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Constructing the quantum Hall system on the Grassmannians $\mathbf{Gr}_2(\mathbb{C}^N)$

F Ballı, A Behtash, S Kürkçüoğlu and G Ünal

Department of Physics, Middle East Technical University, Dumlupinar Boulevard, 06800, Cankaya, Ankara, Turkey

E-mail: fatih.balli@metu.edu.tr, alireza.behtash@metu.edu.tr, kseckin@metu.edu.tr, ugonul@metu.edu.tr

Abstract. In this talk, we give the formulation of Quantum Hall Effects (QHEs) on the complex Grassmann manifolds $\mathbf{Gr}_2(\mathbb{C}^N)$. We set up the Landau problem in $\mathbf{Gr}_2(\mathbb{C}^N)$, solve it using group theoretical techniques and provide the energy spectrum and the eigenstates in terms of the $SU(N)$ Wigner $\mathcal{D}$-functions for charged particles on $\mathbf{Gr}_2(\mathbb{C}^N)$ under the influence of abelian and non-abelian background magnetic monopoles or a combination of these thereof. For the simplest case of $\mathbf{Gr}_2(\mathbb{C}^4)$ we provide explicit constructions of the single and many-particle wavefunctions by introducing the Plücker coordinates and show by calculating the two-point correlation function that the lowest Landau level (LLL) at filling factor $\nu = 1$ forms an incompressible fluid. Finally, we heuristically identify a relation between the $U(1)$ Hall effect on $\mathbf{Gr}_2(\mathbb{C}^4)$ and the Hall effect on the odd sphere S^5, which is yet to be investigated in detail, by appealing to the already known analogous relations between the Hall effects on $\mathbb{C}P^3$ and $\mathbb{C}P^7$ and those on the spheres S^4 and S^8, respectively. The talk is given by S. Kürkçüoğlu at the Group 30 meeting at Ghent University, Ghent, Belgium in July 2014 and based on the article by F.Ballı, A.Behtash, S.Kürkçüoğlu, G.Ünal [1].

1. Introduction

A 4-dimensional generalization of the quantum Hall effect (QHE) was introduced by Hu and Zhang in [2]. They treat the Landau problem on S^4 for charged particles carrying an additional $SU(2)$ degree of freedom which are under the influence of an $SU(2)$ background gauge field. In the thermodynamic limit, the multi-particle problem in the lowest Landau level (LLL) with filling factor $\nu = 1$ may be seen as an incompressible 4-dimensional quantum Hall liquid as demonstrated by these authors. Appearance of massless chiral bosons at the edge of a 2-dimensional quantum Hall droplet [3, 4, 5, 6] generalizes to this setting. It is found that among the edge excitations of this 4-dimensional quantum Hall droplet not only photons and gravitons but also other massless higher spin states occur.

Further developments took place after the work of Hu and Zhang. Other higher-dimensional generalizations of QHE to a variety of manifolds including complex projective spaces $\mathbb{C}P^N$, S^8, S^3, the Flag manifold $\frac{SU(3)}{U(1)\times U(1)}$, as well as quantum Hall systems based on higher dimensional fuzzy spheres have been investigated by several authors [7, 8, 9, 10, 11]. Nair and Karabali examined the QHE on $\mathbb{C}P^N$ [7] and solved the Landau problem on $\mathbb{C}P^N$ using the coset realization of $\mathbb{C}P^N$ over $SU(N+1)$ and performing a suitable restriction of the Wigner $\mathcal{D}$-functions on the latter to obtain the wave functions and the energy spectrum for charged particles under the influence of both $U(1)$ abelian and/or non abelian $SU(N)$ gauge field backgrounds. The degeneracy in each LL is identified with the dimension of the irreducible representation (IRR) to which the wave functions belong. A close connection between the Hall effects on $\mathbb{C}P^3$ and $\mathbb{C}P^7$ with abelian backgrounds and those on the spheres S^4 and S^8 with $SU(2)$ and $SO(8)$ backgrounds, respectively [7, 8, 11] has been revealed in these investigations. These models possess several features which make them interesting in their own right and worthy for further study.

Here we focus on the results of our formulation of QHE on the complex Grassmannians $\mathbf{Gr}_k(\mathbb{C}^N)$ as reported in our article [1]. $\mathbf{Gr}_k(\mathbb{C}^N)$ are generalizations of complex projective spaces $\mathbb{C}P^N$ and share many of their features, such as being a Kähler manifold. Plücker embedding of $\mathbf{Gr}_k(\mathbb{C}^N)$ into $\mathbb{C}P^{\binom{N}{k}-1}$ is quite useful in capturing several of these features. For the case $k=2$, the Plücker embedding describes $\mathbf{Gr}_2(\mathbb{C}^N)$ as a projective algebraic hypersurface in $\mathbb{C}P^N$. For $\mathbf{Gr}_k(\mathbb{C}^4)$ this becomes the well-known Klein Quadric in $\mathbb{C}P^5$ [12]. Employing group theoretical techniques we solved the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ and developed the physics of at the LLL at the filling factor $\nu = 1$. In this proceedings article, we provide a description of the essential features of our methods and give our main results and refer the reader to our article [1] for a full discussion.

It is also worthwhile to remark that Landau problem on two and higher dimensional spaces have close and striking connections to string physics, D-branes and stringy matrix models and to the structure fuzzy spaces such as the fuzzy sphere S_F^2 and fuzzy complex projective spaces $\mathbb{C}P_F^N$. These connections are studied at various levels of sophistication in the literature [13, 14, 15]. Fuzzy spaces arise as quantized versions of their parent manifolds and they are described by finite-dimensional matrix algebras which tend to the algebra of functions over the parent manifolds under a suitable mapping such as the diagonal coherent state map. Quantum field theories are formulated over fuzzy spaces as matrix models with finite degrees of freedom, while preserving the symmetries of the parent space, which makes them appealing for QFT applications (see, [16] and references therein). Construction of fuzzy spaces using geometric quantization methods yields Hilbert spaces $\mathcal{H}_N$ of wave-functions which are holomorphic sections of $U(1)$ bundles over the commutative parent manifold and the matrix algebras Mat_N of linear transformations on $\mathcal{H}_N$'s form the fuzzy spaces [15]. Observables on the fuzzy spaces belong to this matrix algebra. It has been observed that the LLL in Landau problems over S^2, $\mathbb{C}P^N$ in $U(1)$ backgrounds define Hilbert spaces which are identical to $\mathcal{H}_N$ as they are also holomorphic sections of $U(1)$ bundles over these spaces. Similar structural relations between S_F^4 and QHE on S^4 also exists [15]. Building upon this connection, observables of QHE problem are also contemplated as linear transformations in Mat_N acting on $\mathcal{H}_N$. From this angle, we see that there appears almost an immediate connection of our findings for QHE problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ to fuzzy Grassmann spaces which are discussed in some detail in the literature [17, 18, 19].

Landau problem on S^2 , S^4 and in higher dimensions, which may be of interest in the context of string theory, have descriptions in terms of strings interacting with D-branes [13, 14]. In the 2-dimensional case, one considers a $D2$-brane wrapped around an S^2 and with N $D0$-branes dissolved on it. Stack of K $D6$- branes extending in directions perpendicular to $D2$-brane are then moved to the center of the $D2$-brane. Due to Hanany-Witten effect [20], K fundamental strings stretch between a $D2$-brane and $D6$-branes. Each $D0$-brane provides a magnetic flux quantum over the world volume of $D2$-brane while the end points of the string on $D2$-brane play the role of charged particles under the world-volume gauge field. Low energy excitations of this system are described by the QHE system on S^2 with K playing the number of charged particles, N being the magnetic flux and the ratio $\frac{K}{N} = \nu$ being the filling factor. In this picture, background magnetic field may be described as the density of $D0$-branes on the $D2$-brane and $D0$-brane may be viewed to form an incompressible fluid. An alternative point of view is obtained by describing the background magnetic field in terms of a combination of $D0$-

branes and flux due to a background 2-form field $B_{\mu\nu}$. In a similar manner, Fabinger [14] was able to argue that QHE on S^4 describes the low energy dynamics of a configuration of strings interacting with D-branes, where one now wraps a stack of $D4$-branes on S^4 and spreads $D0$-branes on it. Moving flat infinite $D4$-branes to the center of S^4 gives once again fundamental strings connecting the branes at the center and those forming the S^4. Low energy dynamics of this configuration turns out to be the QHE of Hu and Zhang on S^4. Alternatively, one may develop another interpretation of the latter in terms of a certain number of $D0$-branes expanded into a fuzzy four sphere S_F^4 [21]. We consider the possibility that these connections between string physics, fuzzy geometries and QHE systems over two and higher dimensional compact manifolds may be further exploited to give a description of QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ in terms of strings-D-branes configuration, although it may prove very hard to address the stability of the latter. Nevertheless, we hope that our results may be preliminarily conceived as a low energy limit of such a strings-D-branes configuration.

2. Review of QHE on $\mathbb{C}P^1$

In this section we give a short summary of the formulation of quantum Hall problem on $\mathbb{C}P^1$ as a warm up for the developments in the subsequent sections. Treatment of QHE on $\mathbb{C}P^1 \equiv S^2$ is originally due to Haldane [22]. Karabali and Nair [7] have provided a reformulation which is adaptable to higher dimensional spaces which we follow here.

Landau problem on $\mathbb{C}P^1$ can be viewed as electrons on a two-sphere under the influence of a Dirac monopole sitting at the center. We take up the task to construct the Hamiltonian for a single electron under the influence of this Dirac monopole. We use the fact that the functions on the group manifold of $SU(2) \equiv S^3$ may be expanded in terms of the Wigner-$\mathcal{D}$ functions $\mathcal{D}^{(j)}_{L_3 R_3}(g)$ where g is an $SU(2)$ group element and j is an integral or a half-odd integral number labeling the IRR of $SU(2)$. The subscripts L_3 and R_3 are the eigenvalues of the third component of the left- and right-invariant vector fields on $SU(2)$.Throughout this article we sometimes denote the left and right invariant vector fields of $SU(N)$ and their eigenvalues by L_i and R_i, respectively, which one is meant will be clear from the context. The left- and right-invariant vector fields on $SU(2)$ satisfy

$$[L_i \,, L_j] = -\varepsilon_{ijk} L_k \,, \quad [R_i \,, R_j] = \varepsilon_{ijk} R_k \,, \quad [L_i \,, R_j] = 0 \,. \tag{1}$$

Functions and sections of bundles over $\mathbb{C}P^1$ may be obtained from the Wigner-$\mathcal{D}$ functions on $SU(2)$ by a suitable restriction of the latter. The coset realization of $\mathbb{C}P^1$ is

$$\mathbb{C}P^1 \equiv S^2 = \frac{SU(2)}{U(1)} \,. \tag{2}$$

This implies that the sections of $U(1)$ bundle over $\mathbb{C}P^1$ should fulfill

$$\mathcal{D}(g e^{i R_3 \theta}) = e^{i \frac{n}{2} \theta} \mathcal{D}(g) \,, \tag{3}$$

where n is an integer. This condition is satisfied by the functions of the form $\mathcal{D}^{(j)}_{L_3 \frac{n}{2}}(g)$. Eigenvalue $\frac{n}{2}$ of R_3 corresponds to the strength of the Dirac monopole sitting at the center of the sphere and $\mathcal{D}^{(j)}_{L_3 \frac{n}{2}}(g)$ are the desired wave functions as will argue so shortly. Let us observe that $\mathcal{D}^{(j)}_{L_3 0}(g)$ simply correspond to the spherical harmonics on S^2, which are nothing but the wave functions for electrons on a sphere with no background magnetic field.

In the presence of a magnetic field B, the Hamiltonian must involve covariant derivatives whose commutator is proportional to B. We take this commutator as $[D_+ \,, D_-] = B$. We observe that the covariant derivatives $D_\pm$ may be identified by the right invariant vector fields $R_\pm = R_1 \pm i R_2$, as

$$D_\pm = \frac{1}{\sqrt{2}\ell} R_\pm \,, \tag{4}$$

where ℓ denotes the radius of the sphere. Note that $[R_+ , R_-] = 2R_3$, for the eigenvalue $\frac{n}{2}$ of R_3 we have

$$B = \frac{n}{2\ell^2} , \tag{5}$$

for the magnetic monopole field with the magnetic charge $\frac{n}{2}$ in accordance with the Dirac quantization condition. Magnetic flux through the sphere is $2\pi n$.

In terms of the covariant derivatives the Hamiltonian is given as

$$\begin{aligned} H &= \frac{1}{2M}(D_+D_- + D_-D_+) \\ &= \frac{1}{2M\ell^2}(\sum_{i=1}^{3} R_i^2 - R_3^2) , \end{aligned} \tag{6}$$

where M is the mass of the particle. We have $\sum_{i=1}^{3} R_i^2 = \sum_{i=1}^{3} L_i^2 = j(j+1)$, with j labeling the irreducible representations of $SU(2)$. To have $\frac{n}{2}$ occuring as one of the possible eigenvalues of R_3, we need $j = \frac{1}{2}n + q$ where q is an integer. Spectrum of the Hamiltonian is therefore

$$\begin{aligned} E_{q,n} &= \frac{1}{2M\ell^2}\left((\frac{n}{2} + q)(\frac{n}{2} + q + 1) - \frac{n^2}{4} \right) \\ &= \frac{B}{2M}(2q+1) + \frac{q(q+1)}{2M\ell^2} \end{aligned} \tag{7}$$

Eigenfunctions of this Hamiltonian are now clearly seen to be $\mathcal{D}^{(j)}_{L_3 \frac{n}{2}}(g)$ as we have noted earlier. Integer q is readily interpreted as the Landau level (LL) index. The ground state, that is the Lowest Landau Level (LLL), is at $q = 0$ and has the energy $\frac{B}{2M}$. The LLL is separated from the higher LL by finite energy gaps. Degeneracy at an LL controlled by the left invariant vector fields L_i since they commute with the covariant derivatives $[L_i , D_j] = 0$. Each LL is $(2j + 1 = n + 1 + 2q)$-fold degenerate. These are the wavefunctions $\mathcal{D}^{(j)}_{L_3 \frac{n}{2}}(g)$ at a given LL with L_3 eigenvalues ranging from $-j$ to j.

Wave functions $\mathcal{D}^{(j)}_{L_3 \frac{n}{2}}(g)$ may be given in explicit form by choosing a suitable coordinate system. We refer the reader to the original literature [7] where this is done in detail. In [7] it is also shown that the LLL form an incompressible liquid by computing the two-point correlation function for the wave-function density. We investigate this crucial property of the LLL for our case in section ...

3. Landau Problem on the Grassmannian $\mathbf{Gr}_2(\mathbb{C}^4)$

We start with recalling a few facts about the Grassmannians and their geometry. Complex Grassmannians $\mathbf{Gr}_k(\mathbb{C}^N)$ are the set of all k-dimensional linear subspaces of the vector space $\mathbb{C}^N$ with the complex dimension $k(N-k)$. They are smooth and compact complex manifolds and admit Kähler structures. Grassmannians are homogeneous spaces and have the coset realization

$$\mathbf{Gr}_k(\mathbb{C}^N) = \frac{SU(N)}{S[U(N-k) \times U(k)]} \sim \frac{SU(N)}{SU(N-k) \times SU(k) \times U(1)} . \tag{8}$$

We see that $\mathbf{Gr}_1(\mathbb{C}^N) \equiv \mathbb{C}P^N$ and $\mathbf{Gr}_2(\mathbb{C}^4)$ is therefore the simplest Grassmannian that is not a projective space.

To set up and solve the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^4)$, we contemplate that $SU(4)$ Wigner $\mathcal{D}$-functions may be suitably restricted to obtain the harmonics and local sections of bundles over $\mathbf{Gr}_2(\mathbb{C}^4)$. Let $g \in SU(4)$ and let us denote the left- and the right-invariant vector fields on $SU(4)$ by L_α and R_α $(\alpha : 1\,, \cdots, 15)$; they fulfill the Lie algebra commutation relations for $SU(4)$. We introduce the Wigner-$\mathcal{D}$ functions on $SU(4)$ as

$$g \rightarrow \mathcal{D}^{(p,q,r)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15};R^{(1)}R_3^{(1)}R^{(2)}R_3^{(2)}R_{15}}(g) \tag{9}$$

where (p, q, r) are three integers labeling the irreducible representations of $SU(4)$, and the subscripts denote the relevant quantum numbers for the left- and right- rotations. In particular, the left and right generators of $SU(2) \times SU(2)$ subgroup are labeled by $L_\alpha \equiv (L_i^{(1)}, L_i^{(2)})$ and $R_\alpha \equiv (R_i^{(1)}, R_i^{(2)})$ $(i : 1\,, 2\,, 3,\ \alpha : 1\,, \cdots, 6)$ with corresponding $SU(2)\times SU(2)$ quadratic Casimirs $\mathcal{C}_2^L = L^{(1)}(L^{(1)}+1) + L^{(2)}(L^{(2)}+1)$, $\mathcal{C}_2^R = R^{(1)}(R^{(1)}+1) + R^{(2)}(R^{(2)}+1)$.

Hamiltonian on $\mathbf{Gr}_2(\mathbb{C}^4)$ may be written down as

$$\begin{aligned} H &= \frac{1}{2M\ell^2}\sum_{\alpha=7}^{14} R_\alpha^2 \\ &= \frac{1}{2M\ell^2}\left(\mathcal{C}_2(p,q,r) - \mathcal{C}_2^R - R_{15}^2\right) , \end{aligned} \tag{10}$$

where $\mathcal{C}_2(p, q, r)$ is the quadratic Casimir of $SU(4)$ in the IRR (p, q, r) with the eigenvalue

$$\mathcal{C}_2(p,q,r) = \frac{3}{8}(r^2+p^2) + \frac{1}{2}q^2 + \frac{1}{8}(2pr + 4pq + 4qr + 12p + 16q + 12r). \tag{11}$$

The dimension of the IRR $(p\,, q\,, r)$ is

$$\dim(p\,,q\,,r) = \frac{1}{12}(p+q+2)(p+q+r+3)(q+r+2)(p+1)(q+1)(r+1). \tag{12}$$

Coset realization of $\mathbf{Gr}_2(\mathbb{C}^4)$ signals that, there can be both abelian and non-abelian background gauge fields corresponding to the gauging of the $U(1)$ and one or both of the $SU(2)$ subgroups.

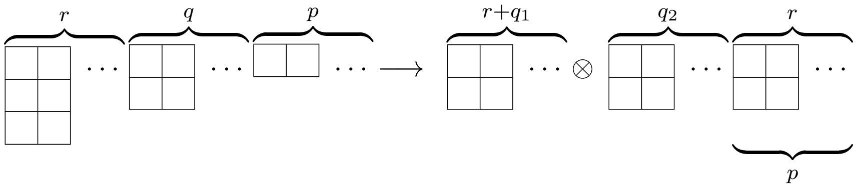
3.1. $U(1)$ gauge field background

In this case we are concerned with the branching

$$SU(4) \supset SU(2) \times SU(2) \times U(1)\,. \tag{13}$$

and obtaining the wave functions with the $U(1)$ background gauge field, requires us to restrict $\mathcal{D}^{(p,q,r)}$ in such a way that they transform trivially (i.e. singlets) under the right action of $SU(2) \times SU(2)$, and carry a right $U(1)$ charge (R_{15} eigenvalue).

$SU(4)$ IRR (p, q, r) has the following branching in Young tableaux notation, which keeps the $SU(2) \times SU(2)$ in the singlet representation,



(14)

Here we have introduced the splitting $q = q_1 + q_2$ in the representation in order to handle the partition of columns labeled by q in the branching. We see that a trivial representation of $SU(2) \times SU(2)$ may be obtained if and only if p is equal to r.

It is possible to show that [1]

$$R_{15} = \frac{n}{\sqrt{2}} = \frac{q_1 - q_2}{\sqrt{2}}\,, \tag{15}$$

with $n \in \mathbb{Z}$ being the $U(1)$ charge of the branching.

Putting these facts together energy spectrum corresponding to the Hamiltonian (10) turns out to be

$$E = \frac{1}{2M\ell^2}\left(p^2 + 3p + np + 2q_2^2 + 4q_2 + 2pq_2 + 2n(1+q_2)\right)\,. \tag{16}$$

The LLL energy at a fixed monopole background n is obtained for $q_2 = p = 0$ and it is

$$E_{LLL} = \frac{n}{M\ell^2} = \frac{2B}{M}\,, \tag{17}$$

with the degeneracy $\dim(0\,,n\,,0) = \frac{1}{12}(n+1)(n+2)^2(n+3)$. In (17), $B = \frac{n}{2\ell^2}$ is the field strength of the $U(1)$ magnetic monopole.

The wave functions corresponding to this energy spectrum are

$$\mathcal{D}^{(p\,,q_1+q_2\,,p)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;\,0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g) \equiv \mathcal{D}^{(p\,,[\frac{q+n}{2}]+[\frac{q-n}{2}]\,,p)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;\,0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g)\,. \tag{18}$$

The degeneracy of each Landau level is given by the dimension of the IRR $(p\,,q\,,p)$ in equation (12). This means that the set of left quantum numbers $\{L^{(1)}\,,L_3^{(1)}\,,L^{(2)}\,,L_3^{(2)}\,,L_{15}\}$ can take on $\dim(p\,,q_1+q_2\,,p)$ different values as a set.

For the many-body problem in which all the states of LLL are filled with the filling factor $\nu = 1$, in the thermodynamic limit $\ell \to \infty, \mathcal{N} \to \infty$ we obtain a finite spatial density of particles

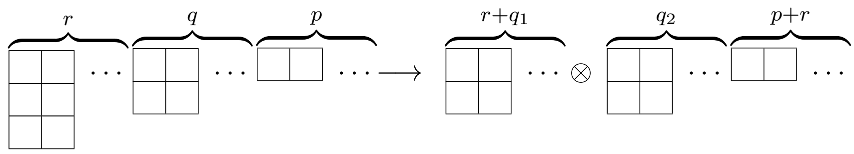
$$\rho = \frac{\mathcal{N}}{\frac{\pi^4\ell^8}{12}} \underset{\ell\to\infty\,,\mathcal{N}\to\infty}{\longrightarrow} \frac{n^4}{\pi^4\ell^8} = \left(\frac{2B}{\pi}\right)^4\,, \tag{19}$$

where we have used $\mathcal{N} = \dim(0\,,n\,,0) = \frac{1}{12}(n+1)(n+2)^2(n+3)$ for the number of fermions in the LLL with $\nu = 1$, and $\mathrm{vol}(\mathbf{Gr}_2(\mathbb{C}^4)) = \frac{\pi^4\ell^8}{12}$.

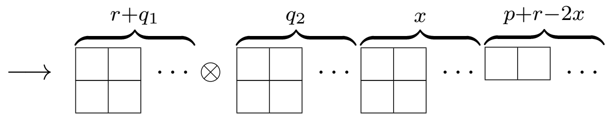
We note that the case $n = 0$ simply reduces the Wigner-$\mathcal{D}$ functions to the harmonics on $\mathbf{Gr}_2(\mathbb{C}^4)$ corresponding to the wave functions of a particle on $\mathbf{Gr}_2(\mathbb{C}^4)$ with vanishing monopole background.

3.2. Single $SU(2)$ gauge field and $U(1)$ gauge field background

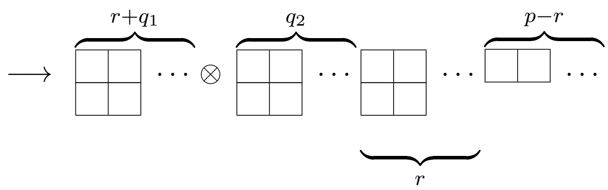
In this case we need to restrict to $\mathcal{D}^{(p,q,r)}$, which transform as a singlet under one or the other $SU(2)$ in the right action of $SU(2)\times SU(2)$, and carry a $U(1)$ charge. There are a range of possibilities within the branching (13) as given in the following Young tableaux decomposition:



(20)



(21)



(22)

Here we have assumed that $p > r$, split $q_1 + q_2 = q$ and introduced the integer x $(0 \leq x \leq r)$ to conveniently represent the generic case. Eigenvalues of R_{15} take the form

$$R_{15} = \frac{n}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \left(2(q_1 - q_2) - (p - r)\right) . \tag{23}$$

We observe from the Young tableaux that the first $SU(2)$ in the branching remains a singlet while the second may take on values over the range

$$R^{(1)} = 0 \,, \quad R^{(2)} = \frac{p-r}{2} \,, \cdots \,, \frac{r+p}{2} \,. \tag{24}$$

$R^{(2)}$ values are integers since n being an integer restricts $p - r$ to even integers.

Some algebra shows that the energy spectrum is

$$E = \frac{1}{2M\ell^2} \left(2q_2^2 + 2q_2(n + R^{(2)} + m + 2) + n(R^{(2)} + m + 2) + (R^{(2)} + m)(2 + m)\right) \,, \tag{25}$$

while the LLL energy at fixed background charges $R^{(2)}$ and n is obtained for $q_2 = m = 0$ and reads

$$E_{LLL} = \frac{1}{2M\ell^2} \left(n(R^{(2)} + 2) + 2R^{(2)}\right) \,. \tag{26}$$

Wave functions with this energy spectrum may be given in the form

$$\mathcal{D}^{(p\,,q_1+q_2\,,r)}_{L^{(1)} L_3^{(1)} L^{(2)} L_3^{(2)} L_{15}\,;\,0\,,0\,,R^{(2)}\,,R_3^{(2)}\,,\frac{n}{\sqrt{2}}} (g) \,. \tag{27}$$

In the thermodynamic limit, for pure $SU(2)$ background $(n = 0\,, R^{(1)} = 0\,, R^{(2)} \neq 0)$, $R^{(2)}$ should scale in the thermodynamic limit as $R^{(2)} \sim \ell^2$. The number of fermions in the LLL with $\nu = 1$ is $\mathcal{N} = \dim(R^{(2)}\,, 0\,, R^{(2)}) \approx \frac{\left(R^{(2)}\right)^5}{6}$ and the corresponding spatial density is $\rho \approx \frac{\left(R^{(2)}\right)^4}{\pi^4 \ell^8}$, which is finite. For case when both $U(1)$ and $SU(2)$ backgrounds are present, we may choose either one of n or $R^{(2)}$ to scale like ℓ^2. Taking $n \sim \ell^2$ and $R^{(2)}$ to be finite in thermodynamic limit, we get $\rho \sim \frac{n^4}{2\pi^4 \ell^8 R^{(2)}}$, which is also finite.

Finally we remark that interchanging the Young tableaux of two $SU(2)$'s amounts to interchanging $R^{(1)}$ and $R^{(2)}$ in (24), and also a flip in the sign of the $U(1)$ charge. In the relevant formulas above, one can compensate for these changes by replacing $R^{(2)}$ with $R^{(1)}$ and substituting $|n|$ for n.

The treatment of the most general case, with $SU(2) \times SU(2)$ gauge field background and generalization of the results of this section to the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ are somewhat lengthy and omitted here. Interested readers can find a full discussion of these in our article [1].

4. Local Form of the Wave Functions and the Gauge Fields

Plücker coordinates for $\mathbf{Gr}_k(\mathbb{C}^N)$ are constructed out of a projective embedding, the so-called Plücker embedding $\mathbf{Gr}_k(\mathbb{C}^n) \hookrightarrow \mathbf{P}\left(\bigwedge^k \mathbb{C}^n\right)$ [12, 24]. For $\mathbf{Gr}_2(\mathbb{C}^4)$ this construction involves the projective space $\mathbf{P}\left(\mathbb{C}^4 \wedge \mathbb{C}^4\right) \equiv \mathbb{C}P^5$. We introduce two sets of complex coordinates $v_\alpha \,, w_\alpha$ ($\alpha = 1 \,, \cdots, 4$) and take the fully antisymmetric basis for the exterior product space $\mathbb{C}^4 \wedge \mathbb{C}^4$ in the form

$$P_{\alpha\beta} = \frac{1}{\sqrt{2}}(v_\alpha w_\beta - v_\beta w_\alpha)\,. \tag{28}$$

$P_{\alpha\beta}$ may be seen as the homogenous coordinates on $\mathbb{C}P^5$ with the identification $P_{\alpha\beta} \sim \lambda P_{\alpha\beta}$ where $\lambda \in U(1)$ and $\sum_{\alpha\,,\beta}^4 |P_{\alpha\beta}|^2 = 1$. Plücker embedding of $\mathbf{Gr}_2(\mathbb{C}^4)$ in $\mathbb{C}P^5$ is given by the homogeneous condition

$$\varepsilon_{\alpha\beta\gamma\delta} P_{\alpha\beta} P_{\gamma\delta} = P_{12}P_{34} - P_{13}P_{24} + P_{14}P_{23} = 0\,. \tag{29}$$

In fact this condition defines the Klein quadric Q_4 in $\mathbb{C}P^5$, which is complex analytically equivalent to $\mathbf{Gr}_2(\mathbb{C}^4)$.

It is possible to use $P_{\alpha\beta}$ to parametrize the columns of $g \in SU(4)$ in the IRR $(0,1,0)$. We choose a parametrization of the form $g_{N6} = P_N := P_{\alpha\beta}$, $g_{N5} = \varepsilon_{NM} P_M^* = \varepsilon_{\alpha\beta\gamma\delta} P_{\gamma\delta}^*$ with $N \equiv [\alpha\beta]$, $N = 1\,, \cdots, 6$ and $\alpha\beta = (12, 13, 14, 23, 24, 34)$.

Wave functions in the $U(1)$ background gauge field are the sections of $U(1)$ bundle over $\mathbf{Gr}_2(\mathbb{C}^4)$. They satisfy the gauge transformation property

$$\mathcal{D}^{(0\,,q_1+q_2\,,0)}(gh) = \mathcal{D}^{(0\,,q_1+q_2\,,0)}(g e^{i\lambda_{15}\theta}) = \mathcal{D}^{(0\,,q_1+q_2\,,0)}(g) e^{i\frac{n}{\sqrt{2}}\theta}\,. \tag{30}$$

In the IRR $(0,1,0)$ we have $\lambda_{15} = \frac{1}{\sqrt{2}} diag(0,0,0,0,-1,1)$. Using this fact and (30), we infer that

$$\mathcal{D}^{(0\,,1\,,0)}(g) \sim P_{\alpha\beta}\,. \tag{31}$$

Since $(0,q,0)$ IRR is the q-fold symmetric tensor product of the IRR $(0,1,0)$ we find that

$$\mathcal{D}^{(0\,,q_1+q_2\,,0)}(g) \sim P_{\alpha_1\beta_1} P_{\alpha_2\beta_2} \cdots P_{\alpha_{q_1}\beta_{q_1}} P^*_{\gamma_1\delta_1} P^*_{\gamma_2\delta_2} \cdots P^*_{\gamma_{q_2}\delta_{q_2}}\,. \tag{32}$$

LLL wave functions are those with $q_2 = 0$ and take the form

$$\mathcal{D}^{(0\,,q_1\,,0)}_{L^{(1)}L_3^{(1)}L^{(2)}L_3^{(2)}L_{15}\,;\,0\,,0\,,0\,,0\,,\frac{n}{\sqrt{2}}}(g) \sim P_{\alpha_1\beta_1} P_{\alpha_2\beta_2} \cdots P_{\alpha_{q_1}\beta_{q_1}}\,, \tag{33}$$

which are holomorphic in the Plücker coordinates.

At filling factor $\nu = 1$ LLL has $\mathcal{N} = \dim(0,1,0) = \frac{1}{12}(n+1)(n+2)^2(n+3)$ number of particles. Its wave-function is given in terms of the Slater determinant as

$$\Psi_{MP} = \frac{1}{\sqrt{\mathcal{N}!}} \varepsilon^{\Lambda_1\Lambda_2\cdots\Lambda_n} \Psi_{\Lambda_1}(P^{(1)}) \Psi_{\Lambda_2}(P^{(2)}) \cdots \Psi_{\Lambda_N}(P^{(N)})\,. \tag{34}$$

Here $P^{\mathbf{i}}$ denotes the i^{th} position fixed in the Hall fluid and correspondingly $\Psi_{\Lambda_j}(P^{\mathbf{i}})$ refers to the wave function of the j^{th} particle located at the position $P^{\mathbf{i}}$. For a one-particle wave function in our notation is

$$\Psi_{\Lambda_i}(P^{\mathbf{i}}) \equiv \Psi^{\mathbf{i}}_{\alpha\beta} \sim P^{\mathbf{i}}_{\alpha\beta}\,. \tag{35}$$

LLL wave function given in (33) is then

$$\Psi_{\Lambda_i}(P^{\mathbf{i}}) \equiv \Psi^{\mathbf{i}}_{\Lambda_i} \sim (P^{\mathbf{i}}_{\alpha\beta})^n\,. \tag{36}$$

Two point correlation function is of the form

$$\Omega(1,2) = \int |\Psi_{MP}|^2 d\mu(3)d\mu(4)\cdots d\mu(\mathcal{N}) = |\Psi^{\mathbf{1}}|^2|\Psi^{\mathbf{2}}|^2 - |\Psi^{*\mathbf{1}}_{\Lambda}\Psi^{\mathbf{2}}_{\Lambda}|^2. \tag{37}$$

wher $d\mu(i)$ denote integration measure of integration on $\mathbf{Gr}_2(\mathbb{C}^4)$ in the coordinates of the i^{th} particle.

Taking the normalized coordinate chart $\gamma_i := \frac{P_{\alpha\beta}}{P_{12}}$ where $P_{12} \neq 0$ and using the notation $\vec{X} = \vec{\gamma}\ell$, yields after some algebra the result in the thermodynamic limit as

$$\Omega(1,2) \approx 1 - e^{-2B\left(\vec{x}^{\mathbf{1}}-\vec{x}^{\mathbf{2}}\right)^2} e^{-2B\ell^2\left(\det\Gamma^{\mathbf{1}}-\det\Gamma^{\mathbf{2}}\right)^2}, \tag{38}$$

where we have used $n = 2B\ell^2$ and introduced $\Gamma^{\mathbf{i}} := \begin{pmatrix} \gamma^{\mathbf{i}}_2 & \gamma^{\mathbf{i}}_1 \\ \gamma^{\mathbf{i}}_4 & \gamma^{\mathbf{i}}_3 \end{pmatrix}$. This result shows the two-point function of the particles located at the positions $\vec{x}^{\mathbf{1}}, \vec{x}^{\mathbf{2}}$ on $\mathbf{Gr}_2(\mathbb{C}^4)$, is extracted from that of the particles on $\mathbb{C}P^5$ at the positions $\vec{X}^{\mathbf{1}}, \vec{X}^{\mathbf{2}}$ by a restriction of these particles to the algebraic variety determined by $X^{\mathbf{i}}_5 \equiv \ell \det\Gamma^{\mathbf{i}}$, as expected. It is apparent from the form of $\Omega(1,2)$ that the probability of finding two particles at the same point goes to zero, signalling the incompressibility of the Hall fluid.

A short calculation show that the $U(1)$ gauge field $A = -\frac{in}{\sqrt{2}}\mathrm{Tr}\left(\lambda^{15}_{(6)} g^{-1}dg\right)$ may be written in terms of Plücker coordinates as

$$A = -inP^*_N dP_N\,, \tag{39}$$

and the associated field strength $F = dA$ is

$$F = -indP^*_N \wedge dP_N\,. \tag{40}$$

We note that F is an antisymmetic, gauge invariant, and closed two-form on $\mathbf{Gr}_2(\mathbb{C}^4)$ and as such it is proportional to the Kähler two-form over $\mathbf{Gr}_2(\mathbb{C}^4)$. It is known from very general considerations [25] that the integral of F over a non-contractable two surface Σ in $\mathbf{Gr}_2(\mathbb{C}^4)$ is an integral multiple of 2π:

$$\frac{1}{2\pi}\int_{\Sigma} F = n\,. \tag{41}$$

In the present context, this result signals an analogue of the Dirac quantization condition with $\frac{n}{2}$ identified as the magnetic monopole charge and $B = \frac{n}{2\ell^2}$.

5. Final Remarks

We have presented some of our essential results on the formulation of the quantum Hall problem on $\mathbf{Gr}_2(\mathbb{C}^N)$. A group theoretical approach has been used to obtain the energy spectrum and the wave function for the Landau problem on $\mathbf{Gr}_2(\mathbb{C}^N)$ and this is supplemented by the local description of the LLL physics at $\nu = 1$ using Plücker coordinates.

It is worthwhile to briefly discuss the following observation about the QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ with $U(1)$ background. The isomorphisms $Spin(6) \cong SU(4)$ and $Spin(4) \cong SU(2) \times SU(2)$, indicate that the Stiefel manifold $\mathbf{St}_2(\mathbb{R}^6) \equiv \frac{Spin(6)}{Spin(4)}$ forms the principal $U(1)$ fibration [23]

$$U(1) \longrightarrow \mathbf{St}_2(\mathbb{R}^6) \longrightarrow \mathbf{Gr}_2(\mathbb{C}^4) \,. \tag{42}$$

There are also a family of fibrations $\mathbf{St}_{k-1}(\mathbb{R}^{n-1}) \longrightarrow \mathbf{St}_k(\mathbb{R}^n) \longrightarrow S^{n-1}$, which for $k=2$ and $n=6$ becomes

$$S^4 \longrightarrow \mathbf{St}_2(\mathbb{R}^6) \longrightarrow S^5 \,. \tag{43}$$

Putting these facts together, we see that $\mathbf{Gr}_2(\mathbb{C}^4)$ has the local structure $\frac{S^5 \times S^4}{U(1)}$. Thus, we think that the QHE on S^5 with the S^4 fibers, associated to a $SO(5)$ gauge field background, may be seen as a QHE on $\mathbf{Gr}_2(\mathbb{C}^4)$ with a $U(1)$ background gauge field. The natural question to answer here is then what we mean by QHE on S^5. This problem may be treated by generalizing the formulation of the QHE on the 3-sphere [9]

$$S^3 = \frac{SU(2) \times SU(2)}{SU(2)_{diag}} \cong \frac{Spin(4)}{Spin(3)} \,, \tag{44}$$

which selects the constant background gauge field as the spin connection. In a construction generalizing this to the QHE on S^5,

$$S^5 = \frac{SO(6)}{SO(5)} = \frac{Spin(6)}{Spin(5)} \,, \tag{45}$$

one will be naturally selecting a constant $SO(5)$ background gauge field taking it again as the spin connection. Such a choice of the gauge field appears to be consistent with our heuristic argument. Our observation is inspired by and bears a resemblance to the relation between the QHE on $\mathbb{C}P^7$ and S^8. The former can be realized locally as $\frac{S^8 \times S^7}{U(1)}$, while the latter forms the base of the 3^{rd} Hopf map $S^7 \longrightarrow S^{15} \longrightarrow S^8$, and S^{15} is a $U(1)$ bundle over $\mathbb{C}P^7$ [8].

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