

On conservation laws in teleparallel gravity

E D Emtsova, A N Petrov and A V Toporensky

Sternberg Astronomical Institute, Lomonosov Moscow State University, Moscow, Russia

E-mail: ed.emcova@physics.msu.ru, alex.petrov55@gmail.com, atopor@rambler.ru

Abstract. Applying directly the Noether theorem in the framework of the Teleparallel Equivalent of General Relativity (TEGR), we construct conserved quantities, currents and superpotentials. They are covariant both under coordinate transformations and under local Lorentz rotations, unlike earlier approaches. This advantage is achieved by a presence in expressions of conservation laws of a displacement vector that can be interpreted as a Killing vector, as a proper vector of an observer, etc. We introduce, as well, a principle for a definition of an inertial spin connection that is an undetermined quantity in TEGR in the original formulation. The new expressions for conserved quantities and the introduced principle are applied to calculate mass for the Schwarzschild black hole and energy density for an observer freely falling in spatially flat Friedmann world.

1. Introduction

Last decades, Teleparallel Equivalent of General Relativity (TEGR) and its numerous modifications, like $f(T)$ theories, are developed very intensively, see [1] and references therein. Dynamic variables in teleparallel gravity are components of the tetrad field instead of metric coefficients. Each of field theories requires a construction conservation laws and conserved quantities. Moller [2] was maybe first who suggested to use tetrad GR for constructing covariant energy-momentum tensor for gravitational field instead of traditional energy-momentum pseudotensors in metric formulation of GR. However, both Moller's Lagrangian and Moller's energy-momentum tensor are not covariant with respect to local Lorentz rotations, although the field equations are left Lorentz covariant. Later, in the frame of TEGR, preserving coordinate covariance of Moller's tetrad GR, it was restored covariance with respect to local Lorentz rotations [1]. This nice property is achieved due to introducing a pure inertial Lorentz special spin connection with zero curvature. In spite of that conservation laws in TEGR in the traditional interpretation [1] have a set of problems, for the recent discussion see [3]. For example, if one constructs Lorentz covariant conserved currents, then one cannot construct conserved charges by an acceptable way [3], and conversely.

The goal of our study is to construct conservation laws and conserved quantities in TEGR resolving problems remarked in [3]. Applying directly Noether's theorem, we obtain conserved currents that are expressed through divergences of superpotentials (antisymmetric tensor densities). Currents describe local quantities, like energy density, whereas superpotentials permit to construct global quantities (charges) in the form of well defined (mathematically acceptable) surface integrals. Essential property of conservation law expressions is a presence of an arbitrary displacement vector. Namely it permits to resolve problems noted in [3] that is to construct local and global conserved quantities trusting a covariance of both the kinds. In the case, if a



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displacement vector is a timelike Killing vector, for example, one can interpret a charge as a mass of a system; in the case of a proper vector of observer one can interpret components of a current as related densities; etc.

Of course, the Noether theorems have been applied in teleparallel gravity. Let us list some of previous results: in [4], it was used the diffeomorphism invariance of the Lagrangian, however, the final results have been simplified to expressions with excluding a displacement vector, and conservation laws acquire the form of them in [1] with their problems; in [5,6], it was used an invariance to a specific group to make an appropriate choice among potential variants of modified teleparallel gravities only; etc. Unlike of the above, applying the Noether theorem, we give well structured expressions for conserved quantities with a clear and standard interpretation, which are ready to examine concrete models.

To apply and check our results we calculate in the framework of TEGR, first, mass of the Schwarzschild black hole, second, the energy density in the frame of a freely falling observer in the spatially flat Friedmann world. To obtain an acceptable result it is necessary to choose a spin connection in a correct way. Our principle for such a choice is quite natural, namely, when one “switches off” the gravity conserved quantities must vanish. As a result, we obtain, first, the standard and acceptable mass for the Schwarzschild black hole; second, the freely falling observer in a non-perturbed spatially flat Friedmann world measures a zero energy density that, at least, coincides with some of previous results, see [4].

2. The main notions in TEGR

2.1. Lagrangian and field equations

Here, we rewrite the main expressions related to TEGR given in the book [1] following its notations. Dynamic variables in the frame of TEGR are components of the tetrad h^a_ρ , which are connected with the metric $g_{\rho\sigma}$ of the metric GR by the relation:

$$g_{\rho\sigma} = \eta_{ab} h^a_\rho h^b_\sigma \quad (1)$$

with the Minkowski metric η_{ab} . The Lagrangian of TEGR is

$$\bullet L = \frac{h}{2\kappa} \left(\frac{1}{4} \bullet T^\rho_{\mu\nu} \bullet T^\mu_\rho \bullet T^\nu_\nu + \frac{1}{2} \bullet T^\rho_{\mu\nu} \bullet T^{\nu\mu} \bullet T^{\mu\nu} - \bullet T^\rho_{\mu\rho} \bullet T^{\nu\mu} \bullet T^{\mu\nu} \right), \quad (2)$$

where h is a determinant of h^a_ρ , and the torsion tensor

$$\begin{aligned} \bullet T^\rho_{\mu\nu} = & h_a^\rho (\partial_\mu h^a_\nu - \partial_\nu h^a_\mu + \bullet A^a_{c\mu} h^c_\nu - \bullet A^a_{c\nu} h^c_\mu) = \\ & h_a^\rho \left(\overset{\circ}{\nabla}_\mu h^a_\nu - \overset{\circ}{\nabla}_\nu h^a_\mu + \bullet A^a_{c\mu} h^c_\nu - \bullet A^a_{c\nu} h^c_\mu \right) \end{aligned} \quad (3)$$

with the covariant derivative $\overset{\circ}{\nabla}_\mu$ compatible with the metric $g_{\mu\nu}$. The pure *inertial Lorentz connection*

$$\bullet A^a_{c\mu} = \Lambda^a_d(x) \partial_\mu \Lambda^d_c(x) \quad (4)$$

with the local Lorentz matrix $\Lambda^a_b(x)$, preserving coordinate invariance, makes the torsion (3) Lorentz invariant. Thus, the Lagrangian (2) is coordinate and Lorentz covariant as well. Because all the components of the inertial spin connection (4) can be suppressed by corresponding local rotations a related curvature $\bullet R^a_{b\mu\nu}$ has to be zero. Indeed, one can easily show by a direct substitution that

$$\bullet R^a_{b\mu\nu} = \partial_\mu \bullet A^a_{b\nu} - \partial_\nu \bullet A^a_{b\mu} + \bullet A^a_{c\mu} \bullet A^c_{b\nu} - \bullet A^a_{c\nu} \bullet A^c_{b\mu} \equiv 0.$$

Lagrangian (2) transforms to the Moller Lagrangian [2] when the connection (4) is equalized to zero:

$${}^{\bullet}L_M = {}^{\bullet}L|_{({}^{\bullet}A)=0}. \quad (5)$$

Both Lagrangians (2) and (5) are equivalent to the usual Hilbert Lagrangian up to a divergence. Thus, both of them give the equations, which are equivalent to the Einstein equations in the usual form. Then, it turns out that:

$${}^{\bullet}L = {}^{\bullet}L_M + \partial_{\alpha}f^{\alpha}(h, {}^{\bullet}A). \quad (6)$$

Thus, equations of motion do not depend on the connection (4) in whole, that is they are invariant to its choice.

To obtain the equations of motion let us vary (2) with respect to dynamical variables h^a_{ρ} :

$$\frac{\delta {}^{\bullet}L}{\delta h^a_{\rho}} \equiv \frac{\partial {}^{\bullet}L}{\partial h^a_{\rho}} - \partial_{\sigma} \left(\frac{\partial {}^{\bullet}L}{\partial h^a_{\rho,\sigma}} \right) = 0. \quad (7)$$

The expression on the left hand side is the Lagrangian derivative and is covariant in the two senses.

2.2. The standard form of conservation laws in TEGR

Following [1], let us denote items of the expression (7) in the form:

$${}^*J_a^{\rho} \equiv -\frac{1}{h} \frac{\delta {}^{\bullet}L}{\delta h^a_{\rho}} = -\frac{h_a^{\rho}}{h} {}^{\bullet}L + \frac{1}{\kappa} h_a^{\pi} {}^{\bullet}S_c^{\beta\rho} {}^{\bullet}T^c_{\beta\pi} + \frac{1}{\kappa} {}^{\bullet}A^c_{\alpha\beta} {}^{\bullet}S_c^{\rho\beta}; \quad (8)$$

$${}^*S_a^{\rho\sigma} \equiv -\frac{\kappa}{h} \left(\frac{\partial {}^{\bullet}L}{\partial h^a_{\rho,\sigma}} \right) = \frac{1}{2} {}^{\bullet}T_a^{\rho\sigma} - {}^{\bullet}T^{[\rho\sigma]}_a - 2 {}^{\bullet}T^{\alpha[\rho} \delta_a^{\sigma]}. \quad (9)$$

Then the equations of motion (7) is rewritten in the form of the conservation law:

$$\kappa h {}^*J_a^{\rho} = \partial_{\beta} \left(h {}^{\bullet}S_a^{\rho\beta} \right) = \overset{\circ}{\nabla}_{\beta} \left(h {}^{\bullet}S_a^{\rho\beta} \right). \quad (10)$$

Indeed, this relation has a sense of a conservation law because the current ${}^*J_a^{\rho}$ is conserved,

$$\partial_{\rho} \left(h {}^*J_a^{\rho} \right) = \overset{\circ}{\nabla}_{\rho} \left(h {}^*J_a^{\rho} \right) = 0 \quad (11)$$

that takes a place due to antisymmetry of the superpotential (9).

We note that all the expressions in (8) – (11) are spacetime covariant, whereas the current (8), being spacetime covariant, *is not covariant with respect to local Lorentz rotations*. In spite of that, we have a possibility to construct conserved charges on hypersurfaces $x^0 = \text{const} \rightarrow \Sigma$ after integrating (10) and (11) by the standard method

$$P_a = \int_{\Sigma} dx^3 (h {}^*J_a^0) = \frac{1}{\kappa} \int_{\partial\Sigma} ds_i (h {}^{\bullet}S_a^{0i}). \quad (12)$$

The Lorentz covariant expressions can be constructed with introducing Lorentz covariant derivatives, ${}^{\bullet}D_{\rho}$, see [1] and related references therein. For example, for the covariant tetrad vector V_a one defines ${}^{\bullet}D_{\rho}V_a = \partial_{\rho}V_a - {}^{\bullet}A^c_{a\rho}V_c$. Thus, one constructs Lorentz and spacetime covariant current

$${}^{\bullet}J_a^{\rho} = {}^*J_a^{\rho} - \frac{1}{\kappa} {}^{\bullet}A^c_{\alpha\beta} {}^{\bullet}S_c^{\rho\beta} = -\frac{h_a^{\rho}}{h} {}^{\bullet}L + \frac{1}{\kappa} h_a^{\pi} {}^{\bullet}S_c^{\beta\rho} {}^{\bullet}T^c_{\beta\pi}; \quad (13)$$

that permits to rewrite the conservation law (10) in both Lorentz and spacetime covariant form:

$$\kappa h^\bullet J_a^\rho = \bullet D_\beta \left(h^\bullet S_a^{\rho\beta} \right). \quad (14)$$

However, with this presentation it is not possible to construct conserved charges, like (12). This problem is remarked in [1], in more details it is discussed in [3]. In the next section, we resolve it.

3. Lorentz and spacetime covariant conservation laws in TEGR

Because Lagrangian (2) is a scalar density we can derive the main Noether's identity:

$$\mathcal{L}_\xi^\bullet L \equiv -\partial_\alpha (\xi^\alpha \bullet L), \quad (15)$$

where \mathcal{L}_ξ is the Lie derivative (with the opposite sign only). The Lagrangian $\bullet L$ depends on geometrical objects, h^a_ρ and $A^a_{b\mu}$, variations of which (Lie derivatives) are

$$\begin{aligned} \delta h^a_\rho &= \mathcal{L}_\xi h^a_\rho = -\xi^\alpha \overset{\circ}{\nabla}_\alpha h^a_\rho - h^a_\alpha \overset{\circ}{\nabla}_\rho \xi^\alpha, \\ \delta^\bullet A^a_{b\mu} &= \mathcal{L}_\xi^\bullet A^a_{b\mu} = -\xi^\alpha \overset{\circ}{\nabla}_\alpha \bullet A^a_{b\mu} - \bullet A^a_{b\alpha} \overset{\circ}{\nabla}_\mu \xi^\alpha. \end{aligned} \quad (16)$$

Then, after standard identical transformations, see books [7,8], the identity (15) can be rewritten in the form of conservation law for the current $\bullet I^\alpha(\xi)$:

$$\partial_\alpha \bullet I^\alpha(\xi) \equiv \overset{\circ}{\nabla}_\alpha \bullet I^\alpha(\xi) \equiv 0. \quad (17)$$

For the Lagrangian (2) the current is presented as

$$\bullet I^\alpha(\xi) = \left[\frac{1}{\kappa} h^\bullet S_a^{\alpha\rho} \left(\overset{\circ}{\nabla}_\sigma h^a_\rho + \bullet A^a_{b\sigma} h^b_\rho \right) - \bullet L \delta_\sigma^\alpha + \frac{\delta^\bullet L}{\delta h^a_\alpha} h^a_\sigma \right] \xi^\sigma + \frac{1}{\kappa} h^\bullet S_a^{\alpha\rho} h^a_\sigma \overset{\circ}{\nabla}_\rho \xi^\sigma. \quad (18)$$

With making the use of the Klein-Noether identities, that follow from the identity [17], the current (18) can be expressed through the superpotential $\bullet I^{\alpha\beta}(\xi)$:

$$\bullet I^\alpha(\xi) \equiv \partial_\beta \bullet I^{\alpha\beta}(\xi) \equiv \overset{\circ}{\nabla}_\beta \bullet I^{\alpha\beta}(\xi), \quad (19)$$

where

$$\bullet I^{\alpha\beta}(\xi) = \frac{1}{\kappa} h^\bullet S_a^{\alpha\beta} h^a_\sigma \xi^\sigma. \quad (20)$$

Let us discuss (17) and (19). They are identities only, they do not bring a physical content because up to now the field equations have not been used. Already vacuum equations have been derived (7), for a more generality we include the matter sources with the matter energy-momentum tensor $\bullet \Theta_a^\rho$, see book [1]:

$$\frac{\delta^\bullet L}{\delta h^a_\rho} \equiv \frac{\partial^\bullet L}{\partial h^a_\rho} - \partial_\sigma \left(\frac{\partial^\bullet L}{\partial h^a_{\rho,\sigma}} \right) = h^\bullet \Theta_a^\rho. \quad (21)$$

After using this equation the current (18) becomes physically sensible and is rewritten as

$$\bullet I^\alpha(\xi) = h [\bullet \theta_\sigma^\alpha + \bullet \Theta_\sigma^\alpha] \xi^\sigma + \frac{1}{\kappa} h^\bullet S_a^{\alpha\rho} h^a_\sigma \overset{\circ}{\nabla}_\rho \xi^\sigma, \quad (22)$$

where the energy-momentum tensor for the gravitational (tetrad) field is

$$\bullet\theta_\sigma^\alpha = \frac{1}{\kappa}\bullet S_a^{\alpha\rho} \left(\overset{\circ}{\nabla}_\sigma h^a_\rho + \bullet A^a_{b\sigma} h^b_\rho \right) - \frac{1}{h} \bullet L \delta_\sigma^\alpha. \quad (23)$$

Thus, for the current (22) and superpotential (20) the identity (19) acquires quite physical sense.

Both, the current (18), the same (22), and the superpotential (20) are spacetime covariant and invariant under local Lorentz rotations. Thus, the conservation law (19) has the same property. However, unlike the conservation law (14), the equality (19) can be integrated to obtain sensible charges:

$$P(\xi) = \int_{\Sigma} dx^3 \bullet I^0(\xi) = \int_{\partial\Sigma} ds_i \bullet I^{0i}(\xi). \quad (24)$$

It is important to accent the following. Let us define the other useful quantity, GR spin connection,

$$\overset{\circ}{A}{}^a_{b\sigma} = -h_b^\rho \overset{\circ}{\nabla}_\sigma h^a_\rho \quad (25)$$

that reflects the presence of gravitational field, unlike inertial spin connection $\bullet A^a_{b\sigma}$ that can be suppressed by appropriate Lorentz rotations. Then, the torsion (3) can be rewritten in the form:

$$\bullet T^\rho_{\mu\nu} = h_a^\rho (h_c^\mu (\overset{\circ}{A}{}^a_{c\nu} - \bullet A^a_{c\nu}) - h_c^\nu (\overset{\circ}{A}{}^a_{c\mu} - \bullet A^a_{c\mu})). \quad (26)$$

Thus, we conclude that the Lagrangian $\bullet L$ in (2) and the superpotential $\bullet S_a^{\rho\sigma}$ in (9) depend essentially on the difference $\bullet K^a_{c\mu} = \bullet A^a_{c\mu} - \overset{\circ}{A}{}^a_{c\mu}$ that is the contortion tensor, see [1]. Thus, the current (18), the same (22), depends essentially on $\bullet K^a_{c\mu}$ as well. We derive explicitly the expression for the superpotential (20):

$$\bullet I^{\alpha\beta}(\xi) = \frac{h}{2\kappa} (\overset{\circ}{A}{}^a_{c\mu} - \bullet A^a_{c\mu}) \left[2h_a^\mu (\delta_\sigma^\alpha h^{c\beta} - \delta_\sigma^\beta h^{c\alpha}) - \delta_\sigma^\mu (h_a^\alpha h^{c\beta} - h_a^\beta h^{c\alpha}) \right] \xi^\sigma. \quad (27)$$

Thus, here, we have constructed all the expressions both coordinate and Lorentz covariant.

Due to (5) and (6) the field equations (7) or (21) do not depend on the inertial spin connection $\bullet A^a_{c\mu}$ at all, whereas the current and superpotential depend on $\bullet A^a_{c\mu}$ essentially. The question arises: How can one to choose $\bullet A^a_{c\mu}$ for a concrete tetrad and GR spin connection defined in (25)? *Our recipe is simple:* in the case when the gravity is “switched off”, the current $\bullet I^\alpha(\xi)$ and the superpotential $\bullet I^{\alpha\beta}(\xi)$ have to be zeroth, as well as all tensor quantities $\bullet T^\alpha_{\mu\nu}$, $\bullet K^a_{c\mu}$, $\bullet S_a^{\rho\sigma}$ responsible for gravity. From the above it is clear that this criterion plays, if and only if $\overset{\circ}{A}{}^a_{c\mu} = \bullet A^a_{c\mu}$. The main property of $\bullet A^a_{c\mu}$ is that the related curvature defined by the Riemannian tensor is zero. Thus, for a concrete solution, one has to construct Riemannian tensor with the use of $\overset{\circ}{A}{}^a_{c\mu}$ and equalize it zero. By this operation, one finds conditions for the “switch off” gravity for parameters of a solution under consideration. Namely these conditions transform components of $\overset{\circ}{A}{}^a_{c\mu}$ to components of $\bullet A^a_{c\mu}$.

4. Applications

4.1. Mass of the Schwarzschild black hole

Consider black hole Schwarzschild metric in spherical coordinates:

$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \left(1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\phi^2). \quad (28)$$

In this case, following (1), it is convenient to choose a tetrad components in the form:

$$h^a_\mu = \begin{bmatrix} (1 - \frac{2M}{r})^{1/2} & 0 & 0 & 0 \\ 0 & (1 - \frac{2M}{r})^{-1/2} & 0 & 0 \\ 0 & 0 & r & 0 \\ 0 & 0 & 0 & r \sin\theta \end{bmatrix}. \quad (29)$$

Then, for (28) and (29) we calculate components of the GR spin connection (25), the non-zero ones of which are

$$\begin{aligned} {}^{\circ}A^0_{10} = {}^{\circ}A^1_{00} &= \frac{M}{r^2}; \quad {}^{\circ}A^1_{22} = -{}^{\circ}A^2_{12} = -\left(1 - \frac{2M}{r}\right)^{1/2}; \\ {}^{\circ}A^1_{33} = -{}^{\circ}A^3_{13} &= -\sin\theta\left(1 - \frac{2M}{r}\right)^{1/2}; \quad {}^{\circ}A^2_{33} = -{}^{\circ}A^3_{23} = -\cos\theta. \end{aligned} \quad (30)$$

Following the principle suggested at the end of previous section, to construct the inertial Lorentz connection ${}^{\bullet}A^a_{c\mu}$ we “switch off” the gravity in the solution (28) by the simple equalizing $M = 0$. Then, indeed, (28) becomes a metric of a flat spacetime, and from (30) we have

$${}^{\bullet}A^1_{22} = -{}^{\bullet}A^2_{12} = -1; \quad {}^{\bullet}A^1_{33} = -{}^{\bullet}A^3_{13} = -\sin\theta; \quad {}^{\bullet}A^2_{33} = -{}^{\bullet}A^3_{23} = -\cos\theta. \quad (31)$$

The difference of (31) and (30) gives non-zero components of the contortion,

$$\begin{aligned} {}^{\bullet}K^0_{10} = {}^{\bullet}K^1_{00} &= -\frac{M}{r^2}; \quad {}^{\bullet}K^1_{22} = -{}^{\bullet}K^2_{12} = \left(1 - \frac{2M}{r}\right)^{1/2} - 1; \\ {}^{\bullet}K^1_{33} = -{}^{\bullet}K^3_{13} &= \sin\theta \left[\left(1 - \frac{2M}{r}\right)^{1/2} - 1 \right]. \end{aligned} \quad (32)$$

The components (29) and (32) permit to present non-zero spacetime components of the torsion tensor (26) which is antisymmetric in lower indexes:

$${}^{\bullet}T^0_{10} = \frac{M}{r^2} \left(1 - \frac{2M}{r}\right)^{-1}; \quad {}^{\bullet}T^2_{12} = {}^{\bullet}T^3_{13} = \frac{1}{r} \left[1 - \left(1 - \frac{2M}{r}\right)^{1/2}\right] \quad (33)$$

and the same with opposite sign for swapped lower indexes. Then, one easily finds non-zero spacetime components of the superpotential (9) which is antisymmetric in upper indexes:

$${}^{\bullet}S_0^{01} = \frac{2}{r} \left[1 - \left(1 - \frac{2M}{r}\right)^{1/2}\right] - \frac{4M}{r^2}; \quad {}^{\bullet}S_2^{12} = {}^{\bullet}S_3^{13} = -\frac{1}{r} \left[1 - \left(1 - \frac{2M}{r}\right)^{1/2}\right] + \frac{M}{r^2} \quad (34)$$

and the same with opposite sign for swapped upper indexes.

Now we can to calculate the total mass/energy of the Schwarzschild black hole. We choose a displacement vector in (24) as the timelike Killing vector $\xi^\alpha = (-1, 0, 0, 0)$, and obtain

$$E = \frac{1}{\kappa} \lim_{r \rightarrow \infty} \int_{\partial\Sigma} dx^2 h {}^{\bullet}S_0^{01} \xi^0 = M \quad (35)$$

where it was used $\kappa = 8\pi$, $h = r^2 \sin\theta$ and the components (34). All the tensors: torsion, ${}^{\bullet}T^\alpha_{\mu\nu}$, contortion, ${}^{\bullet}K^\alpha_{\mu\nu}$, superpotential, ${}^{\bullet}S_\mu^{\rho\sigma}$, do not depend on local Lorentz rotations by the construction. Integrating the same superpotential in (35) we get the same mass $E = M$ after arbitrary Lorentz rotations. Of course, spin connection, like (30) and (31) will change, for example, ${}^{\bullet}A^a_{c\mu}$ in (4) can be equalized to zero ${}^{\bullet}A^a_{c\mu} = 0$. Such a gauge fixation is called as the

Weitzenböck gauge. In the case of the tetrad (29) and spin connection (31) the related Lorentz rotations are described by the matrix

$$\Lambda^a{}_b = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi & \cos \theta \cos \phi & -\sin \phi \\ 0 & \sin \theta \sin \phi & \cos \theta \sin \phi & \cos \phi \\ 0 & \cos \theta & -\sin \theta & 0 \end{bmatrix}. \quad (36)$$

Then, the inertial spin connection components (31), indeed, vanish, whereas the tetrad components (29) become

$$h^a{}_\mu = \begin{bmatrix} (1 - \frac{2M}{r})^{1/2} & 0 & 0 & 0 \\ 0 & \sin \theta \cos \phi (1 - \frac{2M}{r})^{-1/2} & r \cos \theta \cos \phi & -r \sin \theta \sin \phi \\ 0 & \sin \theta \sin \phi (1 - \frac{2M}{r})^{-1/2} & r \cos \theta \sin \phi & r \sin \theta \cos \phi \\ 0 & \cos \theta (1 - \frac{2M}{r})^{-1/2} & -r \sin \theta & 0 \end{bmatrix}. \quad (37)$$

Such a tetrad is called usually as a *proper* one. Arbitrary coordinate transformations leave the condition ${}^*A^a{}_{c\mu} = 0$, whereas the components of the proper tetrad (37) are changed as the components of covariant vector leaving the proper tetrad.

4.2. Energy density in spatially flat FLRW world for a free-falling observer

Let derive the metric of the spatially flat FLRW world in the form:

$$ds^2 = -dt^2 + a^2(t) (dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)). \quad (38)$$

A more convenient choice of the tetrad is

$$h^a{}_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & ar & 0 \\ 0 & 0 & 0 & ar \sin \theta \end{bmatrix}. \quad (39)$$

Setting in (38) $a(t) = 1$ one evidently “switches off” the gravity. Then one easily calculates that components of the inertial spin connection are the same as in (31).

Non-zero components of the contortion tensor calculated for ${}^*K^a{}_{b\mu}$ are

$${}^*K^1{}_{01} = {}^*K^0{}_{11} = {}^*K^2{}_{02} = {}^*K^0{}_{22} = {}^*K^3{}_{03} = {}^*K^0{}_{33} = -H \quad (40)$$

where H is the standard Hubble constant. The related non-zero components of the torsion tensor ${}^*T^\alpha{}_{\beta\mu}$ (antisymmetric in lower indexes) are

$${}^*T^1{}_{10} = {}^*T^2{}_{20} = {}^*T^3{}_{30} = -H \quad (41)$$

and the same with opposite sign for swapped lower indexes. Related non-zero components of the antisymmetric in upper indexes superpotential ${}^*S_\mu{}^{\rho\sigma}$ are

$${}^*S_1{}^{10} = {}^*S_2{}^{20} = {}^*S_3{}^{30} = -2H \quad (42)$$

and with opposite sign for swapped upper indexes. The same as in the black hole case, after the rotations (36) all non-zero components of inertial spin connection, (they are again (31)) are

suppressed. And the tetrad becomes as a proper tetrad but in quite a cumbersome form. Then after transformation from the spherical coordinates to Cartesian ones the proper tetrad becomes

$$h^a_\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \quad (43)$$

with the related metric:

$$ds^2 = -dt^2 + a^2(t) (dx^2 + dy^2 + dz^2). \quad (44)$$

For a gauge (43) and (44) one easily finds that components (40)-(42) are the same.

Now we will calculate the energy density for a freely falling observer with the proper vector $\xi^\alpha = (-1, 0, 0, 0)$ - now it is not a Killing vector. This energy density is a zero component of the current (22). Thus, we need calculate 00-components of $\bullet\theta_\sigma^\alpha$ and $\bullet\Theta_\sigma^\alpha$ that gives $(\bullet\theta_0^0 + \bullet\Theta_0^0) \xi^0 = 6\kappa^{-1} H^2$ for the first term in (22). Calculation of the last term in (22) gives the same quantity with the opposite sign. Thus, finally $\bullet I^0(\xi) = 0$ that means that the freely falling observer measures a zeroth energy density; at least, this result coincides with some of previous ones, see [4]. The same result is valid for the case of $k = -1$.

5. Discussion

Let us analyze the conservation law (19) with the current (18) and superpotential (20). Open a divergence on the right-hand side of (19) and suppress the items with the derivative of ξ^α on the left and right sides. Next, use the vacuum equations (7) and save only the coefficients at ξ^α , removing ξ^α itself. At last, contracting the final expression with h_b^σ one obtains exactly the conservation law in the form (10). Thus, we conclude that the success of constructing the conservation law (19) with the current and the superpotential covariant in the both senses is trusted by the presence of the displacement vector ξ^α incorporated into the Noether theorem from the start.

The other advantage of using ξ^α is that it can be interpreted as a Killing vector, as a proper vector of an observer, etc. It is natural because a definite physical sense can be labeled to related conserved quantities. Frequently, in works where the displacement vector ξ^α is not used, authors identify an observer with a proper vector of tetrad, see, for example [9,10]. Formally it is quite permissible. However, from the point of the observer role, it is an external object created for testing a physical (geometrical) model. On the other hand, any tetrad vector components are dynamical objects and cannot be interpreted as an external structure.

It was noted already that the equations of motion in TEGR [1], the same in the Moller tetrad theory [2] do not depend on the inertial spin connection at all. Thus, solutions to these equations do not depend on a choice of the inertial spin connection at all, as well. However, we have stated that a choice of the inertial spin connection is quite important for calculating conserved quantities, and we suggest a related criterion. In modifications of TEGR, like $f(T)$ theories, fixation of the inertial spin connection becomes crucial for a consistence of the system of equations of motion. In [11-13] the criterion connected with popular symmetries of known solution is suggested. Ideas and results given in the present paper have been presented at the conference PIRT2019 [14] and developed significantly in our recent paper [15] where, in particular, we provide a detail comparison with the results of [11-13].

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