\Theta^{(3)}. \tag{3.117}$$

We see that if we identify $Z_3 \equiv -\frac{1}{4}\mathcal{F}$ and $k \equiv \frac{1}{2}\omega$, then these are exactly the 5d BPS equations (3.12)–(3.14).

Regularity of supertubes in 6d

We hinted earlier that supertubes can be made smooth in 6 dimensions. For a long, straight supertube, take a simple $\mathbb{R}^4$ base and choose cylindrical coordinates on $\mathbb{R}^4$,

$$\mathrm{d}s_4^2 = \mathrm{d}z^2 + \mathrm{d}r^2 + r^2\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{3.118}$$

such that the coordinate z runs along the length of the supertube. Then applying the supertube boundary conditions in Section 3.1.2, we find that $Z_1, Z_2 \sim 1/r$ near the supertube, while Z_3 remains finite. For concreteness and simplicity, we will take

$$Z_1 = Z_2 = \frac{1}{r}, \qquad \mathcal{F} = -1. \tag{3.119}$$

then we find β and ω are given by

$$\beta = -\frac{1}{r}\,\mathrm{d}z - \cos\theta\,\mathrm{d}\phi, \qquad \omega = -\frac{1}{r}\,\mathrm{d}z. \tag{3.120}$$

Putting these into the metric (3.66) results in

$$\mathrm{d}s_6^2 = -2r(\mathrm{d}v+\beta)(\mathrm{d}u+\omega) + r(\mathrm{d}v+\beta)^2 + \frac{1}{r}\,\mathrm{d}z^2 + \frac{1}{r}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\big). \tag{3.121}$$

One can then complete the square, giving

$$\mathrm{d}s_6^2 = -r(\mathrm{d}u+\omega)^2 + r\big(\mathrm{d}v - \mathrm{d}u + \beta - \omega\big)^2 + \frac{1}{r}\,\mathrm{d}z^2 + \frac{1}{r}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\big). \tag{3.122}$$

Defining a new coordinate $\psi \equiv v - u$ and inserting β, ω gives, after some rearrangement

$$\mathrm{d}s_6^2 = -r\,\mathrm{d}u\big(\mathrm{d}u - \frac{2}{r}\mathrm{d}z\big) + r(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2 + \frac{1}{r}\big(\mathrm{d}r^2 + r^2\,\mathrm{d}\Omega_2^2\big). \qquad (3.123)$$

We recognize the last two terms as a 1-center Gibbons-Hawking metric, which is simply another way to write the flat metric on $\mathbb{R}^4$. Therefore at small r we have

$$\mathrm{d}s_6^2 \sim -r\,\mathrm{d}u^2 - 2\,\mathrm{d}u\,\mathrm{d}z + \mathrm{d}s_4^2(\mathbb{R}^4). \qquad (3.124)$$

As promised, the supertube solution is smooth. The coordinate z is spacelike everywhere except at the location of the supertube itself, where it becomes null. We see that an essential aspect of this smooth limit is the 1-form β which provides the KK monopole part $\cos\theta\,\mathrm{d}\phi$. This causes the v circle to fiber nontrivially over the S^2 Gaussian surface around the supertube to create an S^3, which smoothly pinches off at the location of the supertube, just as the S^3 of the angular coordinates in $\mathbb{R}^4$ shrinks to zero size at the origin.

Chapter 4

Non-BPS Microstates in 5d on a LeBrun base

The material in this chapter is taken from two papers, [33, 34] which I authored with collaborators Nikolay Bobev and Nick Warner.

4.1 Motivation

Now that we have set up the mathematical background, we will move on to seek *non-BPS* solutions of the 5-dimensional $\mathcal{N} = 2$ supergravity theory with two vector multiplets.

The BPS story outlined in Section 3.1 is quite remarkable. As shown in [73], starting from a simple brane configuration in M-theory, the problem can be reduced to a system of equations (3.12)–(3.14) with an "upper triangular" structure. At each level the equations are *linear*, and the solutions from one level feed into the next level quadratically as sources. While the full equations of 5d supergravity are very complicated, the BPS equations are simple enough that they can be solved in complete generality, in terms of eight arbitrary harmonic functions on $\mathbb{R}^3$ as in (3.30) and (3.31). The resulting metric (3.33) has, generically, only two isometries (generated by ∂/∂_t and ∂/∂_ψ) and so is "cohomogeneity three". The full solution is a soliton made of pure topological bubbles and fluxes [74, 75, 25, 16]

These BPS solutions can be used to construct smooth BPS microstate geometries that share some key features. First, they are constructed with a time fiber over a hyper-Kähler Gibbons-Hawking (GH) base [69], which contains topologically non-trivial 2-cycles, or "bubbles". These cycles are threaded with self-dual cohomological fluxes (3.34), which can be integrated over a Gaussian surface to reveal "charges dissolved in fluxes". The bubbles pinch off smoothly at either end with no singularities (3.35). The fluxes and topological cycles are further related by the "bubble equations" (3.40)

$$\sum_{\substack{j=1 \\ j\neq i}}^{N} \Pi_{ij}^{(1)} \Pi_{ij}^{(2)} \Pi_{ij}^{(3)} \frac{q_i q_j}{r_{ij}} = -2\left(m_0\, q_i + \frac{1}{2}\sum_{I=1}^{3} k_i^I\right), \tag{4.1}$$

which state that the size r_{ij} of each bubble, roughly speaking, scales in proportion to the product of the three types of flux $\Pi_{ij}^{(I)}$ trapped on it. Hence while gravity alone would tend to compress the bubbles, the fluxes tend to expand them, and they reach an equilibrium between these effects.

Toward non-BPS

This structure gives a hint at what the generic microstates of 5d BPS black holes might look like in the supergravity régime. An important question to answer is what happens when we get away from BPS. Realistic black holes are not supersymmetric, and a satisfactory solution to the information problem requires understanding the microstates of non-BPS black holes (as well as non-extremal and Schwarzschild black holes).

Unfortunately, the non-BPS case is substantially more difficult. Without the tools of supersymmetry, one must confront the full, nonlinear Einstein equations, and reducing them to anything with the simplicity of (3.12)–(3.14) would be rather miraculous. A few isolated examples exist [26, 27, 28, 29] of truly non-BPS, non-extremal smooth geometries, but no infinite familes are yet known (which are necessary for entropy counting).

However, in the non-BPS *extremal* case, there are linear systems which can be solved to obtain infinite families of solutions. One such family are the so-called "almost BPS" solutions [76, 77, 78], where supersymmetry is broken by inverting the orientation of the Gibbons-Hawking base relative to the fluxes. These solutions have been shown to exhibit a rich variety of phenomena not seen in the BPS case [79, 80, 81].

A further avenue of attack was revealed with the "floating brane" ansatz in 5 dimensions [31], which dispenses with supersymmetry, but keeps extremality, imposing a balance between gravitational and electromagnetic forces. It was found that this leads to yet another *linear system* of equations (3.60)–(3.63), this time on a Euclidean-signature Einstein-Maxwell base. The authors of [31] explore solutions based on the Israel-Wilson-Perjés metric [82, 83], which interpolates between the BPS and almost-BPS systems, showing that the almost-BPS solutions are subsumed under the floating brane ansatz. Beyond this, a few solutions are known based on various other Euclidean-Einstein-Maxwell metrics analytically continued from classical GR ones [84], but these are again isolated examples, and do not resemble the BPS story very closely.

What we would like to do is find an infinite family of solutions resembling the BPS ones: smooth metrics with an arbitrary number of bubbles held open by flux. The sequence of papers [33, 34, 35] tell the story of how this is accomplished. Instead of a Ricci-flat, hyperkähler Gibbons-Hawking base, we use a family of Kähler metrics studied by LeBrun [85], who showed that these metrics, in addition to being Kähler, are Euclidean-Einstein-Maxwell solutions [86]. The LeBrun metrics are based not on the Laplace equation in $\mathbb{R}^3$, but on the $SU(\infty)$ Toda equation and its linearization. The Toda equation is a notorious nonlinear PDE which is known to be integrable, but thus far has resisted attempts at a general solution. The same equation has turned up in many contexts in string theory and supergravity, and it seems to have deep geometrical significance [87, 88].

Solutions on a LeBrun-Burns base

By choosing an especially simple solution to the Toda equation, one obtains the LeBrun-Burns class of metrics, which is based on the Laplace equation in hyperbolic space H^3. Here is where we focus our first efforts at non-BPS microstates [33]. We find that the floating brane equations (3.60)–(3.63) are solvable in this context and we present an infinite family of solutions. These solutions share many features with the BPS geometries. The LeBrun-Burns metrics have the structure of a $U(1)$ fiber over H^3. In much the same way as Gibbons-Hawking metrics, this $U(1)$ fiber pinches off at points controlled by a harmonic function on H^3, which allows one to construct solutions with several "bubbles" threaded with cohomological fluxes. We also show that with appropriate choices of parameters, the solutions can be made regular and free of CTCs.

However, these solutions also have a few deficiencies. First, the Maxwell field of the LeBrun-Burns metrics is topologically trivial. This means that, while one can use the $U(1)$ fiber to form 2-cycles, only the fluxes of $\Theta^{(1)}, \Theta^{(2)}$ can be trapped on those 2-cycles; $\Theta^{(3)}$ does not participate. In the BPS story, the bubble equations (4.1) require the product of all three trapped fluxes to hold the bubbles open. What we find in the non-BPS case on LeBrun-Burns is that the "bubble equations" are degenerate and do not constrain the sizes of the bubbles in any way. Second, because of this degeneracy, regular solutions effectively have only two types of dipole charges, so the regularity conditions analogous to (3.35) actually demand that most of the parameters be set to zero. And finally, we show that LeBrun-Burns-based solutions cannot be asymptotically flat, which limits their interpretation as microstates of black holes in flat spacetime. What can be achieved asymptotically, however, is the near horizon geometry of a BMPV-like spinning black hole [89], which hints that there might be solutions of the 5d theory (perhaps violating the simplifying assumptions (3.57) of the floating brane ansatz) that restore the asymptotically-flat region.

5d-6d connection

In [34], we find a curious connection between these 5d non-BPS solutions and the 6d BPS system discussed in Section 3.3. It turns out that after re-organizing the BPS equations in the 6-dimensional IIB frame [70, 71, 72], they can be made to look *identical* to the 5d non-BPS floating brane equations on the LeBrun base. Therefore the exact same family of solutions plays two roles, both supersymmetric and non-supersymmetric. The apparent discrepancy is explained in the dimensional reduction from 6 to 5 dimensions: the Killing spinor in 6 dimensions can be charged under the $U(1)$ on which the reduction occurs, which causes it to vanish in 5 dimensions. This is reminiscent of the Scherk-Schwarz mechanism [90, 91], or also "supersymmetry without supersymmetry" [92].

Generalizing to axisymmetric LeBrun

To repair the deficiencies of the LeBrun-Burns solutions [33], we look back to the general LeBrun metrics governed by the Toda equation. In [35], we tackle this problem by imposing an additional axial symmetry. Subject to this extra $U(1)$ symmetry, the Toda equation can be mapped onto the cylindrically-symmetric Laplace equation in $\mathbb{R}^3$ [87, 88]. This allows us to write generic axisymmetric solutions to the Toda equation and thus explore a much wider variety of LeBrun metrics. In particular, we can find infinite families of Euclidean-Einstein-Maxwell metrics whose self-dual 2-form $\Theta^{(3)}$ has flux trapped on the topological bubbles. Therefore on these backgrounds one has a hope of finding non-BPS 5d supergravity solutions with non-trivial bubble equations (and hence flux v.s gravity interactions) reminiscent of (4.1), and with regularity conditions resembling (3.35).

We find that again we are able to solve the entire floating brane system (3.60)–(3.63) on these axisymmetric LeBrun bases. As we had hoped, we find that non-trivial bubble equations do result, and they resemble (4.1), but with new physical effects not present in the BPS case, such as interactions between non-adjacent bubbles. We also find regularity conditions that look nearly the same as (3.35), which means that unlike in the LeBrun-Burns case, the parameters of the solution are not constrained to zero, and can be chosen with a great deal of freedom. This is exciting and teaches us a lot about the physics of non-BPS extremal microstate geometries.

However, one still does not have asymptotically-flat solutions. It turns out that this is a generic feature of floating-brane solutions based on the LeBrun metrics, and is ultimately due to these metrics being Kähler. Instead one obtains solutions asymptotic to warped, rotating $AdS_2 \times S^3$, which can be thought of as near-horizon limits of black holes, and so these solutions still might tell us something about black hole microstates.

One open question is whether the solutions can be found that restore the asymptotically-flat region by relaxing the Kähler condition on the base $\mathcal{B}$ (which could make the equations significantly more difficult to solve).

Plan for discussion of these results

In this chapter we will present the results of [33, 34]. First in Section 4.2 we will discuss LeBrun metrics in general, how they fit within the floating brane ansatz, and how the asymptotics of resulting 5d solutions are affected. Next in Section 4.3 we specialize to the LeBrun-Burns subclass, which can be written as a $U(1)$ fiber over H^3. We will present the solutions and the key aspects of physics obtained in [33]. In Section 4.4 we will discuss the lift to 6 dimensions found in [34], and then in Section 4.5 we will discuss the implications of these solutions and open problems.

The third paper in this sequence, which finds solutions based on the Toda equation of the more general LeBrun metrics with axial symmetry [35], we will present in Chapter 5.

4.2 LeBrun metrics as a base for floating branes

Our objective is to find non-BPS solutions of the 5d $\mathcal{N}=2$ ungauged supergravity theory described in Section 3.1 based on the bosonic action (3.4):

$$S = \frac{1}{2\kappa_5}\int\bigg(\underset{5}{\star}\mathcal{R} - Q_{IJ}\,\mathrm{d}X^I\wedge\underset{5}{\star}\mathrm{d}X^J \\ - Q_{IJ}\,F^I\wedge\underset{5}{\star}F^J - \frac{1}{6}\,C_{IJK}\,F^I\wedge F^J\wedge A^K\bigg), \tag{4.2}$$

where again, $\mathcal{R}$ is the 5d Ricci scalar, $X^I,\ I\in\{1,2,3\}$ are scalar fields, $F^I\equiv dA^I$ are three Maxwell fields, the constants $C_{IJK} = |\varepsilon_{IJK}|$, and the kinetic terms are coupled via the matrix

$$Q_{IJ}\equiv\frac{1}{2}\,\mathrm{diag}\left((X^1)^{-2},(X^2)^{-2},(X^3)^{-2}\right). \tag{4.3}$$

To obtain non-BPS solutions, we will employ the "floating brane" ansatz described in Section 3.2, where the 5d metric and 1-form potentials take the form (3.41) and (3.42):

$$\mathrm{d}s_5^2 = -Z^{-2}\,(\mathrm{d}t+k)^2 + Z\,\mathrm{d}s_4^2(\mathcal{B}), \qquad Z\equiv(Z_1Z_2Z_3)^{1/3}, \tag{4.4}$$

$$A^I\equiv -Z_I^{-1}\,(\mathrm{d}t+k)+B^I, \qquad F^I\equiv\mathrm{d}A^I, \tag{4.5}$$

The "floating" effect is achieved by matching the warp factors of the metric to the electric potentials in the A^I. This will result in extremal solutions where the gravitational attraction between sources is balanced by their electrostatic repulsion. After defining "magnetic field strengths" via

$$\Theta^{(I)}\equiv\mathrm{d}B^I, \tag{4.6}$$

one can show [31] that, after tedious manipulation and some simplifying assumptions (3.57), the equations of motion reduce to a *linear* system of equations (3.60)–(3.63) to be solved on a background 4-metric $\mathrm{d}s_4^2(\mathcal{B})$ which solves the Euclidean-Einstein-Maxwell equations (3.58). One candidate for such a background metric is given by the LeBrun family of metrics [85, 86].

The LeBrun metrics [85] are given by

$$g \equiv \frac{1}{w}(\mathrm{d}\tau + A)^2 + we^u\,(\mathrm{d}x^2 + \mathrm{d}y^2) + w\,\mathrm{d}z^2, \tag{4.7}$$

which is the expression for the most general Kähler metric in 4 dimensions with a single $U(1)$ isometry. The isometry is along the coordinate τ, which is periodic with period 4π. The functions u and w are independent of τ and solve the $SU(\infty)$ Toda equation[1] and its linearization, respectively (here subscripts are partial derivatives):

$$u_{xx} + u_{yy} + (e^u)_{zz} = 0, \tag{4.8}$$

$$w_{xx} + w_{yy} + (e^u w)_{zz} = 0, \tag{4.9}$$

and the 1-form A satisfies

$$\mathrm{d}A = w_x\,\mathrm{d}y \wedge \mathrm{d}z + w_y\,\mathrm{d}z \wedge \mathrm{d}x + (e^u w)_z\,\mathrm{d}x \wedge \mathrm{d}y. \tag{4.10}$$

The form of the $\mathrm{d}y \wedge \mathrm{d}z$ and $\mathrm{d}z \wedge \mathrm{d}x$ components of $\mathrm{d}A$ guarantee that the almost complex structure,

$$I: \qquad \mathrm{d}x \mapsto \mathrm{d}y, \qquad \mathrm{d}z \mapsto \frac{1}{w}(\mathrm{d}\tau + A), \tag{4.11}$$

is integrable. The $\mathrm{d}x \wedge \mathrm{d}y$ component of $\mathrm{d}A$ further implies that the Kähler 2-form,

$$J \equiv (\mathrm{d}\tau + A) \wedge \mathrm{d}z - e^u w\,\mathrm{d}x \wedge \mathrm{d}y, \tag{4.12}$$

[1] Also known as the Boyer-Finley equation [93].

associated to this complex structure is closed, i.e. $\mathrm{d}J = 0$. The w equation (4.9) is then just the integrability condition for the existence of an A satisfying (4.10). Additionally, whenever u satisfies the Toda equation (4.8), then the Ricci scalar of the metric (4.7) vanishes [85]. In particular, the Ricci scalar must vanish in an Einstein-Maxwell solution in 4 dimensions because the electromagnetic stress-energy tensor in 4d is traceless; therefore equation (4.8) is required.

We choose to introduce the frames,

$$\begin{aligned} e^1 &= w^{-1/2}\,(\mathrm{d}\tau + A), \qquad & e^2 &= e^{u/2} w^{1/2}\,\mathrm{d}x, \\ e^3 &= e^{u/2} w^{1/2}\,\mathrm{d}y, & e^4 &= w^{1/2}\,\mathrm{d}z, \end{aligned} \tag{4.13}$$

with orientation,

$$\mathrm{vol}_4 \equiv e^1 \wedge e^2 \wedge e^3 \wedge e^4 = e^u w\, \mathrm{d}\tau \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z, \tag{4.14}$$

such that J is anti-self-dual. It will also be helpful to define the (anti)-self-dual 2-forms

$$\Omega^{(1)}_{\pm} = e^{-u/2}\left(e^1 \wedge e^2 \pm e^3 \wedge e^4\right) \qquad = (\mathrm{d}\tau + A) \wedge \mathrm{d}x \pm w\,\mathrm{d}y \wedge \mathrm{d}z, \tag{4.15}$$

$$\Omega^{(2)}_{\pm} = e^{-u/2}\left(e^1 \wedge e^3 \pm e^4 \wedge e^1\right) \qquad = (\mathrm{d}\tau + A) \wedge \mathrm{d}y \pm w\,\mathrm{d}z \wedge \mathrm{d}x, \tag{4.16}$$

$$\Omega^{(3)}_{\pm} = e^1 \wedge e^4 \pm e^2 \wedge e^3 \qquad = (\mathrm{d}\tau + A) \wedge \mathrm{d}z \pm w e^u\,\mathrm{d}x \wedge \mathrm{d}y, \tag{4.17}$$

hence we can write $J = \Omega^{(3)}_{-}$.

It is worth noting that taking the function u to be a constant is a trivial solution to (4.8). Then (4.9) becomes the Laplace equation on $\mathbb{R}^3$ and the metric (4.7) reduces to the familiar class of Gibbons-Hawking metrics. Similary, if one takes $w = c\,u_z$ for any constant c, then it also satisfies (4.9), as can be seen by differentiating (4.8) with respect to z. This yields the general class of hyper-Kähler metrics with a non-triholomorphic

$U(1)$ isometry [93, 94, 95, 96], which are based upon the affine Toda equation. In these hyper-Kähler limits, however, the resulting system of equations for non-BPS solutions in five dimensions does *not* reduce to the bubbling BPS equations of Section 3.1, but instead one has the more complicated system of equations (3.60)–(3.63). As we will see in Section 4.2.4, this is because the flux background is a mixture of self-dual and anti-self-dual fluxes and these break supersymmetry. In particular, the anti-self-dual flux is non-normalizable since it is proportional to the complex structure. Thus even the simple Gibbons-Hawking and Toda limits of the LeBrun backgrounds extend the class of solutions considered thus far.

The LeBrun metrics are four-dimensional Euclidean Einstein-Maxwell solutions and it is natural to ask whether some of them preserve supersymmetry. Supersymmetric solutions of four-dimensional Euclidean Einstein-Maxwell theory were classified in [97]. The maximally supersymmetric solutions are $\mathbb{R}^4$ or $H_2 \times S^2$. There are two classes of solutions which preserve half of the supersymmetries—the well-known Gibbons-Hawking solutions and the Euclidean Israel-Wilson-Perjés metrics discussed in [98]. Therefore the classification of [97] also demonstrates that the general LeBrun solutions, although they are Kähler, are non-supersymmetric solutions of Einstein-Maxwell theory.

4.2.1 Topological structure

The LeBrun metrics (4.7) are analogous to Gibbons-Hawking metrics, in that they have the structure of a $U(1)$ fiber over a 3-dimensional base. The 3d base, rather than being flat, is given by the metric

$$h = e^u(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2, \tag{4.18}$$

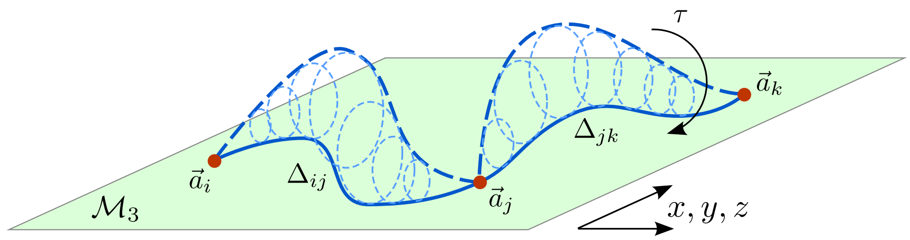

Figure 4.1: *Homological 2-cycles in the LeBrun metric. The τ fiber pinches off at the points $\vec{a}_i$. Sweeping the fiber along a path between any two points forms a homological 2-sphere. Two intersecting 2-cycles are shown.*

which in turn can be thought of as a Riemann surface fibered over a line. If e^u is everywhere finite and non-singular, then the (x, y, z) coordinates can be extended to a topological $\mathbb{R}^3$. In this case, the topology of the 4-manifold can be analyzed in terms of the $U(1)$ fiber parametrized by τ, much like the topology of Gibbons-Hawking manifolds [69].

The function w solves a second-order Laplace-like equation, whose solutions are characterized by a number of points we will call "Gibbons-Hawking points" or "geometric charges", where locally (provided that e^u is smooth),

$$w \sim \frac{1}{r}, \tag{4.19}$$

for some local radial distance r. At these points the τ fiber pinches off to zero size. Hence, if one takes any curve in the 3-dimensional base h that joins two geometric charges, the surface described by the τ fiber over this curve is a homological 2-sphere, as in Figure 4.1.

If e^u is not smooth, it is still possible that g in (4.7) is smooth. One possibility is that z is a radial coordinate, and $e^u(\mathrm{d}x^2 + \mathrm{d}y^2)$ describes a sphere (or perhaps a quotient of a sphere). Another possibility is that $e^u(\mathrm{d}x^2 + \mathrm{d}y^2)$ is a higher-genus Riemann surface, in which case one can have topological cycles that do not involve the τ fiber. Some of these additional topological features will appear in Chapter 5.

4.2.2 As Euclidean-Einstein-Maxwell solutions

Just as in the BPS story of Section 3.1, we will be interested in self-dual harmonic 2-forms. Analogously to (3.23), one can show [33, 34] that self-dual, harmonic 2-forms on LeBrun spaces can be written

$$\Theta \equiv \sum_{a=1}^{3} \partial_a \left(\frac{H}{w} \right) \Omega_+^{(a)} = (\mathrm{d}\tau + A) \wedge \mathrm{d}\frac{H}{w} + w \underset{3}{\star} \mathrm{d}\frac{H}{w}, \tag{4.20}$$

where H solves (4.9),

$$H_{xx} + H_{yy} + (e^u H)_{zz} = 0, \tag{4.21}$$

and $\star_3$ is taken with respect to the 3-metric

$$h = e^u(\mathrm{d}x^2 + \mathrm{d}y^2) + \mathrm{d}z^2. \tag{4.22}$$

By differentiating (4.8) with respect to z, one can show that $H \equiv -(1/2\alpha)\, u_z$ solves (4.21) for any constant α. So define the Maxwell 2-form

$$\mathcal{F} \equiv \Theta + \alpha J, \quad \text{with} \quad H = -\frac{u_z}{2\alpha}, \tag{4.23}$$

where Θ is as in (4.20), J is the Kähler form, and α is a constant. Hence Θ is the self-dual part of $\mathcal{F}$ and αJ is the anti-self-dual part. Because Θ and J are each harmonic, this $\mathcal{F}$ satisfies the Maxwell equations

$$\mathrm{d}\mathcal{F} = \mathrm{d} \underset{4}{\star} \mathcal{F} = 0. \tag{4.24}$$

One can then show [86] that the the Ricci tensor of the LeBrun metric g is given by

$$R_{\mu\nu}(g) = \frac{1}{2}\Big(\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho} - \frac{1}{4}\, g_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma} \Big), \tag{4.25}$$

so that $(g, \mathcal{F})$ are a solution of the Euclidean-Einstein-Maxwell equations (3.58). Therefore the LeBrun metrics give an appropriate base geometry on which to solve the floating brane system (3.60)–(3.63) with the identification (3.59):

$$\mathcal{F} \equiv \Theta^{(3)} - \omega_-^{(3)}. \tag{4.26}$$

For simplicity in matching with this decomposition, we choose $\alpha = -1$, and hence

$$\Theta^{(3)} = \frac{1}{2}(\mathrm{d}\tau + A) \wedge \mathrm{d}\frac{u_z}{w} + \frac{1}{2} w \underset{3}{\star} \mathrm{d}\frac{u_z}{w}, \qquad \omega_-^{(3)} = J. \tag{4.27}$$

4.2.3 Floating branes on a LeBrun base

The next task is to solve the floating brane system (3.60)–(3.63) on the LeBrun base, which we now write with $\omega_-^{(3)} \equiv J$:

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_1 = \Theta^{(2)} \wedge \Theta^{(3)}, \qquad \Theta^{(2)} - \underset{4}{\star}\Theta^{(2)} = 2\, Z_1\, J, \tag{4.28}$$

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_2 = \Theta^{(1)} \wedge \Theta^{(3)}, \qquad \Theta^{(1)} - \underset{4}{\star}\Theta^{(1)} = 2\, Z_2\, J, \tag{4.29}$$

and

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_3 = \Theta^{(1)} \wedge \Theta^{(2)} - J \wedge (\mathrm{d}k - \underset{4}{\star}\mathrm{d}k), \tag{4.30}$$

$$\mathrm{d}k + \underset{4}{\star}\mathrm{d}k = \frac{1}{2}\sum_I Z_I\, (\Theta^{(I)} + \underset{4}{\star}\Theta^{(I)}). \tag{4.31}$$

We will find it convenient to define

$$K^3 \equiv \frac{1}{2}u_z, \quad \text{such that} \quad \Theta^{(3)} = (\mathrm{d}\tau + A) \wedge \mathrm{d}\frac{K^3}{w} + w \underset{3}{\star} \mathrm{d}\frac{K^3}{w}. \tag{4.32}$$

The system (4.28)–(4.31) consists of two "layers" of coupled systems. The first layer (4.28) and (4.29) must be solved for $Z_1, Z_2, \Theta^{(1)}, \Theta^{(2)}$. Once these are obtained, they enter as sources in the second layer (4.30) and (4.31), which we then solve for Z_3 and k. To solve the first layer, one makes the ansätze

$$Z_1 = \frac{K^2 K^3}{w} + L_1, \qquad Z_2 = \frac{K^1 K^3}{w} + L_2, \tag{4.33}$$

$$\Theta^{(1)} = (\mathrm{d}\tau + A) \wedge \mathrm{d}\frac{K^1}{w} + w \star_3 \mathrm{d}\frac{K^1}{w} + Z_2\,(\Omega_-^{(3)} - \Omega_+^{(3)}), \tag{4.34}$$

$$\Theta^{(2)} = (\mathrm{d}\tau + A) \wedge \mathrm{d}\frac{K^2}{w} + w \star_3 \mathrm{d}\frac{K^2}{w} + Z_1\,(\Omega_-^{(3)} - \Omega_+^{(3)}). \tag{4.35}$$

One can then reduce the first layer (4.28) and (4.29) to the following equations for the functions K^1, K^2, L_1, L_2:

$$\partial_x^2 L_1 + \partial_y^2 L_1 + \partial_z^2(e^u L_1) = 0, \tag{4.36}$$

$$\partial_x^2 L_2 + \partial_y^2 L_2 + \partial_z^2(e^u L_2) = 0, \tag{4.37}$$

$$\partial_x^2 K^1 + \partial_y^2 K^1 + \partial_z(e^u \partial_z K^1) = 2\,\partial_z(e^u w\, L_2), \tag{4.38}$$

$$\partial_x^2 K^2 + \partial_y^2 K^2 + \partial_z(e^u \partial_z K^2) = 2\,\partial_z(e^u w\, L_1). \tag{4.39}$$

Next, to solve the second layer, make the ansätze

$$Z_3 = \frac{K^1 K^2}{w} + L_3, \tag{4.40}$$

$$k = \mu\,(\mathrm{d}\tau + A) + \omega, \tag{4.41}$$

$$\mu = -\frac{K^1 K^2 K^3}{w^2} - \frac{1}{2}\sum_{I=1}^{3} \frac{K^I L_I}{w} + M, \tag{4.42}$$

where ω is some unknown 1-form. Then the new functions M and L_3 satisfy the equations

$$\boxed{\begin{aligned} &\partial_x^2 M + \partial_y^2 M + \partial_z(e^u \partial_z M) = \partial_z(e^u L_1 L_2), \\ &\quad \partial_x^2 L_3 + \partial_y^2 L_3 + e^u\, \partial_z^2 L_3 = 4e^u w L_1 L_2 - 4e^u w\, \partial_z M \\ &\qquad\qquad\qquad\qquad - 2e^u (L_1\, \partial_z K^1 + L_2\, \partial_z K^2), \end{aligned}} \tag{4.43, 4.44}$$

and the 1-form ω satisfies

$$\boxed{\begin{aligned} \mathrm{d}\omega = {} & w \underset{3}{\star} \mathrm{d}M - M \underset{3}{\star} \mathrm{d}w - u_z w M \underset{3}{\star} \mathrm{d}z \\ & - 2wL_1L_2 \underset{3}{\star} \mathrm{d}z + \frac{1}{2}\sum_I (L_I \underset{3}{\star} \mathrm{d}K^I - K^I \underset{3}{\star} \mathrm{d}L_I) \\ & - \frac{1}{2} u_z (K^1 L_1 + K^2 L_2) \underset{3}{\star} \mathrm{d}z + \frac{1}{2} u_z K^3 L_3 \underset{3}{\star} \mathrm{d}z. \end{aligned}} \tag{4.45}$$

Therefore, to solve the "floating brane" system on the LeBrun base, one first finds a function u that solves the $SU(\infty)$ Toda equation, which also defines the function $K^3 \equiv \frac{1}{2}u_z$. To determine the base geometry requires a function w that solves (4.9). Once the base is defined, one must solve (4.36)–(4.39) for L_1, L_2, K_1, K_2. Then one must solve (4.43) and (4.44) for M and L_3. Finally, one must solve (4.45) for the 1-form ω. The full supergravity solution is then obtained from these functions and the ansätze (4.4) and (4.5).

4.2.4 Possible asymptotics

The 5-dimensional Einstein equations obtained from (4.2) are

$$\mathrm{Ric}^{(5)} - \frac{1}{2}\mathcal{R}^{(5)}\, g^{(5)} = \frac{1}{2}T^{(5)}, \tag{4.46}$$

where the energy-momentum tensor $T^{(5)}_{\mu\nu}$ comes from varying the matter part of (4.2) with respect to the inverse 5d metric $g^{\mu\nu}_{(5)}$. Since the floating brane system (4.28)–(4.31) is sufficient to guarantee the solution of the Einstein equations (4.46), we can learn about the Ricci curvature of possible asymptotic regions by examining $T^{(5)}_{\mu\nu}$. The full 5-dimensional $T^{(5)} \equiv T^{(5)}_{\mu\nu}\, \mathrm{d}x^\mu \otimes \mathrm{d}x^\nu$ is given by

$$\begin{aligned}
\frac{1}{2}T^{(5)} = e^0 \otimes e^0\, Q_{IJ}\Big[&\frac{1}{2}Z^{-1}\left\langle \mathrm{d}X^I,\, \mathrm{d}X^J\right\rangle_4 + \frac{1}{2}Z\left\langle \mathrm{d}(Z_I^{-1}),\, \mathrm{d}(Z_J^{-1})\right\rangle_4 \\
&+ \frac{1}{2}Z^{-2}\left\langle \Theta^{(I)} - Z_I^{-1}\,\mathrm{d}k,\, \Theta^{(J)} - Z_J^{-1}\,\mathrm{d}k\right\rangle_4\Big] \\
&- e^0 \otimes Q_{IJ} \underset{4}{\star}\Big[\mathrm{d}(Z_I^{-1}) \wedge \underset{4}{\star}\big(\Theta^{(J)} - Z_J^{-1}\,\mathrm{d}k\big)\Big] \\
&- Q_{IJ} \underset{4}{\star}\Big[\mathrm{d}(Z_I^{-1}) \wedge \underset{4}{\star}\big(\Theta^{(J)} - Z_J^{-1}\,\mathrm{d}k\big)\Big] \otimes e^0 \\
&+ Q_{IJ}\Big[\mathrm{d}X^I \otimes \mathrm{d}X^J - \frac{1}{2}\left\langle \mathrm{d}X^I,\, \mathrm{d}X^J\right\rangle_4 g^{(4)} \\
&\quad - Z^2\,\mathrm{d}(Z_I^{-1}) \otimes \mathrm{d}(Z_J^{-1}) + \frac{1}{2}Z^2\left\langle \mathrm{d}(Z_I^{-1}),\, \mathrm{d}(Z_J^{-1})\right\rangle_4 g^{(4)} \\
&\quad + Z^{-1}\,\mathcal{T}_{\hat{a}\hat{b}}\big(\Theta^{(I)} - Z_I^{-1}\,\mathrm{d}k,\, \Theta^{(J)} - Z_J^{-1}\,\mathrm{d}k\big)\, \hat{e}^a \otimes \hat{e}^b\Big],
\end{aligned} \tag{4.47}$$

where the 5d frames are given by

$$e^0 \equiv -Z^{-1}\,(\mathrm{d}t + k), \qquad e^a \equiv Z^{1/2}\,\hat{e}^a, \tag{4.48}$$

and $g^{(4)} = \mathrm{d}s_4^2 = \delta_{ab}\,\hat{e}^a \otimes \hat{e}^b$ is the 4d metric. The contraction $\langle\cdot,\cdot\rangle_4$ is defined on p-forms by

$$\langle X, Y\rangle_4 = \frac{1}{p!}X_{i_1..i_p}Y^{i_1..i_p}, \tag{4.49}$$

where indices are raised with $g^{(4)}$. And $\mathcal{T}_{ab}(X,Y)$ is the bilinear form defined in (3.49) acting on 2-forms X, Y as

$$\mathcal{T}_{ab}(X,Y) \equiv \frac{1}{2}\big(X_{ac}Y_{bc} + X_{bc}Y_{ac}\big) - \frac{1}{4}\delta_{ab}\, X_{cd}Y_{cd}. \tag{4.50}$$

Are there asymptotically flat solutions?

To obtain asymptotically flat solutions, it is necessary that $T^{(5)} \to 0$ at infinity. In particular, remembering that Q_{IJ} is diagonal (4.3), we see that the time-time component $T^{(5)}_{00}$ is positive-definite[2]:

$$\begin{aligned}\frac{1}{2}T^{(5)}_{00} = Q_{IJ}\bigg[\frac{1}{2}Z^{-1}\left\langle \mathrm{d}X^I,\, \mathrm{d}X^J\right\rangle_4 + \frac{1}{2}Z\left\langle \mathrm{d}(Z_I^{-1}),\, \mathrm{d}(Z_J^{-1})\right\rangle_4 \\ + \frac{1}{2}Z^{-2}\left\langle \Theta^{(I)} - Z_I^{-1}\,\mathrm{d}k,\, \Theta^{(J)} - Z_J^{-1}\,\mathrm{d}k\right\rangle_4\bigg].\end{aligned} \tag{4.51}$$

Therefore each term must vanish individually. Hence for each $I \in \{1,2,3\}$, we must have

$$\mathrm{d}X^I \to 0, \qquad \mathrm{d}(Z_I^{-1}) \to 0, \qquad \Theta^{(I)} - Z_I^{-1}\,\mathrm{d}k \to 0. \tag{4.52}$$

The first condition implies that the Z_I must all have the same asymptotic behavior. The second condition implies that this behavior is $Z_I \to$ const. The last condition implies that, in the asymptotic region,

$$\mathrm{d}k = Z_1\Theta^{(1)} = Z_2\Theta^{(2)} = Z_3\Theta^{(3)}. \tag{4.53}$$

However, this last condition cannot be satisfied on the LeBrun class of base spaces. From the definition (4.26) we see that $\Theta^{(3)}$ is self-dual. But the floating brane equations

[2]Recall that Z can only become negative when $g^{(4)}$ is "ambipolar"—the simultaneous flip of $Z \to -Z$ when $g^{(4)}$ goes from $(++++)$ to $(----)$ maintains the positive-definiteness of (4.51).

(4.28) and (4.29) state that $\Theta^{(1)}, \Theta^{(2)}$ each have an anti-self-dual part proportional to the Kähler form J. Since the Kähler form is strictly non-vanishing (it satisfies $J \wedge J = -2\,\mathrm{vol}_4$), then when $Z_I \sim$ const, the 2-forms $\Theta^{(1)}, \Theta^{(2)}$ have a non-vanishing anti-self-dual part. Therefore the last equality in (4.53) cannot be true, and there are *no asymptotically-flat solutions* with a LeBrun base space[3].

Are there asymptotically AdS-like solutions?

Having ruled out asymptotically-flat solutions, we can look for other interesting asymptotics. A logical choice is an AdS-like boundary condition, which is like the near-horizon region of an extremal black hole, and is useful for studying things from the holographic perspective. In ungauged 5d supergravity, the possibilities are $AdS_2 \times S^3$ and $AdS_3 \times S^2$. The $AdS_3 \times S^2$ case involves complicated coordinate transformations [99], so we will leave it for possible future study. However, it is simple to fit $AdS_2 \times S^3$ into the metric ansatz (4.4) by choosing

$$Z_1, Z_2, Z_3 \to \frac{1}{\rho^2}, \qquad \mathrm{d}s_4^2(\mathcal{B}) \to \mathrm{d}\rho^2 + \rho^2\,\mathrm{d}\Omega_3^2. \tag{4.54}$$

Then one has

$$\mathrm{d}s_5^2 \to -\rho^4\,(\mathrm{d}t + k)^2 + \frac{\mathrm{d}\rho^2}{\rho^2} + \mathrm{d}\Omega_3^2, \tag{4.55}$$

where the first two terms give the AdS_2 factor and $\mathrm{d}\Omega_3^2$ gives the S^3 factor. Strict AdS_2 requires also that $k \to 0$. A non-vanishing k will yield a *rotating* AdS_2-like metric, such as the near-horizon region of the BMPV black hole [89].

---

[3]The exception to this is when $u_z/w =$ const. Then $\Theta^{(3)} \equiv 0$ by (4.32) and the Ricci tensor (4.25) vanishes. In this case, we can choose $\omega_-^{(3)} \equiv 0$ which allows asymptotic flatness. However, since we are making a different discrete choice of $\omega_-^{(3)}$, this possibility is not continuously related to the generic LeBrun ansatz.

The rotation vector k is sourced by the off-diagonal components of the energy-momentum tensor:

$$\frac{1}{2}T^{(5)}_{0a} = -e_a \lrcorner \left(Q_{IJ} \underset{4}{\star} \left[\mathrm{d}(Z_I^{-1}) \wedge \underset{4}{\star} \left(\Theta^{(J)} - Z_J^{-1}\,\mathrm{d}k \right) \right] \right). \tag{4.56}$$

Given that $Z_I \to 1/\rho^2$, we see that this can only vanish if (again):

$$\mathrm{d}k = Z_1\Theta^{(1)} = Z_2\Theta^{(2)} = Z_3\Theta^{(3)}. \tag{4.57}$$

But since in this case the Z_I are *not* constant, this constraint is inconsistent with the fact that the $\Theta^{(I)}$ are closed:

$$\mathrm{d}\Theta^{(1)} = \mathrm{d}\Theta^{(2)} = \mathrm{d}\Theta^{(3)} = 0. \tag{4.58}$$

The only way around this problem is to suppose that in fact

$$\mathrm{d}k = Z_1\Theta^{(1)} = Z_2\Theta^{(2)} = Z_3\Theta^{(3)} = 0. \tag{4.59}$$

But *this* condition is (again) inconsistent with the requirement that $\Theta^{(1)}, \Theta^{(2)}$ have anti-self-dual parts proportional to the Kähler form (4.28) and (4.29):

$$\Theta^{(1)} - \underset{4}{\star}\Theta^{(1)} = 2\,Z_2\,J, \qquad \Theta^{(2)} - \underset{4}{\star}\Theta^{(2)} = 2\,Z_1\,J. \tag{4.60}$$

And although the $Z_I \sim 1/\rho^2$ here are tending to zero, there are enough positive powers of ρ in (4.56) that the rotation vector k remains significant in the asymptotic region. Hence we see that while AdS-like asymptotics are possible on the LeBrun base, they will be *strictly rotating* ones, like the near-horizon regions of rotating black holes.

Therefore, in seeking non-BPS 5d microstate geometries using the floating brane ansatz over a LeBrun base, we should not expect to find asymptotically-flat solutions, nor even asymptotically-$AdS_2 \times S^3$ solutions. We see in particular that the obstruction to finding such asymptotics is that $\Theta^{(1)}, \Theta^{(2)}$ have an anti-self-dual part proportional to the Kähler form J, which is non-normalizable. Given this constraint, what we should hope for is to find asymptotically-*rotating*-$AdS_2 \times S^3$ solutions. Such solutions are still useful; they correspond to the near-horizon region of rotating black holes, which indicates that they might have an interpretation as microstates of those black holes.

* * *

4.3 Solutions on the LeBrun-Burns subclass

The simplest non-trivial subclass of LeBrun metrics (4.7) is obtained by choosing the function u (4.8) to be

$$u = \log 2z. \tag{4.61}$$

It is then convenient to reparametrize by defining:

$$z \equiv \frac{1}{2}\zeta^2, \qquad V \equiv e^u w = 2zw = \zeta^2 w. \tag{4.62}$$

The LeBrun metric can then be written as

$$\mathrm{d}s_4^2 = \zeta^2\Big[V^{-1}\left(\mathrm{d}\tau + A\right)^2 + V\Big(\frac{\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}\zeta^2}{\zeta^2}\Big)\Big]. \tag{4.63}$$

These 4-dimensional Kähler metrics were first studied by Burns [100] and so we call them "LeBrun-Burns metrics". We see that under the simplification (4.61), the 3-dimensional metric is now the standard constant-curvature metric on the hyperbolic plane, H^3:

$$ds^2_{H^3} = \frac{dx^2 + dy^2 + d\zeta^2}{\zeta^2}. \tag{4.64}$$

The equations (4.9) and (4.10) that define the four-dimensional base imply that V is a harmonic function on the hyperbolic plane and that A is an appropriate monopole on H^3:

$$d \underset{H^3}{\star} dV = 0, \qquad dA = \underset{H^3}{\star} dV. \tag{4.65}$$

4.3.1 Geometry of the LeBrun-Burns metric

Asymptotics

To avoid a conical singularity at $\zeta = 0$, one must have $V \to 1$ at this point so that the metric in the (ζ, τ) direction limits to that of $\mathbb{R}^2$ in polar coordinates. Thus the metric in the neighborhood of $\zeta = 0$ is that of $\mathbb{R}^4$ and regularity requires that one restrict the space to $\zeta \geq 0$. Similarly, if one requires $V \to 1$ at infinity, the space is asymptotic to $\mathbb{R}^4 = \mathbb{C}^2$. Note that the circle defined by τ lies in an $\mathbb{R}^2$ plane of the $\mathbb{R}^4$, and the associated isometry therefore only commutes with another $U(1)$ factor in the generic $SO(4)$ holonomy of the base metric. This is quite different from the way in which the isometry associated with the $U(1)$ fiber behaves in GH spaces.

The Green functions of the Laplacian on H^3 are the functions:

$$G(x, y, \zeta; a, b, c) \equiv \left(\frac{(x-a)^2 + (y-b)^2 + \zeta^2 + c^2}{\sqrt{((x-a)^2 + (y-b)^2 + \zeta^2 + c^2)^2 - 4c^2\zeta^2}} - 1 \right), \tag{4.66}$$

where one should remember that $\zeta \geq 0$ on H^3 and so this function only has one singularity in the domain of definition. The constant has been added so that G vanishes at infinity and at $\zeta = 0$. Given G, we can then solve for A in (4.65). Putting $A = D(x, y, \zeta; a, b, c)\,\mathrm{d}\phi$, we obtain

$$D(x, y, \zeta; a, b, c) \equiv \frac{(x-a)^2 + (y-b)^2 + \zeta^2 - c^2}{\sqrt{((x-a)^2 + (y-b)^2 + \zeta^2 + c^2)^2 - 4c^2\zeta^2}} \tag{4.67}$$

One can then take:

$$V = \varepsilon_0 + \sum_{j=1}^{N} q_j\, G(x, y, \zeta; a_j, b_j, c_j), \tag{4.68}$$

$$A = \sum_{j=1}^{N} q_j\, D(x, y, \zeta; a_j, b_j, c_j)\,\mathrm{d}\phi. \tag{4.69}$$

With these choices and $\varepsilon_0 = 1$, the LeBrun-Burns metric is a smooth Kähler metric on $\mathbb{C}^2$ blown up at N points. It is thus a Kähler, electrovac generalization of the Gibbons-Hawking metrics.

Near (a_j, b_j, c_j), one has

$$G(x, y, \zeta; a_j, b_j, c_j) \sim \frac{c_j}{\sqrt{(x-a_j)^2 + (y-b_j)^2 + (\zeta - c_j)^2}} \equiv \frac{c_j}{r}, \tag{4.70}$$

$$D(x, y, \zeta; a_j, b_j, c_j) \sim \frac{\zeta - c_j}{\sqrt{(x-a_j)^2 + (y-b_j)^2 + (\zeta - c_j)^2}} \equiv \cos\theta, \tag{4.71}$$

and the metric (4.63) behaves as:

$$\begin{aligned} \mathrm{d}s_4^2 &= c_j\, q_j \Big[q_j^{-2}\, r\, (\mathrm{d}\tau + \cos\theta\, \mathrm{d}\phi)^2 + r^{-1}(\mathrm{d}r^2 + r^2\, \mathrm{d}\theta^2 + r^2 \sin^2\theta\, \mathrm{d}\phi^2)\Big] \\ &= c_j\, q_j \Big[\mathrm{d}\rho^2 + \tfrac{1}{4}\rho^2\big(\mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2 + q_j^{-2}\, (\mathrm{d}\tau + \cos\theta\, \mathrm{d}\phi)^2\big)\Big], \end{aligned} \tag{4.72}$$

where we have introduced spherical polar coordinates about (a_j, b_j, c_j) and made a change of variable $r = \frac{1}{4}\rho^2$. Thus near the singular points of V, the metric is locally $\mathbb{R}^4/\mathbb{Z}_{q_j}$, and hence may be viewed as regular in string theory.

At infinity one has:

$$G(x, y, \zeta; a_j, b_j, c_j) \sim \frac{2\, c_j^2\, \zeta^2}{(x^2 + y^2 + \zeta^2)^2}, \tag{4.73}$$

$$D(x, y, \zeta; a_j, b_j, c_j) \sim 1, \tag{4.74}$$

and hence $V \to \varepsilon_0$ and $A \to d\phi$, and the metric is asymptotic to $\mathbb{R}^4 = \mathbb{C}^2$ for $\varepsilon_0 = 1$.

Homology and periods

Exactly as in Gibbons-Hawking geometries, the LeBrun-Burns metrics have non-trivial two-cycles defined by the $U(1)$ fibers over any curve between the poles of V. More specifically, the $U(1)$ fiber (defined by τ) taken over a generic line interval in the H^3 base describes a cylinder. However, if one runs this interval between two poles of V at points, $\vec{a}^{(i)}$ and $\vec{a}^{(j)}$ then the fiber is pinched off at the ends and the result is essentially a topological two-sphere. The asymptotic behavior of the metric at each end of the interval, (4.72), means that this two-sphere may, in fact, be modded out by some discrete group that depends upon the values of q_i and q_j. The two-cycles defined in this way will be denoted as Δ_{ij} and are depicted in Figure 4.2.

The periods of these cycles are trivial to compute using (7.58):

$$\frac{1}{2\pi}\int_{\Delta_{ij}} J = \frac{1}{2\pi}\int_{\Delta_{ij}} \mathrm{d}\tau \wedge \mathrm{d}z = 2(z_j - z_i) = \zeta_j^2 - \zeta_i^2, \tag{4.75}$$

where $z_i = \frac{1}{2}\zeta_i^2$ denote the z-coordinates of the corresponding poles of V.

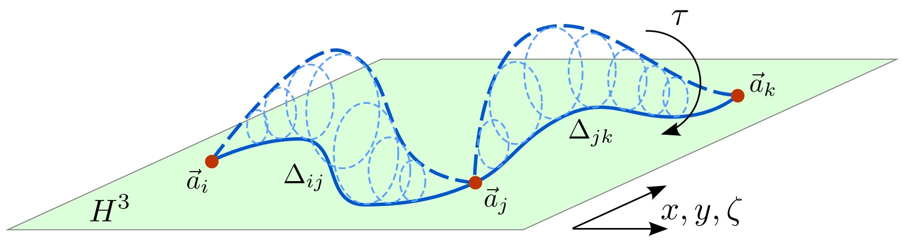

Figure 4.2: *The non-trivial cycles of the LeBrun-Burns metrics are defined by sweeping the $U(1)$ fiber along a path, in H^3, between any two poles of the potential, V. The fiber is pinched off at the poles. Here the fibers sweep out a pair of intersecting two-cycles.*

The Maxwell fields, $\Theta^{(1)}, \Theta^{(2)}, \Theta^{(3)}$ defined in (4.32), (4.34) and (4.35) have components along the fiber of the form

$$\Theta^{(I)} = \mathrm{d}\tau \wedge \mathrm{d}\bigg(\frac{K^{(I)}}{w}\bigg), \qquad I = 1, 2, 3, \tag{4.76}$$

where $K^{(1)}$ and $K^{(2)}$ satisfy (4.38) and (4.39) and

$$K^{(3)} \equiv \frac{1}{2}\,\partial_z u. \tag{4.77}$$

From this it follows that these fields have fluxes

$$\Pi^{(I)}_{ij} \equiv \frac{K^{(I)}}{w}\bigg|_{\vec{a}^{(j)}} - \frac{K^{(I)}}{w}\bigg|_{\vec{a}^{(i)}}, \qquad I = 1, 2, 3. \tag{4.78}$$

Note, in particular, that for the LeBrun-Burns metric $K^{(3)} w^{-1} = V^{-1}$ which vanishes at all the $\vec{a}^{(i)}$. Therefore $\Theta^{(3)}$ has *no non-trivial* fluxes on the compact two cycles.

In summary, the bubbled non-BPS solutions generically have non-vanishing fluxes only for $\Theta^{(1)}, \Theta^{(2)}$, whereas $\Theta^{(3)}$ has trivial fluxes. We will see that this fact enters into the regularity conditions and the bubble equations in Section 4.3.5.

4.3.2 The floating brane equations on the LeBrun-Burns base

In specializing the LeBrun system of equations (4.36)–(4.39) and (4.43)–(4.45) to the LeBrun-Burns metrics, we find two differential operators of interest:

$$\mathcal{L}_1 H \equiv \partial_x^2 H + \partial_y^2 H + \zeta^{-1}\,\partial_\zeta\,(\zeta \partial_\zeta H), \tag{4.79}$$

$$\mathcal{L}_2 G \equiv \partial_x^2 G + \partial_y^2 G + \zeta\,\partial_\zeta\,\big(\zeta^{-1}\,\partial_\zeta G\big). \tag{4.80}$$

Note that $\zeta^2 \mathcal{L}_2$ is simply the Laplacian on H^3. The operator $\mathcal{L}_1$ also appears in the equations of motion and it is useful to note that it has a simple geometric interpretation. Observe that the Laplacian on $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ may be written as

$$\hat{\mathcal{L}}_1 H = \partial_x^2 H + \partial_y^2 H + \zeta^{-1}\,\partial_\zeta\,(\zeta \partial_\zeta H) + \zeta^{-2}\,\partial_\varphi^2\,H, \tag{4.81}$$

where (x, y) are Cartesians on the first $\mathbb{R}^2$ and (ζ, φ) are polars on the second $\mathbb{R}^2$. Thus solving equations that involve $\mathcal{L}_1$ may simply be viewed as looking for φ-independent solutions to the flat Laplacian on $\mathbb{R}^4$. The equations and solutions that involve $\mathcal{L}_1$ are thus extremely familiar from the extensive literature on black rings. In particular, it is useful to note that the following are Green functions for $\mathcal{L}_1$:

$$H(x, y, \zeta; a, b, c) \equiv \frac{1}{\sqrt{((x-a)^2 + (y-b)^2 + \zeta^2 + c^2)^2 - 4\,c^2\,\zeta^2}}, \tag{4.82}$$

At infinity one has:

$$H(x, y, \zeta; a, b, c) \sim \frac{1}{(x^2 + y^2 + \zeta^2)}. \tag{4.83}$$

Solving the linear system

To solve the linear system, one first solves the homogeneous equations:

$$\mathcal{L}_2 V = 0, \qquad \mathcal{L}_2\,(\zeta^2\,L_1) = 0, \qquad \mathcal{L}_2(\zeta^2\,L_2) = 0, \tag{4.84}$$

and then uses these solutions in the equations that define the magnetic fluxes and the angular momentum function, M:

$$\mathcal{L}_1 K^{(1)} = 2\,\zeta^{-1}\partial_\zeta\,(V\,L_2), \qquad \mathcal{L}_1 K^{(2)} = 2\,\zeta^{-1}\partial_\zeta\,(V\,L_1), \tag{4.85}$$

$$\mathcal{L}_1 M = \zeta^{-1}\partial_\zeta\,(\zeta^2\,L_1\,L_2). \tag{4.86}$$

The last step is to use these solutions in:

$$\mathcal{L}_2 L_3 = 4V\big(L_1 L_2 - \zeta^{-1}\partial_\zeta M\big) - 2\zeta\big(L_1\,\partial_\zeta K^{(1)} + L_2\,\partial_\zeta K^{(2)}\big). \tag{4.87}$$

The physical functions now have the form

$$Z_1 = \frac{K^{(2)}}{V} + L_1, \quad Z_2 = \frac{K^{(1)}}{V} + L_2, \quad Z_3 = \frac{\zeta^2\,K^{(1)}\,K^{(2)}}{V} + L_3, \tag{4.88}$$

$$\mu = -\frac{\zeta^2\,K^{(1)}\,K^{(2)}}{V^2} - \frac{1}{2}\,\frac{\zeta^2\,(K^{(1)}\,L_1 + K^{(2)}\,L_2)}{V} - \frac{1}{2}\,\frac{L_3}{V} + M. \tag{4.89}$$

The equations for ω reduce to

$$\begin{aligned}(\partial_y\omega_\zeta - \partial_\zeta\omega_y) = &-\frac{1}{\zeta}(M\partial_x V - V\partial_x M) - \frac{1}{2\zeta}\partial_x L_3 \\ &- \frac{1}{2\zeta}\sum_{j=1}^{2}(K^{(j)}\partial_x(\zeta^2 L_j) - \zeta^2 L_j\partial_x K^{(j)}),\end{aligned} \tag{4.90}$$

$$
\begin{aligned}
(\partial_\zeta \omega_x - \partial_x \omega_\zeta) = -\frac{1}{\zeta}(M\partial_y V - V\partial_y M) - \frac{1}{2\zeta}\partial_y L_3 \\
- \frac{1}{2\zeta}\sum_{j=1}^{2}(K^{(j)}\partial_y(\zeta^2 L_j) - \zeta^2 L_j \partial_y K^{(j)}),
\end{aligned}
\tag{4.91}
$$

$$
\begin{aligned}
(\partial_x \omega_y - \partial_y \omega_x) = -\frac{1}{\zeta}(M\partial_\zeta V - V\partial_\zeta M) - \frac{1}{2\zeta}\partial_\zeta L_3 - 2VL_1L_2 \\
- \frac{1}{2\zeta}\sum_{j=1}^{2}(K^{(j)}\partial_\zeta(\zeta^2 L_j) - \zeta^2 L_j \partial_\zeta K^{(j)}).
\end{aligned}
\tag{4.92}
$$

This system of equations has a gauge invariance that leaves the physical solution completely invariant. See Appendix A.1 for details.

4.3.3 A single-center solution with AdS-like asymptotics

Before writing down the general, multicenter solution to (4.84)–(4.87) and (4.90)–(4.92), it is instructive to consider the simplest possible solution: a single-centered solution on an $\mathbb{R}^4$ background. The purpose of doing this is to reveal what kinds of black-object geometries can be generated from the LeBrun-Burns metric using the solution technique of Section 4.3.2, and what their asymptotic regions will look like. Indeed, we will show that the natural boundary conditions at infinity correspond to the near-horizon regions of black holes.

To make the base space completely flat, we take $V \equiv 1$ in the LeBrun-Burns metric. It is important to note that even though we have thus trivialized the metric on the base, the Maxwell field $\mathcal{F}$ is still non-zero, but is now purely anti-self-dual[4] and proportional to the complex structure J. Similarly, the other Maxwell fields (4.34) and (4.35) have both anti-self-dual and self-dual parts on the base. This will generically mean that supersymmetry is completely broken and that the solutions we get will be non-BPS.

[4]This means that $\mathcal{F}$ has vanishing energy-momentum tensor, consistent with the flatness of the base.

The simplest possible solution

Perhaps the simplest non-trivial solution is a spherically symmetric one, whose sources necessarily lie at $(x, y, \zeta) = (0, 0, 0)$ This example will demonstrate the typical asymptotic behavior[5]. In addition we set some of the electric potentials to zero:

$$L_1 \equiv L_2 \equiv 0. \tag{4.93}$$

It is also convenient to introduce polar coordinates in $\mathbb{R}^2$ and $\mathbb{R}^4$: We already have ζ and τ in one copy of $\mathbb{R}^2$ and so we define[6]

$$x = \eta \cos\phi, \quad y = \eta \sin\phi, \qquad \zeta = \rho\cos\theta, \quad \eta = \rho\sin\theta, \tag{4.94}$$

$$\rho \equiv x^2 + y^2 + \zeta^2. \tag{4.95}$$

The functions $K^{(I)}$ and M are then homogeneous solutions to $\mathcal{L}_1 H = 0$ and the spherically symmetric solutions are proportional to $H(x, y, \zeta; 0, 0, 0) = \rho^{-2}$ (see (4.82)). We therefore take

$$Z_1 = K^{(2)} = \frac{\beta_2}{\rho^2}, \qquad Z_2 = K^{(1)} = \frac{\beta_1}{\rho^2}, \qquad M = \frac{\gamma}{\rho^2}, \tag{4.96}$$

where β_1, β_2 and γ are constant parameters.

It is easy to see that one can satisfy (4.87) by taking:

$$L_3 = \hat{L}_3 + 2\,M, \qquad \mathcal{L}_2 \hat{L}_3 = 0, \tag{4.97}$$

---

[5] The asymptotics are not substantially different if the sources instead lie at $(x, y, \zeta) = (0, 0, c)$.

[6] The coordinate θ here is not the same as the one in (4.72).

for some function, $\hat{L}_3$. The natural choice for $\hat{L}_3$ is the function G in (4.66), but this vanishes for $c = 0$, and one must take a limit:

$$\hat{L}_3 = \beta_3 \lim_{c \to 0} \frac{1}{2\,c^2}\, G(x, y, \zeta; 0, 0, c) = \beta_3\, \frac{\zeta^2}{\rho^4} = \beta_3\, \frac{\cos^2\theta}{\rho^2}, \tag{4.98}$$

One then has

$$Z_3 = \zeta^2\, K^{(1)}\, K^{(2)} + L_3 = (\beta_1\, \beta_2\, + \beta_3)\, \frac{\cos^2\theta}{\rho^2} + \frac{2\,\gamma}{\rho^2}, \tag{4.99}$$

$$\mu = -\,\zeta^2\, K^{(1)}\, K^{(2)} - \frac{1}{2}\, \hat{L}_3 = -\frac{1}{2}\, (2\, \beta_1\, \beta_2 + \beta_3)\, \frac{\cos^2\theta}{\rho^2}. \tag{4.100}$$

The last step is to solve for ω, for which we can choose the gauge $\omega_z = 0$. Equations (4.90)–(4.92) then reduce to:

$$\zeta\, \partial_\zeta \omega_y = \frac{1}{2}\, \partial_x \hat{L}_3, \quad \zeta\, \partial_\zeta \omega_x = -\frac{1}{2}\, \partial_y \hat{L}_3, \qquad \partial_x \omega_y - \partial_y \omega_x = -\frac{1}{2}\, \zeta^{-1}\, \partial_\zeta \hat{L}_3, \tag{4.101}$$

for which the solution is:

$$\omega = -\frac{\beta_3}{2}\, \frac{1}{\rho^4}\, (y\, dx - x\, dy) = \frac{\beta_3}{2}\, \frac{\sin^2\theta}{\rho^2}\, \mathrm{d}\phi, \tag{4.102}$$

where the homogeneous solutions have been chosen so that ω goes to zero at infinity.

The near-horizon limit of a black hole

Taking this simple solution, we obtain the 5-dimensional metric:

$$\boxed{\begin{aligned} \mathrm{d}s_5^2 = -W_0(\theta)^{-2}\, \rho^4 \left(\mathrm{d}t - \frac{1}{2}\, (\beta_3 + 2\, \beta_1\, \beta_2)\, \frac{\cos^2\theta}{\rho^2}\, \mathrm{d}\tau + \frac{\beta_3}{2}\, \frac{\sin^2\theta}{\rho^2}\, \mathrm{d}\phi\right)^2 \\ + W_0(\theta) \left(\frac{\mathrm{d}\rho^2}{\rho^2} + \mathrm{d}\theta^2 + \cos^2\theta\, \mathrm{d}\tau^2 + \sin^2\theta\, \mathrm{d}\phi^2\right), \end{aligned}} \tag{4.103}$$

where

$$W_0(\theta) \equiv \left(\beta_1\beta_2(2\,\gamma + (\beta_1\,\beta_2\, + \beta_3)\,\cos^2\theta)\right)^{\frac{1}{3}}. \tag{4.104}$$

The conditions for absence of causal pathologies for solutions on the LeBrun-Burns base are discussed in Appendix A.2. For the simple solution in this section there is no Dirac-Misner string in ω and the condition for absence of CTC's is that all constants γ, β_1, β_2 are non-negative and

$$8\gamma\beta_1\beta_2 \geq \beta_3, \tag{4.105}$$

which one can obtain, e.g., by setting $t = \text{const}, \theta = \pi/2$ in (4.103).

For a generic choice of parameters satisfying (4.105), the metric (4.103) has the form of a warped rotating $AdS_2 \times S^3$. The general solution has unequal angular momenta in each $\mathbb{R}^2$, and has a distorting warp factor $W_0(\theta)$. For the special choice $\beta_3 = -\beta_1\beta_2$ the function W_0 becomes a constant and the two angular momenta become equal. The metric then is precisely the near horizon limit of the BMPV black hole [89]. It is worth emphasizing that the BMPV black hole (and its near horizon limit) is a supersymmetric solution of supergravity whereas our solution has anti-self-dual flux that breaks supersymmetry. We will see in Section 4.3.4 that these are the typical asymptotics of the family of LeBrun-Burns-based solutions.

4.3.4 Multicenter solutions

To find multicenter solutions, we find that the equations (4.84)–(4.87) and (4.90)–(4.92) are rather difficult to solve without imposing an additional axial symmetry. Therefore we will restrict our search to solutions on an *axisymmetric* LeBrun-Burns base. This provides an infinite class of explicit five-dimensional multi-centered solutions with (at least) one time-like and two space-like Killing vectors $(\partial_t, \partial_\tau, \partial_\phi)$. Amongst our solutions are multi-center generalizations of the solution in Section 4.3.3 as well as a class of regular bubbled geometries that we discuss in some detail in Section 4.3.5 below.

General axisymmetric solutions

We will look for solutions on an axisymmetric LeBrun-Burns base in which the geometry at infinity has the form (4.103). The singular points of the harmonic function V, that determines the LeBrun-Burns base, are located along the ζ axis at points c_j:

$$V = \varepsilon_0 + \sum_{j=1}^{N} q_j \, G_j. \tag{4.106}$$

Where for convenience we have defined

$$G_i \equiv G(x, y, \zeta; 0, 0, c_i) = \frac{\rho^2 + c_i^2}{\sqrt{(\rho^2 + c_i^2)^2 - 4\zeta^2 c_i^2}} - 1, \tag{4.107}$$

$$H_i \equiv H(x, y, \zeta; 0, 0, c_i) = \frac{1}{\sqrt{(\rho^2 + c_i^2)^2 - 4\zeta^2 c_i^2}}, \tag{4.108}$$

$$D_i \equiv D(x, y, \zeta; 0, 0, c_i) = \frac{\rho^2 - c_i^2}{\sqrt{(\rho^2 + c_i^2)^2 - 4\zeta^2 c_i^2}}, \tag{4.109}$$

where we will assume that $c_i \neq 0$. It is trivial to solve (4.84) for the functions L_1 and L_2

$$L_a = \frac{1}{\zeta^2} \left(\ell_a^0 + \sum_{i=1}^{N} \ell_a^i G_i \right), \qquad a = 1, 2. \tag{4.110}$$

Solving (4.85) and (4.86) for $K^{(a)}$ and M one finds

$$K^{(1)} = k_1^0 + \frac{\beta_1}{\rho^2} + \sum_{i=1}^{N} k_1^i H_i - V L_2 + 4\rho^2 \sum_{i,j=1}^{N} q_i \ell_2^j H_i H_j, \tag{4.111}$$

$$K^{(2)} = k_2^0 + \frac{\beta_2}{\rho^2} + \sum_{i=1}^{N} k_2^i H_i - V L_1 + 4\rho^2 \sum_{i,j=1}^{N} q_i \ell_1^j H_i H_j, \tag{4.112}$$

$$M = m_0 + \frac{\gamma}{\rho^2} + \sum_{i=1}^{N} m_i H_i - \frac{\zeta^2}{2} L_1 L_2 + 2\rho^2 \sum_{i,j=1}^{N} \ell_1^i \ell_2^j H_i H_j. \tag{4.113}$$

After a somewhat tedious exercise[7] one can also solve equation (4.87)

$$\begin{aligned}
L_3 = \ell_3^0 + \sum_{i=1}^{N} \ell_3^i G_i - \zeta^2 V L_1 L_2 \\
&+ \sum_{i=1}^{N} \Big(2(\varepsilon_0 - Q) m_i + (\ell_1^0 - \Lambda_1) k_1^i + (\ell_2^0 - \Lambda_2) k_2^i \Big) H_i \\
&+ \beta_3 \frac{\zeta^2}{\rho^4} + \Big(2(\varepsilon_0 - Q)\gamma + (\ell_1^0 - \Lambda_1)\beta_1 + (\ell_2^0 - \Lambda_2)\beta_2 \Big) \frac{1}{\rho^2} \\
&+ 2\gamma \sum_{i=1}^{N} \frac{q_i}{c_i^2} \frac{\rho^{-2} - H_i}{H_i} + \sum_{i=1}^{N} (2 q_i m_i + \ell_1^i k_1^i + \ell_2^i k_2^i)(\eta^2 - \zeta^2 + c_i^2) H_i^2 \\
&+ \sum_{i \neq j=1}^{N} \frac{(2 q_i m_j + \ell_1^i k_1^j + \ell_2^i k_2^j)}{c_i^2 - c_j^2} \frac{H_j - H_i}{H_i} \\
&+ 4 \sum_{i,j=1}^{N} \Big((\varepsilon_0 - Q) \ell_1^i \ell_2^j + (\ell_1^0 - \Lambda_1) q_i \ell_2^j + (\ell_2^0 - \Lambda_2) q_i \ell_1^j \Big) \rho^2 H_i H_j \\
&+ 4 \sum_{i,j,k=1}^{N} q_i \ell_1^j \ell_2^k \rho^2 (3\rho^2 - 4\zeta^2 + c_i^2 + c_j^2 + c_k^2) H_i H_j H_k,
\end{aligned} \tag{4.114}$$

where we have defined

$$Q \equiv \sum_{i=1}^{N} q_i, \qquad \Lambda_1 \equiv \sum_{i=1}^{N} l_1^i, \qquad \Lambda_2 \equiv \sum_{i=1}^{N} l_2^i. \tag{4.115}$$

---

[7] Some of the identities used to solve the equations for $K^{(a)}$, M, L_3 and ω_ϕ are collected in Appendix Appendix A.3.

The one form $\omega = \omega_\phi \, \mathrm{d}\phi$ is given by

$$\begin{aligned}\omega_\phi = \omega_0 &+ \frac{\beta_3}{2}\frac{\sin^2\theta}{\rho^2} - \gamma\sum_{i=1}^{N}\frac{q_i}{c_i^2}D_i - \sum_{j=1}^{N}\left(m_0 q_j + k_1^0\ell_1^j + k_2^0\ell_2^j + \frac{\ell_3^j}{2}\right)D_j \\ &- \sum_{j=1}^{N}(2m_j q_j + k_1^j\ell_1^j + k_2^j\ell_2^j)\eta^2 H_j^2 \\ &- \sum_{i\neq j=1}^{N}\frac{(2q_i m_j + k_1^i\ell_1^j + k_2^i\ell_2^j)}{2(c_i^2 - c_j^2)}(D_iD_j + 4\eta^2 c_i^2 H_iH_j) \\ &- 8\sum_{i,j,k=1}^{N} q_i\ell_1^j\ell_2^k\eta^2\rho^2 H_iH_jH_k,\end{aligned} \tag{4.116}$$

where ω_0 is a constant which should be fixed so as to avoid CTCs and Dirac-Misner strings.

Substituting (4.110)–(4.114) in the expressions for Z_1, Z_2, Z_3 and μ, (4.88) and (4.89), one finds the most general non-BPS solution on an axisymmetric LeBrun-Burns base captured by the floating brane ansatz of [31]. For easy comparison with the solution in Section 4.3.3 we have chosen to single out the terms in the solution which have poles at $\rho = 0$, *i.e.* the terms with coefficients involving β_1, β_2, β_2 and γ.

In addition to the parameters β_1, β_2, β_3 and γ, the solution in general has $(8N + 7)$ parameters: $\{c_i, \varepsilon_0, q_i, \ell_I^0, \ell_I^i, k_a^0, k_a^i, m_0, m_i\}$. As we will see in the next subsection imposing regularity and absence of causal pathologies will greatly reduce the number of independent parameters.

4.3.5 Regularity and bubble equations

The solution we construct here will be asymptotic to the metric (4.103), which can be viewed as the "elementary" solution within our ansatz. These regular solutions on a base with non-trivial topology can be viewed as a non-supersymmetric generalization of the BPS bubbled solutions of [74, 75].

We begin by defining a radial coordinate around each of the poles of the harmonic functions

$$\rho_i^2 = \eta^2 + (\zeta - c_i)^2. \tag{4.117}$$

We will be interested in constructing a solution that is regular at the locations of the poles of the harmonic functions, $\rho_i \to 0$, and is free of CTCs and Dirac-Misner strings.

For $\rho_i \to 0$ we have the following expansion of the harmonic functions

$$G_i \sim \frac{c_i}{\rho_i}, \qquad H_i \sim \frac{1}{2c_i\rho_i}. \tag{4.118}$$

Since we are looking for a regular bubbled solution in five dimensions we will assume that all functions in the solution have the same singular points (excluding the point $\rho = 0$ which, as discussed in the previous section, will be treated separately). The functions Z_1 and Z_2 near a singular point, $\rho_i \to 0$, diverge as

$$Z_1 \sim \frac{\ell_1^i}{c_i\rho_i}, \qquad Z_2 \sim \frac{\ell_2^i}{c_i\rho_i}. \tag{4.119}$$

To ensure regularity we should set

$$\ell_1^i = \ell_2^i = 0, \qquad \forall i. \tag{4.120}$$

The function Z_3 near a singular point, $\rho_i \to 0$, is

$$\begin{aligned} Z_3 \sim \left(\ell_3^i c_i + \frac{k_1^i k_2^i}{4c_i q_i}\right)\frac{1}{\rho_i} + \frac{m_i}{c_i\rho_i}\left(\varepsilon_0 + \sum_{k=1, k\neq i}^{N} q_k \operatorname{sign}\left(c_k^2 - c_i^2\right)\right) \\ + \frac{q_i m_i(\eta^2 - \zeta^2 + c_i^2)}{2c_i^2\rho_i^2}. \end{aligned} \tag{4.121}$$

The last term in the expression above is divergent and can be made to vanish only for $m_i = 0$. Therefore for a regular Z_3 one should set

$$m_i = 0, \qquad \ell_3^i = -\frac{k_1^i k_2^i}{4c_i^2 q_i}, \qquad \text{for all } i. \tag{4.122}$$

It is not hard to show that with this choice of constants the function μ will limit to a constant near a singular point. The condition for absence of CTC's[8] requires that μ should vanish at a singular point of V and this leads to the constraint:

$$m_0 + \frac{\gamma}{c_i^2} - \frac{k_1^i k_2^i}{8c_i^2 q_i^2} = 0, \qquad \text{for all } i. \tag{4.123}$$

Then to summarize, the conditions for regularity and absence of CTC's and Dirac-Misner strings near the poles of the harmonic functions requires that we set:

$$\boxed{\begin{array}{c} m_i = \ell_1^i = \ell_2^i = 0, \qquad \ell_3^i = -\dfrac{k_1^i k_2^i}{4c_i^2 q_i}, \\ m_0 + \dfrac{\gamma}{c_i^2} - \dfrac{k_1^i k_2^i}{8c_i^2 q_i^2} = 0, \end{array} \qquad \text{for all } i.} \tag{4.124}$$

Note that these conditions are quite different from the regularity and causality constraints (3.35) and (3.40) for BPS bubbled solutions with a GH base [25]. In particular for the class of bubbled solutions discussed here there is no analogue of the "bubble equations" (or integrability conditions) familiar from the supersymmetric multi-center solutions [25, 101]. However we still have an equation that fixes the locations of the poles in the harmonic functions (but not the distance between them) in terms of the parameters $\{\gamma, m_0, k_1^i, k_2^i, q_i\}$.

[8]This comes from $\mathcal{Q} \geq 0$, where $\mathcal{Q}$ is defined in Appendix A.2.

Our analysis so far does not guarantee the regularity of the supergravity scalars (*i.e.* the Kähler moduli of the tori in M-theory) and the absence of causal pathologies at asymptotic infinity. To ensure that we should study the behavior of the solution at $\rho \to \infty$. The harmonic functions have the following expansion

$$G_i \sim 2c_i^2 \frac{\zeta^2}{\rho^4}, \qquad\qquad H_i \sim \frac{1}{\rho^2}. \tag{4.125}$$

Imposing the regularity and causality constraints at $\rho \to \infty$ one finds the following constraints on the parameters of the solution:

$$\begin{gathered} m_0 = k_1^0 k_2^0 = 0, \qquad \ell_3^0 - (k_1^0 \ell_1^0 + k_2^0 \ell_2^0) = 0, \\ k_1^0 \beta_2 + k_2^0 \beta_1 + k_1^0 \sum_{i=1}^N k_2^i + k_2^0 \sum_{i=1}^N k_1^i = 0. \end{gathered} \tag{4.126}$$

The constraints are easily solved by imposing $\ell_3^0 = k_1^0 = k_2^0 = m_0 = 0$, however there are in principle other ways to satisfy the relations in (4.126), so we will not commit to a specific solution. We also point out that once we have set $m_0 = 0$ as required here, then the regularity conditions (4.124) actually impose *no constraints* on the locations c_i of the sources. Hence in the solutions based on LeBrun-Burns, there are effectively *no bubble equations*[9].

The asymptotic expansion ($\rho \to \infty$) of the metric functions in the solution is

$$Z_1 \sim \frac{1}{\varepsilon_0}\Big(\beta_2 + \sum_{i=1}^N k_2^i\Big)\frac{1}{\rho^2}, \qquad\qquad Z_2 \sim \frac{1}{\varepsilon_0}\Big(\beta_1 + \sum_{i=1}^N k_1^i\Big)\frac{1}{\rho^2}, \tag{4.127}$$

[9]While our solutions have made the additional assumption of axisymmetry, this conclusion is still reasonable for all LeBrun-Burns solutions because, as seen in (4.1), one needs all three $\Theta^{(I)}$ to have non-trivial cohomological fluxes in order to obtain non-trivial bubble equations, and we have shown that $\Theta^{(3)}$ of the LeBrun-Burns metrics has trivial fluxes (4.78).

$$
\begin{aligned}
Z_3 \sim 2(\varepsilon_0 - Q)\frac{\gamma}{\rho^2} + \frac{1}{\varepsilon_0}\frac{\zeta^2}{\rho^4}\Bigg[\beta_3\varepsilon_0 + \beta_1\beta_2 + \sum_{i=1}^{N}\left(\beta_2 k_1^i + \beta_1 k_2^i - \varepsilon_0 \frac{k_1^i k_2^i}{2q_i}\right) \\
+ \sum_{i,j=1}^{N} k_1^i k_2^j - 4\varepsilon_0\gamma Q\Bigg],
\end{aligned}
\tag{4.128}
$$

$$
\begin{aligned}
\mu \sim -\frac{1}{2\varepsilon_0^2}\frac{\zeta^2}{\rho^4}\Big(\beta_3\varepsilon_0 + 2\beta_1\beta_2 + 2\sum_{i=1}^{N}(\beta_2 k_1^i + \beta_1 k_2^i) \\
+ 2\sum_{i,j=1}^{N} k_1^i k_2^j - \varepsilon_0 \sum_{i=1}^{N}\frac{k_1^i k_2^i}{2q_i} - 4\varepsilon_0\gamma Q\Big).
\end{aligned}
\tag{4.129}
$$

The constraints (4.126) together with (4.124) lead to

$$
\omega = \frac{\beta_3}{2}\frac{\sin^2\theta}{\rho^2}\,\mathrm{d}\phi. \tag{4.130}
$$

It is clear that at $\rho \to \infty$ these regular bubbled solutions are asymptotic to the warped, rotating $AdS_2 \times S^3$ solution (4.103) presented in Section 4.3.3. The parameters of the solution can be arranged such that the warp factor in the metric is a constant and the solution is asymptotic to the near horizon BMPV black hole.

The axisymmetric multi-center solutions have $8N + 11$ parameters. The regularity and causality constraints studied in this section impose $5N + 4$ relations on them, therefore we have a $(3N + 7)$-parameter family of regular solutions with non-trivial topology on the base. It should be emphasized that we have only analyzed in detail the condition for absence of CTC's near the singularity of the harmonic functions and at asymptotic infinity. In principle one needs to ensure that there are no CTC's globally and for this one usually has to rely on numerics [102]. On the other hand, experience with many examples suggests that once one has addressed this at singular points and ensured that the metric coefficients are well-behaved then there are no CTC's globally.

It is interesting to note that there is no analog of the bubble equations [25, 101] for our regular non-BPS solutions. Bubble equations can be viewed as a form of angular momentum balance that constrains the location of sources and with pure flux solutions, non-trivial bubble equations require non-zero sources for all three fluxes. In our solutions, the magnetic flux of $\Theta^{(3)}$ is trivial on the topological two-cycles and the complete Maxwell field $\mathcal{F}$, has no localized sources. Thus one should not be too surprised at the absence of constraints on the location of the remaining flux sources. However, it *is* surprising that the bubble sizes are not constrained to *zero* in the absence of $\Theta^{(3)}$ flux. It is possible this is allowed because of the AdS-like asymptotics.

* * *

4.4 Lift to 6 dimensions

In Section 3.3.3 we saw how the 5d $\mathcal{N}=2$ BPS equations (3.12)–(3.14) can be embedded into the BPS conditions of the $\mathcal{N}=1$ 6d theory of Section 3.3. This fact is unsurprising, because the 5d solutions are dimensional reductions of BPS configurations in 11d supergravity; one should be able to obtain a 6d BPS solution by simply holding onto one of the circles of the T^6 that would otherwise be compactified.

What *is* surprising, however, is that one can lift *non*-BPS solutions of the 5d theory into 6d BPS solutions as well. In [34], we show that this happens for the LeBrun class of solutions. Even though the solutions based upon the LeBrun metrics in five dimensions are not supersymmetric, the solutions *are supersymmetric* in the six-dimensional, IIB duality frame. More generally, the LeBrun solutions are non-supersymmetric in M-theory and are only supersymmetric in the particular IIB frame in which the electromagnetic field of the LeBrun base is used to give the momentum charge to the overall solution. The reason for this is that the surviving supersymmetry necessarily has a charge under the $U(1)$ of the momentum charge fibration in six dimensions. The supersymmetry is broken by the trivial KK reduction of the six-dimensional solution and then any trivial uplift of this solution, such as to M-theory, does not restore the supersymmetry. This is a little reminiscent of Scherk-Schwarz reduction on a circle [91, 90] but the latter explicitly introduces masses through dependence of the fields on the extra dimensions whereas here the dependence on extra dimensions only arises in the supersymmetry and not in the fields themselves (similarly to the "supersymmetry without supersymmetry" of [92]).

4.4.1 The Lebrun metrics as a base for 6d BPS solutions

Here we show that the LeBrun metrics can be used as a four-dimensional base for constructing six-dimensional BPS solutions of the form described in Section 3.3. With fairly simple ansätze one can find how the 5d quantities correspond to 6d quantities, and we show that the entire system of floating brane equations (3.60)–(3.63) is then embeddable within the 6d BPS system (3.101)–(3.107), provided the base space $\mathcal{B}$ is within the LeBrun class of metrics (4.7).

The first step is to find a 1-form β and three 2-forms $J^{(A)}$ that satisfy (3.74), (3.76) and (3.77). We assume that β is independent of the 6th coordinate v. Then (3.77) becomes simply

$$\tilde{\mathrm{d}}\beta = \underset{4}{\star}\,\tilde{\mathrm{d}}\beta, \tag{4.131}$$

and hence we need a self-dual, harmonic 2-form on the base. On the LeBrun backgrounds we are provided a natural choice (4.26), so let us choose

$$\tilde{\mathrm{d}}\beta = \tfrac{1}{2}\,\Theta^{(3)}, \qquad \beta = \tfrac{1}{2}\,B^{(3)} = \frac{1}{4}\left[-\left(\frac{\partial_z u}{w}\right)(d\tau + A) + (\partial_y u)\,dx - (\partial_x u)\,dy\right], \tag{4.132}$$

where the factor of $\frac{1}{2}$ will be useful later.

Next we need three $J^{(A)}$ satisfying the algebra (3.74). One obvious choice is

$$\widetilde{J}^{(1)} \equiv \hat{e}^0 \wedge \hat{e}^1 - \hat{e}^2 \wedge \hat{e}^3 = e^{u/2}\Big((\mathrm{d}\tau + A) \wedge \mathrm{d}x - w\, \mathrm{d}y \wedge \mathrm{d}z\Big), \tag{4.133}$$

$$\widetilde{J}^{(2)} \equiv \hat{e}^0 \wedge \hat{e}^2 - \hat{e}^3 \wedge \hat{e}^1 = e^{u/2}\Big((\mathrm{d}\tau + A) \wedge \mathrm{d}y + w\, \mathrm{d}x \wedge \mathrm{d}z\Big), \tag{4.134}$$

$$J^{(3)} \equiv J = \hat{e}^0 \wedge \hat{e}^3 - \hat{e}^1 \wedge \hat{e}^2 = (\mathrm{d}\tau + A) \wedge \mathrm{d}z - w\, e^u\, \mathrm{d}x \wedge \mathrm{d}y, \tag{4.135}$$

where the frames are defined in (4.13) and J is the original Kähler form. However, $\widetilde{J}^{(1)}, \widetilde{J}^{(2)}$ and $J^{(3)}$ are v-independent and only $J^{(3)}$ is closed and so they do not satisfy the differential constraint (3.76). On the other hand, if one defines a rotating form of these structures:

$$J^{(1)} \equiv \cos(2\,v)\, \widetilde{J}^{(1)} - \sin(2\,v)\, \widetilde{J}^{(2)}, \qquad J^{(2)} \equiv \sin(2\,v)\, \widetilde{J}^{(1)} + \cos(2\,v)\, \widetilde{J}^{(2)}, \tag{4.136}$$

one finds that the $J^{(A)}$ are a set of almost hyper-Kähler structures that do indeed obey (3.76). The fact that this elementary modification works is a very special property of the LeBrun family of metrics and does not work in other familiar examples of four-dimensional metrics, like the Israel-Wilson-Perjés metrics used as a base for five or six-dimensional supergravity solutions in [31].

With this choice for the $J^{(A)}$, it is easy to verify that

$$\hat{\psi} \equiv \tfrac{1}{16}\, \epsilon^{ABC}\, J^{(A)mn}\, \dot{J}^{(B)}{}_{mn}\, J^{(C)} = J^{(3)} = J. \tag{4.137}$$

Thus in particular, $\hat{\psi}$ is v-independent, and hence it will be consistent to assume v-independence of all the fields in the 6d BPS system (3.101)–(3.107)

From (4.26) and (4.137) one immediately sees that the duality conditions (3.101) of the 6d BPS system,

$$\tfrac{1}{2}(\Theta^{(1)} - \underset{4}{\star}\Theta^{(1)}) = Z_2\hat{\psi}, \qquad \tfrac{1}{2}(\Theta^{(2)} - \underset{4}{\star}\Theta^{(2)}) = Z_1\hat{\psi}, \tag{4.138}$$

are precisely the same as non-BPS duality conditions in (4.28) and (4.29). This suggests we identify the 5d non-BPS $Z_1, Z_2, \Theta^{(1)}, \Theta^{(2)}$ with their 6d BPS counterparts $Z_1, Z_2, \Theta^{(1)}, \Theta^{(2)}$, and we may take them all to be v-independent. When all these quantities are v-independent, then equations (3.102) and (3.103),

$$\mathcal{D}\Theta^{(2)} - \dot{\beta} \wedge \Theta^{(2)} = -\tfrac{1}{2}\partial_v\big(\underset{4}{\star}(\mathcal{D}Z_1 + Z_1\dot{\beta})\big), \tag{4.139}$$

$$\mathcal{D}\Theta^{(1)} - \dot{\beta} \wedge \Theta^{(1)} = -\tfrac{1}{2}\partial_v\big(\underset{4}{\star}(\mathcal{D}Z_2 + Z_2\dot{\beta})\big), \tag{4.140}$$

reduce simply to the requirement that $\Theta^{(1)}, \Theta^{(2)}$ be closed,

$$\mathrm{d}\Theta^{(1)} = 0, \qquad \mathrm{d}\Theta^{(2)} = 0, \tag{4.141}$$

which is consistent with the 5d definition $\Theta^{(I)} \equiv \mathrm{d}B^I$.

Equations (3.104) and (3.105),

$$\mathcal{D}\underset{4}{\star}\big(\mathcal{D}Z_1 + Z_1\dot{\beta}\big) = 2\Theta^{(2)} \wedge \mathcal{D}\beta, \tag{4.142}$$

$$\mathcal{D}\underset{4}{\star}\big(\mathcal{D}Z_2 + Z_2\dot{\beta}\big) = 2\Theta^{(1)} \wedge \mathcal{D}\beta, \tag{4.143}$$

reduce, on v-independence, to the other non-BPS equations in (4.28) and (4.29):

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_1 = \Theta^{(2)} \wedge \Theta^{(3)}, \qquad \mathrm{d}\underset{4}{\star}\mathrm{d}Z_2 = \Theta^{(1)} \wedge \Theta^{(3)}. \tag{4.144}$$

Finally, the last two equations (3.106) and (3.107),

$$\underset{4}{\star}\mathcal{D}\underset{4}{\star}\mathcal{L} - 2\langle\dot{\beta},\mathcal{L}\rangle_{\mathcal{B}} = -\tfrac{1}{2}\sqrt{Z_1Z_2}\,h^{ij}\partial_v^2\big(\sqrt{Z_1Z_2}\,h_{ij}\big) - \tfrac{1}{4}\partial_v\big(\sqrt{Z_1Z_2}\,h^{ij}\big)\,\partial_v\big(\sqrt{Z_1Z_2}\,h_{ij}\big) - 2Z_1Z_2\big(\partial_v\phi\big)^2 + 2\underset{4}{\star}\Big(\Theta^{(1)}\wedge\Theta^{(2)} - \hat{\psi}\wedge\mathcal{D}\omega\Big), \tag{4.145}$$

$$\mathcal{D}\omega + \underset{4}{\star}\mathcal{D}\omega = 2Z_1\Theta^{(1)} + 2Z_2\Theta^{(2)} - \mathcal{F}\,\mathcal{D}\beta - 4Z_1Z_2\hat{\psi}, \tag{4.146}$$

reduce to (4.30) and (4.31),

$$\mathrm{d}\underset{4}{\star}\mathrm{d}Z_3 = \Theta^{(1)}\wedge\Theta^{(2)} - J\wedge(\mathrm{d}k - \underset{4}{\star}\mathrm{d}k), \tag{4.147}$$

$$\mathrm{d}k + \underset{4}{\star}\mathrm{d}k = \frac{1}{2}\sum_I Z_I\,(\Theta^{(I)} + \underset{4}{\star}\Theta^{(I)}). \tag{4.148}$$

if one makes the identifications (as in Section 3.3.3)

$$\mathcal{F} = -4\,Z_3, \qquad \omega = 2\,k. \tag{4.149}$$

One can then rewrite the metric (3.66) as a standard fibration of the v-circle over a five dimensional space-time and upon reduction on this v-circle one obtains precisely the metric (4.4) provided one sets $u = 2t$.

Thus the non-BPS "floating brane" solutions in five dimensions based upon the LeBrun metrics found in [33] can be recast as supersymmetric solutions in the six-dimensional framework. This appears to contradict the belief that the non-BPS systems do not have supersymmetry. However it is relatively easy to resolve this apparent inconsistency.

One should note that the constancy of the Killing spinors (3.65) was contingent upon being in a system of frames in which the almost hyper-Kähler forms have constant coefficients. However, the differential constraints on the $J^{(A)}$ required that we pass to the system of rotating structures, (4.136) and so the frames, $\tilde{e}^a$, for the six-dimensional constant spinors must be related to the standard, v-independent frames, $\hat{e}^a$, of the LeBrun base via:

$$\tilde{e}^1 = \cos(2\,v)\,\hat{e}^1 - \sin(2\,v)\,\hat{e}^2, \qquad \tilde{e}^2 = \cos(2\,v)\,\hat{e}^2 + \sin(2\,v)\,\hat{e}^1, \tag{4.150}$$

One could, of course, work in six dimensions with the frames, $\hat{e}^a$, and transform everything using the foregoing frame rotation. One would then find that the supersymmetries *necessarily depend upon* v. It is for this reason that *trivial dimensional reduction* to five dimensions *breaks the supersymmetry*.

More generally, if one works purely in five dimensions, or in any setting, like M-theory, where there is no non-trivial Kaluza-Klein fibration, then there is no way to preserve the supersymmetry because the fiber dependence that is essential to the supersymmetry cannot be realized. Thus it is only in the six-dimensional theory and its IIB uplift that the solutions with a LeBrun base can be rendered supersymmetric.

4.4.2 The solutions: asymptotics and regularity

Given the various identifications found Section 4.4.1, we find that the 6d BPS system (3.101)–(3.107) reduces exactly to the floating brane system (3.60)–(3.63), with the LeBrun metric (4.7) serving as the base space in each. Therefore the solutions we obtain are the same, as given in Section 4.3.4. However, these solutions are now in a different geometrical context, so it is worth re-examining their asymptotic behavior and regularity conditions.

Asymptotics at infinity

To understand the asymptotic behavior of the general multi-center solution of Section 4.3.4 we will again look at the spherically symmetric solution on a flat $\mathbb{R}^4$ base which corresponds to choosing $V = 1$ for the function determining the Burns base. The sources for this solution lie at $(x, y, \zeta) = (0, 0, 0)$. We will also set some of the electric potentials to zero:

$$L_1 \equiv L_2 \equiv 0\,. \tag{4.151}$$

The functions $K^{(I)}$ and M are then homogeneous solutions to $\mathcal{L}_1 H = 0$, where

$$\mathcal{L}_1 H \equiv \partial_x^2 H + \partial_y^2\, H + \zeta^{-1}\, \partial_\zeta \left(\zeta \partial_\zeta H\right), \tag{4.152}$$

and we take

$$Z_1 = K^{(2)} = \frac{\beta_2}{\rho^2}\,, \qquad Z_2 = K^{(1)} = \frac{\beta_1}{\rho^2}\,, \qquad M = \frac{\gamma}{\rho^2}\,, \tag{4.153}$$

where β_1, β_2 and γ are constant parameters.

It is easy to see that the rest of the functions in the solution are

$$Z_3 = \ell_3^0 + \frac{2\,\gamma}{\rho^2} + (\beta_1\,\beta_2\, + \beta_3)\,\frac{\cos^2\theta}{\rho^2}\,, \tag{4.154}$$

$$\mu = -\frac{1}{2}\,(2\,\beta_1\,\beta_2 + \beta_3)\,\frac{\cos^2\theta}{\rho^2}\,, \qquad \omega = \frac{\beta_3}{2}\,\frac{\sin^2\theta}{\rho^2}\,d\phi\,. \tag{4.155}$$

The six-dimensional metric is then

$$\begin{aligned} \mathrm{d}s_6^2 = -\frac{\rho^2}{\sqrt{\beta_1\beta_2}}\,\mathrm{d}v\left(2\mathrm{d}u - 2(2\,\beta_1\,\beta_2 + \beta_3)\,\frac{\cos^2\theta}{\rho^2}\,\mathrm{d}\tau + 2\beta_3\,\frac{\sin^2\theta}{\rho^2}\,\mathrm{d}\phi - 4Z_3\,\mathrm{d}v\right) \\ + \sqrt{\beta_1\beta_2}\,\frac{\mathrm{d}\rho^2}{\rho^2} + \sqrt{\beta_1\beta_2}\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + \cos^2\theta\,\mathrm{d}\tau^2). \end{aligned} \tag{4.156}$$

For generic values of the parameters above (in particular for $\ell_3^0 \neq 0$) this solution is asymptotic to a pp-wave type background at $\rho \to \infty$. However for $\ell_3^0 = 0$ and $\beta_3 = -\beta_1\beta_2$ the metric becomes precisely the near horizon metric of a BPS D1-D5-P black string (see, for example, [103]). To have a precise identification of the parameters of our solution with the charges of the D1-D5-P string we performed a careful comparison with the 3-charge solutions in D1-D5-P frame discussed in [104]. We find the following identification

$$Q_1 = \beta_2, \qquad Q_5 = \beta_1, \qquad Q_P = 8\gamma, \qquad J = \beta_1\beta_2, \tag{4.157}$$

where Q_1, Q_5 and Q_P are D1, D5 and momentum charges of the black string and J is its angular momentum. Note that the entropy of the black string is $S \sim \sqrt{Q_1Q_5Q_P - J^2}$ and we have the bound $Q_1Q_5Q_P \geq J^2$. It is also interesting to note that (4.157) implies $J = Q_1Q_5$ which, for $Q_P = 0$, is the condition for a maximally spinning D1-D5 supertube [20]. In general however we have $Q_P \neq 0$ and the condition $J = Q_1Q_5$ seems less natural. It would be very interesting to understand this relation between J, Q_1 and Q_5 from the point of view of the dual D1-D5-P CFT.

Asymptotics near the charge centers

Having understood the asymptotic structure at infinity in a very simple example, we now return the the generic multi-centered solutions of Section 4.3.4 and examine the physics of solutions near these centers. As one would expect, one can easily recover the solutions for multiple concentric black rings [73, 105, 106] from our general multi-center solutions. The details depend upon the behavior of the solution as $\rho_i \to 0$, where

$$\rho_i \equiv \sqrt{x^2 + y^2 + (\zeta - c_i)^2}\,. \tag{4.158}$$

As $\rho_i \to 0$, one can easily arrange that $Z_1, Z_2 \sim \rho_i^{-1}$, $Z_3 \sim \rho_i^{-2}$, and $V \sim \rho_i^{-1}$, and each such center thus corresponds to a rotating black ring/string. The reason for the differing power of ρ_i in Z_3 is the presence of local dipole charges; recall that a 3-charge black ring solution looks schematically like [73, 105, 106]

$$Z_3 \sim \frac{Q_3}{\Sigma} - \frac{d_1 d_2}{\Sigma^2} \quad \text{and cyclic.} \tag{4.159}$$

Due to the trivial nature of $\Theta^{(3)}$ in the LeBrun-Burns metrics, the $\Theta^{(3)}$ dipole charge is zero, thus removing the more-strongly-divergent term from Z_1, Z_2. We expect that a more general metric in the LeBrun class will have $\Theta^{(3)}$ dipole charges and thus allow centers which open up into rotating $AdS_3 \times S^3$ throats.

Another possibility is that the geometry remain smooth as $\rho_i \to 0$. In five dimensions, this has been thoroughly analyzed in [33]. The local conditions are

$$m_i = \ell_1^i = \ell_2^i = 0, \qquad \ell_3^i = -\frac{k_1^i k_2^i}{4c_i^2 q_i}, \qquad m_0 + \frac{\gamma}{c_i^2} - \frac{k_1^i k_2^i}{8c_i^2 q_i^2} = 0. \tag{4.160}$$

These conditions will also lead to regular geometries in six dimensions. However, one might ask whether the extra $U(1)$ fiber (along the v coordinate) in six dimensions might allow for more general regular solutions. We find that the answer is "no", and so the conditions for regularity remain as in (4.160). The reason for this is because $\Theta^{(3)}$ has no non-trivial fluxes; thus the KK monopole ingredient, which is necessary to make regular supertubes as explained in Section 3.3.3, is absent. More general LeBrun metrics can certainly have such non-trivial fluxes, just as $\Theta^{(3)}$ can have non-trivial fluxes on GH bases, but the structure of LeBrun-Burns metrics precludes such supertubes.

* * *

4.5 Discussion and open problems

The 5-dimensional solutions

Using the floating brane ansatz of [31] we have constructed a large class of non-BPS multi-centered supergravity solutions. The solutions are determined by a four-dimensional Kähler base with non-trivial topology and that is a solution of the Euclidean Einstein-Maxwell equations. To find explicit solutions one has to solve a coupled linear system of inhomogeneous differential equations on this base. We managed to construct the most general explicit solution of these equations on the axisymmetric LeBrun-Burns base. The generic multi-centered solutions will have horizons but we showed explicitly that by a judicious choice of parameters one can make the solutions completely smooth and regular. Due to the Maxwell flux on the four-dimensional base the five-dimensional solutions are not asymptotically flat but can be arranged to look like a warped, rotating $AdS_2 \times S^3$ space at asymptotic infinity. For specific choice of parameters the asymptotic metric is exactly the near horizon throat metric of the BMPV black hole. We have thus constructed "hair in the back of a throat".

There are a number of possible directions for further work in this area. First, it is well-known that BPS supertubes with two electric and one magnetic dipole charge are regular in six dimensions in the D1-D5 duality frame [19, 20, 104]. Such solutions can thus potentially provide richer classes of regular geometries. Indeed, five-dimensional regularity requires that all the Z_I be non-singular but supertubes allow two of the Z_I to have poles and the singularities are resolved as Kaluza-Klein monopoles in six dimensions. The solutions presented in Section 4.3.4, before five-dimensional regularity was imposed, include solutions that correspond to families of concentric supertubes. Removing the singularities as in (4.119) required us to set some of the parameters to zero (see (4.120)) and while we still found regular solutions with microstate structure, it restricted that family of solutions quite strongly and led us to solutions for which the bubble equations were trivial. We expect that for solutions with supertubes there will be some analog of the familiar radius formula arising from the bubble equations, or integrability conditions. We therefore expect there to be even richer classes of bubbles and "hair" if one allows solutions that are regular in six dimensions but not necessarily in five.

It is also worth recalling that there are spectral flow methods that map regular, six-dimensional supertube geometries onto five-dimensional, regular bubbled geometries [107]. For BPS solutions, these transformations do not substantially modify the geometry of the four-dimensional base, though they can modify the asymptotics at infinity. On the other hand, for non-BPS solutions such spectral flows can completely change the geometry of the base, for example, mapping a hyper-Kähler geometry onto an Israel-Wilson electrovac solution [31]. It would be interesting to see how such spectral flows might modify the solutions considered here, particularly if one first includes supertube configurations. It will almost certainly move one beyond the LeBrun class of solutions and perhaps give a richer class of geometries at infinity.

There are other natural generalizations of the solutions considered here. Our solutions can be uplifted to eleven dimensions where they are sourced by intersecting M2 and M5 branes on T^6 [25]. It is fairly evident that there will also be solutions that can be obtained from intersecting M2 and M5 branes wrapping two-cycles and four-cycles in a more general Calabi-Yau three-fold. Going in the opposite direction, any solution with a LeBrun base has a space-like Killing vector (defined by τ-translations) and so one can perform a dimensional reduction along this direction to find supergravity solutions in four dimensions. These solutions will clearly be non-BPS and will represent an infinite class of multi-center four-dimensional solutions that are non-supersymmetric generalizations of the solutions of [101].

It would be interesting to explore the attractor mechanism for our solutions and make connections with discussions on non-BPS attractors [108]. The multi-centered solutions of Section 4.3.4 may realize non-BPS split attractors. It is interesting to note that in a recent discussion on non-supersymmetric split attractor flows the authors of [108] also found that there are no bubble equations for their non-generic solutions (or integrability conditions). This fits with our analysis in Section 4.3.4 and it will be very interesting to make this connection more precise. On the other hand it is known that there could be bubble equations for non-BPS multi-center solutions, as discussed in [77, 109], and it will be interesting to explore how generic are these constraints.

Since our solutions are asymptotic to anti-de Sitter space one can do holographic analysis of the "hair" corresponding to our geometries and understand them as duals to states (or thermal ensembles) in the corresponding CFT. The solutions presented here have a warped and rotating AdS_2 region and while the AdS_2/CFT_1 correspondence is not understood in such detail as its higher-dimensional analogs[10] there might be some effective approach similar to the one in [111]. Alternatively, one might use a series of dualities and transform the solutions to the D1-D5-P IIB duality frame [104] and study the states in the D1-D5 CFT. One might then be able to study the stability of the solutions and make some connection with the recent discussion of Hawking radiation from non-supersymmetric solutions of the D1-D5 system [112, 113, 114].

One would also very much like to find explicit non-supersymmetric solutions that have a throat region that looks like the solutions discussed in this chapter but are asymptotically flat at infinity. To achieve this, one will probably have to find a way of breaking the relationship between the background electromagnetic field and the Kähler form. To achieve this one will probably have to relax some of the simplifying assumptions of

[10]For a discussion on holography for backgrounds with an AdS_2 factor see [110].

the floating brane ansatz [31] and work with more general (and complicated) equations of motion. However, there are almost certainly even broader classes of non-supersymmetric solutions that are determined by linear systems of equations and thus such explicit non-BPS solutions may well be within reach.

The lift to 6 dimensions

By way of the uplift described in Section 4.4, we have also found a new class of BPS solutions of six-dimensional supergravity coupled to a tensor multiplet and these solutions can be trivially uplifted to supersymmeric solutions of IIB supergravity on T^4. A key ingredient in our construction is a four-dimensional Kähler base with a $U(1)$ symmetry and vanishing Ricci scalar studied by LeBrun. For the LeBrun-Burns class of such four-dimensional metrics the 6d BPS equations can be solved explicitly and one can find closed form expressions for the metric and the background fields. It is important to stress that these solutions provide the first examples of BPS backgrounds of six-dimensional supergravity that do not have a hyper-Kähler base. In fact, almost all explicit BPS solutions discussed previously have the very special Gibbons-Hawking base[11].

The supersymmetry conditions of six-dimensional supergravity impose, amongst other things, the constraints (3.76) and (3.77) on the four-dimensional base of the solution. In contrast to the situation in five-dimensional supergravity, where this base has to be hyper-Kähler, it is not clear to us whether there is a simple geometric meaning of the more general constraint in six dimensions. It is quite conceivable that this constraint could be given a very interesting meaning for some suitably arranged five-dimensional

[11] To the best of our knowledge the only solutions with a more-general hyper-Kähler base are the ones constructed in [96].

spatial geometry. Our analysis clearly demonstrates that some Kähler manifolds can satisfy this constraint but we believe there will be a much more general class of geometries that can be used to construct six-dimensional BPS solutions.

For judicious choice of parameters these 6d solutions are asymptotic, at infinity, to the near horizon geometry of the BPS D1-D5-P black string. It is certainly important to understand the microscopic brane configurations that source the solutions in more detail. Since the D1-D5-P black string geometry is asymptotically locally $AdS_3 \times S^3$ one can apply holographic methods to uncover which states in the D1-D5-P CFT are dual to our regular solutions. The technology developed in [115] for the more restricted two-charge D1-D5 geometries will be certainly useful in this regard. It will also be interesting to see if there is an efficient way to count our regular geometries by some generalization of the techniques used in [116, 23] to count two-charge supertubes or the $\frac{1}{2}$-BPS asymptotically $AdS_5 \times S^5$ solutions of Lin-Lunin-Maldacena (LLM) [117].

As we emphasized, the Killing spinors of our backgrounds will not survive a trivial dimensional reduction along the v-fiber and so supersymmetry will be broken in such a reduction. Moreover, a subsequent *trivial* uplift, like embedding the solution in M-theory, will not restore the supersymmetry. Since the six-dimensional solution is BPS, this means that five-dimensional non-BPS solutions are necessarily extremal because their mass is locked to their electric charges. Extremal non-BPS solutions in four and five dimensions have drawn a lot of attention recently and there is a large number of known multi-centered non-BPS solutions (see for example [109]). It would be interesting to reduce our solutions to four dimensions and understand whether the four-dimensional, axi-symmetric solutions fit in one of the known classes of such solutions discussed in [109] or whether the solutions discussed here provide a completely new system. Furthermore it will be interesting to explore the action of spectral flow [107] and more general U-duality symmetries of string theory on our solutions [118].

Like their 5d non-BPS counterparts, our 6d BPS solutions are also not asymptotically flat and it would be nice to understand how to modify them such that we have a supergravity solution asymptotic to $\mathbb{R}^{1,5}$. Although this is certainly an interesting question we expect that it will not be easy to answer it. For example, one does not know how to make the general $\frac{1}{2}$-BPS LLM solutions in IIB asymptotically flat [117].

Future prospects

On the other hand, there are certainly more general solutions within reach that go beyond the ones constructed here. As we remarked earlier, in (4.61) we made an extremely simple, non-singular choice for the solution, u, of the Affine Toda equation and there are much richer possibilities. Indeed, axi-symmetric solutions of the $SU(\infty)$ Toda equation can be obtained by transforming solutions of the Laplace equation on $\mathbb{R}^3$ [88]. It would be interesting to start from such solutions and see to what extent one can generate explicit BPS solutions. In fact, this approach will be studied in much further detail in Chapter 5.

Chapter 5

Solutions on LeBrun metrics with axial symmetry

The material in this chapter is taken from [35], which is a paper on which I am the sole author, and in which I follow up on the results of [33, 34].

5.1 Motivation

In Chapter 4 we discussed solutions to the "floating brane" equations on a 4d base of Kähler Einstein-Maxwell metrics studied by LeBrun [85, 86]. These metrics are determined by two functions which solve the $SU(\infty)$ Toda equation and its linearization. In a pair of papers [33, 34] (discussed in Sections 4.3 and 4.4), we chose an extremely simple solution to the Toda equation (4.61), leading to the LeBrun-Burns subclass of metrics, which are Kähler analogues to Gibbons-Hawking metrics with a hyperbolic base instead of flat $\mathbb{R}^3$. On the LeBrun-Burns base, we solved the floating brane equations and obtained an infinite family of solutions.

These solutions were shown to have a few desirable properties. The LeBrun-Burns metrics have the structure of a $U(1)$ fiber over H^3, in much the same way that Gibbons-Hawking metrics are described by a $U(1)$ fiber over $\mathbb{R}^3$. This $U(1)$ fiber pinches off at controlled points, which allows one to construct solutions with several 2-cycle "bubbles" on which one can put cohomological fluxes. We also showed that with appropriate choices of parameters, the solutions could be made regular and free of CTC's.

However, these solutions also had a few shortcomings. Due to the simplistic choice of Toda solution (4.61), the self-dual 2-form $\Theta^{(3)}$ of the LeBrun-Burns metrics is topologically trivial. Hence, while one can use the $U(1)$ fiber to sweep out 2-cycles, one can only arrange two of the three "flavors" of fluxes to be cohomological on those 2-cycles. The resulting "bubble equations" turn out to be independent of the sizes of the bubbles, and thus the interplay between bubbles and fluxes, analogous to BPS solutions (see Section 3.1), is gone. Furthermore, the solution is very degenerate, because it effectively has only two types of dipole charges. As a result, the regularity conditions actually demand that most of the parameters be set to zero. Finally, the solutions are not asymptotically flat; however, we have shown in Section 4.2.4 that this fact is generically true of all LeBrun-based solutions, and thus we will not resolve this issue here. However, the lack of asymptotic flatness should not be too great a concern. One does obtain solutions whose asymptotics are like the near-horizon limit of a BMPV black hole [89]. So it is not too far a stretch to say that these are BMPV microstate geometries, and probably the asymptotically-flat region can be restored by relaxing the assumptions of the floating brane ansatz.

As explained in Section 3.3, it has been shown that another linear system of equations can be revealed by re-organizing the BPS equations in the 6-dimensional IIB frame [70, 71, 72]. As discussed in Section 4.4, this makes a curious connection to the 5-dimensional story: the 5d non-BPS "floating brane" equations on a LeBrun base are *identical* to the 6d BPS equations where all functions are made independent of the 6th coordinate [34]. Therefore the exact same family of solutions plays two roles, both supersymmetric and non-supersymmetric. The apparent discrepancy is explained in the trivial KK reduction from 6 to 5 dimensions: the Killing spinor in 6 dimensions can be charged under the $U(1)$ on which the reduction occurs, which causes it to vanish in 5 dimensions.

In this chapter, we improve upon the results of Section 4.3 and overcome its major issues. Despite the 5d-6d link mentioned, we work strictly in the 5-dimensional frame, and leave any investigation of new boundary conditions in 6d (such as supertubes, as discussed in Section 4.4.2) to future work. This chapter is organized as follows: In Section 5.2, we solve the $SU(\infty)$ Toda equation explicitly under the assumption of an additional $U(1)$ isometry. We determine the boundary conditions needed for the solutions we wish to build, and we analyze the resulting base manifold in detail to explore its geometric and topological properties. In Section 5.3, we solve the floating brane equations on this base manifold explicitly, thus giving the full supergravity solution (we will make frequent reference to the equations in Section 4.2.3). We will then determine the conditions needed to make solutions regular in 5 dimensions. We derive the no-CTC conditions, or "bubble equations" and analyze them. Finally, we give an explicit, solved example of a 3-center solution. In Section 5.4, we discuss these results and open problems.

5.2 Axisymmetric Kähler base spaces

Before we discuss solutions to the full system, we will explore the base space $\mathcal{B}$ in detail. Our task is to solve the $SU(\infty)$ Toda equation which, while known to be integrable, is also notoriously hard. However, if we impose an additional $U(1)$ symmetry, there is a known method of attack [119, 120, 87, 88].

First let us write the LeBrun metric in an explicitly $U(1) \times U(1)$-invariant form,

$$g = \frac{1}{w}(\mathrm{d}\tau + A)^2 + we^u \left(\mathrm{d}r^2 + r^2 \, \mathrm{d}\phi^2\right) + w \, \mathrm{d}z^2, \tag{5.1}$$

where now all functions depend on r, z only. For completeness, the equations to be solved in these coordinates become

$$\frac{1}{r}\partial_r(ru_r) + (e^u)_{zz} = 0, \tag{5.2}$$

$$\frac{1}{r}\partial_r(rw_r) + (e^u w)_{zz} = 0, \tag{5.3}$$

and

$$\mathrm{d}A = rw_r\,\mathrm{d}\phi \wedge \mathrm{d}z + (e^u w)_z\, r\,\mathrm{d}r \wedge \mathrm{d}\phi. \tag{5.4}$$

At this point, we can solve (5.3) and (5.4) generically. To accomplish this, note that the Laplacian on the 3-dimensional base h is given by

$$e^u\,\Delta_h(\varphi) = \frac{1}{r}\partial_r(r\varphi_r) + (e^u\varphi_z)_z, \tag{5.5}$$

and hence the Laplacian is related to the linearized Toda equation via ∂_z:

$$\partial_z\big(e^u\,\Delta_h(\varphi)\big) = \frac{1}{r}\partial_r(r\partial_r\varphi_z) + (e^u\varphi_z)_{zz}. \tag{5.6}$$

Therefore if we take some $\hat{w}$ which solves the Laplace equation on h,

$$\frac{1}{r}\partial_r(r\hat{w}_r) + (e^u\hat{w}_z)_z = 0, \tag{5.7}$$

then it is easy to show that (5.3) and (5.4) are solved by

$$w = \hat{w}_z, \qquad A = -r\hat{w}_r\,\mathrm{d}\phi. \tag{5.8}$$

One can think of $\hat{w}$ as a "potential" that gives us the solutions for w and A.

5.2.1 Solving the axisymmetric Toda equation

Now let us focus on the Toda equation with an axial symmetry (5.2). The additional $U(1)$ symmetry allows one to make a Bäcklund transformation to new coordinates ρ, η [119, 120, 87, 88]:

$$r^2 e^u = \rho^2, \qquad \log r = V_\eta, \qquad z = -\rho V_\rho. \tag{5.9}$$

The Toda equation can then be mapped onto the axisymmetric Laplace equation in $\mathbb{R}^3$ in cylindrical coordinates:

$$\frac{1}{\rho}\partial_\rho(\rho V_\rho) + V_{\eta\eta} = 0. \tag{5.10}$$

In principle, one must then invert the transformation (5.9) to obtain u. But in practice, for most functions V this is intractable. It is easier to change the metric to the new coordinates ρ, η, which results in[1]

$$g = \frac{1}{w}(\mathrm{d}\tau + A)^2 + w\,h, \tag{5.11}$$

$$h = \rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) + \rho^2\,\mathrm{d}\phi^2. \tag{5.12}$$

We must also change (5.3) and (5.4) into the new coordinates. The Laplacian Δ_h becomes, up to an overall factor, the cylindrically-symmetric Laplacian on $\mathbb{R}^3$,

$$\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)\ \Delta_h(\varphi) = \frac{1}{\rho}\partial_\rho(\rho\varphi_\rho) + \varphi_{\eta\eta}, \tag{5.13}$$

[1] N.B. – As a result of the transformations (5.9), the cylindrical coordinates ρ, η, ϕ inherit the orientation opposite to the usual: $\mathrm{vol}_h = \rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)\,\mathrm{d}\rho \wedge \mathrm{d}\eta \wedge \mathrm{d}\phi$.

and so the potential $\hat{w}$ also solves

$$\frac{1}{\rho}\partial_\rho(\rho\hat{w}_\rho) + \hat{w}_{\eta\eta} = 0, \tag{5.14}$$

whose solutions we know well. Then w and A are given by

$$w = \hat{w}_z = \frac{1}{\rho(V_{\rho\eta}^2 + V_{\eta\eta}^2)}\big(V_{\eta\eta}\,\hat{w}_\rho - V_{\rho\eta}\,\hat{w}_\eta\big). \tag{5.15}$$

and

$$A = -r\hat{w}_r\,\mathrm{d}\phi = -\frac{1}{V_{\rho\eta}^2 + V_{\eta\eta}^2}\big(V_{\rho\eta}\,\hat{w}_\rho + V_{\eta\eta}\,\hat{w}_\eta\big)\,\mathrm{d}\phi. \tag{5.16}$$

Therefore, the geometric data of the base space are determined in terms of two functions $V, \hat{w}$ that solve the axisymmetric Laplace equation in $\mathbb{R}^3$.

5.2.2 Boundary conditions

The task of writing an explicit base space is then reduced to solving cylindrically symmetric electrostatics problems in $\mathbb{R}^3$ [119]. The question is what kinds of electrostatic problems give interesting solutions. We will argue for the specific form of boundary conditions needed; the construction that results is essentially the same as that in [121] for toric Kähler metrics.

By analogy with BPS solutions on Gibbons-Hawking bases [25], we expect to specify a collection of points along the η axis where w and $K^3 \equiv \frac{1}{2}u_z$ have poles. The poles of w control where the τ fiber pinches off, thus creating a series of homology 2-cycles (provided that the 3-dimensional base h remain smooth at these points). The poles of u_z control sources of $\Theta^{(3)}$. If u_z has a pole where w does not, we expect the base metric to be singular. But if u_z has poles coincident with poles of w, we expect that the base geometry is smooth (but possibly with conical singularities), and such poles should control the fluxes of $\Theta^{(3)}$ on the adjacent 2-cycles.

In the simplest case, we consider where w and u_z each have a single, coincident pole. Since both w and u_z solve the same elliptic linear PDE (4.9) (with the same boundary condition at infinity) and have only one "source point", it follows that w and u_z are proportional. Hence $\Theta^{(3)} = 0$ and the metric is Ricci-flat, and therefore hyper-Kähler—thus the metric (5.1) should be a Gibbons-Hawking metric, in alternative coordinates[2]. Therefore we attempt to interpret metric (5.1) as a 1-center Gibbons-Hawking metric:

$$\mathrm{d}s^2(GH) = \frac{R}{q}\Big(\mathrm{d}\psi + q\cos\theta\,\mathrm{d}\phi\Big)^2 + \frac{q}{R}\Big(\mathrm{d}R^2 + R^2\,\mathrm{d}\Omega_2^2\Big), \tag{5.17}$$

where $\mathrm{d}\Omega_2^2$ is the metric on a unit 2-sphere. Comparing to (5.1), we identify z as the radial coordinate, and take $\mathrm{d}\Omega_2^2$ to be written in stereographic coordinates r, ϕ. Hence we identify

$$e^u = \frac{4z^2}{(1+r^2)^2}, \qquad u_z = \frac{2}{z}, \qquad w = \frac{q}{z}, \tag{5.18}$$

where q is any integer, and it is easy to verify that these solve (5.2) and (5.3) as expected. Then as $z \to 0$, the metric (5.1) approaches, as usual, the flat metric on $\mathbb{R}^4/\mathbb{Z}_q$. This gives the canonical example of coincident poles in w, u_z. We expect that near any location where w, u_z both blow up, the metric will locally have this form.

To get a function u_z with many poles, we should choose a cylindrically-symmetric Laplace solution V that gives rise to the behavior in (5.18), and then use linearity to

[2] In the general LeBrun ansatz, taking $w \sim u_z$ gives not a Gibbons-Hawking metric, but a more general hyper-Kähler manifold. However, if we set $w \sim u_z$ in the $U(1)\times U(1)$-invariant ansatz of (5.1), there is always some linear combination of the $U(1)$'s which is tri-holomorphic, hence the manifold must in fact be Gibbons-Hawking but written in a funny way. See also Appendix B.1

combine several solutions at centered at different points. Using the Bäcklund transformation (5.9), we have

$$u_z = \qquad \frac{2V_{\eta\eta}}{\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)} = -\frac{2}{\rho V_\rho} \qquad = \frac{2}{z}, \tag{5.19}$$

where the center equality is the boundary condition we need to satisfy near the source point in order for u_z to have the appropriate singular behavior. We see that while the cylindrically-symmetric Laplace equation for V (5.10) is linear, the boundary condition for V is nonlinear. To solve this boundary condition, one can guess a few known possibilities for V. It turns out the appropriate choice is also the most obvious one to give a pole in the numerator:

$$V_{\eta\eta} = \frac{1}{\sqrt{\rho^2 + \eta^2}}. \tag{5.20}$$

Integrating this twice with respect to η and choosing appropriate integration constants, we find

$$V = -\sqrt{\rho^2 + \eta^2} + \eta \log \frac{\eta + \sqrt{\rho^2 + \eta^2}}{\rho}. \tag{5.21}$$

Then we have

$$z = -\rho V_\rho = \sqrt{\rho^2 + \eta^2}, \qquad V_{\rho\eta} = -\frac{\eta}{\rho} \frac{1}{\sqrt{\rho^2 + \eta^2}}, \tag{5.22}$$

and hence

$$\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2) = 1, \quad \text{which implies} \quad u_z = \frac{2}{z}, \tag{5.23}$$

and the boundary condition is satisfied.

By the superposition principle, we can then write a solution with N such poles as

$$V = k_0^3\, \eta \log \rho + \sum_{i=1}^{N} k_i^3\, H_i(\rho, \eta), \tag{5.24}$$

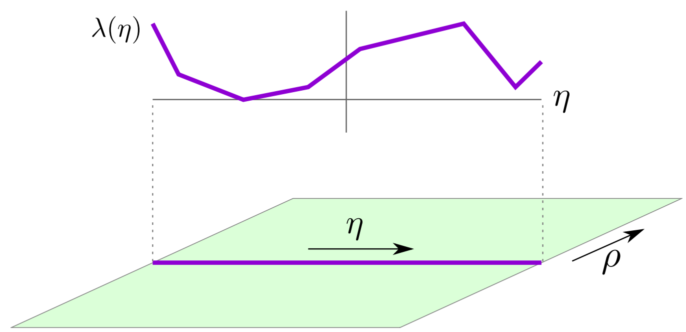

Figure 5.1: *The electrostatics problem corresponding to V. $\lambda(\eta)$ is a line charge density profile along the η axis, which is piecewise linear with "kinks" at each of the η_i.*

where

$$H_i(\rho,\eta) = -\sqrt{\rho^2+(\eta-\eta_i)^2} + (\eta-\eta_i)\log\frac{\eta-\eta_i+\sqrt{\rho^2+(\eta-\eta_i)^2}}{\rho}, \tag{5.25}$$

and the η_i are the locations of the poles on the η axis. Interpreted as an electrostatics problem, this corresponds to the potential of a line charge along the η axis of varying charge density $\lambda(\eta)$. The charge density profile $\lambda(\eta)$ is piecewise linear, with a "kink" at each η_i as in Figure 5.1, such that

$$\lambda''(\eta) = \sum_{i=1}^{N} k_i^3\,\delta(\eta-\eta_i), \tag{5.26}$$

where the parameters k_i^3 represent the amount by which the slope jumps as one moves across the kink at η_i. In V (5.24), we have also put an additional parameter k_0^3, which represents the freedom to choose the value of $\lambda'(\eta)$ at infinity[3].

[3]Specifically, $2\,k_0^3$ is the sum $\lambda'(\infty)+\lambda'(-\infty)$, while the difference $\lambda'(\infty)-\lambda'(-\infty)$ is given by the sum of all the jumps k_i^3.

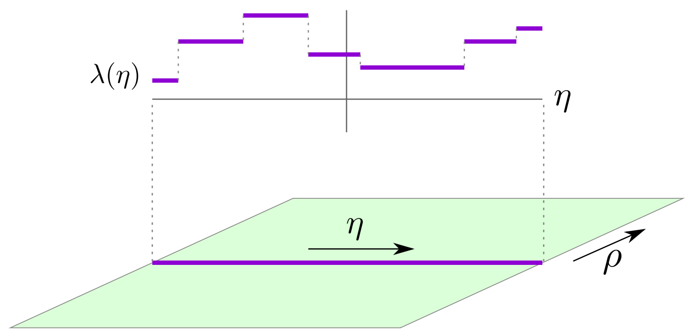

Figure 5.2: *The electrostatics problem corresponding to $\hat{w}$. The line charge profile $\lambda(\eta)$ is piecewise constant, with "jumps" at each η_i.*

We must also choose $\hat{w}$ such that $w = \hat{w}_z$ has $1/z$ type behavior at the source points. It is easy to show that correct choice is

$$\hat{w} = q_0 \log \rho + \sum_{i=1}^{N} q_i \, G_i(\rho, \eta), \tag{5.27}$$

$$G_i(\rho, \eta) = \log \frac{\eta - \eta_i + \sqrt{\rho^2 + (\eta - \eta_i)^2}}{\rho}. \tag{5.28}$$

As an electrostatics problem, this corresponds to a line charge profile $\lambda(\eta)$ which is piecewise constant, with "jumps" at each η_i as in Figure 5.2.

For completeness, it is helpful to write out the η- and ρ-derivatives of these, which appear in all other formulas:

$$V_{\eta\eta} = \sum_{i=1}^{N} \frac{k_i^3}{\Sigma_i}, \qquad V_{\rho\eta} = \frac{k_0^3}{\rho} - \frac{1}{\rho} \sum_{i=1}^{N} \frac{k_i^3 \, (\eta - \eta_i)}{\Sigma_i}, \tag{5.29}$$

$$\hat{w}_\eta = \sum_{i=1}^{N} \frac{q_i}{\Sigma_i}, \qquad \hat{w}_\rho = \frac{q_0}{\rho} - \frac{1}{\rho} \sum_{i=1}^{N} \frac{q_i \, (\eta - \eta_i)}{\Sigma_i}, \tag{5.30}$$

where we have defined $\Sigma_i \equiv \sqrt{\rho^2 + (\eta - \eta_i)^2}$. We note that this is essentially the same construction as in [121] for scalar-flat toric Kähler 4-manifolds (which can always be written in LeBrun form). Thus the base space is defined via the functions (5.29) and (5.30) and the $2N+2$ parameters k_0^3, k_i^3, q_0, q_i.

We will impose one further requirement, which is that the sum of all these parameters be even:

$$\left(k_0^3 + \sum_{i=1}^{N} k_i^3 + q_0 + \sum_{i=1}^{N} q_i\right) \in 2\mathbb{Z}. \tag{5.31}$$

This condition is required such that, at every singular point of the functions (5.29) and (5.30), the metric (5.11) describes (locally) an orbifold point $\mathbb{R}^4/G$ for some finite group G. *Without* this condition, the metric at such points still approaches a conical point, but the cone does not have the right deficit angles to be a *quotient* of $\mathbb{R}^4$, and thus is not an orbifold. This is derived in Appendix B.2, especially Appendix B.2.5.

5.2.3 Near the singularities

The base space is constructed out of N "source points" where the functions V and w are singular. In this section we look in the neighborhood of these points and show that the manifold is perfectly smooth, up to orbifold identifications, in a similar manner to Gibbons-Hawking metrics [69]. Specifically we will find that the metric (5.11) at these points locally approaches the orbifold $\mathbb{R}^4/G$, where $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ is a finite subgroup of the maximal torus[4] $U(1) \times U(1) \subset SO(4)$.

Taking the limit as $(\rho, \eta) \to (0, \eta_\ell)$ for some η_ℓ, we can define new coordinates

$$\rho = R\sin\theta, \qquad \eta - \eta_\ell = R\cos\theta. \tag{5.32}$$

[4]We note that the factors $\mathbb{Z}_m, \mathbb{Z}_n \subset U(1) \times U(1)$ are not necessarily rotations *in a plane* (i.e. fixing every point in the orthogonal plane). One can have, for example, $\mathbb{Z}_m$ acting in the first $U(1)$ and $\mathbb{Z}_n$ acting in the diagonal $U(1)$. Rotations in the diagonal $U(1)$ fix only the origin.

We will find it convenient to define the quantities

$$\bar{K}^3_\ell \equiv \sum_{\substack{i \\ i\neq\ell}} k^3_i \,\mathrm{sign}(\eta_\ell - \eta_i), \qquad \bar{Q}_\ell \equiv \sum_{\substack{i \\ i\neq\ell}} q_i \,\mathrm{sign}(\eta_\ell - \eta_i), \tag{5.33}$$

and also the functions

$$\widetilde{K}(\theta) \equiv (k^3_\ell)^2 + (\bar{K}^3_\ell - k^3_0)^2 + 2\, k^3_\ell(\bar{K}^3_\ell - k^3_0)\cos\theta, \tag{5.34}$$

$$\widetilde{Q}(\theta) \equiv q^2_\ell + (\bar{Q}_\ell - q_0)^2 + 2\, q_\ell(\bar{Q}_\ell - q_0)\cos\theta, \tag{5.35}$$

$$\widetilde{KQ}(\theta) \equiv k^3_\ell q_\ell + (\bar{K}^3_\ell - k^3_0)(\bar{Q}_\ell - q_0) + \big(k^3_\ell(\bar{Q}_\ell - q_0) + q_\ell(\bar{K}^3_\ell - k^3_0)\big)\cos\theta. \tag{5.36}$$

Then for small R, we have

$$\rho^2(V^2_{\eta\eta} + V^2_{\rho\eta}) \to \widetilde{K}(\theta), \qquad w \to \frac{1}{\widetilde{K}(\theta)}\frac{\widetilde{q}_\ell}{R}, \qquad A \to -\frac{\widetilde{KQ}(\theta)}{\widetilde{K}(\theta)}\,\mathrm{d}\phi, \tag{5.37}$$

where we define the determinant:

$$\widetilde{q}_\ell \equiv q_\ell(\bar{K}^3_\ell - k^3_0) - k^3_\ell(\bar{Q}_\ell - q_0). \tag{5.38}$$

The metric becomes

$$\mathrm{d}s^2 = \frac{\widetilde{K}(\theta)R}{\widetilde{q}_\ell}\left(\mathrm{d}\tau - \frac{\widetilde{KQ}(\theta)}{\widetilde{K}(\theta)}\,\mathrm{d}\phi\right)^2 + \frac{\widetilde{q}_\ell}{R}\big(\mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2\big) + \frac{\widetilde{q}_\ell R}{\widetilde{K}(\theta)}\sin^2\theta\,\mathrm{d}\phi^2, \tag{5.39}$$

which, surprisingly enough, is flat. Setting $R = \varrho^2/(4\,\widetilde{q}_\ell)$, this can be rearranged into the more convenient form

$$\mathrm{d}s^2 = \mathrm{d}\varrho^2 + \frac{\varrho^2}{4}\left[\mathrm{d}\theta^2 + \frac{1}{\widetilde{q}_\ell{}^2}\left(\widetilde{K}(\theta)\,\mathrm{d}\tau^2 - 2\widetilde{KQ}(\theta)\,\mathrm{d}\tau\,\mathrm{d}\phi + \widetilde{Q}(\theta)\,\mathrm{d}\phi^2\right)\right]. \tag{5.40}$$

We compare this to a flat metric[5] on $\mathbb{R}^4$:

$$\mathrm{d}s^2 = \mathrm{d}\varrho^2 + \frac{\varrho^2}{4}\left[\mathrm{d}\theta^2 + 2\big(1+\cos\theta\big)\,\mathrm{d}\alpha^2 + 2\big(1-\cos\theta\big)\,\mathrm{d}\beta^2\right], \tag{5.41}$$

where both α, β are identified modulo 2π and $\theta \in [0,\pi]$. The metrics (5.40) and (5.41) are then related by a coordinate transformation

$$\tau = (q_\ell - \bar{Q}_\ell + q_0)\,\alpha - (q_\ell + \bar{Q}_\ell - q_0)\,\beta, \tag{5.42}$$

$$\phi = (k^3_\ell - \bar{K}^3_\ell + k^3_0)\,\alpha - (k^3_\ell + \bar{K}^3_\ell - k^3_0)\,\beta. \tag{5.43}$$

To discover the precise geometry in the neighborhood of the origin, we must carefully follow the identifications of the angular coordinates. This entire process is described in detail in Appendix B.2, with the main results in Appendix B.2.5 which will be used here. We find it is natural to identify the coordinates (τ,ϕ) on the "diamond" lattice,

$$(\tau,\phi): \quad (0,0) \sim (4\pi,0) \sim (2\pi,2\pi) \sim (2\pi,-2\pi), \tag{5.44}$$

which can be written as a matrix Λ_{LB} of column vectors which represent the coordinates where (τ,ϕ) are identified:

$$\Lambda_{LB} = 2\pi\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}, \quad \text{or} \quad \Lambda_{LB} = 2\pi\begin{pmatrix}2 & 1\\ 0 & 1\end{pmatrix}. \tag{5.45}$$

We are free to choose any pair of column vectors that generate the same lattice of identifications; alternatively, Λ_{LB} is defined only up to right action by $GL(2,\mathbb{Z})$[6]. Then

---

[5]This metric is related to the standard spherical coordinates on $\mathbb{R}^4$ by $\theta = 2\vartheta$.

[6]We define $GL(2,\mathbb{Z})$ as the group of 2×2 matrices with integer entries and determinant ± 1, hence invertible over $\mathbb{Z}$. This group is sometimes also called $S^*L(2,\mathbb{Z})$ or $SL^{\pm}(2,\mathbb{Z})$.

applying the coordinate transformation (5.42) and (5.43), we find that the (α, β) coordinates should be identified on the lattice $\widetilde{\Gamma}$, generated by the basis

$$\widetilde{\Lambda} = 2\pi \cdot \frac{1}{2\widetilde{q}_\ell} \begin{pmatrix} k_\ell^3 + \hat{K}_\ell^3 + q_\ell + \hat{Q}_\ell & k_\ell^3 + \hat{K}_\ell^3 - q_\ell - \hat{Q}_\ell \\ k_\ell^3 - \hat{K}_\ell^3 + q_\ell - \hat{Q}_\ell & k_\ell^3 - \hat{K}_\ell^3 - q_\ell + \hat{Q}_\ell \end{pmatrix}, \tag{5.46}$$

where for ease of legibility we have defined

$$\hat{K}_\ell^3 \equiv \bar{K}_\ell^3 - k_0^3, \qquad \hat{Q}_\ell \equiv \bar{Q}_\ell - q_0. \tag{5.47}$$

To determine the group $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$, we then compare this lattice to a "reference" lattice Γ, generated by the basis

$$\Lambda = 2\pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.48}$$

which represents the ordinary 2π identifications that (α, β) would take if there were no conical singularity. Given the lattices $\widetilde{\Gamma}, \Gamma$ generated by (5.46) and (5.48), one can then find the group G by reducing $\widetilde{\Lambda}^{-1}\Lambda$ to *Smith normal form*, where one diagonalizes $\widetilde{\Lambda}^{-1}\Lambda$ by left and right $GL(2, \mathbb{Z})$ actions:

$$R = \widetilde{P}^{-1}\widetilde{\Lambda}^{-1}\Lambda P, \qquad R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \quad \text{where} \quad P, \widetilde{P} \in GL(2, \mathbb{Z}). \tag{5.49}$$

Given the parity condition (5.31), it is always true that $\widetilde{\Lambda}^{-1}\Lambda = 2\pi\widetilde{\Lambda}^{-1}$ has integer entries. Then the numbers r_1, r_2 are integers, and determine G via

$$G = \mathbb{Z}_m \times \mathbb{Z}_n, \quad \text{where} \quad m = r_1, \quad n = r_2. \tag{5.50}$$

Specific details of the groups G

We then find a number of interesting facts (whose detailed derivation can be found in Appendix B.2.5). We will assume the sum of all the parameters of the base space is even as in (5.31), and thus every conical point is an orbifold point.

First, at every orbifold point it is always true that $\Gamma \subseteq \widetilde{\Gamma}$ as a sublattice, and the group G is formally given by the quotient $G \simeq \widetilde{\Gamma}/\Gamma$. The order of the group G is

$$\#G \;=\; |\det(\widetilde{\Lambda}^{-1}\Lambda)| \;=\; |\det(2\pi\widetilde{\Lambda}^{-1})| \;=\; |\widetilde{q}_\ell|, \tag{5.51}$$

and thus the group G is trivial exactly when $\widetilde{q}_\ell = \pm 1$. At such points, the metric approaches flat $\mathbb{R}^4$ with no conical singularity.

Second, we would like to know under what conditions the LeBrun metric approaches an orbifold point whose structure is like that of a charge $m > 1$ Gibbons-Hawking metric. These are points where $G \simeq \mathbb{Z}_m$ and the action of $\mathbb{Z}_m$ is in the diagonal $U(1)$ of the maximal torus $U(1) \times U(1) \in SO(4)$. We find that such orbifold points occur whenever:

$$\widetilde{q}_\ell = \pm m, \qquad \frac{2(\bar{K}^3_\ell - k^3_0)}{\widetilde{q}_\ell} \in \mathbb{Z}, \qquad \text{and} \qquad \frac{2(\bar{Q}_\ell - q_0)}{\widetilde{q}_\ell} \in \mathbb{Z}. \tag{5.52}$$

One can also consider $G \simeq \mathbb{Z}_m$ acting in the *anti*-diagonal $U(1)$, which results in similar conditions:

$$\widetilde{q}_\ell = \pm m, \qquad \frac{2\,k^3_\ell}{\widetilde{q}_\ell} \in \mathbb{Z}, \qquad \text{and} \qquad \frac{2\,q_\ell}{\widetilde{q}_\ell} \in \mathbb{Z}. \tag{5.53}$$

More generally, $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ where each $\mathbb{Z}_k$ acts in some linear combination of the two $U(1)$'s. In the simplest case, the $\mathbb{Z}_k$ act by rotation within a plane; i.e. by rotating (x^1, x^2) and leaving (x^3, x^4) fixed. However, the "diagonal" rotations discussed above act in both planes and do not fix any point aside from the origin. One can also obtain more general rotations that rotate both (x^1, x^2) and (x^3, x^4) planes by unequal amounts.

In any case, an orbifold singularity with a finite group action such as $\mathbb{R}^4/G$ is benign in string theory [122], so in the context of microstate geometries, we will count such points as regular.

5.2.4 At infinity

In the asymptotic region of the base metric, let us define

$$\rho = R\sin\theta, \qquad \eta = R\cos\theta. \tag{5.54}$$

Then as $R \to \infty$, we have

$$\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2) \to (k_0^3)^2 + (K_\star^3)^2 - 2\,k_0^3 K_\star^3 \cos\theta, \tag{5.55}$$

$$w \to \left(\frac{q_0 K_\star^3 - k_0^3 Q_\star}{(k_0^3)^2 + (K_\star^3)^2 - 2\,k_0^3 K_\star^3 \cos\theta}\right)\frac{1}{R}, \tag{5.56}$$

$$A \to \left(\frac{k_0^3 q_0 + K_\star^3 Q_\star - (q_0 K_\star^3 + k_0^3 Q_\star)\cos\theta}{(k_0^3)^2 + (K_\star^3)^2 - 2\,k_0^3 K_\star^3 \cos\theta}\right)\mathrm{d}\phi, \tag{5.57}$$

where the quantities $K_\star^3, Q_\star$ are defined as

$$K_\star^3 \equiv \sum_{i=1}^{N} k_i^3, \qquad Q_\star \equiv \sum_{i=1}^{N} q_i. \tag{5.58}$$

We see that (5.55)–(5.57) have the same structure as (5.37). So at infinity, the base metric approaches a metric with the same structure as (5.39). We can define the determinant

$$\widetilde{q}_\infty \equiv q_0 K_\star^3 - k_0^3 Q_\star, \tag{5.59}$$

and then the conditions (5.51) and (5.52), (5.53) apply in the same way. In particular, one has smooth $\mathbb{R}^4$ at infinity whenever

$$\widetilde{q}_\infty = \pm 1. \tag{5.60}$$

One can obtain $\mathbb{R}^4/\mathbb{Z}_m$, where $\mathbb{Z}_m$ acts on the diagonal $U(1)$ via

$$\widetilde{q}_\infty = \pm m, \qquad \frac{2\,K^3_\star}{\widetilde{q}_\infty} \in \mathbb{Z}, \qquad \text{and} \qquad \frac{2\,Q_\star}{\widetilde{q}_\infty} \in \mathbb{Z}, \tag{5.61}$$

or where $\mathbb{Z}_m$ acts on the anti-diagonal $U(1)$ via

$$\widetilde{q}_\infty = \pm m, \qquad \frac{2\,k^3_0}{\widetilde{q}_\infty} \in \mathbb{Z}, \qquad \text{and} \qquad \frac{2\,q_0}{\widetilde{q}_\infty} \in \mathbb{Z}. \tag{5.62}$$

In general, the geometry approaches $\mathbb{R}^4/G_\infty$, where again $G_\infty \simeq \mathbb{Z}_m \times \mathbb{Z}_n$.

5.2.5 Ambipolar bases

If the base space is considered in isolation, then we must restrict the "charges" $\widetilde{q}_\ell$ at each point to be positive. Otherwise, the function w will change sign[7], and the signature of the metric (5.1) will flip from $(++++)$ to $(----)$.

However, in the context of supergravity solutions, the metric (5.1) appears multiplied by the warp factor $Z = (Z_1 Z_2 Z_3)^{1/3}$ in the full 5-dimensional metric,

$$\mathrm{d}s^2_5 = -Z^{-2}\,(\mathrm{d}t + k)^2 + Z\,\mathrm{d}s^2_4. \tag{5.63}$$

Therefore, we can allow w to change sign, so long as each of the Z_1, Z_2, Z_3 changes sign along the same locus, such that the 5-dimensional metric retains the signature

[7]Caveat: This is not quite true, as we will show in Section 5.2.6.

$(-++++)$. We call such a base space "ambipolar", where the signature is allowed to flip from $(++++)$ to $(----)$, as has been discussed at length in [25, 16]. This justifies the use of $\widetilde{q}_\ell, \widetilde{q}_\infty = \pm 1, \pm m$ in (5.52), (5.53) and (5.60)–(5.62).

With this allowed flexibility in the charges $\widetilde{q}_\ell$, we can construct a wide variety of base spaces. In particular, it should be possible to have both $\widetilde{q}_\ell = \pm 1$ at every point *and* $\widetilde{q}_\infty = \pm 1$ at infinity, thus allowing us to write down supergravity solutions with an arbitrary number of bubbles and no orbifold points anywhere.

5.2.6 Engineering solutions

Here we will describe a simple algorithm for generating solutions with an arbitrary number of points η_ℓ, each of which has trivial orbifold group (and thus is smooth). We will assume that each $\widetilde{q}_\ell = +1$ in order to show an interesting result. It is simple to generalize this algorithm to the more flexible ambipolar case where $\widetilde{q}_\ell = \pm 1$.

To derive this algorithm, we first observe that

$$\bar{Q}_{i+1} - \bar{Q}_i = q_i + q_{i+1}, \tag{5.64}$$

and hence one has

$$(\bar{Q}_{i+1} + q_{i+1}) = (\bar{Q}_i + q_i) + 2q_{i+1}, \tag{5.65}$$

and similarly for $\bar{K}^3_i$. The parity condition (5.31) can also be written

$$k^3_0 + q_0 + (\bar{Q}_i + q_i) + (\bar{K}^3_i + k^3_i) \in 2\mathbb{Z}, \tag{5.66}$$

where $i \in \{1 \dots N\}$ is any of the N points. Since the q_i are integers, (5.65) guarantees that if (5.66) is true for any given i, it is true for all i. Therefore without explicitly writing down the sum of all the parameters, we can describe a recursive algorithm for constructing solutions starting at $i = 1$ and adding as many points as we like.

A second observation we will need is that

$$\widetilde{q}_{i+1} \equiv q_{i+1}(\bar{K}^3_{i+1} - k^3_0) - k^3_{i+1}(\bar{Q}_{i+1} - q_0) \tag{5.67}$$

$$= q_{i+1}(\bar{K}^3_{i+1} + k^3_{i+1} - k^3_0) - k^3_{i+1}(\bar{Q}_{i+1} + q_{i+1} - q_0) \tag{5.68}$$

$$\begin{aligned} &= q_{i+1}(\bar{K}^3_i + k^3_i + 2k^3_{i+1} - k^3_0) \\ &\qquad - k^3_{i+1}(\bar{Q}_i + q_i + 2q_{i+1} - q_0) \end{aligned} \tag{5.69}$$

$$\widetilde{q}_{i+1} = q_{i+1}(\bar{K}^3_i + k^3_i - k^3_0) - k^3_{i+1}(\bar{Q}_i + q_i - q_0), \tag{5.70}$$

where the third line (5.69) follows from (5.65). Since we wish to set each $\widetilde{q}_i = 1$, the last line (5.70) gives us a recurrence relation for the parameters q_i, k^3_i. Then the algorithm proceeds as follows:

1. Define

$$a_i \equiv \bar{K}^3_i + k^3_i - k^3_0, \qquad b_i \equiv \bar{Q}_i + q_i - q_0, \tag{5.71}$$

and choose any a_1, b_1, k^3_1, q_1 such that

$$\widetilde{q}_1 \equiv q_1\, a_1 - k^3_1\, b_1 = 1, \qquad a_1 + b_1 + k^3_1 + q_1 \in 2\mathbb{Z}. \tag{5.72}$$

2. Next, find some k^3_2, q_2 such that (using (5.70))

$$\widetilde{q}_2 = q_2\, a_1 - k^3_2\, b_1 = 1, \tag{5.73}$$

and such that

$$a_2 = a_1 + 2\, k^3_2, \qquad b_2 = b_1 + 2\, q_2 \tag{5.74}$$

are relatively prime[8].

3. Repeat this as many times as desired, finding some k^3_{i+1}, q_{i+1} such that

$$\widetilde{q}_{i+1} = q_{i+1}\, a_i - k^3_{i+1}\, b_i = 1, \tag{5.75}$$

   and

$$a_{i+1} = a_i + 2\, k^3_{i+1}, \qquad b_{i+1} = b_i + 2\, q_{i+1} \tag{5.76}$$

   are relatively prime.

4. After choosing N such k^3_i, q_i, plug them all back into the definitions (5.71) along with a_1, b_1 from the initial step, and solve for the remaining parameters k^3_0, q_0.

It is simple to generalize this algorithm to produce a sequence of points with any desired $\widetilde{q}_i$. In this case, the requirement that each a_i, b_i be relatively prime can be weakened, noting that in general, $\gcd(a_i, b_i)$ must divide both $\widetilde{q}_i$ and $\widetilde{q}_{i+1}$.

We also note that in the final step of the algorithm, there is no longer any freedom to choose parameters, and k^3_0, q_0 must be solved for, from (5.71). Therefore once we have laid down a sequence of N points with given $\widetilde{q}_i$, the orbifold structure at infinity is fixed[9].

If a specific behavior at infinity is required, one can re-write the algorithm to work backwards. The "reverse" algorithm is *not* identical to the one written here, but it is simple to work out from the reasoning in (5.65) and (5.66) along similar lines.

* * *

[8]This is required in order for the next constraint $\widetilde{q}_{i+1} = 1$ to have a solution.

[9]However, the orbifold structure at infinity depends on the specific k^3_i, q_i of the solution, and the same sequence of $\widetilde{q}_i$ can result in different asymptotics!

Using this algorithm it is easy to obtain some interesting solutions. We will give only the solutions and not the details of the algorithm used to obtain them. These two examples show some surprising features which emphasize the difference between LeBrun metrics and Gibbons-Hawking metrics regarding the types of allowed orbifold points:

Example 1: Every interior $\widetilde{q}_i = 1$, but at infinity $\widetilde{q}_\infty = -1$

The first example has three points, and is given by the parameters:

$$q_1 = 4, \qquad q_2 = -3, \qquad q_3 = 2; \qquad q_0 = -2, \tag{5.77}$$

$$k_1^3 = 5, \qquad k_2^3 = -4, \qquad k_3^3 = 1; \qquad k_0^3 = -1. \tag{5.78}$$

For this example, one has

$$\widetilde{q}_1 = 1, \qquad \widetilde{q}_2 = 1, \qquad \widetilde{q}_3 = 1, \qquad \widetilde{q}_\infty = -1. \tag{5.79}$$

Hence at all the source points η_i one has smooth $\mathbb{R}^4$ with trivial orbifold group. However, the minus sign in $\widetilde{q}_\infty$ reveals that it is possible for a LeBrun metric to flip signature $(++++)$ to $(----)$ at infinity even if all the interior points have positive "charges"!

This also implies that the naïve positivity condition mentioned at the beginning of Section 5.2.5 is not quite correct, and requires that one also take into account the numerator of (5.56) to have a metric with positive signature everywhere. Since in the context of higher-dimensional supergravity solutions we do not require the signature of the base to remain $(++++)$ everywhere, we will not worry about this.

Example 2: Every interior $\widetilde{q}_i \geq 1$, but at infinity $\widetilde{q}_\infty = +1$

A second important example is also given by three points:

$$q_1 = -1, \qquad q_2 = 2, \qquad q_3 = 2; \qquad q_0 = 2, \tag{5.80}$$

$$k_1^3 = 0, \qquad k_2^3 = 1, \qquad k_3^3 = 1; \qquad k_0^3 = 1. \tag{5.81}$$

and this example has

$$\widetilde{q}_1 = 3, \qquad \widetilde{q}_2 = 1, \qquad \widetilde{q}_3 = 1, \qquad \widetilde{q}_\infty = 1. \tag{5.82}$$

In this case the metric does not unexpectedly flip signature. However, we do see that it is possible for a LeBrun metric to be asymptotically *flat* (and not just locally flat) even if the interior "charges" are all positive and some of them are greater than 1. This is in contrast to Gibbons-Hawking metrics, where it is a mathematical theorem that the only asymptotically (globally) flat hyper-Kähler metric in 4 dimensions is $\mathbb{R}^4$ [68]. Because LeBrun metrics are merely Kähler and not hyper-Kähler, they are not subject to this restriction, and the set of parameters (5.80) and (5.81) give an explicit example to this effect.

It does not, however, appear to be possible to choose parameters such that all the $\widetilde{q}_i = +1$ *and* $\widetilde{q}_\infty = +1$, although we have not found a way to prove this impossibility in general.

5.2.7 A topological ménagerie

We have shown that the base metric approaches $\mathbb{R}^4/G$, for $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$, near each of the geometric charges where the τ fiber pinches off. As explained in Section 4.2.1, these points control the appearance of homology 2-spheres as the τ fiber sweeps along a path between any two such points.

There are also additional phenomena which appear when we look more carefully at the axis in the 3-dimensional base h:

$$\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) + \rho^2\,\mathrm{d}\phi^2. \tag{5.83}$$

Along the axis, but away from the Gibbons-Hawking points, one has

$$\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2) \to \left(k_0^3 - \sum_{i=1}^{N} k_i^3 \,\mathrm{sign}(\eta - \eta_i)\right)^2 \equiv a^2, \tag{5.84}$$

which is a piecewise-constant function with jumps at each η_i. Whenever $a^2 = 1$, then as $\rho \to 0$, the ϕ circle pinches off smoothly. If instead $a^2 \neq 1$ and $a^2 > 0$, then the ϕ circle pinches off in a conical singularity $\mathbb{R}^2/\mathbb{Z}_a$.

But it is also possible that $a = 0$. Expanding to the next order in ρ^2, and imposing

$$k_0^3 = \sum_{i=1}^{N} k_i^3 \,\mathrm{sign}(\eta - \eta_i), \tag{5.85}$$

one has, as $\rho \to 0$,

$$\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2) \to \rho^2 f(\eta)^2, \qquad w \to \frac{1}{\rho^2}\frac{g(\eta)}{f(\eta)^2}, \qquad A \to -\frac{h(\eta)}{f(\eta)^2}\,\mathrm{d}\phi, \tag{5.86}$$

where the functions $f(\eta), g(\eta), h(\eta)$ are given by

$$f(\eta) = \sum_{i=1}^{N} \frac{k_i^3}{|\eta - \eta_i|}, \tag{5.87}$$

$$g(\eta) = \left(q_0 - \sum_{i=1}^{N} q_i \,\mathrm{sign}(\eta - \eta_i)\right) f(\eta), \tag{5.88}$$

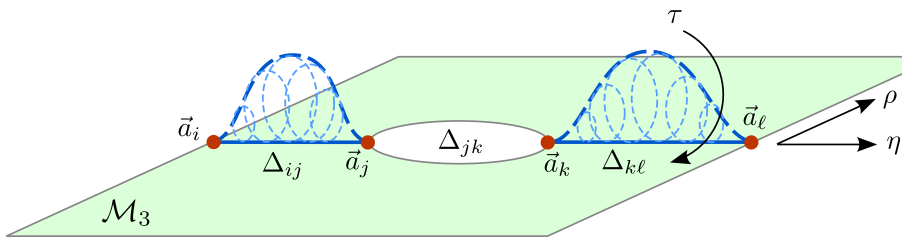

Figure 5.3: *Homology 2-cycles in the axisymmetric base space. Δ_{ij} and $\Delta_{k\ell}$ are cycles formed by sweeping the τ fiber between source points. Δ_{jk} is a cycle formed by the ϕ circle. In the ρ, η coordinates, the ϕ-cycle appears as a line segment between $\vec{a}_j$ and $\vec{a}_k$. However, ϕ does not pinch off there, but approaches a finite size as $\rho \to 0$.*

$$
\begin{aligned}
h(\eta) = &\sum_{i=1}^{N} \frac{q_i}{|\eta - \eta_i|} f(\eta) \\
&+ \frac{1}{2}\left(q_0 - \sum_{i=1}^{N} q_i \operatorname{sign}(\eta - \eta_i)\right) \sum_{j=1}^{N} \frac{k_j^3 \operatorname{sign}(\eta - \eta_j)}{(\eta - \eta_j)^2}.
\end{aligned}
\tag{5.89}
$$

Then as $\rho \to 0$, the 4-metric can be rearranged to give

$$
g \to \frac{g(\eta)}{f(\eta)^2}\, \mathrm{d}\phi + \frac{f(\eta)^2}{g(\eta)} \left[\frac{g(\eta)^2}{f(\eta)^2}(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) + \rho^2\, \mathrm{d}\tau^2\right], \tag{5.90}
$$

where the coordinates τ, ϕ have now exchanged roles. Notably, along the entire segment over which p vanishes, the ϕ circle remains a finite size as $\rho \to 0$, whereas the τ circle pinches off. In particular, we have

$$
\frac{g(\eta)^2}{f(\eta)^2} = \left(q_0 - \sum_{i=1}^{N} q_i \operatorname{sign}(\eta - \eta_i)\right)^2 \equiv 4b^2, \tag{5.91}
$$

so the τ circle is pinching off in a conical singularity $\mathbb{R}^2/\mathbb{Z}_b$ (the factor of 4 in (5.91) is to account for the fact that the period of τ is 4π rather than 2π). This sort of homology 2-cycle, in which ϕ remains finite while τ pinches off along a finite portion of the axis, is illustrated in Figure 5.3.

We also point out that the axisymmetric LeBrun metrics we consider here are toric Kähler manifolds, and there is possibly a more elegant description of what is going on with the various types of 2-cycles using the techniques of toric geometry [121].

5.2.8 Magnetic flux through cycles

A desired property of these new solutions is that the magnetic 2-form $\Theta^{(3)}$ have non-trivial flux through the homological 2-cycles in the base. The 2-form $\Theta^{(3)}$ is given by

$$\Theta^{(3)} = \frac{1}{2}(\mathrm{d}\tau + A) \wedge \mathrm{d}\frac{u_z}{w} + \frac{1}{2} w \star_3 \mathrm{d}\frac{u_z}{w}, \tag{5.92}$$

but it will be more helpful to write it as

$$\Theta^{(3)} = \mathrm{d}B^3 = -\frac{1}{2}\mathrm{d}\Big[\frac{u_z}{w}(\mathrm{d}\tau + A) + r u_r\, \mathrm{d}\phi\Big] \tag{5.93}$$

where

$$\frac{1}{2}u_z = \frac{V_{\eta\eta}}{\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)}, \qquad \frac{1}{2} r u_r = -1 + \frac{1}{\rho(V_{\rho\eta}^2 + V_{\eta\eta}^2)} V_{\rho\eta}, \tag{5.94}$$

$$w = \frac{1}{\rho(V_{\rho\eta}^2 + V_{\eta\eta}^2)}\big(V_{\eta\eta}\, \hat{w}_\rho - V_{\rho\eta}\, \hat{w}_\eta\big). \tag{5.95}$$

On a 2-cycle Δ_{ij} swept out by the τ fiber, the flux can be computed via

$$\Pi_{ij}^{(3)} = \frac{1}{4\pi}\int_{\Delta_{ij}} \Theta^{(3)} = \frac{1}{4\pi}\int_{\Delta_{ij}} \mathrm{d}\tau \wedge \mathrm{d}\frac{K^3}{w} = \frac{k_j^3}{\widetilde{q}_j} - \frac{k_i^3}{\widetilde{q}_i}, \tag{5.96}$$

where $\widetilde{q}_i \equiv q_i(\bar{K}_i^3 - k_0^3) - k_i^3(\bar{Q}_i - q_0)$. This is very reminiscent of the fluxes in the BPS case [25], and is notably different to the previously-known non-BPS solutions [33, 34] where $\Theta^{(3)}$ had no topological fluxes.

On a 2-cycle swept out by the ϕ circle, one has to be considerably more careful. Along a line segment of the η axis between η_i and η_j where the ϕ circle has a finite size, one can show that as $\rho \to 0$,

$$\Theta^{(3)} \to \frac{1}{g_0}\,\mathrm{d}\Big[-\mathrm{d}\tau + \frac{\tilde{f}(\eta)}{f(\eta)}\,\mathrm{d}\phi\Big], \tag{5.97}$$

where

$$f(\eta) = \sum_{i=1}^{N} \frac{k_i^3}{|\eta-\eta_i|}, \qquad \tilde{f}(\eta) = \sum_{i=1}^{N} \frac{q_i}{|\eta-\eta_i|}, \tag{5.98}$$

$$g_0 = \Big(q_0 - \sum_{i=1}^{N} q_i \,\mathrm{sign}(\eta-\eta_i)\Big), \tag{5.99}$$

and we note that along this single line segment between two source points, g_0 is constant. *Outside* this line segment, the approximation (5.97) no longer holds; in particular, we should not be concerned about the $\mathrm{sign}(\eta-\eta_i)$ in g_0, because the full $\Theta^{(3)}$ (5.92) is continuous everywhere and has no jumps. Then using (5.97), the flux of $\Theta^{(3)}$ through a ϕ cycle is given by

$$\Pi_{ij}^{(3)} = \frac{1}{4\pi}\int_{\Delta_{ij}} \frac{1}{g_0}\,\mathrm{d}\frac{\tilde{f}(\eta)}{f(\eta)} \wedge \mathrm{d}\phi = \frac{1}{2g_0}\bigg(\frac{q_j}{k_j^3} - \frac{q_i}{k_i^3}\bigg), \tag{5.100}$$

where, interestingly, the k_i^3, k_j^3 have ended up in the denominator rather than in the numerator as they were in (5.96).

We have thus succeeded in constructing a useful and interesting base space for supergravity solutions. It has the homological 2-spheres we expected, swept out by τ; these have cohomological fluxes which can be adjusted in any desired way by choosing parameters. As a bonus, we also obtain homological 2-spheres swept out by ϕ, which also have cohomological flux.

Interestingly, the fluxes of each type take different forms. If we assign units to the parameters of the solution, then τ fluxes have units of "$1/q$" and ϕ fluxes have units of "$1/k$". This is consistent with the coordinate transformation (5.42), (5.43); if we assume the angles α, β are dimensionless, then the the fluxes $\Pi^{(3)}_{ij}$ will have the same units through both τ cycles and ϕ cycles.

5.3 Multi-centered supergravity solutions

Now that we have an appropriate base space, we must solve the system (4.36) and (4.37), (4.38), (4.39), (4.43), (4.44), and finally (4.45). The route to the solutions is tedious and not particularly illuminating, so we will describe it only briefly.

First, the L_1, L_2 equations (4.36) and (4.37) are simply the linearized Toda equation, which we have already solved to obtain w. We define "potentials" in the same way as in (5.7),

$$L_1 = \partial_z \hat{L}_1, \qquad L_2 = \partial_z \hat{L}_2, \tag{5.101}$$

such that $\hat{L}_1, \hat{L}_2$ solve the cylindrically-symmetric Laplace equation:

$$\hat{L}_1 = \ell_1^0 \log \rho + \sum_i \ell_1^i \, G_i(\rho, \eta), \qquad \hat{L}_2 = \ell_2^0 \log \rho + \sum_i \ell_2^i \, G_i(\rho, \eta), \tag{5.102}$$

$$G_i(\rho, \eta) = \log \frac{\eta - \eta_i + \sqrt{\rho^2 + (\eta - \eta_i)^2}}{\rho}, \tag{5.103}$$

where sums are understood to run from 1 to N. Then L_1, L_2 can be written

$$L_1 = \frac{1}{\rho(V_{\rho\eta}^2 + V_{\eta\eta}^2)} \big(V_{\eta\eta} \, \hat{L}_{1,\rho} - V_{\rho\eta} \, \hat{L}_{1,\eta} \big), \tag{5.104}$$

$$L_2 = \frac{1}{\rho(V_{\rho\eta}^2 + V_{\eta\eta}^2)} \big(V_{\eta\eta} \, \hat{L}_{2,\rho} - V_{\rho\eta} \, \hat{L}_{2,\eta} \big). \tag{5.105}$$

The K^1, K^2, M equations (4.38), (4.39), (4.43) are all similar. On the left-hand side is the cylindrically-symmetric Laplace operator on $\mathbb{R}^3$, and on the right-hand side is a product of two functions that solve the linearized Toda equation. Writing down the obvious homogeneous part, and then making an appropriate guess to match the source terms, the solutions are

$$K^1 = k_0^1 + \sum_i \frac{k_i^1}{\Sigma_i} + \frac{1}{V_{\rho\eta}^2 + V_{\eta\eta}^2}\Big(V_{\eta\eta}\left(\hat{w}_\eta \hat{L}_{2,\eta} - \hat{w}_\rho \hat{L}_{2,\rho}\right) + V_{\rho\eta}\left(\hat{w}_\eta \hat{L}_{2,\rho} + \hat{w}_\rho \hat{L}_{2,\eta}\right)\Big), \tag{5.106}$$

$$K^2 = k_0^2 + \sum_i \frac{k_i^2}{\Sigma_i} + \frac{1}{V_{\rho\eta}^2 + V_{\eta\eta}^2}\Big(V_{\eta\eta}\left(\hat{w}_\eta \hat{L}_{1,\eta} - \hat{w}_\rho \hat{L}_{1,\rho}\right) + V_{\rho\eta}\left(\hat{w}_\eta \hat{L}_{1,\rho} + \hat{w}_\rho \hat{L}_{1,\eta}\right)\Big), \tag{5.107}$$

$$\begin{aligned} M = m_0 + \sum_i \frac{m_i}{\Sigma_i} + \frac{1}{2}\frac{1}{V_{\rho\eta}^2 + V_{\eta\eta}^2}\Big(&V_{\eta\eta}\left(\hat{L}_{1,\eta}\hat{L}_{2,\eta} - \hat{L}_{1,\rho}\hat{L}_{2,\rho}\right) \\ &+ V_{\rho\eta}\left(\hat{L}_{1,\eta}\hat{L}_{2,\rho} + \hat{L}_{1,\rho}\hat{L}_{2,\eta}\right)\Big), \end{aligned} \tag{5.108}$$

where $\Sigma_i \equiv \sqrt{\rho^2 + (\eta - \eta_i)^2}$. We point out that the inhomogeneous parts of these hold automatically given the equations solved by $V, \hat{w}, \hat{L}_1, \hat{L}_2$, and do not depend on the specific forms we have written down in (5.29), (5.30), (5.104) and (5.105).

The L_3 equation offers no shortcuts. After a tedious exercise, one can show its solution is

$$\begin{aligned} L_3 = \ell_3^0 - \ell_3^z\,\rho V_\rho &+ \sum_i \frac{1}{\Sigma_i}\left(k_0^3\ell_3^i + \ell_1^0 k_i^1 + \ell_2^0 k_i^2 + 2q_0 m_i\right) \\ &+ \sum_{\substack{ij \\ i\neq j}} \frac{1}{\eta_i - \eta_j}\frac{\Sigma_i}{\Sigma_j}\left(k_i^3\ell_3^j + \ell_1^i k_j^1 + \ell_2^i k_j^2 + 2q_i m_j\right) \end{aligned} \tag{5.109}$$

$$
\begin{aligned}
&- \sum_i \frac{\eta - \eta_i}{\Sigma_i} \big(k_i^3 \ell_3^i + \ell_1^i k_i^1 + \ell_2^i k_i^2 + 2 q_i m_i \big) \\
&+ \frac{\rho}{V_{\rho\eta}^2 + V_{\eta\eta}^2} \bigg[V_{\rho\eta} \Big(- \hat{w}_\eta \hat{L}_{1,\eta} \hat{L}_{2,\eta} + \hat{w}_\rho \hat{L}_{1,\rho} \hat{L}_{2,\eta} + \hat{w}_\rho \hat{L}_{1,\eta} \hat{L}_{2,\rho} + \hat{w}_\eta \hat{L}_{1,\rho} \hat{L}_{2,\rho} \Big) \\
&\qquad + V_{\eta\eta} \Big(- \hat{w}_\rho \hat{L}_{1,\rho} \hat{L}_{2,\rho} + \hat{w}_\rho \hat{L}_{1,\eta} \hat{L}_{2,\eta} + \hat{w}_\eta \hat{L}_{1,\rho} \hat{L}_{2,\eta} + \hat{w}_\eta \hat{L}_{1,\eta} \hat{L}_{2,\rho} \Big) \bigg],
\end{aligned}
$$

where the parameter ℓ_3^z multiplies $z = -\rho V_\rho$. It is important to note here that the pair k_i^3, ℓ_3^j behaves oppositely to the pairs ℓ_1^i, k_j^1 and ℓ_2^i, k_j^2. And again, the formula in the last term holds automatically given the equations for $V, \hat{w}, \hat{L}_1, \hat{L}_2$.

Finally, one must solve the ω equation (4.45). If we write

$$
\omega = \omega_{(\phi)} \, \mathrm{d}\phi, \tag{5.110}
$$

then (4.45) reduces to the two equations

$$
\begin{aligned}
r\partial_r \big(\omega_{(\phi)} \big) &= \frac{1}{2} \big(\rho^2 L_1 \, \partial_z K^1 - K^1 \, \partial_z (\rho^2 L_1) \big) + \frac{1}{2} \big(\rho^2 L_2 \, \partial_z K^2 - K^2 \, \partial_z (\rho^2 L_2) \big) \\
&\quad + \frac{1}{4} \big(L_3 \, \partial_z^2 (\rho^2) - \partial_z (\rho^2) \, \partial_z L_3 \big) \\
&\qquad + \rho^2 w \, \partial_z M - M \, \partial_z (\rho^2 w) - 2 \rho^2 w L_1 L_2,
\end{aligned} \tag{5.111}
$$

$$
\begin{aligned}
-\partial_z \big(\omega_{(\phi)} \big) &= \frac{1}{2} \big(L_1 \, r\partial_r K^1 - K^1 \, r\partial_r L_1 \big) + \frac{1}{2} \big(L_2 \, r\partial_r K^2 - K^2 \, r\partial_r L_2 \big) \\
&\quad + \frac{1}{4} \big(L_3 \, r\partial_r u_z - u_z \, r\partial_r L_3 \big) + w \, r\partial_r M - M \, r\partial_r w.
\end{aligned} \tag{5.112}
$$

And one can show that these are solved by the following formula:

$$
\begin{aligned}
\omega_{(\phi)} = \omega_0 &+ \frac{1}{\rho^2 (V_{\rho\eta}^2 + V_{\eta\eta}^2)} \bigg\{ \frac{1}{2} \ell_3^z \Big(\rho^2 V_\rho V_{\rho\eta} - \eta \rho^2 (V_{\rho\eta}^2 + V_{\eta\eta}^2) \Big) \\
&+ \frac{1}{2} \big(k_0^1 \ell_1^0 + k_0^2 \ell_1^0 - \ell_3^0 + 2 m_0 q_0 \big) \Big(k_0^3 - \sum_i \frac{\eta - \eta_i}{\Sigma_i} k_i^3 \Big) \\
&- \frac{1}{2} k_0^3 \sum_i \big(k_0^1 \ell_1^i + k_0^2 \ell_2^i + 2 m_0 q_i \big) \frac{\eta - \eta_i}{\Sigma_i}
\end{aligned} \tag{5.113}
$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{ij} k_i^3\big(k_0^1\ell_1^j + k_0^2\ell_2^j + 2m_0 q_j\big)\frac{\rho^2+(\eta-\eta_i)(\eta-\eta_j)}{\Sigma_i\Sigma_j}\\
&+\frac{1}{2}k_0^3\sum_{\substack{ij\\ i\neq j}}\big(k_i^1\ell_1^j + k_i^2\ell_2^j - \ell_3^i k_j^3 + 2m_i q_j\big)\frac{1}{\eta_i-\eta_j}\frac{\rho^2+(\eta-\eta_i)(\eta-\eta_j)}{\Sigma_i\Sigma_j}\\
&-\frac{1}{2}\sum_{\substack{ijk\\ i\neq j}} k_k^3\big(k_i^1\ell_1^j + k_i^2\ell_2^j + 2m_i q_j\big)\frac{1}{\eta_i-\eta_j}\frac{1}{\Sigma_i\Sigma_j\Sigma_k}\times\\
&\qquad\qquad\times\Big[\rho^2\big(\eta-\eta_i+\eta_j-\eta_k\big)+(\eta-\eta_i)(\eta-\eta_j)(\eta-\eta_k)\Big]\\
&+\frac{1}{2}\sum_{ik} k_k^3\big(k_i^1\ell_1^i + k_i^2\ell_2^i + 2m_i q_i\big)\frac{\rho^2}{\Sigma_i^2\Sigma_k}\\
&+\frac{1}{2}\sum_{\substack{ijk\\ i\neq k}} k_i^3 k_j^3\ell_3^k\frac{\eta_i-\eta_j}{\eta_i-\eta_k}\frac{\rho^2}{\Sigma_i\Sigma_j\Sigma_k} - \frac{1}{2}\sum_{ij} k_i^3 k_j^3\ell_3^i\frac{\rho^2}{\Sigma_i^2\Sigma_j} + \frac{1}{2}\sum_i (k_i^3)^2\ell_3^i\frac{\rho^2}{\Sigma_i^3}\\
&+\frac{1}{2}\sum_{\substack{ijk\\ i\neq k}} k_i^3 k_j^3\ell_3^k\frac{1}{\eta_i-\eta_k}\frac{(\eta-\eta_k)\big(\rho^2+(\eta-\eta_i)(\eta-\eta_j)\big)}{\Sigma_i\Sigma_j\Sigma_k}\\
&+\sum_{ijk} q_i\ell_1^j\ell_2^j\frac{\rho^2}{\Sigma_i\Sigma_j\Sigma_k}\Bigg\},
\end{aligned}
$$

where again, all sums are assumed to run over $i, j, k \in \{1 \dots N\}$.

We now have the complete data for constructing supergravity solutions. The solution is characterized by N number of points η_i along the axis in the base space, and by the $8N + 10$ parameters $\{q_0, k_0^1, k_0^2, k_0^3, \ell_1^0, \ell_2^0, \ell_3^0, m_0, \omega_0, \ell_3^z, q_i, k_i^1, k_i^2, k_i^3, \ell_1^i, \ell_2^i, \ell_3^i, m_i\}$, which in general are constrained by the requirement for the absence of CTC's and Dirac-Misner strings. Finally, to complete the supergravity solution, one puts the functions $w, K^1, K^2, K^3, L_1, L_2, L_3, M$ into the ansätze of Sections 4.2 and 4.2.3.

5.3.1 Asymptotics of the 5d metric

Now we look at the behavior of the 5-dimensional metric (4.4) at infinity. The parameters k_0^1, k_0^2, ℓ_3^z lead to terms that blow up at infinity, so we set them to zero for the remainder of our discussion:

$$k_0^1 = 0, \qquad k_0^2 = 0, \qquad \ell_3^z = 0. \tag{5.114}$$

We will use the coordinates R, θ defined via

$$\rho = R\sin\theta, \qquad \eta = R\cos\theta. \tag{5.115}$$

Then the warp factors Z_1, Z_2 go as

$$Z_1 \sim \left(\frac{K_\star^2 K_\star^3 + Q_\star L_1^\star}{q_0 K_\star^3 - k_0^3 Q_\star}\right)\frac{1}{R}, \qquad Z_2 \sim \left(\frac{K_\star^1 K_\star^3 + Q_\star L_2^\star}{q_0 K_\star^3 - k_0^3 Q_\star}\right)\frac{1}{R}, \tag{5.116}$$

where we define

$$\begin{aligned} K_\star^1 &\equiv \sum_{i=1}^N k_i^1, \qquad K_\star^2 \equiv \sum_{i=1}^N k_i^2, \qquad K_\star^3 \equiv \sum_{i=1}^N k_i^3, \qquad Q_\star \equiv \sum_{i=1}^N q_i, \\ L_1^\star &\equiv \sum_{i=1}^N \ell_1^i, \qquad L_2^\star \equiv \sum_{i=1}^N \ell_2^i, \qquad L_3^\star \equiv \sum_{i=1}^N \ell_3^i, \qquad M_\star \equiv \sum_{i=1}^N m_i. \end{aligned} \tag{5.117}$$

To first order, the remaining metric functions $Z_3, \mu, \omega_{(\phi)}$ go as constants:

$$Z_3 \sim \ell_3^0 - \sum_{\substack{ij \\ i\neq j}} \frac{k_i^1\ell_1^j + k_i^2\ell_2^j - k_i^3\ell_3^j + 2m_i q_j}{\eta_i - \eta_j}, \tag{5.118}$$

$$\mu \sim m_0 - \frac{1}{2}\frac{K_\star^3}{q_0 K_\star^3 - k_0^3 Q_\star}\left(\ell_3^0 - \sum_{\substack{ij \\ i\neq j}} \frac{k_i^1\ell_1^j + k_i^2\ell_2^j - k_i^3\ell_3^j + 2m_i q_j}{\eta_i - \eta_j}\right), \tag{5.119}$$

$$
\begin{aligned}
\omega_{(\phi)} \sim \omega_0 + \frac{1}{2}\,\frac{Q_\star}{q_0 K^3_\star - k^3_0 Q_\star}\left(\ell^0_3 - \sum_{\substack{ij\\ i\neq j}} \frac{k^1_i \ell^j_1 + k^2_i \ell^j_2 - k^3_i \ell^j_3 + 2 m_i q_j}{\eta_i - \eta_j}\right) \\
+ \left(\frac{k^3_0 q_0 + K^3_\star Q_\star - (q_0 K^3_\star + k^3_0 Q_\star)\cos\theta}{(k^3_0)^2 + (K^3_\star)^2 - 2\,k^3_0 K^3_\star \cos\theta}\right)\times \\
\times \left[m_0 - \frac{1}{2}\,\frac{K^3_\star}{q_0 K^3_\star - k^3_0 Q_\star}\left(\ell^0_3 - \sum_{\substack{ij\\ i\neq j}} \frac{k^1_i \ell^j_1 + k^2_i \ell^j_2 - k^3_i \ell^j_3 + 2 m_i q_j}{\eta_i - \eta_j}\right)\right].
\end{aligned}
\tag{5.120}
$$

We must have $\mu \to 0, \ \omega_{(\phi)} \to 0$ asymptotically in order to avoid CTC's at infinity. Therefore we must set

$$
m_0 = \frac{1}{2}\frac{K^3_\star}{q_0 K^3_\star - k^3_0 Q_\star}\left(\ell^0_3 - \sum_{\substack{ij\\ i\neq j}} \frac{k^1_i \ell^j_1 + k^2_i \ell^j_2 - k^3_i \ell^j_3 + 2 m_i q_j}{\eta_i - \eta_j}\right), \tag{5.121}
$$

$$
\omega_0 = -\frac{1}{2}\frac{Q_\star}{q_0 K^3_\star - k^3_0 Q_\star}\left(\ell^0_3 - \sum_{\substack{ij\\ i\neq j}} \frac{k^1_i \ell^j_1 + k^2_i \ell^j_2 - k^3_i \ell^j_3 + 2 m_i q_j}{\eta_i - \eta_j}\right). \tag{5.122}
$$

As one can see in Section 3.1, the Z_I must all have the same asymptotic behavior for T^6 in the 11-dimensional metric (3.1) to remain compact at infinity. However, we can also consider solutions where the Z_I behave differently, if we give up the notion of lifting them to 11-dimensional supergravity. As was pointed out in Section 4.4, a natural setting for differing asymptotic behavior of the Z_I is in the 6-dimensional theory obtained by reducing IIB supergravity on T^4.

We first consider the case that all three Z_I have the same asymptotic behavior. Therefore we set

$$
\ell^0_3 = \sum_{\substack{ij\\ i\neq j}} \frac{k^1_i \ell^j_1 + k^2_i \ell^j_2 - k^3_i \ell^j_3 + 2 m_i q_j}{\eta_i - \eta_j}. \tag{5.123}
$$

Imposing (5.121), (5.122), (5.123), we expand $Z_3, \mu, \omega_{(\phi)}$ to the next order. This gives

$$\begin{aligned} Z_3 \sim \frac{1}{R}\Bigg\{ \frac{1}{q_0 K_\star^3 - k_0^3 Q_\star} \bigg[&\Big((k_0^3)^2 + (K_\star^3)^2 - 2\, k_0^3 K_\star^3 \cos\theta \Big) K_\star^1 K_\star^2 \\ &+ \Big(k_0^3 q_0 + K_\star^3 Q_\star - 2\, k_0^3 Q_\star \cos\theta \Big) \big(K_\star^1 L_1^\star + K_\star^2 L_2^\star \big) \\ &+ \Big((q_0)^2 + (Q_\star)^2 - 2\, q_0 Q_\star \cos\theta \Big) L_\star^1 L_\star^2 \bigg] \\ &- \frac{1}{2} \big(K_\star^1 L_1^\star + K_\star^2 L_2^\star + K_\star^3 L_3^\star + 2 Q_\star M_\star \big) \cos\theta \\ &+ \frac{1}{2} \big(k_0^3 L_3^\star + 2 q_0 M_\star \big) \Bigg\} \end{aligned} \tag{5.124}$$

and

$$\begin{aligned} \mu \sim \frac{1}{R}\Bigg\{ \frac{1}{(q_0 K_\star^3 - k_0^3 Q_\star)^2} \bigg[&- K_\star^3 \Big((k_0^3)^2 + (K_\star^3)^2 - 2\, k_0^3 K_\star^3 \cos\theta \Big) K_\star^1 K_\star^2 \\ &- K_\star^3 \Big(k_0^3 q_0 + K_\star^3 Q_\star - 2\, k_0^3 Q_\star \cos\theta \Big) \big(K_\star^1 L_1^\star + K_\star^2 L_2^\star \big) \\ &- Q_\star \Big(k_0^3 q_0 + K_\star^3 Q_\star - (q_0 K_\star^3 + k_0^3 Q_\star) \cos\theta \Big) L_\star^1 L_\star^2 \bigg] \\ + \frac{1}{2} \frac{1}{q_0 K_\star^3 - k_0^3 Q_\star} \bigg[&\big(k_0^3 + K_\star^3 \cos\theta \big) \big(K_\star^1 L_1^\star + K_\star^2 L_2^\star \big) \\ &+ \big(K_\star^3 \cos\theta - k_0^3 \big) \big(K_\star^3 L_3^\star + 2 Q_\star M_\star \big) \bigg] \Bigg\}, \end{aligned} \tag{5.125}$$

and

$$\omega_{(\phi)} \sim \frac{1}{2R} \frac{K_\star^3 \sin^2\theta}{(k_0^3)^2 + (K_\star^3)^2 - 2\, k_0^3 K_\star^3 \cos\theta} \big(K_\star^1 L_1^\star + K_\star^2 L_2^\star + K_\star^3 L_3^\star + 2 Q_\star M_\star \big). \tag{5.126}$$

The 5-dimensional metric (3.41) then becomes

$$
\begin{aligned}
\mathrm{d}s_5^2 = &-\frac{R^2}{f_4(\theta)^2}\Big[\mathrm{d}t + \frac{1}{R}f_5(\theta)\,\mathrm{d}\tau + \frac{1}{R}\Big(f_5(\theta)f_3(\theta) + f_6(\theta)\Big)\,\mathrm{d}\phi\Big]^2 \\
&+ \frac{f_4(\theta)}{f_2(\theta)}\Big(\mathrm{d}\tau + f_3(\theta)\,\mathrm{d}\phi\Big)^2 \\
&\quad + \frac{f_2(\theta)f_4(\theta)}{R^2}\Big[f_1(\theta)(\mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2) + R^2\sin^2\theta\,\mathrm{d}\phi^2\Big],
\end{aligned}
\tag{5.127}
$$

where generically speaking,

$$
\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2) \sim f_1(\theta), \qquad w \sim \frac{1}{R}f_2(\theta), \qquad A \sim f_3(\theta)\,\mathrm{d}\phi \tag{5.128}
$$

$$
Z \sim \frac{1}{R}f_4(\theta), \qquad \mu \sim \frac{1}{R}f_5(\theta), \qquad \omega \sim \frac{1}{R}f_6(\theta)\,\mathrm{d}\phi, \tag{5.129}
$$

and simplifications likely occur in (5.127) if one works these out in more specificity. Due to the $\mathrm{d}R^2/R^2$ term, this metric is something related to $AdS_2 \times S^3$. Specifically, it is a warped, rotating quotient $AdS_2 \times S^3/G_\infty$, where G_∞ is a finite group acting on the S^3 factor as described in Section 5.2.4. If we choose parameters such that $\widetilde{q}_\infty = \pm 1$ as defined in (5.59), then the base space approaches $\mathbb{R}^4$ without orbifold identifications, as described in Section 5.2.5. One can then choose parameters such that

$$
Z_3 \sim \frac{1}{R}, \qquad \mu \sim \frac{1}{R}(c_1 + c_2\cos\theta), \qquad \omega \sim \mathcal{O}(R^{-2}). \tag{5.130}
$$

Then changing coordinates via

$$
R = \frac{1}{4}\varrho^2, \qquad \theta = 2\vartheta, \qquad \tau = \psi + \chi, \qquad \phi = \psi - \chi, \tag{5.131}
$$

(up to shifts in t and τ), one obtains a 5-dimensional metric of the form

$$
\mathrm{d}s_5^2 = -\varrho^4\Big(\mathrm{d}t + J_1\frac{\sin^2\vartheta}{\varrho^2}\,\mathrm{d}\psi + J_2\frac{\cos^2\vartheta}{\varrho^2}\,\mathrm{d}\chi\Big)^2 + \frac{\mathrm{d}\varrho^2}{\varrho^2} + \mathrm{d}\Omega_3^2, \tag{5.132}
$$

which is the metric of the near-horizon region of a BMPV black hole [89].

6d asymptotics

Alternatively, we can choose to allow $Z_3 \sim$ (const) at infinity while $Z_1, Z_2 \sim 1/\varrho^2$, and therefore not impose (5.123). Then the 5-dimensional metric will generically be of the form

$$\mathrm{d}s_5^2 = -\varrho^{8/3}\,(\mathrm{d}t + k)^2 + \varrho^{-4/3}\,(\mathrm{d}\varrho^2 + \varrho^2\,\mathrm{d}\Omega_3^2), \tag{5.133}$$

which looks somewhat strange. As shown in Section 4.4, however, there is a natural lift into 6-dimensional $\mathcal{N} = 1$ supergravity coupled to one anti-self-dual tensor multiplet [70, 71, 72]. The metric ansatz in 6 dimensions can be written in terms of the 5-dimensional quantities as

$$\mathrm{d}s_6^2 = -\frac{2}{\sqrt{Z_1 Z_2}}\left(\mathrm{d}v + B^3\right)\left(\mathrm{d}u + k - \frac{1}{2}Z_3\left(\mathrm{d}v + B^3\right)\right) + \sqrt{Z_1 Z_2}\,\mathrm{d}s_4^2, \tag{5.134}$$

where B^3 is the 1-form potential such at $\Theta^{(3)} = \mathrm{d}B^3$ as in (5.93). In this context, applying the asymptotics at infinity where $Z_3 \sim$ (const) and $Z_1, Z_2 \sim 1/\varrho^2$ gives the result

$$\mathrm{d}s_6^2 = -2\varrho^2\,\mathrm{d}v\left(\mathrm{d}u + k - \frac{1}{2}Z_3\,\mathrm{d}v\right) + \frac{\mathrm{d}\varrho^2}{\varrho^2} + \mathrm{d}\Omega_3^2, \tag{5.135}$$

which is a momentum wave propagating on $AdS_3 \times (S^3/G_\infty)$. Furthermore, nothing prevents us from imposing $Z_3 \sim 1/\varrho^2$ in this lifted metric; in such a case, one would obtain the 6-dimensional lift of the near-horizon BMPV metric (5.132), which is the near-horizon metric of a BPS, rotating D1-D5-P black string [103].

Generally speaking, we see that our solutions are asymptotic to a warped, rotating version of $AdS_2 \times (S^3/G_\infty)$, and for special choices of parameters, to near-horizon BMPV. Alternatively, one can lift to IIB supergravity on T^4, giving a 6-dimensional metric which allows Z_3 to have different asymptotics to Z_1, Z_2. In this case, one can impose $Z_3 \sim$ (const) to obtain a momentum wave solution propagating on $AdS_3 \times (S^3/G_\infty)$; or, imposing $Z_3 \sim 1/\varrho^2$, one obtains the near-horizon metric of a BPS, rotating black string.

We should note, as explained in Section 4.2.4, that the "floating brane" equations [31] on a LeBrun base do not have asymptotically flat solutions, and solutions must generically have nonzero rotation parameters at infinity. The reason for this is that the T_{00} component of the 5-dimensional energy-momentum tensor is a manifestly positive-definite function of the $Z_I, \Theta^{(I)}$. Even if we have $Z_I \sim 1$ at infinity, then $\Theta^{(1)}, \Theta^{(2)}$ still contain a term proportional to the Kähler form J, which contributes a constant to T_{00} and prevents asymptotic flatness. The rotation at infinity comes from the off-diagonal terms T_{0a}, which also do not vanish.

5.3.2 Regularity conditions

The solutions we have obtained generically have a number of singularities at each η_i which act as sources of the electric potentials Z_I and magnetic field strengths $\Theta^{(I)}$. However, in the context of black hole microstate geometries, we are interested in solutions that are everywhere smooth, with no singular sources. This can be accomplished by choosing the parameters in such a way that singularities are eliminated. The necessary condition for smoothness is that each of the functions $Z_1, Z_2, Z_3, \mu, \omega_{(\phi)}$ remain non-singular as the source points are approached.

Looking near a point η_ℓ, we again define a local radial coordinate via

$$\rho = R\sin\theta, \qquad \eta - \eta_\ell = R\cos\theta. \tag{5.136}$$

Then as $R \to 0$, we have

$$Z_1 \to \frac{1}{R}\left(\frac{k_\ell^2 k_\ell^3 + q_\ell \ell_1^\ell}{q_\ell\big(\bar{K}_\ell^3 - k_0^3\big) - k_\ell^3\big(\bar{Q}_\ell - q_0\big)}\right), \tag{5.137}$$

$$Z_2 \to \frac{1}{R}\left(\frac{k_\ell^1 k_\ell^3 + q_\ell \ell_2^\ell}{q_\ell\big(\bar{K}_\ell^3 - k_0^3\big) - k_\ell^3\big(\bar{Q}_\ell - q_0\big)}\right), \tag{5.138}$$

where again,

$$\bar{K}_\ell^3 \equiv \sum_{\substack{i\\ i\neq\ell}} k_i^3 \operatorname{sign}(\eta_\ell - \eta_i), \qquad \bar{Q}_\ell \equiv \sum_{\substack{i\\ i\neq\ell}} q_i \operatorname{sign}(\eta_\ell - \eta_i). \tag{5.139}$$

Therefore, the singular parts of Z_1, Z_2 will vanish if

$$\ell_1^\ell = -\frac{k_\ell^2 k_\ell^3}{q_\ell}, \qquad \ell_2^\ell = -\frac{k_\ell^1 k_\ell^3}{q_\ell}, \tag{5.140}$$

at every source point. Next, imposing (5.140), we have

$$Z_3 \to \frac{1}{R}\Bigg[\frac{k_\ell^1 k_\ell^2}{q_\ell^2}\Big(q_\ell(\bar{K}_\ell^3 - k_0^3) - k_\ell^3(\bar{Q}_\ell - q_0)\Big) - \ell_3^\ell(\bar{K}_\ell^3 - k_0^3) + 2m_\ell(\bar{Q}_\ell - q_0) - \Big(k_\ell^3 \ell_3^\ell + 2m_\ell q_\ell\Big)\cos\theta\Bigg], \tag{5.141}$$

and hence the singular part of Z_3 vanishes if

$$\ell_3^\ell = \frac{k_\ell^1 k_\ell^2}{q_\ell}, \qquad m_\ell = -\frac{k_\ell^1 k_\ell^2 k_\ell^3}{2q_\ell^2}. \tag{5.142}$$

Together, (5.140) and (5.142) are also sufficient to guarantee $\mu \to \text{(const)}$ and $\omega_{(\phi)} \to \text{(const)}$ near η_ℓ; hence we will have a regular solution if we impose these conditions at every source point.

We note that these conditions appear exactly the same (up to signs that result from differing conventions) as the smoothness conditions (3.35) in the original BPS story [25]. However, there is a key difference: In these solutions, the parameters q_ℓ do not directly control the charges at the singularities of w, but as in (5.37), the charges in w are controlled by the determinants

$$\widetilde{q}_\ell \equiv q_\ell(\bar{K}^3_\ell - k^3_0) - k^3_\ell(\bar{Q}_\ell - q_0). \tag{5.143}$$

5.3.3 Fluxes through cycles

It will be useful to have expressions for the magnetic flux threading 2-cycles formed by sweeping the τ fiber between source points in the 4-dimensional base space. We have already calculated the flux of $\Theta^{(3)}$ on these cycles (5.96):

$$\Pi^{(3)}_{ij} \equiv \frac{1}{4\pi}\int_{\Delta_{ij}} \Theta^{(3)} = \frac{k^3_j}{\widetilde{q}_j} - \frac{k^3_i}{\widetilde{q}_i}. \tag{5.144}$$

Before calculating the remaining two fluxes, we will impose the regularity conditions (5.140), (5.142). Then as we approach a source point η_ℓ, we have

$$\frac{K^1}{w} \to \frac{k^1_\ell(\bar{K}^3_\ell - k^3_0)}{q_\ell} - \ell^0_2 + \bar{L}^\ell_2, \qquad \frac{K^2}{w} \to \frac{k^2_\ell(\bar{K}^3_\ell - k^3_0)}{q_\ell} - \ell^0_1 + \bar{L}^\ell_1, \tag{5.145}$$

where we have defined new quantities

$$\bar{L}^\ell_1 \equiv \sum_{\substack{i\\ i\neq\ell}} \ell^i_1 \,\mathrm{sign}(\eta_\ell - \eta_i), \qquad \bar{L}^\ell_2 \equiv \sum_{\substack{i\\ i\neq\ell}} \ell^i_2 \,\mathrm{sign}(\eta_\ell - \eta_i). \tag{5.146}$$

Then the flux through τ cycles can be calculated in a way similar to (5.96):

$$\Pi_{ij}^{(1)} \equiv \frac{1}{4\pi}\int_{\Delta_{ij}} \Theta^{(1)} = \frac{k_j^1(\bar{K}_j^3 - k_0^3)}{q_j} + \bar{L}_2^j - \frac{k_i^1(\bar{K}_i^3 - k_0^3)}{q_i} - \bar{L}_2^i, \tag{5.147}$$

$$\Pi_{ij}^{(2)} \equiv \frac{1}{4\pi}\int_{\Delta_{ij}} \Theta^{(2)} = \frac{k_j^2(\bar{K}_j^3 - k_0^3)}{q_j} + \bar{L}_1^j - \frac{k_i^2(\bar{K}_i^3 - k_0^3)}{q_i} - \bar{L}_1^i. \tag{5.148}$$

One can in principle also calculate the fluxes through the 2-cycles swept out by ϕ, as was done in Section 5.2.8. However, this is tedious and not very illuminating, so we omit it.

5.3.4 Causality conditions: the "bubble equations"

We have determined the conditions that a solution is smooth (up to benign orbifold singularities) as one approaches the various points η_ℓ in the base manifold. However, to construct sensible supergravity solutions, one must also ensure that there are no closed timelike curves.

Looking at the metric (4.4) on a surface of constant t, we can rearrange it as follows:

$$\begin{aligned} ds_5^2 = {} & \frac{\mathcal{Q}}{w^2 Z^2}\left(d\tau + A - \frac{w^2\mu}{\mathcal{Q}}\,\omega\right)^2 + Zw\left(\rho^2\, d\phi^2 - \frac{\omega^2}{\mathcal{Q}}\right) \\ & + Zw\,\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)(d\rho^2 + d\eta^2), \end{aligned} \tag{5.149}$$

where

$$\mathcal{Q} \equiv Z_1 Z_2 Z_3 w - w^2\mu^2, \qquad Z \equiv (Z_1 Z_2 Z_3)^{1/3}. \tag{5.150}$$

In order for CTC's to be absent everywhere, (5.149) must be positive-definite. This requires

$$\mathcal{Q} \geq 0, \qquad Zw \geq 0, \qquad \rho^2\, d\phi^2 \geq \frac{\omega^2}{\mathcal{Q}}. \tag{5.151}$$

It is generally impractical to enforce these global conditions from the local point of view of choosing parameters in the solution; one must write down a solution and then explore it numerically to look for CTC's. However, one can look at *local* causality conditions near the source points, and this leads to a system of equations that must be solved as a necessary (but not sufficient) condition that a solution be causally sensible.

In the BPS context [25], this leads to a system of so-called "bubble equations" (3.40) that relate the distances between the GH centers (as measured in the $\mathbb{R}^3$ base) to the product of the fluxes of the $\Theta^{(I)}$ through the various 2-cycles described by the GH centers. Thus the size of each "bubble" is governed by the amount of flux trapped on it. Importantly, the bubble equations depend upon the product of all three fluxes. In the previous work of Chapter 4 on non-supersymmetric solutions derived from floating branes [33, 34], the third flux $\Theta^{(3)}$ was topologically trivial and contributed no fluxes to the bubble equations. The result was that the causality conditions did not constrain the sizes of the homological 2-cycles. In these new solutions, however, $\Theta^{(3)}$ has non-trivial fluxes on the 2-cycles (as in Section 5.2.8), so we expect to find non-trivial bubble equations.

Looking at (5.149) near the points η_ℓ, one finds two potential sources of CTC's coming from the two angular coordinates τ, ϕ. To eliminate CTC's near these points, we must require that

$$\mu \to 0, \qquad \omega \to 0 \qquad \text{at each } \eta_\ell. \tag{5.152}$$

While these appear to be two different conditions, they are really the same. To see this, we can rearrange the ω equation (4.45) as follows:

$$\begin{aligned} \mathrm{d}\omega = {} & w Z_1 \underset{3}{\star} \mathrm{d}\frac{K^1}{w} + w Z_2 \underset{3}{\star} \mathrm{d}\frac{K^2}{w} + w Z_3 \underset{3}{\star} \mathrm{d}\frac{K^3}{w} - 2 w Z_1 Z_2 \underset{3}{\star} \mathrm{d}z \\ & + w \underset{3}{\star} \mathrm{d}\mu - \mu \, \mathrm{d}A. \end{aligned} \tag{5.153}$$

We choose parameters such that ω vanishes at infinity (5.121), (5.122), so for ω to be non-vanishing somewhere on the axis would require Dirac-Misner strings. Given the regularity conditions (5.140), (5.142), the only term in (5.153) that can source Dirac-Misner strings is $-\mu\,\mathrm{d}A$. Therefore, to eliminate local CTC's near the points, it is enough to demand that μ vanish at each η_ℓ, which results in the following "bubble equations":

$$\begin{aligned}-2m_0\,\widetilde{q}_\ell + \ell_3^0\,k_\ell^3 \;=\; &(k_0^3 - \bar{K}_\ell^3)\sum_{\substack{i\\ i\neq\ell}} \widehat{\Pi}_{\ell i}^{(1)}\widehat{\Pi}_{\ell i}^{(2)}\widehat{\Pi}_{\ell i}^{(3)}\frac{q_\ell q_i}{r_{\ell i}}\\ &+\frac{1}{2}k_\ell^3\sum_{\substack{ij\\ i\neq j}} \widehat{\Pi}_{ij}^{(1)}\widehat{\Pi}_{ij}^{(2)}\widehat{\Pi}_{ij}^{(3)}\frac{q_i q_j}{r_{ij}}\,s(i,j)\,s(\ell,i)\,s(\ell,j),\end{aligned} \tag{5.154}$$

where we have defined

$$r_{ij}\equiv|\eta_i-\eta_j|, \qquad \widehat{\Pi}_{ij}^{(I)}\equiv\left(\frac{k_j^I}{q_j}-\frac{k_i^I}{q_i}\right), \qquad s(a,b)\equiv\mathrm{sign}(\eta_a-\eta_b), \tag{5.155}$$

$$\widetilde{q}_\ell \equiv q_\ell(\bar{K}_\ell^3 - k_0^3) - k_\ell^3(\bar{Q}_\ell - q_0). \tag{5.156}$$

The combinations of parameters $\widehat{\Pi}_{ij}^{(I)}$ which appear in the bubble equations are not the *physical* fluxes $\Pi_{ij}^{(I)}$ calculated in (5.96), (5.147) and (5.148). However, with a little algebra one can show that they are related linearly and homogeneously[10]:

$$\Pi_{\ell i}^{(1)} = (-k_0^3+\bar{K}_\ell^3)\,\widehat{\Pi}_{\ell i}^{(1)} + \sum_{j=1}^{N} k_j^3\,\widehat{\Pi}_{ij}^{(1)}\,\big(s(\ell,j)-s(i,j)\big), \tag{5.157}$$

$$\Pi_{\ell i}^{(2)} = (-k_0^3+\bar{K}_\ell^3)\,\widehat{\Pi}_{\ell i}^{(2)} + \sum_{j=1}^{N} k_j^3\,\widehat{\Pi}_{ij}^{(2)}\,\big(s(\ell,j)-s(i,j)\big), \tag{5.158}$$

$$\widetilde{q}_\ell\widetilde{q}_i\,\Pi_{\ell i}^{(3)} = q_\ell q_i(-k_0^3+\bar{K}_\ell^3)\,\widehat{\Pi}_{\ell i}^{(3)} + k_\ell^3\sum_{j=1}^{N} q_i q_j\,\widehat{\Pi}_{ij}^{(3)}\big(s(\ell,j)-s(i,j)\big). \tag{5.159}$$

[10] Here we again assume the regularity conditions (5.140), (5.142) are imposed.

These look tantalizingly like they might allow a simpler expression of the right-hand side of (5.154); however, the presence of $1/r_{\ell i}, 1/r_{ij}$ in the sums complicates the algebra, and the expression we have written in (5.154) is probably the simplest.

We have thus succeeded in finding a family of non-BPS solutions with *non-trivial* bubble equations which constrain the bubble diameters r_{ij} in terms of the fluxes trapped on the bubbles. We also observe that there is a significant, important difference between these non-BPS bubble equations and the well-known BPS version (3.40). The term on the second line of (5.154) is entirely new: In order to avoid CTC's at η_ℓ, the equations depend not only on the diameters $r_{\ell i}$ of the 2-cycles adjacent to η_ℓ, but also on the diameters r_{ij} of each of the other 2-cycles. This is telling us about *new physics*: these *non*-supersymmetric solutions exhibit a richer variety of $E \times B$ interactions than previously known BPS solutions.

However, while these bubble equations differ from the BPS ones in this very important way, they are similar in another particularly striking way: They are *linear* in the inverse bubble diameters $1/r_{ij}$. This stands in contrast to the so-called "almost BPS" family of solutions where the bubble equations are cubic in the inverse distances [78, 80, 79]. So although these solutions lack supersymmetry, they are in some sense closer to BPS than the "almost BPS" solutions. This is of course because they are trivial KK reductions of 6d geometries which are BPS in the IIB frame, as explained in Section 4.4.

Ultimately, there are only $N-1$ independent r_{ij}, so we expect there to be $N-1$ independent bubble equations. This is easiest to demonstrate by looking directly at the Dirac-Misner strings in ω. This results in the same set of bubble equations as above, but with each multiplied by a constant (which is different at each η_ℓ). Near η_ℓ, the Dirac-Misner string part of ω is given by the jump that occurs in crossing from one side of η_ℓ to the other:

$$\omega\Big|_{\theta=0} - \omega\Big|_{\theta=\pi} = -\left(A\Big|_{\theta=0} - A\Big|_{\theta=\pi}\right)\mu = \frac{2\,\widetilde{q}_\ell}{\left(\bar{K}^3_\ell - k^3_0\right)^2 - \left(k^3_\ell\right)^2}\,\mu\,\mathrm{d}\phi. \tag{5.160}$$

Since ω contains a sequence of Dirac-Misner string sources along the η axis, and vanishes at both positive and negative infinity, then the sum of all the jumps must be zero. Therefore, the weighted sum of all the bubble equations (5.154), each multiplied by the coefficient in (5.160), must give zero. This weighted sum gives

$$m_0 = \frac{1}{2}\frac{K^3_\star}{q_0 K^3_\star - k^3_0 Q_\star}\left(\ell^0_3 - \sum_{\substack{ij\\ i\neq j}} \frac{k^1_i \ell^j_1 + k^2_i \ell^j_2 - k^3_i \ell^j_3 + 2m_i q_j}{\eta_i - \eta_j}\right). \tag{5.161}$$

which is the condition we have already imposed (5.121) in order that $\mu \to 0$ at infinity. Hence as expected, the bubble equations constitute $N-1$ independent equations in the $N-1$ independent variables r_{ij}.

Finally, there is a curious thing that happens if we impose all of the conditions derived in Section 5.3.1 for near-horizon BMPV-like (i.e. warped, rotating $AdS_2 \times S^3$) asymptotics. First we note that the value of ℓ^0_3 in (5.123) is entirely a linear combation of the inverse bubble diameters $1/r_{ij}$. Second, when (5.123) is imposed, then $m_0 = \omega_0 = 0$ as in (5.121), (5.122). Therefore if we insist on near-horizon BMPV-like asymptotics, the bubble equations will take the form, schematically,

$$\sum \widehat{\Pi}^{(1)}\widehat{\Pi}^{(2)}\widehat{\Pi}^{(3)}\frac{qq}{r} = 0. \tag{5.162}$$

If we instead think of this equation as a limiting process where we replace the right-hand side with some δ and let $\delta \to 0$, then the solutions, as we follow this process, are *scaling solutions* [123, 102, 79]. The right-hand side roughly scales as $(\Pi)^3/r$, and thus if we adjust the dipole charges while simultaneously shrinking the bubble diameters, such that $\Pi \sim \lambda, r \sim \lambda$ for λ small, this tends toward zero. In such solutions, the overall size of the bubbled region shrinks (as measured in the 3-dimensional base), while the ratios between the bubble sizes becomes constant. In the full 5-dimensional metric, this represents the appearance of an arbitrarily deep throat, smoothly capped off by topological bubbles at some finite depth. Thus one can see the near-horizon BMPV geometry, and the related rotating-AdS-like metrics with angular dependence as in (5.127), as the result of this limiting procedure.

More generally, if we consider asymptotic conditions where Z_3 behaves differently from Z_1, Z_2 (thus naturally lifting to the 6d IIB metric (5.135) rather than to 11d supergravity), we can set the constant ℓ_3^0 to anything we like. In this case, one can find finite, non-trivial solutions to the bubble equations without subjecting them to a limiting procedure. We demonstrate this in Section 5.3.5.

5.3.5 An explicit numerical example

In this section we will give an explicit, solved example with three source points, illustrating how a smooth, CTC-free solution can be constructed. The solution will be in the class asymptotic to (5.135), where $Z_3 \sim$ (const) and $Z_1 \sim 1/\rho^2, Z_2 \sim 1/\rho^2$. We will focus on satisfying the local conditions near the points, and not delve into exactly what asymptotics result.

We begin by choosing three source points along the η axis and assigning them geometric charges. The parameters of the solution are *ordered* in the manner drawn in

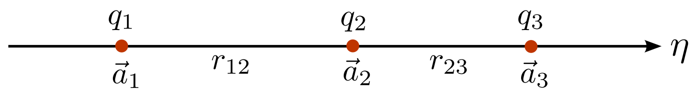

Figure 5.4: *Setup for a 3-center example. Geometric charges q_1, q_2, q_3 are put at the points $\vec{a}_1, \vec{a}_2, \vec{a}_3$ along the η axis. One must then solve the bubble equations to find r_{12}, r_{23}.*

Figure 5.4; thus by hypothesis the bubble diameters r_{12}, r_{23} are positive. At the points $\vec{a}_1, \vec{a}_2, \vec{a}_3$ we put the following charges:

$$\begin{aligned}
&q_0 = 2, \qquad &&q_1 = 3, \qquad &&q_2 = 2, \qquad &&q_3 = 6, \\
&k_0^1 = 0, &&k_1^1 = 5, &&k_2^1 = 2, &&k_3^1 = 3, \\
&k_0^2 = 0, &&k_1^2 = 5, &&k_2^2 = 4, &&k_3^2 = 3, \\
&k_0^3 = 1, &&k_1^3 = 2, &&k_2^3 = 2, &&k_3^3 = 2, \\
&\ell_1^0 = 0, &&\ell_2^0 = 0, &&\ell_3^0 = 10, &&\ell_3^z = 0.
\end{aligned} \tag{5.163}$$

Our particular choices are made to satisfy a few constraints: 1) the parity condition (5.31) such that each point will be an orbifold point; 2) the condition that all the $\widehat{\Pi}_{ij}^{(I)}$ are nonzero; 3) the condition that the $\widetilde{q}_i$ are all "nice" numbers; 4) the condition that the bubble equations yield real, positive solutions for the r_{ij}; and 5) the condition that $\mathcal{Q} > 0$ in order to be free of CTC's. Choosing parameters (5.163) to satisfy all of these properties is a bit of an art, and it would be interesting to better understand the moduli space of *physical* solutions.

The value of ℓ_3^0 sets the overall scale of the solution, as it is the only unconstrained constant sitting on the left-hand side of (5.154). Since we have put $\ell_3^0 \neq 0$, this solution will have asymptotics best described in the 6d IIB frame as in (5.135). Most of the functions w, K^I, L_I, M that make up the solution are too lengthy to write out, but as an example, we have

$$\hat{w}_\eta = \frac{3}{\sqrt{\rho^2+\eta^2}} + \frac{2}{\sqrt{\rho^2+(\eta-r_{12})^2}} + \frac{6}{\sqrt{\rho^2+(\eta-r_{12}-r_{23})^2}}, \tag{5.164}$$

$$\hat{w}_\rho = \frac{2}{\rho} - \frac{3\,\eta}{\rho\sqrt{\rho^2+\eta^2}} - \frac{2\,(\eta-r_{12})}{\rho\sqrt{\rho^2+(\eta-r_{12})^2}} - \frac{6\,(\eta-r_{12}-r_{23})}{\rho\sqrt{\rho^2+(\eta-r_{12}-r_{23})^2}}, \tag{5.165}$$

and so on. There are two remaining constants m_0, ω_0 which we have not set in (5.163). To meet the regularity conditions at infinity, these constants will be set equal to (5.121) and (5.122), and then their numerical values will be determined after the r_{ij} are known via solving the bubble equations (5.154).

At each source point, the base metric approaches $\mathbb{R}^4/G_\ell$, where the order of G_ℓ at the source point η_ℓ is given by $\#G_\ell = |\widetilde{q}_\ell|$, and for the parameters (5.163) these $\widetilde{q}_\ell$ are given by

$$\widetilde{q}_1 = 5, \qquad \widetilde{q}_2 = 8, \qquad \widetilde{q}_3 = 12, \qquad \widetilde{q}_\infty = 1. \tag{5.166}$$

Therefore we see that this is another example of the phenomenon described in Section 5.2.6, where the base metric can be asymptotically *globally* flat, despite having orbifold points on the interior, and without resorting to making it "ambipolar" as described in Section 5.2.5.

We will first analyze the groups at these orbifold points. We find that the lattice generators $\widetilde{\Lambda}_\ell$, calculated from (5.46), are given by

$$\widetilde{\Lambda}_1 = \frac{1}{5}\begin{pmatrix} 2 & -5 \\ -3 & 10 \end{pmatrix}, \qquad \widetilde{\Lambda}_2 = \frac{1}{8}\begin{pmatrix} 2 & -1 \\ -2 & 5 \end{pmatrix}, \qquad \widetilde{\Lambda}_3 = \frac{1}{12}\begin{pmatrix} -2 & 7 \\ -2 & 1 \end{pmatrix}, \tag{5.167}$$

and the corresponding groups are

$$G_1 \simeq \mathbb{Z}_5^{\text{diag}}, \qquad G_2 \simeq \mathbb{Z}_8, \qquad G_3 \simeq \mathbb{Z}_{12} \simeq \mathbb{Z}_3 \times \mathbb{Z}_4, \tag{5.168}$$

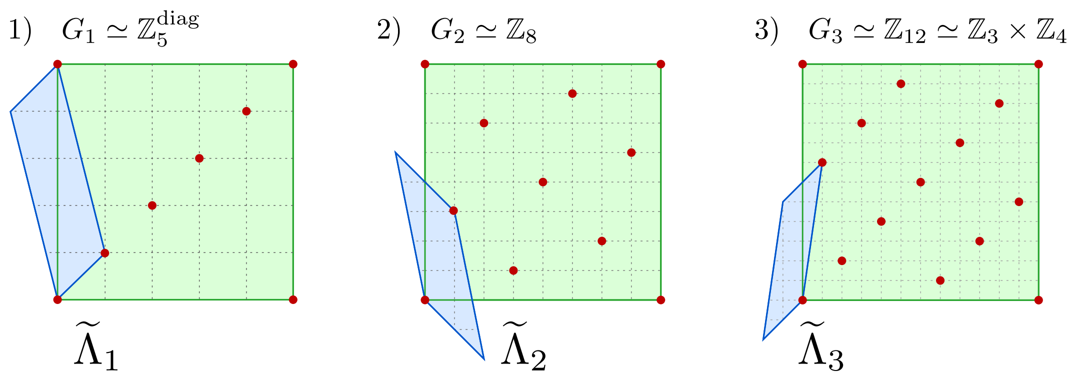

Figure 5.5: *The unit cells $\widetilde{\Lambda}_\ell$ of each lattice $\widetilde{\Gamma}_\ell$ and their corresponding groups $G_\ell \simeq \widetilde{\Gamma}_\ell/\Gamma$. The small parallelograms represent the lattice generators* (5.167) *(where $\widetilde{\Lambda}_1$ has been shifted by a right $GL(2,\mathbb{Z})$ action in order to make it fit in the figure). The heavy red dots represent the members of each group G_ℓ. The corners of the large squares are to be identified; they represent the lattice Γ of the natural 2π identifications of the (α,β) coordinates in $\mathbb{R}^4$.*

where G_1 at point η_1 acts in the diagonal $U(1)$ of $SO(4)$, which one can check using (5.52). These lattice generators $\widetilde{\Lambda}_\ell$, and the groups given by $G_\ell \simeq \widetilde{\Gamma}_\ell/\Gamma$, are illustrated in Figure 5.5.

Next, we put the general expression for m_0 (5.121) into the bubble equations (5.154) and solve them for the r_{ij}, subject to the triangle constraint

$$r_{12} + r_{23} = r_{13}. \tag{5.169}$$

At this point in the process it is quite possible to fail to find a solution. The r_{ij} should be strictly positive (they do not enter the equations in a way that allows them to be treated as "directional"). The bubble equations are linear in $1/r_{ij}$, and (5.169) is linear in r_{ij}, hence one is solving a system of quadratic equations. Thus it is possible to get negative or imaginary r_{ij}, and if this happens, one must adjust some of the dipole charges in (5.163) and try again. For the particular charges used here, we obtain two solution sets of real, positive r_{ij}, from which we select (via hindsight) the following:

$$r_{12} = 2.45827, \qquad r_{23} = 0.891937, \qquad r_{13} = 3.35021. \tag{5.170}$$

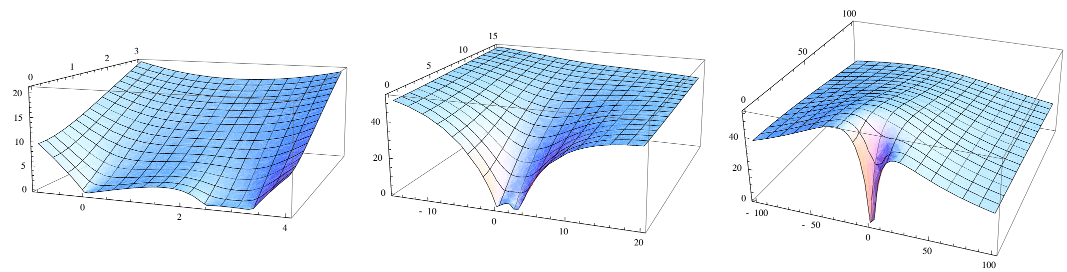

Figure 5.6: *The function $\mathcal{Q} \equiv Z_1 Z_2 Z_3 w - w^2 \mu^2$ plotted near the source points at three different levels of magnification. $\mathcal{Q}$ is everywhere non-negative, and therefore the solution is free of CTC's.*

From this solution and the expressions (5.121) and (5.122), we then find

$$m_0 = 1.96384, \qquad \omega_0 = -3.60037, \tag{5.171}$$

which will then guarantee that there are no CTC's at infinity.

Finally, to show there are no CTC's anywhere, we plot

$$\mathcal{Q} \equiv Z_1 Z_2 Z_3 w - w^2 \mu^2 \tag{5.172}$$

in Figure 5.6. We see that it is positive near the centers as we expect, and appears to be positive everywhere, giving us a supergravity solution which is globally free of closed timelike curves[11].

5.4 Discussion and open problems

Using the floating brane ansatz of [31] we have obtained a new, infinite family of solutions to 5-dimensional $\mathcal{N} = 2$ ungauged supergravity coupled to two vector multiplets. To build the solutions, we start with a LeBrun metric for the 4-dimensional base. These

[11]Naturally, it is not enough just to look at graphs. It is also helpful to plot $\mathcal{Q} - |\mathcal{Q}|$, which quickly reveals any place $\mathcal{Q}$ might go negative. This was checked in this example, and $\mathcal{Q} \geq 0$ everywhere.

metrics are Kähler and solve the Euclidean-Einstein-Maxwell equations, and are specified by two functions that solve the $SU(\infty)$ Toda equation and its linearization. The full supergravity solution is then constructed by solving the "floating brane equations" on this base space. To these equations we obtain general, explicit solutions which generically represent a collection of concentric black rings stabilized by their angular momentum and electromagnetic charges. Under appropriate regularity conditions, the black rings are replaced by topological "bubbles", and the solutions are smooth and horizon-free. Imposing causality conditions, we obtain "bubble equations" which dictate the sizes of topological bubbles in terms of the cohomological fluxes trapped on them.

The 4-dimensional Kähler base space is interesting in its own right, and we spend some time analyzing its properties. Choosing a subclass of LeBrun metrics with $U(1) \times U(1)$ symmetry, we are able to solve the Toda equation and write down an explicit metric. Like the Gibbons-Hawking metrics, these metrics have an explicit $U(1)$ fiber that pinches off at various points along the axis to create a series of homological 2-spheres. However, a new feature of the LeBrun metrics is that homological 2-spheres can also be formed by the other angular coordinate, and we obtain the specific boundary conditions that allow this to happen. We also find a new feature as we approach the Gibbons-Hawking points, or "geometric charges." In the GH metric, the $U(1)$ near these points fibers over the S^2 in the base to give $S^3/\mathbb{Z}_q$, which makes the local metric an orbifold $\mathbb{R}^4/\mathbb{Z}_q$. In the LeBrun metric, however, one generically has $\mathbb{R}^4/G$ at these points, where $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ acts on the two angular coordinates in $\mathbb{R}^4 \simeq \mathbb{R}^2 \times \mathbb{R}^2$. Finally, and perhaps most importantly, the explicit LeBrun metrics obtained have a Maxwell field whose self-dual part $\Theta^{(3)}$ is non-trivially trapped on these topological 2-cycles. This allows rich new phenomena in the full supergravity solution that were not present in the previous work of Sections 4.3 and 4.4.

Looking at the full supergravity solution, we see a striking similarity between these non-supersymmetric solutions and the previous, well-known BPS solutions [25], also explained in Section 3.1. The regularity conditions take virtually the same form. By demanding the absence of CTC's, we also obtain "bubble equations" which have largely the same features as in the BPS solutions: a 2-cycle is held open by the product of the three flavors of fluxes trapped on it. However, the non-BPS bubble equations at a given point involve not only the fluxes on cycles adjacent to that point, but also involve all the fluxes on the nonadjacent cycles (which is a radical departure from the BPS bubble equations). This indicates *new physics* that was not present in the BPS case, involving a richer variety of $E \times B$ type interactions.

It is known from previous work that these 5-dimensional non-supersymmetric solutions on a Kähler base are actually trivial KK reductions of *BPS* solutions in the 6-dimensional IIB frame [70, 71, 72]. This explains some of the features we see, and yet makes others more mysterious. It seems clear that the 5-dimensional solutions are force-balanced by a kind of "supersymmetry without supersymmetry" [92], and in fact might be *closer* to BPS than the so-called "almost BPS" solutions [76, 77, 78]. For example, the bubble equations here and in the traditional 5d BPS solutions are both linear in the inverse distances $1/r_{ij}$, whereas the "almost BPS" bubble equations are cubic. Still, there are important differences between these bubble equations and the 5d BPS bubble equations that must be explained if we are to think of these as "secretly BPS."

Having found the non-BPS bubble equations, we also find that imposing the asymptotics of the near-horizon BMPV metric [89] precludes the existence of any finitely-sized bubbled solutions. However, one can see the near-horizon BMPV-like metrics as the result of a limiting process of scaling solutions [123, 102, 79]. Alternatively, one can lift to the 6d IIB frame where one can allow different asymptotic behavior in one of the warp factors, and in this case one can find an infinite family of smooth geometries, with finitely-sized bubbles held open by their cohomological fluxes, which are asymptotic to a momentum wave solution on $AdS_3 \times S^3$.

It would be interesting to explore further the lift to the 6d IIB frame, as was done with the LeBrun-Burns metrics in Section 4.4. In 6 dimensions, one has the possibility of regular supertubes, and one might also get a better handle on why the bubble equations differ between here and the traditional setting (particularly in containing non-local interactions).

It would also be interesting to look for an asymptotically-flat completion of these solutions in 5 dimensions by relaxing the simplifying assumptions used in the floating brane ansatz [31]. This is certainly a non-trivial thing to do, as one will likely be forced to address the full Einstein equations.

Finally, we also point out that while this work has focused on smooth solutions, one also has within the same solution set an infinite family of singular solutions, representing various collections non-supersymmetric, yet force-balanced, spinning 3-charge black rings.

We have presented in Chapters 4 and 5 a number of results and techniques which we hope yield insight into supergravity and black hole microstates. Recent progress in the ability to find supergravity solutions is very exciting and full of possibilities, and it is clear that there are many avenues waiting to be explored.

Chapter 6

Superstrata and Supersheets

The material in this chapter is taken from [36], which I authored with collaborators Orestis Vasilakis and Nick Warner.

6.1 Motivation

In Section 2.3 we discussed black hole constructions in string theory. We looked at the 2-charge black hole in 5 and 6 dimensions and showed that it has zero classical horizon area. However, with α' corrections it has a Planck-scale horizon, and its entropy can be calculated microscopically from its D-brane construction. This entropy can be partially accounted for by *supertubes*[1], which are objects with 2 electric charges and 1 magnetic dipole charge that can take an arbitrary shape as a function of 1 variable, while maintaining 8 out of 32 supersymmetries (thus being $\frac{1}{4}$-BPS). One can find supergravity solutions corresponding to supertubes of arbitrary shape in IIB reduced to 6 dimensions [19, 20], where such solutions are smooth as shown in Section 3.3.3. It is such geometries that lie at the heart of Mathur's original fuzzball proposal for the microstate structure of 2-charge black holes (see, for example, [15, 21]).

We also looked at 3-charge black holes, and showed they have a macroscopic horizon area that matches their microscopic entropy counting [65], scaling schematically as $S \sim Q^{3/2}$. However, while many smooth 3-charge BPS supergravity solutions are known, we have yet to find the solutions that give enough entropy (see [25, 124, 125, 24] for

[1] That is, the entropy of supertube states makes up a finite fraction of the black hole entropy.

some reviews). The entropy of supertubes scales as $S \sim Q$; and in some 3-charge circumstances this entropy can be enhanced to $S \sim Q^{5/4}$ [67, 104]. But $S \sim Q^{3/2}$ seems unreachable in the supergravity régime without something drastic.

The missing ingredient is of course the *superstratum*, which is a new class of BPS object conjectured to give the $S \sim Q^{3/2}$ scaling in the 3-charge case [32]. These superstrata generalize supertubes in several important ways, and could lead to microstate geometries that provide the dominant semi-classical contribution to the microstate structure of the 3-charge system. The conjectured superstratum carries *three* electric charges and three dipole charges, two of which are independent, and is described by an arbitrary, $(2+1)$-dimensional world-surface. It is expected to be a regular, smooth solution in IIB supergravity reduced to six space-time dimensions.

The argument for the existence of the superstratum has its origins in earlier work [126] that suggested that one should be able to make two independent supertube transitions to produce new BPS solutions that carry three electric charges, two magnetic dipole charges and depend upon functions of two variables. It was originally believed that such objects would be non-geometric and have spatial co-dimension two, but it was shown in [32] that if one does this in the proper manner for the D1-D5-P system in IIB supergravity then the result will not only be a *geometric* BPS object with co-dimension *three* but one that is also completely smooth. Indeed, very near the superstratum the geometry approaches that of the supertube and so the smoothness follows directly from that of the supertube geometry. Thus the superstratum provides a new microstate *geometry* of co-dimension three that carries three electric charges, two independent magnetic dipole charge and depends upon several functions of two variables.

This would all be a mere philosophical exercise if not for some exciting new developments for BPS solutions in six dimensions. First, the BPS equations for six-dimensional, minimal $\mathcal{N}=1$ supergravity [70] coupled to an anti-self-dual tensor multiplet [71] were shown to be linear,[2] as was found in [72] and we have reviewed in Section 3.3.2. This not only provides a huge simplification in solving the equations of Section 3.3.1 but it also enables one to use superposition to obtain multi-component solutions and, more abstractly, analyze the moduli spaces of such solutions. It is not only anticipated that this will lead to interesting new developments in the study of black-hole microstate geometries but that it will also lead to interesting new results for holography on $AdS_3 \times S_3$ geometries.

While the arguments given in [32] for the existence of the superstratum are fairly compelling, it still remains to construct one explicitly and thereby establish its existence beyond all doubt. The fact that the BPS equations in six dimensions are linear gives one hope that the explicit supergravity solution may just be within reach (although it will still be extremely complicated). The construction in [32] has the virtue that it lays out a sequence of steps, via two supertube transitions, to arrive at the superstratum and so a possible route to making a superstratum might be to replicate these intermediate steps in a series of progressively more complicated but exact supergravity solutions. Indeed some initial progress in this direction was achieved in [72] where the D1-D5-P system was pushed through the first supertube transition to obtain a new three-charge, two-dipole charge[3] generalized supertube with an arbitrary profile as a function of one variable. We will refer to such a solution as a *superthread*.

The next step towards a superstratum, which will be the subject of this chapter, requires the construction of a multi-superthread solution that could then be smeared to a continuum and thus obtain a three-charge solution with a two-dimensional spatial profile that is a function of two variables. This solution will still be singular and, like

[2]As with the corresponding result in five dimensions [73, 25], the equations that determine the spatial base geometry are still non-linear.

[3]The two dipole charges in this solution are related to one another and so, to get to the superstratum, a further independent dipole charge must be added via a second supertube transition [32].

the standard supertube, will only become regular after the second supertube transition in which a Kaluza-Klein monopole is combined with the smearing. This last step is probably going to be the most difficult and will not be addressed here.

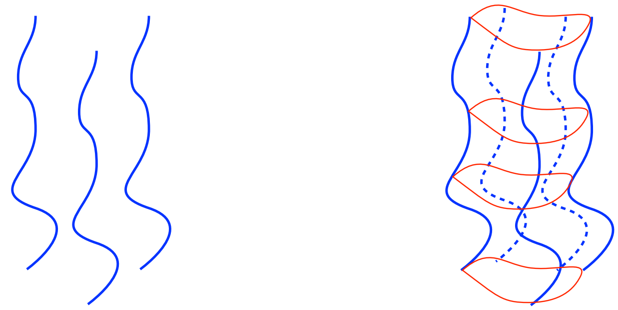

Figure 6.1: *Multi-thread solution in which all the threads are parallel. When smeared the sheet profile is described by a product of functions of one variable: the original thread profile and the thread densities.*

In [72] the step to the multi-superthread was only achieved for the highly restricted situation in which each thread was given exactly the same profile with a rigid translation to each distinct center. The smearing of such a multi-threaded solution will thus produce two-dimensional surface that is determined by a several functions of *one* variable, namely the smearing density and the original superthread profile functions. (See Fig. Figure 6.1). To get a surface that is truly a generic function of two variables one must find the multi-superthread solution in which the threads at each center have independent profile functions so that, in the continuum limit, one obtains a one-parameter family of curves and hence a surface swept out by a generic function of two variables (See Fig. Figure 6.2). The purpose of this chapter is to find this general a multi–thread solution. The difficulty that we overcome here is that multiple superthreads with different profiles have highly non-trivial shape-shape interactions and we show exactly how these contribute to the angular momentum and local momentum charge densities. It should, of course, be stressed that even though our solutions represent only a step towards the ultimate goal of the superstratum, the multi-superthread solutions presented here are completely new BPS solutions that are interesting in their own right.

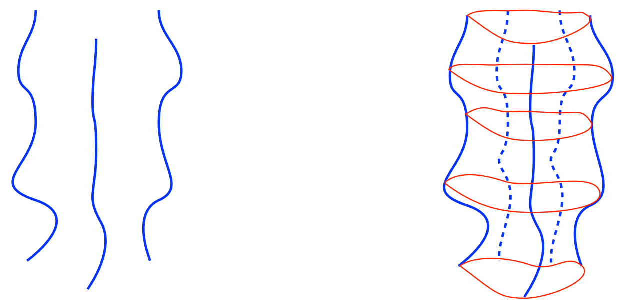

Figure 6.2: *Multi-thread solution in which all the threads have independent profiles. When smeared the sheet profile is described by generic functions of two variables.*

This chapter is organized as follows: In Section 6.2, we briefly summarize the linear BPS system to be solved, and then we will present our new solutions which generalize the parallel superthreads of [72]. In Section 6.3 we will discuss regularity near the superthreads and conditions for the absence of CTCs, and their relation to the asymptotic charges of the superthreads. In Section 6.4 we will discuss *supersheets*, which are arbitrary 2-dimensional objects made of many superthreads, and are an important step on the way to constructing superstrata. Finally in Section 6.5 we will discuss these results and open problems.

6.2 Solving the BPS equations

In the quest for BPS objects fluctuating as a function of two variables, we must solve the 6d BPS system (3.101)–(3.107) where $Z_1, Z_2, \Theta^{(1)}, \Theta^{(2)}, \mathcal{F}, \omega$ are allowed to depend on the 6th coordinate v. With v-dependence turned on, the equations can get quite tricky. To tame these difficulties as much as possible, we will choose a simple base space.

The context for these solutions is the 6d metric (3.66):

$$\mathrm{d}s_6^2 = -2H^{-1}\,(\mathrm{d}v+\beta)\big(\mathrm{d}u+\omega+\frac{1}{2}\mathcal{F}(\mathrm{d}v+\beta)\big) + H\,\mathrm{d}s_4^2(\mathcal{B}), \tag{6.1}$$

where the 4d base space $\mathcal{B}$ is given by

$$\mathrm{d}s_4^2(\mathcal{B}) = h_{ij}(x^k;v)\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{6.2}$$

We will also find it helpful to define a "restricted" exterior derivative $\tilde{\mathrm{d}}$ on $\mathcal{B}$ that acts only on the x^i and treats v as a constant:

$$\tilde{\mathrm{d}}(\varphi_I\,\mathrm{d}x^I) = \left(\frac{\partial\varphi_I}{\partial x^i}\right)_v \mathrm{d}x^i\wedge\mathrm{d}x^I, \tag{6.3}$$

where I is a multi-index and $\varphi_I\,\mathrm{d}x^I$ is a generic p-form. Using this, we define a "covariant" exterior derivative $\mathcal{D}$ via

$$\mathcal{D}\varphi \equiv \tilde{\mathrm{d}}\varphi - \beta\wedge\partial_v\varphi, \tag{6.4}$$

such that the total exterior derivative d (on u-independent fields) can be written

$$\mathrm{d}\varphi = (\mathrm{d}v+\beta)\wedge\dot{\varphi} + \mathcal{D}\varphi, \tag{6.5}$$

where we have used a dot to denote the v-derivative $\dot{\phi}\equiv\partial_v\phi$.

Next we must choose a base space $\mathcal{B}$ with an "almost hyper-Kähler" structure (3.74). We make the simplest possible choice: flat $\mathbb{R}^4$. Then the base metric is

$$\mathrm{d}s_4^2(\mathcal{B}) = h_{ij}(x^k;v)\,\mathrm{d}x^i\,\mathrm{d}x^j \equiv \delta_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j. \tag{6.6}$$

Then the $J^{(A)}$ are just the usual $SU(2)$ structure on $\mathbb{R}^4$, and the 1-form $\beta \equiv 0$, which trivially solves $\mathcal{D}\beta = \star_4 \mathcal{D}\beta$.

With trivial β this means that there will be no Kaluza-Klein monopoles in the solution. The spatial part of the metric is simply flat $\mathbb{R}^4 \times S^1$. In particular, it means that we will *not* find regular, smooth solutions, because the KK monopole of β is necessary for the smoothness of supertubes as explained in Section 3.3.3. However, we will still seek interesting *singular* solutions that fluctuate as a function of two variables.

Since the $J^{(A)}$ are v-independent, we see from (3.81) that we also have $\hat{\psi} \equiv 0$. Then the 6d BPS system can be written

$$\Theta^{(1)} - \underset{4}{\star}\Theta^{(1)} = 0, \qquad \Theta^{(2)} - \underset{4}{\star}\Theta^{(2)} = 0, \tag{6.7}$$

$$\tilde{\mathrm{d}}\Theta^{(1)} = -\tfrac{1}{2}\partial_v \underset{4}{\star}\tilde{\mathrm{d}}Z_2, \qquad \tilde{\mathrm{d}}\Theta^{(2)} = -\tfrac{1}{2}\partial_v \underset{4}{\star}\tilde{\mathrm{d}}Z_1, \tag{6.8}$$

$$\tilde{\mathrm{d}}\underset{4}{\star}\tilde{\mathrm{d}}Z_1 = 0, \qquad \tilde{\mathrm{d}}\underset{4}{\star}\tilde{\mathrm{d}}Z_2 = 0, \tag{6.9}$$

together with

$$\underset{4}{\star}\tilde{\mathrm{d}}\underset{4}{\star}\tilde{\mathrm{d}}\mathcal{F} = 2\underset{4}{\star}\tilde{\mathrm{d}}\underset{4}{\star}\dot{\omega} + 2\,\partial_v^2(Z_1 Z_2) - 2\,\dot{Z}_1\dot{Z}_2 - 4\underset{4}{\star}\big(\Theta^{(1)} \wedge \Theta^{(2)}\big), \tag{6.10}$$

$$\tilde{\mathrm{d}}\omega + \underset{4}{\star}\tilde{\mathrm{d}}\omega = 2Z_1\Theta^{(1)} + 2Z_2\Theta^{(2)}. \tag{6.11}$$

The functions Z_1, Z_2 describe the D1 and D5 electric charges of the solution, whereas the 2-forms $\Theta^{(1)}, \Theta^{(2)}$ describe the D1 and D5 magnetic dipole charges. The 1-form ω gives the angular momentum, and the function $\mathcal{F}$ gives the momentum charge P.

6.2.1 The new solutions

The first steps in our new solution directly parallel those of [72]. The harmonic functions, Z_i, are sourced on the thread profiles, $\vec{F}^{(p)}(v)$:

$$Z_i = 1 + \sum_{p=1}^{n} \frac{Q_{i\,p}}{|\vec{x} - \vec{F}^{(p)}(v)|^2}, \tag{6.12}$$

where we have required that $Z_i \to 1$ at infinity so that the metric is asymptotically Minkowskian. The Maxwell fields, Θ_i, that solve (6.7) and (6.8) are simply given by:

$$\Theta_i = \tfrac{1}{2}\,(1 + \underset{4}{\star})\,\tilde{\mathrm{d}} \left(\sum_{p=1}^{n} \frac{Q_{i\,p}\,\dot{F}^{(p)}_m \mathrm{d}x^m}{|\vec{x} - \vec{F}^{(p)}(v)|^2} \right). \tag{6.13}$$

As noted in [72], the magnetic dipoles of this solution may be thought of as being defined by

$$\vec{d}_1 = Q_1 \dot{\vec{F}}(v), \qquad \vec{d}_2 = Q_2 \dot{\vec{F}}(v), \tag{6.14}$$

and they satisfy the constraint that is familiar from the five-dimensional, generalized supertube [127, 128, 129]:

$$Q_1 \left|\vec{d}_2\right| = Q_2 \left|\vec{d}_1\right|. \tag{6.15}$$

This means that even though the solution has two dipole charges, only one of them is independent of the other charges.

To write the solution for the angular momentum vector and the third function, $\mathcal{F}$, it is useful to define:

$$\vec{R}^{(p)} \equiv \vec{x} - \vec{F}^{(p)}(v), \qquad R_p \equiv \left|\vec{R}^{(p)}\right| \equiv \left|\vec{x} - \vec{F}^{(p)}(v)\right|, \tag{6.16}$$

and for each p and q, introduce the anti-self-dual 2-form area element:

$$\mathcal{A}_{ij}^{(p,q)} \equiv R_i^{(p)} R_j^{(q)} - R_j^{(p)} R_i^{(q)} - \varepsilon^{ijk\ell} R_k^{(p)} R_\ell^{(q)} \,, \tag{6.17}$$

where $\varepsilon^{1234} = 1$. The angular momentum vector can be written in three pieces:

$$\omega = \omega_0 + \omega_1 + \omega_2 \,. \tag{6.18}$$

where the first two parts are very similar to the those in [72]:

$$\begin{aligned} \omega_0 =& \sum_{i=1}^{2} \sum_{p=1}^{n} \frac{Q_{i\,p}\, \dot{F}_m^{(p)} \mathrm{d}x^m}{|\vec{x} - \vec{F}^{(p)}(v)|^2}, \\ \omega_1 =& \frac{1}{2} \sum_{p,q=1}^{n} (Q_{1\,p}\, Q_{2\,q} + Q_{2\,p}\, Q_{1\,q})\, \frac{\dot{F}_m^{(p)} \mathrm{d}x^m}{R_p^2\, R_q^2}. \end{aligned} \tag{6.19}$$

The last part of the solution, ω_2, is part of our new result and arises from the interaction between non-parallel threads:

$$\begin{aligned} \omega_2 = \frac{1}{4} \sum_{\substack{p,q=1 \\ p\neq q}}^{n} (Q_{1p} Q_{2q} + Q_{2p} Q_{1q})\, \frac{\left(\dot{F}_i^{(p)} - \dot{F}_i^{(q)}\right)}{\left|\vec{F}^{(p)} - \vec{F}^{(q)}\right|^2} \times \\ \times \left\{ \left(\frac{1}{R_p^2} - \frac{1}{R_q^2} \right) \mathrm{d}x^i - \frac{2}{R_p^2 R_q^2}\, \mathcal{A}_{ij}^{(p,q)}\, \mathrm{d}x^j \right\}. \end{aligned} \tag{6.20}$$

From this one can easily verify that

$$\vec{\nabla} \cdot \vec{\omega} = -\partial_v (Z_1 Z_2), \tag{6.21}$$

which means that the equation for $\mathcal{F}$ simplifies to

$$
\begin{aligned}
\nabla^2 \mathcal{F} &= -2\left[\dot{Z}_1 \dot{Z}_2 + \underset{4}{\star}(\Theta^{(1)} \wedge \Theta^{(2)})\right] \\
&= -4 \sum_{p,q=1}^{n} (Q_{1p} Q_{2q} + Q_{2p} Q_{1q}) \frac{1}{R_p^4 R_q^4} \times \\
&\qquad\qquad \times \left[(\vec{R}^{(p)} \cdot \vec{R}^{(q)}) \left(\dot{\vec{F}}^{(p)} \cdot \dot{\vec{F}}^{(q)} \right) - \dot{\vec{F}}^{(p)i} \dot{\vec{F}}^{(q)j} \mathcal{A}_{ij}^{(p,q)} \right].
\end{aligned}
\tag{6.22}
$$

This can be solved by the somewhat obvious guess:

$$
\begin{aligned}
\mathcal{F} = -4 - 4 \sum_{p=1}^{n} \frac{Q_{3\,p}}{R_p^2} - \frac{1}{2} \sum_{p,q=1}^{n} & \frac{(Q_{1p} Q_{2q} + Q_{2p} Q_{1q})}{R_p^2 R_q^2} \left(\dot{\vec{F}}^{(p)} \cdot \dot{\vec{F}}^{(q)} \right) \\
& + \sum_{\substack{p,q=1 \\ p \neq q}}^{n} (Q_{1p} Q_{2q} + Q_{2p} Q_{1q}) \frac{1}{R_p^2 R_q^2} \frac{\dot{F}_i^{(p)} \dot{F}_j^{(q)} \mathcal{A}_{ij}^{(p,q)}}{\left| \vec{F}^{(p)} - \vec{F}^{(q)} \right|^2},
\end{aligned}
\tag{6.23}
$$

where the first two terms represent particular choices for the harmonic pieces of $\mathcal{F}$. In normalizing these harmonic pieces we have kept in mind the fact that dimensional reduction to five space-time dimensions yields $\mathcal{F} = -4Z_3$, where Z_3 determines the third electric charge of the solution and is on the same footing (in five dimensions) as Z_1 and Z_2. The terms in ω and $\mathcal{F}$ that contain $\mathcal{A}_{ij}^{(p,q)}$ express the non-trivial interaction between non-parallel superthreads. These terms vanish for solutions with multiple threads of parallel profiles, $\vec{F}(v)$, and hence did not appear in [72].

Finally, there are also possible harmonic pieces that can be added to the angular momentum vector, ω. To define these, introduce the following self-dual harmonic forms on $\mathbb{R}^4$:

$$
\begin{aligned}
\Omega_+^{(1)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4, \\
\Omega_+^{(2)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 - \mathrm{d}x^2 \wedge \mathrm{d}x^4, \\
\Omega_+^{(1)} &= \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3.
\end{aligned}
\tag{6.24}
$$

Then the following are zero modes of the equation (6.11) that defines ω:

$$\omega_{harm} = \sum_{a=1}^{3} \sum_{p=1}^{n} \frac{1}{R_p^4} J_p^{(a)}(v) \, \Omega_{+ij}^{(a)} \, R^{(p)i} \, \mathrm{d}x^j, \tag{6.25}$$

where the $J_p^{(a)}(v)$ are v-dependent angular momentum densities. The one-form in (6.25) is sourced along the profile of the superthread. Moreover, one can easily verify that:

$$\tilde{\mathrm{d}} \underset{4}{\star} \omega_{harm} = 0, \tag{6.26}$$

and so this induces no additional contribution to $\mathcal{F}$ in (6.10).

6.3 Regularity and the near-thread limit

The six-dimensional metric we are considering is:

$$\mathrm{d}s_6^2 = -2(Z_1 Z_2)^{-1/2} \, \mathrm{d}v \left(\mathrm{d}u + \omega + \tfrac{1}{2} \, \mathcal{F} \, \mathrm{d}v \right) + 2(Z_1 Z_2)^{1/2} \, |\mathrm{d}\vec{x}|^2. \tag{6.27}$$

Regularity requires that $Z_1 Z_2 > 0$ and we will ensure this by taking

$$Q_{1p} \, , \, Q_{2p} \geq 0 \qquad \text{for all } p. \tag{6.28}$$

Moreover, if one sets all displacements to zero except along the circular fiber parametrized by v then the metric collapses to $\mathrm{d}s_6^2 = -(Z_1 Z_2)^{-1/2} \mathcal{F} \, \mathrm{d}v^2$, which means that one must require

$$-\mathcal{F} \geq 0 \tag{6.29}$$

everywhere if one is to avoid closed timelike curves. The expression for $\mathcal{F}$ in (6.23) is somewhat complicated but the condition (6.29) can generically be satisfied if one takes Q_{3p} to be positive and large enough. We will discuss this further below.

The near-thread limit is going to be singular because it is locally a three-charge, two-dipole charge object. However we must also ensure that there are no closed time-like curves (CTC's) near the superthreads. To that end we collect all the divergent and finite parts of the metric in the limit $R_p \to 0$:

$$\begin{aligned}
Z_i &\sim \frac{Q_{ip}}{R_p^2} + 1 + \sum_{q\neq p} \frac{Q_{iq}}{F_{pq}^2} + \mathcal{O}(R_p), \qquad\qquad i = 1,2, \\
\mathcal{F} &\sim -\frac{Q_{1p}\,Q_{2p}}{R_p^4}\,\big|\dot{\vec{F}}^{(p)}\big|^2 \\
&\qquad - \frac{1}{R_p^2}\bigg[4\,Q_{3p} + \sum_{q\neq p} \frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\,\big(\dot{\vec{F}}^{(p)}\cdot\dot{\vec{F}}^{(q)}\big)\bigg] + \mathcal{O}(1), \\
\omega &\sim -\frac{Q_{1p}\,Q_{2p}}{R_p^4}\,\big(\dot{\vec{F}}^{(p)}\cdot d\vec{x}\,\big) + \frac{1}{R_p^3}\sum_{a=1}^{3} J_p^{(a)}(v)\,\Omega_{+ij}^{(a)}\,\widehat{R}^{(p)i}\,dx^j \\
&\qquad + \frac{1}{R_p^2}\bigg[(Q_{1p}+Q_{2p}) + \sum_{q\neq p}\frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\bigg]\big(\dot{\vec{F}}^{(p)}\cdot d\vec{x}\,\big) + \mathcal{O}\Big(\frac{1}{R_p}\Big).
\end{aligned} \tag{6.30}$$

where we have included the harmonic pieces, (6.25), of ω and where

$$F_{pq}^2 \equiv \big|\vec{F}^{(p)} - \vec{F}^{(q)}\big|^2, \qquad \widehat{R}^{(p)} \equiv \frac{\vec{R}^{(p)}}{R_p}. \tag{6.31}$$

Setting $du = 0$, one finds, at leading order as $R_p \to 0$,

$$\mathrm{d}s^2 \sim \frac{\sqrt{Q_{1p}Q_{2p}}}{R_p^2}\left[\big|\dot{\vec{F}}^{(p)}\big|^2\left(\mathrm{d}v - \frac{\dot{\vec{F}}^{(p)}\cdot\mathrm{d}\vec{x}}{\big|\dot{\vec{F}}^{(p)}\big|^2}\right)^2 + \mathrm{d}x_\perp^2\right]. \tag{6.32}$$

where

$$\mathrm{d}x_\perp^2 \equiv |\mathrm{d}\vec{x}|^2 - \frac{\left|\dot{\vec{F}}^{(p)} \cdot \mathrm{d}\vec{x}\right|^2}{\left|\dot{\vec{F}}^{(p)}\right|^2}, \tag{6.33}$$

which is the spatial metric in $\mathbb{R}^4$ perpendicular to the tangent, $\dot{\vec{F}}^{(p)}$, to the superthread. The asymptotic metric (6.32) is manifestly positive but not positive-definite: There is a null direction along the supertube. That is, the leading order terms vanish precisely if one takes

$$\mathrm{d}\vec{x} = \dot{\vec{F}}^{(p)}\, \mathrm{d}\lambda, \qquad \mathrm{d}v = \left|\dot{\vec{F}}^{(p)}\right| \mathrm{d}\lambda, \tag{6.34}$$

for some infinitesimal displacement, $\mathrm{d}\lambda$.

For this displacement one finds a leading order term coming from the harmonic pieces of ω:

$$\mathrm{d}s_\lambda^2 = \frac{\mathrm{d}\lambda^2}{R_p} \frac{1}{\sqrt{Q_{1p}Q_{2p}}} \sum_{a=1}^{3} J_p^{(a)}(v)\, \Omega_{+ij}^{(a)}\, \widehat{R}^{(p)i}\, \dot{F}^{(p)j} \tag{6.35}$$

If one looks in the direction $R_i^{(p)} \sim -\sum_{a=1}^{3} J_p^{(a)}(v)\, \Omega_{+ij}^{(a)}\, \dot{F}^{(p)j}$ one finds that $\mathrm{d}s_\lambda^2$ is negative and proportional to $\sum_{a=1}^{3} (J_p^{(a)}(v))^2 \left|\dot{\vec{F}}^{(p)}\right|^2$. Thus for a superthread with $\dot{\vec{F}}^{(p)} \neq 0$ one can only avoid CTC's if one sets

$$J_p^{(a)}(v) = 0; \tag{6.36}$$

that is, the harmonic pieces, (6.25), produce CTC's and so must be discarded. The complete physical solution is thus given by $\omega_0 + \omega_1 + \omega_2$ defined in (6.19) and (6.20).

An important consequence of this analysis is that the angular momentum vector is completely determined by the electric charges and profiles of the configuration. This is slightly different from the five-dimensional solutions in which one has independent choices of harmonic functions in the angular momentum vectors and the angular momenta are then fixed in terms of the charges and positions of the sources via bubble equations, or integrability conditions, that remove CTC's. For the six-dimensional solutions presented here one fixes charges, positions and profiles and the angular-momentum vector is adjusted automatically: there are no bubble equations.

Having now killed the leading order of the metric along the displacement (6.34) it turns out that there is a finite order piece. As $R_p \to 0$ the metric becomes:

$$\mathrm{d}s_\lambda^2 = \frac{\mathrm{d}\lambda^2}{\sqrt{Q_{1p}Q_{2p}}}\Bigg[4\,Q_{3p} - \big|\dot{\vec{F}}^{(p)}\big|^2(Q_{1p}+Q_{2p}) \\ - \sum_{q\neq p}\frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\,\dot{\vec{F}}^{(p)}\cdot\big(\dot{\vec{F}}^{(p)}-\dot{\vec{F}}^{(q)}\big)\Bigg] + \mathcal{O}(R_p). \quad (6.37)$$

Again, to avoid closed timelike curves we require that the quantity in brackets be non-negative, which is equivalent to asking that

$$-\,\mathcal{F} \geq \dot{F}_i^{(p)}\,\omega_i \quad (6.38)$$

near each thread. Hence the positivity of $\mathrm{d}s_\lambda^2$ in (6.37) places a lower bound on each of the charges Q_{3p}:

$$Q_{3p} \geq \frac{1}{4}\,\big|\dot{\vec{F}}^{(p)}\big|^2(Q_{1p}+Q_{2p}) + \frac{1}{4}\sum_{q\neq p}\frac{(Q_{1p}Q_{2q}+Q_{2p}Q_{1q})}{F_{pq}^2}\,\dot{\vec{F}}^{(p)}\cdot\big(\dot{\vec{F}}^{(p)}-\dot{\vec{F}}^{(q)}\big). \quad (6.39)$$

The individual bounds for each p depend upon the detailed geometric layout of the threads but if one sums over all the threads then one obtains a global bound upon the total charges:

$$\sum_{p=1}^{n} Q_{3p} \geq \frac{1}{4}\sum_{p=1}^{n}\big|\dot{\vec{F}}^{(p)}\big|^2(Q_{1p}+Q_{2p}). \quad (6.40)$$

The origins of these bounds can be understood in terms of "charges dissolved in flux" [73]. From (6.14) one sees that the right-hand sides of (6.39) and (6.40) can be thought of as the dipole-dipole interactions that give rise to an effective electric contribution to the momentum charge described by $\mathcal{F}$. As we will describe below, the harmonic charge term, described by Q_{3p} in $\mathcal{F}$, is the charge measured at infinity and so these bounds mean that the only physically sensible solutions are those in which one does indeed correctly account, at infinity, for the charge coming dipole-dipole interactions.

6.3.1 Asymptotic charges

The electric charges measured at infinity come from the asymptotic forms of Z_1, Z_2 and $Z_3 \equiv -\frac{1}{4}\mathcal{F}$. From the leading ($\mathcal{O}(R^{-2})$) terms in (6.12) and (6.23) one can easily read off the D1, D5, and P charges:

$$\text{D1:} \;\; \sum_p Q_{1p}, \qquad \text{D5:} \;\; \sum_p Q_{2p} \qquad \text{P:} \;\; \sum_p Q_{3p}. \tag{6.41}$$

The terms in the tensor, $\mathcal{A}_{ij}^{(p,q)}$, defined (6.17) do not contribute in $\mathcal{F}$ because $\vec{R}^{(p)}$ and $\vec{R}^{(q)}$ become nearly parallel at large distances and so this term vanishes at leading order.

The asymptotic form of ω can be massaged into

$$\begin{aligned}
\omega \sim \frac{1}{R^2} \sum_p \big(Q_{1p} + Q_{2p}\big)\, \dot{\vec{F}}^{(p)} \cdot \mathrm{d}\vec{x} + \frac{2}{R^4} \sum_p \big(Q_{1p} + Q_{2p}\big)\big(\vec{R} \cdot \vec{F}^{(p)}\big)\, \dot{\vec{F}}^{(p)} \cdot \mathrm{d}\vec{x} \\
+ \frac{1}{2}\frac{1}{R^4} \sum_{\substack{p,q \\ p \neq q}} \frac{Q_{1p}Q_{2q} + Q_{2p}Q_{1q}}{F_{pq}^2} R^i \Big[F_i^{(pq)} \big(\dot{\vec{F}}^{(pq)} \cdot \mathrm{d}\vec{x}\big) \\
- \dot{F}_i^{(pq)} \big(\vec{F}^{(pq)} \cdot \mathrm{d}\vec{x}\big) + \varepsilon_{ijk\ell} F_j^{(pq)} \dot{F}_k^{(pq)}\, \mathrm{d}x^\ell \Big] \\
+ \frac{1}{2}\frac{1}{R^4} \vec{R} \cdot \mathrm{d}\vec{x} \sum_{\substack{p,q \\ p \neq q}} \frac{Q_{1p}Q_{2q} + Q_{2p}Q_{1q}}{F_{pq}^2} \big(\vec{F}^{(pq)} \cdot \dot{\vec{F}}^{(pq)}\big),
\end{aligned} \tag{6.42}$$

where $\vec{F}^{(pq)} \equiv \vec{F}^{(p)} - \vec{F}^{(q)}$. The first term falls of as R^{-1} and is perhaps somewhat unexpected. Mathematically it arises through the contribution of the constant terms in the Z_i to the source for ω in (6.11). These source terms mean that, to leading order,

$(1+\star_4)\,\tilde{\mathrm{d}}\omega$ limits to $2\,(\Theta_1+\Theta_2)$ and thus ω inherits an asymptotic behavior given by the vector fields in parentheses in (6.13). In five dimensions, $(\Theta_1 + \Theta_2)$ falls off faster and leads to standard expansions for angular momenta in ω. The presence of the $\mathcal{O}(R^{-1})$ terms in six-dimensions comes because of the v-dependent sources in (6.8). The fact that this term is a total v-derivative means it will always vanish when we reduce to five dimensions. This is because, in order to reduce to five dimensions, the sources must be smeared in a way that kills all v dependence; hence the unusual $\mathcal{O}(R^{-1})$ term disappears and one recovers the standard behavior of five-dimensional solutions. We will illustrate this in the next section.

Physically, the $\mathcal{O}(R^{-1})$ terms represent a *linear* momentum for the configuration. The somewhat unusual feature of the six-dimensional linear system is that all the equations are solved on a constant-v slice and that, for a given value of v, the solution is insensitive to the configuration at other values of v and so, slice-by-slice, the solution sees the superthread as indistinguishable from the thread that carries a linear momentum. It is only when one smears the solution along a closed profile that the solution combines different sections of the solution with different orientations so that the leading momentum behavior cancels and leaves one with a more standard angular momentum.

The second term in (6.42) is purely rotational, and expresses the difference $J_T \equiv J_1 - J_2$. The third term is the potential of a purely anti-self-dual 2-form, and so it expresses the sum $J_1 + J_2$. The last term is a total derivative, and may be viewed as pure gauge.

6.4 Supersheets

6.4.1 General supersheets

It is straightforward to take the continuum limit of the multi-superthread solution. The set of profiles, $\vec{F}^{(p)}(v)$, are replaced by a function of two variables, $\vec{F}(\sigma, v)$, the discrete charges, $Q_{i\,p}$, are replaced by density functions, $\rho_i(\sigma)$ and the sums are replaced by integrals. Thus we have

$$Z_i = 1 + \int_0^{2\pi} \frac{\rho_i(\sigma)\, d\sigma}{|\vec{x} - \vec{F}(\sigma, v)|^2}, \tag{6.43}$$

$$\Theta_i = \tfrac{1}{2}\,(1 + \star_4)\,\tilde{\mathrm{d}}\left(\int_0^{2\pi} \frac{\rho_i(\sigma)\,\partial_v \vec{F}(\sigma, v)\cdot \mathrm{d}\vec{x}}{|\vec{x} - \vec{F}(\sigma, v)|^2}\, d\sigma\right), \tag{6.44}$$

where we have chosen to normalize the smearing over the interval $[0, 2\pi]$. Following (6.16) and (6.17) we define

$$\vec{R}(\sigma) \equiv \vec{x} - \vec{F}(\sigma, v), \qquad R(\sigma) \equiv \left|\vec{R}(\sigma, v, \vec{x})\right|, \tag{6.45}$$

and the tensor

$$\mathcal{A}_{ij}(\sigma_1, \sigma_2) \equiv R_i(\sigma_1) R_j(\sigma_2) - R_j(\sigma_1) R_i(\sigma_2) - \varepsilon^{ijk\ell} R_k(\sigma_1) R_\ell(\sigma_2)\,, \tag{6.46}$$

With these definitions, the rest of the continuum solution can be written

$$\omega_0 = \sum_{i=1}^{2} \int_0^{2\pi} \frac{\rho_i(\sigma)\,\partial_v \vec{F}(\sigma, v)\cdot \mathrm{d}\vec{x}}{|\vec{x} - \vec{F}(\sigma, v)|^2}\, d\sigma, \tag{6.47}$$

$$\begin{aligned}\omega_1 = \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi} \Big(\rho_1(\sigma_1)\rho_2(\sigma_2) + \rho_2(\sigma_1)\rho_1(\sigma_2)\Big)\times \\ \times\, \frac{\partial_v \vec{F}(\sigma_1, v)\cdot \mathrm{d}\vec{x}}{R(\sigma_1, v, \vec{x})^2\, R(\sigma_2, v, \vec{x})^2}\, d\sigma_1 d\sigma_2,\end{aligned} \tag{6.48}$$

$$\omega_2 = \frac{1}{4}\int_0^{2\pi}\int_0^{2\pi}\Big(\rho_1(\sigma_1)\rho_2(\sigma_2)+\rho_2(\sigma_1)\rho_1(\sigma_2)\Big)\frac{\big(\partial_v F_i(\sigma_1,v)-\partial_v F_i(\sigma_2,v)\big)}{\big|\vec{F}(\sigma_1,v)-\vec{F}(\sigma_2,v)\big|^2}\times$$
$$\times\left\{\left(\frac{1}{R(\sigma_1)^2}-\frac{1}{R(\sigma_2)^2}\right)\mathrm{d}x^i-\frac{2}{R(\sigma_1)^2R(\sigma_2)^2}\,\mathcal{A}_{ij}(\sigma_1,\sigma_2)\,\mathrm{d}x^j\right\}d\sigma_1 d\sigma_2, \tag{6.49}$$

$$\begin{aligned}\mathcal{F} = -4 - 4\int_0^{2\pi}\frac{\rho_3(\sigma)}{R(\sigma)^2}\,d\sigma \\ -\int_0^{2\pi}\int_0^{2\pi}\left(\rho_1(\sigma_1)\rho_2(\sigma_2)+\rho_2(\sigma_1)\rho_1(\sigma_2)\right)\frac{1}{R(\sigma_1)^2R(\sigma_2)^2}\times \\ \times\left[\frac{1}{2}\big(\partial_v\vec{F}(\sigma_1,v)\big)\cdot\big(\partial_v\vec{F}(\sigma_2,v)\big)\right. \\ \left.-\frac{\partial_v F_i(\sigma_1,v)\,\partial_v F_j(\sigma_2,v)\;\mathcal{A}_{ij}(\sigma_1,\sigma_2)}{\big|\vec{F}(\sigma_1,v)-\vec{F}(\sigma_2,v)\big|^2}\right]d\sigma_1 d\sigma_2.\end{aligned} \tag{6.50}$$

The integrals for ω_2 and $\mathcal{F}$ have potential singularities at the coincidence limits, $\sigma_1 = \sigma_2$, with a double pole coming from the denominator factor of $|\vec{F}(\sigma_1,v)-\vec{F}(\sigma_2,v)|^2$. However, the tensor $\mathcal{A}_{ij}$ has a simple zero as $\sigma_1\to\sigma_2$ and this skew tensor is further contracted with factors that have simple zeroes in the coincidence limit. Thus there is also a double zero in the numerator leading to a finite contribution in the coincidence limit.

While we have smeared the multi-superthread solution into a single supersheet, it is also clear that one can smear the multi-superthread solutions into multiple supersheets and such solutions will be given by straightforward generalizations of (6.43)–(6.50).

Finally, we note that one can, of course, recover the multi-superthread solutions from this continuum solution by replacing the density functions, ρ_a, by sums over delta functions:

$$\rho_a(\sigma) = \sum_{j=1}^{N} Q_{a\,p}\,\delta\big(\sigma-\sigma^{(p)}\big), \qquad a=1,2,3, \tag{6.51}$$

and where the individual profile functions are specified by the sampled values of $\vec{F}(\sigma, v)$:

$$\vec{F}^{(p)}(v) = \vec{F}\big(\sigma^{(p)}, v\big). \tag{6.52}$$

6.4.2 The five-dimensional generalized supertube as a supersheet

The supersheets described above are sourced by sheet profiles described by arbitrary functions of *two* variables and are thus much more general than previously-known solutions. However, it is worthwhile to smear our solutions in a more trivial way in order to see exactly how five-dimensional solutions emerge. Therefore we give an example that produces a v-independent sheet profile, allowing us to reduce on the v fiber and obtain a standard five-dimensional solution.

A useful, non-trivial way to accomplish this is to choose any profile $\vec{F}(\sigma)$ in $\mathbb{R}^4$, and define $\vec{F}(\sigma, v) = \vec{F}(\sigma + \kappa v)$. The result should then be a solution of the standard, linear BPS system in five dimensions [73, 25]. One should also directly recover physical constraints like radius relations. For simplicity, we will take the charge densities to be constant and we will smear a simple helical configuration that will produce a cylinder along v and a ring in $\mathbb{R}^2 \subset \mathbb{R}^4$:

$$\vec{F}(\sigma, v) = \Big(0, 0, a\cos(\kappa v + \sigma), a\sin(\kappa v + \sigma)\Big), \tag{6.53}$$

where κ and a are constants with $\kappa = \frac{2n\pi}{L}$, for $n \in \mathbb{Z}$. Each thread will have a constant charge distribution, given by

$$\rho_i(\sigma) \equiv \frac{Q_i}{2\pi}. \tag{6.54}$$

To carry out the integrals (6.43) and (6.47)–(6.50), it is easiest to work in polar coordinates on $\mathbb{R}^2 \times \mathbb{R}^2$ given by:

$$x^1 = \eta \cos\phi, \qquad x^2 = \eta \sin\phi, \qquad x^3 = \zeta \cos\psi, \qquad x^4 = \zeta \sin\psi. \tag{6.55}$$

Note that $R^2 = \eta^2+\zeta^2$. From these coordinates we can easily go to spherical coordinates by defining $\eta = R\cos\theta$ and $\zeta = R\sin\theta$. Then, for example, we obtain Z_1 by integrating

$$\begin{aligned} Z_1 &= 1 + \frac{Q_1}{2\pi}\int_0^{2\pi} d\sigma \, \frac{1}{\eta^2+\zeta^2+a^2-2a\zeta\cos(\sigma+\kappa v)} \\ &= 1 + \frac{Q_1}{\sqrt{(\eta^2+\zeta^2+a^2)^2-4a^2\zeta^2}}. \end{aligned} \tag{6.56}$$

The rest of the integrals are tedious, but straightforward. The result is

$$Z_{1,2} = 1 + \frac{Q_{1,2}}{\Sigma}, \qquad \mathcal{F} = -4 - \frac{4\,Q_3}{\Sigma} - \kappa^2\, Q_1 Q_2\, \frac{1}{\Sigma}\bigg(\frac{\eta^2+\zeta^2}{\Sigma} - 1\bigg), \tag{6.57}$$

$$\omega = \frac{\kappa}{2}\,(Q_1+Q_2)\bigg(\frac{\eta^2+\zeta^2+a^2}{\Sigma} - 1\bigg)\,\mathrm{d}\psi + \kappa\, Q_1 Q_2\, \frac{1}{\Sigma^2}\,\big(\eta^2\,\mathrm{d}\phi + \zeta^2\,\mathrm{d}\psi\big), \tag{6.58}$$

where we have defined

$$\Sigma \equiv \sqrt{(\eta^2+\zeta^2+a^2)^2 - 4a^2\zeta^2} = \sqrt{(R^2+a^2)^2 - 4a^2R^2\cos^2\theta}. \tag{6.59}$$

At infinity, these behave as

$$Z_{1,2} \sim 1 + \frac{Q_{1,2}}{R^2}\,, \qquad\qquad Z_3 = -\tfrac{1}{4}\,\mathcal{F} \sim 1 + \frac{Q_3}{R^2}, \tag{6.60}$$

$$\omega \sim \frac{\kappa}{R^2}\big[\big((Q_1+Q_2)\,a^2 + Q_1Q_2\big)\sin^2\theta\,\mathrm{d}\psi + Q_1Q_2\,\cos^2\theta\,\mathrm{d}\phi\big] \tag{6.61}$$

$$= \frac{1}{2R^2}\big[J_1\,\sin^2\theta\,\mathrm{d}\psi + J_2\,\cos^2\theta\,\mathrm{d}\phi\big], \tag{6.62}$$

where the five-dimensional angular momentum vector, k, is related to ω via $\omega = 2k$. This explains the factor of 2 in (6.62).

This solution corresponds, as expected, to the three-charge, two-dipole-charge generalized supertube [129], with charges Q_1, Q_2, Q_3, and dipole charges

$$q^1 \equiv -\frac{\kappa Q_2}{2}, \qquad q^2 \equiv -\frac{\kappa Q_1}{2}, \qquad q^3 \equiv 0. \tag{6.63}$$

We define $\widetilde{Q}_3$ as

$$\widetilde{Q}_3 \equiv Q_3 - \tfrac{1}{4}\,\kappa^2 Q_1 Q_2. \tag{6.64}$$

Note that $\widetilde{Q}_3$ is the constituent electric charge while the charge measured at infinity, Q_3, also contains the charge arising from the dipole-dipole interaction.

From (6.62) one can read off the angular momenta and one can also check that the radius relation:

$$J_T \equiv J_1 - J_2 = -\tfrac{1}{2}\,\kappa\, a^2\,(Q_1 + Q_2) = (q^1 + q^2 + q^3)\, a^2 \tag{6.65}$$

is satisfied automatically.

The condition that one has $\mathcal{F} \leq 0$ globally implies that $\widetilde{Q}_3 \geq 0$ and hence:

$$Q_3 \geq \tfrac{1}{4}\,\kappa^2\, Q_1 Q_2 = q^1\, q^2. \tag{6.66}$$

This is simply the continuum analog of (6.40).

Near the ring, we find that to avoid CTC's one must have:

$$Q_1 Q_2 \left(\widetilde{Q}_3 - \tfrac{1}{4}\,\kappa^2 a^2 (Q_1 + Q_2)\right) \geq 0, \tag{6.67}$$

and hence

$$\widetilde{Q}_3 \geq \tfrac{1}{4}\,\kappa^2 a^2\,(Q_1 + Q_2) = \tfrac{1}{2}\,\kappa\,J_T. \tag{6.68}$$

This is not quite the same as the continuum limit of (6.39) because the latter bound was derived assuming that R_q remained finite as $R_p \to 0$ whereas the continuum limit gets other important terms in from the coincidence limits when two threads approach one another. This is evident from the fact that the general integrals in Section 6.4 are finite in the coincidence limit but the continuum limit of (6.39) involves a divergent integral.

We have thus recovered one of the standard five-dimensional solutions. The process of obtaining a solution in five dimensions usually involves choosing some harmonic functions and then adjusting the coefficients so as to avoid closed timelike curves. These choices are already implicit in our six-dimensional solution and emerge directly in the smeared solution.

6.5 Discussion and open problems

The BPS equations in six-dimensional, minimal $\mathcal{N} = 1$ supergravity coupled to one tensor multiplet have been shown to be a linear system [72] once an appropriate base geometry has been determined. This allows one to use superposition to create a wide variety of solutions and such solutions could lead to interesting new developments in the study of black hole microstate geometries, as well has holography on $AdS_3 \times S_3$. It has also been conjectured [32] that a new class of BPS microstate geometries, *superstrata*, may exist. Such objects carry three electric charges and two *independent* dipole charges, depend on arbitrary functions of two variables and are expected to be *regular* solutions in the IIB duality frame. They are thus a sheetlike, three-charge generalization of the supertube. The fact that they depend upon functions of two variables suggests that they should be able to store large amounts of entropy in their shape modes, indeed the superstrata microstate geometries are expected to give the dominant semi-classical contribution to the entropy of the three-charge system.

While compelling arguments have been given for the existence of superstrata [32], it remains to explicitly construct one. The results we present here are a very significant step in that direction.

The non-trivial aspect of our new solutions is that they take into account the shape-shape interactions of the separate superthreads. It was evident in [72] that superthreads interact non-trivially with one another when the threads have different profiles and so the completely general multi-superthread was not constructed. Indeed, as depicted in Fig. Figure 6.1, the multi-centered solutions found in [72] only involves parallel threads, shifted by rigid translation in $\mathbb{R}^4$. Such solutions can only be smeared together into a sheet depending on arbitrary functions of *one* variable with one set of functions describing the thread profile and another defining the smearing densities. To get a solution that is genuinely a function of two variables by smearing, it is essential to construct the multi-superthread solution in which all the threads can have independent profiles and so the smeared threads yields a thread-density profile, $\vec{F}(\sigma, v)$. This is depicted in Fig. Figure 6.2.

In this chapter we have analyzed the effect of this shape-shape interaction and presented the general solution with multiple threads of completely arbitrary and independent shapes at each center. These solutions were then smeared to obtain new solutions sourced by a two-dimensional sheet of completely arbitrary profile, described by arbitrary functions of *two* variables. It is also evident that our results can easily be generalized to multi-supersheet solutions.

We also checked our results against a known five-dimensional solution by taking a simple helical profile and smearing it to a cylindrical sheet and dimensionally reducing.

We thus recovered the generalized supertube solution with three-charges and two-dipole charges [127, 128, 129]. We found that CTC conditions, like the radius relation, which usually require an additional constraint on the five dimensional solution, emerge automatically from our six-dimensional solutions.

The solutions presented in this chapter are completely new geometries and are interesting in their own right as three-charge solutions sourced by arbitrary two-dimensional surfaces. To obtain the superstratum we will need to do exactly what we have achieved here but with an additional KKM magnetic charge smeared along the profile thereby providing the required second *independent* dipole moment [32].

In the following chapter, we will attempt this in a restricted way such that $\partial_v \beta = 0$ in (3.76) and (3.77), thus making these equations a linear system. The more general case of $\partial_v \beta \neq 0$ we will leave to future work.

Chapter 7

Superstrata with v-indepentent KKM charge

The material in this chapter is taken from [37], which I authored with my advisor and collaborator Nick Warner.

7.1 Motivation

In Chapter 6 we discussed *supersheets*, which are BPS solutions to 6d $\mathcal{N} = 1$ supergravity coupled to an anti-self-dual tensor multiplet. Supersheets have three electric charges and one (independent) magnetic dipole charge, and are capable of taking a 2-dimensional shape described by arbitrary functions of two variables. As such, they are an important step along the way to finding a *superstratum* solution. However, unlike superstrata, supersheets are lacking a KKM dipole charge that would allow them to be smooth solutions of supergravity, and so they do not make proper microstate geometries. In this chapter, we will discuss another angle of attack on the 6d BPS system (3.101)–(3.107), where we will turn on KKM charge, but will keep it v-independent; the hope is that one can still have fluctuations in the other charges as functions of two variables, thus demonstrating the existence of superstrata as supergravity solutions, even if only in this restricted case.

The general classification of BPS solutions in supergravity is expected to have a wide range of applications, ranging from holography to the description of black-hole microstates. This issue has become particularly significant in six dimensions for several reasons. First, it is perhaps the simplest setting of the D1-D5-P system, which lies at the heart of the stringy description of BPS black holes with macroscopic horizons [65] and the possible construction of microstate geometries (for reviews, see [15, 25, 125, 24, 21]). Secondly, six-dimensional supergravity underlies the study of $AdS_3 \times S_3$ holography (see, for example, [59, 130, 131, 132, 115]). Thirdly, it has become evident that while five-dimensional microstate geometries can resolve black-hole singularities and provide rich families of solutions that sample the typical sector of the black-hole conformal field theory, there are not enough such microstate geometries to sample the states of the black hole with sufficient density so as to yield a semi-classical description of the thermodynamics [133, 123, 134, 135]. The five-dimensional microstate geometries are trivial compactifications of IIB supergravity and M-theory and it is hoped that the incorporation of fluctuations in six, or more, dimensions will greatly extend the phase-space coverage of the microstate geometries. Finally, there was something of a breakthrough in the analysis of six-dimensional supergravity in that the BPS equations of the simplest, but probably most important class of such supergravities are substantially linear [72]. This raises the possibility of finding new classes and families of solution and analyzing the phase space structure more completely.

The jump from the five-dimensional to the six-dimensional BPS system is not expected to be merely incremental in terms of solutions and structure. While five-dimensions is just enough to resolve black-hole singularities, one of the key messages in [133, 123, 134] is that this formulation is still too rigid. The index computations of [134, 135] show that fluctuations around five-dimensional backgrounds involving a simple "graviton gas" can produce a denser but still inadequate (for semi-classical thermodynamics) sampling of microstates. One way to evade this conclusion is to put fluctuations on non-perturbative structures in higher dimensions and for this even the humble, fluctuating supertube could, in principle, be sufficient particularly if the entropy enhancement mechanism [67, 99] can be fully realized. A somewhat more radical proposal was made in [32], where it was proposed that there should be a new kind of solitonic object in six dimensions: the *superstratum*. This was conjectured to be a smooth, six-dimensional microstate geometry whose shape and density modes can be general functions of two variables. These objects also carry three charges and two independent dipole charges and are thus very natural, fundamental constituents of the three-charge black hole. The construction of superstrata, even in a restricted form, is one of the major motivations for this chapter.

The theory that underlies much of the work on five-dimensional microstate geometries is $\mathcal{N} = 2$ supergravity coupled to two vector multiplets in five dimensions. The three vector fields (including the graviphoton) are sourced by the essential three charges of the system. This theory, when uplifted becomes minimal supergravity coupled to an anti-self-dual tensor multiplet in six dimensions. One may also think of this in terms of a compactification of the D1-D5-P system in IIB supergravity on a four-torus. The BPS equations of this system were first extracted in [70, 71] but the form seemed hopelessly non-linear at all levels. In spite of this, one could construct a limited set of non-trivial solutions using spectral flow techniques [136]. However, it was shown in [72] that while the requirements on the five-dimensional spatial background are essentially non-linear, the BPS equations that determine all the charges, two sets of the magnetic fluxes and the angular momentum are, in fact, linear. This means that there are certainly interesting new classes of BPS solution within reach and some of these have already been obtained [72, 36, 34, 137]. These solutions typically start with a simple background geometry which is then decorated with non-trivial charges, magnetic fluxes and sometimes shape modes. Such solutions, while interesting, are usually singular.

Our purpose here is to start with non-trivial geometric backgrounds involving Kaluza-Klein monopoles (KKM's), analyze the BPS equations and then find new, smooth microstate geometries that fluctuate non-trivially as a function of two variables. In our analysis of the BPS equations, we find features that fit very well with the constructive algorithm outlined in [32]: We see that the BPS system can accommodate the tilting and boosting of the parallel D1 and D5 branes, and the addition of momentum and angular momentum densities, in such a manner that one induces d1-d5 dipole densities by reorienting the D1-D5 charge densities and that all of this can be achieved in a way that makes the densities into functions of two variables. It was further argued in [32] that such solutions can be made smooth via the addition of an appropriately varying KKM configuration. This last step generically involves solving non-linear equations and, to date, the only known non-trivial solutions come from freezing fluctuations of the KKM configuration[1], which reduces the problem to a linear system. We use this strategy here and freeze the KKM configuration. In this sense, our new solutions may be thought of as a form of semi-rigid superstrata in that the KKM's are rigid but the charge densities fluctuate. As we will see, the fluctuations are limited by the rigidity of the Kaluza-Klein monopoles and are ultimately sourced by an arbitrary, but finite, number of functions of one variable. One the other hand, the one variable that appears in each such source can be a different linear combination of the two variables that parametrize the fluctuations and so our solutions are indeed doubly-fluctuating.

This chapter is organized as follows: In Section 7.2 we start with some important conventions that differ between this chapter and the original exposition of 6d BPS solutions in Section 3.3. In Section 7.2.1 we review the various background fields of minimal supergravity coupled to an anti-self-dual tensor multiplet in six dimensions, and

[1]There have been some interesting recent attempts to use string amplitude calculations to determine at least the perturbative form of generic fluctuating KKM's [138, 139, 140, 141].

in Section 7.2.2 we give the BPS equations in the conventions of (7.6). Next in Section 7.3 we specialize to a background that involves a collection of KKM's fibered over a four-dimensional Gibbons-Hawking (GH) base manifold. This is then recast as a torus fibration over a flat, $\mathbb{R}^3$ base with the circles of the torus fibered non-trivially as a set of doubled, independent KKM's. The general, doubly fluctuating solution will depend non-trivially on both directions on the torus. In Section 7.4 we reduce the BPS equations on this double KKM fibration and find a large class of fluctuating solutions that are governed by a single differential operator. This operator is particularly interesting because it is the six-dimensional Laplacian reduced on the "time coordinate." The appearance of such an operator is not surprising because BPS solutions necessarily have some form of time-translation invariance, but the new feature here is that the form of the six-dimensional supersymmetry means that this invariance is actually a null translation and the reduction of the Laplacian thus produces a degenerate operator. In Section 7.5 we examine the regularity of such solutions, first by working with a particularly simple example and then using this to infer the structure of the general solutions. We find that the rigidity of the doubled set of KKM monopoles restricts the fluctuations to be sourced by functions of one variable but that the particular variable in the density functions can slice the torus in different ways depending upon the KKM charges. Finally in Section 7.6 we discuss these results and open problems.

In addition, in Appendix C there are some technical generalizations of our BPS analysis.

7.2 The BPS solutions in six dimensions

The six-dimensional system we study is $\mathcal{N} = 1$ minimal supergravity coupled to one anti-self-dual tensor multiplet. This theory, upon trivial dimensional reduction, gives rise to $\mathcal{N}=2$, five-dimensional supergravity coupled to two vector multiplets and thus contains three independent electromagnetic fields. In the six-dimensional theory, the graviton multiplet contains a self-dual tensor field and so the entire bosonic sector consists of the graviton, the dilaton and an unconstrained 2-form gauge field with a 3-form field strength. The BPS equations of this theory were constructed in [70, 71] and in [72] it was shown that the BPS system could be dramatically simplified and that most of the equations could be reduced to a linear system. The 6d BPS system is reviewed in Section 3.3; however, in this chapter we will adopt some slightly different conventions.

Conventions

In this chapter we will adopt some conventions slightly different from the presentation of the 6d BPS system in Section 3.3. These are explained as follows:

We will proceed as usual with the 6d metric tensor in $(-+++++)$ signature,

$$ds_6^2 = -2H^{-1}\,(dv+\beta)\big(du+\omega+\frac{1}{2}\mathcal{F}(dv+\beta)\big) + H\,ds_4^2(\mathcal{B}), \tag{7.1}$$

where everything is independent of the coordinate u. The 4d base space $\mathcal{B}$ is given by

$$ds_4^2(\mathcal{B}) = h_{ij}(x^k;v)\,dx^i\,dx^j. \tag{7.2}$$

As before, we will find it useful to define a "restricted" exterior derivative $\tilde{d}$ on $\mathcal{B}$ that acts only on the x^i and treats v as a constant:

$$\tilde{d}(\varphi_I\,dx^I) = \left(\frac{\partial\varphi_I}{\partial x^i}\right)_v dx^i\wedge dx^I, \tag{7.3}$$

where I is a multi-index and $\varphi_I \, \mathrm{d}x^I$ is a generic p-form. Using this, we define a "covariant" exterior derivative $\mathcal{D}$ via

$$\mathcal{D}\varphi \equiv \tilde{\mathrm{d}}\varphi - \beta \wedge \partial_v \varphi, \tag{7.4}$$

such that the total exterior derivative d (on u-independent fields) can be written

$$\mathrm{d}\varphi = (\mathrm{d}v + \beta) \wedge \dot{\varphi} + \mathcal{D}\varphi, \tag{7.5}$$

where we have used a dot to denote the v-derivative $\dot{\phi} \equiv \partial_v \phi$.

In contrast to the conventions in Section 3.3, we will define

$$\begin{gathered} \tilde{\Theta}^{(1)} = 2\Theta^{(1)}, \qquad \tilde{\Theta}^{(2)} = 2\Theta^{(2)}, \qquad \tilde{\Theta}^{(3)} \equiv \mathcal{D}\beta, \\ Z_3 = -\mathcal{F}, \qquad k = \omega. \end{gathered} \tag{7.6}$$

The functions Z_1, Z_2 remain the same as in Section 3.3. The changes (7.6) merely represent another way to relate the 6d solutions to 5d solutions. Note however that we will omit writing the tildes; so the $\Theta^{(j)}$ of this chapter correspond to $2\Theta^{(j)}$ in Section 3.3 and Chapter 6.

On the metric (7.1) we can define a null-orthonormal frame

$$\mathrm{d}s_6^2 = -2e^+ \, e^- + \delta_{ij} \, e^i \, e^j, \tag{7.7}$$

by defining

$$e^+ \equiv H^{-1}\big(\mathrm{d}v + \beta\big), \qquad e^- \equiv \mathrm{d}u + \omega + \frac{1}{2}\mathcal{F}H \, e^+, \qquad e^i = H^{1/2} \, \tilde{e}^i, \tag{7.8}$$

where $\tilde{e}^i$, $i \in 1,2,3,4$ are an orthonormal frame on the 4d base $\mathcal{B}$. In contrast to the conventions of [70, 37], we will work using the more standard definition (like that of [142]) of the Hodge dual,

$$\underset{n}{\star}(e^{i_1} \wedge \ldots \wedge e^{i_p}) = \frac{1}{(n-p)!}\, \varepsilon^{i_1 \ldots i_p}{}_{j_1 \ldots j_{n-p}}\, (e^{j_1} \wedge \ldots \wedge e^{j_{n-p}}). \tag{7.9}$$

7.2.1 The 6d $\mathcal{N}=1$ background fields

As outlined in Section 3.3, the first step to finding 6d BPS solutions is to find three "almost hyper-Kähler structures" $J^{(A)}$ on the 4d base $\mathcal{B}$ that satisfy the quaternion algebra (3.74). Together with the 1-form β in (7.1), these $J^{(A)}$ must solve the equations

$$\tilde{\mathrm{d}} J^{(A)} = \partial_v(\beta \wedge J^{(A)}), \tag{7.10}$$

$$\mathcal{D}\beta = \underset{4}{\star}\mathcal{D}\beta, \qquad \text{or} \qquad \Theta^{(3)} = \underset{4}{\star}\Theta^{(3)}. \tag{7.11}$$

The self-dual condition on $\mathcal{D}\beta$ is nonlinear (in fact, it can be thought of as self-dual Yang-Mills based on the group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle as explained in Section 3.3), and this is the main source of difficulty in constructing superstratum solutions. β represents the Kaluza-Klein monopole charge of the solution, and this KKM charge is necessary for the solution to be smooth (as in Section 3.3.3). Hence a completely general superstratum solution must find a non-trivial solution to (7.11).

The tensor gauge field and the dilaton

The three-form tensor gauge field satisfies a Bianchi identity and has the equation of motion:

$$\mathrm{d}G = 0, \qquad \mathrm{d}\big(e^{2\sqrt{2}\phi}\, \underset{6}{\star} G\big) = 0, \tag{7.12}$$

However, supersymmetry imposes strong constraints on the form of G and these constraints can be significantly simplified by writing G and its dual in terms of electric and magnetic parts [72]:

$$G = \mathrm{d}\big[-\tfrac{1}{2}\, Z_1^{-1}\,(\mathrm{d}u + k) \wedge (\mathrm{d}v + \beta)\big] + \widehat{G}_1, \tag{7.13}$$

$$e^{2\sqrt{2}\phi} \underset{6}{\star} G = \mathrm{d}\big[\tfrac{1}{2}\, Z_2^{-1}\,(\mathrm{d}u + k) \wedge (\mathrm{d}v + \beta)\big] + \widehat{G}_2, \tag{7.14}$$

where

$$\widehat{G}_1 \equiv -\tfrac{1}{2} \underset{4}{\star} (\mathcal{D}Z_2 + Z_2\,\dot{\beta}) + \tfrac{1}{2}\,(dv + \beta) \wedge \Theta^{(1)}, \tag{7.15}$$

$$\widehat{G}_2 \equiv \tfrac{1}{2} \underset{4}{\star} (\mathcal{D}Z_1 + Z_1\,\dot{\beta}) - \tfrac{1}{2}\,(dv + \beta) \wedge \Theta^{(2)}. \tag{7.16}$$

Thus the flux is defined in terms of two electrostatic potentials, Z_j, and two two-forms, $\Theta^{(j)}$. Note that we have rescaled $\Theta^{(j)} \to \frac{1}{2}\Theta^{(j)}$ relative to the conventions of [72] and Section 3.3. This new choice of normalization is probably the simplest way to map the six-dimensional BPS equations onto the standard form of the five-dimensional BPS equations. The Bianchi identities and Maxwell equations (7.12) require the closure of the $\widehat{G}_j$, which shows that these quantities do indeed measure a conserved magnetic charge.

The two-form fluxes are required to be self-dual up to shifts by the two-form, ψ:

$$\underset{4}{\star}\Theta^{(1)} = \Theta^{(1)} - 4\,e^{-\sqrt{2}\phi}\,\psi = \Theta^{(1)} - 4\,Z_2\,\hat{\psi}, \tag{7.17}$$

$$\underset{4}{\star}\Theta^{(2)} = \Theta^{(2)} - 4\,e^{\sqrt{2}\phi}\,\psi = \Theta^{(2)} - 4\,Z_1\,\hat{\psi}, \tag{7.18}$$

and so if one defines

$$\widehat{\Theta}^{(1)} \equiv \Theta^{(1)} - 2\,Z_2\,\hat{\psi}, \qquad \widehat{\Theta}^{(2)} \equiv \Theta^{(2)} - 2\,Z_1\,\hat{\psi}, \tag{7.19}$$

then these two-forms are self-dual:

$$\widehat{\Theta}^{(j)} \equiv \underset{4}{\star}\,\widehat{\Theta}^{(j)}, \qquad j = 1, 2. \tag{7.20}$$

Supersymmetry also requires that the electric potentials be related to the warp factor and dilaton in a simple generalization of the "floating brane ansatz" [31]:

$$Z_1 \equiv H\, e^{\sqrt{2}\phi}, \qquad Z_2 \equiv H\, e^{-\sqrt{2}\phi}. \tag{7.21}$$

The form of G required by supersymmetry makes its self-dual part (in six dimensions) the same as the self-dual part of spin connection. This means that the supersymmetry variations become trivial and that the supersymmetry parameters are constant in the frames and coordinates introduced above:

$$\partial_M \epsilon = 0. \tag{7.22}$$

The angular momentum vector and the momentum potential

It is convenient to define:

$$\mathcal{L} \equiv \dot{k} + \tfrac{1}{2}\,\mathcal{F}\dot{\beta} - \tfrac{1}{2}\,\mathcal{D}\mathcal{F} = \tfrac{1}{2}\,\mathcal{D}Z_3 - \tfrac{1}{2}\,\dot{\beta}\,Z_3 + \dot{k}, \tag{7.23}$$

where we have introduced the momentum potential,

$$Z_3 \equiv -\mathcal{F}, \tag{7.24}$$

so as to make direct contact with the five-dimensional formulation [74, 75, 25].

The quantity, L, is gauge invariant under the transformation:

$$\mathcal{F} \to \mathcal{F} + 2\,\partial_v f, \qquad k \to k + Df, \tag{7.25}$$

for any function, $f(v, x^m)$. This transformation is induced by a coordinate change $u \to u + f(v, x^m)$ in the metric (7.1).

7.2.2 The BPS equations

Once one has constructed a background that satisfies the conditions stipulated in Section 7.2.1 then the remaining BPS equations are linear. According to the conventions defined in (7.6), the 6d BPS system of Section 3.3.2 can be written as follows. First, the electrostatic potentials are related to the magnetic fluxes via:

$$\mathcal{D}\Theta^{(2)} - \dot{\beta} \wedge \Theta^{(2)} = -\partial_v \big(\star_4 (\mathcal{D}Z_1 + Z_1\dot{\beta}) \big), \tag{7.26}$$

$$\mathcal{D}\Theta^{(1)} - \dot{\beta} \wedge \Theta^{(1)} = -\partial_v \big(\star_4 (\mathcal{D}Z_2 + Z_2\dot{\beta}) \big), \tag{7.27}$$

$$\mathcal{D} \star_4 \big(\mathcal{D}Z_1 + Z_1\dot{\beta} \big) = \Theta^{(2)} \wedge \mathcal{D}\beta, \tag{7.28}$$

$$\mathcal{D} \star_4 \big(\mathcal{D}Z_2 + Z_2\dot{\beta} \big) = \Theta^{(1)} \wedge \mathcal{D}\beta. \tag{7.29}$$

These constitute linear systems in $(Z_1, \Theta^{(2)})$ and $(Z_2, \Theta^{(1)})$ independently.

The last layer of the BPS equations then relate the angular momentum vector to the momentum potential, $Z_3 = -\mathcal{F}$:

$$\begin{aligned} \star_4 \mathcal{D} \star_4 \mathcal{L} - 2\langle \dot{\beta}, \mathcal{L} \rangle_{\mathcal{B}} = &-\tfrac{1}{2}\sqrt{Z_1 Z_2}\, h^{ij} \partial_v^2 \big(\sqrt{Z_1 Z_2}\, h_{ij} \big) \\ &- \tfrac{1}{4} \partial_v \big(\sqrt{Z_1 Z_2}\, h^{ij} \big)\, \partial_v \big(\sqrt{Z_1 Z_2}\, h_{ij} \big) \\ &- 2 Z_1 Z_2 \big(\partial_v \phi \big)^2 + \star_4 \Big(\tfrac{1}{2} \Theta^{(1)} \wedge \Theta^{(2)} - 2\hat{\psi} \wedge \mathcal{D}k \Big), \end{aligned} \tag{7.30}$$

and finally

$$\begin{aligned}\mathcal{D}k + \underset{4}{\star}\mathcal{D}k &= Z_1\Theta^{(1)} + Z_2\Theta^{(2)} - \mathcal{F}\,\mathcal{D}\beta - 4Z_1Z_2\hat{\psi}\\ &= Z_1\big(\Theta^{(1)} - 2Z_2\,\hat{\psi}\big) + Z_2\big(\Theta^{(2)} - 2Z_1\,\hat{\psi}\big) - \mathcal{F}\,\mathcal{D}\beta,\\ &= Z_1\widehat{\Theta}^{(1)} + Z_2\widehat{\Theta}^{(2)} + Z_3\Theta^{(3)}.\end{aligned} \tag{7.31}$$

Once again, this is a linear system for (Z_3, k).

Note that if the background and fields are all v-independent, then these BPS equations reduce to:

$$\tilde{\mathrm{d}}\Theta^{(1)} = \tilde{\mathrm{d}}\Theta^{(1)} = 0, \qquad \Theta^{(j)} = \underset{4}{\star}\Theta^{(j)}, \tag{7.32}$$

$$\nabla^2_{(4)} Z_1 = \underset{4}{\star}\big(\Theta^{(2)} \wedge \Theta^{(3)}\big), \tag{7.33}$$

$$\nabla^2_{(4)} Z_2 = \underset{4}{\star}\big(\Theta^{(1)} \wedge \Theta^{(3)}\big), \tag{7.34}$$

$$\nabla^2_{(4)} Z_3 = \underset{4}{\star}\big(\Theta^{(1)} \wedge \Theta^{(2)}\big), \tag{7.35}$$

and finally

$$\tilde{\mathrm{d}}k + \underset{4}{\star}\tilde{\mathrm{d}}k = \sum_{I=1}^{3} Z_I\Theta^{(I)}. \tag{7.36}$$

These are, of course, the canonically normalized five-dimensional BPS equations [74, 75, 25], discussed in Section 3.1.

7.3 BPS solutions with a Gibbons-Hawking base

We now simplify the BPS system by considering background geometries that are completely independent of v and in which one has a generic, multi centered Gibbons-Hawking (GH) metric on the base. We will assume that the vector field, β, defining the fibration is also v independent. While we will simplify the base and the fiber potential in this manner, we will allow the fluxes, warp factors and dilaton to be v dependent.

7.3.1 The background geometry

We start by taking

$$ds_4^2 = V^{-1} \left(d\psi + A\right)^2 + V\, d\vec{y} \cdot d\vec{y}, \tag{7.37}$$

where, on the flat $\mathbb{R}^3$ defined by the coordinates $\vec{y}$, one has:

$$\nabla^2 V = 0, \qquad \vec{\nabla} \times \vec{A} = \vec{\nabla} V. \tag{7.38}$$

We take V to have the form

$$V = h + \sum_{j=1}^{N} \frac{q_j}{|\vec{y} - \vec{y}^{(j)}|}, \tag{7.39}$$

for some fixed points, $\vec{y}^{(j)} \in \mathbb{R}^3$ and some charges, $q_j \in \mathbb{Z}$.

We use the following set of frames:

$$\tilde{e}^1 = V^{-1/2}\,(d\psi + A), \qquad \tilde{e}^{a+1} = V^{1/2}\, dy^a, \quad a = 1, 2, 3. \tag{7.40}$$

and define two associated sets of two-forms:

$$\Omega_{\pm}^{(a)} \equiv \tilde{e}^1 \wedge \tilde{e}^{a+1} \pm \tfrac{1}{2}\, \epsilon_{abc}\, \tilde{e}^{b+1} \wedge \tilde{e}^{c+1}, \qquad a = 1, 2, 3. \tag{7.41}$$

The two-forms $\Omega_{-}^{(a)}$ are anti-self-dual, harmonic and non-normalizable and they define the hyper-Kähler structure on the base. We therefore identify the $J^{(A)}$ in (3.74) and (7.10) with the $\Omega_{-}^{(A)}$. These are manifestly v-independent and thus, from (3.81), we have $\hat{\psi} \equiv 0$. Moreover, from (7.11) we see that $\Theta^{(3)} \equiv \mathcal{D}\beta$ must be harmonic.

The forms $\Omega_+^{(a)}$ are self-dual and can be used to construct harmonic fluxes that are dual to the two-cycles. In particular, we will take

$$\Theta^{(3)} \equiv -\sum_{a=1}^{3} \big(\partial_a \big(V^{-1} K^3\big)\big)\, \Omega_+^{(a)}, \tag{7.42}$$

where $\nabla^2 K^3 = 0$ on $\mathbb{R}^3$. The vector potential, β, is then given by:

$$\beta \equiv \frac{K^3}{V}\,(\mathrm{d}\psi + A) + \vec{\xi} \cdot \mathrm{d}\vec{y}, \tag{7.43}$$

where

$$\vec{\nabla} \times \vec{\xi} = -\vec{\nabla} K^3. \tag{7.44}$$

The one-form

$$\alpha \equiv V\,(\mathrm{d}v + \beta) = V\,(\mathrm{d}v + \xi) + K^3\,(\mathrm{d}\psi + A), \tag{7.45}$$

appears throughout the metric and flux and will play a significant role in that it has a manifest symmetry between the v and ψ fibers, and between the flux potential and the GH potential. This symmetry lies at the heart of the spectral flow transformations [107]. In particular, note that $\alpha \to -\alpha$ under the mapping

$$V \leftrightarrow K^3; \qquad v \to -\psi,\;\; \psi \to -v; \qquad A \to -\xi,\;\; \xi \to -A. \tag{7.46}$$

The negative signs are required so as to respect the relations (7.38) and (7.44) between the harmonic functions and the vector potentials. We will refer to the transformation (7.46) as *spectral inversion*.

If one rewrites the solution by making the interchange above, one must also send $u \to -u$ so as to preserve the terms of the form $\alpha\,\mathrm{d}u$ in the metric and the electric potential terms in the flux. One can then rewrite the entire supersymmetric form in terms of functions $\widetilde{Z}_I$, $\tilde{\mu}$ and a one form $\tilde{\omega}$. A straightforward calculation akin to that of [107] shows that

$$\widetilde{Z}_i = \frac{V}{K^3}\, Z_i, \quad i = 1,2; \qquad \widetilde{Z}_3 = \frac{K^3}{V}\, Z_3 + \frac{Z_1 Z_2}{K^3} - 2\,\mu,$$
$$\tilde{k} = -k, \qquad \tilde{\mu} = -\frac{V}{K^3}\,\mu + \frac{Z_1 Z_2 V}{(K^3)^2}, \qquad \tilde{\omega} = -\omega. \tag{7.47}$$

Under this transformation, the magnetic fluxes, $\Theta^{(j)}$ are also mapped to $\widetilde{\Theta}^{(j)} = -\Theta^{(j)}$ because of the flip in the sign of α.

The goal is to use this symmetry between two fibers to generate new classes of solutions and to formulate the theory in such a manner as to make this symmetry more apparent in BPS conditions.

7.3.2 The simplified BPS equations

With our choice of background geometry, the BPS equations simplify significantly. The equations for the fluxes and potentials reduce to

$$\mathcal{D}\Theta^{(2)} = -\underset{4}{\star}\mathcal{D}\dot{Z}_1, \qquad \mathcal{D}\underset{4}{\star}\mathcal{D}Z_1 = \Theta^{(2)} \wedge \Theta^{(3)}, \tag{7.48}$$

and

$$\mathcal{D}\Theta^{(1)} = -\underset{4}{\star}\mathcal{D}\dot{Z}_2, \qquad \mathcal{D}\underset{4}{\star}\mathcal{D}Z_2 = \Theta^{(1)} \wedge \Theta^{(3)}. \tag{7.49}$$

The equations for k and Z_3 become:

$$\underset{4}{\star}\mathcal{D}\underset{4}{\star}(\mathcal{D}Z_3 + 2\,\dot{k}) = \underset{4}{\star}\left(\Theta^{(1)} \wedge \Theta^{(2)}\right) - 2\left[Z_1\partial_v^2 Z_2 + Z_2\partial_v^2 Z_1 + (\partial_v Z_1)(\partial_v Z_2)\right], \tag{7.50}$$

$$\mathcal{D}k + \underset{4}{\star}\mathcal{D}k = \sum_{I=1}^{3} Z_I\, \Theta^{(I)}. \tag{7.51}$$

7.3.3 The five-dimensional solutions

The "classic" solutions

To motivate the construction of the six-dimensional solutions, it is useful to begin by recalling the well-known form of the five-dimensional BPS solutions as described in Section 3.1. These solutions are independent of the GH fiber coordinate, ψ [106, 74, 75, 129, 143].

The fluxes, $\Theta^{(j)}$ are harmonic and are given by expressions of the form (7.42):

$$\Theta^{(J)} = -\sum_{a=1}^{3} \big(\partial_a\big(V^{-1}\, K^J\big)\big)\, \Omega_+^{(a)}, \qquad J = 1, 2, 3, \tag{7.52}$$

where $\nabla^2 K^j = 0$ on $\mathbb{R}^3$.

The potentials, Z_I, are given by

$$Z_I = \frac{K^J\, K^K}{V} + L_I, \qquad I = 1, 2, 3, \tag{7.53}$$

where $\{I, J, K\} = \{1, 2, 3\}$ are all distinct and where $\nabla^2 L_I = 0$ on $\mathbb{R}^3$. The angular momentum vector has the form:

$$k = \mu(\mathrm{d}\psi + A) + \vec{\omega}\cdot \mathrm{d}\vec{y}, \tag{7.54}$$

with

$$\mu = \frac{K^1\, K^2\, K^3}{V^2} + \frac{1}{2}\sum_{I=1}^{3} \frac{K^I\, L_I}{V} + M, \tag{7.55}$$

and $\nabla^2 M = 0$ on $\mathbb{R}^3$. The angular momentum on $\mathbb{R}^3$ is then given by

$$\vec{\nabla} \times \vec{\omega} = V\vec{\nabla}M - M\vec{\nabla}V + \frac{1}{2}\sum_{I=1}^{3}(K^I\vec{\nabla}L_I - L_I\vec{\nabla}K^I), \tag{7.56}$$

and (7.55) guarantees the integrability of this equation for $\vec{\omega}$.

The fact that these solutions are both ψ and v independent means that their form must be invariant under spectral inversion (7.46). Indeed, the transformation (7.47) can be rewritten as:

$$\begin{aligned} \widetilde{V} =& K^3, \quad \widetilde{K}^3 = V, \quad \widetilde{K}^1 = L_2, \quad \widetilde{K}^2 = L_1, \\ \widetilde{L}_1 =& K^2, \quad \widetilde{L}_2 = K^1, \quad \widetilde{L}_3 = -2\,M, \quad \widetilde{M} = -\tfrac{1}{2}\,L_3. \end{aligned} \tag{7.57}$$

More general five-dimensional solutions

The solutions above are independent of both v and ψ and since the most general, five-dimensional BPS solutions have to satisfy (7.32)–(7.36), it is thus natural to ask about generalizations that are still independent of v but depend upon ψ.

If the base metric, $\mathrm{d}s_4^2$, is smooth and Euclidean, it is easy to see that (7.42) represents the most general smooth solution to (7.32). Specifically, the equations, (7.32), imply that the $\Theta^{(j)}$ are harmonic and the possible choices for K^J given by:

$$K^J = k_0^J + \sum_{i=1}^{N} \frac{k_i^J}{|\vec{y} - \vec{y}^{(i)}|}, \tag{7.58}$$

for some parameters, k_i^J, form a basis for the harmonic forms. Thus, in five-dimensions, the $\Theta^{(J)}$ are necessarily ψ-independent.

One can see this more explicitly by taking $\Theta^{(j)}$ and subtracting its harmonic part, ${\Theta^{(J)}}_{harm}$ to yield $\Theta'^{(j)} \equiv \Theta^{(j)} - {\Theta^{(J)}}_{harm}$. This is necessarily exact and so $\Theta'^{(j)} \wedge \Theta'^{(j)}$ (no sum on j) is also exact. Hence

$$0 = \int_{\mathcal{B}} \Theta'^{(j)} \wedge \Theta'^{(j)} = \int_{\mathcal{B}} \Theta'^{(j)} \wedge \underset{4}{\star} \Theta'^{(j)}. \tag{7.59}$$

However, the last integrand is necessarily non-negative and so one must have $\Theta'^{(j)} \equiv 0$.

There is, however, a gap in this argument for general BPS backgrounds: If the base space is ambipolar [144, 74, 75, 25]. then the metric, ds_4^2, is singular and so the Hodge decomposition theorem no longer applies. It is therefore quite possible that in ambipolar bases the Maxwell fields, $\Theta^{(j)}$, might be able to have a ψ-dependence. This dependence would, however, have to be sourced in some manner associated with the critical surfaces where V vanishes. We will, however, not pursue this possibility here.

The simplest way to generate five-dimensional solutions that depend upon ψ was discussed in [99]. These solutions are important because they represent an infinite family of smooth microstate geometries in six-dimensions and thus can be used to generate smooth microstate geometries in five dimensions by spectral flow [107]. These solutions start with the fluxes exactly as in (7.52) and (7.58) and then introduce the ψ-dependence in the next layer of BPS equations by letting the functions, L_I, in (7.53) depend upon ψ. Equations (7.33)–(7.35) then imply that the L_I must be harmonic in four dimensions:

$$\nabla^2_{(4)} L_I = 0. \tag{7.60}$$

The last BPS equation, (7.36), can now be written as

$$(\mu \vec{\mathfrak{D}} V - V \vec{\mathfrak{D}} \mu) + \vec{\mathfrak{D}} \times \vec{\omega} + V \partial_\psi \vec{\omega} = -V \sum_{I=1}^{3} Z_I \vec{\nabla}\big(V^{-1} K^I\big), \tag{7.61}$$

where

$$\vec{\mathfrak{D}} \equiv \vec{\nabla} - \vec{A}\,\partial_\psi. \tag{7.62}$$

The BPS equation, (7.36), has a gauge invariance: $k \to k + df$ and this reduces to:

$$\mu \to \mu + \partial_\psi f, \qquad \vec{\omega} \to \vec{\omega} + \vec{\mathfrak{D}} f. \tag{7.63}$$

It is simplest to use a Lorentz gauge-fixing condition, $\mathrm{d} \star_4 k = 0$, which reduces to

$$V^2\,\partial_\psi \mu + \vec{\mathfrak{D}} \cdot \vec{\omega} = 0. \tag{7.64}$$

The four-dimensional Laplacian can be written:

$$\nabla^2_{(4)} F = V^{-1}\big[V^2\,\partial_\psi^2 F + \vec{\mathfrak{D}} \cdot \vec{\mathfrak{D}} F\big]. \tag{7.65}$$

Now take the covariant divergence, using $\vec{\mathfrak{D}}$, of (7.61) and use the Lorentz gauge choice, and one obtains:

$$V^2\,\nabla^2_{(4)} \mu = \vec{\mathfrak{D}} \cdot \Big(V \sum_{I=1}^{3} Z_I\, \vec{\mathfrak{D}}\big(V^{-1} K^I\big)\Big). \tag{7.66}$$

Remarkably enough, this equation is still solved by:

$$\mu = V^{-2} K^1 K^2 K^3 + \tfrac{1}{2} \sum_{I=1}^{3} V^{-1} K_I L_I + M, \tag{7.67}$$

where, once again, M is a harmonic function in four dimensions. Finally, we can use this solution back in (7.61) to simplify the right-hand side to obtain:

$$\vec{\mathfrak{D}} \times \vec{\omega} + V \partial_\psi \vec{\omega} = V \vec{\mathfrak{D}} M - M \vec{\mathfrak{D}} V + \frac{1}{2} \sum_{I=1}^{3} \big(K^I \vec{\mathfrak{D}} L_I - L_I \vec{\mathfrak{D}} K^I\big). \tag{7.68}$$

Once again one sees the emergence of the familiar symplectic form on the right-hand side of this equation. One can also verify that the covariant divergence (using $\vec{\mathfrak{D}}$) generates an identity that is trivially satisfied as a consequence of (7.38), (7.64), (7.67) and

$$\nabla^2_{(4)} L_I = \nabla^2_{(4)} M = 0. \tag{7.69}$$

Spectral inversion revisited

One can now use the spectral interchange symmetry generated by (7.46) and realized in (7.47) create convert the ψ-dependent solutions into new v-dependent solutions of the BPS equations. In particular, the generalization of (7.47):

$$K^3\,\widetilde{Z}_1 = V\,Z_1 = K^3\,K^2 + V\,L_1, \qquad K^3\,\widetilde{Z}_2 = V\,Z_2 = K^3\,K^1 + V\,L_2, \tag{7.70}$$

now means that the new solutions have v-dependent fluxes governed by:

$$\widetilde{K}^1(v,\vec{y}) = \left. L_2(\psi,\vec{y})\right|_{\psi=-v}, \qquad \widetilde{K}^2(v,\vec{y}) = \left. L_1(\psi,\vec{y})\right|_{\psi=-v}. \tag{7.71}$$

These solutions will then obey the more general BPS equations (7.48)–(7.51). It is also important to note that because (7.46) is simply induced by a coordinate change, the new solution will also be smooth.

One can now hybridize this observation with the strategy of Section 7.3.3. That is, one can start by using the new v-dependent fluxes and then, once again, allow the solutions to develop a ψ-dependence allowing the L functions to depend upon ψ as well as $\vec{y}$. One then solves the remaining BPS equations. Rather than pursue this course here we use this to motivate a significantly more general class of solutions that will be developed in the next section.

7.4 Families of doubly-fluctuating solutions

Based on the observations in the previous section, it is relatively easy to formulate an ansatz that will capture at least all the solutions proposed in Section 7.3.3. We will see that it captures a far more general class of solutions.

We will keep the geometry exactly as in Section 7.3: A GH base with the six-dimensional geometry being completely independent of v. The background therefore has the spectral inversion symmetry (7.46). We also introduce a generalization of the operator (7.62):

$$\vec{\mathfrak{D}} \equiv \vec{\nabla} - \vec{A}\,\partial_\psi - \vec{\xi}\,\partial_v, \tag{7.72}$$

and define the second order operator:

$$\mathcal{L}F \equiv \vec{\mathfrak{D}} \cdot \vec{\mathfrak{D}}\,F + (V\,\partial_\psi - K^3\,\partial_v)^2 F. \tag{7.73}$$

Both of these operators are invariant under (7.46). Also note that

$$\mathcal{L}F = V \underset{4}{\star}\mathcal{D}\underset{4}{\star}\mathcal{D}F, \tag{7.74}$$

where $\mathcal{D}$ is the operator defined in (7.4).

The appearance of the operator (7.73) is very easy to understand because it is essentially the six-dimensional Laplacian for the metric (7.1) acting on u-independent functions. That is, one can easily verify that

$$\nabla^2_{(6)} F(v,\psi,\vec{y}) = -\frac{1}{H\,V}\,\mathcal{L}F(v,\psi,\vec{y}). \tag{7.75}$$

7.4.1 The first layers of BPS equations

We generalize the expression for fluxes, (7.52), to

$$\Theta^{(j)} = -\sum_{a=1}^{3} \big(\mathfrak{D}_a\big(V^{-1}\,K^j\big)\big)\,\Omega_+^{(a)}, \qquad j=1,2, \tag{7.76}$$

where, in principle, K^j is a function of v, ψ and $\vec{y}$. We also use, without loss of generality, the ansatz

$$Z_1 = \frac{K^3\,K^2}{V} + L_1, \qquad Z_2 = \frac{K^3\,K^1}{V} + L_2, \tag{7.77}$$

where the functions, L_j, are, as yet, general functions of v, ψ and $\vec{y}$. For completeness, the Appendix C contains the analysis of the BPS equations for the most general form of the $\Theta^{(j)}$.

If one substitutes this ansatz into the first BPS equations (7.48) and (7.49) one obtains the following linear equations for K^j and L_j, $j=1,2$:

$$\mathcal{L}K^j = \mathcal{L}L_j = 0, \qquad j=1,2, \tag{7.78}$$

$$\partial_\psi K^1 + \partial_v L_2 = 0, \qquad \partial_\psi K^2 + \partial_v L_1 = 0, \tag{7.79}$$

where $\mathcal{L}$ is defined in (7.73).

The general solution to the constraints (7.79) is simply:

$$K^j = \hat{K}^j(v,\vec{y}) + \partial_v H_j(v,\psi,\vec{y}), \qquad j=1,2, \tag{7.80}$$

$$L_1 = \hat{L}_1(\psi,\vec{y}) - \partial_\psi H_2(v,\psi,\vec{y}), \qquad L_2 = \hat{L}_2(\psi,\vec{y}) - \partial_\psi H_1(v,\psi,\vec{y}), \tag{7.81}$$

where

$$\mathcal{L}\hat{K}^j = \mathcal{L}\hat{L}_j = \mathcal{L}H_j = 0. \tag{7.82}$$

Note that the parts of the solution that involve $\hat{K}^j$ and $\hat{L}_j$ are actually redundant because they can be absorbed into various zero-mode parts of the H_j. We have specifically exhibited $\hat{K}^j$ and $\hat{L}_j$ here because they represent precisely what would have been generated by the procedure outlined in Section 7.3.3. This also makes it evident that the functions, H_j, represent something completely new and much more general. We will discuss this more below.

7.4.2 The last layer of BPS equations

Without loss of generality, we can, once again, use the ansätze:

$$Z_3 = \frac{K^1\,K^2}{V} + L_3, \qquad\qquad k = \mu(\mathrm{d}\psi + A) + \vec{\omega}\cdot\mathrm{d}\vec{y}, \tag{7.83}$$

with

$$\mu = \frac{K^1\,K^2\,K^3}{V^2} + \frac{1}{2}\sum_{I=1}^{3}\frac{K^I\,L_I}{V} + M, \tag{7.84}$$

where L_3 and M are general functions that depend upon $(v, \psi, \vec{y})$.

Equation (7.51) for the angular momentum vector reduces to a straightforward generalization of (7.68):

$$\vec{\mathfrak{D}} \times \vec{\omega} + (V\partial_\psi - K^3\partial_v)\vec{\omega} = V\vec{\mathfrak{D}}M - M\vec{\mathfrak{D}}V + \frac{1}{2}\sum_{I=1}^{3}\left(K^I\vec{\mathfrak{D}}L_I - L_I\vec{\mathfrak{D}}K^I\right). \tag{7.85}$$

To simplify this and (7.50) we need to make a suitable generalization of the gauge choice (7.64). To express this gauge, it is useful to introduce the spectral inversion of the function, μ, under the effect of (7.57):

$$\tilde{\mu} \equiv \frac{L_1\,L_2\,V}{(K^3)^2} + \frac{1}{2}\left(\frac{K^1\,L_1 + K^2\,L_2}{K^3}\right) - \frac{V\,M}{K^3} - \frac{1}{2}\,L_3, \tag{7.86}$$

and define

$$\begin{aligned}\Phi &\equiv V^2\,\partial_\psi\mu + (K^3)^2\,\partial_v\tilde{\mu} \\ &= K^3\partial_\psi(K^1K^2) + V\partial_v(L_1L_2) + \tfrac{1}{2}\,(V\partial_\psi + K^3\partial_v)(K^1L_1 + K^2L_2) \\ &\quad + (V\partial_\psi - K^3\partial_v)(\tfrac{1}{2}\,K^3L_3 + VM).\end{aligned} \tag{7.87}$$

The gauge choice that simplifies all the equations is to take

$$\vec{\mathfrak{D}}\cdot\vec{\omega} + \Phi = 0, \tag{7.88}$$

but we will not impose this and we will retain Φ in our equations.

If one uses the equations (7.78) and (7.79) one finds that (7.50) collapses to

$$\mathcal{L}L_3 = -2\,\partial_v\big[\vec{\mathfrak{D}}\cdot\vec{\omega} + \Phi\big], \tag{7.89}$$

while the covariant divergence of (7.85) becomes:

$$\tfrac{1}{2}\,K^3\,\mathcal{L}L_3 + V\,\mathcal{L}M = (V\partial_\psi - K^3\partial_v)\big[\vec{\mathfrak{D}}\cdot\vec{\omega} + \Phi\big]. \tag{7.90}$$

Combining this with (7.89) yields

$$\mathcal{L}M = \partial_\psi\big[\vec{\mathfrak{D}}\cdot\vec{\omega} + \Phi\big]. \tag{7.91}$$

Therefore, with the gauge choice (7.88), the functions K^j and L_j are fixed by (7.80), (7.81) and (7.82) and the remaining parts of the solution are given by (7.83), (7.84) and (7.85) where L_3 and M are general functions of $(v, \psi, \vec{y})$ satisfying

$$\mathcal{L}L_3 = \mathcal{L}M = 0. \tag{7.92}$$

These conditions then guarantee the integrability of (7.85) for $\vec{\omega}$.

7.4.3 The metric and its regularity

One can write the metric in a more symmetric form that is manifestly invariant under spectral inversion. First recall that in (7.45) we defined the one form:

$$\alpha \equiv V\,(dv+\beta) = V\,(dv+\xi) + K^3(d\psi + A). \tag{7.93}$$

Now define the functions:

$$\hat{H} \equiv \sqrt{(VZ_1)(VZ_2)}, \qquad \mathcal{Q} \equiv Z_1 Z_2 Z_3 V - \mu^2 V^2, \tag{7.94}$$

and the 1-form

$$\gamma \equiv (K^3)^2\,\tilde{\mu}\,(d\psi + A) - V^2\,\mu\,(dv+\xi). \tag{7.95}$$

All of these quantities are invariant under the spectral inversion transformation (7.57) and the quantity $\mathcal{Q}$ is simply the $E_{7(7)}$ quartic invariant constructed out of the functions V, K^I, L_I and M [25]:

$$\begin{aligned} \mathcal{Q} = &- V^2\,M^2 - 2\,K^1\,K^2\,K^3\,M - M\,V\sum_{I=1}^{3} K^I\,L_I - \frac{1}{4}\sum_{I=1}^{3}(K^I L_I)^2 \\ &+ V\,L_1\,L_2\,L_3 + (K^1K^2L_1L_2 + K^1K^3L_1L_3 + K^2K^3L_2L_3). \end{aligned} \tag{7.96}$$

The six dimensional metric, (7.1) can now be written as:

$$ds_6^2 = -2\hat{H}^{-1}\,(\mathrm{d}u+\omega)\,\alpha + \hat{H}^{-3}\left[\mathcal{Q}\,\alpha^2 + \gamma^2\right] + \hat{H}\,\mathrm{d}\vec{y}\cdot\mathrm{d}\vec{y}. \tag{7.97}$$

The standard, bubbled microstate geometries [74, 75, 25, 107, 104, 143] (discussed in Section 3.1) allow singularities at points in the $\mathbb{R}^3$ defined by $\vec{y}$. Indeed, near such a singular point, P, one has $V \sim \frac{q_p}{r_p}$ while one also requires that the Z_I are finite as $r_p \to 0$ and the bubble equations require that $\mu(r_p) = 0$. This means that, as $r_p \to 0$, one has

$$\hat{H}, \mathcal{Q}, \alpha, \gamma \sim \mathcal{O}\big(r_p^{-1}\big). \tag{7.98}$$

Since supertubes can be mapped onto microstate geometries by spectral flow [107], it follows that supertube solutions have identical asymptotics to that of (7.98). One can also check this directly. It then follows that in all such configurations the metric (7.97) remains smooth (up to orbifold points). The apparent singularity on the $\mathbb{R}^3$ base can be resolved by the standard coordinate change: $r_p = \frac{1}{4}R^2$.

For supertubes one can also get smooth solutions with slightly more singular behavior than that allowed by (7.98) [2]. Indeed, $\mathcal{Q}$ and γ can have double poles that cancel in (7.97) so that the metric remains regular. The simplest, and perhaps only, examples of this are coordinate transformations of solutions that satisfy (7.98) but in which constants in $\mathcal{F} = -Z_3$ are removed via a re-definition of u. This can, in turn, move singular terms between the functions that define the solution. To fix this ambiguity we will require that our asymptotically flat solutions have $Z_3 \to c_3$ at infinity, where $c_3 \neq 0$ is a constant. Upon completing the square in (7.1) one can then rewrite the six-dimensional metric as:

$$ds_6^2 = -(HZ_3)^{-1}(\mathrm{d}u + k)^2 + Z_3 H^{-1}\big[\mathrm{d}v + \beta - Z_3^{-1}(\mathrm{d}u + k)\big]^2 + H\,\mathrm{d}s_4^2(\mathcal{B}). \tag{7.99}$$

Since Z_3 and H go to non-zero constants at infinity, this means that in Kaluza-Klein reduction on the v circle to five dimensions, u becomes the time coordinate. It is in this description that supertube solutions are most easily related to five-dimensional bubbled

[2] We are very grateful to Stefano Giusto for pointing this out and helping to clarify this point.

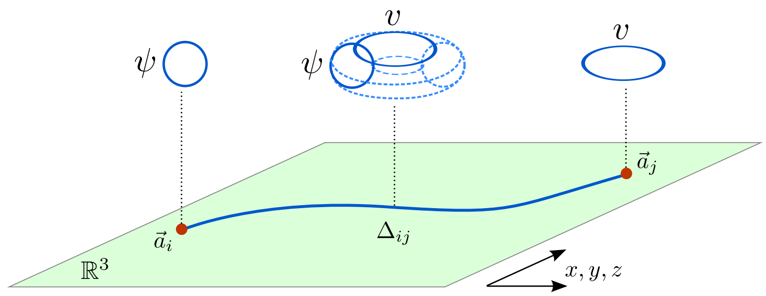

Figure 7.1: *The 5d base of the metric* (7.99) *has the structure of a T^2 fibered over $\mathbb{R}^3$. At $\vec{a}_i$, the function K^3 is singular, and thus the v fiber pinches off there, leaving the ψ fiber a finite size. At $\vec{a}_j$, the function V is singular, and this the ψ fiber pinches off, leaving the v fiber a finite size. If both V, K^3 are singular at some $\vec{a}_k$, then some linear combination of ψ, v shrinks, while the orthogonal linear combination stays finite. The fiber over a curve in the base between $\vec{a}_i$ and $\vec{a}_j$ gives a homology 3-sphere Δ_{ij}.*

solutions and both sets of solutions obey (7.98). Henceforth, we will assume that our solutions, including supertubes, obey (7.98).

Finally, it is important to note that near a singular point of the harmonic functions, the metric on the (v, ψ) torus has a pre-factor of $\hat{H}^{-3} \sim r_p^3$ and, given the asymptotic behavior in (7.98), this will pinch off the circle defined by $r_p \gamma$ while the circle defined by $r_p \alpha$ will remain finite. For the standard five-dimensional bubbled microstate geometries this corresponds to pinching off the ψ-circle while the v-circle remains finite and for the standard supertube [107] the v-circle pinches off and the ψ-circle remains finite. This fiber structure of the 5d base of (7.99) is illustrated in Figure 7.1.

7.4.4 The physical structure underlying the BPS system

In conjecturing the existence of a superstratum [32], one of the crucial first steps was to argue that this D1-D5-P system would retain the same supersymmetries if the D1-D5 system were "tilted and boosted" so as to lay it out along an arbitrary closed profile in $(v, \psi, \vec{y})$. These configurations were then to be smeared along v so as to make a supersheet and it was proposed that if this configuration also had a KKM dipole charge that was arranged in the proper manner then the whole BPS configuration would remain smooth.

In the simplest, standard D1-D5-P configuration, the common direction of the D1-D5 system lies along the circle defined by v while the momentum modes excite oscillations in the transverse four-manifold described by $(\psi, \vec{y})$. If one tilts and boosts this configuration in the manner described in [32] then some of the D1 and D5 electric charges are tilted into $d1$ and $d5$ magnetic dipole charges and some of the momentum, P, is tilted into angular momentum around the profile.

Now recall that the functions Z_1 and Z_2 encode the charge densities associated with the D1 and D5 branes respectively and so the pairs, (L_1, K^2) and (L_2, K^1), encode the $(electric, magnetic)$ charge densities of the D1 and D5 branes respectively. The new feature of the solutions presented here are the functions, $H_j(v, \psi, \vec{y})$, appearing in (7.80) and (7.81). These functions tie the D1-d1 and D5-d5 charges together in a manner that reflects precisely the tilting process described in [32] and the fact that this arises directly from the BPS conditions provides further support for the arguments given in [32]. It should, of course, be remembered that we have frozen the background geometry so that it is independent of (v, ψ) and so the shape of the profile that we are trying to generate does not fluctuate directly. Instead, we are fluctuating the charge densities within the (v, ψ)-independent profile and it is these densities that are being tilted and boosted. The effect of these fluctuating densities back-react in the full metric and will thus change the physical size of the configuration as a function of (v, ψ) and so the shape will indeed ultimately fluctuate.

The fact that the functions, $H_j(v, \psi, \vec{y})$, are general solutions of the reduced Laplacian (7.75) is a very natural generalization of the harmonic charge sources that are part of the five-dimensional system. Indeed, one should recall that if there is a supersymmetry, ϵ, then one can construct the vector[3]:

$$T^\mu \equiv \bar{\epsilon}\gamma^\mu\epsilon \tag{7.100}$$

and this will generically be a time-like, or null Killing vector. In five dimensions this is time-like and hence the BPS solutions have a time-translation invariance. However, in six dimensions, T^μ is a null vector [70] and this accounts for the u-independence of the solution. The fact that T^μ is null means that the hypersurfaces of constant u are null and the induced metric on these surfaces is degenerate.

This accounts for the somewhat degenerate form of (7.73): While it involves derivatives with respect to small five spatial variables, $(v, \psi, \vec{y})$, it is written in a diagonal form that is only the sum of four squares. This fact will have a significant impact on the space of solutions. Indeed, one can imagine trying to find the Green functions for $\mathcal{L}$ by following the approach of [99] and integrating out the time direction in the propagator of the full Laplacian. In six-dimensions, this will involve integrating out the null coordinate u in the Green function for the six-dimensional Laplacian. This is not a well-defined procedure in the six-dimensional theory because the null initial-value problem may not be well-posed and one will potentially be integrating along singularities corresponding to propagating data.

7.5 Multi-centered configurations

Multi-centered are those solutions that start by taking a multi-centered geometry in which V and K^3 have the form (7.38) and (7.58). Initially, we will not make any

[3]The supersymmetry may have internal indices and this expression may involve some contractions over these indices.

assumptions about the form of the other functions, K^j, L_j, $(j = 1, 2)$, L_3 and M. We start by analyzing, in detail, one of the simplest, non-trivial microstate geometries: the two-centered solution and then use this to describe what we believe will be the structure of a generic multi-centered solution.

7.5.1 The general two-centered configuration and $AdS_3 \times S^3$

To define the general two-centered solution it is simplest to introduce cylindrical polar coordinates, (z, ρ, ϕ), on the $\mathbb{R}^3$ base and define

$$r_\pm \equiv \sqrt{\rho^2 + (z \mp a)^2}. \tag{7.101}$$

The key geometric elements are then given by:

$$V = \frac{q_+}{r_+} + \frac{q_-}{r_-}, \qquad A = \Big(q_+ \frac{(z-a)}{r_+} + q_- \frac{(z+a)}{r_-}\Big)\, \mathrm{d}\phi, \tag{7.102}$$

$$K^3 = \frac{k_+}{r_+} + \frac{k_-}{r_-}, \qquad \xi = -\Big(k_+ \frac{(z-a)}{r_+} + k_- \frac{(z+a)}{r_-}\Big)\, \mathrm{d}\phi, \tag{7.103}$$

The two-centered system is greatly simplified by working with bipolar coordinates:

$$\rho = a\, \sinh\xi\, \sin\theta, \qquad z = a\, \cosh\xi\, \cos\theta, \tag{7.104}$$

and then one has:

$$r_\pm = a\,(\cosh\xi \mp \cos\theta). \tag{7.105}$$

One can then reparametrize the T^2 of (v, ψ) by introducing the angles, χ and η, defined by:

$$\begin{aligned} \partial_\chi &= -(q_+ + q_-)\,\partial_\psi + (k_+ + k_-)\,\partial_v, \\ \partial_\eta &= (q_+ - q_-)\,\partial_\psi - (k_+ - k_-)\,\partial_v, \end{aligned} \tag{7.106}$$

which is equivalent to

$$\chi = \frac{1}{2\,\Delta}\big((k_+ - k_-)\,\psi + (q_+ - q_-)\,v\big),$$
$$\eta = \frac{1}{2\,\Delta}\big((k_+ + k_-)\,\psi + (q_+ + q_-)\,v\big), \tag{7.107}$$

where $\Delta \equiv (q_+k_- - q_-k_+)$.

The operator $\mathcal{L}$ then has the relatively simple form:

$$\mathcal{L}F = a^{-2}(\sinh^2\xi + \sin^2\theta)^{-2}\Bigg[\frac{1}{\sinh\xi}\partial_\xi\big(\sinh\xi\,\partial_\xi F\big) + \frac{1}{\sinh^2\xi}(\partial_\eta + \cosh\xi\,\partial_\phi)^2 F + \frac{1}{\sin\theta}\partial_\theta\big(\sin\theta\,\partial_\theta F\big) + \frac{1}{\sin^2\theta}(\partial_\chi + \cos\theta\,\partial_\phi)^2 F\Bigg]. \tag{7.108}$$

The manifolds, S^3 and AdS_3, can be thought of as a unit sphere in $\mathbb{C}^2$ and as a unit hyperboloid in $\mathbb{C}^{1,1}$ respectively:

$$\mathrm{d}s^2_{S^3} = |dw_1|^2 + |dw_2|^2, \qquad |w_1|^2 + |w_2|^2 = 1; \tag{7.109}$$

$$\mathrm{d}s^2_{AdS_3} = |dz_1|^2 - |dz_2|^2, \qquad |z_1|^2 - |z_2|^2 = 1. \tag{7.110}$$

Parametrizing these surfaces in standard fashion:

$$w_1 = \cos\frac{\theta}{2}\,e^{i(\chi-\phi)}, \qquad w_2 = \sin\frac{\theta}{2}\,e^{i(\chi+\phi)}, \tag{7.111}$$

$$z_1 = \cosh\frac{\xi}{2}\,e^{i(\eta-t)}, \qquad z_2 = \sinh\frac{\xi}{2}\,e^{i(\eta+t)}, \tag{7.112}$$

leads to the polar forms

$$\mathrm{d}s^2_{S^3} = \tfrac{1}{4}\Big(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2 + (\mathrm{d}\chi - \cos\theta\,\mathrm{d}\phi)^2\Big)$$
$$= \tfrac{1}{4}\,\mathrm{d}\theta^2 + \sin^2\frac{\theta}{2}\,(\mathrm{d}\chi + \mathrm{d}\phi)^2 + \cos^2\frac{\theta}{2}\,(\mathrm{d}\chi - \mathrm{d}\phi)^2; \tag{7.113}$$

$$
\begin{aligned}
\mathrm{d}s^2_{AdS_3} &= \tfrac{1}{4}\Big(\mathrm{d}\xi^2 + \sinh^2\xi\,\mathrm{d}\eta^2 - (\mathrm{d}t - \cosh\xi\,\mathrm{d}\eta)^2\Big) \\
&= \tfrac{1}{4}\,\mathrm{d}\xi^2 + \sinh^2\frac{\xi}{2}\,(\mathrm{d}\eta + \mathrm{d}t)^2 - \cosh^2\frac{\xi}{2}\,(\mathrm{d}\eta - \mathrm{d}t)^2. \qquad (7.114)
\end{aligned}
$$

The Laplacians on these spaces are $4\mathcal{L}_j$ where:

$$
\mathcal{L}_1\,F \equiv \frac{1}{\sinh\xi}\partial_\xi\big(\sinh\xi\,\partial_\xi F\big) + \frac{1}{\sinh^2\xi}(\partial_\eta + \cosh\xi\,\partial_t)^2 F - \partial_t^2 F, \qquad (7.115)
$$

$$
\mathcal{L}_2\,F \equiv \frac{1}{\sin\theta}\partial_\theta\big(\sin\theta\,\partial_\theta F\big) + \frac{1}{\sin^2\theta}(\partial_\chi + \cos\theta\,\partial_\phi)^2 F + \partial_\phi^2 F. \qquad (7.116)
$$

The important point is that the operator, $\mathcal{L}$, that defines the solutions of interest is simply given by:

$$
\mathcal{L} = (\mathcal{L}_1 + \mathcal{L}_2)\big|_{\phi=t}. \qquad (7.117)
$$

Thus, even though the final six-dimensional metric is not ultimately going to be $AdS_3 \times S^3$, the differential operator, $\mathcal{L}$, is precisely the Laplacian of $AdS_3 \times S^3$ acting on u-independent modes, where $u \equiv \phi - t$.

7.5.2 Fluctuations

Generalities

It is rather straightforward to see that there are no non-singular BPS fluctuations that fall off suitably rapidly at infinity on $AdS_3 \times S^3$. Suppose that $F(\xi,\theta,\xi,\eta,\phi)$ were such a fluctuating mode and observe that:

$$
\begin{aligned}
0 &= -\int_{AdS_3\times S^3} a^2(\sinh^2\xi + \sin^2\theta)^2 \sinh^2\xi\,(F\mathcal{L}\,F) \\
&= \int_{AdS_3\times S^3} \sin^2\theta\big[(\sinh\xi\,\partial_\xi F)^2 + (\partial_\eta F + \cosh\xi\,\partial_\phi F)^2\big] \\
&\qquad\qquad + \sinh^2\xi\big[(\sin\theta\,\partial_\theta F)^2 + (\partial_\chi F + \cos\theta\,\partial_\phi F)^2)\big]. \qquad (7.118)
\end{aligned}
$$

where we have assumed that F vanishes fast enough so that the boundary terms at infinity that behave as $\sim\ e^{2\xi} F \partial_\xi F$ can be discarded. All the terms in the integrand are positive definite and so they must all vanish. This means that F must be constant. Given the fact that the general operator, $\mathcal{L}$, in (7.73) is the sum of squares, even when V is ambipolar, one would expect a similar conclusion for a generic, multi-centered solution.

Thus fluctuating modes must either be non-normalizable or must have singularities. While there might be interesting solutions that involve the former, as we discussed in Section 7.4.3, earlier work shows there are huge families of smooth solutions in which the harmonic functions are sourced on surfaces of spatial co-dimension 3. The simplest of these has singular surfaces that are points in $\mathbb{R}^3$ and are swept out by (v, ψ). Thus we will further specialize our notion of multi-centered solutions to those in which the harmonic functions, K^j, L_j, L_3 and M have singularities of order $\mathcal{O}(|\vec{y} - \vec{y}^{(i)}|^{-1})$ at the points, $\vec{y}^{(i)}$, where V and K^3 are similarly singular. However, unlike V and K^3, the functions K^j, L_j, L_3 and M will be allowed to depend upon (v, ψ). As noted earlier, the effect of the singular behavior of the harmonic functions involves pinching off a direction in the (v, ψ) torus, thereby creating a topological cycle that can then support non-trivial, smooth cohomological fluxes.

We now examine perhaps the simplest example.

A specific example

We start with the solution in Section 7.5.1 with

$$q_- = k_+ = 1, \qquad q_+ = k_- = 0. \tag{7.119}$$

The base manifold, $\mathcal{B}$, is then simply flat $\mathbb{R}^4 = \mathbb{C}^2$ and the coordinate change relating this to the GH form is then

$$\zeta_1 = \sqrt{r_-}\,\cos(\tfrac{1}{2}\theta_-)\, e^{i(\psi+\phi)/2}, \qquad \zeta_2 = \sqrt{r_-}\,\sin(\tfrac{1}{2}\theta_-)\, e^{i(\psi-\phi)/2}, \tag{7.120}$$

where (r_-, θ_-, ϕ) are polar coordinates with r_- defined in (7.101). The Green function of the Laplacian $\mathbb{R}^4$ for a source located at $r_+ = 0$ is simply a constant multiple of Λ where:

$$\begin{aligned} \Lambda^{-1} =& \big|\zeta_1 - \sqrt{2a}\big|^2 + \big|\zeta_2\big|^2 \\ =& (r_- + 2a) - 2\sqrt{2ar_-}\,\cos(\tfrac{1}{2}\theta_-)\,\cos\tfrac{1}{2}(\psi+\phi). \end{aligned} \tag{7.121}$$

One can then integrate this against any Fourier mode $e^{im(\psi+\phi)/2}$ to get a fluctuating harmonic source. This is an elementary contour integral and it yields:

$$\begin{aligned} F_m^+ &\equiv \frac{1}{r_+}\left(\frac{\cos(\frac{1}{2}\theta)}{\cosh(\frac{1}{2}\xi)}\right)^{|m|} e^{im(\psi+\phi)/2} \\ &= \frac{1}{a\,(\cosh\xi - \cos\theta)}\left(\frac{\cos(\frac{1}{2}\theta)}{\cosh(\frac{1}{2}\xi)}\right)^{|m|} e^{-im(\chi+\eta-\phi)/2}. \end{aligned} \tag{7.122}$$

where we have written the results in terms of the coordinates (7.104) and the angles (χ, η) defined in (7.107). By construction, these are harmonic functions on the four-dimensional base and thus satisfy $\mathcal{L}F_m^+ = 0$, for all values of m.

One can similarly verify that

$$\begin{aligned} F_m^- &\equiv \frac{1}{r_-}\left(\frac{\sin(\frac{1}{2}\theta)}{\cosh(\frac{1}{2}\xi)}\right)^{|m|} e^{-im(v+\phi)/2} \\ &= \frac{1}{a\,(\cosh\xi + \cos\theta)}\left(\frac{\sin(\frac{1}{2}\theta)}{\cosh(\frac{1}{2}\xi)}\right)^{|m|} e^{-im(\chi-\eta+\phi)/2} \end{aligned} \tag{7.123}$$

also satisfy $\mathcal{L} F^-_m = 0$. However, these functions now have explicit mode dependence on v and thus go beyond the standard harmonic ansätze.

The functions, $F^\pm_m = 0$, have several very important properties. First, they are smooth except except a $\mathcal{O}(r_\pm^{-1})$ singularity as $r_\pm \to 0$. The zero-modes are precisely $F^\pm_0 = \frac{1}{r_\pm}$ and, for $m \neq 0$, one has:

$$F^+_m\big|_{\theta=\pi} = 0, \qquad F^-_m\big|_{\theta=0} = 0, \qquad m \neq 0. \tag{7.124}$$

Finally, the second expressions in (7.122) and (7.123) show that the $F^\pm_m$, are smooth as functions on $AdS_3 \times S_3$, independent of the choice (7.119). One can therefore immediately generalize our discussion by working in $AdS_3 \times S_3$ and dropping the condition (7.119) and leaving the parameters $q_\pm$ and $k_\pm$ completely generic. It should be noted that we have not made a careful discussion of the proper periodicities of the angles, χ and η and so one might be dealing with an orbifold of $AdS_3 \times S_3$. We will, however, continue to impose (7.119) so that we can easily relate our results to earlier work on bubbled geometries and supertubes.

One can now generate solutions by taking H_j, L_3 and M to have Fourier expansions:

$$\sum_{m=-\infty}^{\infty} (b^+_m \, F^+_m + b^-_m \, F^-_m), \tag{7.125}$$

where reality requires $b^\pm_{-m} = (b^\pm_m)^*$. The fact that all the functions only have singularities of order $r_\pm^{-1}$ means that one can create completely smooth geometries. At $r_+ = 0$, the function K^3 is singular and the v fiber and the ϕ-circle pinch off. This point corresponds to $\xi = 0$, $\theta = 0$ and so (7.124) implies that the only non-trivial fluctuations come from F^+_m and so lie along the non-collapsing ψ fiber. The construction of regular solutions exactly follows the discussion of the wiggling supertubes in [99]. As discussed in detail in [99], regularity near the supertube will impose constraints on the charge densities in the H_j, L_3 and M and there will be one remaining, freely choosable charge density function, $\rho_+(\psi)$, at this point. Similarly, at $r_- = 0$ or to $\xi = 0$, $\theta = \pi$, the function V is singular and the ψ fiber and the ϕ-circle pinch off. Again, (7.124) implies that the only non-trivial fluctuations come from F_m^- and so lie along the non-collapsing v fiber. The regularity at $r_- = 0$ will be the spectral inversion of the regularity at $r_+ = 0$ and one will be left with another freely choosable charge density function, $\rho_-(v)$.

To be more specific, one can find solutions with

$$K^j = \hat{K}^j(v, \vec{y}), \qquad L_j = \hat{L}_j(\psi, \vec{y}), \qquad j = 1, 2; \tag{7.126}$$

$$L_3 = \hat{L}_3(v, \vec{y}), \qquad M = M(\psi, \vec{y}), \tag{7.127}$$

where a dependence on $(\psi, \vec{y})$ implies an expansion in F_m^+ alone and a dependence on $(v, \vec{y})$ implies an expansion in F_m^- alone. In principle there are six freely choosable charge density functions, three at each point: $\rho_J^+(\psi)$, $\rho_J^-(v)$. At $r_+ = 0$ one has $\theta = 0$ and so the K^j and L_3 collapse to their zero modes. The analysis of regularity then exactly follows the analysis of [99], which means that the $\rho_J^+(\psi)$ can all be parametrized in terms of one function, $\rho^+(\psi)$. At $r_- = 0$ one has $\theta = \pi$ and so the L_j and M collapse to their zero modes and analysis of regularity is the spectral inversion of the analysis at $r_+ = 0$, which means that the $\rho_J^-(v)$ can all be parametrized in terms of one function, $\rho^-(v)$.

If one removes the condition (7.119), and works with general $q_\pm, k_\pm$, then the foregoing discussion goes through as before except that the collapsing circles and density functions are parametrized by $(\phi, \chi \pm \eta)$ and $\chi \mp \eta$ respectively. Using (7.107), one obtains

$$\eta + \chi = \frac{1}{\Delta}\big(k_+ \, \psi + q_+ \, v\big), \qquad \eta - \chi = \frac{1}{\Delta}\big(k_- \, \psi + q_- \, v\big), \tag{7.128}$$

and these define the modes along the circles of finite size at $r_+ = 0$ and $r_- = 0$ respectively. This is, of course, consistent with the observation that the finite circle and its modes are defined by $r_\pm \alpha$ as in Section 7.4.3.

7.5.3 The general form of these solutions

Perhaps the most important lesson of the last section is that in the multi-centered solutions one can have have special classes of singular sources in the solution and, at these sources, one circle in the (v, ψ) torus pinches off and the source charge can then be spread in a general line distribution along the other direction.

This is also evident in the structure of the differential operator, $\mathcal{L}$, defined in (7.73). The (v, ψ) modes contribute to the following terms to this operator:

$$(V \, \partial_\psi - K^3 \, \partial_v), \qquad (\vec{A} \, \partial_\psi + \vec{\xi} \, \partial_v). \tag{7.129}$$

Suppose that V and K^3 have their generic forms (7.39) and (7.58). Then the foregoing terms will have singularities of the form $\mathcal{O}(|\vec{y} - \vec{y}^{(j)}|^{-1})$ except for modes $e^{i(nv+p\psi)}$ where the contributions for the two fibers cancel; that is, when:

$$p \, q_j - n \, k_3^j = 0. \tag{7.130}$$

Thus the nature of the differential equation, and its solutions, will be quite different for generic modes and for special modes satisfying (7.130). This identity implies $q_j(nv +$

$p\psi) = n(q_j v + k_3^j \psi)$ and so these special modes at $\vec{y}^{(j)}$ define the Fourier series of functions of one variable that depend upon $\sigma_j \equiv (q_j v + k_3^j \psi)$. Now recall that the circle that remains of finite size is defined by $r_j \, \alpha \to q_j \, \mathrm{d}v + k_3^j \, \mathrm{d}\psi = \mathrm{d}\sigma_j$.

The modes satisfying this relationship lie along the circle that remains large at $\vec{y}^{(j)}$ and a linear charge distribution in these Fourier modes will only give rise to the required $\mathcal{O}(|\vec{y} - \vec{y}^{(j)}|^{-1})$ singularity in the solutions to $\mathcal{L}F = 0$. This identity implies $q_j(nv + p\psi) = n(q_j v + k_3^j \psi)$ and so this means that the modes at $\vec{y}^{(j)}$ must depend upon $\sigma_j \equiv (q_j v + k_3^j \psi)$. Thus in a general solution we expect to be able to introduce line sources at every point, $\vec{y}^{(j)}$, and the source densities will be functions of one variable, σ_j.

We therefore expect that a generic fluctuating BPS solution based upon the ansatz of Section 7.3.1 can depend in a highly non-trivial manner on both variables v and ψ, however this dependence is generated by source functions of one variable located at the points $\vec{y}^{(j)}$. The modes introduced at $\vec{y}^{(j)}$ depend upon the KKM and GH charges at that point and so by varying these charges between points one can get broad classes of fluctuations.

7.6 Discussion and open problems

We have analyzed the BPS equations of minimal six-dimensional supergravity coupled to one anti-self-dual tensor multiplet. In particular, we have focussed upon a simple class of five-dimensional spatial backgrounds that may be thought of as T^2 fibration over a flat $\mathbb{R}^3$ base. This fibration is non-trivial because the fibration of the circles involves two independent sets of KKM's. The generic BPS configuration we considered could fluctuate with densities that depend freely on both directions of the torus, T^2. However, we found that requiring smooth configurations restricts these densities to functions of one variable, albeit a different torus circle depending on each pair of KKM charges.

Thus, by choosing different combinations of KKM charges one can obtain rich families of doubly fluctuating microstate geometries that depend non-trivially on all directions within the T^2.

It was conjectured that the general superstratum [32] will be a smooth solution of the supergravity theory studied here and yet has shape and density modes that are general functions of two variables. Our analysis here lends support to the construction outlined in [32]. In particular, the first step of this construction involves tilting and boosting the D1-D5-P system to generate d1-d5 dipole moments and angular momentum along a new profile. We showed here that the six-dimensional BPS equations admit solutions that precisely represent this tilting and boosting procedure.

The next and most difficult step in the construction of a generic superstratum is to add KKM's along the new profile so as to desingularize the tilted and boosted D1-D5-P system. Here we have managed to realize this is a limited manner: Our solutions may be thought of as semi-rigid superstrata in that they are not sourced by generic functions of two variables. This seems to be a direct result of the rigidity of our array of Kaluza-Klein monopoles: regularity at each KKM selects the direction of the charge density dependence within the T^2. If the KKM charge configuration could be made to vary non-trivially as a function of some combination of the T^2 fibers, v and ψ, then the smooth configurations might indeed involve density functions that are generic functions of two variables. This will, however, involve solving the non-linear system (7.10) and (7.11) for a general vector field, β. While this is challenging, it may not be impossibly difficult because it is a form of self-dual Yang Mills equation, as pointed out in Section 3.3.

Most of the focus of the latter part of this chapter has been upon microstate geometries and smooth solutions. One should not forget that there are very interesting singular solutions, like black holes and black rings. Our analysis of the BPS equations will certainly provide interesting new families of such solutions in which there are fluctuations along the T^2. More generally, the ultimate simplicity of the BPS system based on the T^2 fibration suggests that it might be used as more general "floating-brane ansatz" as in [31]. This might lead to six-dimensional generalizations of the whole class of almost-BPS and non-BPS configurations.

Chapter 8

Conclusions

It has been a long-standing question whether string theory, being a quantum theory of gravity, can explain the puzzles of the information paradox discovered by Bekenstein and Hawking [9, 10]. By examining the microscopic dynamics of the D-branes used to construct black holes, it was shown that the classical Bekenstein entropy can be reproduced by the Cardy formula for the entropy of the CFT on the D-branes [65, 64]. This tantalizing discovery shows that a string-theoretical explanation of the paradox is probably possible.

An important development in this story is the "fuzzball proposal" of Mathur [15], who argues that string theory must make $\mathcal{O}(1)$ corrections at the *horizon scale* of black holes in order for information to be extracted from them [12]. Therefore, we must conclude that the horizon and the interior of a black hole are a classical fiction—even though the curvature at the horizon might be arbitrary small, string theory must somehow be sensitive to it. A full, stringy black hole must be some astronomical-scale ball of "fuzz", which has a complicated, but unitary, dynamics, and re-radiates out all the information that falls in.

It is natural to ask whether any aspect of these "fuzzballs" can be seen in the supergravity limit of string theory. Are there classical solutions that represent black hole microstates? Such solutions need to have asymptotic charges like a black hole, but in the center they must remain smooth and free of horizons. Furthermore, one must have a vast multiplicity of them—e^S different states for the entropy S—to correspond to a given set of external charges. In 4 dimensions there are black hole uniqueness theorems that prevent this; however, in higher dimensions one has many well-known infinite families of solutions (see [25, 145] for reviews).

An important class of solutions are *supertubes*, which are BPS objects with 2 electric charges and 1 magnetic dipole charge [17, 18], whose shape modes can fluctuate as an arbitrary function of one variable while retaining supersymmetry. Supertubes are found to be smooth in IIB supergravity reduced to 6 dimensions [19, 20, 18], and as microstate geometries they are found to account for a finite fraction[1] of the entropy of the 2-charge BPS black hole [15, 21, 22, 23, 24].

Classically, however, the 2-charge black hole in 5 or 6 dimensions has a microscopic horizon, so it would be much more interesting to find the supergravity microstates of the 3-charge black hole. In addition, since real astrophysical black holes are quite far from being BPS, it would be interesting to understand non-BPS microstates.

Non-BPS microstates

In this thesis, we set out to quest after both of these prizes. First, we look at non-BPS solutions in 5 dimensions. For these we employ the "floating brane ansatz" of [31]. This gives a method for finding extremal, yet non-BPS, solutions of 5d $\mathcal{N} = 2$ supergravity coupled to two vector multiplets. It involves first finding an Einstein-Maxwell base space, and then solving a linear system of equations on top of it.

For our base space we use the LeBrun family of metrics, which are Kähler and also solve the Euclidean-Einstein-Maxwell equations [85, 86]. These metrics are defined by two functions which solve the $SU(\infty)$ Toda equation and its linearization. The nonlinear Toda equation is notoriously hard, so for our first family of solutions, we

[1] That is, the entropy of supertubes scales as $\sqrt{Q_1 Q_5}$, just as the entropy of a 2-charge black hole.

choose an extremely simple (nearly trivial) solution of it. This results in the LeBrun-Burns class of metrics, which are described by a $U(1)$ fiber over hyperbolic space H^3 (in contrast to Gibbons-Hawking metrics which are a $U(1)$ fiber over $\mathbb{R}^3$).

Based on the LeBrun-Burns metrics, we find an infinite family of supergravity solutions in Chapter 4, and we analyze their properties. We find that it is possible to have smooth, horizon-free solutions. However, due to the simplistic choice of Toda solution, we find that the "bubble equations", which should constrain the sizes of topological 2-cycles, are trivial. This is due to one of the fluxes (defined by the Toda solution) being topologically trivial. We also find that one cannot have asymptotically-flat supergravity solutions in the LeBrun class, but instead we find solutions asymptotic to warped, rotating $AdS_2 \times S^3$.

Next we discuss how our LeBrun-Burns solutions can be lifted into 6d supergravity, where they turn out to be BPS! This was at first a surprising fact, since the solutions are non-BPS in 5 dimensions. However, the lift to 6d is not trivial; it involves defining a set of rotating complex structures that depend on the new 6th coordinate (which parametrizes a $U(1)$ fiber). In turn this means that the 6d Killing spinor depends on this fiber coordinate. Looking at this process in reverse, since the supersymmetry in 6d depends on the fiber coordinate, when we do a trivial KK reduction to 5 dimensions, we kill the supersymmetry. This provides a realization of the Scherk-Schwarz mechanism [90, 91], or also "supersymmetry without supersymmetry" [92].

In Chapter 5, we improve upon these results by generalizing to all LeBrun metrics with an extra axial symmetry (hence $U(1) \times U(1)$ symmetric). In this case, the Toda equation reduces to a PDE in two independent variables, and there is a method to map it onto the (linear) Laplace equation in 3 dimensions, also with axial symmetry [119, 120, 87, 88]. Employing this method, we find the family of all axisymmetric LeBrun metrics that have the type of boundary conditions we wish to consider. We discuss all sorts of interesting new features in these metrics, such as the appearance of homological 2-cycles that are not swept out by the fiber coordinate. In particular, the self-dual flux of these metrics is non-trivial on the homological 2-cycles, so we are able to derive non-trivial bubble equations. This provides a satisfying picture of non-BPS microstate geometries that is very analogous to the BPS story in [25].

Toward superstrata

While the 5d BPS story in [25] is fascinating for its ability to build a wide variety of "bubbling" solutions via linear superposition, these solutions are far too restrictive to get enough microstates to account for the entropy of a 3-charge BPS black hole. A slightly more general idea is the supertube, which depends upon an arbitrary function of one variable, and hence has a (classically) infinite-dimensional moduli space. However, as we mentioned above, supertubes turn out to give the correct microstate counting for the *two* charge system, whose entropy scales as $S \sim Q$. This will fall short of the entropy needed in the 3-charge system, which scales as $S \sim Q^{3/2}$. In some cases, the entropy of supertubes can be enhanced to $S \sim Q^{5/4}$ [67, 104], but for a full understanding of 3-charge black holes, one needs something more.

The *superstratum* is a prime candidate for solving this puzzle. Superstrata are BPS objects conjectured to exist [32, 126], which have 3 electric charges and 2 magnetic dipole charges, and have shape modes which can fluctuate as functions of *two* variables while maintaining supersymmetry. This doubly infinite set of modes is expected to give the $S \sim Q^{3/2}$ entropy scaling needed in the 3-charge case. Furthermore, superstrata are expected to be regular supergravity solutions in IIB reduced to 6 dimensions, in much the same way that supertubes are. It has even been shown that the BPS equations in 6 dimensions can be reduced to a linear system [70, 71, 72], once the conditions defining a 4d base space have been satisfied. The main difficulty is that the conditions defining the

4d base are nonlinear. These same conditions define the Kaluza-Klein monopole charge of the solution, which is necessary for the solution to be smooth, so this is a major hurdle to be overcome.

To attack this problem, we have proceeded in two steps. First, in Chapter 6, we turned off the KKM charge and focused on getting solutions which allowed for shape modes given by arbitrary functions of two variables. We obtained solutions consisting of several *superthreads*, which are 1-dimensional, singular objects; and by weaving together several superthreads of varying shape, one can obtain *supersheets*, which have an arbitrary 2-dimensional shape. Because they lack KKM charge, supersheets are singular solutions. However, they do represent very interesting supergravity solutions due to their arbitrary shape modes depending on 2 variables.

The next step, presented in Chapter 7, is to turn on KKM charge, but let it be independent of the fiber coordinates. This allows one to determine the base space easily (in our case, we choose it to be Gibbons-Hawking). Then hopefully it will be possible to find solutions on top of this base space that do fluctuate as functions of two variables. We do not quite accomplish this, although we do get some rather interesting solutions. We find that the 5d spatial slices of the solution can be given the structure of a T^2 fibered over $\mathbb{R}^3$, and then the solution describes various points in the $\mathbb{R}^3$ where some circle in the T^2 pinches off as a KK monopole. Where one circle pinches off, its dual circle in the T^2 stays a finite size, and we can put fluctuations on this circle. Therefore solutions will contain many arbitrary functions of *one* variable, but at each (smooth) source point, one can choose how this one variable is oriented within the T^2 fiber. One can think of these solutions as containing many supertubes of two different "flavors", and their linear combinations, each coming with an arbitrary function of one variable.

Outlook and future prospects

We have found many supergravity solutions that are interesting both in their own right, and as pieces of the information-paradox puzzle. We have pushed into the realm of non-BPS solutions, which should point the way toward understanding non-supersymmetric black holes. And we have made great progress on the quest for the superstratum, which may be the dominant source of entropy in the classical régime of the 3-charge black hole. There remain many unanswered questions to address in future work:

First, in the process of finding solutions on the LeBrun class of base metrics, we showed that such solutions can never be asymptotically flat. The reason is that the fluxes acquire an anti-self-dual part proportional to the Kähler form on the base, which is non-normalizable. This non-vanishing part of the fluxes sources energy-momentum at infinity and prevents asymptotic flatness. It would be very interesting in future research to find a way to obtain asymptotically-flat solutions. This would truly complete the analogy to the 5d BPS solutions in [25] and would allow for some useful comparisons.

More generally, one would like to know about *non-extremal* microstates, and this problem is very hard. Only isolated examples exist [26, 27, 28, 29, 30], and it is not clear how to proceed. Non-extremal solutions will certainly not be captured by the floating brane ansatz [31], since extremality is what provided the very notion of "floating".

Regarding superstrata, the future direction of work is clear: a proper, smooth superstratum fluctuating as a function of two variables remains to be constructed. An integral part of this construction will be to find some non-trivial solution to the nonlinear equations that define the KKM charge on the 4-dimensional base. The full problem, including the fluctuations one wants to find, is "cohomogeneity five", and quite difficult. Perturbative approaches have been tried [141, 146], and this tactic may be enough to prove existence. But it would be more satisfying to find a full solution, and perhaps there is a way to do this.

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Appendix A

Details of the LeBrun-Burns solutions

A.1 Gauge invariance

The general solutions on a LeBrun-Burns base discussed in Section 4.3.2 have a "gauge invariance" similar to the one present in multi-centered BPS solutions with a GH base (see equation (94) in [25])). It is easy to check that the following transformation leaves (4.88) and (4.89) invariant

$$K^{(1)} \to K^{(1)} + \gamma_1 V, \qquad K^{(2)} \to K^{(2)} + \gamma_2 V, \tag{A.1}$$

$$L_1 \to L_1 - \gamma_2, \qquad L_2 \to L_2 - \gamma_1, \tag{A.2}$$

$$L_3 \to L_3 - \gamma_1 \zeta^2 K^{(2)} - \gamma_2 \zeta^2 K^{(1)} - \gamma_1 \gamma_2 \zeta^2 V, \tag{A.3}$$

$$M \to M + \tfrac{1}{2}\gamma_1 \zeta^2 L_1 + \tfrac{1}{2}\gamma_2 \zeta^2 L_2 - \tfrac{1}{2}\gamma_1 \gamma_2 \zeta^2, \tag{A.4}$$

One can also show that the equations for the one-form ω, (4.90)–(4.92), are invariant, therefore the transformation above is a symmetry of the full solution.

A.2 Causality

A supergravity background is causal only if there are no CTCs and Dirac-Misner strings. To study the constraints imposed by these conditions one should study the five-dimensional metric at a constant time slice:

$$\mathrm{d}s^2 = \mathcal{Q}\left(\mathrm{d}\tau + A - \frac{\mu V^2}{\mathcal{Q}}\omega\right)^2 + W^2 V\left(\eta^2 \mathrm{d}\phi^2 - \frac{\zeta^2}{\mathcal{Q}}\omega^2\right) + W^2 V(\mathrm{d}\eta^2 + \mathrm{d}\zeta^2), \quad \text{(A.5)}$$

where

$$\mathcal{Q} \equiv W^6\zeta^2 V - \mu^2 V^2, \qquad\qquad W^2 \equiv (Z_1 Z_2 Z_3)^{1/3}. \quad \text{(A.6)}$$

For absence of CTC's we need to impose the following conditions

$$\mathcal{Q} \geq 0, \qquad W^2 V \geq 0, \qquad Z_I^{-1} W^2 \geq 0, \qquad I = 1, 2, 3. \quad \text{(A.7)}$$

The last inequality comes from imposing positive definite metric in the six internal directions along T^6 upon uplift of our solutions to eleven-dimensional supergavity. The expression for $\mathcal{Q}$ resembles quite closely the one for solutions with a GH base (see equation (102) in [25])

$$\begin{aligned}\mathcal{Q} = &-M^2V^2 + 2M\zeta^2 K^{(1)} K^{(2)} + MV\left(\zeta^2 K^{(1)} L_1 + \zeta^2 K^{(2)} L_2 + L_3\right) \\ &- \tfrac{1}{4}\left(\zeta^2 K^{(1)} L_1 + \zeta^2 K^{(2)} L_2 + L_3\right)^2 + \zeta^2 V L_1 L_2 L_3 \\ &+ \left(\zeta^4 K^{(1)} L_1 K^{(2)} L_2 + L_3 \zeta^2 K^{(1)} L_1 + L_3 \zeta^2 K^{(2)} L_2\right) \end{aligned} \quad \text{(A.8)}$$

There is also the possibility of having Dirac-Misner strings in ω. To ensure that this does not happen one has to require that ω_ϕ vanishes for $\eta = 0$.

A.3 Useful identities

Here we collect some identities used in Section 4.3.4. We used the following identities to solve the equations for $K^{(a)}$ and M

$$\mathcal{L}_1\left(\frac{1}{\zeta^2}\right) = \frac{4}{\zeta^4}, \tag{A.9}$$

$$\mathcal{L}_1\left(-\frac{1+G_i}{\zeta^2}\right) = \frac{2}{\zeta}\partial_\zeta\left(\frac{1+G_i}{\zeta^2}\right), \tag{A.10}$$

$$\mathcal{L}_1\left(-\frac{(1+G_i)(1+G_j)}{\zeta^2} + 4\rho^2 H_i H_j\right) = \frac{2}{\zeta}\partial_\zeta\left(\frac{(1+G_i)(1+G_j)}{\zeta^2}\right). \tag{A.11}$$

The following identities are useful when one solves the equation for L_3

$$\mathcal{L}_2\left(\frac{1}{\zeta^2}\right) = \frac{8}{\zeta^4}, \qquad \mathcal{L}_2\left(-\frac{1+G_i}{\zeta^2}\right) = \frac{4}{\zeta}\partial_\zeta\left(\frac{1+G_i}{\zeta^2}\right), \qquad \mathcal{L}_2\left(-H_i\right) = \frac{2}{\zeta}\partial_\zeta H_i, \tag{A.12}$$

$$\begin{aligned}
\mathcal{L}_2\left(-\frac{(1+G_i)(1+G_j)}{\zeta^2} + 4\rho^2 H_i H_j\right) &= \frac{4}{\zeta}\partial_\zeta\left(\frac{(1+G_i)(1+G_j)}{\zeta^2}\right) - \frac{8}{\zeta}\partial_\zeta(\rho^2 H_i H_j), \\
\mathcal{L}_2\left(-\frac{1}{c_i^2 - c_j^2}\frac{H_j - H_i}{H_i}\right) &= \frac{2}{\zeta}(1+G_i)\partial_\zeta H_j, \qquad i \neq j \\
\mathcal{L}_2\left(-(\rho^2 + c_i^2 - 2\zeta^2)H_i^2\right) &= \frac{2}{\zeta}(1+G_i)\partial_\zeta H_i,
\end{aligned} \tag{A.13}$$

$$\begin{aligned}
&\mathcal{L}_2\left(-\frac{(1+G_i)(1+G_j)(1+G_k)}{\zeta^2} + 4\rho^2(3\rho^2 - 4\zeta^2 + c_i^2 + c_j^2 + c_k^2)H_i H_j H_k\right) \\
&\quad = \frac{4}{\zeta}\partial_\zeta\left(\frac{(1+G_i)(1+G_j)(1+G_k)}{\zeta^2}\right) - \frac{8}{\zeta}\Big[(1+G_i)\partial_\zeta(\rho^2 H_j H_k) \\
&\qquad + (1+G_j)\partial_\zeta(\rho^2 H_k H_i) + (1+G_k)\partial_\zeta(\rho^2 H_i H_j)\Big].
\end{aligned} \tag{A.14}$$

There are similar identities involving D_i, H_i and G_i that we have used to solve the equation for ω_ϕ, however they are pretty lengthy and we refrain from presenting them explicitly.

A.4 Black ring coordinates

To facilitate comparison of our solutions with the more standard black-ring and super-tube solutions, it is useful to recall the canonical separable bipolar coordinates on $\mathbb{R}^4$, [147] (we set $a = b = 0$ below):

$$\tilde{x} \equiv -(G + 1 - 2c^2 H) = -\frac{x^2 + y^2 + \zeta^2 - c^2}{\sqrt{((\zeta - c)^2 + x^2 + y^2)((\zeta + c)^2 + x^2 + y^2)}}, \tag{A.15}$$

$$\tilde{y} \equiv -(G + 1) = -\frac{x^2 + y^2 + \zeta^2 + c^2}{\sqrt{((\zeta - c)^2 + x^2 + y^2)((\zeta + c)^2 + x^2 + y^2)}}, \tag{A.16}$$

In these coordinates the flat metric on $\mathbb{R}^4$ takes the form:

$$\mathrm{d}s^2_{\mathbb{R}^4} = \frac{R^2}{(\tilde{x} - \tilde{y})^2}\left(\frac{\mathrm{d}\tilde{y}^2}{\tilde{y}^2 - 1} + (\tilde{y}^2 - 1)\,\mathrm{d}\tau^2 + \frac{\mathrm{d}\tilde{x}^2}{1 - \tilde{x}^2} + (1 - \tilde{x}^2)\,\mathrm{d}\phi^2\right). \tag{A.17}$$

where $R = c$. In particular, note that the canonical coordinates, $\tilde{x}$ and $\tilde{y}$, are simply related to the Green functions that we have been using and thus the solutions of Sections 4.3.3 and 4.3.4 can easily be expressed as rational functions of $\tilde{x}$ and $\tilde{y}$.

Appendix B

Additional details for axisymmetric LeBrun solutions

B.1 Relation of LeBrun to Gibbons-Hawking metrics

The LeBrun metric (4.7) is the most general scalar-flat Kähler metric with a $U(1)$ isometry (generated by ∂_τ). As discussed in Section 4.2.2, they are also Euclidean-Einstein-Maxwell solutions, with Ricci tensor

$$R_{\mu\nu}(g) = \frac{1}{2}\Big(\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho} - \frac{1}{4}\, g_{\mu\nu}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\Big), \tag{B.1}$$

where the Maxwell field $\mathcal{F} \equiv \Theta^{(3)} - \omega_-^{(3)}$ has self-dual and anti-self-dual parts

$$\Theta^{(3)} = \frac{1}{2}(\mathrm{d}\tau + A) \wedge \mathrm{d}\frac{u_z}{w} + \frac{1}{2} w \star_3 \mathrm{d}\frac{u_z}{w}, \qquad \omega_-^{(3)} = J. \tag{B.2}$$

In the case that $u_z/w \equiv \alpha$ for some constant α, one has that $\mathcal{F}$ is purely anti-self-dual, and hence $R_{\mu\nu} = 0$. Then the LeBrun metric is actually hyper-Kähler (although not Gibbons-Hawking, because the $U(1)$ isometry is not tri-holomorphic in general). One has in this case

$$w = \frac{1}{\alpha}\, u_z, \qquad A = \frac{1}{\alpha}(u_y\,\mathrm{d}x - u_x\,\mathrm{d}y). \tag{B.3}$$

One can then give the hyper-Kähler structure explicitly in terms of the basic anti-self-dual 2-forms,

$$\Omega_-^{(1)} = e^1 \wedge e^2 - e^3 \wedge e^4 \qquad = e^{u/2}\Big[(\mathrm{d}\tau + A)\wedge \mathrm{d}x - w\,\mathrm{d}y \wedge \mathrm{d}z\Big], \tag{B.4}$$

$$\Omega_-^{(2)} = e^1 \wedge e^3 - e^4 \wedge e^1 \qquad = e^{u/2}\Big[(\mathrm{d}\tau + A)\wedge \mathrm{d}y - w\,\mathrm{d}z \wedge \mathrm{d}x\Big], \tag{B.5}$$

$$\Omega_-^{(3)} = e^1 \wedge e^4 - e^2 \wedge e^3 \qquad = (\mathrm{d}\tau + A)\wedge \mathrm{d}z - we^u\,\mathrm{d}x \wedge \mathrm{d}y, \tag{B.6}$$

which satisfy

$$\mathrm{d}\Omega_-^{(1)} = \frac{1}{2}\mathrm{d}u \wedge \Omega_-^{(1)}, \qquad \mathrm{d}\Omega_-^{(2)} = \frac{1}{2}\mathrm{d}u \wedge \Omega_-^{(2)}, \qquad \mathrm{d}\Omega_-^{(3)} = 0. \tag{B.7}$$

The Kähler 2-forms $J^{(A)}$ are then given by

$$J^{(1)} = \cos\frac{\alpha\tau}{2}\,\Omega_-^{(1)} - \sin\frac{\alpha\tau}{2}\,\Omega_-^{(2)}, \tag{B.8}$$

$$J^{(2)} = \sin\frac{\alpha\tau}{2}\,\Omega_-^{(1)} + \cos\frac{\alpha\tau}{2}\,\Omega_-^{(2)}, \tag{B.9}$$

$$J^{(3)} = \Omega_-^{(3)}. \tag{B.10}$$

These satisfy $\mathrm{d}J^{(A)} = 0$ and the quaternion algebra (3.74) according to the prescription (3.75) in Section 3.3.

B.1.1 The $U(1) \times U(1)$-invariant case

If a LeBrun metric meets the hyper-Kähler conditions (B.3) and has *two* commuting $U(1)$ isometries (given by $\partial_\tau, \partial_\phi$), then some linear combination of these $U(1)$'s is tri-holomorphic, and the metric is actually Gibbons-Hawking written in some alternative coordinates. In this section we show this explicitly.

Transforming A in (B.3) to cylindrical coordinates r, ϕ yields $A = -\frac{1}{\alpha}\, r u_r \,\mathrm{d}\phi$. However, we will find it convenient to introduce a gauge parameter λ and write (B.3) in the form

$$w = \frac{1}{\alpha}\, u_z, \qquad A = \frac{1}{\alpha}\Big[2\lambda - (2 + r u_r)\Big]\mathrm{d}\phi. \tag{B.11}$$

The reasons for this are twofold. First is that the combination $2 + r u_r$ is particularly simple after the Bäcklund transformation (5.9):

$$2 + r u_r = \frac{2\, V_{\rho\eta}}{\rho(V_{\rho\eta}^2 + V_{\eta\eta}^2)}. \tag{B.12}$$

Second is that the gauge parameter λ (with conventional factor of 2) will be convenient in matching up to the near-singularity limit of the LeBrun metrics in (5.37).

With these conventions and in r, ϕ coordinates, it is simplest to write the expressions for the $J^{(A)}$ in full:

$$\begin{aligned} J^{(1)} = e^{u/2}\Bigg\{&(\mathrm{d}\tau + A) \wedge \left[\cos\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}r - r\sin\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}\phi\right] \\ &+ w\,\mathrm{d}z \wedge \left[\sin\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}r + r\cos\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}\phi\right]\Bigg\} \end{aligned} \tag{B.13}$$

$$\begin{aligned} J^{(2)} = e^{u/2}\Bigg\{&(\mathrm{d}\tau + A) \wedge \left[\sin\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}r + r\cos\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}\phi\right] \\ &+ w\,\mathrm{d}z \wedge \left[-\cos\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}r + r\sin\left(\frac{\alpha\tau}{2} + \lambda\phi\right)\mathrm{d}\phi\right]\Bigg\} \end{aligned} \tag{B.14}$$

$$J^{(3)} = (\mathrm{d}\tau + A) \wedge \mathrm{d}z - e^{u} w\, r\,\mathrm{d}r \wedge \mathrm{d}\phi. \tag{B.15}$$

It is then straightforward to show that the particular linear combination

$$Y \equiv -\lambda\,\partial_\tau + \frac{\alpha}{2}\,\partial_\phi \tag{B.16}$$

is the tri-holomorphic Killing vector:

$$\mathcal{L}_Y\, J^{(A)} = \mathrm{d}\,\iota_Y\, J^{(A)} = 0, \qquad \text{for} \quad J^{(A)} = J^{(1)}, J^{(2)}, J^{(3)}. \tag{B.17}$$

Therefore an axisymmetric LeBrun metric satisfying (B.11) can be re-written in Gibbons-Hawking form

$$\mathrm{d}s^2(GH) = \frac{1}{V}(\mathrm{d}\psi + \tilde{A}_\chi\,\mathrm{d}\chi)^2 + V\,(\mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2 + R^2\sin^2\theta\,\mathrm{d}\chi^2). \tag{B.18}$$

In particular, the vector Y in (B.16) generates the tri-holomorphic isometry, which means it must be proportional to ∂_ψ. Therefore, Y is orthogonal to $\mathrm{d}\chi$,

$$\iota_Y\,\mathrm{d}\chi = 0, \qquad Y \equiv -\lambda\,\partial_\tau + \frac{\alpha}{2}\,\partial_\phi. \tag{B.19}$$

Hence the coordinate χ *orthogonal* to the Gibbons-Hawking fiber is

$$\chi \sim \frac{\alpha\tau}{2} + \lambda\phi, \tag{B.20}$$

which we ought to have suspected from the form of (B.13)–(B.15). We should note that the coordinate ψ *along* the Gibbons-Hawking fiber is not uniquely determined, as a shift in ψ by any constant multiple of χ can be absorbed in the definition of $\tilde{A} \equiv \tilde{A}_\chi\,\mathrm{d}\chi$ in (B.18).

B.1.2 Near-singularity limit of LeBrun as Gibbons-Hawking

In the near-singularity limit discussed in (5.37), the metric becomes a "one point" LeBrun metric and therefore one has $u_z/w = \alpha$ automatically. Therefore the LeBrun metric in the neighborhood of the source points becomes (locally) a Gibbons-Hawking metric. In the Bäcklund-transformed LeBrun metric (5.11),

$$g = \frac{1}{w}(\mathrm{d}\tau + A)^2 + w\Big[\rho^2(V_{\rho\eta}^2 + V_{\eta\eta}^2)(\mathrm{d}\rho^2 + \mathrm{d}\eta^2) + \rho^2\,\mathrm{d}\phi^2\Big], \tag{B.21}$$

this can be shown by setting $\rho = R\sin\theta$, $\eta - \eta_\ell = R\cos\theta$ as usual, and then sending the angular coordinates (τ, ϕ) to the (ψ, χ) of Gibbons-Hawking via a *linear* map acting only on these coordinates.

The previous section gives us $\chi \sim (\alpha\tau/2) + \lambda\phi$ for free; we need only identify the parameters α, λ. In the small R limit, we have

$$\rho^2(V_{\eta\eta}^2 + V_{\rho\eta}^2) \to \widetilde{K}(\theta), \qquad w \to \frac{1}{\widetilde{K}(\theta)}\frac{\widetilde{q}_\ell}{R}, \qquad A \to -\frac{\widetilde{KQ}(\theta)}{\widetilde{K}(\theta)}\,\mathrm{d}\phi, \tag{B.22}$$

as well as

$$u_z \to \frac{2k_\ell}{\widetilde{K}(\theta)}\frac{1}{R}, \qquad 2 + ru_r \to -\frac{2}{\widetilde{K}(\theta)}(k_\ell\cos\theta + \bar{K}_\ell), \tag{B.23}$$

where $\widetilde{q}_\ell$ is a determinant:

$$\widetilde{q}_\ell \equiv q_\ell(\bar{K}_\ell^3 - k_0^3) - k_\ell^3(\bar{Q}_\ell - q_0). \tag{B.24}$$

A short bit of algebra then reveals

$$\alpha = \frac{2k_\ell}{\widetilde{q}_\ell}, \qquad \lambda = -\frac{q}{\widetilde{q}_\ell}, \qquad \chi \sim \frac{\alpha\tau}{2} + \lambda\phi = \frac{k_\ell\,\tau - q_\ell\,\phi}{\widetilde{q}_\ell}. \tag{B.25}$$

Next, we compare the near-center axisymmetric LeBrun metric (5.40)

$$\mathrm{d}s^2(LB) = \mathrm{d}\varrho^2 + \frac{\varrho^2}{4}\left[\mathrm{d}\theta^2 + \frac{1}{\widetilde{q}_\ell{}^2}\left(\widetilde{K}(\theta)\,\mathrm{d}\tau^2 - 2\widetilde{KQ}(\theta)\,\mathrm{d}\tau\,\mathrm{d}\phi + \widetilde{Q}(\theta)\,\mathrm{d}\phi^2\right)\right], \tag{B.26}$$

to a 1-center Gibbons-Hawking metric. Specifically we choose a GH metric with "charge" 1:

$$\mathrm{d}s^2(GH) = R\big(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\chi\big)^2 + \frac{1}{R}\big(\mathrm{d}R^2 + R^2\,\mathrm{d}\theta^2 + R^2\sin^2\theta\,\mathrm{d}\chi^2\big), \qquad \text{(B.27)}$$

and by setting $R = \varrho^2/4$ this can be written

$$\mathrm{d}s^2(GH) = \mathrm{d}\varrho^2 + \frac{\varrho^2}{4}\left[\mathrm{d}\theta^2 + \mathrm{d}\psi^2 + \mathrm{d}\chi^2 + 2\cos\theta\,\mathrm{d}\psi\,\mathrm{d}\chi\right]. \qquad \text{(B.28)}$$

One can then find the coordinate change relating (B.26) and (B.28):

$$\psi = \frac{1}{\widetilde{q}_\ell}\Big((\bar{K}^3_\ell - k^3_0)\,\tau - (\bar{Q}_\ell - q_0)\,\phi\Big), \qquad \chi = \frac{1}{\widetilde{q}_\ell}\Big(k^3_\ell\,\tau - q_\ell\,\phi\Big), \qquad \text{(B.29)}$$

which matches our expectations in (B.25).

We should note that depending on the parameters $q_0, q_\ell, \bar{Q}_\ell, k^3_0, k^3_\ell, \bar{K}^3_\ell$ it is possible for ψ, χ to become identified in many different ways, giving a conical point at $\varrho \to 0$ with group structure $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ for some integers m, n. In Appendix B.2 we will develop an algorithm to compute m, n and thus determine G from the parameters $q_0, q_\ell, \bar{Q}_\ell, k^3_0, k^3_\ell, \bar{K}^3_\ell$.

B.2 Groups at conical points from lattices in $SO(4)$

In this section we discuss how to compute the group structure at the conical singularities of the LeBrun metrics. We stress that there are *two* possible types of conical singularities that may occur: orbifold points where the geometry approaches $\mathbb{R}^4/G$ for some finite group $G \subset SO(4)$, and more general conical singularities that cannot be locally modeled as a quotient space of $\mathbb{R}^4$. To illustrate the difference, consider two different 2-dimensional cone metrics:

$$\mathrm{d}s_A^2 = \mathrm{d}r^2 + r^2\frac{\mathrm{d}\theta^2}{n^2}, \qquad \mathrm{d}s_B^2 = \mathrm{d}r^2 + r^2\frac{m^2\,\mathrm{d}\theta^2}{n^2}, \qquad \theta \sim \theta + 2\pi, \tag{B.30}$$

for $m, n > 0 \in \mathbb{Z}$ relatively prime[1]. In the first metric $\mathrm{d}s_A^2$, a circuit around the tip of the cone subtends $2\pi/n$ radians; hence an n-fold cover of this space will fill out the standard $\mathbb{R}^2$, and this is the quotient space $\mathbb{R}^2/\mathbb{Z}_n$. In the second metric $\mathrm{d}s_B^2$, however, a path enclosing the origin subtends $2\pi m/n$ radians, and there is no p-fold cover of this space that gives us $\mathbb{R}^2$; hence it is not a quotient of $\mathbb{R}^2$. Nevertheless, it has a group structure which can be defined via the lattice inside $U(1)$ generated by $e^{2\pi m i/n}$. Since m and n are relatively prime, this is again $\mathbb{Z}_n$.

In a similar manner, we will analyze the conical singularities of the LeBrun metrics. However, since the metric (B.26) allows *two* different angular coordinates to be identified in a non-standard way, computing the group structure at such points is no longer obvious by inspection as it was in (B.30). We will describe in detail how to define and compute these groups, and we will derive a simple condition to restrict our metrics to have orbifold singularities only, without the presence of more general conical singularities. We will proceed somewhat pedantically; the more practical computations can be found in Appendices B.2.4 and B.2.5.

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[1] If $m < n$ the cone $\mathrm{d}s_B^2$ has a deficit angle, and if $m > n$ it has an *excess* angle. A flat d-dimensional cone with excess angle can always be isometrically embedded in $(d+1)$-dimensional Minkowski space $\mathbb{R}^{1,d}$ as a cone outside the lightcone. The lightcone itself is approached in the limit of *infinite* excess angle.

B.2.1 Coordinates on $\mathbb{R}^4$ and on $U(1) \times U(1) \subset SO(4)$

The group structure at each conical point in the LeBrun metric is some group $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ which is a finite subgroup of $SO(4)$ acting in the maximal torus $U(1) \times U(1)$[2]. We will deduce G by looking at how coordinates in the LeBrun metric are identified, and this makes use of a canonical map between two of the angular coordinates of $\mathbb{R}^4$ and coordinates on $U(1) \times U(1) \subset SO(4)$. This map is defined as follows:

Choosing standard $U(1) \times U(1)$-invariant spherical coordinates on $\mathbb{R}^4$,

$$\mathrm{d}s^2(\mathbb{R}^4) = \mathrm{d}\rho^2 + \rho^2\Big(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\alpha^2 + \sin^2\theta\,\mathrm{d}\beta^2\Big), \tag{B.31}$$

the orbits of the maximal torus $T^2_{SO(4)} \subset SO(4)$ are precisely the tori $\rho =$ const, $\theta =$ const, giving a natural relation between $T^2_{SO(4)}$ and the coordinates (α, β). Specifically, for every $\rho \in (0, \infty), \theta \in (0, \pi/2)$ one has a smooth embedding[3]

$$\varepsilon_{\rho,\theta} : T^2_{SO(4)} \to \mathbb{R}^4, \quad \text{given by} \quad (\alpha, \beta) \mapsto (\rho, \theta, \alpha, \beta), \tag{B.32}$$

and hence we can regard the coordinates (α, β) equally well as coordinates on $T^2_{SO(4)}$. Furthermore, since $\mathbb{R}^4$ is invariant under 2π rotations along either of these coordinates, one has the same identifications of these coordinates in both $\mathbb{R}^4$ and $T^2_{SO(4)}$:

$$(\alpha, \beta) : \quad (0,0) \sim (2\pi, 0) \sim (0, 2\pi) \sim (2\pi, 2\pi), \tag{B.33}$$

and hence the embedding $\varepsilon_{\rho,\theta}$ is a *global* embedding (i.e. an embedding of the whole $T^2_{SO(4)} \hookrightarrow \mathbb{R}^4$). If we choose the particular image at $\rho = 1, \theta = \pi/4$ (i.e. the Clifford

---

[2] $\mathbb{Z}_m, \mathbb{Z}_n$ can act on linear combinations of the $U(1)$'s; e.g. $\mathbb{Z}_m$ might act on the diagonal $U(1)$ and $\mathbb{Z}_n$ on the anti-diagonal $U(1)$, etc.

[3] The embedding $\varepsilon_{\rho,\theta}$ degenerates at the endpoints $\rho = 0$ or $\theta = 0, \pi/2$; however, this will not be important here.

torus in $\mathbb{R}^4$), then $\varepsilon_{\rho,\theta}$ is also globally *isometric* (given a suitable normalization of the Haar measure on $SO(4)$).

The existence of this canonical isometric embedding $\varepsilon_{\rho,\theta} : T^2_{SO(4)} \to \mathbb{R}^4$ will justify our treatment of $T^2_{SO(4)} \subset SO(4)$ and the Clifford torus $T^2_{(\alpha,\beta)} \subset \mathbb{R}^4$ as the same, and thereby use the (identifications of) coordinates along $T^2_{(\alpha,\beta)} \subset \mathbb{R}^4$ to determine the group structure $G \subset T^2_{SO(4)}$ of conical points in the LeBrun metric.

B.2.2 Tori from lattices

It is useful to think of a torus T^2_Γ as the quotient of $\mathbb{R}^2$ by the action of some lattice Γ. Here Γ is some additive group with generators $\vec{a}_1, \vec{a}_2 \in \mathbb{R}^2$:

$$\Gamma = \{\vec{g} \in \mathbb{R}^2 : \vec{g} = \Lambda\vec{s},\ \vec{s} \in \mathbb{Z}^2\}, \qquad \Lambda \equiv \begin{pmatrix} \vec{a}_1 & \vec{a}_2 \end{pmatrix}, \tag{B.34}$$

where the basis Λ is a matrix whose columns are the generators of Γ. The choice of basis is not unique—the same lattice results if we send $\Lambda \to \Lambda P$ for any $P \in GL(2,\mathbb{Z})$[4]. The group action α_Γ of Γ on $\mathbb{R}^2$ is the usual

$$\alpha_\Gamma : \Gamma \times \mathbb{R}^2 \to \mathbb{R}^2, \quad \text{given by} \quad (\vec{g}, \vec{x}) \mapsto \vec{x} + \vec{g}, \tag{B.35}$$

and then the torus can be written $T^2_\Gamma \simeq \mathbb{R}^2/\alpha_\Gamma$, or by the standard abuse of notation, $T^2_\Gamma \simeq \mathbb{R}^2/\Gamma$, when it is clear from context what group action we are talking about. We will also need the quotient map ρ_Γ:

$$\rho_\Gamma : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma \simeq T^2_\Gamma, \quad \text{given by} \quad \vec{x} \mapsto [\vec{x}], \tag{B.36}$$

[4]We define $GL(2,\mathbb{Z})$ as the group of 2×2 matrices with integer entries and determinant ± 1, hence invertible over $\mathbb{Z}$. This group is sometimes also called $S^*L(2,\mathbb{Z})$ or $SL^\pm(2,\mathbb{Z})$.

where $[\vec{x}]$ is the equivalence class of $\vec{x}$ under the group action α_Γ.

B.2.3 Lattices *within* tori

In a similar manner to $\mathrm{d}s_B^2$ in (B.30) we will define the group action $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ at a conical point of the LeBrun metric as the finite subgroup of the maximal torus $U(1) \times U(1) \subset SO(4)$ which is generated by the lattice $\widetilde{\Gamma}$ of coordinate identifications, allowing this lattice to "wrap around" $U(1) \times U(1)$ multiple times if needed[5]. Just as $e^{2\pi m/n} \in U(1)$ generates the same subgroup as $e^{2\pi i/n}$ (for m, n relatively prime), it is always possible to choose a new basis in which the lattice wraps around only once. Now we will make this notion more precise:

Take some torus $T^2_\Gamma \simeq \mathbb{R}^2/\Gamma$ defined by a lattice Γ (with basis Λ) which acts on $\mathbb{R}_2$ via the group action α_Γ, and where ρ_Γ is the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\Gamma$. Consider a second lattice $\widetilde{\Gamma}$ (with basis $\widetilde{\Lambda}$) acting on $\mathbb{R}^2$ via the group action $\alpha_{\widetilde{\Gamma}}$. This lattice group action then descends along ρ_Γ to the group action α_G of the group G on T^2_Γ such that the following diagram commutes:

$$\begin{array}{ccc} \widetilde{\Gamma} \times \mathbb{R}^2 & \xrightarrow{\alpha_{\widetilde{\Gamma}}} & \mathbb{R}^2 \\ {\scriptstyle \rho_\Gamma \times \rho_\Gamma} \downarrow & & \downarrow {\scriptstyle \rho_\Gamma} \\ G \times (\mathbb{R}^2/\Gamma) & \xrightarrow{\alpha_G} & \mathbb{R}^2/\Gamma, \end{array} \tag{B.37}$$

where $\rho_\Gamma(\widetilde{\Gamma})$ is defined via the natural inclusion $\widetilde{\Gamma} \hookrightarrow \mathbb{R}^2$. The group action α_G is

$$\alpha_G : G \times (\mathbb{R}^2/\Gamma) \to \mathbb{R}^2/\Gamma, \quad \text{given by} \quad \big([\tilde{\vec{g}}], [\vec{x}]\big) \mapsto [\vec{x}] \, +_\Lambda \, [\tilde{\vec{g}}], \tag{B.38}$$

[5] One can also consider the more general problem of finding finite subgroups of Lie groups, which are also referred to as "lattices", and on which there is a great deal of mathematical literature (see e.g. [148, 149, 150], as well as [151]). The simplest examples are the cyclic groups $\mathbb{Z}_m \subset SO(2)$ and the regular polyhedra in $SO(3)$.

where $+_\Lambda$ is addition modulo the unit cell Λ of the lattice Γ which defines the torus. We are then interested in understanding the group $G \equiv \rho_\Gamma(\widetilde{\Gamma})$:

We point out that G is *not* given by the lattice quotient $\widetilde{\Gamma}/\Gamma$, because this operation is only defined when $\Gamma \subseteq \widetilde{\Gamma}$; that is, when all the points in Γ are also in $\widetilde{\Gamma}$. Rather, the quotient map ρ_Γ acts on $\mathbb{R}^2$ and carries $\widetilde{\Gamma} \subset \mathbb{R}^2$ along with it. The group G then contains the image under ρ_Γ of every $\tilde{\vec{g}} \in \widetilde{\Gamma}$:

$$G = \left\{ \left[\tilde{\vec{g}}\right] \in \mathbb{R}^2/\Gamma : \left[\tilde{\vec{g}}\right] = \left[\widetilde{\Lambda}\vec{s}\right],\, \vec{s} \in \mathbb{Z}^2 \right\}. \tag{B.39}$$

Under certain nice conditions[6], G is a lattice in $\mathbb{R}^2/\Gamma$, and its preimage $\rho_\Gamma^{-1}(G)$ is a lattice in $\mathbb{R}^2$. The preimage $\rho_\Gamma^{-1}(G)$ is actually the "smallest" lattice that contains both Γ and $\widetilde{\Gamma}$ as sublattices, so one typically has that $\widetilde{\Gamma}$ is strictly a sublattice of $\rho_\Gamma^{-1}(G)$ (with equality if and only if $\Gamma \subseteq \widetilde{\Gamma}$). In fact, there is a sense in which we can write $\rho_\Gamma^{-1}(G) = \mathrm{lcm}(\Gamma, \widetilde{\Gamma})$, where a lattice Γ' is considered a "multiple" of another lattice Γ if Γ' *contains* Γ, such that the lattice quotient Γ'/Γ makes sense[7].

We have said that LeBrun metrics can have both orbifold points and more general conical singularities, and we can now clarify the conditions under which one or the other occurs. The torus of angular coordinates near a conical point in the LeBrun metric is $\mathbb{R}^2/\widetilde{\Gamma}$, whereas the torus of standard $\mathbb{R}^4$ is $\mathbb{R}^2/\Gamma$. One has a quotient of $\mathbb{R}^4$ (and thus an orbifold point) whenever $\mathbb{R}^2/\Gamma$ is some p-fold cover of $\mathbb{R}^2/\widetilde{\Gamma}$. This happens precisely when $\Gamma \subseteq \widetilde{\Gamma}$ as a sublattice, or equivalently when $\rho_\Gamma^{-1}(G) = \widetilde{\Gamma}$. Otherwise one has a conical singularity that is not an orbifold.

---

[6] Γ and $\widetilde{\Gamma}$ must be commensurable. If they are not, then G will be dense in $\mathbb{R}^2/\Gamma$ and thus fail to be a lattice. For sensible results one must have $\Lambda^{-1}\widetilde{\Lambda} \in GL(2, \mathbb{Q})$. In the LeBrun metrics with integer parameters, we will only see rational lattices, so this is not a problem.

[7] Note that as lattices, $\mathbb{Z}$ is a "multiple" of $2\mathbb{Z}$ and not the other way around, because $\mathbb{Z}$ contains $2\mathbb{Z}$, and one can sensibly define the quotient $\mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}_2$.

By analogy with the $U(1)$ case in (B.30), the basis of the preimage lattice $\rho_\Gamma^{-1}(G)$ can be written

$$\Lambda_G \equiv \gcd(\Lambda, \widetilde{\Lambda}), \tag{B.40}$$

where we define the (left) $\gcd$ of two matrices $\Lambda, \widetilde{\Lambda} \in GL(2, \mathbb{Q})$ as the matrix $\Lambda_G \in GL(2, \mathbb{Q})$ of largest determinant such that $\Lambda_G^{-1}\Lambda$ and $\Lambda_G^{-1}\widetilde{\Lambda}$ are both integer. It turns out the answer is unique up to right multiplication $\Lambda_G \to \Lambda_G Q$ by $Q \in GL(2, \mathbb{Z})$, and it is invariant under right multiplications of the arguments $(\Lambda, \widetilde{\Lambda}) \to (\Lambda P, \widetilde{\Lambda}\widetilde{P})$ by $P, \widetilde{P} \in GL(2, \mathbb{Z})$. So it has all the required properties of a lattice basis. As we will show, $\gcd(\Lambda, \widetilde{\Lambda})$ can be computed by reducing $\widetilde{\Lambda}^{-1}\Lambda$ to *Smith normal form* using an algorithm analogous to the Euclidean algorithm.

Having found a basis for the preimage $\rho_\Gamma^{-1}(G)$, it is then simple to write down a more useful form of $G \simeq \rho_\Gamma(\rho_\Gamma^{-1}(G))$. In particular, since $\Gamma \subseteq \rho_\Gamma^{-1}(G)$, the lattice quotient $G \simeq \rho_\Gamma^{-1}(G)/\Gamma$ makes sense, and it is given by (omitting brackets $[\,\cdot\,]$ for equivalence classes):

$$G = \{\vec{g} \in \mathbb{R}^2/\Gamma : \ (\vec{g} \cong \Lambda_G\, \vec{s} \mod \Lambda), \ \vec{s} \in \mathbb{Z}^2\}, \qquad \Lambda_G \equiv \gcd(\Lambda, \widetilde{\Lambda}). \tag{B.41}$$

In the basis Λ_G, the structure of G as a direct product of cyclic groups $\mathbb{Z}_m \times \mathbb{Z}_n$ will be more obvious. In the following section, we will show how to compute $\gcd(\Lambda, \widetilde{\Lambda})$ and find precisely the cyclic groups for which $G \simeq \mathbb{Z}_m \times \mathbb{Z}_n$.

B.2.4 Orbifold points and more general conical singularities

Near each conical point in the LeBrun metric, one finds that the (local) metric approaches that of flat $\mathbb{R}^4$, but with the $U(1) \times U(1)$ coordinates identified on a lattice $\widetilde{\Gamma}$ different from the usual one Γ. One can define a group structure G, which is a finite subgroup of $U(1) \times U(1) \subset SO(4)$, by comparing the two lattices $\Gamma, \widetilde{\Gamma}$. The conical point is an *orbifold* point precisely when $\Gamma \subseteq \widetilde{\Gamma}$ as a sublattice, and then the local geometry approaches $\mathbb{R}^4/G$. In this section we will compute G.

Let Γ be the *standard* lattice on which to identify the $U(1) \times U(1)$ coordinates of $\mathbb{R}^4$. In the coordinates

$$\mathrm{d}s^2(\mathbb{R}^4) = \mathrm{d}\rho^2 + \rho^2\Big(\mathrm{d}\theta^2 + \cos^2\theta\,\mathrm{d}\alpha^2 + \sin^2\theta\,\mathrm{d}\beta^2\Big), \tag{B.42}$$

one has $(\alpha, \beta) \sim (\alpha + 2\pi, \beta) \sim (\alpha, \beta + 2\pi)$, and hence the basis Λ of Γ can be written

$$\Lambda = 2\pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{B.43}$$

In other coordinates, one must transform the lattice basis accordingly. Given a (linear) coordinate transformation $\mathcal{M} : (\alpha, \beta) \to (\alpha', \beta')$, the new lattice basis Λ' is given by

$$\begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = \mathcal{M} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \qquad \Lambda' = \mathcal{M}\Lambda, \tag{B.44}$$

where $\mathcal{M}$ acts only from the left, because Λ is a collection of column vectors. For example, if we transform to a 1-center Gibbons-Hawking chart with coordinates $\psi = \alpha + \beta, \chi = \alpha - \beta$, the new lattice Γ_{GH} is given by the basis

$$\Lambda_{GH} = 2\pi \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \text{or} \quad \Lambda_{GH} = 2\pi \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}, \tag{B.45}$$

as these are equivalent under right action by $GL(2, \mathbb{Z})$. In any case, given a standard lattice Γ such that $\mathbb{R}^2/\Gamma \simeq U(1) \times U(1) \subset SO(4)$ in our particular choice of coordinates, we can then compare this to the lattice $\widetilde{\Gamma}$ of coordinate identifications in (the near-singularity limit of) the LeBrun metric.

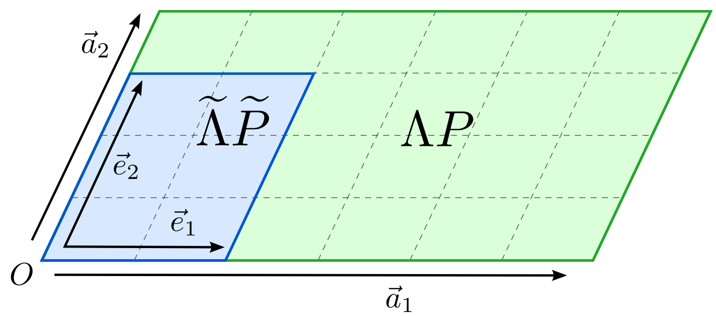

Figure B.1: *The lattice bases ΛP and $\widetilde{\Lambda}\widetilde{P}$ are parallel. There exist rational numbers r_1, r_2 such that $\vec{a}_1 = r_1\,\vec{e}_1$ and $\vec{a}_2 = r_2\,\vec{e}_2$. In this case $r_1 = 3$ and $r_2 = 4/3$.*

Reduction to Smith normal form

The lattices $\Gamma, \widetilde{\Gamma}$ have unit cells which are parallelograms of any dimensions and oriented in any directions. Let $\Lambda, \widetilde{\Lambda}$ be a choice of basis for each of $\Gamma, \widetilde{\Gamma}$. Since the lattices are rational to each other, we can always make a change of basis via right action by $P, \widetilde{P} \in GL(2, \mathbb{Z})$ such that the new bases $\Lambda P, \widetilde{\Lambda}\widetilde{P}$ are *parallel*, by which we mean

$$\widetilde{\Lambda}\widetilde{P}R = \Lambda P, \qquad \text{where} \qquad R = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix}, \tag{B.46}$$

for some rational numbers $r_1, r_2 > 0$. This is shown in Figure B.1.

The rational numbers r_1, r_2 give the factors by which each leg of ΛP is larger than the same leg of $\widetilde{\Lambda}\widetilde{P}$. It is easy to see that each leg of $\widetilde{\Lambda}\widetilde{P}$ generates a cyclic group modulo the unit cell ΛP, and hence one has

$$G \simeq \mathbb{Z}_m \times \mathbb{Z}_n, \quad \text{where} \quad m = \frac{r_1}{\gcd(1, r_1)}, \quad n = \frac{r_2}{\gcd(1, r_2)}. \tag{B.47}$$

Then the basis $\Lambda_G \equiv \gcd(\Lambda, \widetilde{\Lambda})$ of the preimage lattice $\rho_\Gamma^{-1}(G)$ (defined as in (B.37)) is given by

$$\Lambda_G = \begin{pmatrix} \vec{g}_1 & \vec{g}_2 \end{pmatrix}, \quad \text{where} \quad \vec{g}_1 = \gcd(1, r_1)\,\vec{e}_1, \quad \vec{g}_2 = \gcd(1, r_2)\,\vec{e}_2. \tag{B.48}$$

An *orbifold* point occurs precisely when r_1, r_2 are integers, in which case the lattice cell $\widetilde{\Lambda}$ already "fits into" Λ evenly. Then (B.47) can be written simply

$$G \simeq \mathbb{Z}_m \times \mathbb{Z}_n, \quad \text{where} \quad m = r_1, \quad n = r_2. \tag{B.49}$$

That is, at an orbifold point, the entries in the diagonal matrix R give the orders of $\mathbb{Z}_m, \mathbb{Z}_n$.

What is left is to find r_1, r_2 in the first place. To do this, one takes (B.46) and isolates the diagonal matrix R:

$$R = \widetilde{P}^{-1}\widetilde{\Lambda}^{-1}\Lambda P. \tag{B.50}$$

We do not need to know $P, \widetilde{P} \in GL(2, \mathbb{Z})$ explicitly; we merely need to describe an algorithm for diagonalizing $\widetilde{\Lambda}^{-1}\Lambda$ by independent actions of $GL(2, \mathbb{Z})$ from both the left and the right. This is precisely the algorithm for finding the *Smith normal form* of a matrix. Since we have available both left and right $GL(2, \mathbb{Z})$ actions, we may apply any sequence of elementary row *or* column operations which are invertible over $\mathbb{Z}$.

Hence to obtain R we diagonalize $\widetilde{\Lambda}^{-1}\Lambda$ via the following process. At every step of the algorithm, we may

1. Swap any two rows or any two columns, or
2. Multiply any row, or any column, by -1, or
3. Add an integer multiple of any row (column) to another row (column).

The objective is to reach a diagonal matrix (this is always possible). The full algorithm for the Smith normal form continues until the matrix is not only diagonal, but each entry along the diagonal divides the next, i.e. $r_1 | r_2$ in this case. For our purposes, however, any diagonal matrix will do (and the result may not be unique).

In the case where the result is not unique, different possible results R yield different ways of writing the *same* group G. For example, a given matrix might be diagonalized in two different ways to give $G \simeq \mathbb{Z}_4 \times \mathbb{Z}_6$ or $G \simeq \mathbb{Z}_2 \times \mathbb{Z}_{12}$, but these groups are isomorphic. The same matrix cannot also be diagonalized to give, e.g. $\mathbb{Z}_3 \times \mathbb{Z}_8$—the algorithm as constructed preserves the group structure[8].

Once we have obtained R, we can then calculate the group G via (B.47). We note that the order of G is

$$\#G = mn = \frac{r_1}{\gcd(1, r_1)} \times \frac{r_2}{\gcd(1, r_2)} \geq \frac{r_1 r_2}{\gcd(1, r_1 r_2)}. \tag{B.51}$$

But $r_1 r_2 = \det R = \det(\widetilde{\Lambda}^{-1}\Lambda)$. Hence in terms of our lattice bases, we can put a lower bound on $\#G$:

$$\#G \geq \frac{\det \Lambda}{\gcd(\det \Lambda,\, \det \widetilde{\Lambda})}, \tag{B.52}$$

where we assume, without loss of generality, that $\det \Lambda, \det \widetilde{\Lambda} > 0$ (which can always be arranged by the right action of $GL(2, \mathbb{Z})$). We note further that, at an *orbifold* point where $r_1, r_2 \in \mathbb{Z}$, the inequality (B.52) is saturated, and then we can calculate the order of the group G directly from $\Lambda, \widetilde{\Lambda}$.

B.2.5 The conical points of LeBrun metrics

In this section we will find the groups G at the conical points of the LeBrun metric using the methods outlined in the previous section.

[8]Specifically, the reduction to Smith normal form of a square matrix M preserves the sequence of *invariant factors* $r_1 | r_2 | \ldots | r_n$ such that $\det M = r_1 r_2 \ldots r_n$ and each $r_i | r_{i+1}$. It is precisely this sequence that distinguishes when the direct product of cyclic groups $\mathbb{Z}_{r_1} \times \mathbb{Z}_{r_2} \times \ldots \times \mathbb{Z}_{r_n}$ is isomorphic to another direct product of the same order.

Near the conical points, the LeBrun metric approaches the form (5.40) and (B.26)

$$\mathrm{d}s^2(LB) = \mathrm{d}\varrho^2 + \frac{\varrho^2}{4}\left[\mathrm{d}\theta^2 + \frac{1}{\widetilde{q}_\ell{}^2}\Big(\widetilde{K}(\theta)\,\mathrm{d}\tau^2 - 2\widetilde{KQ}(\theta)\,\mathrm{d}\tau\,\mathrm{d}\phi + \widetilde{Q}(\theta)\,\mathrm{d}\phi^2\Big)\right], \quad \text{(B.53)}$$

and one must then compare it to a standard flat metric on $\mathbb{R}^4$. One choice is a 1-center Gibbons-Hawking metric (B.28):

$$\mathrm{d}s^2(GH) = \mathrm{d}\varrho^2 + \frac{\varrho^2}{4}\left[\mathrm{d}\theta^2 + \mathrm{d}\psi^2 + \mathrm{d}\chi^2 + 2\cos\theta\,\mathrm{d}\psi\,\mathrm{d}\chi\right]. \quad \text{(B.54)}$$

The coordinate change to (ψ, χ) of Gibbons-Hawking is (B.29)

$$\psi = \frac{1}{\widetilde{q}_\ell}\Big((\bar{K}^3_\ell - k^3_0)\,\tau - (\bar{Q}_\ell - q_0)\,\phi\Big), \qquad \chi = \frac{1}{\widetilde{q}_\ell}\Big(k^3_\ell\,\tau - q_\ell\,\phi\Big). \quad \text{(B.55)}$$

Alternatively, we can compare to the more standard $\mathbb{R}^4$ metric,

$$\mathrm{d}s^2(\mathbb{R}^4) = \mathrm{d}\varrho^2 + \varrho^2\Big(\mathrm{d}\vartheta^2 + \cos^2\vartheta\,\mathrm{d}\alpha^2 + \sin^2\vartheta\,\mathrm{d}\beta^2\Big), \quad \text{(B.56)}$$

related to (B.54) via $\psi = \alpha + \beta, \chi = \alpha - \beta$ and $\theta = 2\vartheta$. From the LeBrun coordinates (τ, ϕ), one can go to (α, β) via

$$\alpha = \frac{1}{2\widetilde{q}_\ell}\Big((k^3_\ell + \bar{K}^3_\ell - k^3_0)\,\tau - (q_\ell + \bar{Q}_\ell - q_0)\,\phi\Big), \quad \text{(B.57)}$$

$$\beta = \frac{1}{2\widetilde{q}_\ell}\Big((k^3_\ell - \bar{K}^3_\ell + k^3_0)\,\tau - (q_\ell - \bar{Q}_\ell + q_0)\,\phi\Big). \quad \text{(B.58)}$$

While the transformation (B.55) to (ψ, χ) looks simpler, we will generally find it less confusing to work with (α, β), with the exception of the following paragraph:

In order to apply the method of the previous section, we first need to identify a "standard" lattice Γ, which means we need to sort out how the LeBrun coordinates (τ, ϕ) should be identified in the first place. This is actually an arbitrary choice (it will merely affect how we interpret the various parameters q_ℓ, k^3_ℓ). However, we can use the transformation to Gibbons-Hawking (B.55) as a guide to make a "nice" choice. We observe that the coordinate transformation (B.55) has determinant $-1/\widetilde{q}_\ell$, so let us choose some parameters such that

$$\widetilde{q}_\ell \equiv q_\ell(\bar{K}^3_\ell - k^3_0) - k^3_\ell(\bar{Q}_\ell - q_0) = 1. \tag{B.59}$$

Making the choice $q_\ell = \bar{K}^3_\ell - k^3_0 = 1, k^3_\ell = \bar{Q}_\ell - q_0 = 0$, we obtain very simply

$$\psi = \tau, \qquad \chi = -\phi. \tag{B.60}$$

So, we will find it very natural to identify τ, ϕ on a diamond:

$$(\tau,\phi): \quad (0,0) \sim (4\pi, 0) \sim (2\pi, 2\pi) \sim (2\pi, -2\pi). \tag{B.61}$$

and then the above choice of parameters corresponds to flat $\mathbb{R}^4$ with trivial orbifold group.

From here forward we will stick to the (α, β) coordinates. By following the identifications (B.61) along the coordinate transformation (B.57), (B.58), we obtain the lattice $\widetilde{\Gamma}$ in the coordinates (α, β) given by the basis

$$\widetilde{\Lambda} = 2\pi \cdot \frac{1}{2\widetilde{q}_\ell} \begin{pmatrix} k^3_\ell + \hat{K}^3_\ell + q_\ell + \hat{Q}_\ell & k^3_\ell + \hat{K}^3_\ell - q_\ell - \hat{Q}_\ell \\ k^3_\ell - \hat{K}^3_\ell + q_\ell - \hat{Q}_\ell & k^3_\ell - \hat{K}^3_\ell - q_\ell + \hat{Q}_\ell \end{pmatrix}, \tag{B.62}$$

where for ease of legibility we have defined

$$\hat{K}^3_\ell \equiv \bar{K}^3_\ell - k^3_0, \qquad \hat{Q}_\ell \equiv \bar{Q}_\ell - q_0. \tag{B.63}$$

The standard lattice Γ in the coordinates (α, β) is given simply by the basis

$$\Lambda = 2\pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{B.64}$$

which makes the calculations easy, as $\widetilde{\Lambda}^{-1}\Lambda$ is just $2\pi\widetilde{\Lambda}^{-1}$.

From (B.52), we see that the order of the group G is at least $|\widetilde{q}_\ell|$:

$$\det(\widetilde{\Lambda}^{-1}\Lambda) = -\widetilde{q}_\ell, \quad \text{and hence} \quad \#G \geq |\widetilde{q}_\ell|, \tag{B.65}$$

And if $r_1, r_2 \in \mathbb{Z}$, we have simply

$$\#G = |\widetilde{q}_\ell| \quad \text{at orbifold points.} \tag{B.66}$$

When is a conical point an orbifold point?

As we have pointed out, an orbifold point occurs when $r_1, r_2 \in \mathbb{Z}$, or alternatively, when $\widetilde{\Lambda}^{-1}\Lambda \in \mathrm{Mat}_2(\mathbb{Z})$, the set (not group) of 2×2 matrices with integer entries. This yields the condition

$$\frac{1}{2} \begin{pmatrix} k_\ell^3 - \hat{K}_\ell^3 - q_\ell + \hat{Q}_\ell & -k_\ell^3 - \hat{K}_\ell^3 + q_\ell + \hat{Q}_\ell \\ -k_\ell^3 + \hat{K}_\ell^3 - q_\ell + \hat{Q}_\ell & k_\ell^3 + \hat{K}_\ell^3 + q_\ell + \hat{Q}_\ell \end{pmatrix} \in \mathrm{Mat}_2(\mathbb{Z}), \tag{B.67}$$

where notably the $1/\widetilde{q}_\ell$ in (B.62) has dropped out. Thus a LeBrun metric contains *only* orbifold points, and no generic conical points, when the sum of all the parameters is even:

$$\left(k_0^3 + \sum_{i=1}^{N} k_i^3 + q_0 + \sum_{i=1}^{N} q_i \right) \in 2\mathbb{Z}. \tag{B.68}$$

Conversely, *none* of the conical points have the quotient structure $\mathbb{R}^4/G$ if the sum of parameters is odd. We will assume this sum is even such that each conical point is an orbifold point with structure $\mathbb{R}^4/G$.

When is the group G trivial?

The group G is trivial whenever $\widetilde{\Gamma}, \Gamma$ are the *same* lattice. This happens whenever $\widetilde{\Lambda}^{-1}\Lambda \in GL(2,\mathbb{Z})$. That is,

$$\frac{1}{2}\begin{pmatrix} k_\ell^3 - \hat{K}_\ell^3 - q_\ell + \hat{Q}_\ell & -k_\ell^3 - \hat{K}_\ell^3 + q_\ell + \hat{Q}_\ell \\ -k_\ell^3 + \hat{K}_\ell^3 - q_\ell + \hat{Q}_\ell & k_\ell^3 + \hat{K}_\ell^3 + q_\ell + \hat{Q}_\ell \end{pmatrix} \in GL(2,\mathbb{Z}), \tag{B.69}$$

Thus again the sum of the parameters k_0^3, k_i^3, q_0, q_i must be even. The determinant of this matrix is $\widetilde{q}_\ell \equiv q_\ell \hat{K}_\ell^3 - k_\ell^3 \hat{Q}_\ell$. Therefore for the metric to locally look like $\mathbb{R}^4$ with no conical singularity requires

$$\widetilde{q}_\ell = \pm 1, \quad \text{and} \quad \left(k_0^3 + \sum_{i=1}^{N} k_i^3 + q_0 + \sum_{i=1}^{N} q_i\right) \in 2\mathbb{Z}. \tag{B.70}$$

When is the group G like a Gibbons-Hawking orbifold group?

A 1-center Gibbons-Hawking metric with "charge" m, written

$$\mathrm{d}s^2(GH) = \frac{r}{m}\Big(\mathrm{d}\psi + m\cos\theta\,\mathrm{d}\chi\Big)^2 + \frac{m}{r}\Big(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\chi^2\Big), \tag{B.71}$$

is a metric on the orbifold $\mathbb{R}^4/\mathbb{Z}_m$, where $\mathbb{Z}_m$ acts in the *diagonal* $U(1)$ of the maximal torus $U(1)\times U(1) \in SO(4)$. In (α,β) coordinates, this corresponds to the lattice Γ_{GH} with basis

$$\Lambda_{GH} = 2\pi \begin{pmatrix} 1 & \frac{p}{m} \\ 0 & \frac{p}{m} \end{pmatrix}, \tag{B.72}$$

where p and m are relativaly prime. The LeBrun metric then has a "diagonal" orbifold point whenever $\widetilde{\Lambda}^{-1}\Lambda_{GH} \in GL(2,\mathbb{Z})$, or equivalently, whenever $\Lambda_{GH}^{-1}\widetilde{\Lambda} \in GL(2,\mathbb{Z})$, since the determinant is ± 1 in any case. This requires first that

$$\det(\Lambda_{GH}^{-1}\widetilde{\Lambda}) = -\frac{m}{p\widetilde{q}_\ell} = \pm 1, \quad \text{or} \quad m = \pm p\widetilde{q}_\ell. \tag{B.73}$$

But since p and m are relatively prime, we must have $p = 1$ and $\widetilde{q}_\ell = m$. Next, writing out $\Lambda_{GH}^{-1}\widetilde{\Lambda}$ we have

$$\frac{1}{2\widetilde{q}_\ell}\begin{pmatrix} 2(\hat{K}_\ell^3 + \hat{Q}_\ell) & 2(\hat{K}_\ell^3 - \hat{Q}_\ell) \\ \widetilde{q}_\ell(k_\ell^3 - \hat{K}_\ell^3 + q_\ell - \hat{Q}_\ell) & \widetilde{q}_\ell(k_\ell^3 - \hat{K}_\ell^3 - q_\ell + \hat{Q}_\ell) \end{pmatrix} \in GL(2,\mathbb{Z}). \tag{B.74}$$

So again, the sum of all the parameters must be even, and one gets a "diagonal" orbifold point wherever

$$\frac{2(\bar{K}_\ell^3 - k_0^3)}{\widetilde{q}_\ell} \in \mathbb{Z} \quad \text{and} \quad \frac{2(\bar{Q}_\ell - q_0)}{\widetilde{q}_\ell} \in \mathbb{Z}. \tag{B.75}$$

One may also consider $\mathbb{Z}_m$ acting in the "anti-diagonal" $U(1)$, which in (α,β) coordinates corresponds to the lattice $\Gamma_{\overline{GH}}$ with basis

$$\Lambda_{\overline{GH}} = 2\pi\begin{pmatrix} 1 & -\frac{1}{m} \\ 0 & \frac{1}{m} \end{pmatrix}. \tag{B.76}$$

One can similarly show that these points occur for $\widetilde{q}_\ell = m$ and

$$\frac{2\,k_\ell^3}{\widetilde{q}_\ell} \in \mathbb{Z} \quad \text{and} \quad \frac{2\,q_\ell}{\widetilde{q}_\ell} \in \mathbb{Z}. \tag{B.77}$$

Appendix C

More general equations for 6d solutions with KK monopoles

In Section 7.4, we made certain simplifying assumptions that result in a system of equations constrained by (7.79). This gives a particularly simple set of equations $\mathcal{L}f = 0$, where f is any of $K_1, K_2, L_1, L_2, L_3, M$. However, this is not the most general form of the equations. Revisiting (7.76), we can easily make a general ansatz at least for the Θ_j and do it in a manner that leads to a similar simplification of the source terms. We simply introduce vector fields, $\vec{\lambda}_j$, into (7.76):

$$\Theta_j \;=\; -\sum_{a=1}^{3} \Big(\mathfrak{D}_a\big(V^{-1}\,K_j\big) + \lambda_{j\,a}\Big)\,\Omega_+^{(a)}\,, \qquad j = 1, 2\,. \tag{C.1}$$

Then the constraints (7.79) are replaced by

$$\begin{aligned} \vec{\mathfrak{D}}(\partial_\psi K_1 + \partial_v L_2) &= \vec{\mathfrak{D}} \times \vec{\lambda}_1 - (V\partial_\psi - K_3\partial_v)\vec{\lambda}_1, \\ \vec{\mathfrak{D}}(\partial_\psi K_2 + \partial_v L_1) &= \vec{\mathfrak{D}} \times \vec{\lambda}_2 - (V\partial_\psi - K_3\partial_v)\vec{\lambda}_2. \end{aligned} \tag{C.2}$$

The remainder of the equations can be organized (after some manipulation) into pairs that exhibit manifest symmetry under spectral interchange. The first layer are given by

$$\mathcal{L}K_1 = -V\vec{\mathfrak{D}}\cdot\vec{\lambda}_1 - 2\,\vec{\nabla}V\cdot\vec{\lambda}_1 + V\,(V\partial_\psi - K_3\partial_v)(\partial_\psi K_1 + \partial_v L_2), \tag{C.3}$$

$$\mathcal{L}L_2 = K_3\vec{\mathfrak{D}}\cdot\vec{\lambda}_1 + 2\,\vec{\nabla}K_3\cdot\vec{\lambda}_1 - K_3\,(V\partial_\psi - K_3\partial_v)(\partial_\psi K_1 + \partial_v L_2), \tag{C.4}$$

and a similar pair under exchanging the subscripts $(1 \leftrightarrow 2)$. The second layer becomes

$$\begin{aligned}\mathcal{L}L_3 = -2\partial_v\big(\vec{\mathfrak{D}} \cdot \vec{\omega} + \Phi\big)&\\ + 2V\,\vec{\lambda}_1 \cdot \vec{\lambda}_2 + K_2\,\vec{\mathfrak{D}} \cdot \vec{\lambda}_1 + 2\,\vec{\mathfrak{D}}K_2 \cdot \vec{\lambda}_1 + K_1\,\vec{\mathfrak{D}} \cdot \vec{\lambda}_2 + 2\,\vec{\mathfrak{D}}K_1 \cdot \vec{\lambda}_2&\\ - K_2(V\partial_\psi - K_3\partial_v)(\partial_\psi K_1 + \partial_v L_2) - K_1(V\partial_\psi - K_3\partial_v)(\partial_\psi K_2 + \partial_v L_1)&\\ - 2\,(V\partial_\psi - K_3\partial_v)K_2\,(\partial_\psi K_1 + \partial_v L_2) - 2\,(V\partial_\psi - K_3\partial_v)K_1\,(\partial_\psi K_2 + \partial_v L_1)&\\ + 2V\,(\partial_\psi K_1 + \partial_v L_2)(\partial_\psi K_2 + \partial_v L_1),&\end{aligned} \tag{C.5}$$

and

$$\begin{aligned}\mathcal{L}M = \partial_\psi\big(\vec{\mathfrak{D}} \cdot \vec{\omega} + \Phi\big)&\\ - K_3\,\vec{\lambda}_1 \cdot \vec{\lambda}_2 + \tfrac{1}{2}L_1\,\vec{\mathfrak{D}} \cdot \vec{\lambda}_1 + \vec{\mathfrak{D}}L_1 \cdot \vec{\lambda}_1 + \tfrac{1}{2}L_2\,\vec{\mathfrak{D}} \cdot \vec{\lambda}_2 + \vec{\mathfrak{D}}L_2 \cdot \vec{\lambda}_2&\\ - \tfrac{1}{2}L_1(V\partial_\psi - K_3\partial_v)(\partial_\psi K_1 + \partial_v L_2) - \tfrac{1}{2}L_2(V\partial_\psi - K_3\partial_v)(\partial_\psi K_2 + \partial_v L_1)&\\ - (V\partial_\psi - K_3\partial_v)L_1\,(\partial_\psi K_1 + \partial_v L_2) - (V\partial_\psi - K_3\partial_v)L_2\,(\partial_\psi K_2 + \partial_v L_1)&\\ - K_3\,(\partial_\psi K_1 + \partial_v L_2)(\partial_\psi K_2 + \partial_v L_1),&\end{aligned} \tag{C.6}$$

which show the spectral interchange symmetry and the dependence on $\partial_\psi K_1 + \partial_v L_2$ and $\partial_\psi K_2 + \partial_v L_1$. Again, Φ is defined as in (7.87). Finally, for $\vec{\omega}$, we have

$$\begin{aligned}\vec{\mathfrak{D}} \times \vec{\omega} + (V\partial_\psi - K_3\partial_v)\vec{\omega} = V\vec{\mathfrak{D}}M - M\vec{\mathfrak{D}}V + \frac{1}{2}\sum_{I=1}^{3}\big(K^I\vec{\mathfrak{D}}L_I - L_I\vec{\mathfrak{D}}K^I\big)&\\ - (K_2K_3 + VL_1)\,\vec{\lambda}_1 - (K_1K_3 + VL_2)\,\vec{\lambda}_2,&\end{aligned} \tag{C.7}$$

which is invariant under spectral interchange.