Particle Accelerators, 1996, Vol. 55, pp. [499–530] / 253–284
Reprints available directly from the publisher
Photocopying permitted by license only

SUMMARY OF THE WORKING GROUP ON MAPS

ALEX J. DRAGT

Physics Department, University of Maryland, College Park, Maryland 20742, USA

(Received 8 March 1996; in final form 8 March 1996)

This note summarizes the uses of maps and reports on the Map Working Group that comprised part of the LHC95 International Workshop on Single-Particle Effects in Large Hadron Colliders.

Keywords: Maps; normal forms; nonlinear dynamics; long-term stability; tracking.

1 INTRODUCTION

Maps now play a major role in the description of the linear and nonlinear motion involved in charged particle beam transport. Several important advances in the use of maps have been made in the past few years, and were reported on in the map portion of the LHC95 International Workshop on Single-Particle Effects in Large Hadron Colliders. The purpose of this note is to present a brief summary of the history of maps, their current use, recent developments, and possible future developments.

2 BRIEF HISTORY OF MAPS

The current use of maps in accelerator physics represents the confluence of two mathematical/physical streams of thought. The first of these streams is that of Dynamics, and originates largely in the discoveries of Newton (1642–1727). The second stream is Geometry, and dates back to the ancient Greeks.

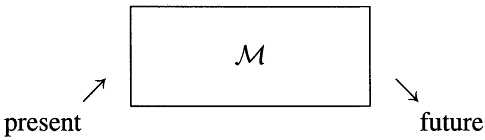

FIGURE 1 In Dynamics the future can be determined by performing a certain operation, called a mapping $\mathcal{M}$, on the present.

Let us begin with the stream of Dynamics. Newton's basic discovery was that motion is governed by laws, and the nature of these laws is such that the *future* can be determined from a knowledge of the *present.* We illustrate this fact with the sketch in Figure 1. Suppose we think of the present as a set of *initial conditions*, and regard the future as a set of *final conditions*. Newton's laws, when appropriately formulated, can be regarded as a set of first-order ordinary differential equations. There is a mathematical theorem about first-order ordinary differential equations to the effect that they have solutions. Moreover, these solutions are unique, and are completely determined by the initial conditions. Thus there is a rule, or *mapping* $\mathcal{M}$, that sends the initial conditions (the present) into the final conditions (the future): one simply integrates Newton's equations in first-order form, perhaps numerically on a computer. Laplace (1749–1827) subsequently made this concept quite explicit when he wrote:

> We ought then to regard the present state of the universe as the effect of its anterior state and as the cause of the one which is to follow. Given for one instant an intelligence which could comprehend all the forces by which nature is animated and the respective situation of the beings who compose it — an intelligence sufficiently vast to submit these data to analysis — it would embrace in the same formula the movements of the greatest bodies of the universe and those of the lightest atom; for it, nothing would be uncertain and the future, as the past, would be present to its eyes. The human mind offers, in the perfection which it has been able to give to astronomy, a feeble idea of this intelligence. Its discoveries in mechanics and geometry, added to that of universal gravity, have enabled it to comprehend in the same analytical expressions the past and future states of the system of the world. Applying the same method to some other objects of its knowledge, it has succeeded in referring to general laws observed phenomena and in foreseeing those

> which given circumstances ought to produce. All these efforts in the search for truth tend to lead it back continually to the vast intelligence which we have just mentioned, but from which it will always remain infinitely removed. This tendency, peculiar to the human race, is that which renders it superior to animals; and their progress in this respect distinguishes nations and ages and constitutes their true glory.

In modern terminology, Laplace was describing what we call a transfer map.

Lagrange (1736–1813) and others discovered that for many systems of physical interest all the differential equations of motion could be generated by (derived from) a *single* master function now called the Lagrangian. These equations of motion were second order. Building on this work, Hamilton (1805–1865) showed that it was possible to write a related set of first-order equations, and that all these equations could also be generated by a single master function now called the Hamiltonian.

Being well aware of the properties of first-order differential equations as described above, Hamilton made a detailed study of the nature of the relation between initial and final conditions (the transfer map $\mathcal{M}$) for Hamiltonian systems. In modern language, he showed that such maps must be symplectic (canonical). He also discovered mixed-variable generating functions, and showed that they can be used at will to produce symplectic maps. Finally, he and Jacobi (1804–1851) studied how symplectic maps could be used to transform Hamiltonians with the aim of simplifying them, and thus also the flows (differential equations) they generate. In modern terminology, their work was the beginning of the Theory of Normal Forms for Hamiltonians and for Symplectic Maps.

Poincaré (1854–1912) was the next person to champion the use of maps and explore their properties: He introduced what we now call stroboscopic maps and Poincaré surface-of-section maps. He showed that the discovery of periodic orbits in the gravitational three-body problem is equivalent to proving the existence of fixed points for a certain symplectic map of the plane into itself. He discovered what we now call the Poincaré invariants and showed they are preserved by symplectic maps. He showed that attempts to use symplectic maps to bring certain classes of Hamiltonians to a certain kind of normal form, which if successful would prove the existence of integrals of motion, were doomed to failure. Indeed, he demonstrated that this failure arises from the appearance of small denominators that spoil the convergence of the series designed to construct the desired normal form.

Birkhoff (1884–1944), in addition to making other outstanding contributions to mathematics, extended the program of Poincaré. In a celebrated early paper he was able to prove what, despite considerable effort, had eluded Poincaré: the map for the 3-body problem developed by Poincaré did indeed have fixed points. He also studied the possibility of using symplectic maps to bring certain classes of Hamiltonians to what is now called Birkhoff normal form. Again, he found that the appearance of small denominators destroyed convergence. Finally, Birkhoff made other significant contributions to Dynamics including fundamental work on ergodic theory and what we now call bifurcation theory.

We now turn to a further discussion of the small denominator problem. Consider an analytic function w in the complex variable z having a Taylor series of the form

$$w(z) = \lambda z + \sum_{n=2}^{\infty} c_n z^n. \tag{1}$$

Observe that (1) may be viewed as a map $\mathcal{M}$ of the complex plane into itself, and this map evidently has the origin as a fixed point,

$$w(0) = 0. \tag{2}$$

Let $\mathcal{A}$ be another analytic map also having the origin as a fixed point. Use this map to transform $\mathcal{M}$ into another map $\mathcal{N}$ by the relation

$$\mathcal{N} = \mathcal{A}\mathcal{M}\mathcal{A}^{-1}. \tag{3}$$

This operation (which really amounts to a change of variables) is called *conjugation*, and we say that $\mathcal{M}$ and $\mathcal{N}$ belong to the same *conjugacy class*. In analogy to the operation of matrix diagonalization, the goal of (3) is to select the conjugating map $\mathcal{A}$ in such a way that $\mathcal{N}$ has a simple form, called a (or the) *normal* form. Poincaré and others studied this problem. For the case $|\lambda| \neq 1$, they showed that an $\mathcal{A}$ could be found such that the map $\mathcal{N}$ took the simple form

$$w = \lambda z. \tag{4}$$

That is, when $|\lambda| \neq 1$ all the nonlinear terms in (1) can be removed by a suitable analytic change of variables. They also studied the case $|\lambda| = 1$ but were unable to arrive at (4) in this case because they again encountered problems with small denominators.

To Siegel (1896–) we owe the honor of first overcoming the small denominator problem. He considered the case

$$\lambda = \exp(i\theta), \tag{5}$$

and for $(\theta/2\pi)$ suitably *irrational* was again able to arrive at (4). Note that when (5) holds, the relation (4) simply amounts to a rotation of the complex plane about the origin by an angle θ. That is, circles about the origin are *invariant* (preserved and not changed) under the action of $\mathcal{N}$. Correspondingly, because the relation (3) preserves topology, the original map $\mathcal{M}$ given by (1) must then also have closed invariant curves (which are the images of these circles under $\mathcal{A}$) whenever $|\lambda| = 1$ and (4) can be achieved.

Inspired by this result, Siegel's student Moser tackled the far more difficult problem of studying symplectic maps of the (real) plane into itself that have a twist property — maps of the form originally studied first by Poincaré and then by Birkhoff for the three-body problem. Under the assumption of certain (eventually mild) differentiability requirements (but not analyticity), he was able to also overcome the small denominator problem that arises in this case, and prove the existence of closed invariant curves. At about the same time, Kolmogorov (1903–) and his student Arnold studied the problem of bringing slightly perturbed integrable Hamiltonians to integrable normal form. Under the (stronger) assumption of suitable analyticity, they also succeeded in overcoming the small denominator problem, and demonstrated the existence of invariant tori, which are the generalization of closed invariant curves to the case of higher dimensional phase spaces.

Let us momentarily turn our attention to accelerator physics. Courant and Snyder pioneered the use of matrices to characterize transverse beam behavior in the linear (first-order or paraxial) approximation. These matrices were enlarged by Penner to include chromatic effects. In subsequent work Brown made the important step of extending the linear matrix formalism to include nonlinear effects through second order. From the perspective of maps, we may view the use of a matrix as making a linear approximation to the underlying transfer map $\mathcal{M}$, and the inclusion of second-order effects as introducing the first nonlinear terms that appear in a Taylor expansion of $\mathcal{M}$ about the origin. It is now relatively easy to compute the terms in a Taylor expansion of $\mathcal{M}$ to very high order. This computation is made possible by two tools. The first is that of Truncated Power Series Algebra (TPSA) computer programs which manipulate very high-order polynomials in several variables.[1,2] The second tool is the use of Lie methods (to be described shortly) to represent transfer maps for individual beamline elements. If H is the Hamiltonian for some beamline element, if τ is the independent "time-like" variable, and if H is autonomous, then the transfer map $\mathcal{M}$ for this element is given (in Lie algebraic notation still to be explained) by the relation

$$\mathcal{M} = \exp(:-\tau H:). \tag{6}$$

We now turn to the second stream, the stream of Geometry. A fundamental notion in geometry as conceived by Euclid (c. 300 B.C.) is that of congruence. Roughly speaking, we consider two triangles as congruent if one can be placed over the other with a resulting perfect fit. From the perspective of maps, we have in mind the operations of translations and rotations which map Euclidean space into itself. Together these operations form a group. Thus we may say that two triangles are congruent if one can be *transformed* into the other under the action of the Euclidean group.

The concepts underlying the Euclidean group were subsequently broadened to include the idea of general transformation groups that map various kinds of spaces or various classes of objects into themselves. Lie (1842–1899), and others both before and after him, studied transformation groups for their applications to both geometry and (in what amounts to a systematic procedure for transforming variables) the simplification and perhaps even solution of certain classes of differential equations. Lie in particular studied the properties of what we now call Lie groups: groups that can be generated by near-identity operations. The generators of these near-identity operations form algebras which we now call Lie algebras. For example, in the case of the rotation group there exist small (infinitesimal) rotations, and any group element can be constructed from these near identity operations. When a matrix representation is used, the generators of the infinitesimal rotations are three matrices, call them L_x, L_y, and L_z, that obey the commutation (Lie algebraic multiplication) rules

$$[L_x, L_y] = L_z, \quad \text{etc.} \tag{7}$$

The elements of the set of all continuous and invertible maps of a space into itself are called *homeomorphisms*. Topology (another area largely pioneered by Poincaré) is the study of those properties of spaces, and objects in these spaces, that are invariant under homeomorphisms. Homeomorphisms that are differentiable are called *diffeomorphisms*. The set of all diffeomorphisms forms a group that is a Lie group. Differential geometry is the study of those properties of spaces, and objects in these spaces, that are invariant under diffeomorphisms.

The set of all symplectic maps (sometimes called *symplectomorphisms*) also forms a Lie group, and this Lie group is a subgroup of the Lie group of diffeomorphisms. In both the group of all diffeomorphisms and the group of all symplectic maps, *Lie transformations* are those group elements produced by a single generator. Hori and Deprit were the first (in the context of Dynamics) to use Lie transformations for the production of symplectic maps.[3,4] They employed these maps to try to bring to normal form various Hamiltonians that arise in celestial mechanics, and showed that the use of Lie transformations is often much more convenient than the method of mixed-variable generating functions developed earlier by Hamilton and Jacobi. The use of Lie algebraic methods in accelerator physics was first proposed by Dragt.[5] In this case Lie transformations, and products of Lie transformations, were used to represent symplectic transfer maps, and Lie algebraic formulas (the Baker-Campbell-Hausdorff and Zassenhaus formulas) were used to multiply and factorize maps. Subsequently Lie methods were also used to bring transfer maps to normal form.[6,7]

The study of maps and, more recently, the study of their properties under iteration are recognized as important subjects. Both have broad application to many areas and are also important in their own right. Consider, for example, the biological subject of insect population growth. Let P_n be the population in year n (of some insect species), and let P_{n+1} be the population the following year. Then we might imagine that there is some kind of rule (or map) $\mathcal{M}$ that relates the population in successive years as shown schematically in Figure 2.

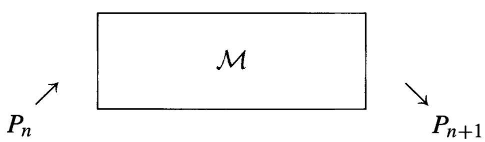

FIGURE 2 The insect populations in successive years are related by a map $\mathcal{M}$.

The simplest form for the map $\mathcal{M}$ is a relation of the kind

$$P_{n+1} = \alpha P_n, \tag{8}$$

where α is viewed as some *fixed* growth rate. However, depending on the size of α, the recursion relation (8) has only exponentially damped or exponentially growing solutions; and both these possibilities are unphysical. An improved model would be to assume that the growth rate itself depends on the current population. For example, we might imagine that if the population were small, then food would be plentiful, and the growth rate should be high. Conversely, if the population were at some maximum value $P_{\max}$, then the food might be in such short supply that there would be no reproduction at all. A simple form for α having this property is obtained by writing

$$\alpha(P) = \beta(P_{\max} - P). \tag{9}$$

With this improved model the map $\mathcal{M}$ takes the form

$$P_{n+1} = \beta(P_{\max} - P_n)P_n. \tag{10}$$

Finally, for mathematical convenience, let us introduce the *fractional* population x defined by the rule

$$x = P/P_{\max}. \tag{11}$$

In terms of this variable the relation (10) takes the form known as the *logistic map*,

$$x_{n+1} = \lambda x_n(1 - x_n). \tag{12}$$

Let us solve (12) for an *equilibrium* value (fixed point) x_e. By definition this value must satisfy the relation

$$\mathcal{M}x_e = x_e, \tag{13}$$

from which we find the result

$$x_e = \lambda x_e(1 - x_e) \tag{14}$$

with the solutions

$$x_e = 0, \tag{15}$$

$$x_e = (\lambda - 1)/\lambda. \tag{16}$$

Suppose we select some value x_0 for an initial (fractional) population and apply the map $\mathcal{M}$ repeatedly for a total of m times to find the result

$$x_m = \mathcal{M}^m x_0. \tag{17}$$

That is, we carry out the operation (12) for a total of m times. Then we might wonder what happens in the limit of large m. For example, does x_m approach x_e, or does something else happen?

Figure 3 shows, as a function of λ, the limiting values, called x_∞, that occur as $m \to \infty$. The calculations for this graphic were made using $x_0 = 1/2$, but other choices in the interval (0,1) would have given the same result. We see that x_∞ is unique for $\lambda < 3$, and can be verified to have the value x_e given by (16). That is, the fixed point x_e is attracting (stable) for $\lambda < 3$. However, x_e is repelling (unstable) for $\lambda > 3$ and no longer appears in the figure for these λ values. Instead period doubling occurs at $\lambda = 3$ so that $\mathcal{M}^2$ has two stable fixed points for λ slightly larger than 3. Insects living in this regime experience alternating fat and lean years!

Inspection of Figure 3 shows that there is a cascade of period doublings as λ increases beyond 3, and that an infinite number of doublings have occurred by the time λ reaches the *critical* value $\lambda_{cr} \simeq 3.57$. For λ slightly beyond λ_{cr} the set of x_∞ points is infinite, and the action of $\mathcal{M}$ on these points is chaotic. Then, remarkably, as λ is increased still further, there are occasional *windows of stability* again followed by period doublings and subsequent chaotic regimes. For example, there is a period-three window (a regime having three values for x_∞) for λ slightly larger than 3.8.

Mathematical experience has taught us that sometimes to *complexify* is to *simplify*. For example, the behavior of power series is understood more simply using complex variables rather than real variables. With this lesson in mind, suppose we extend both x and λ in (12) to complex values. Then the map takes the form form

$$z_{n+1} = \gamma z_n (1 - z_n) \tag{18}$$

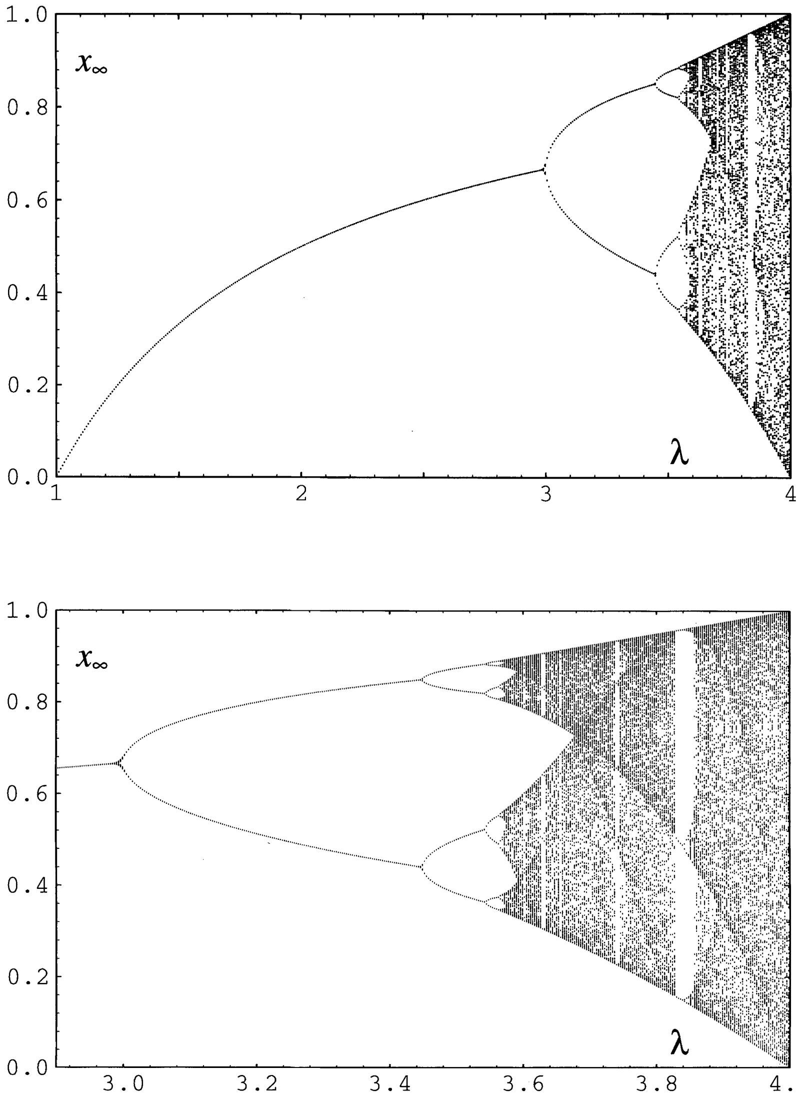

FIGURE 3 Limiting values x_∞ as a function of λ for the logistic map.

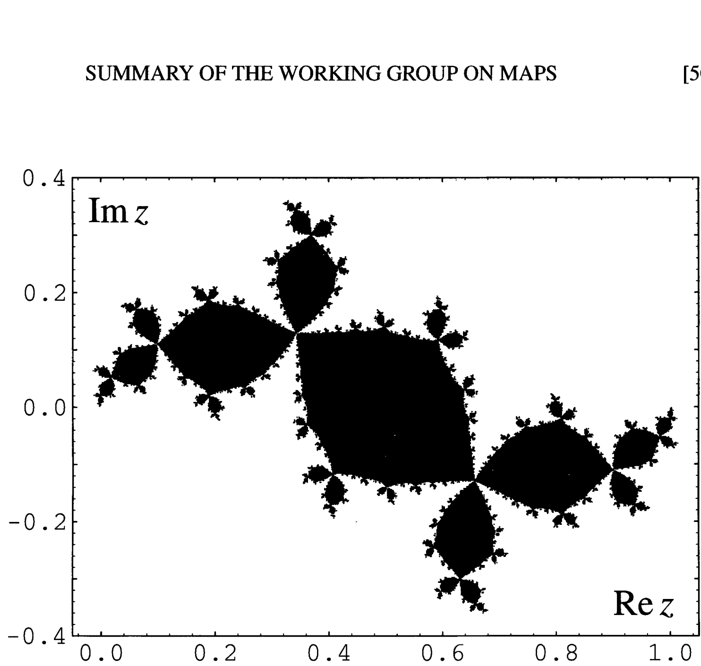

FIGURE 4 Douady's rabbit, the dynamic aperture in the mapping plane for the case $\gamma = 2.55268 - 0.959456i$.

where z is the complex extension of x, and γ is the complex extension of λ. Associated with the map (18) are two complex planes. One of these, the z plane, will be called the *mapping* plane since the map sends this plane into itself. The other, the γ plane, will be called the *control* plane. The nature of what happens under repeated iteration in the mapping plane depends sensitively on where γ is in the control plane. For example, Figure 4 shows the nature of the map for $\gamma = 2.55268 - 0.959456i$. Points in the black area of the mapping plane remain there indefinitely under repeated application of (18). Any point launched in the white area eventually iterates away to infinity. In accelerator language, we would call the black area the *dynamic aperture*. It can be shown that the boundary of the dynamic aperture (the *Julia* set) is fractal. Remarkably, it is nevertheless possible to name in a precise way every point on the boundary.[8,9]

If γ is changed, the dynamic aperture also is changed. For example, Figure 4 shows what is called *Douady's* rabbit; for other values of γ the dynamic aperture disintegrates into a cloud of isolated points called *Fatou* dust. Since the nature of what happens under repeated interation in the mapping plane depends sensitively on the location of γ in the control plane,

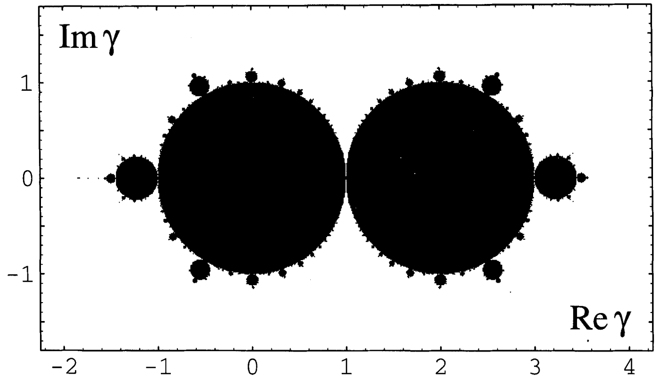

FIGURE 5 The Mandelbrot set M in the control plane.

we may turn the matter around. That is, we may characterize points in the γ plane by the behavior (under repeated iteration) of points in the mapping plane. Suppose we consider those points M in the control plane for which the dynamic aperture in the mapping plane is a *connected* set. This set M in the control plane is called the *Mandelbrot* set. It is shown in Figure 5.[10]

When viewed from a distance, the Mandelbrot set M appears as two back-to-back circles with sprouts. The circles are tangent at the point $\gamma = (1, 0)$, and M has reflection symmetry about both the lines $\text{Re}\,\gamma = 1$ and $\text{Im}\,\gamma = 0$. Closer examination reveals the presence of what appear to be very small islands around the mainland. (In fact, these islands resemble the mainland, and the whole structure of the Mandelbrot set is fractal.) Since γ is the complexification of λ, one can see that λ values in the range $(1, \lambda_{cr})$ correspond to *real* γ values lying in the right-hand circle and its sprouts and its subsprouts. In addition, it can be shown that λ values for the windows of stability seen in Figure 3 correspond to real γ values lying in small islands on the real γ axis to the right of the mainland. Finally, contrary to superficial appearances, it can be shown that the Mandelbrot set is *simply connected.* There are thin island chains, too small to be seen in Figure 5, that connect the visible apparent islands to the mainland. Thus, there is really only a mainland!

Consider the value of γ for Douady's rabbit. It lies in the sprout located at the five-o'clock position of the right-hand circle in Figure 5. It can be shown that for this γ value the map (18) has three *attracting* complex period-three fixed points. Indeed, Douady's rabbit turns out to be the basin of attraction for these fixed points. In addition, there exist another three *repelling* complex period-three fixed points that lie on the boundary of the rabbit. Now continuously vary the value of γ until it enters the island for the period-three window, and eventually takes on a real value corresponding to a λ value lying in the period-three window of Figure 3. As γ varies, the period-three fixed points move. They may change their nature (*e.g.* they all become repellors when γ leaves the sprout), but they cannot disappear. It can be shown that in this case, as γ changes from the Douady-rabbit value in the sprout to a real value in the period-three window, all the associated period-three fixed points of Douady's rabbit move from their original complex values to the real line. Furthermore, the three period-three fixed points that begin as repellors when γ lies in the sprout become the three attractors x_∞ when γ reaches the island. The other three period-three fixed points, which begin as attractors when γ lies in the sprout and become repellors when γ leaves the sprout, remain repellors when γ reaches the island. Thus, by extending the logistic map to the complex domain, we have learned that seemingly isolated phenomena are in fact related.

The complex logistic map (18) may be viewed as the simplest nonlinear two-dimensional *analytic* map. In the same spirit, the simplest nonlinear two-dimensional *symplectic* map is the Hénon map. It too is a quadratic map. We take the opportunity here to describe it in Lie algebraic terms. To do this, we will need to introduce some Lie algebraic tools.

We begin by redefining the symbol z; it will now stand for a canonically conjugate pair of position and momentum variables q and p,

$$z = (q, p). \tag{19}$$

Next, let $f(z)$ denote any function of q, p. We will associate with each such function a *differential* operator, called a *Lie operator* and denoted by the symbol $:f:$, according to the definition

$$:f: = (\partial f/\partial q)(\partial/\partial p) - (\partial f/\partial p)(\partial/\partial q). \tag{20}$$

Then if g is any other function of the phase-space variables z, we have the result

$$:f: g = (\partial f/\partial q)(\partial g/\partial p) - (\partial f/\partial p)(\partial g/\partial q) = [f, g], \tag{21}$$

where [∗, ∗] denotes the familiar Poisson (1781–1840) bracket. Powers of $:f:$ are defined by repeated application of (20) or (21),

$$:f:^2 g = [f, [f, g]],$$

$$:f:^3 g = [f, [f, [f, g]]], \quad \text{etc.} \tag{22}$$

Finally, we define $:f:^0$ to be the identity operator,

$$:f:^0 g = g. \tag{23}$$

Now that powers of Lie operators have been defined, we can also define power series. Of particular interest is the power series for the exponential function,

$$\exp(:f:) = \sum_{k=0}^{\infty} :f:^k /k!. \tag{24}$$

This object is referred to as a *Lie transformation*, and $:f:$ (or f) is called its *generator*. Specifically, if g is any function, we have the result

$$\exp(:f:)g = g + [f, g] + [f, [f, g]]/2! + \cdots . \tag{25}$$

At this point the reader should verify the results

$$\exp(:q^3:)q = q, \tag{26}$$

$$\exp(:q^3:)p = p + 3q^2. \tag{27}$$

We see from (26) and (27) that $\exp(:q^3:)$ produces the phase-space mapping associated with a thin sextupole kick. Similarly, the reader should verify the results

$$\exp[-(\theta/2) :p^2 + q^2:]q = q \cos\theta + p \sin\theta, \tag{28}$$

$$\exp[-(\theta/2) :p^2 + q^2:]p = p \cos\theta - q \sin\theta. \tag{29}$$

We see from (28) and (29) that $\exp[-(\theta/2) : p^2 + q^2 :]$ produces the phase-space mapping for a simple phase advance.

With this background in mind, the Hénon map is simply the product

$$\mathcal{M} = \exp[-(\theta/2) : p^2 + q^2 :] \exp(: q^3 :). \tag{30}$$

It consists of a simple phase advance followed by a sextupole kick. As seen from (26) through (29), the map (30) does indeed consist of linear and quadratic terms, as advertised.[11]

The Hénon map has been studied in detail. It is known to have very complicated properties: these include homoclinic points, chaotic behavior, and period bifurcations. In view of these complicated properties, and in analogy with what has been learned in the case of the logistic map, one might wonder if further insight could be gained by complexifying the Hénon map, i.e. by making both q and p complex. Then (30) would become a mapping of C^2 (the space of two complex variables) into itself. Also, the control parameter θ could be made complex. By such a study one might hope, for example, to better understand the boundary of the dynamic aperture. Hubbard has begun such a study.[12] What he finds to date is that the complex Hénon map is a remarkably complicated object. This should be a sobering thought to accelerator physicists, because we are interested in knowing the behavior of far more complicated symplectic maps in more (four and six) dimensions. When complexified, four- and six-dimensional phase spaces become C^4 and C^6. Thus it is no wonder that questions of dynamic aperture for realistic accelerators are so complicated. Nor, in analogy to the properties of the Mandelbrot set, should we be surprised that the dynamic aperture depends sensitively on the choice of accelerator parameters such as tunes, local phase advances, multipole strengths, etc. What we are observing in all these instances is that complicated properties can arise as a result of an infinite process, namely that of indefinite iteration.

3 CURRENT USE

Maps currently have at least five uses in accelerator physics:

(a) Element, beamline, and machine description;
(b) Analysis;
(c) Providing useful bounds on long-term orbit stability;

(d) Ray tracing;
(e) Long-term tracking.

These various uses will be described briefly below. Further detail may be found in some of the references.

Let $\mathcal{M}$ be a transfer map that sends the phase-space initial conditions z^i into the final conditions z^f. There is a theorem of Poincaré to the effect that if the right-hand side of a set of differential equations is analytic, then the final conditions are analytic functions of the initial conditions. Thus, we quite generally expect to be able to make Taylor expansions of the kind

$$z_a^f = c_a + \sum_b R_{ab} z_b^i + \sum_{bc} T_{abc} z_b^i z_c^i + \sum_{bcd} U_{abcd} z_b^i z_c^i z_d^i + \cdots . \tag{31}$$

Equation (31) is the form for a general diffeomorphism, which can include dissipative processes such as the effects of synchrotron radiation. Since diffeomorphisms form a Lie group, they can also be written in Lie form. In the case of Hamiltonian systems, the transfer map is symplectic and can be written in the factored-product form

$$\mathcal{M} = \exp(:f_1:)\exp(:f_2:)\exp(:f_3:)\exp(:f_4:)\cdots . \tag{32}$$

Here each f_j is a homogeneous polynomial of degree j in the phase-space variables. The factor $\exp(:f_1:)$ describes misalignment, misplacement, and mispowering errors; the factor $\exp(:f_2:)$ describes the linear part $\mathcal{L}$ of the map; and the remaining factors $\exp(:f_3:)\exp(:f_4:)\cdots$ describe nonlinear effects. In single-pass machines, such as linacs or the SLC or the NLC, the nonlinear effects are aberrations. In rings the nonlinear effects are chromaticities, anharmonicities, and nonlinear resonance strengths. If the system described by the transfer map is dissipative, then the transfer map can also be written in factored-product form, but the Lie generators must be more general *vector fields* than those given by expressions of the form (20).

The advantages of the representation (32) include its compactness, the ease with which it can be manipulated, and the fact that each generator has a simple physical interpretation. The interpretation of the various terms in the various f_n has been described in detail by Irwin for single-pass machines; and this approach has been used by him and others to improve the performance of the SLC and to make detailed low-aberration designs for the NLC.[13] He has also made a similar analysis for circular machines.[13] Finally, Irwin has exploited the ability to move factors around in products of maps (using a calculus for noncommuting operators) so that all nonlinearities can be viewed as acting at one common point.

The concept of normal forms has been outlined in Section 2 in connection with (3). Normal forms for symplectic maps and their uses in accelerator physics were first developed by Dragt and Forest.[6,7,14] Given any map $\mathcal{M}$ of the form (32), there is a systematic way of computing both the normal form $\mathcal{N}$ and the normalizing map $\mathcal{A}$. With appropriate computer codes, some of which more or less exist, it is now possible to compute $\mathcal{M}$, $\mathcal{A}$, and $\mathcal{N}$ to any desired order for realistic machine models. The normal form $\mathcal{N}$ contains all information about tunes, chromaticities, and anharmonicities to arbitrary order and with arbitrary coupling. The normalizing map $\mathcal{A}$ contains all information about the closed orbit, dispersion, lattice functions, phase-space distortion, generalized Courant-Snyder invariants, smear, etc. to all orders and with arbitrary coupling. Thus, the use of normal forms provides an ideal tool for analysis. Their use makes routine the kind of calculations only dreamed of in the past when using the older methods of Hamiltonian perturbation theory.

Maps can also be used to produce useful lower bounds on long-term orbit stability in storage rings. Warnock and Berg have shown how to construct invariants from maps represented by mixed-variable generating functions using Fourier series for angle variables and spline fits for action variables. They have studied numerically the long-term behavior of these invariants, and from their behavior have been able to prove long-term orbit stability.[15] Hoffstatter and Berz have used normal forms to construct generalized Courant-Snyder invariants, and interval arithmetic to analyze (analytically) their turn to turn variation.[16] Based on this procedure, which is a kind *Nekhoroshev* analysis, they are able to place long-term bounds on orbit stability by analytical means (but numerically implemented due to their complexity). The use of interval arithmetic is a new contribution to accelerator physics, and its promise and power warrants further investigation.

Both single-pass and circular machines are composed of many beamline elements. The effect at each element can be described by maps of the form (31) or (32), and these maps can then be used to carry out element-by-element ray tracing. Alternatively, the maps for various sections of the machine can be concatenated together to form lumped maps for the various sections. These lumped maps can also be used for ray tracing. In the limit, all maps may be concatenated together to form a single-pass map or a one-turn map, and this map could be used for ray tracing. To the extent that this can be done successfully without loss of necessary accuracy (the method can be self checking), there can be a considerable savings in computer time. There can also be improved understanding since this approach can be used to isolate offensive beamline elements.

Finally, one-turn circular-machine maps can often be used for long-term tracking studies. (And if the use of a one-turn map is too crude, use of a few lumps generally suffices.) When a map is employed for very long-term tracking, it is essential that it be used in a symplectic form.[17] The use of mixed-variable generating functions (along with Newton's method) for this purpose was first proposed and carried out by Dragt and Douglas, and subsequently used by Yan for the SSC.[18,19] Cremona factorization can also be used to represent symplectic maps (see the next section), and an early version of this procedure was used by Schmidt to study the long-term dynamic aperture of the LHC. Finally, Irwin (again see the next section) has used maps for tracking studies in the design of the PEP II B factory and to study combined nonlinear ring and nonlinear beam-beam effects. Whenever one-map (or few-map) long-term tracking can be used in place of element-by-element long-term tracking, one gains a significant savings in computer time.

4 RECENT DEVELOPMENTS

Three recent developments in the use of maps were described at the workshop:

(a) Evaluation of maps using only n Poisson brackets (n-PB tracking);
(b) Beam-beam simulations for circular machines using high-order nonlinear maps for the ring and all-order maps for the beam-beam effect;
(c) Optimal Cremona symplectification.

Each will be described briefly. Readers are referred to papers presented at the workshop and other references for more detail.

Suppose a symplectic map is written in the form (32), and we wish to evaluate

$$z^f = \mathcal{M} z^i. \tag{33}$$

Evaluation of the effect of the factors $\exp(:f_1:)$ and $\exp(:f_2:)$ presents no particular problem since they correspond, respectively, to translations and linear transformations in phase space, and these operations can easily be carried out to machine precision. However, evaluating the effects of the nonlinear factors $\exp(:f_3:)\exp(:f_4:)\cdots$ is more problematic. Irwin has found it useful to proceed as follows: First, use the Baker-Campbell-Hausdorff formula to combine (up to some higher order) the factors $\exp(:f_3:)\exp(:f_4:)\cdots$ into a single exponent,

$$\exp(:f_3:)\exp(:f_4:)\cdots = \exp(:g_3+g_4+\cdots:) = \exp(:g:). \tag{34}$$

Second, evaluate $\exp(:g:)$ by retaining only n Poisson brackets in the expansion (24),

$$\exp(:g:)z^i \simeq \sum_{k=0}^{n}(:g:^k/k!)z^i. \tag{35}$$

Third, if one wishes to be even bolder, expand g in a resonance basis (action-angle variables) and retain only those few terms that are believed to be important, and then use (35). Analysis shows that it is quite easy to evaluate $:g:^k z^i$ in action-angle variables if only a few terms are present in g. Experience has shown that this procedure works well for PEP II dynamic aperture determinations (tracking for a few damping times) even for modest values of n (say $n=3$) and the retention of relatively few resonant terms in g. Indeed, calculations and design cycles that would take weeks using element-by-element tracking could be, and were, done in a single evening.[20]

It is interesting to observe that when maps are used for tracking, some significant fraction of the computational burden consists of computing the map itself. Let us write the one-turn map in the form

$$\mathcal{M} = \exp(:f_1:)\exp(:f_2:)\exp(:g:). \tag{36}$$

The tunes of many machines are controlled by adjusting relatively few quadrupoles. For some machines, adjusting these quadrupoles has relatively little effect on g. In this case, it is not necessary to recompute the whole one-turn map when tunes are changed. One merely needs to adjust the factors $\exp(:f_1:)\exp(:f_2:)$, and this can be done by hand. Indeed, if $\mathcal{M}$ is brought to partial normal form by using an $\mathcal{A}$ of the form $\exp(:h_1:)\exp(:h_2:)$, then the $\exp(:f_1:)$ term is removed or made small enough to be neglected, and the $\exp(:f_2:)$ term consists of simple phase advances of the form (28) and (29).

By proceeding in this manner, it is feasible with only a modest computing effort to make detailed tune-plane scans (dynamic aperture as a function of horizontal and vertical tunes) of a kind never possible before. If still more accuracy is desired, it is possible to store the maps for those portions of the lattice that are between tune adjusting quadrupoles, and then rebuild the full one-turn map simply by concatening the maps for the tune quadrupoles and the stored maps.

The beam-beam interaction has two interesting features: it contains very high-order nonlinearities, and it is relatively weak in the sense that its effect is fairly small on any given pass. This circumstance makes it possible to represent the beam-beam interaction by a map, call it $\mathcal{M}_b$, of the form

$$\mathcal{M}_b = \exp(:b:), \tag{37}$$

where b is expanded in powers of the beam-beam interaction strength, but need not be a polynomial. Let $\mathcal{M}_r$ be the map for the ring. Then the full one-turn map $\mathcal{M}$, including both the ring and the beam-beam interaction, can be written in the form

$$\mathcal{M} = \mathcal{M}_r \mathcal{M}_b. \tag{38}$$

Irwin has shown that this map can be used efficiently and effectively for electron machines (often with an n-PB evaluation of each factor) to study the interplay between effects arising from nonlinearities in the ring map and nonlinearities in the beam-beam interaction, and has found both effects are important.[21]

The last topic to be discussed in this section is optimal Cremona symplectification. Consider element-by-element tracking in the thin-lens kick approximation as illustrated in Figure 6. In this approach, the one-turn transfer map is given by the approximation

$$\mathcal{M} \simeq \mathcal{D}_1 \mathcal{K}_1 \mathcal{D}_2 \mathcal{K}_2 \mathcal{D}_3 \mathcal{K}_3 \cdots \mathcal{D}_{N_e} \mathcal{K}_{N_e} \tag{39}$$

where N_e is the number of elements (required kicks if elements are subdivided) in the lattice. Typically N_e ranges from 1000 to 10,000. Let us instead attempt to find an approximation for $\mathcal{M}$ of the form

$$\mathcal{M} \simeq (\mathcal{L}_1 \hat{\mathcal{K}}_1 \mathcal{L}_1^{-1})(\mathcal{L}_2 \hat{\mathcal{K}}_2 \mathcal{L}_2^{-1}) \cdots (\mathcal{L}_N \hat{\mathcal{K}}_N \mathcal{L}_N^{-1}). \tag{40}$$

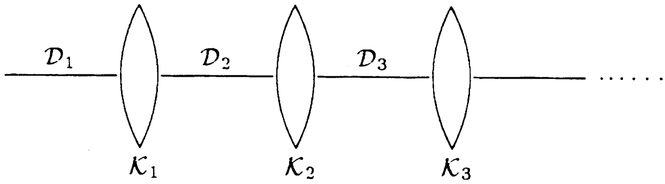

FIGURE 6 Approximation of the one-turn transfer map by drifts $\mathcal{D}_i$ and thin-lens kicks $\mathcal{K}_i$.

Here each $\mathcal{L}_j$ is a general linear transformation of the form exp(: f_2 :), and the $\hat{\mathcal{K}}_j$ are kicks that wear a hat to indicate that they are not necessarily directly related to elements in the lattice, but rather are mathematical objects selected to make (40) work for the given map $\mathcal{M}$. The factors $(\mathcal{L}_j \hat{\mathcal{K}}_j \mathcal{L}_j^{-1})$ are called *jolts*. It is easily verified that each jolt is a map that is both *polynomial* and exactly *symplectic*. Such maps are called *Cremona* maps. Moreover, products of Cremona maps are again Cremona maps. Consequently the entire map given by the right side of (40) is a Cremona map. The goal is to select the $\mathcal{L}_j$ and the $\hat{\mathcal{K}}_j$ in such a way that $N \ll N_e$, and yet $\mathcal{M}$ is well approximated by (40). Indeed, what we would like is to have (40) be a good approximation even when $\mathcal{M}$ is a realistic map including all fringe-field and thick-element-with-errors effects.

Factorizations of the kind (40), with a specific prescription for the choice of the $\mathcal{L}_j$, were first proposed by Irwin.[22] Subsequently Forest explored the use of (40) with the $\mathcal{L}_j$ chosen at random, and this procedure was used by Schmidt to make a map study of the long-term dynamic aperture of the LHC[23]. Inspired by this work, Abell and Dragt examined the question of how to choose the $\mathcal{L}_j$ in an optimal fashion.[17,24] Pursuit of this matter rapidly leads one into difficult questions of group theory and cubature formulas on manifolds. Although many questions remain to be answered, they were able to show that both prescriptions for choosing the $\mathcal{L}_j$ (that of Irwin and random selection) are far from optimal. They were also able to find what appears to be an optimal procedure. Table I below shows (for the case of 4-dimensional phase space) the number of jolts N required in their optimal procedure to produce any given $\mathcal{M}$ through terms of some given order.

When applicable, the use of Cremona maps can be very fast. Let us examine the ratio N_e/N. Since N_e lies in the range 1000 to 10,000 and N lies in the range 32 to 72, we find that the ratio N_e/N lies in the range 13 to 300.

TABLE I Number of jolts N required to produce an arbitary $\mathcal{M}$ (for the case of 4-dimensional phase space) through some given order of the Lie generators

Order	N
5	12
7	24
9	32
11	50
14	72

We therefore expect the use of Cremona maps to be one to two orders of magnitude faster than element-by-element tracking. The use of Cremona maps also permits better modelling. The map $\mathcal{M}$ can include all fringe-field and thick-element-with-errors effects, and even so the approximation (40) continues to hold. Thus, unlike the case of element-by-element tracking where N_e must be made very large to achieve improved modelling, the Cremona method can use optimal models with no loss in computational speed.

As an illustration of the use of Cremona maps, Figure 7 shows the results in horizontal phase space (the vertical phase space is equally excited, but not shown) of tracking the Berkeley Advanced Light Source (set up to have considerable skew coupling) with a 9^{th} -order one-turn truncated Taylor map. Figure 8 shows the results of tracking with the same map, but truncated at 4^{th} order. Evidently the results of Figure 8 are unsatisfactory due to the violations of the symplectic condition produced by truncating the Taylor map. Figure 9 shows the results of tracking with a Cremona symplectification of the 4^{th} -order Taylor map. We see that this relatively low-order, but exactly symplectic, map is able to model many of the features of the 9^{th} -order Taylor map. A detailed study of the use of Cremona one-turn maps for tracking the LHC is currently in progress.

We close this section by remarking that, because of their polynomial nature, Cremona maps have a natural complexification. Thus, if map complexification proves to be useful as described in Section 2, we already have the tools to do so.

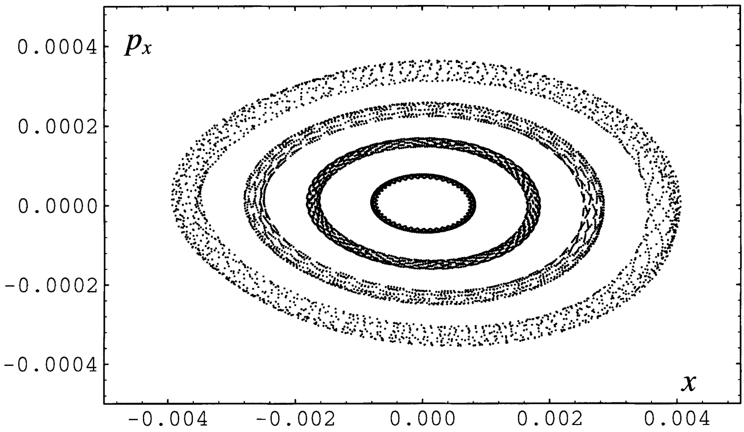

FIGURE 7 Horizontal phase-space tracking results for a model of the Berkeley Advanced Light Source with considerable skew coupling. The vertical phased space is equally excited, but not shown. Tracking is done using a 9 th -order one-turn truncated Taylor map.

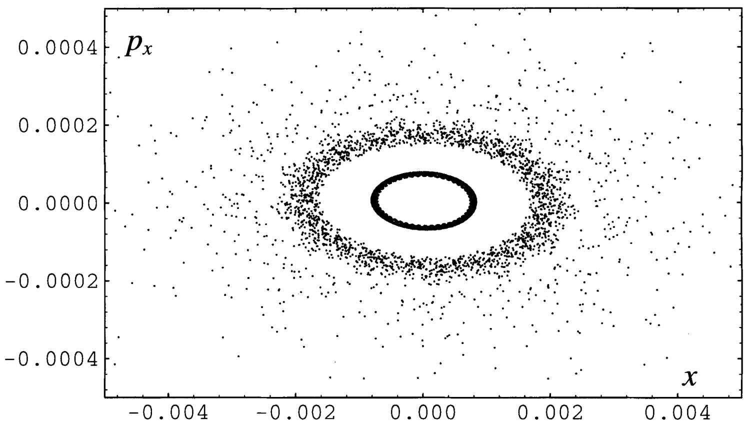

FIGURE 8 Horizontal phase-space results for the same case as Figure 7 except that the Taylor map is truncated at 4^{th} order.

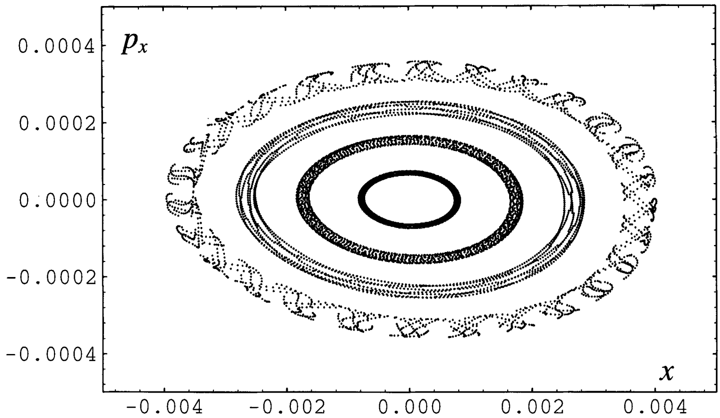

FIGURE 9 Horizontal phase-space tracking results for a Cremona map constructed from the 4th order Taylor map of Figure 8.

5 FUTURE NEW USE

There are at least three possible new uses for maps:

(a) Improved Fourier and Lyapunov analysis of tracking data;
(b) Faster determination of power supply ripple effects with the additional possibility of analytic insight;
(c) Study of noise effects.

Each of these will be sketched below. Although none have yet been tried, items (a) and (b) seem quite plausible, and item (c) seems to at least merit further study.

The uses of normal forms $\mathcal{N}$ and their associated normalizing maps $\mathcal{A}$ were described briefly in Sections 2 and 3 in the context of analyzing maps. The map $\mathcal{A}$ can also be useful for analyzing tracking data. Recall that tracking data is generated by selecting a set of initial conditions in phase-space, and then repeatedly applying $\mathcal{M}$ to them to obtain a sequence of points related by the operation

$$z_{n+1} = \mathcal{M} z_n. \tag{41}$$

The operation (41) can be carried out using element-by-element tracking, lump-by-lump tracking, or one-turn map tracking, depending on what is deemed appropriate. In analyzing tracking data it is customary to examine, usually by plotting, the sequence of phase-space points z_n arising from the various initial conditions. However, it is far better to examine the *transformed* sequence of points $\tilde{z}_n$ given by the relation

$$\tilde{z}_n = \mathcal{A}^{-1} z_n. \tag{42}$$

Here $\mathcal{A}$ is the normalizing (conjugating) map that brings $\mathcal{M}$ to the normal form $\mathcal{N}$ as in (3). The transformation (42) has the effect of removing all the gross nonlinear features from the tracking data, such as smear, without any loss of information. For example, even if the original phase-space points z_n lie on distorted and badly oriented (with respect to the phase-space axes used for plotting) tori, the transformed points $\tilde{z}_n$ will lie nearly on perfect tori (the topological product of circles in the canonically conjugate pairs q_i, p_i).

Since the transformed tracking data are free of gross nonlinear features, they can be examined with greater resolution for signs of instability such as chaotic behavior or resonance streaming. First of all, the transformed data can be analyzed by eye or by looking at the behavior of the generalized Courant-Snyder invariants. In addition, they can also be subjected to Fourier analysis or Lyapunov analysis.[25] In the case of Fourier analysis, one should find the higher harmonics greatly reduced in the transformed data. Correspondingly, the spectrum should have much sharper peaks at the fundamental frequencies, and therefore it should be possible to extract more precise frequency values from the transformed data. In the case of Lyapunov analysis, the absence of gross nonlinear features in the transformed data should make it possible to extract weak long-term trends with greater reliability or with the use of fewer turns. Indeed, Schmidt has already found this to be true.[23]

There is one further aspect of the use of $\mathcal{A}$ that is worth exploring. In conventional tracking it can happen that phase-space distortions present in $\mathcal{M}$ cause regularly spaced initial conditions to sample phase space unevenly. This problem can be overcome by placing the initial conditions on a regular grid (as is usually done), then applying $\mathcal{A}$ to them to produce a set of transformed initial conditions, and then tracking these transformed initial conditions. Thus, the proposed improved tracking and analysis procedure is that illustrated in Figure 10 below.

$$\left\{\begin{matrix}\text{set of initial}\\ \text{conditions on}\\ \text{a regular grid}\end{matrix}\right\} \xrightarrow{\mathcal{A}} \left\{\begin{matrix}\text{transformed}\\ \text{initial}\\ \text{conditions}\end{matrix}\right\} \xrightarrow[\text{turns using } \mathcal{M}]{\text{track for } N} \left\{\begin{matrix}\text{tracking}\\ \text{data}\end{matrix}\right\} \text{———}$$

$$\xrightarrow{\mathcal{A}^{-1}} \left\{\begin{matrix}\text{transformed}\\ \text{tracking}\\ \text{data}\end{matrix}\right\} \xrightarrow[\text{Lyapunov analysis, etc.}]{\text{invariant analysis, Fourier analysis,}} \left\{\begin{matrix}\text{cleaner}\\ \text{results}\end{matrix}\right\}$$

FIGURE 10 Improved generation and analysis of tracking data using the normalizing map $\mathcal{A}$.

We next turn to the treatment of power supply ripple in the LHC using maps. We begin by making the approximation that the circulation frequency of particles in an LHC ring is an integer multiple N of the ripple frequency,

$$(\text{ circulation frequency })/(\text{ ripple frequency }) \simeq (10,000)/(50) = 200 = N. \qquad (43)$$

Let $\mathcal{M}$ be the one-turn transfer map in the absence of ripple, and let $\mathcal{M}_k$ denote the one-turn transfer map for the k^{th} turn in the presence of ripple. Let us write $\mathcal{M}_k$ in the form

$$\mathcal{M}_k = \mathcal{M}\mathcal{E}_k \qquad (44)$$

where the *error* map $\mathcal{E}_k$ is the correction to $\mathcal{M}$ on the k^{th} turn due to ripple. Evidently $\mathcal{E}_k$ is defined by the relation

$$\mathcal{E}_k = \mathcal{M}^{-1}\mathcal{M}_k. \qquad (45)$$

Since the effect of ripple is expected to be small on any given turn, we expect each $\mathcal{E}_k$ to be near the identity. Next let $\mathcal{M}_c$ be the map for one full *cycle* of ripple. It is defined by the product

$$\mathcal{M}_c = \mathcal{M}_1\mathcal{M}_2\mathcal{M}_3\cdots\mathcal{M}_N. \qquad (46)$$

From (44) and (46) and algebraic manipulation we find the result

$$\begin{aligned}
\mathcal{M}_c &= \mathcal{M}\mathcal{E}_1\mathcal{M}\mathcal{E}_2\mathcal{M}\mathcal{E}_3\cdots\mathcal{M}\mathcal{E}_N \\
&= \mathcal{M}^N\left[\left(\mathcal{M}^{N-1}\right)^{-1}\mathcal{E}_1\mathcal{M}^{N-1}\right]\left[\left(\mathcal{M}^{N-2}\right)^{-1}\mathcal{E}_2\mathcal{M}^{N-2}\right]\cdots \\
&\quad \cdots\mathcal{M}^{-1}\mathcal{E}_{N-1}\mathcal{M}\mathcal{E}_N \\
&= \mathcal{M}^N\mathcal{E}_1^{\text{tr}}\mathcal{E}_2^{\text{tr}}\mathcal{E}_3^{\text{tr}}\cdots\mathcal{E}_N^{\text{tr}}.
\end{aligned} \tag{47}$$

Here we have introduced the *transformed* error maps $\mathcal{E}_k^{\text{tr}}$ defined by the rule

$$\mathcal{E}_k^{\text{tr}} = \left(\mathcal{M}^{N-k}\right)^{-1}\mathcal{E}_k\mathcal{M}^{N-k}. \tag{48}$$

Now define the *net* error map $\mathcal{E}$ for one full cycle by the product

$$\mathcal{E} = \mathcal{E}_1^{\text{tr}}\mathcal{E}_2^{\text{tr}}\cdots\mathcal{E}_N^{\text{tr}}. \tag{49}$$

With this definition, $\mathcal{M}_c$ takes the form

$$\mathcal{M}_c = \mathcal{M}^N\mathcal{E}. \tag{50}$$

The operations (45), (48), and (49) can all be carried out by Lie algebraic means (on a computer) so that the map $\mathcal{E}$ can be calculated. (We note that since $\mathcal{E}$ is calculated explicitly, it can be examined in detail with the possibility of obtaining some analytic insight.) Finally, we perform long-term tracking using $\mathcal{M}_c$; that is, according to (50), we track for N turns as if there were no ripple, then apply the error map $\mathcal{E}$, and then repeat, etc. Since N is relatively large, we conclude that (once $\mathcal{E}$ is known) long-term tracking with ripple need be no more time consuming than usual tracking. Moreover, if the usual tracking can be speeded up by the use of one-turn or few-lump maps, then so can tracking with ripple.

The treatment of noise using maps is similar to that of ripple. In this case we let N be a number of turns that is large, but not so large that the effect of noise over this number of turns is substantial. Let us refer to N turns as an *interval*. Then the map $\mathcal{M}_I$ for an interval is given by the relation

$$\mathcal{M}_I = \mathcal{M}^N\mathcal{E} \tag{51}$$

where $\mathcal{E}$ is defined as before by (45), (48), and (49). However in this case, unlike the case of ripple, the net interval error map $\mathcal{E}$ varies from interval to interval. It is possible that this complication can be overcome by the following strategy: Suppose, as is presumably the case, that the $\mathcal{E}_k$ (or perhaps the error maps for individual beamline elements) are to be drawn from some ensemble with some prescribed statistics. Repeatedly drawing from this ensemble a set of N error maps $\mathcal{E}_k$ and using (45), (48), and (49) produces a collection of *net interval error maps* of the form $\mathcal{E}$. Suppose we can determine the statistics of the net interval error maps. Then, once their statistics are known, we can equally well generate net interval error maps simply by drawing them from an ensemble having the proper statistics. Now we have all the tools we need to perform long-term tracking using the maps $\mathcal{M}_I$. According to (51), we track for N turns as if there were no noise, draw and apply a net interval error map $\mathcal{E}$, and then repeat.

We observe that if the strategy just described succeeds, then long-term tracking with noise also need be no more time consuming than usual tracking. And again, if the usual tracking can be speeded up by the use of one-turn or few lump maps, then so can tracking with noise.

6 FUTURE WORK

There is much future work to be done on map methods which ranges from continuing and extending past work, through developing better tools, to fantastic dreams.

With regard to the development of better tools, it appears possible to develop better TPSA algorithms and better map production and manipulation algorithms that will be at least an order of magnitude faster than those in current wide use.[26] Such routines will greatly shorten the relatively large time currently required for the computation of one-turn maps for large machines. They will also foster the production of better maps that treat more realistic fields (including fringe fields) and more realistic errors.

In addition to the development of better algorithms, it should be possible to develop computer programs that are more modular, easier to maintain, and easier to use. The CLASSIC project (a joint undertaking of SLAC, CERN, Fermilab, Maryland, Colorado, and others) is intended to produce computer programs (written in C^{++} and perhaps other languages) that have all these features and also employ the best possible algorithms.[27]

A third fruitful area for tool development is in the area of producing parallel algorithms and code for vector and multiprocessor computers. Such tools would be of great use in both many-particle simulations and long-term tracking studies.[28]

Let us now dream: Thanks to past advances, we currently know how to *compute* maps to high order and (if enough effort is expended) to what we believe is a high degree of realism for a variety of machines. It is highly desirable, both as a reality check and as a diagnostic tool, to also *measure* machine maps. The feasibility of such measurements, based (at least initially) on good-quality position-monitor data, is currently being explored.[29] Another method being studied is modulating the strengths of various beamline elements to see what effect they have on various measurable beam quantities.[30]

If we can compute nonlinear effects (and believe such computations based on measurements), then we might hope to tailor nonlinear effects in such a way as to produce machines described by nearly integrable maps or by stable maps. This approach is being explored from several points of view.[31,32]

Finally, there is the dream that it will some day be possible to compute all desired quantities, and particularly long-term properties such as dynamic aperture, directly from a knowledge of the one-turn map. The first step in this direction is to compare (after suitable normalization) the Lie coefficients of various maps, and to try to correlate the sizes of these coefficients with observed or computed long-term behavior. This is an empirical approach, and some progress has been made in this direction.[33] A yet more ambitious goal is to develop a fundamental theory that relates the sizes of Lie coefficients for a map to its long-term properties under iteration.

7 CONCLUDING SUMMARY

The long-term goal of map methods is to be able to describe, predict, and control nonlinear properties with the same facility with which we now handle linear properties. Much has been accomplished in this direction, particularly with regard to single-pass systems and short-to-moderate-term behavior in circulating systems. It is known that once-differentiable symplectic maps (and probably even analytic symplectic maps) *generically* have simultaneously hyperbolic fixed points, elliptic fixed points, and homoclinic points that are all *everywhere dense* in phase space.[34] Consequently, the detailed long-term behavior of most symplectic maps under repeated iteration must be complicated beyond comprehension. However, there is still the hope that it may be possible to compute gross long-term properties: the rough size of the dynamic aperture, approximate (but useful) lower bounds on the life time for some sizeable fraction of a circulating beam, etc.

Acknowledgements

The author thanks Dan Abell for preparing the figures and many useful suggestions. This work was supported in part by DOE grant DEFG0296ER40949.

References

[1] Giorgilli, A. (1979). *Comp. Phys. Comm.*, **16**, 331. The various MARYLIE charged particle beam transport codes from their inception in 1982 have employed TPSA similar to that described by Giorgilli.

[2] Berz, M. (1989). *Part. Accel.*, **24**, 109.

[3] Hori, G. (1966). Theory of general perturbations with unspecified canonical variables, Publications of the *Astronomical Society of Japan*, **18**, 287.

[4] Deprit, A. (1969). *Celest. Mech.*, **1**, 12.

[5] Dragt, A.J. (1979). A Method of Transfer Maps for Linear and Nonlinear Beam Elements, *IEEE Transactions on Nuclear Science*, **NS-26**, No. 3, 3601; Lectures on Nonlinear Orbit Dynamics, Physics of High Energy Particle Accelerators, edited by Carrigan, R.A. *et al.*, AIP Conf. Proc. No. 87 (1982); *Ann. Rev. Nuc. Part. Sci.*, **38**, 455 (1988); Lectures on Nonlinear Dynamics and Lie Methods with Applications to Accelerator Physics, U. of Maryland Physics Department (1996); Forest, E., A New Attitude and Framework for Beam Dynamics, monograph in preparation (1996).

[6] Forest, E. (1984). Lie Algebraic Methods for Charged Particle Beams and Light Optics, University of Maryland Physics Department Ph.D. Thesis.

[7] Dragt, A.J. (1985). Nonlinear Lattice Functions, Proceedings of 1984 Summer Study on the Design and Utilization of the Superconducting Super Collider, Snowmass, Colorado, Donaldson, R. and Morfin, J. (eds.), published by the Division of Particles and Fields of the American Physical Society.

[8] Douady, A. and Hubbard, J. (1984–85). Étude dynamique des polynômes complexes, Publications mathématiques d'Orsay, Université de Paris-Sud.

[9] Carleson, L. and Gamelin, T.W. (1993). *Complex Dynamics*, Springer-Verlag.

[10] The reader has probably seen the Mandelbrot set shown as a cardiod-shaped mainland with a fractal structure of surrounding "islands". That shape arises from iterating the equation $\bar{z}_{n+1} = \bar{z}_n^2 + c$ (and looking at the plane of complex c). But this equation and (18) are related by the simple rules $\bar{z} = -\gamma(z - 1/2)$ and $c = (\gamma/2)(1 - \gamma/2)$.

[11] More precisely, (30) is a variant of the Hénon map that is related to the usual Hénon map by a linear change of variables.

[12] Hubbard, J.H. and Oberste-Vorth, R.W. (1994). Hénon Mappings in the Complex Domain I: The Global Topology of Dynamical Space, Institut Des Hautes Études Scientifiques Publications Mathématiques 79; II: Projective and Inductive Limits of Polynomials, in Real and Complex Dynamical Systems, Branner, B. and Hjorth, P. (eds.), NATO ASI Series C: Mathematical and Physical Sciences Vol. 464 (Kluwer, Amsterdam 1995).

[13] Irwin, J. (1996). Using Lie Algebra Maps for the Design and Operation of Colliders, this issue of Particle Accelerators.

[14] Forest, E., Berz, M. and Irwin, J. (1989). Normal Form Methods for Complicated Periodic Systems: A Complete Solution Using Differential Algebra and Lie Operators, *Particle Accelerators*, **24**, 91.

[15] Warnock, R.L. and Berg, J.C. (1996). Fast Symplectic Mapping, Quasi-Invariants, and Long-Term Stability in the LHC, this issue of Particle Accelerators.

[16] Hoffstätter, G.H. (1994). Rigorous estimate of survival times in storage rings and efficient computation of fringe-field transfer maps, Michigan State University Department of Physics and Astronomy Ph.D. Thesis; Berz, M. and Hoffstätter, G.H., Exact estimates of the long term stability of weakly nonlinear systems applied to the design of large storage rings, *Interval Computations*, **2**, 68 (1994); Berz, M. and Hoffstätter, G.H., Computation and Application of Taylor Polynomials with Interval Remainder Bounds, Interval Computations, submitted (1995); Fuchi, K. and Berz, M., Remainder Differential Algebras and their Applications, Computational Differentiation: Techniques, Applications and Tools, Berz, M., Bischof, C., Corliss, G. and Griewank, A. (eds.), SIAM (1996); Berz, M., Hoffstätter, G., Wan, W., Shamesddine, K. and Fuchi, K., COSY Infinity and its Applications to Nonlinear Dynamics, Computational Differentiation: Techniques, Applications and Tools, Berz, M., Bischof, C., Corliss, G. and Griewank, A. (eds.), SIAM (1996).

[17] Dragt, A.J. and Abell, D.T. (1996). Symplectic Maps and Computation of Orbits in Particle Accelerators, to appear in Fields Institute Communications Volume on Integration Algorithms for Classical Mechanics.

[18] Douglas, D.R. and Dragt, A.J. (1983). Lie Algebraic Methods for Particle Tracking Calculations, in *Proceedings of the 12$^{\text{th}}$ International Conference on High Energy Accelerators*, Cole, F.T. *et al.* (eds.), Fermilab. See also reference 17.

[19] Yan, Y.T. *et al.* (1993). Performance of an Algorithm for Symplectic Implicit One-Turn Map Tracking, SSC Laboratory Report SSCL-157.

[20] Yan, Y.T., Irwin, J. and Chan, T. (1996). Resonance Basis Maps and n-PB Tracking, this issue of Particle Accelerators.

[21] Irwin, J. private communication.

[22] Irwin, J., A Multi-Kick Factorization Algorithm for Nonlinear Maps, in Accelerator Physics at the Superconducting Super Collider, (Yan, Y. Naples, J. and Syphers, M. (eds.), *AIP Conf. Proc.*, 326 (1995). Originally written in 1989 as report SSC-228.

[23] Schmidt, F. private communication.

[24] Abell, D.T. (1995). Analytic Properties and Cremona Approximation of Transfer Maps for Hamiltonian Systems, University of Maryland Physics Department Ph.D. Thesis.

[25] Laskar, J. and Robin, D. (1996). Application of Frequency Map Analysis to the ALS, this issue of Particle Accelerators.

[26] Dragt, A.J. *et al.* (1995). Notes from SLAC Discussion and Working Group on Map and Lie Algebraic Algorithms; Computation of Maps for Particle and Light Optics by Scaling, Splitting, and Squaring, *Phys. Rev. Lett.*, **75**, 1946; Design of Optimal Truncated Power Series Algebra Routines: I. Monomial Indexing Schemes and Linked Lists and II. Computing Sums and Ordinary and Lie Products of Polynomials Using Monomial Indexing or Linked Lists, University of Maryland Physics Department preprints (1996).

[27] Iselin, F.Ch. (1996). The CLASSIC Project, this issue of Particle Accelerators.

[28] Ryne, R.D. and Habib, S. (1996). Beam Dynamics Calculations and Particle Tracking Using Massively Parallel Processors, this issue of Particle Accelerators.

[29] Irwin, J. and Wang, C.-X. (1996). Private communication and manuscript in progress.

[30] Ziemann, V. (1996). Measuring Hamiltonian Coefficients with a Wobbling Method, this issue of Particle Accelerators.

[31] Cary, J.R. (1996). Reducing Chaos in Four-Dimensional Symplectic Maps, this issue of Particle Accelerators.

[32] Channell, P.S. (1991). Stable Symplectic Maps, *J. Math. Phys.*, **32**, 408; Stable Symplectic Maps Near Arbitrary Lattice Maps, in Accelerator Physics at the Superconducting Super Collider (Yan, Y., Naples, J. and Syphers, M. (eds.), *AIP Conf. Proc.*, 326 (1995).

[33] Irwin, J., Private communication. See also references 13 and 20.

[34] Xia, Z. (1994). Arnold Diffusion and Oscillatory Solutions in the Planar Three-Body Problem, J. of Differential Equations **110**, 289; How Complicated is a Typical Hamiltonian System?, University of Maryland Monroe Martin Prize Lecture (1996).