



$SW_{1+\infty}$ symmetries of $N = 2$ supersymmetric CKP hierarchy and its multicomponent generalization

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Abstract

In this paper, we define a $N = 2$ supersymmetric CKP(SCKP) hierarchy and construct its additional symmetries. These additional flows constitute a C type $SW_{1+\infty}$ Lie algebra. Further we generalize the $N = 2$ SCKP hierarchy to a $N = 2$ supersymmetric multi-component CKP hierarchy equipped with a C type $\otimes SW_{1+\infty}$ Lie algebra.

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1. Introduction

In the theory of integrable systems, it is important to construct their symmetries and identify their algebraic structure. Among these symmetries, the additional symmetry is an important one. Additional symmetries of the Kadomtsev-Petviashvili (KP) hierarchy were introduced by Orlov and Shulman [1] which contain one of the most significant additional symmetries is the Virasoro constraints on partition functions of matrix models of string theory under the additional non-isospectral symmetries in the underlying integrable hierarchies. There are two important sub-hierarchies as the BKP hierarchy and CKP hierarchy [2–7].

There are various supersymmetric extensions [8] of the KP hierarchy. One important supersymmetric extension is the supersymmetric Manin-Radul Kadomtsev-Petviashvili (MR-SKP) hierarchy [9] which contains a lot of integrable super solitary equations. Mulase supersymmetrize the KP hierarchy by constructing a hierarchy called the Jacobian SKP hierarchy [10] which has strict Jacobian flows and it preserves the super Riemann surface. The additional symmetries for super hierarchies were firstly found in the paper [11] and the additional symmetry of the MR-SKP hierarchy was studied by Stanciu [12]. The ghost symmetries, hamiltonian structures and extensions of the MR-SKP hierarchy were studied as well as reductions of the MR-SKP hierarchy [13,14]. Later the supersymmetric BKP (SBKP) hierarchy was constructed in [15]. After that this series of super hierarchies attracts more attentions [16–19] including their Darboux transformations and symmetries. Particularly in [17], we construct supersymmetric C type Kadomtsev-Petviashvili (CKP) hierarchy and six-reduced supersymmetric CKP hierarchy and their additional symmetries. These additional flows of the supersymmetric CKP hierarchy constitute a C type $SW_{1+\infty}$ Lie algebra. Meanwhile, we find the six-reduced supersymmetric CKP hierarchy contains the super Sawada-Kotera equation as a primary equation. The additional C type $SW_{1+\infty}$ Lie algebraic structure is kept after doing the six-reduction from the supersymmetric CKP hierarchy. Bosonic hierarchies and their connection with physical models (2D and topological field theories) is known, while the knowledge of $N = 2$ supersymmetric integrable hierarchies maybe lead to a link with physical models, e.g. with untwisted $N = 2$ conformal field theories.

In the paper [20], we construct the generalized additional symmetries of the two-component BKP hierarchy and identify its algebraic structure. Besides, the D type Drinfeld-Sokolov hierarchy was found to be a good differential model to derive a Block algebra. About the Block algebra related to integrable systems, we did a series of works in [21,22]. In [19], we construct the additional symmetries of the supersymmetric BKP(SBKP) hierarchy and these additional flows constitute a B type $SW_{1+\infty}$ Lie algebra. Further we generalize the SBKP hierarchy to a supersymmetric multi-component BKP (SMBKP) hierarchy equipped with a B type $SW_{1+\infty} \times SW_{1+\infty}$ Lie algebra. As a Bosonic reduction of the S2BKP hierarchy, we defined a new constrained system called the supersymmetric Drinfeld-Sokolov hierarchy of type D which admits a supersymmetric Block type symmetry. The main topic of this article is the study of additional symmetries of scalar and multicomponent $N = 2$ supersymmetric CKP hierarchies. These symmetries are shown to form an infinite-dimensional non-Abelian superloop superalgebra.

This paper is arranged as follows. In the next section we define the $N = 2$ supersymmetric CKP hierarchy. In Sections 3, we will give the additional symmetries for the $N = 2$ supersymmetric CKP hierarchy. Further we define the $N = 2$ multicomponent supersymmetric CKP hierarchy

in Sections 4, and in Section 5 we will give the additional symmetries for the $N = 2$ multicomponent supersymmetric CKP hierarchy.

2. The $N = 2$ supersymmetric CKP hierarchy

Let us define a $N = 2$ supersymmetric CKP system now. \mathcal{A} is assumed as an algebra of smooth functions of a spatial coordinate x , a grassmann variable θ_{\pm} and their super-derivation denoted as $D_{\pm} = \partial_{\theta_{\pm}} + \theta_{\pm}\partial$. This algebra \mathcal{A} has the following multiplying rule

$$D_{\pm}^n \Phi = \sum_{i=0}^{\infty} \begin{bmatrix} n \\ n-i \end{bmatrix} (-1)^{|\Phi|(n-i)} \Phi^{\pm[i]} D_{\pm}^{n-i}, \tag{2.1}$$

$$\begin{bmatrix} n \\ n-i \end{bmatrix} = \begin{cases} 0 & i < 0 \text{ or } (n, i) = (0, 1) \pmod{2}; \\ \begin{pmatrix} \lfloor \frac{n}{2} \rfloor \\ \lfloor \frac{n-i}{2} \rfloor \end{pmatrix} & i \geq 0, (n, i) \neq (0, 1) \pmod{2}. \end{cases} \tag{2.2}$$

Here the value $|\Phi|$ means the super degree of the operator Φ which shows the operator Φ is Fermionic or Bosonic. The supersymmetric derivative D_{\pm} satisfies the supersymmetric analog of the Leibniz rule

$$D_{\pm}(ab) = D_{\pm}(a)b + (-1)^{|a|}aD_{\pm}(b), \tag{2.3}$$

where a is a homogeneous element of \mathcal{A} . We introduce the even and odd time variables $(t_1^{\pm}, t_2, t_5^{\pm}, t_6, \cdot)$ and the following definition of even and odd flows

$$D_{4i-2} = \frac{\partial}{\partial t_{4i-2}}, \quad D_{4i-3}^{\pm} = \frac{\partial}{\partial t_{4i-3}^{\pm}} + \sum_{j=1}^{\infty} t_{4j-3}^{\pm} \frac{\partial}{\partial t_{4i+4j-6}}. \tag{2.4}$$

We recall that the supercommutator is defined as $[X, Y] = XY - (-1)^{|X||Y|}YX$. The bracket has a property as $[X, YZ] = [X, Y]Z + (-1)^{|X||Y|}Y[X, Z]$. Then $D_{\pm}^2 = \frac{1}{2}[D_{\pm}, D_{\pm}] = \partial$, $\{D_+, D_-\} = 0$. This family of infinite odd and even flows satisfy a nonabelian Lie superalgebra whose commutation relations are

$$\begin{aligned} [D_{4i-2}, D_{4j-2}] &= 0, & [D_{4i-2}, D_{4j-3}^{\pm}] &= 0, & [D_{4i-3}^{\pm}, D_{4j-3}^{\pm}] &= -2 \frac{\partial}{\partial t_{4i+4j-6}}, \\ [D_{4i-3}^{\pm}, D_{4j-3}^{\mp}] &= 0, & [D_{4i-2}, D_{\pm}] &= 0, & [D_{4i-3}^{\pm}, D_{\pm}] &= 0. \end{aligned} \tag{2.5}$$

For any operator $A = \sum_{i \in \mathbb{Z}} (f_i^{(0)} + f_i^+ D^+ + f_i^- D^- + f_i^{(1)} D^+ D^-) \partial^i \in \mathcal{A}$ and homogeneous operators P, Q , its nonnegative projection, negative projection, adjoint operator are respectively defined as

$$A_+ = \sum_{i \geq 0} (f_i^{(0)} + f_i^+ D_+ + f_i^- D_- + f_i^{(1)} D_+ D_-) \partial^i, \quad A_- = A - A_+, \tag{2.6}$$

$$(PQ)^* = (-1)^{|P||Q|} Q^* P^*, \quad (P^{-1})^* = (-1)^{|P|} (P^*)^{-1}. \tag{2.7}$$

Also for the operator D^k , the adjoint operator is defined as

$$(D_{\pm}^k)^* = (-1)^{\frac{k(k+1)}{2}} D_{\pm}^k. \tag{2.8}$$

The rules of conjugation within the super-pseudo-differential formalism are as follows: $(AB)^* = (-1)^{|A||B|} B^* A^*$ for any two elements with Grassmann parities $|A|$ and $|B|$; $(\partial^k)^* = (-1)^k \partial^k$ and $u^* = u$ for any superfield. The Lax operator of the $N = 2$ supersymmetric CKP hierarchy has a form as

$$L = \Phi D_- \Phi^{-1}, \quad L^* = -L, \tag{2.9}$$

where

$$\Phi = 1 + \sum_{i \geq 1} (a_i^{(0)} + a_i^+ D_+ + a_i^- D_- + a_i^{(1)} D_+ D_-) \partial^{-i}, \tag{2.10}$$

satisfy

$$\Phi^* = \Phi^{-1}. \tag{2.11}$$

The $N = 2$ supersymmetric CKP hierarchy is defined by the following Lax equations

$$D_{4k-2} L = [(L^{4k-2})_+, L], \quad D_{4k-3}^- L = [(L^{4k-3})_+, L] - 2L^{4k-2}, \quad k \geq 1, \tag{2.12}$$

$$D_{4k-3}^+ L = [(\Lambda^{4k-3})_+, L], \quad \Lambda = \Phi D_+ \Phi^{-1}, \quad k \geq 1. \tag{2.13}$$

We call the eq. (2.11) the C type condition of the $N = 2$ supersymmetric CKP hierarchy. The $N = 2$ supersymmetric CKP hierarchy (2.12) can also be redefined as

$$D_{4k-2} \Phi = -(L^{4k-2})_- \Phi, \quad D_{4k-3}^- \Phi = -(L^{4k-3})_- \Phi, \quad D_{4k-3}^+ \Phi = -(\Lambda^{4k-3})_- \Phi, \tag{2.14}$$

with $k \geq 1$.

With the above preparation, it is time to construct additional symmetries for the $N = 2$ supersymmetric CKP hierarchy in the next section.

3. Additional symmetries of the N=2 supersymmetric CKP hierarchy

In this section, we are to construct additional symmetries for the N=2 supersymmetric CKP hierarchy by using the Orlov–Schulman operators whose coefficients depend explicitly on the time variables of the hierarchy. The Orlov–Schulman operators M_i^\pm and auxiliary operator Q_\pm are constructed in the following dressing structure

$$M_i^\pm = \Phi \Gamma_i^\pm \Phi^{-1}, \quad i = 0, 1; \quad Q_\pm = \Phi Q_\pm \Phi^{-1},$$

where

$$\Gamma_0^\pm = x + \frac{1}{2} \sum_{k \geq 1} (4k - 2) t_{4k-2} D_\pm^{4k-4} + \frac{1}{2} (4k - 3) t_{4k-3}^\pm D_\pm^{4k-5} - \frac{1}{2} \sum_{k \geq 1} t_{4k-3}^\pm \partial^{2k-3} Q_\pm + \sum_{i, j \geq 1} (i - j) t_{4i-3}^\pm t_{4j-3}^\pm \partial^{2i+2j-4}, \tag{3.1}$$

$$\Gamma_1^\pm = \theta_\pm + \sum_{k \geq 1} t_{4k-3}^\pm \partial^{2k-2}, \tag{3.2}$$

where $Q_\pm = \partial_{\theta_\pm} - \theta_\pm \partial$.

Then one can get the following lemma.

Lemma 3.1. *The operators Γ_j^\pm, Q_\pm satisfy*

$$[D_{4i-2} - D_\pm^{4i-2}, \Gamma_j^\pm] = [D_{4i-3}^\pm - D_\pm^{4i-3}, \Gamma_j^\pm] = 0; \quad j = 0, 1, \tag{3.3}$$

$$[D_{4i-2} - D_\pm^{4i-2}, Q_\pm] = [D_{4i-3}^\pm - D_\pm^{4i-3}, Q_\pm] = 0, \tag{3.4}$$

$$[Q_\pm, \Gamma_0^\pm] = -\Gamma_1^\pm, \quad [Q_\pm, \Gamma_1^\pm] = 1, \quad [\partial, \Gamma_0^\pm] = 1. \tag{3.5}$$

Proof. The proof is quite similar as the proof in [18] as following

$$[D_{4i-2} - D_\pm^{4i-2}, \Gamma_0^\pm] = \frac{1}{2}(4i - 2)D_\pm^{4i-4} - [D_\pm^{4i-2}, x] = 0, \tag{3.6}$$

$$\begin{aligned} & [D_{4i-3}^\pm - D_\pm^{4i-3}, \Gamma_0^\pm] \\ &= \left[\frac{\partial}{\partial t_{4i-3}^\pm} - \sum_{j=1}^\infty t_{4j-3}^\pm \frac{\partial}{\partial t_{4i+4j-6}} - D_\pm^{4i-3}, \Gamma_0^\pm \right] \end{aligned} \tag{3.7}$$

$$= \frac{1}{2}(4i - 3)D_\pm^{4i-3} - \frac{1}{2}\partial^{2i-2}Q_\pm + 2 \sum_{j \geq 1} (i - j)t_{4j-3}^\pm \partial^{2i+2j-4} \tag{3.8}$$

$$\begin{aligned} & - \sum_{j=1}^\infty (2i + 2j - 1)t_{4j-3}^\pm D^{4i+4j-8} - [D_\pm^{4i-3}, x - \frac{1}{2} \sum_{k \geq 1} (4k - 3)t_{4k-3}^\pm D_\pm^{4k-3}] \\ &= 0, \end{aligned} \tag{3.9}$$

in which we used

$$[D_\pm^{4i-3}, x] = \frac{1}{2}(4i - 3)D_\pm^{4i-3} - \frac{1}{2}\partial^{2i-2}Q_\pm,$$

and

$$[D_\pm^{4i-3}, \frac{1}{2} \sum_{k \geq 1} (4k - 3)t_{4k-3}^\pm D_\pm^{4k-3}] = - \sum_{j \geq 1} (4j - 3)t_{4j-3}^\pm D_\pm^{4i+4j-8}. \tag{3.10}$$

For Γ_1^\pm, Q_\pm we can prove it similarly. Also we do a direct calculation as

$$[Q_\pm, \Gamma_0^\pm] = [Q_\pm, x] - [Q_\pm, \frac{1}{2} \sum_{k \geq 1} t_{4k-3}^\pm \partial^{2k-2} Q_\pm] = -\theta_\pm - \sum_{k \geq 1} t_{4k-3}^\pm \partial^{2k-3} = -\Gamma_1^\pm,$$

and other identities can be proved similarly. \square

Then it is easy to get the following lemma by dressing structures.

Lemma 3.2. *The operators M_j^\pm, Q_\pm, L satisfy*

$$[Q_\pm, M_0^\pm] = -M_1^\pm, \quad [Q_\pm, M_1^\pm] = 1, \quad [L^2, M_0^\pm] = 1, \tag{3.11}$$

$$D_k^- M_j^\pm = [(L^k)_+, M_j^\pm], \quad D_k^- Q_\pm = [(L^k)_+, Q_\pm], \quad k = 4i - 2, 4i - 3, \quad i \in \mathbb{Z}_+, \tag{3.12}$$

$$D_k^+ M_j^\pm = [(\Lambda^k)_+, M_j^\pm],$$

$$D_k^+ Q_\pm = [(\Lambda^k)_+, Q_\pm], \quad k = 4i - 2, 4i - 3, \quad i \in \mathbb{Z}_+. \tag{3.13}$$

Here $D_{4i-2}^+ = D_{4i-2}^- = D_{4i-2}$.

Proof. The dressing structure

$$\Phi[D_{4i-3}^\pm - D_\pm^{4i-3}, \Gamma_1^\pm]\Phi^{-1} = 0; \tag{3.14}$$

will lead to

$$[\Phi D_{4i-3}^\pm \Phi^{-1} - \Phi D_\pm^{4i-3} \Phi^{-1}, M_1^\pm] = 0. \tag{3.15}$$

Here we only consider

$$[D_{4i-3}^+ - (D_{4i-3}^+ \Phi)\Phi^{-1} + \sum_{j=0}^\infty t_{4j-3}^+ \Phi_{4i+4j-6} \Phi^{-1} - \Lambda^{4i-3}, M_1^+] = 0. \tag{3.16}$$

Then we get

$$[D_{4i-3}^+ - (D_{4i-3}^+ \Phi)\Phi^{-1} - \Lambda^{4i-3}, M_1^+] = 0; \tag{3.17}$$

and using eq. (2.14) we can derive

$$[D_{4i-3}^+ - (\Lambda^{4i-3})_+, M_1^+] = 0. \tag{3.18}$$

Similarly we can derive

$$[D_{4i-3}^- - (L^{4i-3})_+, M_1^-] = 0. \tag{3.19}$$

The other identities can be proved using the similar dressing techniques. \square

We will introduce the following operator B_{mklp}^\pm defined as

$$B_{mklp}^\pm = M_0^{\pm k} M_1^{\pm l} Q_\pm^p L^{2m} - (-1)^{pl+m+p+l} L^{2m} (Q_\pm^p) M_1^{\pm l} M_0^{\pm k}, \tag{3.20}$$

where $k, m \geq 0; l, p = 0, 1$. This operator is the generator of the additional symmetry of the $N = 2$ SCKP hierarchy.

Then the following proposition can be derived.

Proposition 3.3. *The operator B_{mnlp}^\pm satisfies the following flow equations*

$$D_{4k-2} B_{mnlp}^\pm = -[(L^{4k-2})_-, B_{mnlp}^\pm], \quad D_{4k-3}^- B_{mnlp}^\pm = -[(L^{4k-3})_-, B_{mnlp}^\pm]. \tag{3.21}$$

$$D_{4k-3}^+ B_{mnlp}^\pm = -[(\Lambda^{4k-3})_-, B_{mnlp}^\pm]. \tag{3.22}$$

Proof. The lemma can be proved by dressing the following identities by Φ

$$[D_{4k-2} - D_\pm^{4k-2}, \Gamma_0^{\pm n} \Gamma_1^{\pm l} Q_\pm^p \partial^m] = [D_{4k-3}^\pm - D_\pm^{4k-3}, \Gamma_0^{\pm n} \Gamma_1^{\pm l} Q_\pm^p \partial^m] = 0. \quad \square \tag{3.23}$$

To prove that B_{mnlp}^\pm satisfies the C type condition, we need the following lemma.

Lemma 3.4. *The operators M_i^\pm satisfy the following conjugate identities,*

$$M_i^{\pm*} = (-1)^i M_i^\pm, \quad Q_\pm^* = -Q_\pm. \tag{3.24}$$

Proof. Using

$$\Phi^* = \Phi^{-1}, \Gamma_i^{\pm*} = (-1)^i \Gamma_i^{\pm}, Q_{\pm}^* = -Q_{\pm}, \tag{3.25}$$

the following calculations

$$M_i^{\pm*} = \Phi^{*-1} \Gamma_i^{\pm*} \Phi^* = (-1)^i \Phi \Gamma_i^{\pm} \Phi^{-1} = (-1)^i M_i^{\pm},$$

will lead to the first identity of this lemma. The other identities can be proved in a similar way. \square

It is easy to check the following proposition holds basing on the Lemma 3.4 above.

Proposition 3.5. *The operator B_{mklp}^{\pm} satisfies a C type condition, namely*

$$B_{mklp}^{\pm*} = -B_{mklp}^{\pm}. \tag{3.26}$$

Proof. Using the Proposition 3.4, the following calculation

$$\begin{aligned} B_{mklp}^{\pm*} &= (M_0^{\pm k} M_1^{\pm l} Q_{\pm}^p L^{2m} - (-1)^{pl+m+p+l} L^{2m} (Q_{\pm}^p) M_1^{\pm l} M_0^{\pm k})^* \\ &= (-1)^{pl} L^{2m*} (Q_{\pm}^p)^* M_1^{\pm l*} M_0^{\pm k*} + (-1)^{m+p+l} M_0^{\pm k*} M_1^{\pm l*} (Q_{\pm}^p)^* L^{2m*} \\ &= (-1)^{pl+m+p+l} L^{2m} Q_{\pm}^p M_1^{\pm l} M_0^{\pm k} - M_0^{\pm k} M_1^{\pm l} Q_{\pm}^p L^{2m} \\ &= -(M_0^{\pm k} M_1^{\pm l} Q_{\pm}^p L^{2m} - (-1)^{pl+m+p+l} L^{2m} Q_{\pm}^p M_1^{\pm l} M_0^{\pm k}), \end{aligned}$$

will lead to this proposition. \square

Basing on above proposition, it is reasonable to define additional flows of the N=2 supersymmetric CKP hierarchy as

$$D_{mklp}^{\pm} L = [-(B_{mklp}^{\pm})_-, L], \quad k, m \geq 0; l, p = 0, 1. \tag{3.27}$$

Proposition 3.6. *The flows (3.27) commute with the flows of the N=2 supersymmetric CKP hierarchy as*

$$\left[D_{mnlp}^{\pm}, D_{4i-2}^{\pm} \right] = \left[D_{mnlp}^{\pm}, D_{4i-3}^{\pm} \right] = 0, \quad m, n \geq 0; l, p = 0, 1, i \in \mathbb{Z}_+, \tag{3.28}$$

which holds in the sense of acting on Φ .

Proof. The proposition when $k = 4i - 3$ can be checked as

$$\begin{aligned} &\left[D_{mnlp}^{\pm}, D_k^{\pm} \right] \Phi \\ &= D_{mnlp}^{\pm} D_k^{\pm} \Phi - (-1)^{(l+p)k} D_k^{\pm} D_{mnlp}^{\pm} \Phi \\ &= (-1)^{(l+p)k} [(L^k)_-, (B_{mnlp}^{\pm})_-] \Phi + [(B_{mnlp}^{\pm})_-, L^k]_- \Phi + (-1)^{(l+p)k} [(L^k)_+, (B_{mnlp}^{\pm})_-] \Phi \\ &= 0. \quad \square \end{aligned} \tag{3.29}$$

That means that the additional flows of the $N = 2$ supersymmetric CKP hierarchy are symmetries whose algebraic structure can be shown in the following proposition.

Proposition 3.7. *The algebra of additional symmetries of the $N = 2$ SCKP hierarchy given by eq. (3.27) is isomorphic to the $N = 2$ super Lie algebra $SW_{1+\infty}$.*

Proof. The isomorphism is given by

$$z \mapsto \partial, \quad \xi \mapsto Q_+ + \Gamma_1^+ \partial, \quad \eta \mapsto Q_- + \Gamma_1^- \partial, \tag{3.30}$$

$$\partial_z \mapsto \Gamma_0^\pm, \quad \partial_\xi \mapsto \Gamma_1^+, \quad \partial_\eta \mapsto \Gamma_1^-, \tag{3.31}$$

which further lead to

$$z \mapsto L^2, \quad \xi \mapsto Q_+ + M_1^+ L^2, \quad \eta \mapsto Q_- + M_1^- L^2, \tag{3.32}$$

$$\partial_z \mapsto M_0^\pm, \quad \partial_\xi \mapsto M_1^+, \quad \partial_\eta \mapsto M_1^-. \tag{3.33}$$

One can find the above construction keeps ξ, η commuting with z, ξ anticommuting with η . \square

4. The $N = 2$ multicomponent supersymmetric CKP hierarchy

Let us define a $N = 2$ multicomponent supersymmetric CKP system in this section.

\hat{A} is assumed as an algebra of smooth matrix-valued functions of a spatial coordinate x , a grassmann variable θ_\pm and their super-derivation denoted as $D_\pm = \partial_{\theta_\pm} + \theta_\pm \partial$. We introduce the even and odd time variables $(t_{1,\alpha}^\pm, t_{2,\alpha}, t_{5,\alpha}^\pm, t_{6,\alpha}, \cdot)$ with $1 \leq \alpha \leq s$ and the following definition of even and odd flows

$$D_{4i-2,\alpha} = \frac{\partial}{\partial t_{4i-2,\alpha}}, \quad D_{4i-3,\alpha}^\pm = \frac{\partial}{\partial t_{4i-3,\alpha}^\pm} + \sum_{j=1}^\infty t_{4j-3,\alpha}^\pm \frac{\partial}{\partial t_{4i+4j-6,\alpha}}. \tag{4.1}$$

This family of infinite odd and even flows satisfy a nonabelian Lie superalgebra whose commutation relations are

$$\begin{aligned} [D_{4i-2,\alpha}, D_{4j-2,\beta}] &= 0, & [D_{4i-2,\alpha}, D_{4j-3,\beta}^\pm] &= 0, \\ [D_{4i-3,\alpha}^\pm, D_{4j-3,\beta}^\pm] &= -2 \frac{\partial}{\partial t_{4i+4j-6,\alpha}} \delta_{\alpha,\beta}, \\ [D_{4i-3,\alpha}^\pm, D_{4j-3,\beta}^\mp] &= 0, & [D_{4i-2,\alpha}, D_\pm] &= 0, & [D_{4i-3,\alpha}^\pm, D_\pm] &= 0. \end{aligned} \tag{4.2}$$

The rules of conjugation is $u^* = u^T$ for any superfield where u^T means the transpose of the matrix-valued superfield u . The Lax operator of the $N = 2$ multicomponent supersymmetric CKP hierarchy has a form as

$$\mathbf{L} = \Psi D_- \Psi^{-1}, \quad \mathbf{L}^* = -D_- \mathbf{L} D_-^{-1}, \quad R_\alpha = \Psi E_\alpha \Psi^{-1}, \tag{4.3}$$

where

$$\Psi = 1 + \sum_{i \geq 1} (a_i^{(0)} + a_i^+ D_+ + a_i^- D_- + a_i^{(1)} D_+ D_-) \partial^{-i}, \tag{4.4}$$

satisfy

$$\Psi^* = D_- \Psi^{-1} D_-^{-1}, \tag{4.5}$$

and E_α is the matrix with element at the position of α -th row and α -th column being 1 and other ones being zeroes. The $N = 2$ multicomponent supersymmetric CKP hierarchy is defined by the following Lax equations

$$D_{4k-2,\alpha} \mathfrak{L} = [(\mathfrak{L}^{4k-2} R_\alpha)_+, \mathfrak{L}], \quad D_{4k-3,\alpha}^- \mathfrak{L} = [(\mathfrak{L}^{4k-3} R_\alpha)_+, \mathfrak{L}] - 2L^{4k} R_\alpha, \quad (4.6)$$

$$D_{4k-3,\alpha}^+ \mathfrak{L} = [(\Lambda^{4k-3} R_\alpha)_+, \mathfrak{L}], \quad \Lambda = \Psi D_+ \Psi^{-1}, \quad k \geq 1. \quad (4.7)$$

We call the eq. (4.5) the C type condition of the $N = 2$ multicomponent supersymmetric CKP hierarchy. The $N = 2$ multicomponent supersymmetric CKP hierarchy (4.6) can also be redefined as

$$D_{4k-2,\alpha} \Psi = -(\mathfrak{L}^{4k-2} R_\alpha)_- \Psi, \quad D_{4k-3,\alpha}^- \Psi = -(\mathfrak{L}^{4k-3} R_\alpha)_- \Psi, \quad (4.8)$$

$$D_{4k-3,\alpha}^+ \Psi = -(\Lambda^{4k-3} R_\alpha)_- \Psi, \quad (4.9)$$

with $k \geq 1$.

With the above preparation, it is time to construct additional symmetries for the $N = 2$ multicomponent supersymmetric CKP hierarchy in the next section.

5. Additional symmetries of the N=2 supersymmetric CKP hierarchy

In this section, we are to construct additional symmetries for the $N = 2$ s -component supersymmetric CKP hierarchy by using the Orlov–Schulman operators. The Orlov–Schulman operators M_i^\pm and auxiliary operator Q_\pm are constructed in the following dressing structure

$$M_{i,\alpha}^\pm = \Psi \Gamma_{i,\alpha}^\pm \Psi^{-1}, \quad i = 0, 1; \quad 1 \leq \alpha \leq s; \quad \hat{Q}_\pm = \Psi Q_\pm \Psi^{-1},$$

where

$$\begin{aligned} \Gamma_{0,\alpha}^\pm &= x E_\alpha + \frac{1}{2} \sum_{k \geq 1} (4k - 2) t_{4k-2,\alpha} E_\alpha D_\pm^{4k-4} + \frac{1}{2} (4k - 3) t_{4k-3,\alpha}^\pm E_\alpha D_\pm^{4k-3} \\ &\quad - \frac{1}{2} \sum_{k \geq 1} t_{4k-3,\alpha}^\pm E_\alpha \partial^{2k-2} Q_\pm + \sum_{i,j \geq 1} (i - j) t_{4i-3,\alpha}^\pm t_{4j-3,\alpha}^\pm E_\alpha \partial^{2i+2j-2}, \end{aligned} \quad (5.1)$$

$$\Gamma_{1,\alpha}^\pm = \theta_\pm E_\alpha + \sum_{k \geq 1} t_{4k-3,\alpha}^\pm E_\alpha \partial^{2k-2}, \quad (5.2)$$

where $Q_\pm = \partial \theta_\pm - \theta_\pm \partial$.

Then one can get the following lemma.

Lemma 5.1. *The operators $\Gamma_{j,\alpha}^\pm, Q_\pm$ satisfy*

$$[D_{4i-2,\alpha} - D_\pm^{4i-2} E_\alpha, \Gamma_{j,\beta}^\pm] = [D_{4i-3,\alpha}^\pm - D_\pm^{4i-3} E_\alpha, \Gamma_{j,\beta}^\pm] = 0; \quad j = 0, 1, \quad (5.3)$$

$$[D_{4i-2,\alpha} - D_\pm^{4i-2} E_\alpha, Q_\pm] = [D_{4i-3,\alpha}^\pm - D_\pm^{4i-3} E_\alpha, Q_\pm] = 0, \quad (5.4)$$

$$[Q_\pm, \Gamma_{0,\alpha}^\pm] = -\Gamma_{1,\alpha}^\pm, \quad [Q_\pm, \Gamma_{1,\alpha}^\pm] = E_\alpha, \quad [\partial, \Gamma_{0,\alpha}^\pm] = E_\alpha. \quad (5.5)$$

Proof. The proof is quite similar as the proof in [18] as following

$$\begin{aligned} [D_{4i-2,\alpha} - D_\pm^{4i-2} E_\alpha, \Gamma_{0,\beta}^\pm] &= \frac{1}{2} (4i - 2) D_\pm^{4i-4} E_\alpha \delta_{\alpha,\beta} - [D_\pm^{4i-2}, x] E_\alpha \delta_{\alpha,\beta} = 0, \\ [D_{4i-3,\alpha}^\pm - D_\pm^{4i-3} E_\alpha, \Gamma_{0,\beta}^\pm] & \\ = [\frac{\partial}{\partial t_{4i-3,\alpha}^\pm} - \sum_{j=1}^\infty t_{4j-3,\alpha}^\pm \frac{\partial}{\partial t_{4i+4j-6,\alpha}} - D_\pm^{4i-3} E_\alpha, \Gamma_{0,\beta}^\pm] & \end{aligned} \quad (5.6)$$

$$\begin{aligned}
 &= \frac{1}{2}(4i - 3)D_{\pm}^{4i-3,\alpha} E_{\alpha}\delta_{\alpha,\beta} - \frac{1}{2}\partial^{2i-2} Q_{\pm} E_{\alpha}\delta_{\alpha,\beta} \\
 &\quad + 2 \sum_{j \geq 1} (i - j)t_{4j-3,\alpha}^{\pm} E_{\alpha}\delta_{\alpha,\beta} \partial^{2i+2j-4} \tag{5.7}
 \end{aligned}$$

$$\begin{aligned}
 &- \sum_{j=1}^{\infty} (2i + 2j - 1)t_{4j-3,\alpha}^{\pm} E_{\alpha}\delta_{\alpha,\beta} D^{4i+4j-8} \\
 &\quad - E_{\alpha}\delta_{\alpha,\beta} [D_{\pm}^{4i-3}, x - \frac{1}{2} \sum_{k \geq 1} (4k - 3)t_{4k-3,\alpha}^{\pm} D_{\pm}^{4k-3}] \\
 &= 0, \tag{5.8}
 \end{aligned}$$

in which we used

$$[D_{\pm}^{4i-3}, x] = \frac{1}{2}(4i - 3)D_{\pm}^{4i-3} - \frac{1}{2}\partial^{2i-2} Q_{\pm},$$

and

$$[D_{\pm}^{4i-3}, \frac{1}{2} \sum_{k \geq 1} (4k - 3)t_{4k-3,\alpha}^{\pm} D_{\pm}^{4k-3}] = - \sum_{j \geq 1} (4j - 3)t_{4j-3,\alpha}^{\pm} D_{\pm}^{4i+4j-8}. \tag{5.9}$$

For $\Gamma_{1,\alpha}^{\pm}, Q_{\pm}$, we can prove it similarly. Also we do a direct calculation as

$$[Q_{\pm}, \Gamma_{0,\beta}^{\pm}] = [Q_{\pm}, x E_{\beta}] - [Q_{\pm}, \frac{1}{2} \sum_{k \geq 1} t_{4k-3,\beta}^{\pm} E_{\beta} \partial^{2k-2} Q_{\pm}] = -\Gamma_{1,\beta}^{\pm},$$

and other identities can be proved similarly. \square

Then it is easy to get the following lemma by dressing structures.

Lemma 5.2. *The operators $M_{j,\alpha}^{\pm}, \hat{Q}_{\pm}, \mathcal{L}$ satisfy*

$$[\hat{Q}_{\pm}, M_{0,\alpha}^{\pm}] = -M_{1,\alpha}^{\pm}, \quad [\hat{Q}_{\pm}, M_{1,\alpha}^{\pm}] = E_{\alpha}, \quad [\mathcal{L}^2, M_{0,\alpha}^{\pm}] = E_{\alpha}, \tag{5.10}$$

$$\begin{aligned}
 D_{k,\alpha}^{-} M_{j,\beta}^{\pm} &= [(\mathcal{L}^k R_{\alpha})_+, M_{j,\beta}^{\pm}], \\
 D_{k,\alpha}^{-} \hat{Q}_{\pm} &= [(\mathcal{L}^k R_{\alpha})_+, \hat{Q}_{\pm}], \quad k = 4i - 2, 4i - 3, i \in \mathbb{Z}_+. \tag{5.11}
 \end{aligned}$$

$$\begin{aligned}
 D_{k,\alpha}^{+} M_{j,\beta}^{\pm} &= [(\Lambda^k R_{\alpha})_+, M_{j,\beta}^{\pm}], \\
 D_{k,\alpha}^{+} \hat{Q}_{\pm} &= [(\Lambda^k R_{\alpha})_+, \hat{Q}_{\pm}], \quad k = 4i - 2, 4i - 3, i \in \mathbb{Z}_+. \tag{5.12}
 \end{aligned}$$

Here $D_{4i-2,\alpha}^{+} = D_{4i-2,\alpha}^{-} = D_{4i-2,\alpha}$.

Proof. The dressing structure

$$\Psi[D_{4i-3,\alpha}^{\pm} - D_{\pm}^{4i-3} E_{\alpha}, \Gamma_{1,\beta}^{\pm}] \Psi^{-1} = 0; \tag{5.13}$$

will lead to

$$[\Psi D_{4i-3,\alpha}^{\pm} \Psi^{-1} - \Psi D_{\pm}^{4i-3} E_{\alpha} \Psi^{-1}, M_{1,\beta}^{\pm}] = 0. \tag{5.14}$$

Here we only consider

$$[D_{4i-3,\alpha}^+ - (D_{4i-3,\alpha}^+ \Psi)\Psi^{-1} + \sum_{j=0}^{\infty} t_{4j-3,\alpha}^+ \Psi_{4i+4j-6} \Psi^{-1} - \Lambda^{4i-3} R_{\alpha}, M_{1,\beta}^+] = 0.$$

Then we get

$$[D_{4i-3,\alpha}^+ - (D_{4i-3,\alpha}^+ \Psi)\Psi^{-1} - \Lambda^{4i-3} R_{\alpha}, M_{1,\beta}^+] = 0; \tag{5.15}$$

and using eq. (4.8) we can derive

$$[D_{4i-3,\alpha}^+ - (\Lambda^{4i-3} R_{\alpha})_+, M_{1,\beta}^+] = 0. \tag{5.16}$$

Similarly we can derive

$$[D_{4i-3,\alpha}^- - (\mathcal{L}^{4i-3} R_{\alpha})_+, M_{1,\beta}^-] = 0. \tag{5.17}$$

The other identities can be proved using the similar dressing techniques. \square

We will introduce the following operator $B_{mklp}^{\pm\alpha\beta\gamma}$ defined as

$$B_{mklp}^{\pm\alpha\beta\gamma} = M_{0,\alpha}^{\pm k} M_{1,\beta}^{\pm l} \hat{Q}_{\pm}^p \mathcal{L}^{2m} R_{\gamma} - (-1)^{pl+m+p+l} R_{\gamma} \mathcal{L}^{2m} (\hat{Q}_{\pm}^p) M_{1,\beta}^{\pm l} M_{0,\alpha}^{\pm k}, \tag{5.18}$$

where $k, m \geq 0; l, p = 0, 1; 1 \leq \alpha, \beta, \gamma \leq s$. This operator is the generator of the additional symmetry of the $N = 2$ multicomponent SCKP hierarchy.

Then the following proposition can be derived.

Proposition 5.3. *The operator $B_{mklp}^{\pm\alpha\beta\gamma}$ satisfies the following flow equations*

$$D_{4k-2,\rho} B_{mklp}^{\pm\alpha\beta\gamma} = -[(\mathcal{L}^{4k-2} R_{\rho})_-, B_{mklp}^{\pm\alpha\beta\gamma}],$$

$$D_{4k-3,\rho}^- B_{mklp}^{\pm\alpha\beta\gamma} = -[(\mathcal{L}^{4k-3} R_{\rho})_-, B_{mklp}^{\pm\alpha\beta\gamma}], \tag{5.19}$$

$$D_{4k-3,\rho}^+ B_{mklp}^{\pm\alpha\beta\gamma} = -[(\Lambda^{4k-3} R_{\rho})_-, B_{mklp}^{\pm\alpha\beta\gamma}]. \tag{5.20}$$

Proof. The lemma can be proved by dressing the following identities by Ψ

$$[D_{4k-2,\rho} - D_{\pm}^{4k-2} E_{\rho}, \Gamma_{0,\alpha}^{\pm n} \Gamma_{1,\beta}^{\pm l} Q_{\pm}^p E_{\gamma} \partial^m]$$

$$= [D_{4k-3,\rho}^{\pm} - D_{\pm}^{4k-3} E_{\rho}, E_{\gamma} \Gamma_{0,\alpha}^{\pm n} \Gamma_{1,\beta}^{\pm l} Q_{\pm}^p \partial^m] = 0. \quad \square \tag{5.21}$$

To prove that $B_{mklp}^{\pm\alpha\beta\gamma}$ satisfies the C type condition, we need the following lemma.

Lemma 5.4. *The operators M_i^{\pm} satisfy the following conjugate identities,*

$$M_{i,\alpha}^{\pm*} = (-1)^i M_{i,\alpha}^{\pm}, \quad \hat{Q}_{\pm}^* = -\hat{Q}_{\pm}. \tag{5.22}$$

Proof. Using

$$\Psi^* = \Psi^{-1}, \quad \Gamma_{i,\alpha}^{\pm*} = (-1)^i \Gamma_{i,\alpha}^{\pm}, \quad Q_{\pm}^* = -Q_{\pm}, \tag{5.23}$$

the following calculations

$$M_{i,\alpha}^{\pm*} = \Psi^{*-1} \Gamma_{i,\alpha}^{\pm*} \Psi^* = (-1)^i \Psi \Gamma_{i,\alpha}^{\pm} \Psi^{-1} = (-1)^i M_{i,\alpha}^{\pm},$$

will lead to the first identity of this lemma. The other identities can be proved in a similar way. \square

It is easy to check the following proposition holds basing on the Lemma 5.4 above.

Proposition 5.5. *The operator $B_{mklp}^{\pm\alpha\beta\gamma}$ satisfies a C type condition, namely*

$$B_{mklp}^{\pm\alpha\beta\gamma*} = -B_{mklp}^{\pm\alpha\beta\gamma}. \tag{5.24}$$

Proof. The following calculation

$$\begin{aligned} B_{mklp}^{\pm\alpha\beta\gamma*} &= (M_{0,\alpha}^{\pm k} M_{1,\beta}^{\pm l} \hat{Q}_{\pm}^p \mathbb{L}^{2m} R_{\gamma} - (-1)^{pl+m+p+l} R_{\gamma} \mathbb{L}^{2m} (\hat{Q}_{\pm}^p) M_{1,\beta}^{\pm l} M_{0,\alpha}^{\pm k})^* \\ &= (-1)^{pl} R_{\gamma}^* \mathbb{L}^{2m*} (\hat{Q}_{\pm}^p)^* M_{1,\beta}^{\pm l*} M_{0,\alpha}^{\pm k*} + (-1)^{m+p+l} M_{0,\alpha}^{\pm k*} M_{1,\beta}^{\pm l*} (\hat{Q}_{\pm}^p)^* \mathbb{L}^{2m*} R_{\gamma}^* \\ &= (-1)^{pl+m+p+l} R_{\gamma} \mathbb{L}^{2m} \hat{Q}_{\pm}^p M_{1,\beta}^{\pm l} M_{0,\alpha}^{\pm k} - M_{0,\alpha}^{\pm k} M_{1,\beta}^{\pm l} \hat{Q}_{\pm}^p \mathbb{L}^{2m} R_{\gamma} \\ &= -(M_{0,\alpha}^{\pm k} M_{1,\beta}^{\pm l} \hat{Q}_{\pm}^p \mathbb{L}^{2m} R_{\gamma} - (-1)^{pl+m+p+l} R_{\gamma} \mathbb{L}^{2m} \hat{Q}_{\pm}^p M_{1,\beta}^{\pm l} M_{0,\alpha}^{\pm k}), \end{aligned}$$

will lead to this proposition. \square

Basing on above proposition, it is reasonable to define additional flows of the N=2 supersymmetric CKP hierarchy as

$$D_{mklp}^{\pm\alpha\beta\gamma} L = [-(B_{mklp}^{\pm\alpha\beta\gamma})_-, L], \quad k, m \geq 0; l, p = 0, 1; 1 \leq \alpha, \beta, \gamma \leq s. \tag{5.25}$$

Proposition 5.6. *The flows (5.25) commute with the flows of the multicomponent N=2 supersymmetric CKP hierarchy. Namely, one has*

$$[D_{mnlp}^{\pm\alpha\beta\gamma}, D_{4i-2,\rho}] = [D_{mnlp}^{\pm\alpha\beta\gamma}, D_{4i-3,\rho}] = 0, \quad m, n \geq 0; l, p = 0, 1, i \in \mathbb{Z}_+; \tag{5.26}$$

which holds in the sense of acting on Ψ and $1 \leq \alpha, \beta, \gamma, \rho \leq s$.

Proof. The proposition when $k = 4i - 3$ can be checked as

$$\begin{aligned} &[D_{mnlp}^{\pm\alpha\beta\gamma}, D_{k,\rho}^{\pm}] \Psi \\ &= D_{mnlp}^{\pm\alpha\beta\gamma} D_{k,\rho}^{\pm} \Psi - (-1)^{(l+p)k} D_{k,\rho}^{\pm} D_{mnlp}^{\pm\alpha\beta\gamma} \Psi \\ &= (-1)^{(l+p)k} [(\mathbb{L}^k R_{\rho})_-, (B_{mnlp}^{\pm\alpha\beta\gamma})_-] \Psi + [(B_{mnlp}^{\pm\alpha\beta\gamma})_-, \mathbb{L}^k R_{\rho}]_- \Psi \end{aligned} \tag{5.27}$$

$$\begin{aligned} &+ (-1)^{(l+p)k} [(\mathbb{L}^k R_{\rho})_+, B_{mnlp}^{\pm\alpha\beta\gamma}]_- \Psi \\ &= 0. \quad \square \end{aligned} \tag{5.28}$$

That means that the additional flows of the $N = 2$ multicomponent supersymmetric CKP hierarchy are its symmetries whose algebraic structure can be shown in the following proposition.

Proposition 5.7. *The algebra of additional symmetries of the $N = 2$ multicomponent SCKP hierarchy given by eq. (5.25) is isomorphic to the Lie algebra $multi\ N = 2 \otimes SW_{1+\infty}$.*

Proof. The isomorphism is given by

$$z \mapsto \partial, \quad \xi \mapsto Q_+ E_{\beta} + \Gamma_{1,\beta}^+ \partial, \quad \eta E_{\beta} \mapsto Q_- E_{\beta} + \Gamma_{1,\beta}^- \partial, \tag{5.29}$$

$$\partial_z E_{\beta} \mapsto \Gamma_{0,\beta}^{\pm}, \quad \partial_{\xi} E_{\beta} \mapsto \Gamma_{1,\beta}^+, \quad \partial_{\eta} E_{\beta} \mapsto \Gamma_{1,\beta}^-, \tag{5.30}$$

which further lead to

$$z \mapsto \mathbb{L}^2, \quad \xi E_\beta \mapsto \hat{\mathbb{Q}}_+ R_\beta + M_{1,\beta}^+ \mathbb{L}^2, \quad \eta E_\beta \mapsto \hat{\mathbb{Q}}_- R_\beta + M_{1,\beta}^- \mathbb{L}^2, \quad (5.31)$$

$$\partial_z E_\beta \mapsto M_{0,\beta}^\pm, \quad \partial_\xi E_\beta \mapsto M_{1,\beta}^+, \quad \partial_\eta E_\beta \mapsto M_{1,\beta}^-, \quad (5.32)$$

where keeps ξ, η commuting with z , ξ anticommuting with η . \square

CRedit authorship contribution statement

Chuanzhong Li is the sole author of the manuscript. He did all the contributions.

Declaration of competing interest

We declare that we have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

- [1] A. Yu. Orlov, E.I. Schulman, Additional symmetries of integrable equations and conformal algebra representation, *Lett. Math. Phys.* 12 (1986) 171–179.
- [2] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, Transformation groups for soliton equations, in: *Nonlinear Integrable Systems—Classical Theory and Quantum Theory*, Kyoto, 1981, World Sci. Publishing, Singapore, 1983, pp. 39–119.
- [3] E. Date, M. Kashiwara, M. Jimbo, T. Miwa, Transformation groups for soliton equations. IV. A new hierarchy of soliton equations of KP-type, *Physica D* 4 (1982) 343–365.
- [4] K. Takasaki, Quasi-classical limit of BKP hierarchy and W-infinity symmetries, *Lett. Math. Phys.* 28 (1993) 177–185.
- [5] M.H. Tu, On the BKP hierarchy: additional symmetries, Fay identity and Adler–Shiota–van Moerbeke formula, *Lett. Math. Phys.* 81 (2007) 91–105.
- [6] J.S. He, K.L. Tian, A. Foerster, W.X. Ma, Additional symmetries and string equation of the CKP hierarchy, *Lett. Math. Phys.* 81 (2007) 119–134.
- [7] J.P. Cheng, J.S. He, S. Hu, The “Ghost” symmetry of the BKP hierarchy, *J. Math. Phys.* 51 (2010) 053514.
- [8] K. Ueno, H. Yamada, A supersymmetric extension of nonlinear integrable systems, in: *Symposium on Topological and Geometrical Methods in Field Theory*, World Scientific, Espoo, Finland, 1986, pp. 59–72.
- [9] Yu. Manin, A. Radul, A supersymmetric extension of the Kadomtsev–Petviashvili hierarchy, *Commun. Math. Phys.* 98 (1985) 65–77.
- [10] M. Mulase, A new super KP system and a characterization of the Jacobians of arbitrary algebraic super curves, *J. Differ. Geom.* 34 (1991) 651.
- [11] A. Yu Orlov, Vertex operator, $\bar{\partial}$ -problem, variational identities and Hamiltonian formalism for 2+1 integrable systems, in: V. Bar’yakhtar, V. Chernousenko, N. Erokhin, A. Sitenko, V.E. Zakharov (Eds.), *Nonlinear and Turbulent Processes in Physics*, Proc Intern Workshop on Plasma Theory and Nonlinear and Turbulent Processes in Physics, Kiev, 1987, World Scientific, Singapore, 1988, pp. 116–134.
- [12] S. Stanciu, Additional symmetries of supersymmetric KP hierarchies, *Commun. Math. Phys.* 165 (1994) 261–279.
- [13] H. Aratyn, E. Nissimov, S. Pacheva, Supersymmetric Kadomtsev–Petviashvili hierarchy: “Ghost” symmetry structure, reductions, and Darboux–Bačklund solutions, *J. Math. Phys.* 40 (1999) 2922–2932.
- [14] J.S. He, Y. Cheng, The KW theorem for the supersymmetric KP hierarchy, *Phys. Lett. B* 449 (1999) 194–200.
- [15] E. Ramos, S. Stanciu, On the supersymmetric BKP-hierarchy, *Nucl. Phys. B* 427 (1994) 338–350.
- [16] C.Z. Li, Darboux transformations of the supersymmetric BKP hierarchy, *J. Nonlinear Math. Phys.* 3 (2016) 306–313.

- [17] C.Z. Li, R.L. Ge, Symmetries of supersymmetric CKP hierarchy and its reduction, *J. Geom. Phys.* 158 (2020) 103894.
- [18] E. Ramos, S. Stanciu, On the supersymmetric BKP-hierarchy, *Nucl. Phys. B* 427 (1994) 338–350.
- [19] C.Z. Li, J.S. He, Supersymmetric BKP systems and their symmetries, *Nucl. Phys. B* 896 (2015) 716–737.
- [20] C.Z. Li, J.S. He, Block algebra in two-component BKP and D type Drinfeld-Sokolov hierarchies, *J. Math. Phys.* 54 (2013) 113501.
- [21] C.Z. Li, J.S. He, Y.C. Su, Block type symmetry of bigraded Toda hierarchy, *J. Math. Phys.* 53 (2012) 013517.
- [22] C.Z. Li, J.S. He, Quantum Torus symmetry of the KP, KdV and BKP hierarchies, *Lett. Math. Phys.* 104 (2014) 1407–1423.