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On the von Neumann algebras associated to Yang–Baxter operators

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Bożejko and Speicher associated a finite von Neumann algebra M_T to a self-adjoint operator T on a complex Hilbert space of the form $\mathcal{H} \otimes \mathcal{H}$ which satisfies the Yang–Baxter relation and $\|T\| < 1$. We show that if $\dim(\mathcal{H}) \geqslant 2$, then M_T is a factor when T admits an eigenvector of some special form.

Keywords: Yang–Baxter operator; von Neumann algebras

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1. Introduction

In the nineties, Bożejko and Speicher in [**5**] constructed von Neumann algebras, which generalize Voiculescu's construction of the free group factors in [**17**]. To be precise, they associate a von Neumann algebra M_T to a Yang–Baxter operator T (see definition 2.1) on a complex Hilbert space of the form $\mathcal{H} \otimes \mathcal{H}$, where $\mathcal{H}$ is the complexification of a real Hilbert space. The von Neumann algebra M_T acts on a deformed Fock space and the matrix representation of T with respect to an appropriately chosen orthonormal basis of $\mathcal{H}$ allows one to provide a manifestation of the generalized deformed commutation relations:

$$l_i^* l_j - \sum_{r,s\in\Lambda} t_{js}^{ir} l_r l_s^* = \delta_{ij} 1, \text{ for all } i, j \in \Lambda,$$

where t_{js}^{ir} for $i, j, r, s \in \Lambda$ are scalars. Under suitable conditions (depending on T), the von Neumann algebra M_T is tracial and acts on the aforesaid (deformed) Fock space in standard form.

Voiculescu's construction of the free group factors, Bożejko-Speicher's construction of the q-Gaussian and the mixed q-Gaussian von Neumann algebras are all subsumed in the above construction. Notably, the Yang–Baxter operator associated with Voiculescu's construction is zero, and that associated with the construction of q-Gaussian von Neumann algebra is $q\mathfrak{F}$, where $q \in (-1, 1)$ and $\mathfrak{F}$ is the flip unitary on $\mathcal{H} \otimes \mathcal{H}$.

There have been significant efforts to understand the free group factors and q-Gaussian algebras and their type III counterparts over the last 30 years. The efforts to understand free group factors gave birth to Free Probability [17], which has grown to be an independent discipline. Without being encyclopedic, if $\dim(\mathcal{H}) \geqslant 2$, then the q-Gaussian von Neumann algebras are known to be non-injective factors (cf. [14] and [16]), which are strongly solid and have the w^*-completely contractive approximation property (cf. [1], also see [15] on solidity of free group factors) and are weakly amenable [8]. For the latest isomorphism theorem on the same topic, see [13]. This scanty account is by no means justified given the huge existing literature. However, the class of von Neumann algebras arising out of Yang–Baxter deformations, in general, have received less attention. The very basic question of factoriality is still open. We only concentrate on the case when the standard vacuum state is a trace. In [9], Królak proved that M_T is a factor when $\dim(\mathcal{H}) = \aleph_0$ and T satisfies certain condition known as the 'Wick product condition' (for details see theorem 4.9). In a later paper [10], factoriality of M_T was further investigated and established depending on the number of self-adjoint generators. Moreover, M_T is non-injective when $\dim(\mathcal{H}) \geqslant 2$ (see theorem 2, [14]). Beyond this, nothing is known.

In this paper, we investigate the factoriality of M_T. Following the approach in [2] and [16], in this paper we show that if $\dim(\mathcal{H}) < \infty$ and T admits a special eigenvector of the form $\xi_0 \otimes \xi_0$, where ξ_0 corresponds to a self-adjoint generator of M_T, then the aforesaid self-adjoint generator generates a strongly mixing maximal abelian self-adjoint algebra (masa in the sequel) of M_T. Furthermore, M_T is a factor. A similar conclusion holds even in the case $\dim(\mathcal{H}) = \aleph_0$, but we need to assume the existence of an appropriate orthonormal basis of $\mathcal{H}$ for which Wick product expansions are tractable.

Thus, our line of investigation is different than that in [9, 10]. Using the examples cited in [9], we show that our assumptions are satisfied in uncountably many examples. We exploit the fact that under the hypothesis of a special eigenvector of T as above, the self-adjoint generator associated with ξ_0 has properties analogous to the generator masas of the q-Gaussian von Neumann algebras, which plays out well when one considers the standard Hilbert space as a bimodule over this generating abelian algebra.

Now, we discuss the layout of this paper. In § 2, we describe the Bożejko–Speicher's construction of von Neumann algebras associated to Yang–Baxter operators and recall some preliminary material that will be used throughout the paper. In § 3, we discuss the analytical properties of the generating abelian subalgebras associated with special eigenvectors of the Yang–Baxter operator. We investigate the bimodule structure of the aforesaid abelian subalgebras in theorems 4.11 and 4.13 and conclude that the generating abelian subalgebras associated with the special eigenvectors of the Yang–Baxter operators are strongly mixing masas.

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In § 5, we establish factoriality of the associated von Neumann algebras under the assumption of the existence of a special eigenvector of the associated Yang–Baxter operator (theorem 5.1). Finally, in § 6, we discuss some examples that satisfy our conditions by borrowing ideas from [**9**]. This allows constructing new non-injective II_1 factors which generalize mixed q-Gaussian von Neumann algebras.

2. Construction and basic facts

All Hilbert spaces in this paper are assumed to be separable and inner products are linear in the *second variable*. Also, all inclusions of subalgebras are assumed to be unital. Materials in this section are taken from [**5**].

Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space and let $\mathcal{H} = \mathcal{H}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ be its complexification. Identify $\mathcal{H}_{\mathbb{R}}$ in $\mathcal{H}$ as $\mathcal{H}_{\mathbb{R}} \otimes 1$. Thus, $\mathcal{H} = \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}}$, and, as a real Hilbert space, the inner product of $\mathcal{H}_{\mathbb{R}}$ in $\mathcal{H}$ is given by $\Re\langle\cdot,\cdot\rangle_{\mathcal{H}}$. The norm on $\mathcal{H}$ will be denoted by $\|\cdot\|_{\mathcal{H}}$.

Definition 2.1. A Yang–Baxter operator $T \in \mathbf{B}(\mathcal{H} \otimes \mathcal{H})$ is a self-adjoint contraction which satisfies the braid relation, i.e.,

(i) $T = T^*$;

(ii) $\|T\| \leqslant 1$;

(iii) $(1 \otimes T)(T \otimes 1)(1 \otimes T) = (T \otimes 1)(1 \otimes T)(T \otimes 1)$, where the aforesaid amplifications are regarded as bounded operators on $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$.

The last relation in definition 2.1 is referred to as the Yang–Baxter equation or the braid relation and has its origins in statistical mechanics.

Associated to a Yang–Baxter operator T as in definition 2.1 such that $\|T\| < 1$, Bożejko and Speicher constructed the 'T-Fock space' and a family of operators $\{l_i\}_{i\in\Lambda}$ on the T-Fock space satisfying the generalized T-commutation relations:

$$l_i^* l_j - \sum_{r,s\in\Lambda} t_{js}^{ir} l_r l_s^* = \delta_{ij} 1, \quad \text{for all } i, j \in \Lambda, \tag{2.1}$$

where t_{js}^{ir}, $i, j, r, s \in \Lambda$, are scalars depending on T. This representation is known as the generalized Fock representation [**5**]. The authors also constructed a von Neumann algebra on the T-Fock space, which we now proceed to describe.

Hereafter, let $T \in \mathbf{B}(\mathcal{H} \otimes \mathcal{H})$ be a Yang–Baxter operator such that $\|T\| = \lambda < 1$. Define

$$T_i := \underbrace{1 \otimes \cdots \otimes 1}_{(i-1)\text{ times}} \otimes T \text{ acting on } \mathcal{H}^{\otimes(i+1)}, \quad i \geqslant 1.$$

Thus, $T_1 := T$. Extend T_i to $\mathcal{H}^{\otimes n}$ for all $n > i+1$ by $T_i \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-i-1}$ and denote the extensions again by T_i with slight abuse of notations.

The following observations are immediate.

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Proposition 2.2. *The following hold.*

(*i*) $T_i = T_i^*$ *for all* $i \in \mathbb{N}$;

(*ii*) $\|T_i\| = \lambda < 1$ *for all* $i \in \mathbb{N}$;

(*iii*) $T_iT_j = T_jT_i$ *when* $|i-j| \geqslant 2$ *and* $T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$ *for all* $i \in \mathbb{N}$.

Proof. The proof follows easily from definition 2.1. □

Let S_n denote the symmetric group of n elements. Note that S_1 is trivial. For $n \geqslant 2$ and $1 \leqslant i \leqslant n-1$, let ς_i be the transposition between i and $i+1$. It is well known that $\{\varsigma_i\}_{i=1}^{n-1}$ is a generating set of S_n with respect to the following conditions:

$$\varsigma_i\varsigma_j = \varsigma_j\varsigma_i, \qquad \text{whenever } |i-j| \geqslant 2, \text{ and,}$$
$$\varsigma_i\varsigma_{i+1}\varsigma_i = \varsigma_{i+1}\varsigma_i\varsigma_{i+1}, \qquad \text{for } 1 \leqslant i \leqslant n-2.$$

For $n \geqslant 2$ and each $1 \neq \sigma \in S_n$, the length of σ with respect to the set of generators $\{\varsigma_i\}_{i=1}^{n-1}$ is defined as:

$$|\sigma| := \min\left\{k \in \mathbb{N} : \exists\ \varsigma_{i(1)}, \ldots, \varsigma_{i(k)} \text{ with } \sigma = \varsigma_{i(1)} \cdots \varsigma_{i(k)}\right\}.$$

By convention, $|1| = 0$.

For $n \in \mathbb{N}$, let $\Phi : S_n \to \mathbf{B}(\mathcal{H}^{\otimes n})$ be the map given by

$$\Phi(\sigma) = T_{i(1)} \cdots T_{i(k)}, \text{ where } \sigma = \varsigma_{i(1)} \cdots \varsigma_{i(k)} \text{ with } |\sigma| = k, \text{ and,} \qquad (2.2)$$
$$\Phi(1) = 1.$$

In particular, we have

$$\Phi(\varsigma_i) = T_i, \qquad i = 1, \ldots, n-1.$$

Also, Φ is quasi-multiplicative, i.e., for all $\sigma_1, \sigma_2 \in S_n$,

$$\Phi(\sigma_1\sigma_2) = \Phi(\sigma_1)\Phi(\sigma_2), \text{ whenever } |\sigma_1\sigma_2| = |\sigma_1| + |\sigma_2|.$$

There is an alternative to calculate the length of permutations which will be helpful. For $\sigma \in S_n$, define the *inversion* of σ as:

$$\operatorname{Inv}(\sigma) := \#\{(i,j) : i < j, \sigma(i) > \sigma(j)\}.$$

Then, $|\sigma| = \operatorname{Inv}(\sigma)$ for $\sigma \in S_n$.

Lemma 2.3. *For* $n \geqslant 2$, $\Phi : S_n \to \mathbf{B}(\mathcal{H}^{\otimes n})$ *is well defined.*

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Proof. Note that (iii) of proposition 2.2 forces that for $1 \leqslant i, j \leqslant n-1$ one has

$$T_iT_j = T_jT_i \quad \text{when } |i-j| \geqslant 2, \quad \text{and, } T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1}$$

for $1 \leqslant i \leqslant n-2$. Similarly, S_n is generated by the set $S = \{\varsigma_i\}_{i=1}^{n-1}$ subject to the conditions:

$$\varsigma_i\varsigma_j = \varsigma_j\varsigma_i \quad \text{when } |i-j| \geqslant 2 \quad \text{for } 1 \leqslant i, j \leqslant n-1, \text{ and,}$$
$$\varsigma_i\varsigma_{i+1}\varsigma_i = \varsigma_{i+1}\varsigma_i\varsigma_{i+1}, \quad 1 \leqslant i \leqslant n-2.$$

Hence, following the proof of [4, pp. 16, proposition 5], Φ is well defined. □

Let

$$\mathcal{F}(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},$$

be the full Fock space of $\mathcal{H}$, where $\mathcal{H}^0 = \mathbb{C}\Omega$ and $\Omega \in \mathbb{C}$ is a distinguished unit vector usually referred to as the vacuum vector.

For $n \in \mathbb{N}$, define

$$P^{(n)} := \sum_{\sigma \in S_n} \Phi(\sigma) \in \mathbf{B}(\mathcal{H}^{\otimes n}), \text{ and,}$$
$$P^{(0)} := 1 \in \mathbf{B}(\mathcal{H}^0).$$

PROPOSITION 2.4. *For $n \in \mathbb{N}$, the operator $P^{(n)}$ is invertible and strictly positive.*

Proof. For the proof, we refer the reader to [5, theorems 2.3, 2.4]. □

Consequently, by the virtue of proposition 2.4, the association

$$\langle \xi, \eta \rangle_T := \delta_{nm} \langle \xi, P^{(n)}\eta \rangle_{\mathcal{H}^{\otimes n}}, \quad \text{for } \xi \in \mathcal{H}^{\otimes m},\ \eta \in \mathcal{H}^{\otimes n},$$

defines a definite inner product on $\mathcal{F}(\mathcal{H})$ as well as on $\mathcal{H}^{\otimes k}$ for every $k \in \mathbb{N}$. Let $\mathcal{F}_T(\mathcal{H})$ be the completion of $\mathcal{F}(\mathcal{H})$ with respect to the norm $\|\cdot\|_T$ on $\mathcal{F}(\mathcal{H})$ induced by $\langle \cdot, \cdot \rangle_T$. The norm on $\mathcal{F}_T(\mathcal{H})$ will be denoted by $\|\cdot\|_T$ as well.

For $n \in \mathbb{N}$, let $\mathcal{H}^{\otimes_T^n}$ denote the closure of $\mathcal{H}^{\otimes n}$ with respect to $\|\cdot\|_T$. Note that $\overline{\mathcal{H}}^{\|\cdot\|_T} = \mathcal{H}$, $\langle \cdot, \cdot \rangle_{\mathcal{H}} = \langle \cdot, \cdot \rangle_T$ and $\|\cdot\|_{\mathcal{H}} = \|\cdot\|_T$ on $\mathcal{H}$. Thus, $\mathcal{H}_{\mathbb{R}}$ embeds in $\overline{\mathcal{H}}^{\|\cdot\|_T}$ as a real Hilbert space.

We define two sets of creation and annihilation operators: one on $\mathcal{F}(\mathcal{H})$ and other on $\mathcal{F}_T(\mathcal{H})$ as follows.

DEFINITION 2.5. For $\xi \in \mathcal{H}$, the canonical creation (denoted by $d(\xi)$) and annihilation (denoted by $d^*(\xi)$) operators on $\mathcal{F}(\mathcal{H})$ are defined as follows.

(i) $d(\xi)\Omega = \xi$, and,
$d(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n$, $\xi_i \in \mathcal{H}$, $1 \leqslant i \leqslant n$, for $n \in \mathbb{N}$.

(ii) $d^*(\xi)\Omega = 0$, and,
$d^*(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \langle \xi, \xi_1 \rangle_{\mathcal{H}}(\xi_2 \otimes \cdots \otimes \xi_n)$, $\xi_i \in \mathcal{H}$, $1 \leqslant i \leqslant n$, for $n \in \mathbb{N}$.

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Again, for $\xi \in \mathcal{H}$, moving to the T-deformed Fock space $\mathcal{F}_T(\mathcal{H})$, the T-deformed creation (resp. T-deformed annihilation) denoted by $l(\xi)$ (resp. $l^*(\xi)$) is defined as follows:

(i) $l(\xi) := d(\xi)$;

(ii) $l^*(\xi)\Omega = 0$, and,
$l^*(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = d^*(\xi)R^{(n)}(\xi_1 \otimes \cdots \otimes \xi_n)$, $\xi_i \in \mathcal{H}$, for $1 \leqslant i \leqslant n$ and $n \in \mathbb{N}$, where $R^{(n)} = 1 + T_1 + T_1T_2 + \cdots + T_1T_2 \cdots T_{n-1} \in \mathbf{B}(\mathcal{H}^{\otimes n})$ for all $n \in \mathbb{N}$.

Note that the operators defined in definition 2.5 are all densely defined. But the following holds.

THEOREM 2.6. *For $\xi \in \mathcal{H}$, the operators $d(\xi), d^*(\xi)$ (resp. $l(\xi), l^*(\xi)$) admit bounded extensions on $\mathcal{F}(\mathcal{H})$ (resp. $\mathcal{F}_T(\mathcal{H})$). Further,*

(i) *$(d(\xi))^* = d^*(\xi)$ on $\mathcal{F}(\mathcal{H})$, and $(l(\xi))^* = l^*(\xi)$ on $\mathcal{F}_T(\mathcal{H})$;*

(ii) *$\|l(\xi)\| = \|l^*(\xi)\| \leqslant \frac{\|\xi\|_{\mathcal{H}}}{\sqrt{1-\lambda}}$.*

Proof. The statement regarding the standard creation and annihilation operators on $\mathcal{F}(\mathcal{H})$ is well known [**17**]. For statements regarding operators on $\mathcal{F}_T(\mathcal{H})$, we refer to [**5**, theorem 3.1]. □

Fix $\xi_1, \xi_2 \in \mathcal{H}$. Note that by definition 2.5, it follows that $l(\xi_1) - l(\xi_2) = l(\xi_1 - \xi_2)$. Thus, taking adjoints $l^*(\xi_1) - l^*(\xi_2) = l^*(\xi_1 - \xi_2)$ and hence by theorem 2.6, it follows that

$$\|l(\xi_1) - l(\xi_2)\| \leqslant \frac{\|\xi_1 - \xi_2\|_{\mathcal{H}}}{\sqrt{1-\lambda}} \quad \text{and} \quad \|l^*(\xi_1) - l^*(\xi_2)\| \leqslant \frac{\|\xi_1 - \xi_2\|_{\mathcal{H}}}{\sqrt{1-\lambda}}.$$

For $\xi \in \mathcal{H}$, let $s(\xi) = l(\xi) + l^*(\xi)$. Then, $s(\xi)$ is self-adjoint.

DEFINITION 2.7. Let $M_T = \{s(\xi) : \xi \in \mathcal{H}_{\mathbb{R}}\}''$. Then, M_T is said to be the von Neumann algebra associated to the Yang–Baxter operator T. Further, consider the vacuum state φ on M_T defined as

$$\varphi(x) := \langle \Omega, x\Omega \rangle_T, \qquad \text{for } x \in M_T.$$

THEOREM 2.8. *The following hold.*

1. *The vacuum state φ is a trace on M_T if and only if*

$$\langle \xi_r \otimes \xi_s, T(\xi_i \otimes \xi_j) \rangle_{\mathcal{H}\otimes\mathcal{H}} = \langle \xi_s \otimes \xi_j, T(\xi_r \otimes \xi_i) \rangle_{\mathcal{H}\otimes\mathcal{H}}$$

for all $\xi_i, \xi_j, \xi_r, \xi_s \in \mathcal{H}_{\mathbb{R}}$.

2. *Ω is cyclic for M_T. If φ is tracial, then Ω is also separating for M_T; in particular, φ is faithful.*

Proof. For the proof, we refer the reader to [**5**, theorems 4.3 and 4.4]. □

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REMARK 2.9.

1. From theorem 2.8, it follows that if φ is tracial, then $(M_T, \mathcal{F}_T(\mathcal{H}), \Omega)$ is the GNS triple for the vacuum state φ and in this case φ will be denoted by τ. Henceforth, we will assume that φ is a trace.

2. An important consequence of theorem 2.8 is the existence of the T-Wick product map. For $\xi \in M_T\Omega$, there exists a unique operator $W(\xi) \in M_T$ such that $W(\xi)\Omega = \xi$. Following [**6**], if $\xi_i \in \mathcal{H}_{\mathbb{R}}$ for $1 \leqslant i \leqslant n$, then $W(\xi_1 \otimes \cdots \otimes \xi_n)$ is said to be the T-Wick product of $s(\xi_1), \ldots, s(\xi_n)$.

Let $J : \mathcal{F}_T(\mathcal{H}) \to \mathcal{F}_T(\mathcal{H})$ given by extending $J(x\Omega) = x^*\Omega$, $x \in M_T$, denote the Tomita's conjugation operator so that $JM_TJ = M_T'$.

PROPOSITION 2.10. *$J(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_n \otimes \cdots \otimes \xi_1$, for $\xi_i \in \mathcal{H}_{\mathbb{R}}$, $1 \leqslant i \leqslant n$, $n \in \mathbb{N}$.*

Proof. By virtue of definition 2.5, we have

$$\xi_1 \otimes \cdots \otimes \xi_n = s(\xi_1)s(\xi_2)\cdots s(\xi_n)\Omega - \eta, \quad \xi_i \in \mathcal{H}_{\mathbb{R}},\ 1 \leqslant i \leqslant n, \tag{2.3}$$

where $\eta \in \bigoplus_{k=0}^{n-1} \mathcal{H}^{\otimes k}$.

Note that $J\Omega = \Omega$, $\overline{\mathcal{H}}^{\|\cdot\|_T} = \mathcal{H}$ and $J\mathcal{H} = \mathcal{H}$. Suppose that $J(\mathcal{H}^{\otimes_T^k}) \subseteq \oplus_{j=0}^{k} \mathcal{H}^{\otimes_T^j}$ for $1 \leqslant k \leqslant n$. Then, by definition 2.5 and equation (2.3), it follows that $J(\mathcal{H}^{\otimes_T^{(n+1)}}) \subseteq \oplus_{j=0}^{n+1} \mathcal{H}^{\otimes_T^j}$. Thus, by induction $J(\mathcal{H}^{\otimes_T^n}) \subseteq \oplus_{j=0}^{n} \mathcal{H}^{\otimes_T^j}$ for all $n \in \mathbb{N}$.

Now, we show that $J(\mathcal{H}^{\otimes_T^n}) \subseteq \mathcal{H}^{\otimes_T^n}$ for all n. Fix $n \geqslant 2$. Indeed,

$$\begin{aligned} & J\Big(\mathop{\oplus}_{l=0}^{n} \mathcal{H}^{\otimes_T^l}\Big) \perp \mathop{\oplus}_{j=n+1}^{\infty} \mathcal{H}^{\otimes_T^j} \\ & \quad \Longrightarrow\ J^2\Big(\mathop{\oplus}_{l=0}^{n} \mathcal{H}^{\otimes_T^l}\Big) \perp J\Big(\mathop{\oplus}_{j=n+1}^{\infty} \mathcal{H}^{\otimes_T^j}\Big) \\ & \quad \Longrightarrow\ \mathop{\oplus}_{l=0}^{n} \mathcal{H}^{\otimes_T^l} \perp J\Big(\mathop{\oplus}_{j=n+1}^{\infty} \mathcal{H}^{\otimes_T^j}\Big), \text{ as } J^2 = 1. \end{aligned}$$

Therefore,

$$J(\mathcal{H}^{\otimes_T^n}) \perp \mathop{\oplus}_{l=0}^{n-1} \mathcal{H}^{\otimes_T^l} \text{ and } J(\mathcal{H}^{\otimes_T^n}) \perp \mathop{\oplus}_{j=n+1}^{\infty} \mathcal{H}^{\otimes_T^j}.$$

Consequently, $J(\mathcal{H}^{\otimes_T^n}) \subseteq \mathcal{H}^{\otimes_T^n}$ for all $n \in \mathbb{N}$.

Now applying J to both sides of equation (2.3) and using equation (2.3) in reverse order again, we get

$$\begin{aligned} J(\xi_1 \otimes \cdots \otimes \xi_n) &= s(\xi_n)s(\xi_{n-1})\cdots s(\xi_1)\Omega - J\eta \\ &= (\xi_n \otimes \cdots \otimes \xi_1) + \eta' - J\eta, \end{aligned} \tag{2.4}$$

where $\eta' \in \bigoplus_{k=0}^{n-1} \mathcal{H}^{\otimes k}$.

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From the first part of the argument and from equation (2.4), it follows that

$$J(\xi_1 \otimes \cdots \otimes \xi_n) - (\xi_n \otimes \cdots \otimes \xi_1) = \eta' - J\eta = 0.$$

This completes the proof. □

REMARK 2.11. Let $\mathcal{J} : \mathcal{H} \to \mathcal{H}$ denote the standard complex conjugation, *i.e.*, $\mathcal{J}(\xi_1 + i\xi_2) = \xi_1 - i\xi_2$, for $\xi_1, \xi_2 \in \mathcal{H}_{\mathbb{R}}$. We will write $\mathcal{J}\xi = \bar{\xi}$ for $\xi \in \mathcal{H}$. Then, for $n \in \mathbb{N}$, we have

$$J(\xi_1 \otimes \cdots \otimes \xi_n) = \bar{\xi}_n \otimes \cdots \otimes \bar{\xi}_1, \quad \xi_i \in \mathcal{H}, \ 1 \leqslant i \leqslant n. \tag{2.5}$$

Since $M_T\Omega = M_T'\Omega$, we write

$$W_r(\xi) = JW(J\xi)J, \qquad \text{for } \xi \in M_T\Omega. \tag{2.6}$$

Thus, $W_r(\xi) \in M_T'$.

Now, following [10], we define the T-deformed right creation and annihilation operators on $\mathcal{F}_T(\mathcal{H})$ as follows.

DEFINITION 2.12. For $\xi \in \mathcal{H}$, the T-deformed right creation operator is defined as the bounded extension of the following:

$$r(\xi)\Omega = \xi,$$
$$r(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi_1 \otimes \cdots \otimes \xi_n \otimes \xi, \quad \xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}, \ n \in \mathbb{N}.$$

Note that $r(\xi) = Jl(\bar{\xi})J$.

The T-deformed right annihilation operator is the bounded extension of:

$$r^*(\xi)\Omega := 0,$$
$$r^*(\xi) := d_r^*(\xi)(1 + T_{n-1} + T_{n-1}T_{n-2} + \cdots + T_{n-1}T_{n-2}\cdots T_1), \text{ on } \mathcal{H}^{\otimes n}, \ n \in \mathbb{N},$$

where $d_r^*(\xi)$ is the bounded extension to $\mathcal{F}(\mathcal{H})$ of:

$$d_r^*(\xi)\Omega = 0,$$
$$d_r^*(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \langle \xi, \xi_n \rangle_{\mathcal{H}}(\xi_1 \otimes \cdots \otimes \xi_{n-1}), \quad \xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}, \ n \in \mathbb{N}.$$

Note that $r^*(\xi) = Jl^*(\bar{\xi})J$.

REMARK 2.13. Note that for $\xi \in \mathcal{H}$, one can also define the right creation as a bounded operator on the full Fock space $\mathcal{F}(\mathcal{H})$ analogously following the construction of free group factors [17]. We denote the same by $r^0(\xi)$.

Define $s_r(\xi) := r(\xi) + r^*(\xi) \quad$ for $\xi \in \mathcal{H}$.

REMARK 2.14. From definition 2.12, it follows that $Jl(\xi)J = r(\xi)$ and $Jl^*(\xi)J = r^*(\xi)$, for every $\xi \in \mathcal{H}_{\mathbb{R}}$. Thus, $Js(\xi)J = r(\xi) + r^*(\xi) = s_r(\xi)$ for all $\xi \in \mathcal{H}_{\mathbb{R}}$. Hence, $s_r(\xi) \in M_T'$ for all $\xi \in \mathcal{H}_{\mathbb{R}}$.

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Remark 2.15.

1. Let $\{e_i\}_{i\in\Lambda}$ be a fixed orthonormal basis of $\mathcal{H}_{\mathbb{R}}$. Note that $\{e_i\}_{i\in\Lambda}$ is also an orthonormal basis of $\mathcal{H}$. Write
$$T(e_i \otimes e_j) = \sum_{r,s\in\Lambda} t_{ij}^{rs}(e_r \otimes e_s),$$
for $i, j, r, s \in \Lambda$ and $t_{ij}^{rs} \in \mathbb{C}$. Then, the T-deformed creation operators $l(e_i),\ i \in \Lambda$, and the annihilation operators $l^*(e_i),\ i \in \Lambda$, fulfil the relations:
$$l^*(e_i)l(e_j) - \sum_{r,s\in\Lambda} t_{js}^{ir} l(e_r)l^*(e_s) = \delta_{ij}1, \text{ for all } i, j \in \Lambda,$$
as described in equation (2.1).

2. Note that if $\mathcal{H}_{\mathbb{R}}$ and $\mathcal{K}_{\mathbb{R}}$ are real Hilbert spaces with complexifications $\mathcal{H}$ and $\mathcal{K}$ respectively, $T \in \mathbf{B}(\mathcal{H}\otimes\mathcal{H})$, $S \in \mathbf{B}(\mathcal{K}\otimes\mathcal{K})$ are Yang–Baxter operators with $\|T\|, \|S\| < 1$ and $U : \mathcal{H}_{\mathbb{R}} \to \mathcal{K}_{\mathbb{R}}$ is an orthogonal operator such that $(V \otimes V)T(V \otimes V)^* = S$, where $V \in \mathbf{B}(\mathcal{H}, \mathcal{K})$ is the complexification of U, then M_T and M_S are spatially isomorphic via the unitary $\mathcal{F}_{T,S}(V)$, the second quantization of U.

3. Generating abelian subalgebras

In this section, we study the properties of generating abelian subalgebras of M_T that arise from special eigenvectors of T.

Lemma 3.1. *Suppose there exists $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ with $\|\xi_0\|_{\mathcal{H}} = 1$ such that $T(\xi_0 \otimes \xi_0) = q(\xi_0 \otimes \xi_0)$ for some $|q| \leqslant \lambda$. Then, the following assertions hold:*

(*i*)
$$\underbrace{s(\xi_0)s(\xi_0)\cdots s(\xi_0)}_{n\ times}\Omega = \sum_{\nu=\{\{i(r),j(r)\}_{1\leqslant r\leqslant l},\{k(p)\}_{1\leqslant p\leqslant m}\}} q^{a(\nu)}(\underbrace{\xi_0 \otimes \cdots \otimes \xi_0}_{m\ times}),$$
where the summation is over all partitions $\nu = \{\{i(r), j(r)\}_{1\leqslant r\leqslant l}, \{k(p)\}_{1\leqslant p\leqslant m}\}$ of $\{1, \ldots, n\}$ having blocks of one or two elements such that
$$l, m \geqslant 0,\ 2l + m = n,\ i(r) < j(r) \text{ for } 1 \leqslant r \leqslant l,\ k(1) < \cdots < k(m),$$
and $a(\nu)$ is given by
$$\begin{aligned} a(\nu) =&\#\{(r,s) :\ 1 \leqslant r, s \leqslant l,\ i(r) < i(s) < j(r) < j(s)\}\\ &+ \#\{(r,p) :\ 1 \leqslant r \leqslant l,\ 1 \leqslant p \leqslant m,\ i(r) < k(p) < j(r)\}.\end{aligned}$$

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(ii)

$$\tau((s(\xi_0))^n) = \begin{cases} 0, & \textit{if } n \textit{ is odd,} \\ \sum_{\nu=\{i(r),j(r)\}_{1\leqslant r\leqslant n/2}} q^{b(\nu)}, & \textit{if } n \textit{ is even,} \end{cases}$$

where the summation is over all pair partitions $\nu = \{i(r), j(r)\}_{1\leqslant r\leqslant n/2}$ *of* $\{1, \ldots, n\}$ *with* $i(r) < j(r)$ *and* $b(\nu)$ *is the number of crossings of* ν*, i.e.,*

$$b(\nu) = \#\{(r, s) : i(r) < i(s) < j(r) < j(s)\}.$$

Proof. (i) The proof is by induction. The case $n = 1$ is trivial. Suppose the formula is true for $k = 1, 2, \ldots, (n-1)$. Therefore, we have

$$\underbrace{s(\xi_0)s(\xi_0)\cdots s(\xi_0)}_{(n-1)\text{ times}}\Omega \tag{3.1}$$
$$= \sum_{\nu=\{\{i(r),j(r)\}_{1\leqslant r\leqslant l},\{k(p)\}_{1\leqslant p\leqslant m}\}} q^{a(\nu)}(\underbrace{\xi_0\otimes\cdots\otimes\xi_0}_{m\text{ times}}),$$

where the summation is over all partitions $\nu = \{\{i(r), j(r)\}_{1\leqslant r\leqslant l}, \{k(p)\}_{1\leqslant p\leqslant m}\}$ of $\{1, \ldots, (n-1)\}$ having blocks of one or two elements such that

$$l, m \geqslant 0,\ 2l + m = n - 1,\ i(r) < j(r) \text{ for } 1 \leqslant r \leqslant l,\ k(1) < \cdots < k(m).$$

Now, by applying $s(\xi_0)$ on both sides of equation (3.1), one has

$$\underbrace{s(\xi_0)s(\xi_0)\cdots s(\xi_0)}_{n\text{ times}}\Omega \tag{3.2}$$
$$= \sum_{\nu=\{\{i(r),j(r)\}_{1\leqslant r\leqslant l},\{k(p)\}_{1\leqslant p\leqslant m}\}} q^{a(\nu)}(\underbrace{\xi_0\otimes\cdots\otimes\xi_0}_{(m+1)\text{ times}})$$
$$+ \sum_{\nu=\{\{i(r),j(r)\}_{1\leqslant r\leqslant l},\{k(p)\}_{1\leqslant p\leqslant m}\}} q^{a(\nu)}(1 + q + \cdots + q^{(m-1)})(\underbrace{\xi_0\otimes\cdots\otimes\xi_0}_{(m-1)\text{ times}}),$$

where the index of the summation is the same as that in equation (3.1). Note that, each partition $\nu = \{\{i(r), j(r)\}_{1\leqslant r\leqslant l}, \{k(p)\}_{1\leqslant p\leqslant m}\}$ of $\{1, \ldots, (n-1)\}$ in the above corresponds to two partitions of $\{1, \ldots, n\}$ as described in the lemma as follows:

$$\nu_0 := \{\{i(r), j(r)\}_{1\leqslant r\leqslant l}, \{n\}, \{k(p)\}_{1\leqslant p\leqslant m}\},$$
$$\nu_u := \{\{k(u), n\}, \{i(r), j(r)\}_{1\leqslant r\leqslant l}, \{k(p)\}_{1\leqslant p\leqslant m,\ p\neq u}\} \text{ for } 1 \leqslant u \leqslant m.$$

It is easy to see that the partitions thus obtained altogether exhausts all partitions of $\{1, \ldots, n\}$ as described in the lemma. Since $n > i(r), j(r)$ for all

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$1 \leqslant r \leqslant l$, one has $a(\nu_0) = a(\nu)$, and, by the choice of ν_u, one has $a(\nu_u) = a(\nu) + m - u$ for $1 \leqslant u \leqslant m$. Therefore, from equation (3.2), it follows that

$$\underbrace{s(\xi_0)s(\xi_0)\cdots s(\xi_0)}_{n \text{ times}}\Omega$$
$$= \sum_{\nu_0} q^{a(\nu_0)} (\underbrace{\xi_0 \otimes \cdots \otimes \xi_0}_{(m+1) \text{ times}}) + \sum_{u=1}^{m} \sum_{\nu_u} q^{a(\nu_u)} (\underbrace{\xi_0 \otimes \cdots \otimes \xi_0}_{(m-1) \text{ times}}).$$

This completes the proof.

(ii) The proof follows by an immediate application of (i) with $m = 0$ in the constraint. We omit the details.

□

Let $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ with $\|\xi_0\|_{\mathcal{H}} = 1$ and $T(\xi_0 \otimes \xi_0) = q(\xi_0 \otimes \xi_0)$ for some $|q| \leqslant \lambda$. Let $M_{\xi_0} = vN(s(\xi_0))$. From part (ii) of lemma 3.1, it follows that the moments of $s(\xi_0)$ satisfy those of the q-semicircular law ν_q, which is absolutely continuous with respect to the uniform measure on the interval

$$\left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right].$$

Consequently, the abelian von Neumann algebra M_{ξ_0} is isomorphic to

$$L^{\infty}\left(\left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right], \nu_q\right)$$

and hence M_{ξ_0} is diffuse. Note that the associated orthogonal polynomials for ν_q are the q-Hermite polynomials H_n^q, $n \geqslant 0$. For more details about the density of ν_q and the recurrence relations defining the q-Hermite polynomials (which we use below), we refer the reader to [**6**].

Lemma 3.2. *Let* $\mathcal{E}_{\xi_0} = \{\xi_0^{\otimes n} : n \geqslant 0\}$, *where* $\xi_0^{\otimes 0} = \Omega$. *Then,* $\overline{M_{\xi_0}\Omega}^{\|\cdot\|_T} = \overline{\text{span } \mathcal{E}_{\xi_0}}^{\|\cdot\|_T}$.

Proof. First, we claim that $H_n^q(s(\xi_0))\Omega = \xi_0^{\otimes n}$ for all $n \in \mathbb{N} \cup \{0\}$. We prove it by induction.

The claim is obvious for $n = 0$. Since

$$H_1^q(x) = x, \quad x \in \left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right],$$

we have $H_1^q(s(\xi_0))\Omega = \xi_0$.

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Suppose the claim is true for all natural numbers $1 \leqslant n \leqslant k$. We want to calculate $H^q_{k+1}(s(\xi_0))\Omega$. By [6, definition 1.9], we have

$$xH^q_k(x) = H^q_{k+1}(x) + (1 + q + \cdots + q^{k-1})H^q_{k-1}(x), \qquad x \in \left[-\frac{2}{\sqrt{1-q}}, \frac{2}{\sqrt{1-q}}\right].$$

Therefore, by functional calculus and definition 2.5, one has

$$\begin{aligned} H^q_{k+1}(s(\xi_0))\Omega &= s(\xi_0)H^q_k(s(\xi_0))\Omega - \left(\sum_{i=0}^{k-1} q^i\right) H^q_{k-1}(s(\xi_0))\Omega \\ &= s(\xi_0)\xi_0^{\otimes k} - \left(\sum_{i=0}^{k-1} q^i\right)\xi_0^{\otimes(k-1)} \\ &= (l(\xi_0) + l^*(\xi_0))\xi_0^{\otimes k} - \left(\sum_{i=0}^{k-1} q^i\right)\xi_0^{\otimes(k-1)} \\ &= \xi_0^{\otimes(k+1)} + d^*(\xi_0)(1 + T_1 + \cdots + T_1T_2\cdots T_{k-1})\xi_0^{\otimes k} \\ &\quad - \left(\sum_{i=0}^{k-1} q^i\right)\xi_0^{\otimes(k-1)} \\ &= \xi_0^{\otimes(k+1)} + d^*(\xi_0)\Big(\xi_0^{\otimes k} + T_1(\xi_0^{\otimes k}) + \cdots + T_1T_2\cdots T_{k-1}(\xi_0^{\otimes k})\Big) \\ &\quad - \left(\sum_{i=0}^{k-1} q^i\right)\xi_0^{\otimes(k-1)} \\ &= \xi_0^{\otimes(k+1)} + (1 + q + \cdots + q^{k-1})\,\langle \xi_0, \xi_0\rangle_{\mathcal{H}}\,\xi_0^{\otimes(k-1)} \\ &\quad - \left(\sum_{i=0}^{k-1} q^i\right)\xi_0^{\otimes(k-1)} \\ &= \xi_0^{\otimes(k+1)}. \end{aligned}$$

This establishes the claim.

Hence, it follows that $\overline{\text{span}\ \mathcal{E}_{\xi_0}}^{\|\cdot\|_T} \subseteq \overline{M_{\xi_0}\Omega}^{\|\cdot\|_T}$.

To prove the reverse inclusion, note that τ restricted to M_{ξ_0} coincides with $\int \cdot d\nu_q$ (see (ii) of lemma 3.1). Therefore, $\overline{M_{\xi_0}\Omega}^{\|\cdot\|_T}$ can be identified with $L^2(\nu_q)$ via a unitary that maps $\xi_0^{\otimes n}$ to H^q_n for all $n \geqslant 0$. Since H^q_n, $n \geqslant 0$, is an orthogonal basis of $L^2(\nu_q)$, it readily follows that $\overline{M_{\xi_0}\Omega}^{\|\cdot\|_T} \subseteq \overline{\text{span}\ \mathcal{E}_{\xi_0}}^{\|\cdot\|_T}$.

This completes the proof. □

4. Singularity of singly generated subalgebra

In this section, we show that if $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ is such that $\|\xi_0\|_{\mathcal{H}} = 1$ and $T(\xi_0 \otimes \xi_0) = q(\xi_0 \otimes \xi_0)$ for some $|q| \leqslant \lambda$, then the associated singly generated abelian subalgebra M_{ξ_0} of M_T generated by the operator $s(\xi_0)$ is a strongly mixing masa in M_T.

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Hence, M_{ξ_0} is singular in M_T. However, in the case $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$, we need to impose a mild hypothesis on existence of appropriate orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ so that calculations involving the T-Wick products are tractable.

Let M be a finite von Neumann algebra equipped with a faithful normal tracial state τ. Let M act in the standard form on the GNS space $\mathcal{H}_\tau := L^2(M, \tau)$ via left multiplication. Let J_τ and Ω_τ respectively denote the Tomita's modular operator and the standard vacuum vector associated to τ. Further, let $\|\cdot\|_{2,\tau}$ and $\langle \cdot, \cdot \rangle_\tau$ respectively denote the norm and the inner product of $\mathcal{H}_\tau$.

Let $A \subseteq M$ be a diffuse abelian subalgebra and let $\mathbb{E}_A : M \to A$ denote the τ-preserving faithful normal conditional expectation from M onto A. Let e_A denote the Jones' projection associated to A.

For $x, y \in M$, consider the densely defined operator

$$T_{x,y} : L^2(A, \tau) \to L^2(A, \tau) \text{ defined by } T_{x,y}(a\Omega_\tau) = \mathbb{E}_A(xay)\Omega_\tau, \ a \in A.$$

Note that,

$$\begin{aligned}
\|\mathbb{E}_A(xay)\Omega_\tau\|_{2,\tau} &\leqslant \|xay\Omega_\tau\|_{2,\tau} \\
&\leqslant \|x\| \, \|ay\Omega_\tau\|_{2,\tau} \\
&\leqslant \|x\| \left(\langle ay\Omega_\tau, ay\Omega_\tau \rangle_\tau\right)^{\frac{1}{2}} \\
&= \|x\| \, \tau(y^*a^*ay)^{\frac{1}{2}} \\
&= \|x\| \, \tau(ayy^*a^*)^{\frac{1}{2}} \\
&\leqslant \|x\| \, \|y\| \, \|a\Omega_\tau\|_{2,\tau}, \text{ for all } a \in A.
\end{aligned}$$

Consequently, $T_{x,y}$ admits a bounded extension to $L^2(A, \tau)$ which will also be denoted by $T_{x,y}$ with a slight abuse of notation.

Definition 4.1. For a masa A of M, the *normalizer* of A, denoted by $N_M(A)$ is defined as, $N_M(A) = \{u \in \mathcal{U}(M) : uAu^* = A\}$. The subalgebra A is called *singular*, if $N_M(A) = \mathcal{U}(A)$.

Definition 4.2 [7]. A diffuse abelian subalgebra $A \subseteq M$ is said to be strongly mixing in M, if $\|\mathbb{E}_A(xa_ny)\Omega_\tau\|_{2,\tau} \to 0$ for all $x, y \in M$ with $\mathbb{E}_A(x) = 0 = \mathbb{E}_A(y)$, whenever $\{a_n\}$ is a bounded sequence in A that goes to 0 in the w^*-topology.

It is easy to see that a strongly mixing (diffuse) abelian subalgebra is automatically a masa in M. Further, strongly mixing masas are singular [7].

Identify $A \cong L^\infty(X, \mu)$, where X is a standard Borel space with X being compact and metrizable, and μ is a non-atomic probability measure on X such that $\tau_{\upharpoonright A} = \int_X \cdot d\mu$. The *left–right* measure of A is the measure η on $X \times X$ (strictly speaking the measure class of η) obtained from the direct integral decomposition of $L^2(M, \tau) \ominus L^2(A, \tau)$ over $X \times X$, so that $(A \vee J_\tau AJ_\tau)(1 - e_A)$ is the algebra of diagonalizable operators with respect to the decomposition (see [12] for details).

The following result will be crucial for our purpose.

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Theorem 4.3 [**2**, theorem 5.2]. *Let $A \subseteq M$ be a diffuse abelian algebra such that the left–right measure of A is absolutely continuous with respect to $\mu \times \mu$. Then, A is a strongly mixing masa in M. In particular, A is a singular masa in M.*

Again, from the results of [**11**, § 2] it follows that the *left–right* measure of A is absolutely continuous with respect to $\mu \times \mu$, whenever $T_{x,y}$ is Hilbert–Schmidt for x, y varying over a set $S \subseteq M$ such that $\mathbb{E}_A(x) = 0 = \mathbb{E}_A(y)$ for all $x, y \in S$, and the span of $S\Omega_\tau$ is dense in $L^2(M,\tau) \ominus L^2(A,\tau)$.

Now, we proceed to apply these techniques to the von Neumann algebra M_T constructed in § 2.

For the next lemma we need some facts on permutations.

Let σ be a bijection on the set $\{1,\dots,n\}$ and let $\gamma : \{2,\dots,(n+1)\} \to \{1,\dots,n\}$ be the function defined by $\gamma(j) = j-1$, $2 \leqslant j \leqslant n+1$. Clearly, γ is a bijection and hence, we get another realization of σ as a bijection σ' on the set $\{2,\dots,(n+1)\}$ by $\sigma' = \gamma^{-1}\sigma\gamma$. We denote $\sigma \times Id$ as a bijection on the set $\{1,\dots,(n+1)\}$, where $(\sigma \times Id)_{\upharpoonright\{1,2,\dots,n\}} = \sigma$ and $\sigma \times Id$ keeps $(n+1)$ fixed. In a similar fashion, we can define $Id \times \sigma'$.

Now, if $\rho = \varsigma_1 \cdots \varsigma_n$, where $\varsigma_i \in S_{n+1}$ is the transposition given by $(i, i+1)$, $1 \leqslant i \leqslant n$, then note that $\rho(\sigma \times Id) = (Id \times \sigma')\rho$. Indeed, when $j \in \{1,\dots,n\}$, note that,

$$\begin{aligned} &\rho(j) = \varsigma_1 \cdots \varsigma_j \cdots \varsigma_n(j) = j+1; \\ \Rightarrow\ &\rho(\sigma \times Id)(j) = \rho(\sigma(j)) = \sigma(j)+1 = \sigma'(j+1) = (Id \times \sigma')\rho(j). \end{aligned}$$

The last equality holds because $\sigma'(j+1)$ is an element of the set $\{2,\dots,(n+1)\}$ and hence $\sigma'(j+1) = (Id \times \sigma')(j+1) = (Id \times \sigma')\rho(j)$. Also,

$$\rho(\sigma \times Id)(n+1) = 1 = (Id \times \sigma')\rho(n+1).$$

Proposition 4.4. *Let $\mathcal{H}_i$, $i = 1, 2$, be Hilbert spaces and let $P_i : \mathfrak{D}(P_i) \subseteq \mathcal{H}_i \to \mathcal{H}_i$ be densely defined strictly positive self-adjoint operators for $i = 1, 2$. Let $B_i : \mathfrak{D}(P_i) \times \mathfrak{D}(P_i) \to \mathbb{C}$ be a sesquilinear form given by $B_i(\eta,\xi) = \langle \eta, P_i\xi\rangle_{\mathcal{H}_i}$, $\xi \in \mathfrak{D}(P_i)$ and $\eta \in \mathcal{H}_i$, for $i = 1, 2$. Let $\mathcal{H}_{P_i}$ denote the Hilbert space completion of $\mathfrak{D}(P_i)$ with respect to B_i, $i = 1, 2$. Let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded operator such that $AP_1 \subseteq P_2A$. Then, A admits a unique extension $\widetilde{A} : \mathcal{H}_{P_1} \to \mathcal{H}_{P_2}$ such that $\left\|\widetilde{A}\right\| = \|A\|$.*

Proof. For a proof see [**3**, proposition A.1]. □

Lemma 4.5. *For $n \geqslant 1$ and $e, f \in \mathcal{H}_\mathbb{R}$ with $\|e\|_\mathcal{H} = \|f\|_\mathcal{H} = 1$, consider the operator $d^*(e)T_1T_2\cdots T_n r^0(f) \in \mathbf{B}(\mathcal{H}^{\otimes n})$. Then,*

(i) *$d^*(e)T_1T_2\cdots T_n r^0(f)$ has a bounded extension to $\mathcal{H}^{\otimes_T^n}$.*

(ii) *Denoting $d^*(e)T_1T_2\cdots T_n r^0(f)$ to be the extension on $\mathcal{H}^{\otimes_T^n}$ again, for $\xi \in \mathcal{H}^{\otimes_T^n}$, one has*

$$\left\|d^*(e)T_1T_2\cdots T_n r^0(f)(\xi)\right\|_T \leqslant \lambda^n \|\xi\|_T.$$

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Proof. First of all, note that in the statement we have by abuse of notation taken the restriction of $r^0(f)$ on $\mathcal{H}^{\otimes n}$ (cf. remark 2.13). Fix $\sigma \in S_n$. Then,

$$d^*(e)T_1T_2\cdots T_nr^0(f)\Phi(\sigma) = d^*(e)\Phi(\rho)\big(\Phi(\sigma)\otimes 1\big)r^0(f).$$

Note that $Inv(\rho) = n$. Again,

$$\begin{aligned} &Inv(\rho(\sigma\times Id)) \\ &= \#\{(i,j) : i<j,\, (\rho(\sigma\times Id))(i) > (\rho(\sigma\times Id))(j)\} \\ &= \#\{(i,n+1) : i<n+1,\, (\rho(\sigma\times Id))(i) > (\rho(\sigma\times Id))(n+1)\} \\ &\qquad + \#\{(i,j) : i<j\neq n+1,\, (\rho(\sigma\times Id))(i) > (\rho(\sigma\times Id))(j)\} \\ &= \#\{(i,n+1) : i<n+1,\, \sigma(i)+1 > \rho(n+1) = 1\} \\ &\qquad + \#\{(i,j) : i<j\neq n+1,\, \sigma(i)+1 > \sigma(j)+1\} \\ &= \#\{(i,n+1) : i<n+1,\, \sigma(i)\geqslant 1\} \\ &\qquad + \#\{(i,j) : i<j\neq n+1,\, \sigma(i) > \sigma(j)\} \\ &= n + Inv(\sigma) \\ &= Inv(\rho) + Inv(\sigma\times Id). \end{aligned} \tag{4.1}$$

Since, $\rho(\sigma\times Id) = (Id\times\sigma')\rho$ (as discussed above), from equation (4.1), we have

$$\begin{aligned} Inv((Id\times\sigma')\rho) = Inv(\rho(\sigma\times Id)) &= Inv(\rho) + Inv(\sigma\times Id) \\ &= Inv(\rho) + Inv(\sigma) \\ &= Inv(\rho) + Inv(\sigma') \\ &= Inv(\rho) + Inv(Id\times\sigma'). \end{aligned}$$

As Φ is quasi-multiplicative, we get

$$\Phi(\rho)\Phi(\sigma\times Id) = \Phi(\rho(\sigma\times Id)) = \Phi((Id\times\sigma')\rho) = \Phi(Id\times\sigma')\Phi(\rho). \tag{4.2}$$

Hence, if $\xi \in \mathcal{H}^{\odot n}$, we have

$$\begin{aligned} d^*(e)T_1T_2\cdots T_nr^0(f)\Phi(\sigma)\xi &= d^*(e)\Phi(\rho)r^0(f)\Phi(\sigma)\xi \\ &= d^*(e)\Phi(\rho)(\Phi(\sigma)\otimes 1)r^0(f)\xi \quad \text{(by equation (2.2))} \\ &= d^*(e)\Phi(\rho)\Phi(\sigma\times Id)r^0(f)\xi \\ &= d^*(e)\Phi(Id\times\sigma')\Phi(\rho)r^0(f)\xi \quad \text{(by equation (4.2))} \\ &= d^*(e)(1\otimes\Phi(\sigma'))\Phi(\rho)r^0(f)\xi \\ &= \Phi(\sigma')d^*(e)\Phi(\rho)r^0(f)\xi \\ &= \Phi(\sigma)d^*(e)\Phi(\rho)r^0(f)\xi \qquad (\Phi(\sigma)=\Phi(\sigma')) \\ &= \Phi(\sigma)d^*(e)T_1T_2\cdots T_nr^0(f)\xi. \end{aligned}$$

Now by the density of $\mathcal{H}^{\odot n}$ in $\mathcal{H}^{\otimes n}$ we get, $d^*(e)T_1T_2\cdots T_nr^0(f)$ commutes with $P^{(n)} = \sum_{\sigma\in S_n}\Phi(\sigma)$ on $\mathcal{H}^{\otimes n}$.

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Note that $d^*(e)$ and $r^0(f)$ are contractions on $\mathcal{F}(\mathcal{H})$ (see theorem 2.6). The rest is a direct application of proposition 4.4 and the fact that $\|T_i\| = \lambda < 1$ for $1 \leqslant i \leqslant n$ (see proposition 2.2). □

Proposition 4.6. *Fix $n \in \mathbb{N} \cup \{0\}$ and let $e, f \in \mathcal{H}_{\mathbb{R}}$ with $\|e\|_{\mathcal{H}} = \|f\|_{\mathcal{H}} = 1$. Let*

$$B_{e,f}^{(n)} = (l^*(e)r(f) - r(f)l^*(e))_{\restriction \mathcal{H}^{\otimes_T^n}}.$$

Then, $\left\|B_{e,f}^{(n)}\right\| \leqslant \lambda^n$.

Proof. First, let $n = 0$. From definition (2.5), it follows that $B_{e,f}^{(0)} = \langle e, f\rangle_{\mathcal{H}} 1$. Hence, $\left\|B_{e,f}^{(0)}\right\| \leqslant 1 = \lambda^0$.

Now fix $n \in \mathbb{N}$. Let $\xi_1 \otimes \cdots \otimes \xi_n \in \mathcal{H}^{\odot n}$. Then,

$$\begin{aligned} l^*(e)r(f)(\xi_1 \otimes \cdots \otimes \xi_n) &= l^*(e)(\xi_1 \otimes \cdots \otimes \xi_n \otimes f) \\ &= d^*(e)R^{(n+1)}(\xi_1 \otimes \cdots \otimes \xi_n \otimes f) \quad \text{(definition (2.5))} \\ &= d^*(e)\big(R^{(n)} \otimes 1 + T_1T_2\cdots T_n\big)(\xi_1 \otimes \cdots \otimes \xi_n \otimes f) \\ &= d^*(e)(R^{(n)} \otimes 1)(\xi_1 \otimes \cdots \otimes \xi_n \otimes f) \\ &\qquad + d^*(e)T_1T_2\cdots T_n r^0(f)(\xi_1 \otimes \cdots \otimes \xi_n) \\ &= d^*(e)\big(R^{(n)}(\xi_1 \otimes \cdots \otimes \xi_n) \otimes f\big) \\ &\qquad + d^*(e)T_1T_2\cdots T_n r^0(f)(\xi_1 \otimes \cdots \otimes \xi_n). \end{aligned}$$

Also,

$$\begin{aligned} r(f)l^*(e)(\xi_1 \otimes \cdots \otimes \xi_n) &= r(f)d^*(e)R^{(n)}(\xi_1 \otimes \cdots \otimes \xi_n) \quad \text{(definition 2.5)} \\ &= \big(d^*(e)R^{(n)}(\xi_1 \otimes \cdots \otimes \xi_n)\big) \otimes f \\ &= d^*(e)\big(R^{(n)}(\xi_1 \otimes \cdots \otimes \xi_n) \otimes f\big). \end{aligned}$$

Consequently,

$$B_{e,f}^{(n)} = d^*(e)T_1T_2\cdots T_n r^0(f), \text{ on } \mathcal{H}^{\odot n}.$$

By the density of $\mathcal{H}^{\odot n}$ in $\mathcal{H}^{\otimes_T^n}$ and by lemma 4.5, one has

$$B_{e,f}^{(n)} = d^*(e)T_1T_2\cdots T_n r^0(f), \quad \text{on } \mathcal{H}^{\otimes_T^n}.$$

Again by lemma 4.5, it follows that $\left\|B_{e,f}^{(n)}\right\| \leqslant \lambda^n$. This completes the proof. □

The following lemma from [18] will be useful.

Lemma 4.7 [18, Lemma 3]. *Let $(H_n)_{n\geqslant 1}$ be a sequence of Hilbert spaces and let $H = \oplus_{n\geqslant 1} H_n$. Let $r, s \in \mathbb{N}$ and let $(a_i)_{1\leqslant i\leqslant r}$, $(b_j)_{1\leqslant j\leqslant s}$ be two families of operators*

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on H which send each H_n into H_{n+1} or H_{n-1} ($H_0 = 0$ by convention), such that there exists $0 < \mu < 1$ with

$$\left\|(a_i b_j - b_j a_i)_{\restriction H_n}\right\| \leqslant \mu^n \text{ for all } n \geqslant 1 \text{ and for all } i, j.$$

For $n \geqslant 1$, let $K_n \subseteq H_n$ be a finite dimensional subspace and let $K = \oplus_{n\geqslant 1} K_n$. Suppose that

$$a_i(K) \subseteq K, \quad 1 \leqslant i \leqslant r-1, \text{ and } a_{r\restriction K} = 0.$$

Then, there exists a constant $C > 0$ independent of n, such that

$$\left\|(a_r \cdots a_1 b_1 \cdots b_s)_{\restriction K_n}\right\| \leqslant C\mu^n \text{ for all } n \geqslant 0.$$

Next we prove that M_{ξ_0} is a strongly mixing masa in M_T. Our analysis is divided into two theorems depending on $\dim(\mathcal{H}_\mathbb{R})$. Let $\mathbb{E}_{\xi_0} : M_T \to M_{\xi_0}$ denote the τ-preserving conditional expectation onto M_{ξ_0}. The Jones' projection associated to M_{ξ_0} will be denoted by e_{ξ_0}.

Definition 4.8. For $k \geqslant 0$, define $U_k :\ \mathrm{span}_\mathbb{C}\{\mathcal{H}^{\odot m}, m \geqslant k, m \neq 0\} \to \mathbf{B}(\mathcal{F}_T(\mathcal{H}))$ by linearly extending:

$$U_k(f_1 \otimes \cdots \otimes f_m) = l(f_1)\cdots l(f_k) l^*(\overline{f}_{k+1}) \cdots l^*(\overline{f}_m), \quad \text{for} \quad m \geqslant k,$$

where, $\overline{\xi + i\eta} = \xi - i\eta$ for $\xi, \eta \in \mathcal{H}_\mathbb{R}$.

By [**14**, pp. 23], for $k \geqslant 0$ and $m \geqslant k$, U_k admits a bounded extension to $\mathcal{H}^{\otimes k} \otimes \mathcal{H}^{\otimes(m-k)}$.

Theorem 4.9 [**9**, theorem 1]. *Let $\{e_\mu : \mu \in \Lambda\}$ be an orthonormal basis of $\mathcal{H}_\mathbb{R}$ and let $T \in \mathbf{B}(\mathcal{H} \otimes \mathcal{H})$ be a Yang–Baxter operator with matrix representation $[t^{rs}_{ij} = \langle e_r \otimes e_s, T(e_i \otimes e_j)\rangle_{\mathcal{H}\otimes\mathcal{H}}]$, $i, j, r, s \in \Lambda$. Let $t^{ir}_{js} = t^{rs}_{ij}$ for all $i, j, r, s \in \Lambda$ and let the set $\{(r, s) : t^{ir}_{js} \neq 0\}$ be finite for every $i, j \in \Lambda$. Then,*

$$W(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{k=0}^{n} \sum_{\sigma \in S_n/(S_k \times S_{n-k})} U_k[\Phi(\sigma)(\xi_1 \otimes \cdots \otimes \xi_n)],$$

where $\xi_1, \ldots, \xi_n \in \mathcal{H}_\mathbb{R}$ and $n \in \mathbb{N}$.

Theorem 4.10. *Let $\dim(\mathcal{H}_\mathbb{R}) < \infty$, and suppose that there exists $\xi_0 \in \mathcal{H}_\mathbb{R}$ with $\|\xi_0\|_\mathcal{H} = 1$ such that $T(\xi_0 \otimes \xi_0) = q(\xi_0 \otimes \xi_0)$ for some $|q| \leqslant \lambda$. Let $x = W(\xi_1 \otimes \cdots \otimes \xi_m)$ and $y = W(\eta_1 \otimes \cdots \otimes \eta_k)$ for $\xi_i, \eta_j \in \mathcal{H}_\mathbb{R}$, $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant k$, $m, k \in \mathbb{N}$, be such that $\mathbb{E}_{\xi_0}(x) = 0 = \mathbb{E}_{\xi_0}(y)$. Then, $T_{x,y} : L^2(M_{\xi_0}, \tau) \to L^2(M_{\xi_0}, \tau)$ defined as*

$$T_{x,y}(a\Omega) = \mathbb{E}_{\xi_0}(xay)\Omega, \ a \in M_{\xi_0},$$

is a Hilbert–Schmidt operator.

Proof. If either x or y is 0, then the result is trivial. Thus, we assume that both x and y are non-zero. Let $v_n = W(\xi_0^{\otimes n})$ for all $n \in \mathbb{N} \cup \{0\}$ ($\xi_0^{\otimes 0} = \Omega$). From lemma 3.2, we

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note that $\{v_n\Omega/\|v_n\Omega\|_T : n \in \mathbb{N} \cup \{0\}\}$ forms an orthonormal basis of $L^2(M_{\xi_0}, \tau)$. Therefore, to show $T_{x,y}$ is Hilbert–Schmidt, it is enough to show that

$$\sum_{n \geqslant 0} \frac{\|T_{x,y}(v_n\Omega)\|_T^2}{\|v_n\Omega\|_T^2} < \infty.$$

Note that for $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
T_{x,y}(v_n\Omega) &= \mathbb{E}_{\xi_0}(xW(\xi_0^{\otimes n})y)\Omega \\
&= \mathbb{E}_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})W(\eta_1 \otimes \cdots \otimes \eta_k)\big)\Omega \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})W(\eta_1 \otimes \cdots \otimes \eta_k)\Omega\big) \\
&= e_{\xi_0}(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})(\eta_1 \otimes \cdots \otimes \eta_k)) \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})J(\eta_k \otimes \cdots \otimes \eta_1)\big) \\
&\quad \text{(by proposition 2.10)} \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})JW(\eta_k \otimes \cdots \otimes \eta_1)\Omega\big) \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})JW(J(\eta_1 \otimes \cdots \otimes \eta_k))\Omega\big) \\
&\quad \text{(by proposition 2.10)} \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})JW(J(\eta_1 \otimes \cdots \otimes \eta_k))J\Omega\big) \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W(\xi_0^{\otimes n})W_r(\eta_1 \otimes \cdots \otimes \eta_k)\Omega\big) \\
&\quad \text{(by equation (2.6))} \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W_r(\eta_1 \otimes \cdots \otimes \eta_k)W(\xi_0^{\otimes n})\Omega\big) \\
&= e_{\xi_0}\big(W(\xi_1 \otimes \cdots \otimes \xi_m)W_r(\eta_1 \otimes \cdots \otimes \eta_k)\xi_0^{\otimes n}\big).
\end{aligned}$$

Since $\mathbb{E}_{\xi_0}(W(\xi_1 \otimes \cdots \otimes \xi_m)) = 0$, we have $e_{\xi_0}(W(\xi_1 \otimes \cdots \otimes \xi_m)\Omega) = 0$. By the T-Wick product formula in theorem 4.9, we have

$$W(\xi_1 \otimes \cdots \otimes \xi_m) = \sum_{a=0}^{m} U_a\Bigg(\sum_{\sigma \in S_m/S_a \times S_{m-a}} [\Phi(\sigma)(\xi_1 \otimes \cdots \otimes \xi_m)]\Bigg).$$

Now, we extend ξ_0 to an orthonormal basis $\{f_\mu : \mu \in \Lambda\}$ of $\mathcal{H}_\mathbb{R}$. Write $\mathcal{F}_T(\mathbb{C}\xi_0) := L^2(M_{\xi_0}, \tau)$ and $l_\mu = l(f_\mu)$, $r_\mu = r(f_\mu)$, $\mu \in \Lambda$. Note that, $\mathbb{E}_{\xi_0}(x) = 0$ implies $\xi_1 \otimes \cdots \otimes \xi_m \in \mathcal{F}_T(\mathbb{C}\xi_0)^\perp$. That is,

$$0 = \langle \xi_1 \otimes \cdots \otimes \xi_m, \xi_0^{\otimes m}\rangle_T = \sum_{\sigma \in S_m} q^{Inv(\sigma)} \langle \xi_1 \otimes \cdots \otimes \xi_m, \xi_0^{\otimes m}\rangle_{\mathcal{H}^{\otimes m}}.$$

Hence, $\xi_1 \otimes \cdots \otimes \xi_m \in \mathcal{H}^{\otimes m} \ominus \mathbb{C}\xi_0^{\otimes m}$. Also, we have

$$\mathcal{H}^{\otimes m} \ominus \mathbb{C}\xi_0^{\otimes m} = \overline{\text{span}_\mathbb{C}\{f_{i_1} \otimes \cdots \otimes f_{i_m} : i_j \in \Lambda, 1 \leqslant j \leqslant m, \text{ at least one } f_{i_j} \neq \xi_0\}}^{\|\cdot\|_{\mathcal{H}^{\otimes m}}}.$$

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As $\mathcal{H}_{\mathbb{R}}$ is finite dimensional (hence $\mathcal{H}$ is also finite dimensional), we have

$$\mathcal{H}^{\otimes m} \ominus \mathbb{C}\xi_0^{\otimes m} = \text{span}_{\mathbb{C}}\{f_{i_1} \otimes \cdots \otimes f_{i_m} : i_j \in \Lambda, 1 \leqslant j \leqslant m, \text{ at least one } f_{i_j} \neq \xi_0\}.$$

Let $\sigma \in S_m$. Note that $\Phi(\sigma)(\mathcal{H}^{\otimes m}) \subseteq \mathcal{H}^{\otimes m}$, $\Phi(\sigma)(\mathbb{C}\xi_0^{\otimes m}) = \mathbb{C}\xi_0^{\otimes m}$ and $\Phi(\sigma)(\mathcal{H}^{\otimes m} \ominus \mathbb{C}\xi_0^{\otimes m}) \subseteq \mathcal{H}^{\otimes m} \ominus \mathbb{C}\xi_0^{\otimes m}$. Therefore, we have

$$\begin{aligned} &\sum_{\sigma \in S_m / S_a \times S_{m-a}} [\Phi(\sigma)(\xi_1 \otimes \cdots \otimes \xi_m)] \\ &\quad = \sum_{\underline{i}=(i_1,\ldots,i_a,\ldots,i_m)\in F} \alpha_{\underline{i}}\, f_{i_1} \otimes \cdots \otimes f_{i_a} \otimes \cdots \otimes f_{i_m}, \end{aligned}$$

where, $F \subseteq \Lambda^m$ (depending on $\xi_1, \ldots, \xi_m$), $\alpha_{\underline{i}}$'s are scalars, and, for each $\underline{i} \in F$ there exists i_α with $1 \leqslant \alpha \leqslant m$ such that $f_{i_\alpha} \neq \xi_0$. Hence, we have

$$W(\xi_1 \otimes \cdots \otimes \xi_m) = \sum_{a=0}^{m} \sum_{\underline{i}=(i_1,\ldots,i_a,\ldots,i_m)\in F} \alpha_{\underline{i}}\, l_{i_1} \cdots l_{i_a} l^*_{i_{a+1}} \cdots l^*_{i_m}. \tag{4.3}$$

Also, by the definition of $W_r(\eta_1 \otimes \cdots \otimes \eta_k)$, remark 2.14 and a similar argument as above, we can write

$$W_r(\eta_1 \otimes \cdots \otimes \eta_k) = \sum_{b=0}^{k} \sum_{\underline{j}=(j_1,\ldots,j_b,\ldots,j_k)\in G} \beta_{\underline{j}}\, r_{j_1} \cdots r_{j_b} r^*_{j_{b+1}} \cdots r^*_{j_k}, \tag{4.4}$$

where $G \subseteq \Lambda^k$ (depending on $\eta_1, \ldots, \eta_k$), $\beta_{\underline{j}}$'s are scalars, and, for each $\underline{j} \in G$ there exists j_β with $1 \leqslant \beta \leqslant k$ such that $f_{j_\beta} \neq \xi_0$.

Fix $\underline{i} \in F$ and $\underline{j} \in G$. In view of equations (4.3) and (4.4), it is enough to show that $\sum_{n \geqslant 0} \frac{\|\zeta_n\|_T^2}{\|v_n\Omega\|_T^2} < \infty$, where

$$\zeta_n := e_{\xi_0}(l_{i_1} \cdots l_{i_a} l^*_{i_{a+1}} \cdots l^*_{i_m} r_{j_1} \cdots r_{j_b} r^*_{j_{b+1}} \cdots r^*_{j_k} \xi_0^{\otimes n}), \quad n \geqslant 0,$$

and, $f_{i_\alpha} \neq \xi_0$ and $f_{i_\beta} \neq \xi_0$.

Let $\gamma = \max\{1 \leqslant u \leqslant m : f_{i_u} \neq \xi_0\}$. If $\gamma \leqslant a$, then there is at least one left creation amongst $\{l_{i_1}, \ldots, l_{i_a}\}$ namely l_{i_γ} such that $f_{i_\gamma} \neq \xi_0$. In this case, $\zeta_n = 0$ for all $n \geqslant 0$ and the argument is complete.

On the other side, let $\gamma \geqslant (a+1)$. From proposition 4.6, it follows that $\left\| [l^*_\mu, r_{\mu'}]_{\restriction \mathcal{H}^{\otimes n}_T} \right\| \leqslant \lambda^n$, for $\mu, \mu' \in \Lambda$ and $n \geqslant 0$. Also, note that $[l^*_\mu, r^*_{\mu'}] = 0$ (by remark 2.14) for all $\mu, \mu' \in \Lambda$, $l^*_{i_\gamma \restriction \mathcal{F}_T(\mathbb{C}\xi_0)} = 0$, and, $l^*_{i_t}(\mathcal{F}_T(\mathbb{C}\xi_0)) \subseteq \mathcal{F}_T(\mathbb{C}\xi_0)$ for $\gamma < t \leqslant m$. Hence, by applying lemma 4.7 to the operators $(l^*_{i_\gamma}, \ldots, l^*_{i_m})$ and $(r_{j_1}, \ldots, r_{j_b}, r^*_{j_{b+1}}, \ldots, r^*_{j_k})$, $K_n := \mathbb{C}\xi_0^{\otimes n}$, $n \geqslant 0$, and $K := \mathcal{F}_T(\mathbb{C}\xi_0)$, we get that there exists a constant $C > 0$ such that

$$\|\zeta_n\|_T \leqslant C\lambda^n \left\|\xi_0^{\otimes n}\right\|_T = C\lambda^n \|v_n\Omega\|_T, \quad \text{for all } n \geqslant 0.$$

Consequently, $\sum_{n \geqslant 0} \frac{\|\zeta_n\|_T^2}{\|v_n\Omega\|_T^2} < \infty$, as required. This completes the proof. □

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THEOREM 4.11. *Let $2 \leqslant \dim(\mathcal{H}_{\mathbb{R}}) < \infty$, and suppose there exists $\xi_0 \in \mathcal{H}_{\mathbb{R}}$ with $\|\xi_0\|_{\mathcal{H}} = 1$ such that $T(\xi_0 \otimes \xi_0) = q(\xi_0 \otimes \xi_0)$ for $|q| \leqslant \lambda$. Then, $M_{\xi_0} \subseteq M_T$ is a strongly mixing masa in M_T whose left–right measure is Lebesgue absolutely continuous. In particular, M_{ξ_0} is singular in M_T.*

Proof. Let $S = \{W(\xi_1 \otimes \cdots \otimes \xi_m) : \xi_i \in \mathcal{H}_{\mathbb{R}}, 1 \leqslant i \leqslant m, \text{and at least one } \xi_i \perp \xi_0, m \in \mathbb{N}\}$. From lemma 3.2, it follows that $\mathbb{E}_{\xi_0}(x) = 0$ for all $x \in S$ and $\text{span}_{\mathbb{C}}\, S\Omega$ is dense in $L^2(M_T, \tau) \ominus L^2(M_{\xi_0}, \tau)$. Further, from theorem 4.10, it follows that $T_{x,y}$ is a Hilbert–Schmidt operator for all $x, y \in S$. Therefore, the result follows from theorem 4.3 and the discussion surrounding it. □

REMARK 4.12. Now we turn to the case $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$. Before we proceed, we need to take a careful look into the T-Wick product, which in the case when $\mathcal{H}_{\mathbb{R}}$ was finite dimensional played a crucial role.

Simple calculations show that the operator $W(\xi_1 \otimes \cdots \otimes \xi_n)$, $\xi_i \in \mathcal{H}_{\mathbb{R}}$, $1 \leqslant i \leqslant n$, depends on T and may not lie inside the $*$-algebra generated by $s(\xi)$, $\xi \in \mathcal{H}_{\mathbb{R}}$, when $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$. This poses challenges in the investigation of structural properties of M_T. Further, for $n \in \mathbb{N}$ and $1 \neq \sigma \in S_n$, $\Phi(\sigma)$ may not take simple tensors to a finite linear combination of simple tensors.

In the context of the mixed q-Gaussian von Neumann algebras (which covers the case of q-Gaussian von Neumann algebras), $W(\xi_1 \otimes \cdots \otimes \xi_n)$ indeed lies in the $*$-algebra generated by $s(\xi)$, $\xi \in \mathcal{H}_{\mathbb{R}}$, regardless of $\dim(\mathcal{H}_{\mathbb{R}})$, and this fact has bearing on studying the aforesaid algebras [**6**, **16**].

Theorem 4.9 assumes the existence of an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ which resolves the aforesaid hurdles and in that sense theorem 4.9 is mostly valuable when $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$. Thus, we first assume the existence of such an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$. (Note that M_T is tracial.)

The special eigenvector ξ_0 in theorem 4.10 (i.e., in the case when $\dim(\mathcal{H}_{\mathbb{R}}) < \infty$) enables the construction of a convenient orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ that behaves well as long as one considers their interaction with elements of M_{ξ_0}. Such interaction played a crucial role in [**2**] as well. Since the analysis of free groups factors, q-Gaussian von Neumann algebras, mixed q-Gaussian von Neumann algebras and their type III counterparts rely heavily on Wick product expansion as above [**1**, **2**, **6**, **14**, **17**], in order to exploit this convenient interaction in the case $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$, we assume the existence of an orthonormal basis $\{e_\mu : \mu \in \Lambda\}$ of $\mathcal{H}_{\mathbb{R}}$, so that the T-Wick product expansion (as above in theorem 4.9) is valid and there exists $\mu_0 \in \Lambda$ such that $T(e_{\mu_0} \otimes e_{\mu_0}) = q(e_{\mu_0} \otimes e_{\mu_0})$ for some $|q| \leqslant \lambda$.

In § 6, we provide examples for which such hypotheses are naturally satisfied.

Let $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$. Suppose that there exists an orthonormal basis $\{e_\mu : \mu \in \Lambda\}$ of $\mathcal{H}_{\mathbb{R}}$ satisfying the hypothesis of theorem 4.9. Then, we have the following theorem.

THEOREM 4.13. *Suppose there exists $\mu_0 \in \Lambda$ such that $T(e_{\mu_0} \otimes e_{\mu_0}) = q(e_{\mu_0} \otimes e_{\mu_0})$ for some $|q| \leqslant \lambda$. Let $x = W(e_{\mu_1} \otimes \cdots \otimes e_{\mu_m})$ and $y = W(e_{\nu_1} \otimes \cdots \otimes e_{\nu_k})$ for $\mu_i, \nu_j \in \Lambda, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant k$, $m, k \in \mathbb{N}$, be such that $\mathbb{E}_{e_{\mu_0}}(x) = 0 = \mathbb{E}_{e_{\mu_0}}(y)$. Then, $T_{x,y} : L^2(M_{e_{\mu_0}}, \tau) \to L^2(M_{e_{\mu_0}}, \tau)$ defined as*

$$T_{x,y}(a\Omega) = \mathbb{E}_{e_{\mu_0}}(xay)\Omega,\ a \in M_{e_{\mu_0}},$$

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is a Hilbert–Schmidt operator. Moreover, $M_{e_{\mu_0}} \subseteq M_T$ is a strongly mixing masa in M_T whose left–right measure is Lebesgue absolutely continuous. In particular, $M_{e_{\mu_0}}$ is singular in M_T.

Proof. Let $v_n = W(e_{\mu_0}^{\otimes n})$ for $n \in \mathbb{N} \cup \{0\}$ $(e_{\mu_0}^{\otimes 0} = \Omega)$. Then, as in the proof of theorem 4.10, we have

$$T_{x,y}(v_n\Omega) = e_{e_{\mu_0}}\big(W(e_{\mu_1} \otimes \cdots \otimes e_{\mu_m})W_r(e_{\nu_1} \otimes \cdots \otimes e_{\nu_k})e_{\mu_0}^{\otimes n}\big), \quad n \geqslant 0.$$

The hypothesis entails that, for $\sigma \in S_m$, $\Phi(\sigma)(e_{\mu_1} \otimes \cdots \otimes e_{\mu_m})$ splits into a finite linear sum of simple tensors consisting of elements of $\{e_\mu : \mu \in \Lambda\}$. Similar is the case for $\Phi(\sigma')(e_{\nu_1} \otimes \cdots \otimes e_{\nu_k})$ with $\sigma' \in S_k$. Therefore, by theorem 4.9, it follows that $W(e_{\mu_1} \otimes \cdots \otimes e_{\mu_m})$ (resp. $W_r(e_{\nu_1} \otimes \cdots \otimes e_{\nu_k})$) splits into a finite sum of products of left creation operators followed by left annihilation operators (resp. right creation operators followed by right annihilation operators). Therefore, proceeding along the similar lines of calculations in the proof of theorem 4.10, it follows that $T_{x,y}$ is a Hilbert–Schmidt operator.

Let $S = \{W(e_{\mu_1} \otimes \cdots \otimes e_{\mu_m}) : \mu_i \in \Lambda, 1 \leqslant i \leqslant m,$ and at least one $e_{\mu_i} \neq e_{\mu_0}$, $m \in \mathbb{N}\}$. From lemma 3.2, it follows that $\mathbb{E}_{e_{\mu_0}}(x) = 0$ for all $x \in S$ and $S\Omega$ is dense in $L^2(M_T, \tau) \ominus L^2(M_{e_{\mu_0}}, \tau)$. Further, $T_{x,y}$ is Hilbert–Schmidt operator for all $x, y \in S$. Hence, the result follows from theorem 4.3 and the discussion surrounding it. □

5. Factoriality of M_T

In this section, we prove the factoriality of M_T under the assumption of the existence of a special eigenvector of T. This is the main result of this paper.

Theorem 5.1. *Suppose $2 \leqslant \dim(\mathcal{H}_\mathbb{R}) < \infty$, and, there exists a non-zero vector $\xi_0 \in \mathcal{H}_\mathbb{R}$ such that $\|\xi_0\|_\mathcal{H} = 1$ and $T(\xi_0 \otimes \xi_0) = q(\xi_0 \otimes \xi_0)$ for some $|q| \leqslant \lambda$. Then, M_T is a factor.*

Proof. By theorem 4.11, M_{ξ_0} is a masa in M_T. Therefore, $\mathcal{Z}(M_T) \subseteq M_{\xi_0}$. Since $\dim(\mathcal{H}_\mathbb{R}) \geqslant 2$, pick $\xi_1 \in \mathcal{H}_\mathbb{R}$ such that $\langle \xi_0, \xi_1 \rangle_\mathcal{H} = 0$.

Let $0 \neq z \in \mathcal{Z}(M_T)$. Choose a sequence $\{z_n\}$ of polynomials in $s(\xi_0)$ such that $z_n \to z$ in the *s.o.t.* Hence, $z_n\Omega \to z\Omega$ in $\|\cdot\|_T$.

Therefore, $s(\xi_1)z_n\Omega \to s(\xi_1)z\Omega$ in $\|\cdot\|_T$. But $s(\xi_1)z_n\Omega = l(\xi_1)z_n\Omega + l^*(\xi_1)z_n\Omega$. Note that $z_n\Omega \in M_{\xi_0}\Omega \subseteq L^2(M_{\xi_0}, \tau)$ and $\langle \xi_0, \xi_1 \rangle_\mathcal{H} = 0$. Therefore, by definition 2.5, we have $l^*(\xi_1)z_n\Omega = 0$ for all n. Since $l(\xi_1)$ is continuous, we have $s(\xi_1)z_n\Omega = \xi_1 \otimes z_n\Omega \to \xi_1 \otimes z\Omega$ in $\|\cdot\|_T$.

On the other hand, $z_n s(\xi_1)\Omega \to z s(\xi_1)\Omega$ in $\|\cdot\|_T$. From remark 2.14, it follows that

$$z_n s(\xi_1)\Omega = z_n\xi_1 = z_n s_r(\xi_1)\Omega = s_r(\xi_1)z_n\Omega = r(\xi_1)z_n\Omega + r^*(\xi_1)z_n\Omega.$$

Since $z_n\Omega \in L^2(M_{\xi_0}, \tau)$ and $\langle \xi_0, \xi_1 \rangle_\mathcal{H} = 0$, so $r^*(\xi_1)z_n\Omega = 0$ (see definition 2.12). Hence, $z_n s(\xi_1)\Omega = z_n\Omega \otimes \xi_1 \to z\Omega \otimes \xi_1$ in $\|\cdot\|_T$, since $r(\xi_1)$ is continuous.

Since $s(\xi_1)z\Omega = zs(\xi_1)\Omega$, we must have $\xi_1 \otimes z\Omega = z\Omega \otimes \xi_1$. Consequently, $z \in \mathbb{C}1$.

This completes the proof. □

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Remark 5.2.

1. Note that the hypothesis of the existence of special eigenvector is crucially used in the proof of theorem 5.1. Even if it were true that $s(\xi)$ generates a masa for some $\xi \in \mathcal{H}_{\mathbb{R}}$, it is not clear how to conclude factoriality without such hypothesis.

2. In the case $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$, the factoriality of M_T was established in [**9**, theorem 3]. However, we can provide a second proof of the same under appropriate hypotheses.

   Suppose that $\dim(\mathcal{H}_{\mathbb{R}}) = \aleph_0$. Let $\{e_\mu : \mu \in \Lambda\}$ be an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ satisfying the conditions of theorem 4.9 and there exists $\mu_0 \in \Lambda$ such that $T(e_{\mu_0} \otimes e_{\mu_0}) = q(e_{\mu_0} \otimes e_{\mu_0})$ for some $|q| \leqslant \lambda$. Then, M_T is a factor. Indeed, by theorem 4.13, $M_{e_{\mu_0}}$ is a masa in M_T. Then, the result follows by a similar argument as in theorem 5.1.

6. Examples

In this section, we construct an uncountable family of Yang–Baxter operators each of which satisfies the traciality condition in theorem 2.8, the sufficient condition for T-Wick product expansion in theorem 4.9 and possess a special eigenvector of the form $\xi_0 \otimes \xi_0$. Examples appearing in this section are not new and are borrowed from [**9**], thus we claim no credit for it. Combining with [**14**, theorem 2], this yields a new collection of non-injective factors.

Let $\mathcal{H}_{\mathbb{R}}$ be a real Hilbert space with $\dim(\mathcal{H}_{\mathbb{R}}) \geqslant 2$ and let U be a self-adjoint orthogonal operator on $\mathcal{H}_{\mathbb{R}}$. Let $\mathcal{H}$ be the complexification of $\mathcal{H}_{\mathbb{R}}$ and let $\widetilde{U}$ be the complexification of U on $\mathcal{H}$. Let $T : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ be the flip unitary. Fix $\lambda \in (-1, 1)$. Define $T_\lambda : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ by $T_\lambda = \lambda(\widetilde{U} \otimes 1)T(\widetilde{U} \otimes 1)$.

Proposition 6.1. *For $\lambda \in (-1, 1)$, the operator $T_\lambda : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a Yang–Baxter operator with $\|T_\lambda\| = |\lambda| < 1$. Further, we have the following.*

1. $$\langle \xi_r \otimes \xi_s, T_\lambda(\xi_i \otimes \xi_j)\rangle_{\mathcal{H}\otimes\mathcal{H}} = \langle \xi_s \otimes \xi_j, T_\lambda(\xi_r \otimes \xi_i)\rangle_{\mathcal{H}\otimes\mathcal{H}},$$
   for all $\xi_i, \xi_j, \xi_r, \xi_s \in \mathcal{H}_{\mathbb{R}}$.

2. $T_\lambda(\xi_0 \otimes \xi_0) = \lambda(\xi_0 \otimes \xi_0)$ *whenever $U\xi_0 = \xi_0$ for some $\xi_0 \in \mathcal{H}_{\mathbb{R}}$.*

3. *Let $\{e_\mu : \mu \in \Lambda\}$ be an orthonormal basis of $\mathcal{H}_{\mathbb{R}}$ consisting of eigenvectors of U. Then, $\{e_\mu : \mu \in \Lambda\}$ satisfies the T-Wick product condition in theorem 4.9.*

Proof. T_λ is self-adjoint, since $\widetilde{U}$ is self-adjoint. Clearly, $\|T_\lambda\| = |\lambda| < 1$. Note that $T_\lambda(\xi \otimes \eta) = \lambda(\widetilde{U}\eta \otimes \widetilde{U}\xi)$ for all $\xi, \eta \in \mathcal{H}$. Then, for all $\xi_1, \xi_2, \xi_3 \in \mathcal{H}$, one has

$$\begin{aligned}(T_\lambda \otimes 1)(1 \otimes T_\lambda)(T_\lambda \otimes 1)(\xi_1 \otimes \xi_2 \otimes \xi_3) &= \lambda^3(\xi_3 \otimes \xi_2 \otimes \xi_1)\\ &= (1 \otimes T_\lambda)(T_\lambda \otimes 1)(1 \otimes T_\lambda)(\xi_1 \otimes \xi_2 \otimes \xi_3).\end{aligned}$$

It follows that T_λ satisfies the braid relation.

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(1). Fix $\xi_i, \xi_j, \xi_r, \xi_s \in \mathcal{H}_{\mathbb{R}}$. We have,

$$\begin{aligned}\langle \xi_r \otimes \xi_s, T_\lambda(\xi_i \otimes \xi_j)\rangle_{\mathcal{H}\otimes\mathcal{H}} &= \lambda\langle \xi_r \otimes \xi_s, U\xi_j \otimes U\xi_i\rangle_{\mathcal{H}\otimes\mathcal{H}} \\ &= \lambda\langle \xi_r, U\xi_j\rangle_{\mathcal{H}}\langle \xi_s, U\xi_i\rangle_{\mathcal{H}} \\ &= \lambda\langle \xi_s, U\xi_i\rangle_{\mathcal{H}}\langle U\xi_r, \xi_j\rangle_{\mathcal{H}} \quad (\text{since } U = U^*) \\ &= \lambda\langle \xi_s, U\xi_i\rangle_{\mathcal{H}}\langle \xi_j, U\xi_r\rangle_{\mathcal{H}} \quad (\text{since } \xi_j, \xi_r \in \mathcal{H}_{\mathbb{R}}) \\ &= \lambda\langle \xi_s \otimes \xi_j, U\xi_i \otimes U\xi_r\rangle_{\mathcal{H}\otimes\mathcal{H}} \\ &= \langle \xi_s \otimes \xi_j, T_\lambda(\xi_r \otimes \xi_i)\rangle_{\mathcal{H}\otimes\mathcal{H}}.\end{aligned}$$

(2) and (3) are easy consequences of the construction. This completes the proof. □

Now, let $\dim(\mathcal{H}_{\mathbb{R}}) \geqslant 3$. Write $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2 \oplus \mathcal{K}_{\mathbb{R}}$, where $\mathcal{K}_{\mathbb{R}} \neq 0$ is a real Hilbert space. Let $\xi = 1 \oplus 0 \oplus 0$ and $\eta = 0 \oplus 1 \oplus 0$. Define U on $\mathcal{H}_{\mathbb{R}}$ as

$$U := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1_{\mathcal{K}_{\mathbb{R}}}.$$

Then, $T_\lambda(\xi \otimes \xi) = \lambda(\eta \otimes \eta)$. Consequently, T_λ is not a scalar multiple of T and hence M_{T_λ} cannot be canonically isomorphic to the q-Gaussian von Neumann algebras for all $\lambda \in (-1, 1)$.

Finally, by choosing $\mathcal{H}_{\mathbb{R}} = \oplus_{i\in I}\, \mathbb{R}^2 \oplus \mathcal{K}_{\mathbb{R}}$, where I is a finite or countable index set and $\mathcal{K}_{\mathbb{R}} \neq 0$ (as before), and defining U on $\mathcal{H}_{\mathbb{R}}$ as

$$U := \bigoplus_{i\in I} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus 1_{\mathcal{K}_{\mathbb{R}}},$$

we obtain many examples of non-injective type II_1 factors (as λ and $\mathcal{H}_{\mathbb{R}}$ vary).

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References

1 S. Avesec. *Strong solidity of the q-Gaussian algebras for all* $-1 < q < 1$, preprint, arXiv:1110.4918, (2011).

2 P. Bikram and K. Mukherjee. Generator masas in q-deformed Araki–Woods von Neumann algebras and factoriality. *J. Funct. Anal.*, **273** (2017), 1443–1478.

3 P. Bikram, R. Kumar and K. Mukherjee. *Mixed q-deformed Araki-Woods von Neumann algebras*, preprint, (2020).

https://doi.org/10.1017/prm.2020.62 Published online by Cambridge University Press

4 N. Bourbaki. *Éléments de mathématique*. Fasc. XXXIV. Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: systèmes de racines, Hermann, Paris, 1968, 288 pp. (loose errata).

5 M. Bożejko and R. Speicher. Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Math. Ann.*, **300** (1994), 97–120.

6 M. Bożejko, B. Kümmerer and R. Speicher. q -Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.,* **185** (1997), 129–154.

7 J. Cameron, J. Fang and K. Mukherjee. Mixing subalgebras of finite von Neumann algebras, New York. *J. Math.*, **19** (2013), 343–366).

8 M. Junge and Q. Zeng. *Mixed q-Gaussian algebras*, unpublished, (2015).

9 I. Królak. Wick product for commutation relations connected with Yang–Baxter operators and new constructions of factors. *Comm. Math. Phys.,* **210** (2000), 685–701.

10 I. Królak. Factoriality of von Neumann algebras connected with general commutation relations-finite dimensional case. *Quantum Probability, Banach Center Publ.,* **73** (2006), 277–284.

11 K. Mukherjee. Singular masas and measure-multiplicity invariant. *Houston J. Math.,* **39** (2013), 561–598.

12 K. Mukherjee. Masas and bimodule decompositions of II_1 factors. *Quart. J. Math.,* **62** (2009), 451–486.

13 B. Nelson and Q. Zeng. An application of free transport to mixed q-Gaussian algebras. *Proc. Amer. Math. Soc.,* **144** (2016), 4357–4366.

14 A. Nou. Non injectivity of the q-deformed von Neumann algebra. *Math. Ann.,* **330** (2004), 17–38.

15 N. Ozawa. Solid von Neumann algebras. *Acta Math.* **192** (2004), 111–117.

16 É. Ricard. Factoriality of q-Gaussian von NeumannAlgebras. *Comm. Math. Phys.* **257** (2005), 659–665.

17 D. V. Voiculescu, K. J. Dykema and A. Nica. *Free Random Variables*, CRM Monograph Series, vol. 1, (Providence, RI: American Mathematical Society, 1992).

18 S. Wang. Singularity of the generator subalgebra in mixe d q-Gaussian algebras. *Colloq. Math.* **158** (2019), 39–43.

https://doi.org/10.1017/prm.2020.62 Published online by Cambridge University Press