A NONPERTURBATIVE CALCULATION OF THE SPECTRUM OF A NONHERMITE FOKKER-PLANCK HAMILTONIAN

Hiromichi Nakazato and Takeshi Yamashiro

Dept. of Phys., Univ. of the Ryukyus, Okinawa 903-01, Japan

ABSTRACT

On the basis of a δ-expansion technique proposed by Bender *et al.*, eigenvalues of a nonhermite Fokker-Planck hamiltonian which appears in the context of stochastic quantization of a system with a complex action are calculated up to the second order. The calculation is considered to be nonperturbative w.r.t. physical coupling constants in the sense that their nontrivial effects have been taken into account. We focus our attention on the sign of the real part of eigenvalues to clarify the approach to the equilibrium state.

Introduction

In the application of the stochastic quantization method [1] to a system with a complex Euclidean action [2] or in its Minkowski space-time formulation, one of the most important problems is to clarify whether the stationary solution of the Fokker-Planck (F-P) hamiltonian is indeed its true equilibrium solution or not. In the usual case with a real Euclidean action, the positivity of eigenvalues of the F-P hamiltonian (we define their eigenvalue equation as in (5) below) is easily shown from the fact that the hamiltonian can be transformed into a hermite one $-A^\dagger A$ by a similarity transformation. Contrary to this, very little is known about the spectrum of the F-P hamiltonian with a complex action or in the Minkowski space-time formulation, because in these cases the hamiltonian is essentially nonhermite; nobody has yet found any transformation to make it hermite. Eigenvalues of such a hamiltonian are in general complex valued, but to clarify asymptotic behaviour of the system in the large fictitious time limit $t \to \infty$ we only have to know the sign of the real part of the eigenvalues. That is, if we can show its positive semi-definiteness the approach to the desired equilibrium states of the form e^{-S_E} or e^{iS_M} will be guaranteed.

The purpose of this paper is to solve the eigenvalue equation of the F-P hamiltonian with a complex action nonperturbatively to get an analytic form of the eigenvalues. We adopt here a δ-expansion technique as a nonperturbative method. This technique has recently been introduced by Bender *et al.* [3] to solve nonlinear equations nonperturbatively and is said to have several advantages over conventional perturbative methods including the property of rapid convergence and the preservation of nontrivial functional dependence on physical parameters. Eigenvalues will be calculated up to the second order in δ and the sign of their real part is discussed following the above mentioned line of thought.

In this paper we consider two simple cases:

(i) a system with quartic interaction as well as quadratic one in the zero dimensional (Minkowski) space-time

$$iS_M = -\frac{i}{2}(\mu^2 - i\epsilon)x^2 - \frac{ig}{4!}x^4 \tag{1}$$

and

(ii) a system with similar interactions represented by a complex action in the zero dimensional (Euclidean) space-time

$$-S_E = -\sigma x^2 - \frac{g'}{2}x^4. \tag{2}$$

Here μ^2, ϵ, g and g' are real positive parameters, while σ takes a complex value. The ϵ-term in (1) represents a damping term and the case Klauder and Petersen have taken in their numerical calculation [4] is represented by the above S_E with $g' = 1$.

Preliminaries

Let us consider the following "effective" F-P equation for the "complex probability" distribution $P(x,t)$

$$\frac{\partial}{\partial t}P(x,t) = H(x)P(x,t) \tag{3}$$

with the F-P hamiltonian $H(x)$

$$H(x)=\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}+S'(x)\right) \qquad (4)$$

where the action S is to be replaced by $-iS_M$ and S_E in cases (i) and (ii) respectively [5]. Owing to the nonhermiticity of H, $H^\dagger \neq H$, which is a consequence of $S^* \neq S$, both right- and left-eigenvalue equations, i.e. both eigenvalue problems for H and $H^\dagger$ should be considered here. Let $|u_n\rangle$ and $|v_n\rangle$ be eigenvectors of H and $H^\dagger$ belonging to eigenvalues $-\lambda_n$ and $-\xi_n^*$, respectively,

$$H|u_n\rangle=-\lambda_n|u_n\rangle, \quad H^\dagger|v_n\rangle=-\xi_n^*|v_n\rangle. \qquad (5)$$

It is easy to show the equality $\lambda_n=\xi_n$ and the orthogonality $\langle v_n|u_m\rangle \propto \delta_{nm}$. We assume here that $\{|v_n\rangle\}$ and $\{|u_n\rangle\}$ form a complete orthonormal set characterized by an orthonormal relation and a completeness relation

$$\langle v_n|u_m\rangle=\delta_{nm}, \quad \sum_n |u_n\rangle\langle v_n|=1. \qquad (6)$$

We introduce an artificial expansion parameter δ into the action S by replacing every 4 appearing in S with $2+2\delta$ [3]. Expanding every quantity as a series in powers of δ

$$H=H_0+H_1+H_2+\cdots, \qquad (7)$$

$$|u_n\rangle=|u_n^{(0)}\rangle+|u_n^{(1)}\rangle+|u_n^{(2)}\rangle+\cdots, \qquad (8a)$$

$$|v_n\rangle=|v_n^{(0)}\rangle+|v_n^{(1)}\rangle+|v_n^{(2)}\rangle+\cdots, \qquad (8b)$$

$$\lambda_n=\lambda_n^{(0)}+\lambda_n^{(1)}+\lambda_n^{(2)}+\cdots, \qquad (9)$$

and equating terms of the same order in δ on both sides of the eigenvalue equations (5), we get higher order corrections to $\lambda_n^{(0)}$ as in the usual perturbation theory.

Calculation of eigenvalues of H

First let us consider case (i). The action S_M is now regarded as a function of the parameter δ

$$S_M(x;\delta)=-\frac{1}{2}(\mu^2-i\epsilon)x^2-\frac{g}{(2+2\delta)!}x^{2+2\delta} \qquad (10)$$

where $(2+2\delta)! \equiv \Gamma(3+2\delta)$. S_M' is expressed as a series in powers of δ and we substitute $S'(x)$ in (4) by this $-iS_M'$. The lowest order eigenvalue equation is easily solved using two operators Q and $\bar{Q}^\dagger$ defined by

$$Q=\frac{\partial}{\partial x}+\lambda_1^{(0)}x, \quad \bar{Q}^\dagger=\frac{\partial}{\partial x} \qquad (11)$$

and we easily construct all eigenvectors $|u_n^{(0)}\rangle$ and $|v_n^{(0)}\rangle$ in the following form

$$\begin{aligned}|u_n^{(0)}\rangle&=\frac{1}{\sqrt{n!}}i^n(\lambda_1^{(0)})^{-\frac{n}{2}}(\bar{Q}^\dagger)^n|u_0^{(0)}\rangle,\\ |v_n^{(0)}\rangle&=\frac{1}{\sqrt{n!}}(-i)^n(\lambda_1^{(0)*})^{-\frac{n}{2}}(Q^\dagger)^n|v_0^{(0)}\rangle\end{aligned} \qquad (12)$$

from the "vacuum states" $|u_0^{(0)}\rangle$ and $|v_0^{(0)}\rangle$ which satisfy $Q|u_0^{(0)}\rangle=0$ and $\bar{Q}|v_0^{(0)}\rangle=0$, respectively. Left- and right-eigenvectors of H_0 are orthonormal to each other and belong to the same eigenvalue $-\lambda_n^{(0)}$, i.e.,

$$\langle v_n^{(0)}|u_m^{(0)}\rangle=\delta_{nm} \qquad (13)$$

$$\begin{aligned}H_0|u_n^{(0)}\rangle&=-\lambda_n^{(0)}|u_n^{(0)}\rangle,\\ \langle v_n^{(0)}|H_0&=-\lambda_n^{(0)}\langle v_n^{(0)}|\end{aligned} \qquad (14)$$

with integral $n \geq 0$

$$\lambda_n^{(0)}=n\lambda_1^{(0)}=n\{\epsilon+i(\mu^2+g)\}. \qquad (15)$$

Note that the real part of $\lambda_n^{(0)}$ is positive semi-definite and proportional to ϵ.

Let us proceed to calculate the first order correction $\lambda_n^{(1)}$, which is explicitly given by

$$\lambda_n^{(1)}=-i\delta g\langle v_n^{(0)}|\frac{\partial}{\partial x}x[\ln x^2-2\{1+\psi(1)\}]|u_n^{(0)}\rangle. \qquad (16)$$

To treat a logarithm we make use of the following formula with a regularization parameter $\alpha\,(>0)$

$$\ln\frac{b}{a}=\lim_{\alpha\to 0}\int_0^\infty dt\, t^{\alpha-1}\left(e^{-at}-e^{-bt}\right)$$

for $a,b>0$. Owing to this regularization parameter α each term in the above integrand is well defined. After somewhat a long calculation [6], we arrive at the final form of $\lambda_n^{(1)}$

$$\lambda_n^{(1)}=i\delta gn\Big\{1+2\sum_{\ell=1}^{[\frac{n-1}{2}]}\frac{1}{2\ell+1}+\frac{(-1)^n-1}{2n}+\gamma-\ln 2|\lambda_1^{(0)}|-i\theta\Big\}. \qquad (17)$$

Here the variable θ stands for an argument of $\lambda_1^{(0)}$, i.e., $\lambda_1^{(0)}=|\lambda_1^{(0)}|e^{i\theta}$ and $\gamma=0.577\cdots$. Notice that the real part of $\lambda_n^{(1)}$ given by

$$\Re(\lambda_n^{(1)})=\delta gn\theta \qquad (18)$$

is positive semi-definite for any positive ϵ as long as $\mu^2 + g > 0$

$$0 < \theta = \arctan \frac{\mu^2 + g}{\epsilon} \lesssim \frac{\pi}{2}. \qquad (19)$$

It is very interesting to note that this property is preserved even in the $\epsilon \to 0$ limit where $\Re(\lambda_n^{(1)}) = \delta g n \pi / 2 > 0$.

The above procedure is also applicable to the calculation of the second order correction $\lambda_n^{(2)}$, whose real part is calculated to be

$$\Re(\lambda_n^{(2)}) = \delta^2 \frac{\pi}{2} n g \Big\{ -\frac{g}{\mu^2 + g} + \gamma - \ln 2(\mu^2 + g) + \sum_{\ell=1}^{[\frac{n-1}{2}]} \frac{1}{2\ell + 1} + \frac{(-1)^n - 1}{2n} + o(\epsilon) \Big\}. \qquad (20)$$

Here ϵ is supposed to be sufficiently small and only the lowest order (ϵ independent) terms are presented. Detailed calculations are found elsewhere [6].

Thus we have calculated the eigenvalue λ_n up to the second order in δ. If we set $\delta = 1$ and consider ϵ small it becomes

$$\Re(\lambda_n) \simeq n \Big\{ \frac{\mu^2}{\mu^2 + g} \epsilon + \frac{\pi}{2} g \Big[\frac{\mu^2}{\mu^2 + g} + \gamma - \ln 2(\mu^2 + g) + \sum_{\ell=1}^{[\frac{n-1}{2}]} \frac{1}{2\ell + 1} + \frac{(-1)^n - 1}{2n} \Big] \Big\} \qquad (21)$$

which may imply that the positivity of $\Re(\lambda_n)$ for $n \geq 1$ is not always guaranteed though it is likely to hold if μ^2 and g are small, specifically $\mu^2 + g < 1/2$.

Next we turn to case (ii). Just as in case (i), we can calculate the eigenvalue of the F-P hamiltonian. Calculations are parallel to the previous case and we easily arrive at the results

$$\lambda_n^{(0)} = n \lambda_1^{(0)} = n(2\sigma + g'), \qquad (22)$$

$$\lambda_n^{(1)} = \delta g' n \Big\{ 4 + 2 \sum_{\ell=1}^{[\frac{n-1}{2}]} \frac{1}{2\ell + 1} - \ln 2|\lambda_1^{(0)}| - \gamma + \frac{(-1)^n - 1}{2n} - i\theta \Big\} \qquad (23)$$

where $|\lambda_1^{(0)}| = \sqrt{(2\Re(\sigma) + g')^2 + (2\Im(\sigma))^2}$ and $\theta = \arg(\sigma)$. If we set $\delta = 1$ and $g' = 1$, the real part of λ_n turns out to be

$$\Re(\lambda_n) = n \Big[5 + 2 \sum_{\ell=1}^{[\frac{n-1}{2}]} \frac{1}{2\ell + 1} + 2\Re(\sigma) - \ln 2|\lambda_1^{(0)}| - \gamma + \frac{(-1)^n - 1}{2n} \Big] \qquad (24)$$

with $|\lambda_1^{(0)}| = \sqrt{4|\sigma|^2 + 4\Re(\sigma) + 1}$.

It is not at all difficult to obtain the second order correction $\lambda_n^{(2)}$ in this case, though we could not find such a compact expression for $\Re(\lambda_n^{(2)})$ with a general n (≥ 0) as that in case (i) [6]. Here we only present the explicit expressions of the real part of the two lowest eigenvalues λ_1 and λ_2 (Needless to say, $\lambda_0 = 0$.):

$$\begin{aligned}
&\Re(\lambda_1) \\
&= 2\Re(\sigma) + g' + g'\delta(3 - \ln 2|\lambda_1^{(0)}| - \gamma) \\
&\quad + \frac{g'^2 \delta^2}{|\lambda_1^{(0)}|} \{ \cos\theta (\ln 2|\lambda_1^{(0)}| + \gamma - 4 + a_1) + \theta \sin\theta \} \\
&\quad + g'\delta^2 \{ \frac{1}{2} (\ln 2|\lambda_1^{(0)}| + \gamma)^2 - \frac{1}{2}\theta^2 \\
&\quad - 2(\ln 2|\lambda_1^{(0)}| + \gamma) + 2 + a_2 \}, \qquad (25)
\end{aligned}$$

$$\begin{aligned}
&\Re(\lambda_2) \\
&= 2(2\Re(\sigma) + g') + 2g'\delta(4 - \ln 2|\lambda_1^{(0)}| - \gamma) \\
&+ \frac{g'^2 \delta^2}{|\lambda_1^{(0)}|} \{ \cos\theta (2(\ln 2|\lambda_1^{(0)}| + \gamma) - \frac{26}{3} + a_3) + 2\theta \sin\theta \} \\
&\quad + g'\delta^2 \{ (\ln 2|\lambda_1^{(0)}| + \gamma)^2 - \theta^2 - 6(\ln 2|\lambda_1^{(0)}| + \gamma) \\
&\quad + 8 + a_4 \} \qquad (26)
\end{aligned}$$

where a_i are numerically evaluated as

$$a_1 \simeq 0.049,$$
$$a_2 \simeq 0.13,$$
$$a_3 \simeq 0.067,$$
$$a_4 \simeq 0.27.$$

If we put $\delta = g' = 1$ in the above we can verify again that the positivity of the real part of the eigenvalues is still retained as long as $\Re(\sigma) > 0$. This case corresponds to the one Klauder and Petersen have studied in detail numerically [4]. They have found strong numerical evidence that $\Re(\lambda_n) > 0$ ($n \geq 1$) for $\Re(\sigma) > 0$. The above expressions may be considered to verify their claim, this time analytically up to the second order in δ.

Summary

We have calculated eigenvalues of the essentially nonhermite F-P hamiltonian in two cases (i) and (ii). The calculations are based on the δ-expansion technique and are regarded as nonperturbative in the sense that the nontrivial dependence on the coupling constants has been taken into account. In case (i) (Minkowski space-time formulation), we have found that the real part of the eigenvalues acquires higher order corrections in powers of δ some of which are not dependent on the damping parameter ϵ and that their positivity is not always guaranteed. On the other hand, for a system with a complex Euclidean action, case (ii), we have shown that the real part of the eigenvalues is positive semi-definite if $\Re(\sigma) \geq 0$. This may be considered as an analytic proof of Klauder and Petersens' claim. Of course, these statements are based on the perturbative calculations w.r.t. δ and their validity is in general restricted by the approximation. Nevertheless we believe they may reflect the true nature of the systems and we hope that these new approaches will shed light on this field of the stochastic quantization.

Acknowledgements

One of the authors (H. N.) would like to thank Dr. K. Okano, Prof. L. Schülke and Dr. S. Marculescu for warm hospitality and helpful discussions at Siegen Univ.. Discussions with Dr. H. Hüffel and Prof. J. R. Klauder are also acknowledged.

References

1. G. Parisi and Y.-S. Wu, *Sci. Sin.* **24** (1981) 483; P. H. Damgaard and H. Hüffel, *Phys. Rep.* **152C** (1987) 227 and references therein.
2. G. Parisi, *Phys. Lett.* **131B** (1983) 393; J. R. Klauder, in *Recent Developments in High-Energy Physics,* ed. H. Mitter and C. B. Lang (Springer-Verlag, Vienna, 1983) 251 and *Phys. Rev.* **A24** (1984) 2036.
3. C. M. Bender, K. A. Milton, M. Moshe, S. S. Pinsky and L. M. Simmons, Jr., *Phys. Rev. Lett.* **58** (1987) 2615 and *Phys. Rev.* **D37** (1988) 1472; C. M. Bender, K. A. Milton, S. S. Pinsky and L. M. Simmons, Jr., *Phys. Lett.* **B205** (1988) 493; N. Brown, *Phys. Rev.* **D38** (1988) 723; C. M. Bender and K. A. Milton, *Phys. Rev.* **D38** (1988) 1310; S. S. Pinsky and L. M. Simmons, Jr., *Phys. Rev.* **D38** (1988) 2518; C. M. Bender and H. F. Jones, *Phys. Rev.* **D38** (1988) 2526 and *J. Math. Phys.* **29** (1988) 2659; C. M. Bender, F. Cooper and K. A. Milton, *Phys. Rev.* **D39** (1989) 3684; C. M. Bender, K. A. Milton, S. S. Pinsky and L. M. Simmons, Jr., *J. Math. Phys.* **30** (1989) 1447.
4. J. R. Klauder and W. P. Petersen, *J. Stat. Phys.* **39** (1985) 53.
5. H. Nakazato, *Prog. Theor. Phys.* **77** (1987) 20.
6. H. Nakazato and T. Yamashiro, *Prog. Theor. Phys.* **82** (1989) 1201 and *Univ. of the Ryukyus preprint* (May 1990) (unpublished).

DISCUSSION

Q. **Yong-Shi Wu** *(Univ. Utah)*: What is the expected convergence of the δ-expansion?

A. **H. Nakazato**: In their papers, Bender *et al.* investigated the range of convergence using a soluble simple model. They found that it is characterized by the condition $\delta < 1$ and $\delta = 1$ lies on the border of the convergence range.

In my case, δ is finally set equal to 1, but I think we can expect that the nonperturbative property of the solution may appear, even at the lowest order in δ.